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Approximating C^∞ Nowhere Analytic Functions

Contents

1 Introduction

At the heart of approximation theory is the idea that the smoother a function, the faster the rate of convergence of its approximant. For the case of Chebyshev approximation we have theorems regarding the convergence of Chebyshev interpolants and Chebyshev projections for functions of varying smoothness. The first degree of smoothness we consider is that of finite differentiability and the second is the stronger notion of analyticity. For these we review the literature regarding the rate of convergence of their respective Chebyshev interpolants and projections. We note early that differentiable functions have algebraic convergence, and analytic functions have geometric convergence. These first sections are largely a review of results by Trefethen [7] and we use them as a basis to consider the convergence of functions of a third, in between, degree of smoothness, that is functions which are infinitely differentiable, say of class C^∞, but nowhere analytic, meaning at no points in their domain do their Taylor series representations converge to the correct function. For such functions we adopt the name *'Smoothies'*, as given by Trefethen [8]. We will present a detailed review of the literature regarding the construction of these function, and provide both deterministic and probabilistic examples. We finally go on to observe numerically the convergence of Chebyshev interpolants of these functions using Chebfun. We implement several examples of Smoothies and analyse errors in the standard infinity norm. We observe and conclude that the examples we consider do indeed display a nature of convergence that is fitting of an in-between degree of smoothness between the previous two types, that is root-exponential. For numerical simulations we make use of the MATLAB based software Chebfun, see `https://www.chebfun.org/`.

Before analysing any functions, we provide some definitions that will be fundamental. It should be noted that for a unique Chebyshev series representation of a function to exist on $[-1, 1]$ we require only Lipschitz continuity [7], which we shall assume from now on since we are focusing on functions which are at least continuously differentiable on this interval and so their derivatives must also be bounded. So supposing Lipschitz continuity of a function $f(x)$ on $[-1, 1]$, we know it can be written as the following absolutely and uniformly convergent infinite sum

$$f(x) = \sum_{k=0}^{\infty} a_k T_k(x) \tag{1}$$

for Chebyshev coefficients

$$a_k = \frac{2}{\pi} \int_{-1}^{1} \frac{f(x) T_k(x)}{\sqrt{1-x^2}}\, dx \ , \quad k \geq 1 \tag{2}$$

where for $k = 0$ we have the same formula with an extra $1/2$ prefactor. And $T_k(x)$ is the kth Chebyshev polynomial of the first kind, that is

$$T_k(x) = \cos(k \arccos(x)). \tag{3}$$

We first present the approximation given by the truncation of (1) to finitely many points.

Definition 1.1 (Chebyshev projection/truncation). *We define the Chebyshev projection to* (1) *by the following finite sum*

$$f_n(x) = \sum_{k=0}^{n} a_k T_k(x) \tag{4}$$

Though this is certainly an important definition, the neccesisty to calculate the integral (2) at each iteration makes it unfavourable for computations. Instead, for computations, we employ the default type of approximation in Chebfun, that is interpolation.

Definition 1.2 (Chebyshev interpolation). *We define the Chebyshev interpolant to* (1) *by the following finite sum*

$$p_n(x) = \sum_{k=0}^{n} c_k T_k(x) \tag{5}$$

For these we employ an aliasing formula to find the coefficients c_k, and this can be done conveniently by, say, the Fast Fourier Transform. The next two results we state are fundamental to many of the arguments we present. We have that the errors in projection and interpolation are given by the following absolutely convergent series respectively [7]:

$$f(x) - f_n(x) = \sum_{k=n+1}^{\infty} a_k T_k(x) \tag{6}$$

$$f(x) - p_n(x) = \sum_{k=n+1}^{\infty} a_k (T_k(x) - T_m(x)) \tag{7}$$

where in (7) m can be predetermined and $0 \leq m \leq n$.

2 Convergence for Differentiable Functions

In this section we motivate and state the theorems by Trefethen [7] regarding the covergence of approximants to functions which are finitely differentiable. We must be careful when defining such functions; one could naively let $f(x) \in C^\nu([-1,1])$ for finite ν and proceed to bound the Chebyshev coefficients and hence find the convergence for these functions, however in doing so we may lose an order of convergence. To demonstrate this we shall look at a theorem regarding the convergence of the best approximation rather than interpolation or projection, as this is what is heavily considered in the literature, though we would expect the same order of convergence.

We consider Jackson's Fourth Theorem [3,5], which states that for $f \in C^\nu([-1,1])$, we have the following bound on the best approximation f^* to f.

$$||f - f_n^*|| \leq \frac{\pi}{2}(\frac{1}{n+1})^\nu ||f^{(\nu)}|| \tag{8}$$

Now consider the following example. Let $f(x) = |x|^p$ for $x \in [-1,1]$ and p an odd number. This function is $p-1$ times continuously differentiable on $[-1,1]$ with discontinuity in the pth derivative, and so we would classify this function as C^{p-1}. So by (8) we would have $\mathcal{O}(n^{1-p})$ convergence for the best approximation. This, however, is not what is observed in numerical simulations. As Figure 1 shows we have $\mathcal{O}(n^{-p})$ convergence for Chebyshev interpolation, and we would expect similar results for the best approximation.

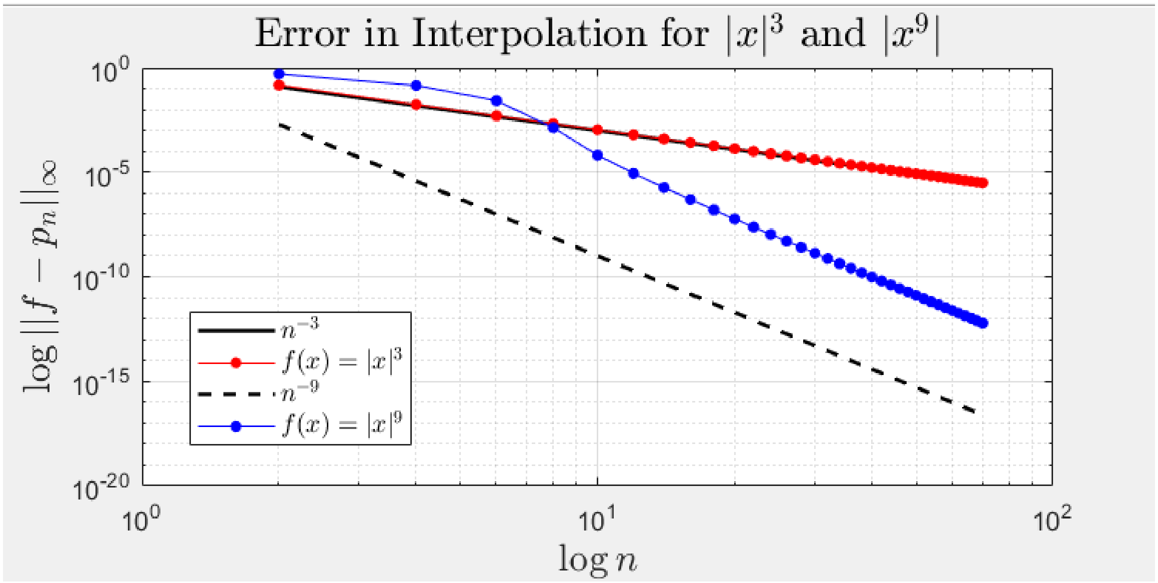

Figure 1: Errors for interpolation of $f(x) = |x|^3$ and $f(x) = |x|^9$ in the infinity norm. We see $\mathcal{O}(n^{-3})$ and $\mathcal{O}(n^{-9})$ convergence respectively, one order better in each case than what Jackson's Theorem would predict.

In general, Jackson's Theorem gives an underestimate in the rate of convergence of approximations to some C^ν functions whose $(\nu+1)$th derivative is discontinuous. This is because even if this derivative is discontinuous, it having a well behaved oscillatory behaviour is enough to ensure the extra degree of convergence for the function. We now state a more rigorous definition of this.

Definition 2.1 (Total Variation)**.** *Let f be defined on $[a, b]$. The Total Variation $V(f)$ of f is defined by*

$$V(f) := \sup \sum_i |f(x_{i+1}) - f(x_i)|$$

over all sequences of ordered points $a = x_1 < x_2 < \ldots < x_n = b$.

Definition 2.2 (Bounded Variation)**.** *Let f be defined on $[a, b]$. f is of bounded variation if $V(f)$ is finite.*

It turns out that the supposition that the function we are considering is $\nu - 1$ times continuously differentiable, with the νth derivative being of bounded variation is enough to ensure $\mathcal{O}(n^{-\nu})$ convergence of interpolation and projection.
With this notion we can now state the theorems by Trefethen [7] regarding the algebraic convergence of interpolation and projection of such functions. We first need to bound the Chebyshev coefficients so we can employ (6) and (7) to bound the errors.

Theorem 2.1 (Chebyshev coefficients of differentiable functions)**.** *For some $\nu \in \mathbb{N}_0$, let $f, f', f'', \ldots f^{(\nu-1)}$ be absolutely continuous on $[-1, 1]$ and suppose $V(f^{(\nu)}) = V < \infty$. Then for $k \geq \nu + 1$, the Chebyshev coefficients of f satisfy*

$$|a_k| \leq \frac{2V}{\pi(k-\nu)^{\nu+1}} \tag{9}$$

The proof of this theorem relies on transplantation to the unit circle and using the Fourier analogue of the Chebyshev representation of the function. This gives a more simple integral than (2) over the unit circle which can be found by repeated integration by parts, until the integral involves the νth derivative which is assumed to be of bounded variation. The bound follows by invoking Hölder's inequality on the resulting integral representation of the coefficient.

Following from this bound, we can now state the theorem regarding the convergence of such functions [7].

Theorem 2.2 (Convergence for analytic functions). *For some $\nu \in \mathbb{N}_0$, let $f, f', f'', \ldots f^{(\nu-1)}$ be absolutely continuous on $[-1,1]$ and suppose $V(f^{(\nu)}) = V < \infty$. Then for any $n > \nu$, the Chebyshev projections satisfy*

$$||f - f_n|| \leq \frac{2V}{\pi\nu(n-\nu)^{\nu}} \tag{10}$$

and the Chebyshev interpolants satisfy

$$||f - p_n|| \leq \frac{4V}{\pi\nu(n-\nu)^{\nu}} \tag{11}$$

The proof of this theorem follows from (6), (7) and (9). Noting that fact that $|T_k(x)| \leq 1$ and $|T_k(x) - T_m(x)| \leq 2$ gives light to the extra factor of 2 in (11).
So we can conclude that these differentiable functions have algebraic convergence rates, $\mathcal{O}(Vn^{-\nu})$, for their Chebyshev interpolants and projections, and the smoother the function, the faster the rate of convergence. We next look at the ideal functions, namely those that are analytic.

3 Convergence for Analytic Functions

Now we not only let f be infinitely differentiable, but also analytic on $[-1,1]$. We refer to analyticity at a single point to mean the function has a convergent Taylor series at that point and in a neighbourhood of that point. The fact that convergence is also required in a neighbourhood of the point is a restatement of the idea of a function being analytically continuable - we can stretch the region of analyticity. We next define the Bernstein Ellipse which extends the idea of analytic continuation of a point to that of an interval. Analogously to a point having a circular region in which it is analytic, we can say that an interval has an ellipsoidal region. Using the notions from [7], we define such a region for $[-1,1]$.

Definition 3.1 (Bernstein Ellipse). *For some $\rho \in \mathbb{R}$, $\rho > 1$, we define the Bernstein Ellipse E_ρ to be the open region bounded by the image of the circle $|z| = \rho$ under the Joukowsky map $x = \frac{z+z^{-1}}{2}$ in the complex x-plane.*

This describes an ellipse centred at the origin, with foci at $x = \pm 1$, and whose semi-minor axis and semi-major axis sum to ρ. As ρ shrinks to 1 the region becomes the interval $[-1,1]$ and as ρ grows to ∞ the region becomes the entire complex plane. As such we can control the value ρ to be arbitrarily close to the interval or infinitely far from it. We now present the theorems, again by Trefethen [7] regarding the bounds on the Chebyshev coefficients of such functions and on the errors in their interpolants and projections.

Theorem 3.1 (Chebyshev coefficients of analytic functions)**.** *Let f be analytic on $[-1,1]$ and analytically continuable to the open Bernstein ellipse E_ρ. Also, let f satisfy $|f(x)| \leq M$ for some $M \in \mathbb{R}$. Then its Chebyshev coefficients satisfy*

$$|a_0| \leq M$$

$$|a_k| \leq 2M\rho^{-k}, \quad k \geq 1 \tag{12}$$

The general principle that is used to bound the Chebyshev coefficients is to exploit the analyticity of f on $[-1,1]$. More precisely, if f is analytic on $[-1,1]$, then it must also be analytic on some neighbourhood of $[-1,1]$ in the complex plane, and hence analytically continuable to some Bernstein Ellipse E_ρ because we can make ρ arbitrarily close to 1 and hence make the ellipse arbitrarily close to the interval. The key factor is the size of ellipse that the function be analytically continued to.

Theorem 3.2 (Convegence for analytic functions)**.** *Let f be analytic on $[-1,1]$ and analytically continuable to the open Bernstein ellipse E_ρ. Also, let f satisfy $|f(x)| \leq M$ for some $M \in \mathbb{R}$. Then, for each $n \geq 0$ its Chebyshev projections satisfy*

$$||f - f_n|| \leq \frac{2M\rho^{-n}}{\rho - 1} \tag{13}$$

and its Chebyshev interpolants satisfy

$$||f - p_n|| \leq \frac{4M\rho^{-n}}{\rho - 1} \tag{14}$$

In a similar fashion the previous section, the proof of this follows from (6), (7) and (12). From this we can conclude that analytic functions have geometric convergence rates, $\mathcal{O}(M\rho^{-n})$, for their Chebyshev interpolants and projections, and the larger an ellipse the function can be analytically continued into, the greater this exponential rate. Consequently functions which have poles closer to $[-1,1]$ show slower geometric convergence than those with poles further away, it follows that entire functions are a best case scenario.

4 Constructing Smoothies

To consider the notion of non analyticity at all points in the domain we must consider what it means for a C^∞ function to not be analytic at a single point. For a function

$f(x) \in C^\infty$, real analyticity at a point x_0 means that the Taylor series representation of $f(x)$ about x_0 converges to $f(x)$ not only at x_0, but also in a neighbourhood of x_0. There are two ways in which $f(x)$ fails to be analytic at x_0, Bilodeau [2] classifies these as *Type I* and *Type II* singularities[1]:

Type I: The Taylor series representation of $f(x)$ converges in some neighbourhood of x_0, but never to $f(x)$

Type II: The Taylor series representation of $f(x)$ is divergent in any neighbourhood of x_0

One can also think in terms of the radius of convergence of the Taylor series of $f(x)$ about x_0. Functions with a *Type I* singularity have a non zero radius of convergence at x_0, but this convergence is not to $f(x)$ except at $x = x_0$. Functions with a *Type II* singularity have a zero radius of convergence at x_0, so the Taylor series about x_0 only converges at $x = x_0$.

Example 4.1. *[Cauchy 1823, cited in [2]] Consider the standard example of a C^∞ function that fails to be analytic at a single point*

$$f(x) = \begin{cases} \exp(\frac{-1}{x^2}) & x \neq 0 \\ 0 & x = 0. \end{cases} \tag{15}$$

This function has a type I singularity at $x = 0$. It is easily seen that all derivatives vanish at $x = 0$, but the Taylor series about $x_0 = 0$ converges to the zero function for all x, hence it only converges to $f(x)$ at $x = 0$ and certainly not in any neighbourhood of this point.

Example 4.2 (Pringsheim 1893, cited in [2])**.** *Consider the C^∞ function*

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!(1 + a^n x)}, \quad a > 1. \tag{16}$$

[1]Common in the literature is to classify these instead as C and P points respectively, after Cauchy who gave the first example of a function with the first type, and Pringsheim who gave the first example of a function with the second kind whose derivatives was easily calculable (Du Bois-Reymond gave the first example of the second type [2]).

This function has a type II singularity at $x = 0$, since

$$\begin{aligned} f^{(k)}(x) &= \sum_{n=0}^{\infty} \frac{(-1)^k k! a^{kn}}{n!(1 + a^n x)^{k+1}} \\ \Rightarrow \quad f^{(k)}(0) &= \sum_{n=0}^{\infty} \frac{(-1)^k k! a^{kn}}{n!} \\ &= (-1)^k k! \sum_{n=0}^{\infty} \frac{(a^k)^n}{n!} \\ &= (-1)^k k! e^{a^k} \end{aligned}$$

and hence its Taylor series about $x_0 = 0$ is given by

$$\begin{aligned} T(x) &= \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k \\ &= \sum_{k=0}^{\infty} (-1)^k e^{a^k} x^k \end{aligned}$$

which diverges for all $x \neq 0$ by the root test and the condition on a, hence $x_0 = 0$ is a singularity of Type II.

4.1 Condensation of Singularities

This method of constructing Smoothies is arguably the most intuitive however the resulting functions are possibly the least applicable in terms of computing, we therefore briefly outline the literature regarding this formulation.

Bilodeau notes that this method was initially coined by Hankel in 1870 but generalised by Cantor in 1882 [2]. The idea is to consider a function which is known to be C^∞ everywhere, and analytic on the entire domain except at a single point. We then take a linear sum of translations of this function in a convenient way so as to increase the density of (condense) the number of singularities on the relevant interval, until an arbitrary neighbourhood of each singularity contains another singularity, and hence the function can no longer be analytic anywhere on the interval.
We consider the theorem by Walczak [9] which formalises this construction. Let $r = \{r_1, r_2, \ldots\} \subseteq \mathbb{R}$ be a countable set of distinct elements and let $\{a_n\}$ be a sequence of non-zero real numbers such that $a = \sum_{n=1}^{\infty} |a_n|$ converges. Then we have the following theorem.

Theorem 4.1. *Let $\varphi(x)$ be a bounded C^∞ function which is analytic everywhere except at $x = 0$. If there are real numbers $\delta, N, M > 0$ and $k \in \mathbb{N}_0$ such that*

$$|\varphi^{(k)}(x)| < \frac{Lk!}{\delta^k} \tag{17}$$

for all $|x| > N$ and k, then the function

$$f(x) = \sum_{n=1}^{\infty} a_n \varphi(x - r_n) \tag{18}$$

is C^∞ but not analytic at every point in the set r.

Here, the condition (17) is made to ensure that there is uniform convergence of the derivatives of the series to those of f. One could tighten the requirement on r so that it is also dense on an interval $I \subset \mathbb{R}$. Then for some $\varphi(x)$ that meets the conditions in the theorem, (18) would describe a Smoothie on I. An example of such a Smoothie would be to take $\varphi(x)$ as (16) and r as an enumeration of the rational numbers, the resulting function would be a Smoothie on $\mathbb{R}$ since it would have *type II* singularities at each rational number, and these form a countable dense set in R.

Though it does not strictly follow the condensation of singularities formulation, the following Smoothie is convenient to place here as it also has *type II* singularities at all points in a dense set in $\mathbb{R}$.

Example 4.3 (Cellérier 1890, cited in [2])**.** *Consider the function*

$$f(x) = \sum_{n=1}^{\infty} \frac{\sin(a^n x)}{n!}, \quad a > 1. \tag{19}$$

This is a Smoothie since it has type II singularities at all points of the form

$$x = \frac{2\pi m}{a^k}, \quad m \in \mathbb{Z} \text{ and } k \in \mathbb{N}_0$$

We could verify that there are indeed type II singularities at such points by finding the derivatives, and inserting these into the Taylor series to see that it diverges for $x \neq (2\pi m)/a^k$ in a similar fashion to Example 4.2. We instead prove the important fact that such points form a dense set in $\mathbb{R}$, which will also be of use when we discuss Lacunary Fourier Series.

Theorem 4.2. *The set of rational numbers of the form*

$$\frac{m}{a^k} \tag{20}$$

for $m \in \mathbb{Z}$, $a \in \mathbb{N}_{\geq 2}$ and $k \in \mathbb{N}$ is dense in $\mathbb{R}$

To prove this we will show that given an arbitrary interval $[A, B] \in \mathbb{R}$, we can always find a rational of the form (20) within it.

Proof. Given an arbitrary interval $[A, B] \in \mathbb{R}$, we can always find $k \in \mathbb{N}$ such that $0 < 1/k < B - A$, but for all $a \geq 2$ we have that $a^k > k$, so in particular we have $0 < 1/a^k < B - A \Rightarrow 1 < (B - A)a^k$, so the interval $[a^k A, a^k B]$ has length greater than 1 and so it must contain an integer, say m. Then we have $a^k A < m < a^k B$ which gives us the required result that $A < m/a^k < B$, hence such numbers are dense in $\mathbb{R}$. $\square$

4.2 Lacunary Fourier Series

We now consider Smoothies defined by Lacunary[2] Fourier Series. The fact that these are defined by Fourier series makes them very convenient for computing. Unfortunately the lack of literature on these types of Fourier series means a general form has yet to be proved to be C^∞ and nowhere analytic. We hypothesise a generalisation of these functions and consider the special case for powers of two for which proofs are available. We define a Lacunary Fourier Series in this context in the following way.

Definition 4.1 (Lacunary Fourier Series)**.** *A Lacunary Fourier Series is defined by the following Fourier Series*

$$f(x) = \sum_{k=0}^{\infty} \alpha^{-\sqrt{\beta^k}} \cos(\beta^k x) \qquad x \in [-1, 1] \tag{21}$$

for $\alpha \in \mathbb{R}$, $\alpha > 1$ *and* $\beta \in \mathbb{Z}$, $\beta \geq 2$.

Theorem 4.3. *The Lacunary Fourier Series described by* (21) *is a Smoothie on* $[-1, 1]$ *for* $\alpha = e$ *and* $\beta = 2$

An example of a proof for this theorem is given in [6]. The proof uses the Weierstrass M-test inductively to show that f is of class C^∞ and then shows that there are *type II* singularities at all $x = (\pi m)/2^k$. It then makes use of Theorem 4.2 for the case $a = 2$ to use that such points are dense in $\mathbb{R}$ and hence in $[-1, 1]$ and so the function fails to be analytic anywhere, therefore it is a Smoothie.

[2] In general the term Lacunary refers to the fact that the function has a natural boundary, meaning it cannot be analytically continued beyond its radius of convergence [1]. The actual term means 'to have gaps', an appropriate definition as the series by which the function is defined has gaps if we think of indexing by powers of β, in which case the coefficients are only non zero when the index is of the form β^k, and they are zero on the gaps between these values.

One could argue that (21) is in fact a Smoothie in general, by using the general case of 4.2, however care needs to be taken in showing that these points indeed give rise to singularities. We could hypothesise that (21) is a Smoothie when α causes the summand to decrease sufficiently fast, and β results in sufficiently fast oscillations of the summand, though the meaning of 'sufficient' here is certainly ambiguous.

4.3 Probabilistic Smoothies

4.3.1 Fabius Function

Described by Jasper Fabius in his 1965 paper [4], the Fabius function is an example of a Smoothie that has a probabilistic definition.

Definition 4.2 (Fabius Function)**.** *The Fabius Function is defined on the interval* $[0, 1]$ *by the cumulative distribution function of random variables of the form*

$$X = \sum_{n=1}^{\infty} 2^{-n} \xi_n \tag{22}$$

where $\xi_1, \xi_2, \xi_3, \ldots$ *are independent random variables uniformly distributed on* $[0, 1]$ *More explicitly we have that*

$$F(x) = P(X \leq x) \tag{23}$$

The statistical definition of this function allows for reasonably easy computation. As such we shall make use of it for numerical simulations. It should be noted that this is not the only way one could define the Fabius function.

4.3.2 Random Smoothies

These Smoothies are defined in a similar fashion to the Lacunary Fourier series. They are defined by a standard Fourier series with conveniently chosen coefficients, the key property of these coefficients are that they decay faster than the reciprocal of any polynomial but slower than any exponential [8]. In the Chebfun implementation of these, the Fourier coefficients decrease root exponentially.

Definition 4.3 (Random Smoothie [6])**.** *For coefficients* a_k *and* b_k *distributed normally in* $[0, 1]$*, and for real* $C > 1$*, a real-valued random smoothie is given by*

$$f(x) = \sum_{k=0}^{\infty} C^{-\sqrt{k}} (a_k \cos kx + b_k \sin kx) \tag{24}$$

In his dissertation, Park proves that (24) converges almost surely to a Smoothie for the case $C = e$, see [6].

5 Convergence for Smoothies

We now consider the convergence rates of interpolation of some of these different Smoothies. We compute $e_n = ||f - p_n||_\infty$, where p_n denotes the degree n Chebyshev interpolant through $n+1$ Chebyshev points, and f denotes the function in question. We first consider the existing *smoothie.m* function in Chebfun, and observe the interpolation convergence for these types of functions. For functions defined by Lacunary Fourier series (particularly for the $\alpha = e$, $\beta = 2$ case) we predict their convergence similarly to the theorems in the second and third sections. Finally we consider the case of the Fabius function, and see if it fits the results of the previous types. We predict a degree of convergence in between algebraic $\mathcal{O}(n^{-v})$ and geometric $\mathcal{O}(\rho^{-n})$, in particular we fit curves of the type $c_1^{-n^{c_2}}$ for $c_1 > 1$ and $0 < c_2 < 1$ to e_n, which in the high n limit is guaranteed to be in between algebraic and geometric, and this provides a more solid meaning to root exponential convergence, i.e. $\mathcal{O}(c_1^{-n^{c_2}})$.

5.1 Random Smoothies

We first consider the convergence for Random Smoothies as these are readily available in the Chebfun package with the *smoothie.m* function. By (24), we could predict that the Chebyshev coefficients of this function decrease like $C^{-\sqrt{k}}$. The Chebfun implementation opts for the case $C = e$ and offers options for complex valued functions and both periodic and non-periodic functions. The complex case calls the real, periodic case twice to obtain a real and imaginary part and the non-periodic case calls the periodic case for a larger domain and restricts it to the required domain, $[-1, 1]$ by default. Though both the periodic and complex cases give interesting results both mathematically and visually, we consider the convergence of the non-periodic case on $[-1, 1]$. We expect decay of Fourier coefficients like $e^{-\sqrt{k}}$, the implementation actually normalises these with respect to the length of the domain L, so we have

$$a_k \sim \frac{e^{-\sqrt{\frac{k}{L}}}}{\sqrt{L}}$$

and this result is exact up to a random prefactor between 0 and 1 because of the random choice of coefficients in this construction. It should also be noted that for a non-periodic function on $[-1, 1]$, *smoothie.m* actually generates a periodic function on $[-1, 1.4]$ and restricts this to $[-1, 1]$ so $L = 2.4$. Indeed we see this root exponential decay in Figure 2.

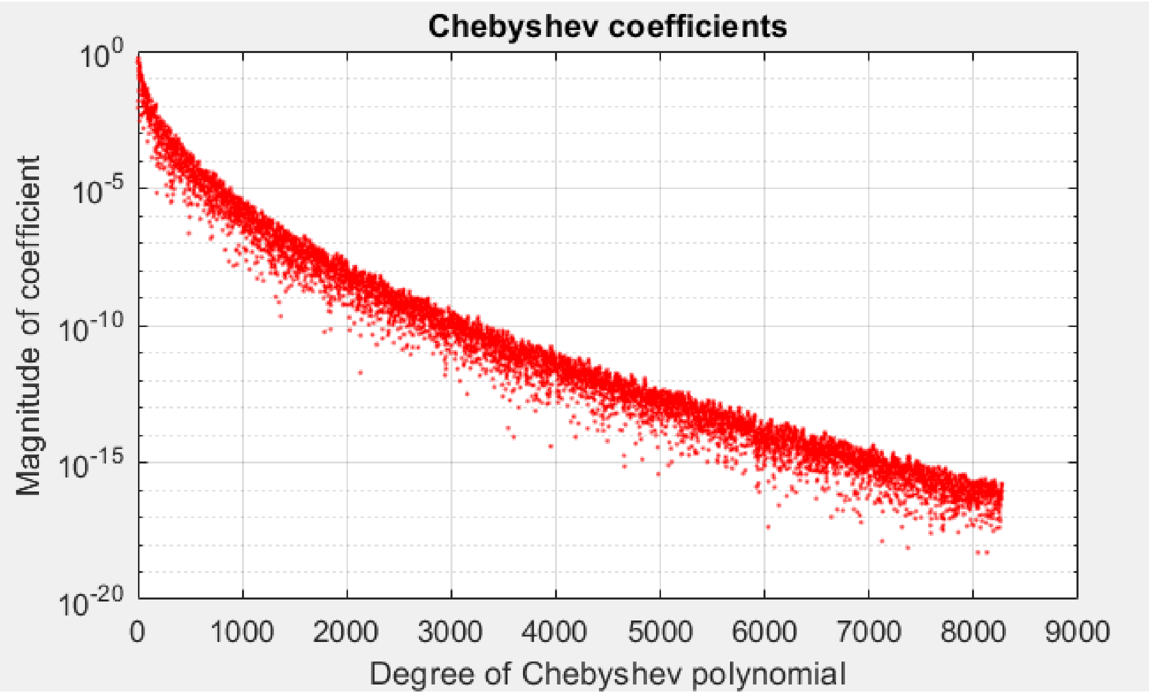

Figure 2: Root exponential decay of the Chebyshev coefficients of a random Smoothie on $[-1, 1]$.

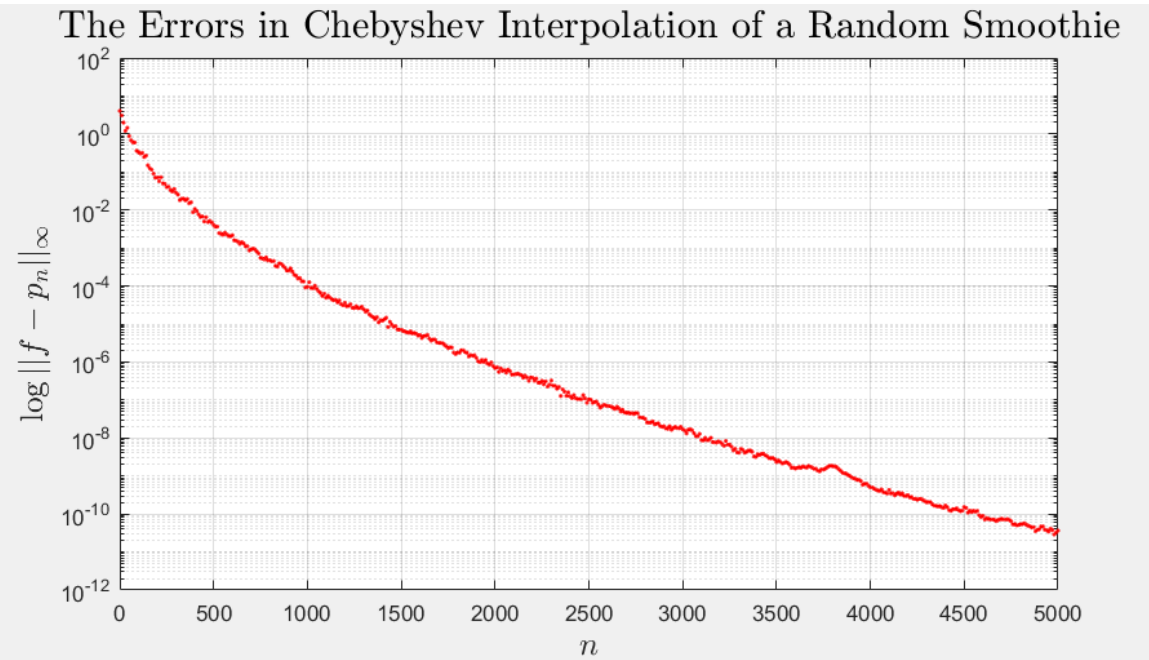

Figure 3: Root exponential decay of the error in the infinity norm of the random Smoothie and its degree n interpolants.

Following from this, we also see root exponential convergence of the Chebyshev interpolants of this function, as shown in 3. This is because the interpolant is bounded by a sum of root exponentially decaying amplitudes.

5.2 Lacunary Fourier Series

We next look at the Lacuary Fourier Series representation of Smoothies. Without even knowing if the general function in (21) is a Smoothie, we can indeed predict the rate of convergence of its Chebyshev interpolants and projections.

Theorem 5.1 (Chebyshev coefficients of Lacunary Fourier Series)**.** *Let f be defined as in* (21)*. Then, for each $k \in \{\beta^0, \beta^1, \beta^2, \cdots\}$ the Chebyshev coefficients of f are given by*

$$|a_k| = \alpha^{-\sqrt{k}} \tag{25}$$

Proof. We denote by $F(\theta)$ the extension of $f(x)$ where instead of $x \in [-1, 1]$, we have $\theta \in [-\pi, \pi]$. We have

$$\begin{aligned} F(\theta) &= \sum_{k=0}^{\infty} \alpha^{-\sqrt{\beta^k}} \cos\left(\beta^k \theta\right) \\ &= \sum_{k=0}^{\infty} \widetilde{a}_k \cos\left(k\theta\right) \end{aligned}$$

where

$$\widetilde{a}_k = \begin{cases} \alpha^{-\sqrt{k}} & k \in \{\beta^0, \beta^1, \beta^2, \cdots\} \\ 0 & \text{else} \end{cases}$$

are the Fourier coefficients of F. We then use the relationship between Fourier series and Chebyshev series, namely

$$F(\theta) = f(\cos(x)) \Leftrightarrow f(x) = F(\cos^{-1}(x))$$

and hence, for $x \in [-1, 1]$ we have

$$\begin{aligned} f(x) &= \sum_{k=0}^{\infty} \widetilde{a}_k \cos\left(k \cos^{-1}(x)\right) \\ &= \sum_{k=0}^{\infty} a_k T_k(x) \end{aligned}$$

So the Chebyshev coefficients a_k and the Fourier coefficients $\widetilde{a}_k$ are equivalent.
Hence, for each $k \in \{\beta^0, \beta^1, \beta^2, \cdots\}$, the Chebyshev coefficients of f are given by

$$a_k = \alpha^{-\sqrt{k}}$$

Noting that the right hand side is always positive and taking absolute values gives the required result. □

Theorem 5.2 (Convergence for Lacunary Fourier Series). *Let f be defined as in* (21). *Then its Chebyshev projections satisfy*

$$||f - f_n|| \leq \sum_{k \in S_n} \alpha^{-\sqrt{k}} \tag{26}$$

and its Chebyshev interpolants satisfy

$$||f - p_n|| \leq 2 \sum_{k \in S_n} \alpha^{-\sqrt{k}} \tag{27}$$

where $S_n = \{\beta^r, \beta^{r+1}, \beta^{r+2}, \cdots | \beta^{r-1} < n+1 \leq \beta^r\}$

Proof. For Chebyshev projections we have

$$\begin{aligned} f - f_n &= \sum_{k=n+1}^{\infty} a_k T_k \\ \Rightarrow \quad ||f - f_n||_\infty &\leq \sum_{k=n+1}^{\infty} |a_k| \end{aligned}$$

Where we have used the fact that $|T_k(x)| \leq 1$. The sum of coefficients start from index $k = n+1$, which means we need to sum $\alpha^{-\sqrt{k}}$ starting from the first power of β larger than or equal to $n+1$, that is over the set

$$S_n = \{\beta^r, \beta^{r+1}, \beta^{r+2}, \cdots | \beta^{r-1} < n+1 \leq \beta^r\}.$$

So we have

$$||f - f_n||_\infty \leq \sum_{k \in S_n} \alpha^{-\sqrt{k}}.$$

By noting that for interpolants we have

$$f - p_n = \sum_{k=n+1}^{\infty} a_k (T_k - T_m)$$

for $0 \leq m \leq n$, and that $-2 \leq T_k - T_m \leq 2$, we gain an extra factor of 2 but the rest of the calculations remain the same. □

So the error in interpolation and projection of these types of functions are given by a sum of root exponential functions, which is also root exponential. First, let us consider the case with $\alpha = e$ and $\beta = 2$, as we know this is a Smoothie,

$$f(x) = \sum_{k=0}^{\infty} e^{-\sqrt{2^k}} \cos(2^k x) \qquad x \in [-1, 1]. \tag{28}$$

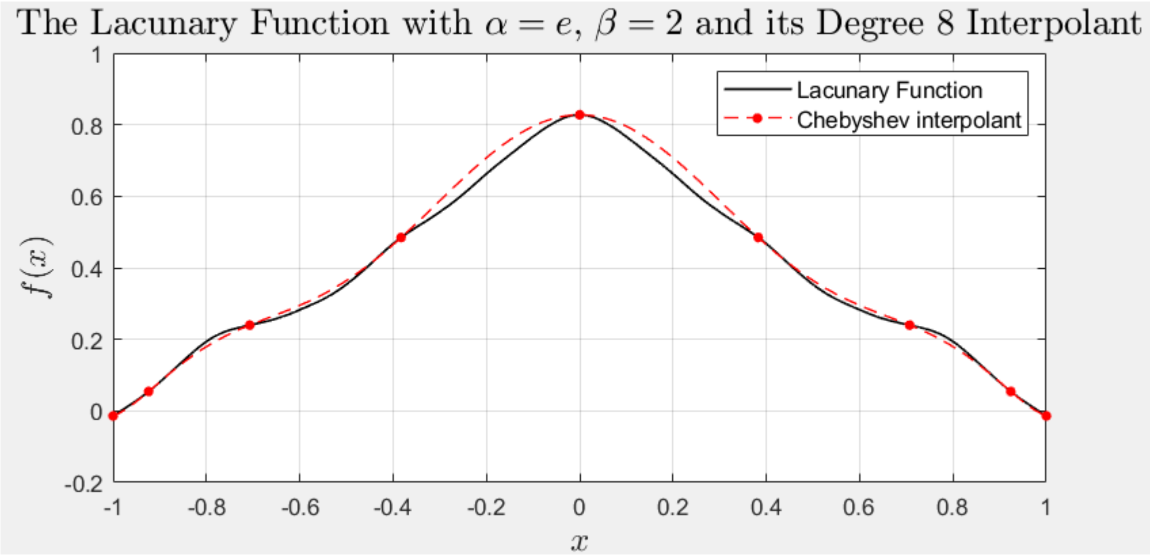

Figure 4: The Lacunary Fourier Series on $[-1, 1]$ for $\alpha = e$ and $\beta = 2$ truncated to $k = 500$, along with its degree 8 Chebyshev interpolant through 9 Chebyshev points.

As before, we now look at the convergence rates of interpolation of (28). We compute the error on a semi-log scale for $0 \leq n \leq 200$.

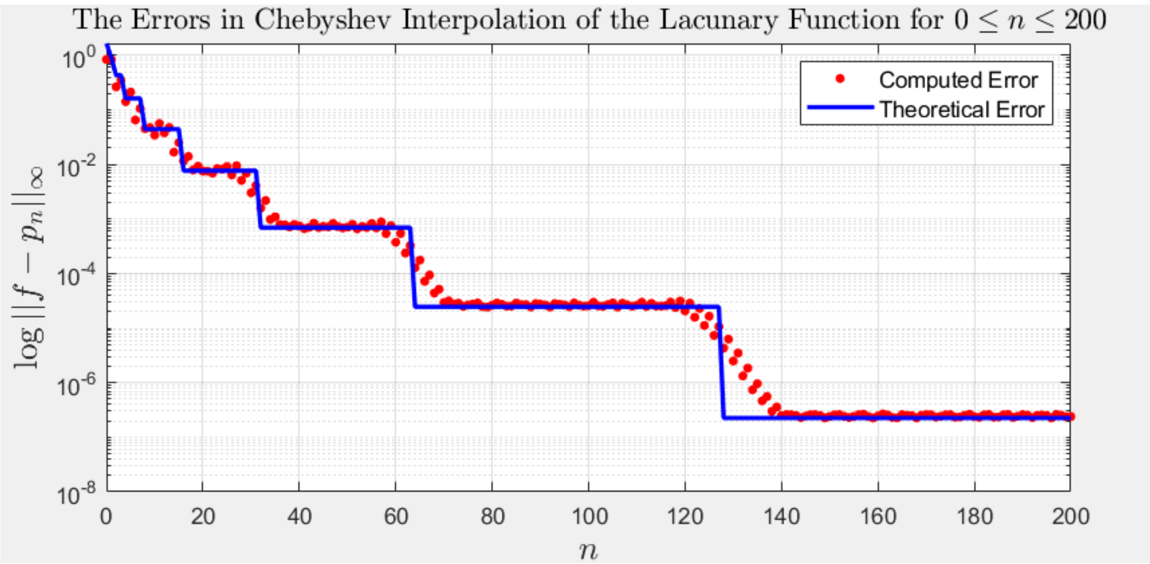

Figure 5: The error plot with an overlay of the theoretical error given by a truncation of (29) from r to $r + 500$ at each n.

We notice steps in what seems to be an otherwise normal root exponentially decaying error. It becomes clear that these are characteristic of Lacunary Fourier series once we examine the expression for the Chebyshev interpolants[3] in (27). We first rewrite the inequality for our special case.

$$||f - p_n|| \leq 2 \sum_{k \in S_n} e^{-\sqrt{k}}, \quad S_n = \{2^r, 2^{r+1}, 2^{r+2}, \cdots | 2^{r-1} < n+1 \leq 2^r\}. \tag{29}$$

From (29) we see that $r = 1$ for $n \in \{0, 1\}$, $r = 2$ for $n \in \{2, 3\}$, $r = 3$ for $n \in \{4, 5, 6, 7\}$ and so on. So the error in interpolation is not unique to each n, but rather to the sets of values of n between powers of two[4]. So in theory a degree 1024 interpolant is just as accurate as a degree 2047 interpolant, but these are much worse than a degree 2048 interpolant . This can easily be seen once we overlay a plot of the right hand side of (29) as a function of n for the same $0 \leq n \leq 200$. These ideas are not unique to the Smoothie defined by (28), they also apply to the general form of the Lacunary Fourier Series (21). We could vary $\alpha > 1$ to change the steepness of the steps and we could vary $\beta \geq 2$ to get longer gaps between the steps, this can be seen in Appendix B.

5.3 Fabius Function

We next consider the Fabius Function, this has been implemented in Chebfun as *fabius.m* and conveniently generates N data points on the first call and compares x values with these. This data consists of N random variables which are truncated sums of the form (22), we truncate to $n = 10^3$ as this gives both an accurate random variable at each point due to the scaling 2^{-n} and also relatively fast computation times.

[3]Of course the projections would give a very similar result but we are considering interpolation as it is the default type of approximation in Chebfun

[4]This echoes the previously mentioned idea of gaps between powers of β.

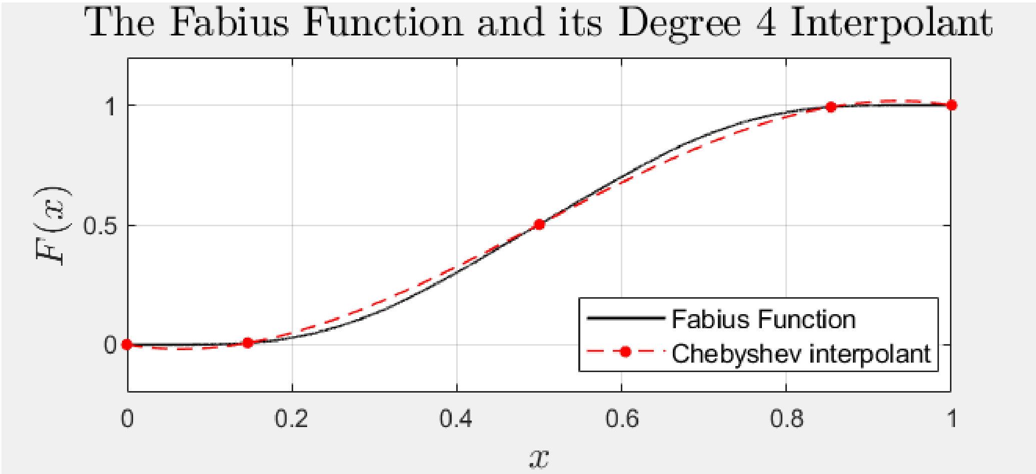

Figure 6: The Fabius function on $[0, 1]$ from a sample size of $N = 10^5$, along with its degree 4 Chebyshev interpolant through 5 Chebyshev points.

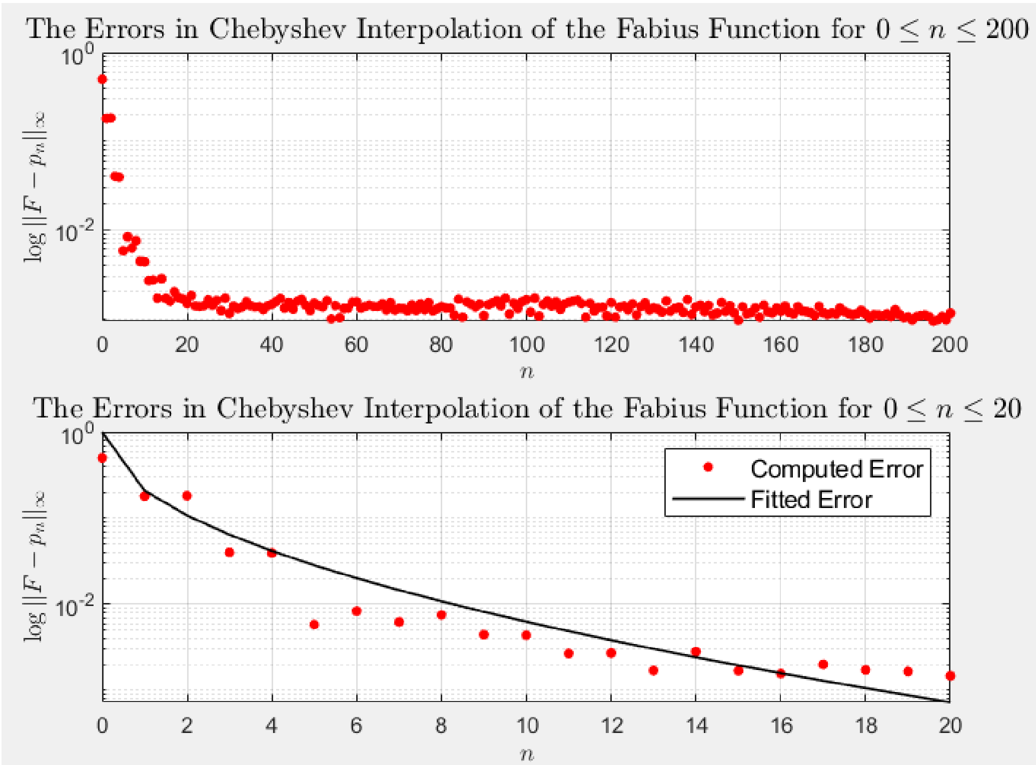

Figure 7: A semi-log plot of n against e_n for the Fabius Function, along with a fitted root exponential curve to 20 data points.

The sample size here is a limiting factor when observing the error. When we determine x values from a sample size $N = 10^4$ the errors begin to level off at degree ~ 15, and this is improves only slightly to ~ 20 with a ten times larger sample. Fitting a curve to these data points, we find that the error decays like $4.78^{-n^{0.51}} \sim 5^{-\sqrt{n}}$. Though a larger data sample would be required to increase our confidence in this, it is certainly a good start in showing that approximations to Smoothies converge root exponentially.

We have seen that both random Smoothies and those defined by Lacunary Fourier series display this root-exponentially decaying error in interpolation (and projection in theory). Also the more abstract example of a Smoothie, the Fabius function, seems to exhibit similar behaviour when being approximated, and though this has not been proven, we hypothesise that Smoothies do in fact show such convergence rates when being approximated, as is beffitting of them being an in between degree of smoothness between finitely differentiable functions and analytic functions.

6 Conclusion

We have reviewed existing theorems regarding the convergence results for both finitely differentiable functions, and analytic functions - they show algebraic and geometric convergence respectively. Through a detailed review of the literature we have motivated the notion of Smoothies as an in between degree of smoothness between the previously mentioned types of functions, and have shown various different types of constructions of these Smoothies. Through numerical simulations we have shown that interpolants to Smoothies tend to exhibit root exponential decay, that is faster than algebraic but slower than geometric, as one would expect for functions with an in bewteen degree of smoothness. More specifically we have shown that the random Smoothies existing in Chebfun show such behaviour. Furthermore we have proved that some specific types of Lacunary Fourier series also follow this type of convergence, and have suggested that perhaps these are Smoothies in general. Finally we considered the Fabius function and implemented it into MATLAB. We have shown that with some leniency regarding the samples sizes used to determine the function, it generally shows similar behaviour to the previously mentioned functions. Thus we conclude that this is likely a general case for appoximants of Smoothies. A possible area of extension would be to see if other such functions also display this convergence, or to prove this rigorously in the general case.

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A MATLAB Code

A.1 Random Smoothie

We make use of the existing *smoothie.m* function in Chebfun, below we have the code used to compute the errors and plot the graphs.

```
%% Initialise the smoothie using the smoothie.m function, plot ...
    Chebyshev coefficients
f=smoothie;
figure;
plotcoeffs(f,'r.');
%
%%  Computing the errors in interpolation of the random smoothie
nn = 0:10:5000;
ee = 0*nn;
for j = 1:length(nn)
    n =nn(j);
    fn=chebfun(f,n+1,'splitting','on');
    ee(j)=norm(f-fn,inf);
end
%% Semi-log plot of the error of interpolant against degree
semilogy(nn,ee,'r.','MarkerSize',4);
xlabel('$n$','Interpreter','latex','FontSize',20)
ylabel('$\log{||f-p_n||_\infty}$','Interpreter','latex','FontSize',20)
title('The Errors in Chebyshev Interpolation of a Random ...
    Smoothie' ,'Interpreter','latex','FontSize',24);
grid on;
```

A.2 Lacunary Fourier Series

This code shows the general Lacunary Fourier Series used in section 5, it allows for varying alpha and beta parameters and graphs for these can be found in Appendix B. Also included is the function used to calculate the theoretical bound from (27). Finally we also include the code used to plot graphs for these functions.

```
function f  = lacunary_smoothie(x,a,b)
% LACUNARY_SMOOTHIE Evaluate Real Valued Lacunary Fourier Series
```

```
% f= lacunary_smoothie returns sum a^(-sqrt(b^k))cos(b^k x), ...
    where a is a
% real number greater than 1 and b is an integer greater than 2. The
% infinite sum is truncated to index 500.
%
%Example:
%
% x = linspace(-1,1,1e5)
% plot(x,lacunary_smoothie(x,exp(1),2)

jlim = 500; %truncating to 500 points
f = 0;
    for j = 0:jlim
        s =(a.^(-(b.^(j/2)))).*cos(x.*b.^j);
        s(isnan(s))=0; % so we don't sum NaN values
        f = f + s;
    end
end
```

```
function S = lacunary_bound(n,a,b)
% LACUNARY_SMOOTHIE Evaluates the theoretical bound in error for ...
    interpolation of
% real valued Lacunary Fourier Series
    p=0;
    while (n+1 > b^p)   % indexing starts from the first power of ...
        b greater than n
        p = p+1;
    end
    S=0;
    for i=p:p+100
        S = S + 2*a^(-sqrt(b^i));
    end
end
```

```
%% Plotting the Lacunary Function and its degree 8 interpolant
a=exp(1);
b=2;
x=linspace(-1,1,5000);
p = @(x) lacunary_smoothie(x,a,b);
f=chebfun(p,9);
```

```
plot(x,lacunary_smoothie(x,a,b),'k','LineWidth',1)
hold on;
plot(f,'r--.','MarkerSize',15,'LineWidth',0.8)
grid on
legend('Lacunary Function ','Chebyshev interpolant','FontSize',20)
xlabel('$x$','Interpreter','latex','FontSize',20)
ylabel('$f(x)$','Interpreter','latex','FontSize',20)
title('The Lacunary Function with $\alpha=e$, $\beta = 2$ and its ...
    Degree $8$ Interpolant' ,'Interpreter','latex','FontSize',24);
%%  Computing the errors in interpolation of the function
f = chebfun(p,'splitting','on');
nn = 0:1:200;
ee = 0*nn;
for j = 1:length(nn)
    n =nn(j);
    fn=chebfun(f,n+1);
    ee(j)=norm(f-fn,inf);
end
%% Computing the theoretical error via the lacunary_bound function
ebound=0*nn;
for j=1:length(nn)
    ebound(j) = lacunary_bound(nn(j),a,b);
end
%% Semi-log plot of error of interpolant against degree, ...
    including the theoretical error
figure;
semilogy(nn,ee,'r.','MarkerSize',14);
hold on;
semilogy(nn,ebound,'b','LineWidth',2);
xlabel('$n$','Interpreter','latex','FontSize',20)
ylabel('$\log{||f-p_n||_\infty}$','Interpreter','latex','FontSize',20)
title('The Errors in Chebyshev Interpolation of the Lacunary ...
    Function for $0\leq n \leq 200$' ...
    ,'Interpreter','latex','FontSize',24);
legend('Computed Error ','Theoretical Error','FontSize',20)
grid on;
```

A.3 Fabius Function

This code shows the implementation of the Fabius function used in section 5. It includes the *fabius.m* file as well as the code used to find errors and plot all graphs.

```
function X = fabius(x,N)
%FABIUS Evaluate the Fabius Function
% X= fabius(x,N) returns the Fabius function evaluated at x, ...
    corresponding
% to a sample size of random variables of size N. Size of N is ...
    recommended
% to be 10^4 < N < 10^5. The data set corresponding to size N ...
    will only be
% calulcated on the first call of the function - this will take ...
    longer for
% larger N on the first call but will be instant on subsequent ...
    calls. any x
% input is allowed but note that the function is typically ...
    defined on [0,1]
% so x<0 returns 0 and x>1 returns 1.
%
% NOTE: The function will take a long time on the first call for ...
    a given N
%
%Examples:
% fabius(0.5,1e4)
%
% x=linspace(0,1,1e4);
% plot(x,fabius(x,1e5));
    persistent fabiusdata;     % we define the data set as a ...
        persistent variable
    if isempty(fabiusdata) || length(fabiusdata) ≠ N % Computing ...
        the data
        % We only want to compute the data to compare x with if ...
            either it
        % has not been computed yet, or the length of N has ...
            changed since
        % the last call.
        fabiusdata=zeros(1,N);
        for i=1:N
            fabiusdata(i)=infsum(); % fills in the data vector ...
                with RVs of the form
            % sum r_n * 2^(-n) where r_n is a uniformly ...
                distributed RV on [0,1]
        end
    end
    X = zeros(1,length(x));
```

```
    if isvector(x)  % case for vector input of x - compute scalar ...
        case at each x
        for i =1:length(x)
            logiclx = fabiusdata< x(i); % logical vector marking ...
                entries that are less than x
            numlx = sum(logiclx(:)); % number of entries less ...
                than x
            X(i) = numlx/(N);  % proportion of entries less than x
        end
    elseif isscalar(x)
            logiclx = fabiusdata < x;   % logical vector marking ...
                entries that are less than x
            numlx = sum(logiclx(:)); % number of entries less ...
                than x
            X = numlx/(length(N)); % proportion of entries less ...
                than x
    end
    function y = infsum()   % used to calulcate the data in line 24
        y = 0;
        for n = 1:1e3
            y = y + (2^(-n))*(rand);
        end
    end
end
```

```
%% Plotting the Fabius Function alongside its degree 4 interpolant
x=linspace(0,1,1e4);
p = @(X) fabius(X,1e5);
f= chebfun(p,[0,1],5,'splitting','on');
figure;
plot(x,fabius(x,1e5),'k','LineWidth',1)
hold on;
plot(f,'r--.','MarkerSize',15,'LineWidth',0.8)
grid on
legend('Fabius Function ','Chebyshev interpolant','FontSize',20)
xlabel('$x$','Interpreter','latex','FontSize',20)
ylabel('$F(x)$','Interpreter','latex','FontSize',20)
title('The Fabius Function and its Degree $4$ Interpolant' ...
    ,'Interpreter','latex','FontSize',24);
%% Computing the errors in interpolation of the function
f = chebfun(p,[0,1],'splitting','on');
nn = 0:1:200;
```

```
ee = 0*nn;
for j = 1:length(nn)
    n =nn(j);
    fn=chebfun(f,[0,1],n+1);
    ee(j)=norm(f-fn,inf);
end
%% Semi-log plot of error against degree of interpolant, ...
   including fitted curve to
% 20 data points
ft = fittype(@(a,b,x)  a.^(-(x).^b));
[ftcurve,gof] = fit(nn(1:21)',ee(1:21)',ft,'StartPoint',[2 0.5]);
figure;
subplot(2,1,1);
semilogy(nn,ee,'r.','MarkerSize',14);
xlabel('$n$','Interpreter','latex','FontSize',10)
ylabel('$\log{||F-p_n||_\infty}$','Interpreter','latex','FontSize',10)
title('The Errors in Chebyshev Interpolation of the Fabius ...
   Function for $0\leq n \leq 200$' ...
   ,'Interpreter','latex','FontSize',12);
grid on;
subplot(2,1,2)
semilogy(nn(1:21),ee(1:21),'r.','MarkerSize',14);
hold on;
semilogy(nn(1:21),ftcurve(nn(1:21)),'k','LineWidth',1);
xlabel('$n$','Interpreter','latex','FontSize',10)
ylabel('$\log{||F-p_n||_\infty}$','Interpreter','latex','FontSize',10)
legend('Computed Error','Fitted Error', 'FontSize',10)
title('The Errors in Chebyshev Interpolation of the Fabius ...
   Function for $0\leq n \leq 20$' ...
   ,'Interpreter','latex','FontSize',12);
grid on;
```

B Varying parameters of the Lacunary Fourier series

Here we show the graphs omitted from section 5 regarding the variation of parameters α and β. First we consider the case $\alpha = e$ with varying β.

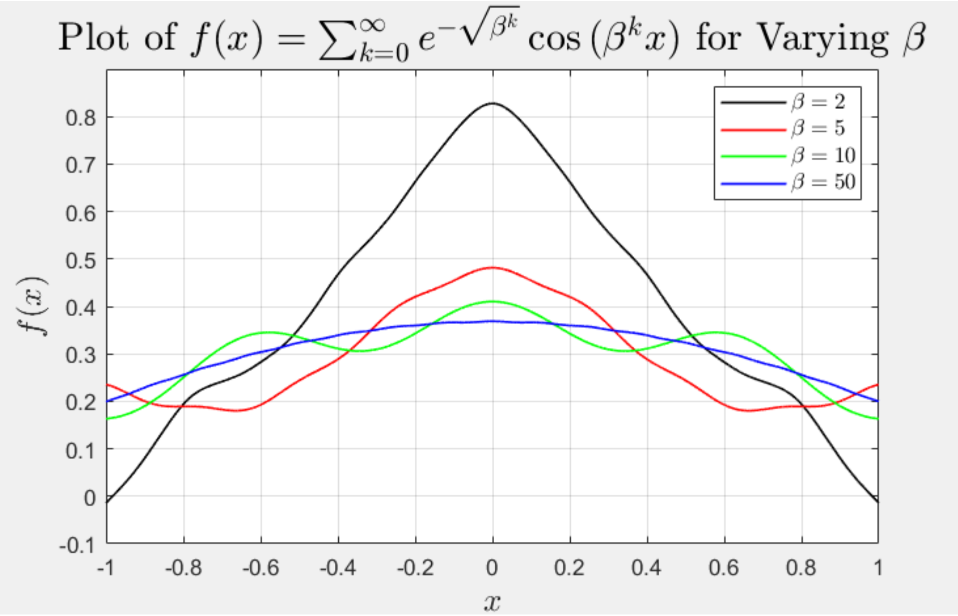

Figure 8: A plot of the Lacunary Fourier series with varying β

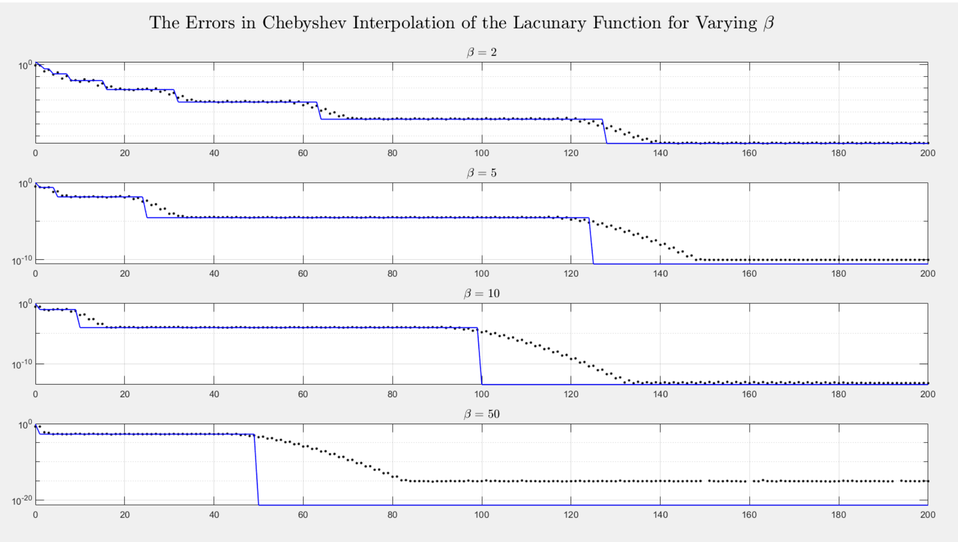

Figure 9: A plot of the errors of the Lacunary Fourier series with varying β

As we can see from Figure 9 the step length increases for higher values of β. It should be noted that the error for $\beta = 50$ becomes machine epsilon before it meets with its $p = 1$ step.
We can consider a similar argument for $\beta = 2$ and varying α, Figure 12 shows the plot for some arbitrary values, these show no particular distinguishing features from 8.

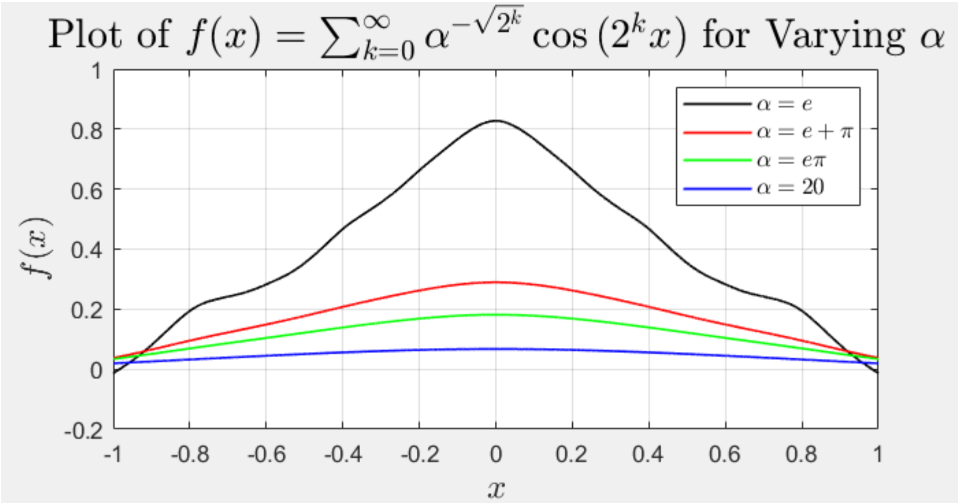

Figure 10: A plot the Lacunary Fourier series with some specific values of α

A more notable case for α is $\alpha = 1 + \epsilon$, for small ϵ.

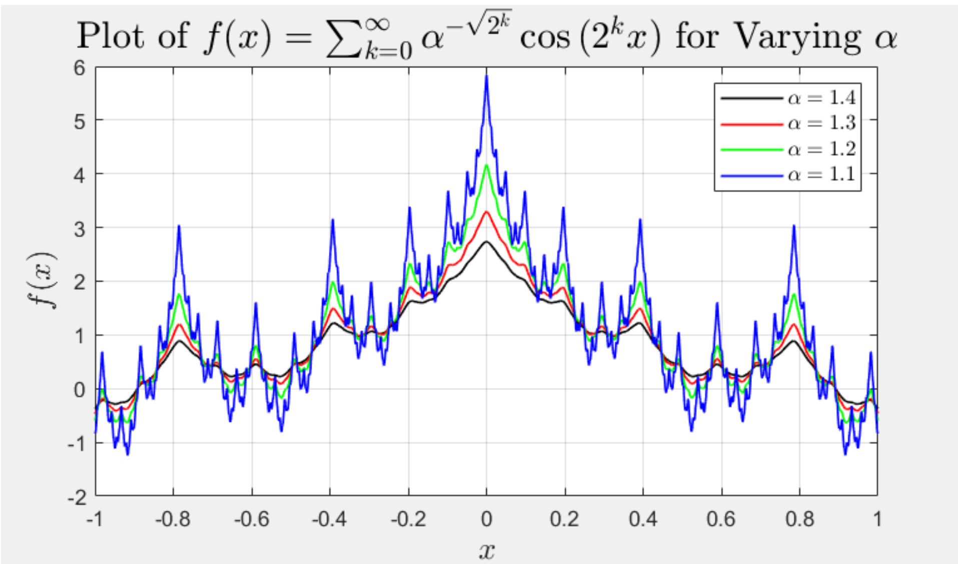

Figure 11: A plot the Lacunary Fourier series with $\alpha = 1 + \epsilon$

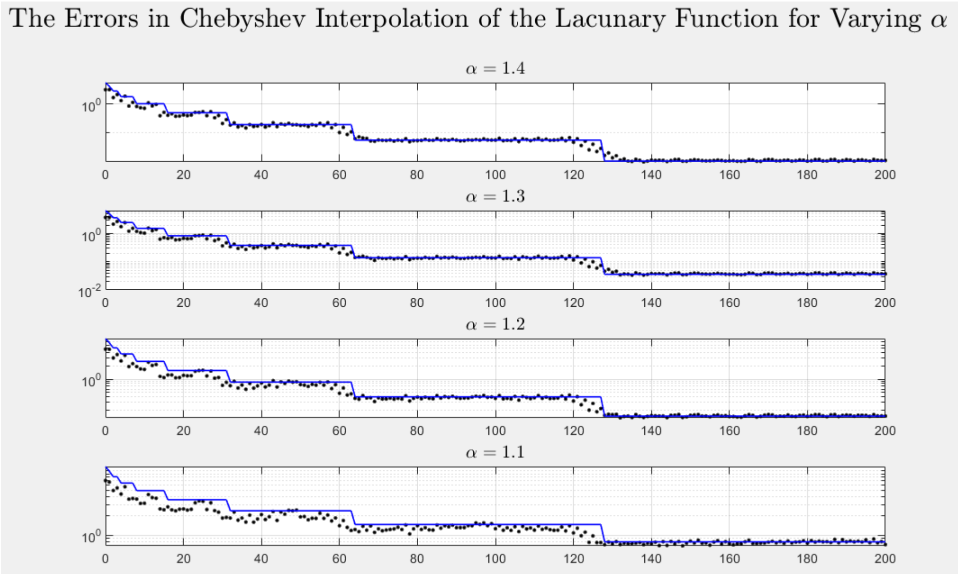

Figure 12: A plot of the errors of the Lacunary Fourier series with $\alpha = 1 + \epsilon$