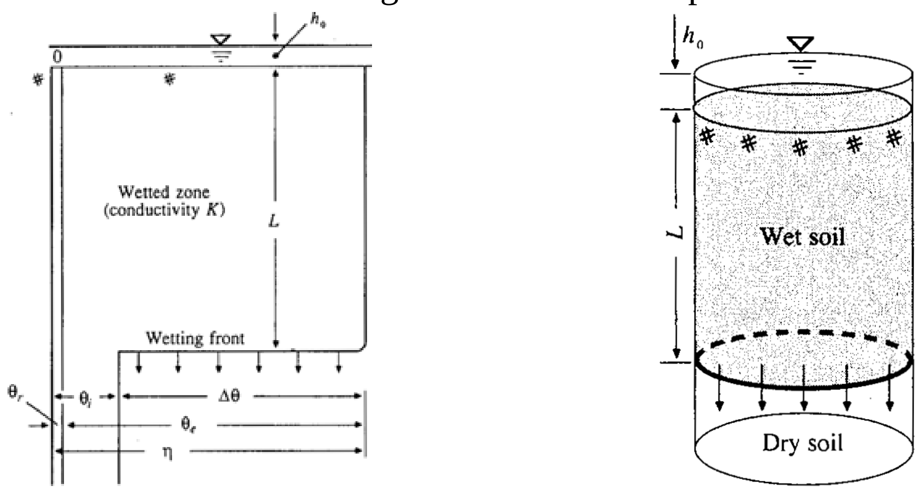
**Lecture 13**
**07-September-2016**

# Green-Ampt Equation

- In the last class, we discussed that there are analytical and empirical solutions to the one dimensional Richards equation.
- Green-Ampt provided a method that can be treated as analytical solution to the Richards equation for a simplified case.
- You know that the wetting front in Green-Ampt's method was sharp.




(The snapshots taken as it is from Chow et al. 1988)

- The water flows through the soil with Darcy velocity, $q$.
- This $q$ is nothing but the negative of infiltration rate $f$.

$$q = -K\frac{\partial H}{\partial z} \approx -K\frac{\Delta H}{\Delta z} \qquad \text{(Eqn.1)}$$

- The datum is assumed at the ground surface.
- $z_1$ at ground surface = 0.
- $z_2$ at position just below the wetting front in dry soil = -L.
- The hydraulic head $H_1 = h_0$

$$H_2 = -|\psi| - L$$

(The reason we are taking absolute value is to infer that the data of suction head will be available as positive value in the data and we need to take its negative value for fluid flow energy head).

- The Darcy velocity can be, therefore, given as:

$$q = -K\left[\frac{H_2 - H_1}{z_2 - z_1}\right]$$
$$= -K\left[\frac{(-|\psi| - L) - h_0}{-L - 0}\right]$$
$$q = -K\left[\frac{|\psi| + L + h_0}{L}\right]$$

(Eqn.2)

Therefore, the infiltration rate will be:

$$f = K\left[\frac{|\psi| + L + h_0}{L}\right]$$

(Eqn.3)

- If the depth of ponding is negligible, then the infiltration rate is:

$$f = K\left[\frac{|\psi| + L}{L}\right]$$

- Recall, cumulative infiltration (F) is accumulated depth of water infiltrated

$$F(t) = L\Delta\theta$$

Therefore,

$$L = \frac{F(t)}{\Delta\theta}$$

Hence,

$$f = K\left[\frac{|\psi| + \frac{F(t)}{\Delta\theta}}{\frac{F(t)}{\Delta\theta}}\right] = K\left[\frac{|\psi|\Delta\theta + F}{F}\right]$$

(Eqn.4)

- In the above expression, you have both f and F appearing, which are unknowns.

- Again recall, the infiltration rate can be given as:

$$f = \frac{dF}{dt}$$

Therefore,

$$\frac{dF}{dt} = K\left[\frac{|\psi|\Delta\theta + F}{F}\right]$$

(Eqn.5)

- This is a differential equation in *F* with respect to time *t*.
- On solving this equation, the solution *F(t)* will be obtained for the infiltration.
- Using method of variable separation,

$$\left[\frac{F}{|\psi|\Delta\theta + F}\right] dF = Kdt$$

i.e.

$$\left[\frac{|\psi|\Delta\theta + F}{|\psi|\Delta\theta + F} - \frac{|\psi|\Delta\theta}{|\psi|\Delta\theta + F}\right] dF = Kdt$$

Integrating up to time *t*,

$$\int_0^{F(t)} \left[1 - \frac{|\psi|\Delta\theta}{|\psi|\Delta\theta + F}\right] dF = \int_0^t Kdt$$

Use the mathematical methods to solve such an integration. E.g. the integration of $\int_0^{F(t)} \frac{a}{a+x} dx = a\ln(|x + a|) + C$ .

Therefore, the solution is:

$$F(t) - |\psi|\Delta\theta \ln\left(\left|1 + \frac{F(t)}{|\psi|\Delta\theta}\right|\right) = Kt$$

(Eqn.6)

- Equation 6 is the Green-Ampt's equation for cumulative infiltration.
- The expression is non-linear and requires methods to find roots of non-linear equations (e.g. like Bisection method; Newton-Raphson method; trial and error; etc.)
- On obtaining *F(t)*, we can evaluate $f = K\left[\frac{|\psi|\Delta\theta + F}{F}\right]$
- While solving, there are some other technical terms that need to be taken care of:
- Effective saturation, $\sigma_e = \frac{\theta_i - \theta_r}{\theta_s - \theta_r}$
- Effective porosity, $\theta_e = \theta_s - \theta_r$
- Therefore, $\theta_i - \theta_r = \sigma_e(\theta_s - \theta_r) = \sigma_e\theta_e$
- Again, $\theta_s - \theta_i = \Delta\theta = \theta_e - \sigma_e\theta_e = \theta_e(1 - \sigma_e)$

As discussed earlier, there are intrinsic relations between suction head and water content that are empirically expressed by many scientists like Brooks and Corey, van-Genuchten, etc.

**Example**

Use Green-Ampt method to compute infiltration rate and cumulative infiltration at every 0.25-hr from the beginning of infiltration for a silty clay soil with porosity, ε = 0.479; Suction head, Ψ = 29.22 cm; K = 0.05 cm/hr. Assume the initial effective saturation is 30 percent and it have continuous ponding. The effective porosity for the silty clay soil is 0.423.

**Solution**

For the silty soil,

$\varepsilon = 0.479$

$\psi = 29.22$ cm

$K = 0.05$ cm/hr

$\sigma_e = \frac{\theta_i - \theta_r}{\theta_s - \theta_r} = 0.30$

$\theta_e = 0.423$

Therefore, $\Delta\theta = \theta_e(1 - \sigma_e) = 0.423 \times (1 - 0.30) = 0.2961$

$$|\psi|\Delta\theta = 29.22 \times 0.2961 = 8.652$$

The non-linear equation in F(t) will be:

$$F(t) - |\psi|\Delta\theta \ln\left(\left|1 + \frac{F(t)}{|\psi|\Delta\theta}\right|\right) = Kt$$

$$F - 8.652 \ln\left(1 + \frac{F}{8.652}\right) = 0.05t$$

(Eqn.1)

$$f = K\left[\frac{|\psi|\Delta\theta}{F} + 1\right] = 0.05 \times \left(\frac{8.652}{F} + 1\right)$$

For each time instant, say $t_1$ = 0.25 hr, $t_2$ = 0.50 hr, $t_3$ = 0.75 hr, and $t_4$ = 1.0 hr, the above non-linear equation (1) is formulated.

For $t$ = 0.25 hr, equation (1) is:

- $F - 8.652\ln\left(1 + \frac{F}{8.652}\right) - 0.0125 = 0$. Solve this equation.
- That is you get, $F$ = 0.4735 cm and $f$ = 0.963 cm/hr

For t = 0.50 hr,

- $F - 8.652\ln\left(1 + \frac{F}{8.652}\right) - 0.025 = 0$. Solve this equation.
- That is you get, $F$ = 0.6745 cm and $f$ = 0.691 cm/hr

For t = 0.75 hr,

- $F - 8.652\ln\left(1 + \frac{F}{8.652}\right) - 0.0375 = 0$. Solve this equation
- That is you get, $F$ = 0.8307 cm and $f$ = 0.5707 cm/hr

For t = 1.0 hr,

- $F - 8.652\ln\left(1 + \frac{F}{8.652}\right) - 0.05 = 0$. Solve this equation
- That is you get, $F$ = 0.9638 cm and $f$ = 0.4988 cm/hr

For t = 1.25 hr,

- $F - 8.652\ln\left(1 + \frac{F}{8.652}\right) - 0.0625 = 0$. Solve this equation
- That is you get, $F$ = 1.082 cm and $f$ = 0.4498 cm/hr