

# Geometry Processing and Geometric Deep Learning

MVA Course, Lecture 3, part 2

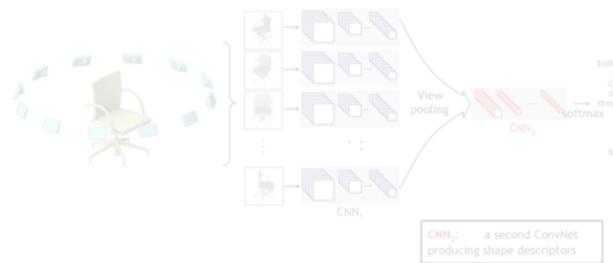
16 / 10 / 2024

Maks Ovsjanikov

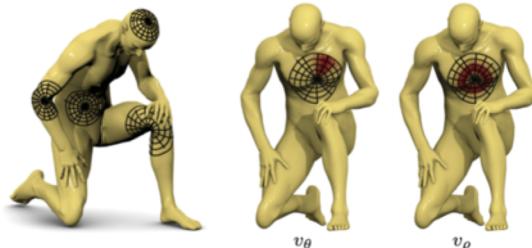


Google DeepMind

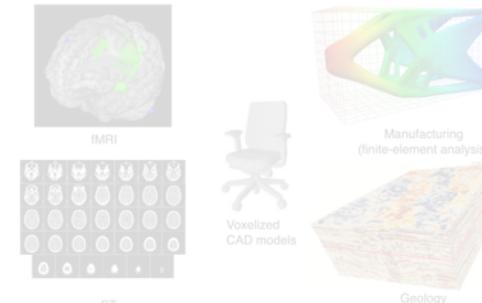
# Approaches for 3D Deep-Learning



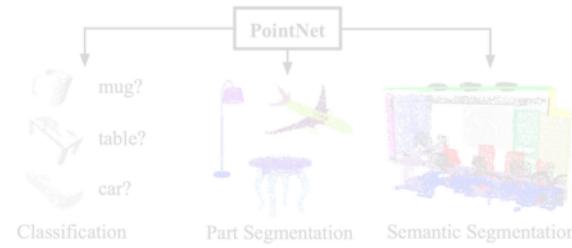
View-based



Intrinsic (surface-based)

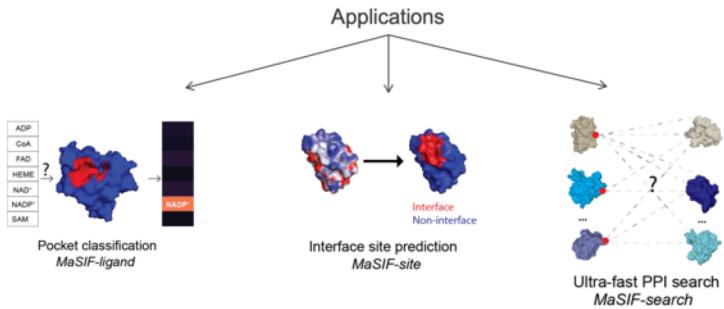
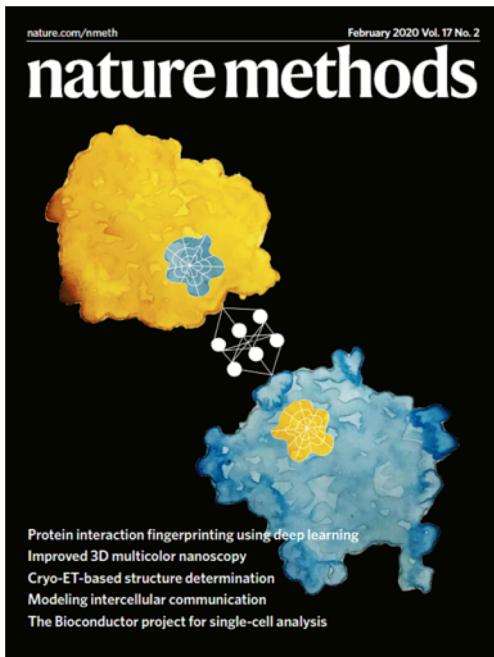


Volumetric

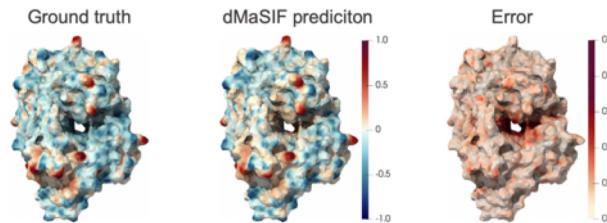


Point-based

# Tasks for intrinsic deep learning



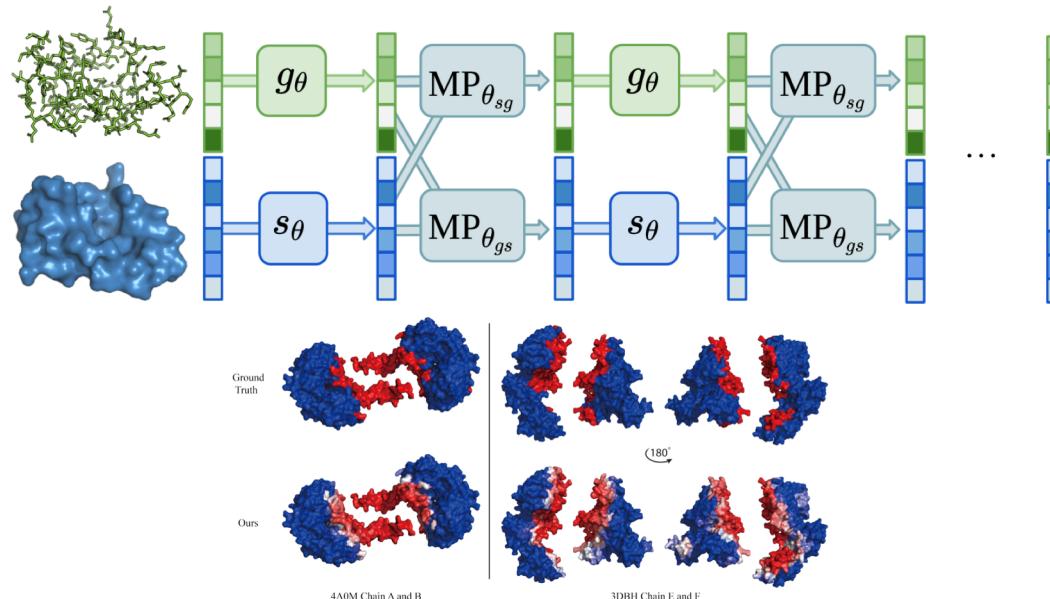
Electrostatic potentials of the protein surface



“Deciphering interaction fingerprints from protein molecular surfaces using geometric deep learning”, P. Gainza et al. *Nature Methods* 2020

# Some Applications

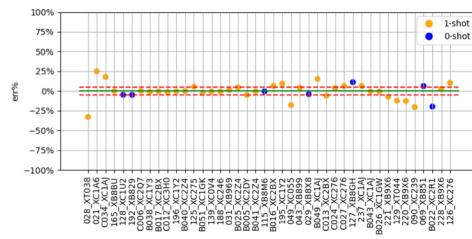
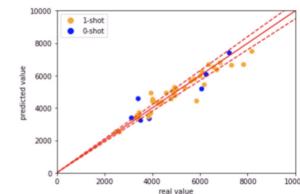
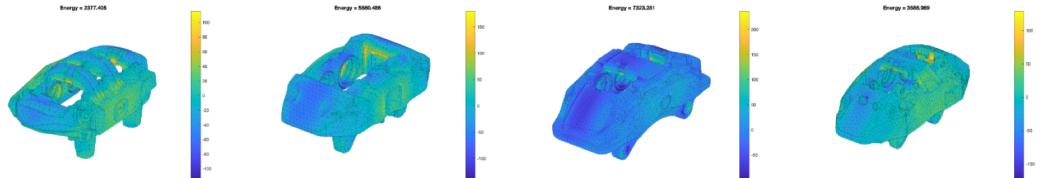
Simultaneously using graphs and surfaces to learn protein-related tasks.



V. Mallet, S. Attaiki, Y. Miao, B. Correia and M Ovsjanikov. "AtomSurf: Surface Representation for Learning on Protein Structures." arXiv preprint arXiv:2309.16519 (2023).

# Some Applications

Surface-based learning for predicting physical properties of complex shapes.



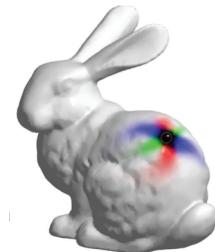
Data-driven energy prediction technique with DiffusionNet

# Today: Deep Learning on 3D shapes

- Recap of CNNs and their properties
- Multi-view, extrinsic, projection-based approaches
- Spectral methods, pros and cons
- Intrinsic approaches
- Learning via diffusion

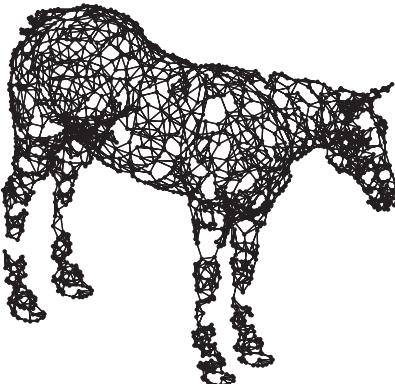
**Main question (for the rest of the lecture):**

How to enable neural networks to  
operate *directly on 3D surfaces?*



Spectral domain Deep Learning methods

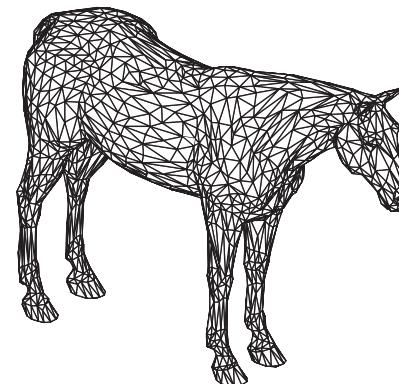
# Discrete surfaces



Nearest neighbor graph

Vertices  $\mathcal{V} = \{1, \dots, n\}$

Edges  $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$



Triangular mesh

Vertices  $\mathcal{V} = \{1, \dots, n\}$

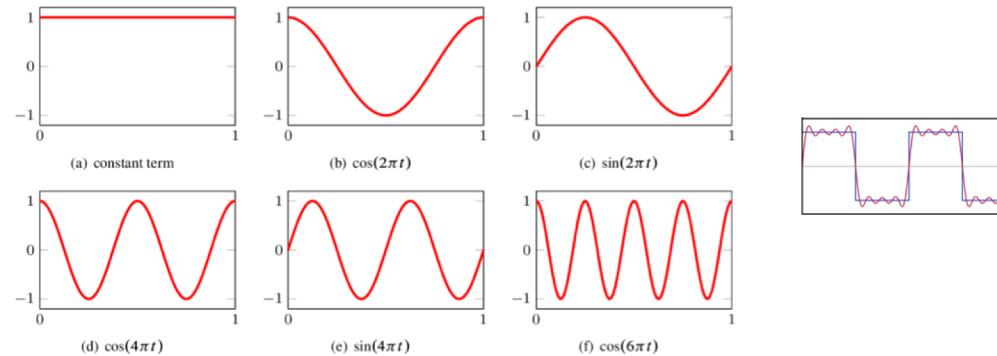
Edges  $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$

Faces  $\mathcal{F} = \{(i, j, k) \in \mathcal{V} \times \mathcal{V} \times \mathcal{V} : (i, j), (j, k), (k, i) \in \mathcal{E}\}$

Manifold mesh = each edge is shared  
by 2 faces + each vertex has 1 loop

# Fourier Bases

*Fourier basis on a periodic domain  $[0, 1]$ :*

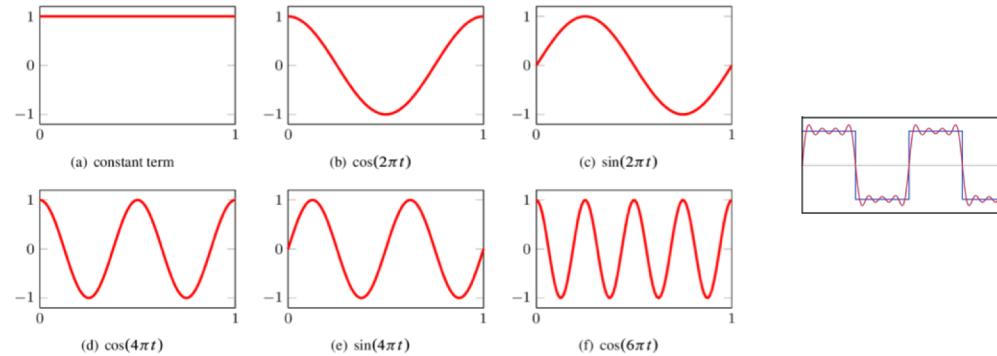


*Classical result:* any (sufficiently “nice”) function can be written as a linear combination of Fourier basis functions:

$$f(t) = a_0 + \sum_{k=1}^{\infty} (a_k \sin(2\pi kt) + b_k \cos(2\pi kt))$$

# Fourier Bases

*Fourier basis on a periodic domain  $[0, 1]$ :*



*Classical result:* any (sufficiently “nice”) function can be written as a linear combination of Fourier basis functions:

$$f(t) = \sum_k a_k e^{ikt}$$

# Laplace-beltrami eigenfunctions

Eigenfunctions of the *Laplacian* matrix.

$$L\phi_i = \lambda_i\phi_i$$

- Generalization of Fourier to graphs and surfaces.
- Ordered by eigenvalues and provide a natural notion of *frequency* (scale).

$$\lambda_0 = 0$$

$$\lambda_1 = 2.39$$

$$\lambda_2 = 3.11$$

$$\lambda_3 = 5.00$$

$$\lambda_4 = 7.31$$



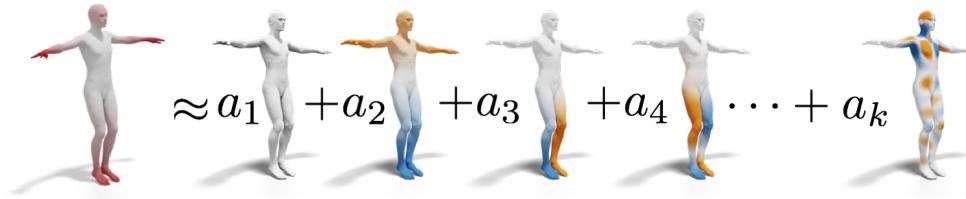
Bruno Levy "Laplace-beltrami eigenfunctions towards an algorithm that" understands" geometry." IEEE International Conference on Shape Modeling and Applications 2006 (SMI'06).

# Signal Processing on Surfaces

Eigenfunctions of the *Laplacian* matrix.

$$L\phi_i = \lambda_i\phi_i$$

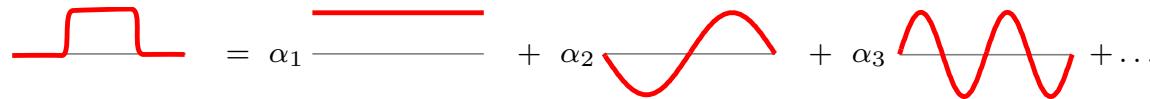
- ➊ Generalization of Fourier to graphs and surfaces.
- ➋ Ordered by eigenvalues and provide a natural notion of *frequency* (scale).
- ➌ Can be used to express functions on a surface:



# Fourier analysis: Euclidean Space

A function  $f : [-\pi, \pi] \rightarrow \mathbb{R}$  can be written as Fourier series

$$f(x) = \sum_{k \geq 0} \underbrace{\frac{1}{2\pi} \int_{-\pi}^{\pi} f(x') e^{-ikx'} dx'}_{\hat{f}_k = \langle f, e^{ikx} \rangle_{L^2([-\pi, \pi])}} e^{ikx}$$

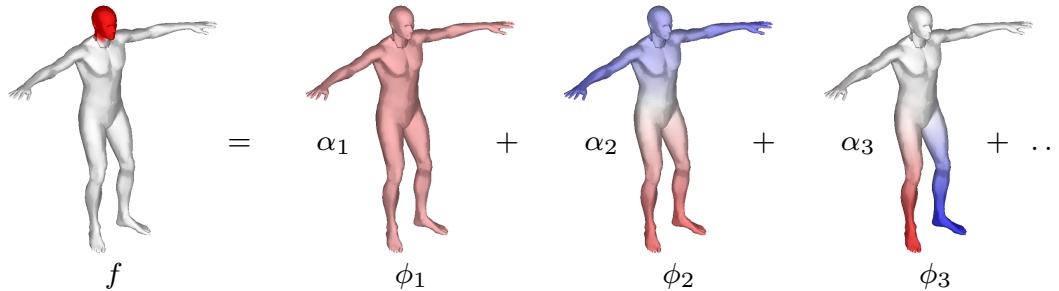


Fourier basis = Laplacian eigenfunctions:  $-\frac{d^2}{dx^2} e^{ikx} = k^2 e^{ikx}$

# Fourier analysis: Non-Euclidean Space

A function  $f : \mathcal{X} \rightarrow \mathbb{R}$  can be written as Fourier series

$$f(x) = \sum_{k \geq 1} \underbrace{\int_{\mathcal{X}} f(x') \phi_k(x') dx'}_{\hat{f}_k = \langle f, \phi_k \rangle_{L^2(\mathcal{X})}} \phi_k(x)$$



Fourier basis = Laplacian eigenfunctions:  $\Delta \phi_k(x) = \lambda_k \phi_k(x)$

# Convolution: Euclidean Space

Given two functions  $f, g : [-\pi, \pi] \rightarrow \mathbb{R}$  their convolution is a function

$$(f \star g)(x) = \int_{-\pi}^{\pi} f(x')g(x - x')dx'$$

- Shift-invariance:  $f(x - x_0) \star g(x) = (f \star g)(x - x_0)$
- Convolution operator commutes with Laplacian:  $(\Delta f) \star g = \Delta(f \star g)$
- Convolution theorem: Fourier transform diagonalizes the convolution operator  $\Rightarrow$  convolution can be computed in the Fourier domain as

$$\widehat{(f \star g)} = \hat{f} \cdot \hat{g}$$

- Efficient computation using FFT

# Generalized Convolution

Generalized convolution of  $f, g \in L^2(\mathcal{X})$  can be defined by analogy

$$f \star g = \sum_{k \geq 1} \underbrace{\langle f, \phi_k \rangle_{L^2(\mathcal{X})} \langle g, \phi_k \rangle_{L^2(\mathcal{X})}}_{\text{product in the Fourier domain}} \phi_k$$

inverse Fourier transform

# Generalized Convolution

Generalized convolution of  $f, g \in L^2(\mathcal{X})$  can be defined by analogy

$$f \star g = \sum_{k \geq 1} \langle f, \phi_k \rangle_{L^2(\mathcal{X})} \langle g, \phi_k \rangle_{L^2(\mathcal{X})} \phi_k$$

In matrix-vector notation

$$f \star g = \Phi (\Phi^+ g \circ \Phi^+ f)$$

$$\Phi^+ = \Phi^T \text{ if } \Phi^T \Phi = Id$$

$$\Phi^+ = \Phi^T A \text{ if } \Phi^T A \Phi = Id$$

- Elementwise multiplication. I.e.,  $(a \circ b)_i = a_i b_i$

# Generalized Convolution

Generalized convolution of  $f, g \in L^2(\mathcal{X})$  can be defined by analogy

$$f \star g = \sum_{k \geq 1} \langle f, \phi_k \rangle_{L^2(\mathcal{X})} \langle g, \phi_k \rangle_{L^2(\mathcal{X})} \phi_k$$

In matrix-vector notation

$$\mathbf{f} \star \mathbf{g} = \mathbf{\Phi} \operatorname{diag}(\hat{g}_1, \dots, \hat{g}_n) \mathbf{\Phi}^\top \mathbf{f}$$

# Spectral Convolution

Convolutional layer expressed in the **spectral domain**

$$\mathbf{g}_l = \xi \left( \sum_{l'=1}^p \Phi \mathbf{W}_{l,l'} \Phi^\top \mathbf{f}_{l'} \right) \quad l = 1, \dots, q \\ l' = 1, \dots, p$$

where  $\mathbf{W}_{l,l'} = n \times n$  diagonal matrix of filter coefficients

# Spectral Convolution

Convolutional layer expressed in the **spectral domain**

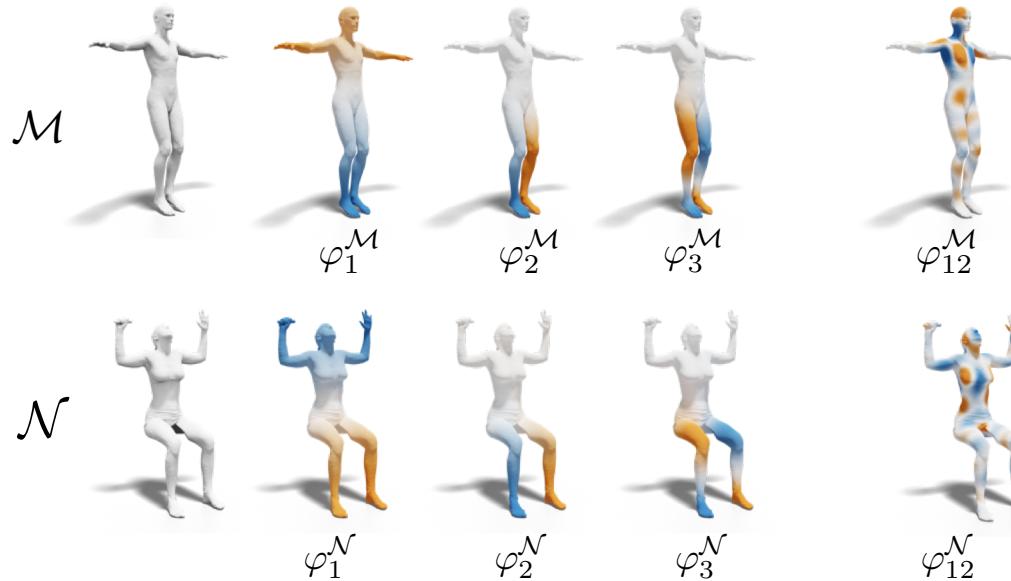
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where  $\mathbf{W}_{l,l'} = n \times n$  diagonal matrix of filter coefficients

- ⊖ Filters are basis-dependent  $\Rightarrow$  does not generalize across graphs!
- ⊖  $\mathcal{O}(n)$  parameters per layer
- ⊖  $\mathcal{O}(n^2)$  computation of forward and inverse Fourier transforms  
 $\Phi^\top, \Phi$  (no FFT on graphs)
- ⊖ No guarantee of spatial localization of filters

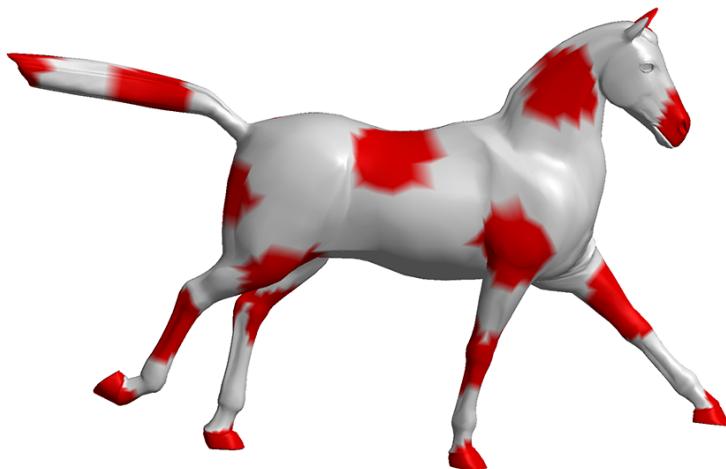
# Domain-dependence of the basis

The Fourier (Laplacian) basis on a surface *depends on the surface*



The filters applied on one shape *do not* translate to a new shape

# Domain-dependence of the basis



Function  $f$

Image credit: E. Rodolà

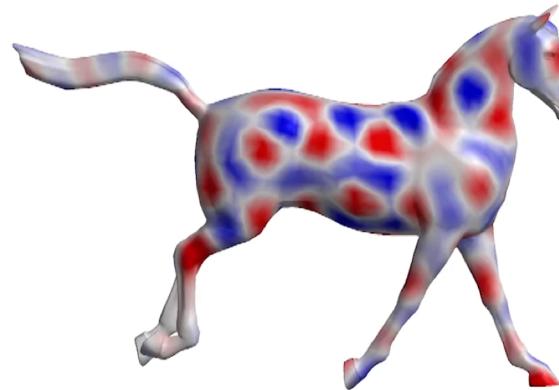
# Domain-dependence of the basis



'Edge detecting' spectral filter  $\Phi \mathbf{W} \Phi^\top \mathbf{f}$

Image credit: E. Rodolà

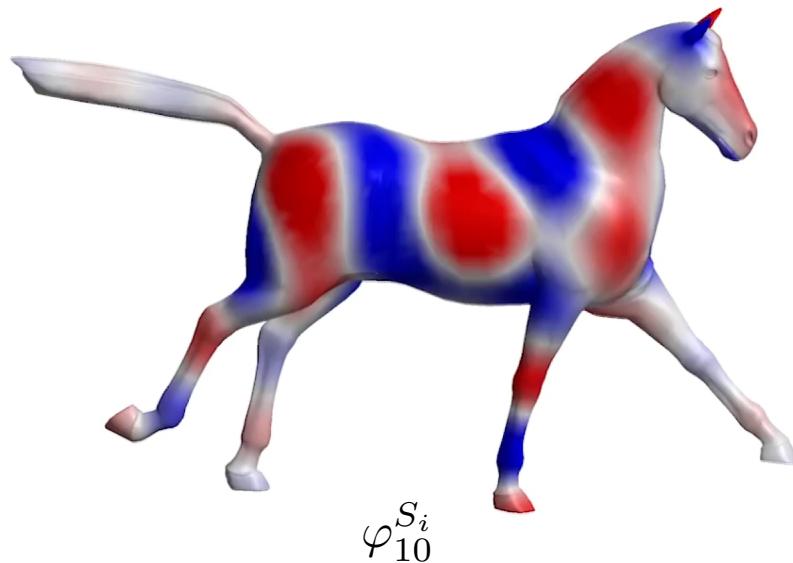
# Domain-dependence of the filter



Applying the same filter on different shapes

$$\Phi \mathbf{W} \Phi^\top \mathbf{f}$$

# Domain-dependence of the basis

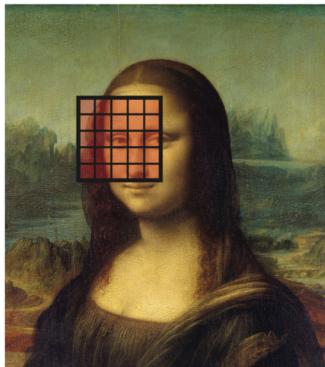


Laplacian eigenfunction on different shapes

Spatial domain Deep Learning methods

# Non-Euclidean learning

Idea: apply kernels directly on the surface!



Euclidean



Non-Euclidean

image credit M.Bronstein

*Geometric Deep Learning: Going Beyond Euclidean Data* Bronstein MM, Bruna J, LeCun Y, Szlam A, Vandergheynst P.. IEEE Signal Processing Magazine. 2017

*A Comprehensive Survey on Geometric Deep Learning.* Cao W, Yan Z, He Z, He Z. 2020

# Non-Euclidean learning

Standard approach: texture mapping on 3D shapes.

A	B	C	D	E	A	B	C	D	E
F	G	H	I	J	F	G	H	I	J
K	L	M	N	O	K	L	M	N	O
P	R	S	T	U	P	R	S	T	U
V	W	X	Y	Z	V	W	X	Y	Z
A	B	C	D	E	A	B	C	D	E
F	G	H	I	J	F	G	H	I	J
K	L	M	N	O	K	L	M	N	O
P	R	S	T	U	P	R	S	T	U
V	W	X	Y	Z	V	W	X	Y	Z

Euclidean



Non-Euclidean

## Challenges:

- No canonical *global system* of coordinates
- No grid structure for convolution
- No shift-invariance

# Non-Euclidean learning

Key idea: parameterize the shape *locally*. Define a **patch operator**:

$$(f \star g)(x) = \int_{(D(x)f)(\mathbf{u})}^{g(\mathbf{u})} d\mathbf{u}$$



The kernel  $g(\mathbf{u})$  is defined in the Euclidean plane. It is *applied* on the surface.

# Geodesic convolutional neural networks

**Key idea:** parameterize the shape *locally*.

Using local polar coordinates on the surface can multiply the signal  $f$  with a trainable kernel  $g$ :

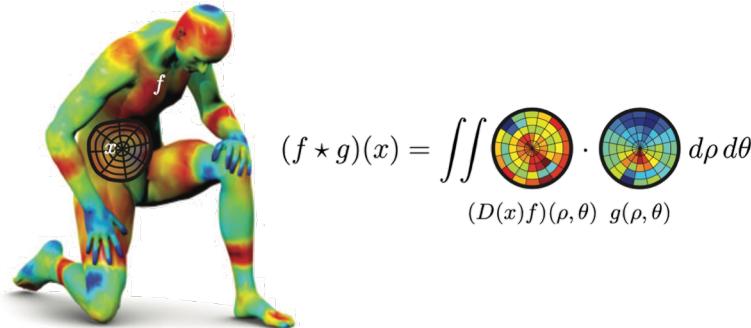


image credit M.Bronstein

Product is a scalar per point  $\Rightarrow$  a real-valued function on the surface.  
Can stack in layers.

- Geodesic convolutional neural networks on Riemannian manifolds, Masci et al., 2015
- Geometric deep learning on graphs and manifolds using mixture model CNNs, Monti et al., 2017
- CNNs on Surfaces using Rotation-Equivariant Features, Wiersma et al. 2020. ...

# Geodesic convolutional neural networks

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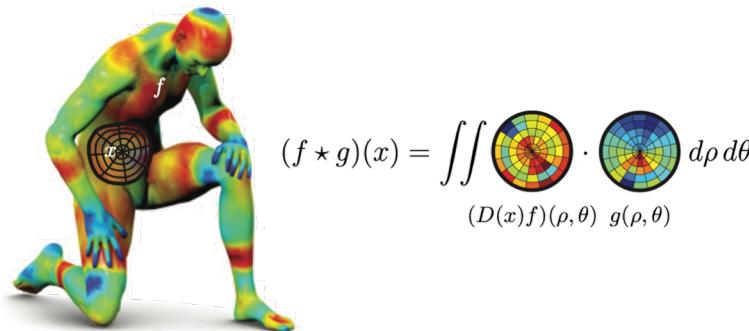


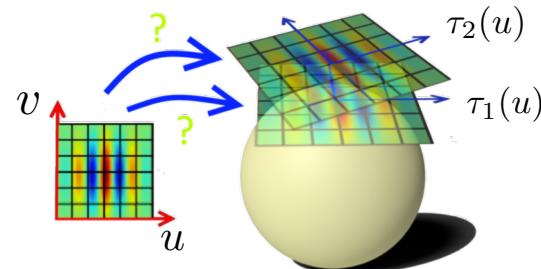
image credit M.Bronstein

Product is a scalar per point  $\Rightarrow$  a real-valued function on the surface.  
Can stack in layers. **Intrinsic = pose invariant!**

- Geodesic convolutional neural networks on Riemannian manifolds, Masci et al., 2015
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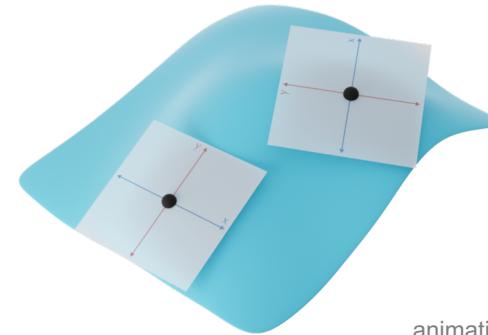
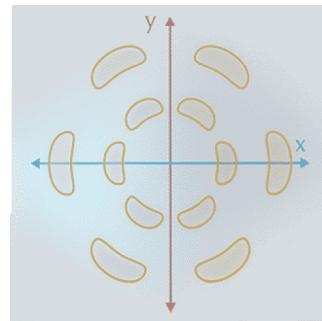
# Intrinsic Learning on Surfaces

**Challenge:** parameterization only defined up to rotation, even locally:



# Intrinsic Learning on Surfaces

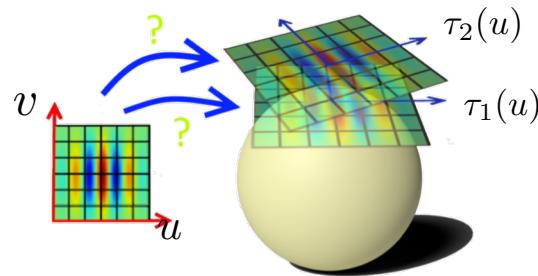
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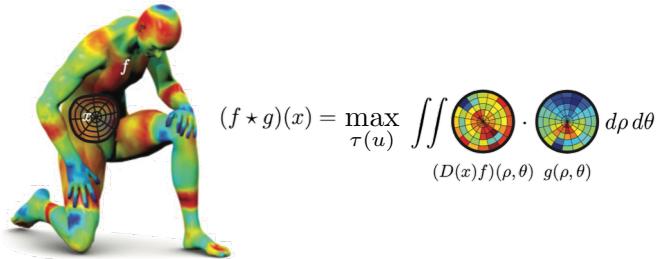
animation by R Wiersma et al.

# Non-Euclidean learning

**Challenge:** parameterization only defined up to rotation, even locally:

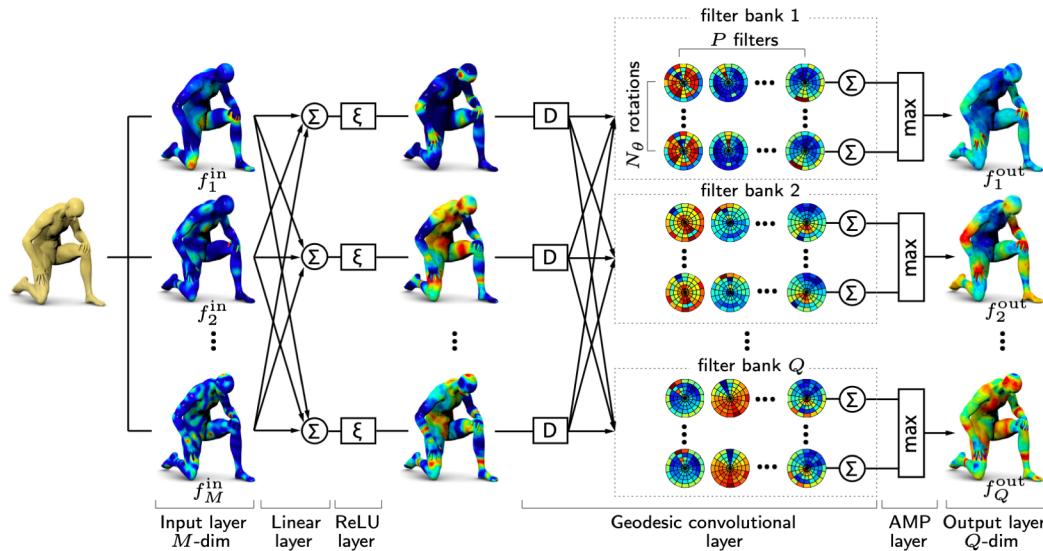


**Early solution:** take the maximal response *across all rotations*.



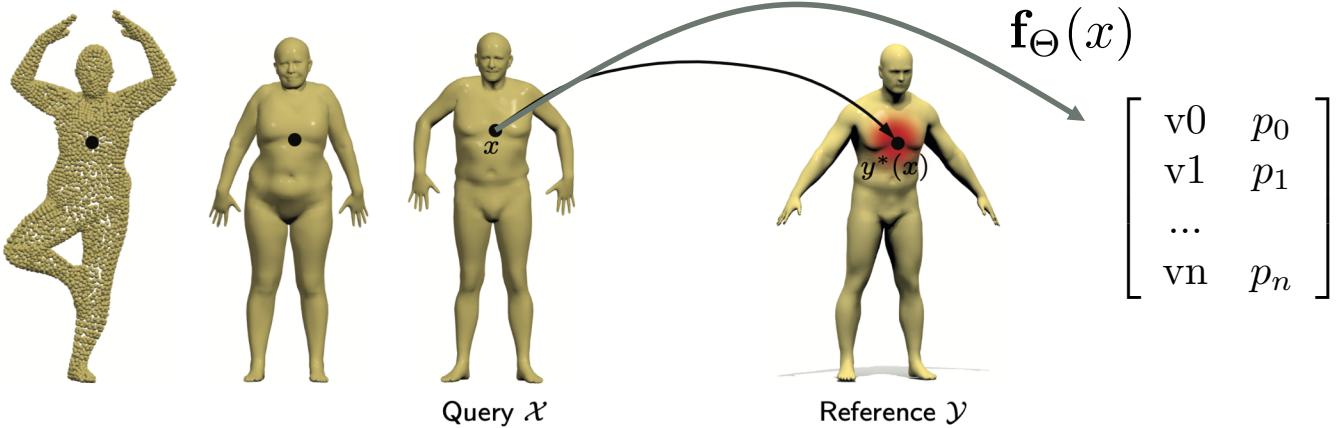
# Intrinsic Learning on Surfaces

Basic GCNN (geodesic CNN) architecture:



slide credit E. Rodolà

# Learning Correspondences

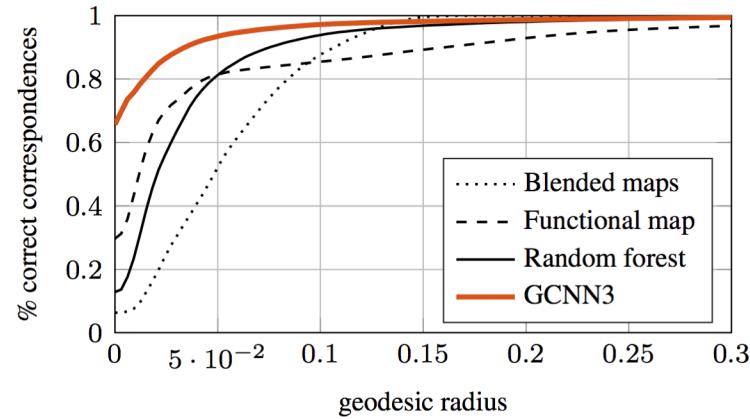
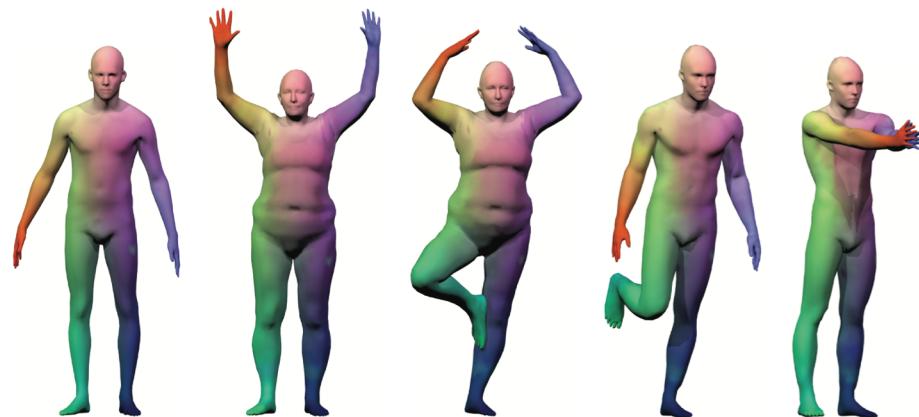


- Correspondence = labeling problem
- GCNN output  $\mathbf{f}_\Theta(x)$  = probability distribution on reference  $\mathcal{Y}$
- Minimize logistic regression cost w.r.t. GCNN parameters  $\Theta$

$$\ell(\Theta) = - \sum_{(x, y^*(x)) \in \mathcal{T}} \langle \delta_{y^*(x)}, \log \mathbf{f}_\Theta(x) \rangle_{L^2(\mathcal{Y})}$$

image credit E. Rodolà

# Learning Correspondences with GCNN



Correspondence found using GCNN (similar colors encode corresponding points)

slide credit E. Rodolà

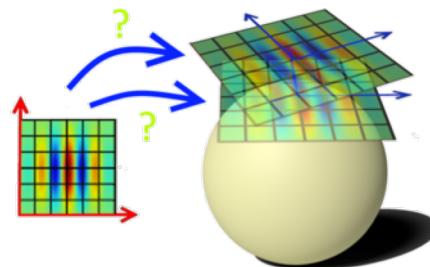
# GCNN: Key challenges

## Problem 1:

Local parametrization only defined up to rotation

## Problem 2:

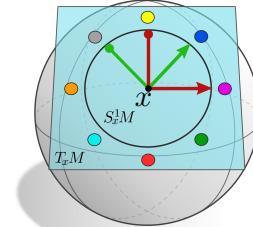
Angular max pooling *loses directional information.*



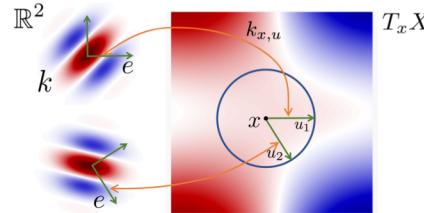
# Parametrization Ambiguity

More recent solutions to kernel rotation ambiguity :  
Designing *equivariant* networks

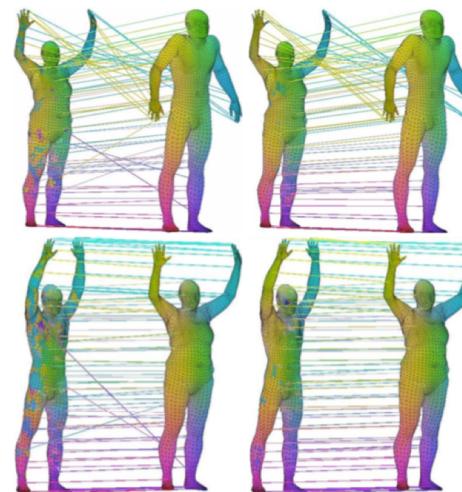
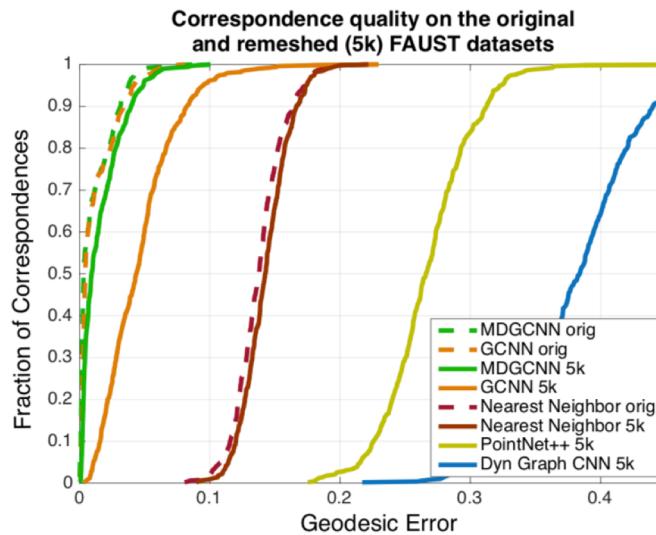
- Multi-directional Geodesic Neural Networks (MDGCNN)



1. Work with *directional functions*  $\varphi(x, v) \rightarrow \mathbb{R}, v \in T_x$
2. Keep responses of all kernel rotations:  $(\varphi \star k)(x, v) = \varphi \star k_{x,v}$
3. Apply angular max pooling once at the end of the network



# MDGCNN Results



GCNN

MDGCNN

- \* remeshed datasets released as part of: *Continuous and Orientation-preserving Correspondences via Functional Maps*, J. Ren, A. Poulenard, P. Wonka, M. O, SIGGRAPH Asia 2018

# Issues

## Technical Problem:

Local parametrization only defined up to rotation

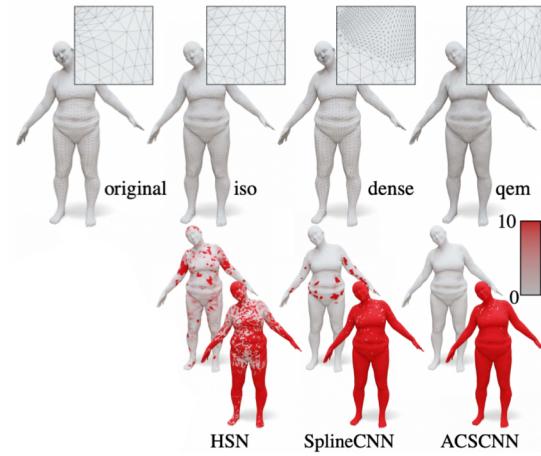
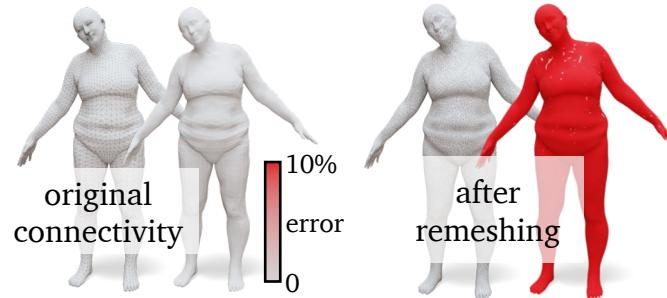
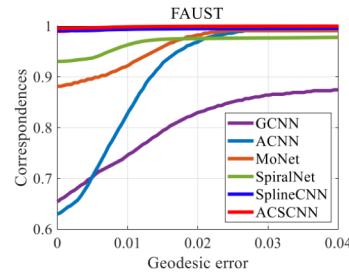
## Even Bigger Problem:

Local patch operations – slow and not robust!

## Main Question:

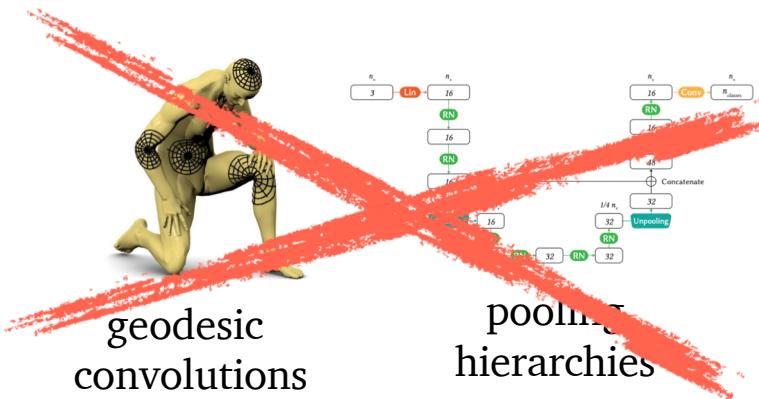
Can we avoid local patch parametrization and yet enable local *communication* on the surface?

# Learning Correspondences – Results



Many existing Geometric Deep Learning methods break under even mild remeshing.

# Common Intrinsic Surface Learning



difficult on surfaces

source of non-robustness

use diffusion instead!

# Alternative: Simple diffusion-based network

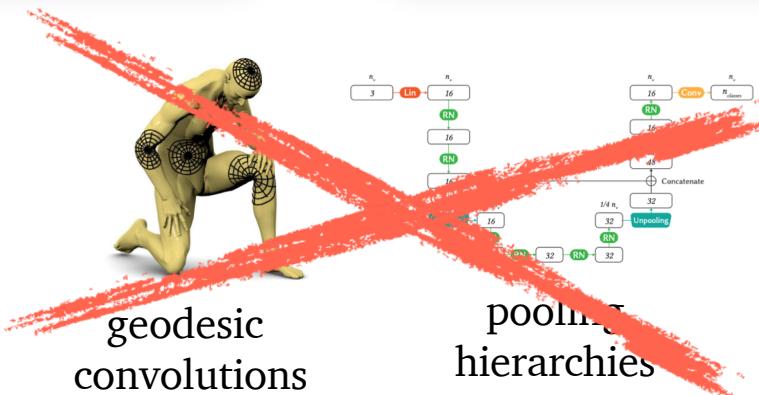
pointwise MLP

+

learned diffusion

+

gradient features



difficult on surfaces

source of non-robustness

use diffusion instead!

# Basics: Laplacians and Diffusion

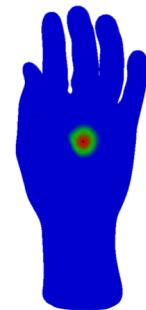
$$\frac{d}{dt} f(t) = \Delta f(t)$$

↳ Basic linear PDE

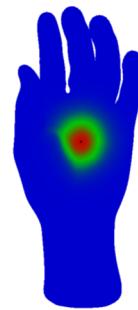
↳ defined on surfaces via the Laplace-Beltrami operator  $\Delta$

↳ implemented & well-studied on many domains

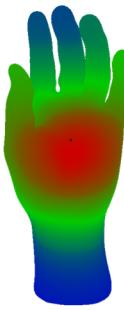
diffusion of a point value →



$t = 0.001$



$t = 0.02$



$t = 0.1$



$t = 1$

$$\begin{aligned}f(t) &= \mathcal{H}_t f_0 \\&= \exp(\Delta t) f_0 \\ \mathcal{H}_t : \quad &\text{Heat operator}\end{aligned}$$

# Learned diffusion

~~convolution  
Pooling~~

✓  
diffusion!

learned diffusion layer

$$h_t : \mathbb{R}^{\Omega \times k} \rightarrow \mathbb{R}^{\Omega \times k}$$

↳ parameterized by  $t \in \mathbb{R}_{\geq 0}^k$

**Key idea:** the *diffusion time* is a learned parameter

- ↳ variable per-channel spatial support
- ↳ ranges from purely local to totally global
- ↳ automatically optimized during training

**Lemma:** diffusion + pointwise MLPs can represent all (radially symmetric) convolutions.

# Evaluating diffusion

There are many ways to evaluate diffusion...

In Laplacian eigenbasis, diffusion is just multiplying by  $e^{-\lambda t}$

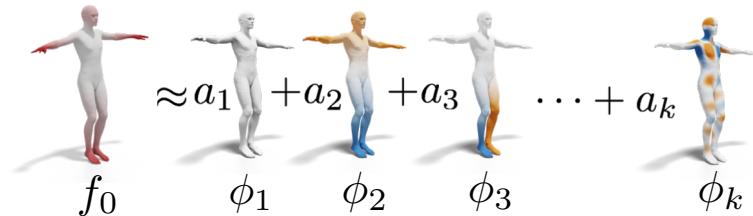
I.e., if a function at time 0 is:  $f_0 = \sum_i a_i \phi_i$

Then at time  $t$  (after diffusion):

$$f_t = \sum_i a_i e^{-t\lambda_i} \phi_i$$

Differentiable with respect to  $t$ .

**Key idea:** the *diffusion time t* is a learnable parameter.

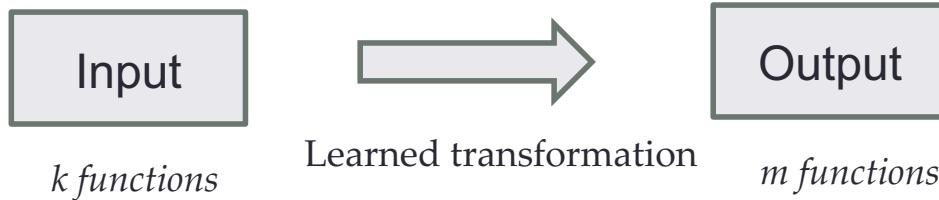


**Note:** this is not spectral learning (we don't learn any *frequency-dependent* filters)!

# Spatial gradient features

Typical architectures process input in blocks (of channels).

For DiffusionNet each channel *is a function on the shape*.



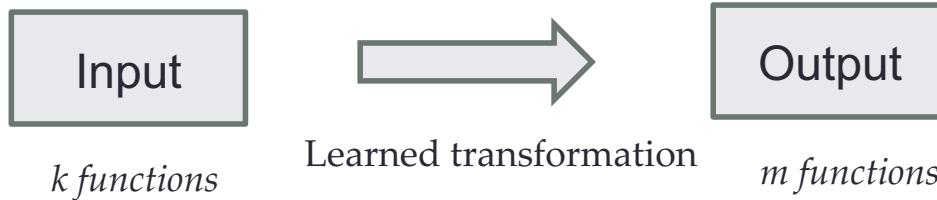
E.g. for diffusion  $f_j^{\text{out}} = \mathcal{H}_{t_j} f_j^{\text{in}}$   $t_j$  learnable diffusion for channel  $j$ .

$$f_j^{\text{out}} = \exp(\Delta t_j) f_j^{\text{in}}$$

# Spatial gradient features

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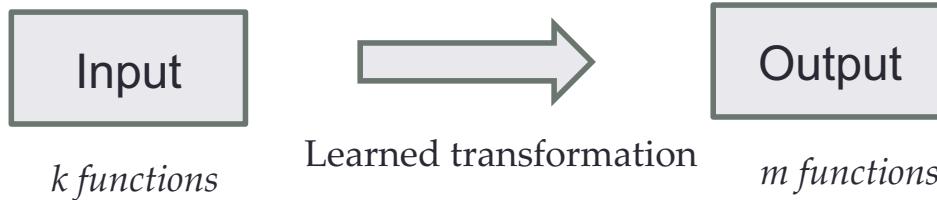


E.g. for a *linear transformation*  $f_j^{\text{out}}(x) = \sum_i A_{ij} f_i^{\text{in}}(x)$   $A_{ij}$  are learned parameters (independent of  $x$ ).

# Spatial gradient features

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DiffusionNet gradient features are similar:

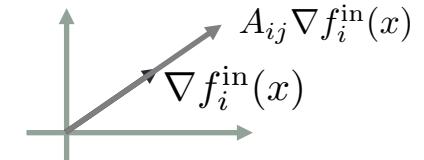
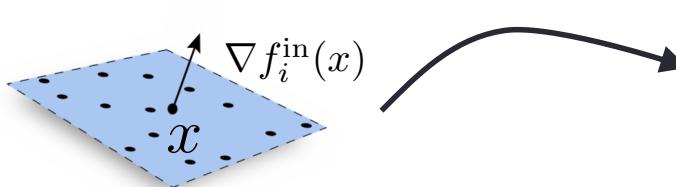
$$f_j^{\text{out}}(x) = \sum_i A_{ij} < \nabla f_i^{\text{in}}(x), \nabla f_j^{\text{in}}(x) >$$

# Spatial gradient features

One more trick:

$$f_j^{\text{out}}(x) = \sum_i A_{ij} < \nabla f_i^{\text{in}}(x), \nabla f_j^{\text{in}}(x) >$$

We can represent  $\nabla f_i^{\text{in}}(x)$  as a 2D vector in the tangent plane of  $x$ :



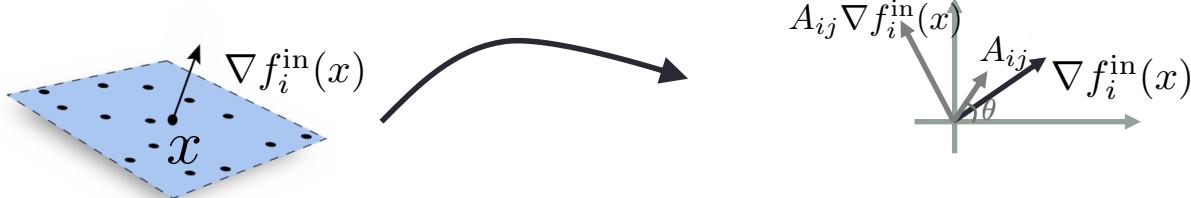
If  $A_{ij}$  is a *real number*, then  $A_{ij} \nabla f_i^{\text{in}}(x)$  is just a *scaled* version of  $\nabla f_i^{\text{in}}(x)$ .

# Spatial gradient features

One more trick:

$$f_j^{\text{out}}(x) = \sum_i A_{ij} < \nabla f_i^{\text{in}}(x), \nabla f_j^{\text{in}}(x) >$$

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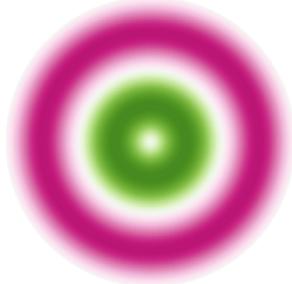
Instead of simply *scaling*  $\nabla f_i^{\text{in}}(x)$ , we can *scale and rotate* it in the 2D plane. Useful for injecting directional information into the pipeline. Can be conveniently done via *complex multiplication*.

# Spatial gradient features

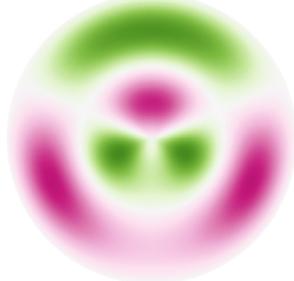
**Challenge:** we want to go beyond radially-symmetric filters

**Solution:** append extra features, dot products of spatial gradients

✗ radially symmetric filters only



✓ beyond radially symmetric filters

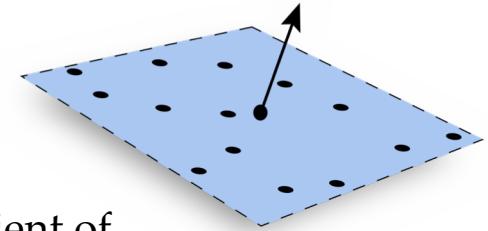


spatial gradient of scalar features

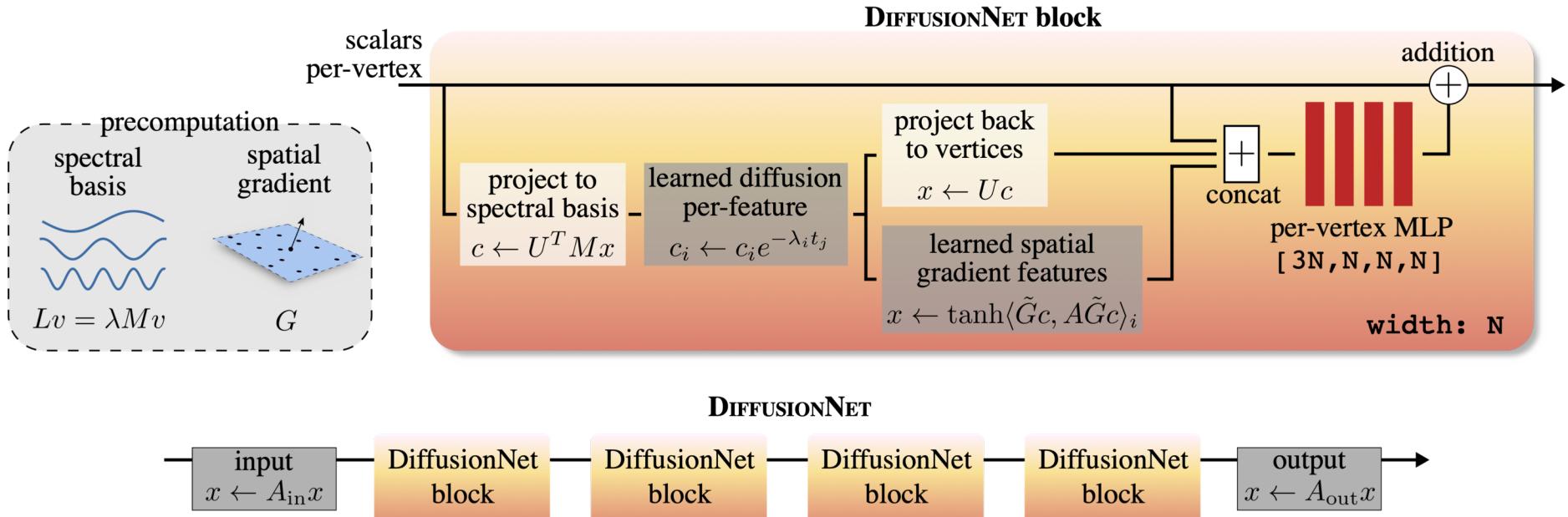
learned scaling  $A$

Our gradient features:  $\tanh(\langle z, Az \rangle)$

$$output(i) = \tanh\left(\sum_j \langle z(i), A_{ij}z(j) \rangle\right)$$



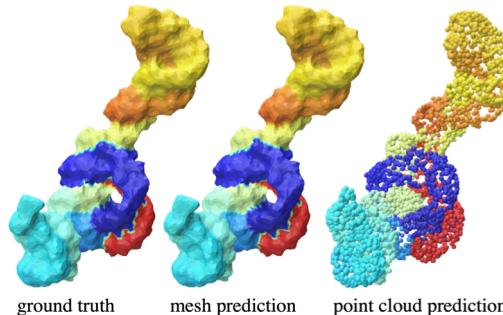
# DiffusionNet Architecture



# DiffusionNet Results

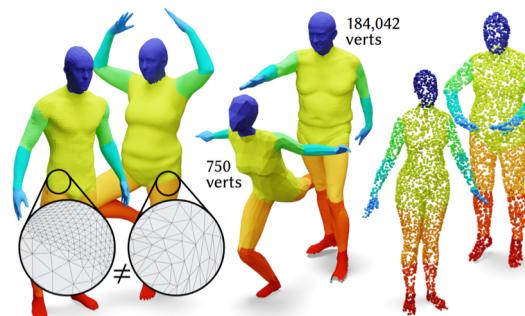
State-of-the-art on several benchmarks!

## RNA segmentation



Method	Accuracy
cloud	PointNet++ [Qi et al. 2017a] 74.4%
	PCNN [Atzmon et al. 2018] 78.0%
	SPHNet [Poulenard et al. 2019] 80.1%
	DiffusionNet - hks 84.0%
	<b>DiffusionNet - xyz 85.0%</b>
mesh	SplineCNN [Fey et al. 2018] 53.6%
	SurfaceNetworks [Kostrikov et al. 2018] 88.5%
	DiffusionNet - hks 91.0%
	<b>DiffusionNet - xyz 91.5%</b>

## Human part segmentation



<sup>†</sup>see note

<sup>‡</sup>see note

Method	Accuracy
GCNN [Masci et al. 2015]	86.4%
ACNN [Boscaini et al. 2016]	83.7%
Toric Cover [Maron et al. 2017]	88.0%
PointNet++ [Qi et al. 2017a]	90.8%
MDGCNN [Poulenard et al. 2018]	88.6%
DGCNN [Wang et al. 2019]	89.7%
SNGC [Haim et al. 2019]	91.0%
HSN [Wiersma et al. 2020]	91.1%
MeshWalker [Lahav and Tal 2020]	<b>92.7%</b>
CGConv [Yang et al. 2021]	89.9%
FC [Mitchel et al. 2021]	92.5%
DiffusionNet - xyz	90.6%
DiffusionNet - hks	91.7%
MeshCNN [Hanocka et al. 2019]	92.3%
MeshWalker [Lahav and Tal 2020]	94.8%
DiffusionNet - xyz	<b>95.5%</b>
DiffusionNet - hks	<b>95.5%</b>
PD-MeshNet [Milano et al. 2020]	85.6%
HodgeNet [Smirnov and Solomon 2021]	85.0%
DiffusionNet - xyz	<b>90.3%</b>
DiffusionNet - hks	<b>90.8%</b>

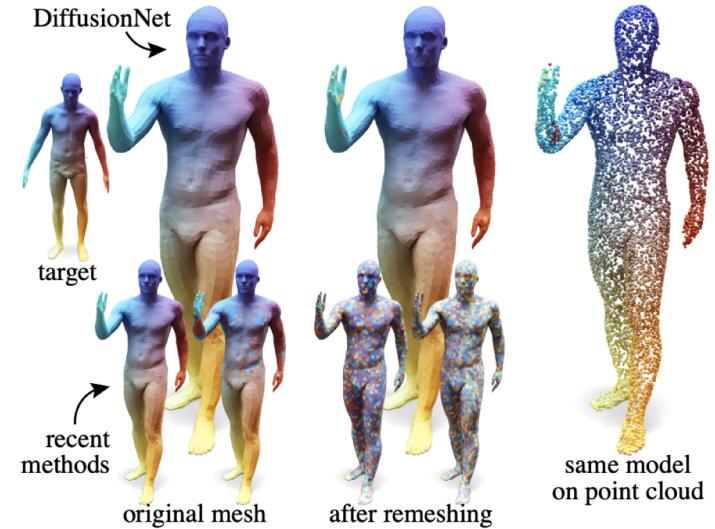
DiffusionNet: Discretization Agnostic Learning on Surfaces, N. Sharp, S. Attaiki, K. Crane, M.O., Trans. On Graph., 2022

\*data released as part of Poulenard et al. Effective rotation-invariant point cnn with spherical harmonics kernels, 3DV 2019

# DiffusionNet Results

## Shape Correspondence with *DiffusionNet*:

Method / Dataset	FAUST	SCAPE	FonS	SonF
KPConv [Thomas et al. 2019]	3.1	4.4	11.0	6.0
KPConv - hks	2.9	3.3	10.6	5.5
HSN [Wiersma et al. 2020]	3.3	3.5	25.4	16.7
ACSCNN [Li et al. 2020c]	<b>2.7</b>	3.2	8.4	6.0
DiffusionNet - hks	<b>2.7</b>	<b>3.0</b>	3.8	<b>3.0</b>
DiffusionNet - xyz + ZoomOut	<b>2.7</b>	<b>3.0</b>	<b>3.3</b>	<b>3.0</b>



# Performance

(all on single RTX2070, in PyTorch)

- ↳ training at 42ms/input on 15k vertex/point inputs
- ↳ < 2.5 GB memory usage for training
- ↳ ~3 sec precompute per input on CPU (scipy)

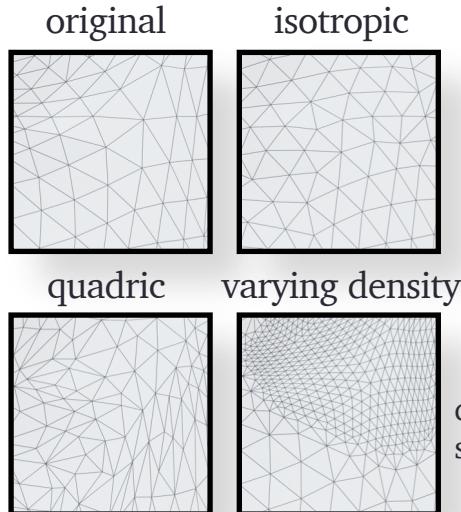
**Key property in practice:**  
runs easily on full-size meshes/clouds!  
(no remeshing/downsampling)



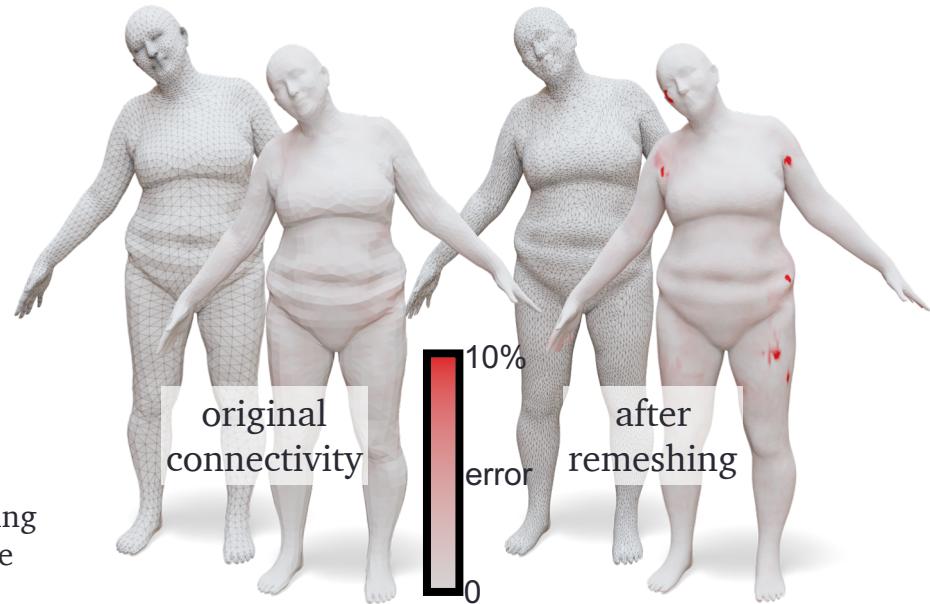
# Benefits

## Sampling

DiffusionNet is quite robust to resampling by default



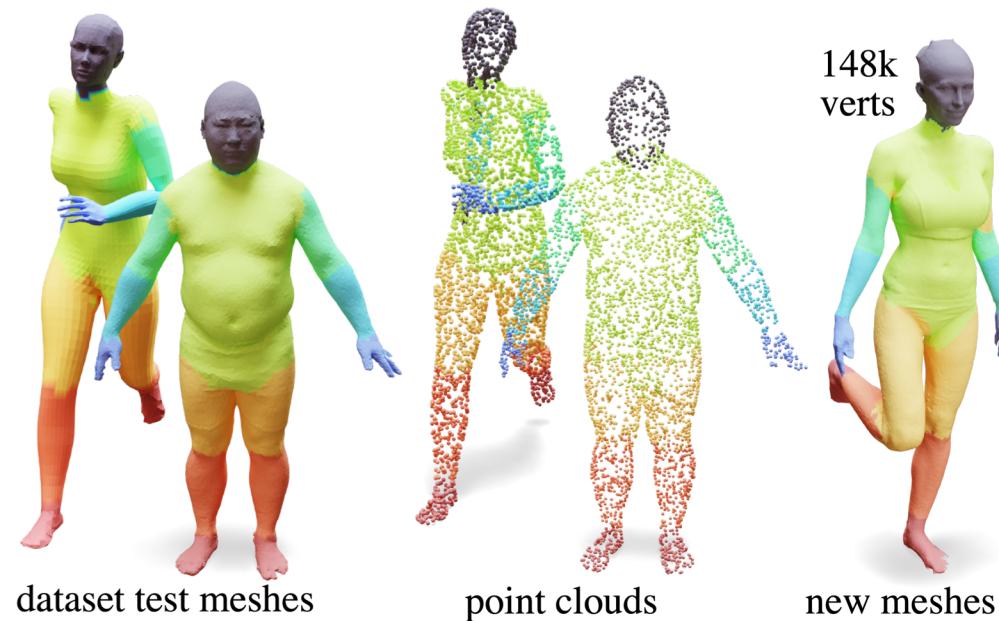
& point cloud sampling



# Benefits

## Transferability

train on meshes,  
infer on a point cloud!



# Conclusion

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- ➊ Spectral filtering is efficient but not easy to generalize across domains
- ➋ Geodesic CNNs generalize well, but can be unstable and not robust.
- ➌ DiffusionNet a robust method for learning on surfaces.