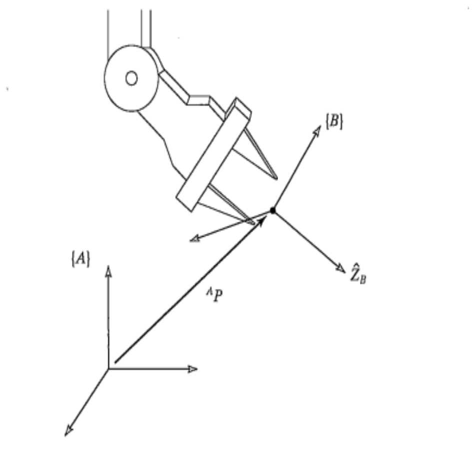
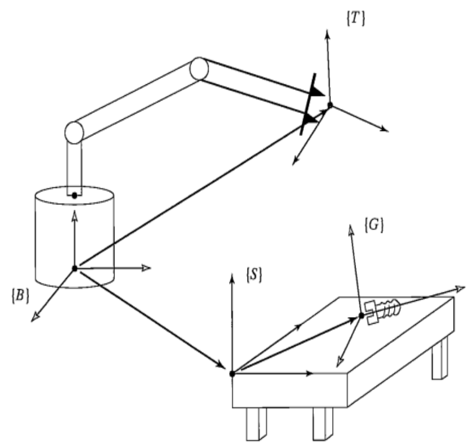
Introduction to Robotics

[Introduction to Robotics book PDF](), Third Edition. *John J. Craig*,

Youtube : 고려대학교 송재복 교수님

Chapter 1. Spatial descriptions and Transformations

1-1. Descriptions: Pose, Position, Orientation

 FIGURE 2.2: Locating an object in position and orientation.	 FIGURE 2.16: Manipulator reaching for a bolt.

- Pose = position(위치) + orientation(방위, 자세, ...)

- example : [ROS Pose Message]()

- Position

- **Description of a position**
  - ${}^{A}P$: position vector of P written in {A}

$$ {}^{A}P = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} $$

- Orientation

- **Description of a orientation**

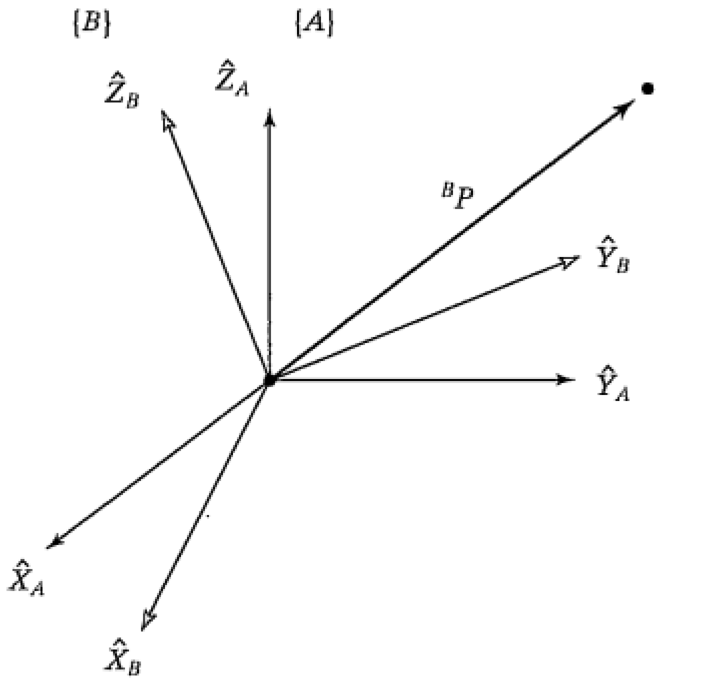
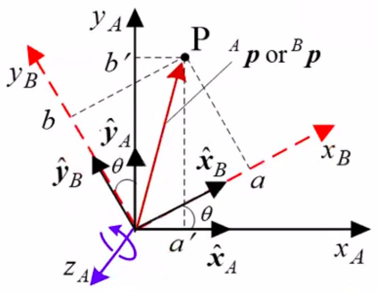


FIGURE 2.5: Rotating the description of a vector.



$$^{A}\boldsymbol{p} = a'\hat{\boldsymbol{x}}_A + b'\hat{\boldsymbol{y}}_A \;\; \& \;\; ^{B}\boldsymbol{p} = a\hat{\boldsymbol{x}}_B + b\hat{\boldsymbol{y}}_B$$

$$\begin{aligned} ^{A}\boldsymbol{p} &= (^{B}\boldsymbol{p}\cdot\hat{\boldsymbol{x}}_A)\hat{\boldsymbol{x}}_A + (^{B}\boldsymbol{p}\cdot\hat{\boldsymbol{y}}_A)\hat{\boldsymbol{y}}_A \\ &= [(a\hat{\boldsymbol{x}}_B + b\hat{\boldsymbol{y}}_B)\cdot\hat{\boldsymbol{x}}_A]\hat{\boldsymbol{x}}_A + [(a\hat{\boldsymbol{x}}_B + b\hat{\boldsymbol{y}}_B)\cdot\hat{\boldsymbol{y}}_A]\hat{\boldsymbol{y}}_A \\ &= [a\hat{\boldsymbol{x}}_B\cdot\hat{\boldsymbol{x}}_A + b\hat{\boldsymbol{y}}_B\cdot\hat{\boldsymbol{x}}_A]\hat{\boldsymbol{x}}_A + [a\hat{\boldsymbol{x}}_B\cdot\hat{\boldsymbol{y}}_A + b\hat{\boldsymbol{y}}_B\cdot\hat{\boldsymbol{y}}_A]\hat{\boldsymbol{y}}_A \end{aligned}$$

$$^{A}\boldsymbol{p} = \begin{bmatrix} a' \\ b' \end{bmatrix} = \begin{bmatrix} \hat{\boldsymbol{x}}_B\cdot\hat{\boldsymbol{x}}_A & \hat{\boldsymbol{y}}_B\cdot\hat{\boldsymbol{x}}_A \\ \hat{\boldsymbol{x}}_B\cdot\hat{\boldsymbol{y}}_A & \hat{\boldsymbol{y}}_B\cdot\hat{\boldsymbol{y}}_A \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \hat{\boldsymbol{x}}_B\cdot\hat{\boldsymbol{x}}_A & \hat{\boldsymbol{y}}_B\cdot\hat{\boldsymbol{x}}_A \\ \hat{\boldsymbol{x}}_B\cdot\hat{\boldsymbol{y}}_A & \hat{\boldsymbol{y}}_B\cdot\hat{\boldsymbol{y}}_A \end{bmatrix} {}^{B}\boldsymbol{p}$$

  - $^{A}\hat{X}_B$: the vector whose components are the projections of that vector onto the unit directions of its reference frame.

$$^{A}_{B}R = [\, ^{A}\hat{X}_B \;\; ^{A}\hat{Y}_B \;\; ^{A}\hat{Z}_B \,] = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

$$^{A}_{B}R = [\, ^{A}\hat{X}_B \;\; ^{A}\hat{Y}_B \;\; ^{A}\hat{Z}_B \,] = \begin{bmatrix} \hat{X}_B\cdot\hat{X}_A & \hat{Y}_B\cdot\hat{X}_A & \hat{Z}_B\cdot\hat{X}_A \\ \hat{X}_B\cdot\hat{Y}_A & \hat{Y}_B\cdot\hat{Y}_A & \hat{Z}_B\cdot\hat{Y}_A \\ \hat{X}_B\cdot\hat{Z}_A & \hat{Y}_B\cdot\hat{Z}_A & \hat{Z}_B\cdot\hat{Z}_A \end{bmatrix}$$

- **Physical meaning of rotation matrix**
  - $^{A}\mathrm{R}_B$: rotation matrix from the coordinates of P relative to {B} to coordinates relative to {A}

$$^{A}P = {}^{A}_{B}R \, {}^{B}P.$$

Example

Figure 2.6 shows a frame $\{B\}$ that is rotated relative to frame $\{A\}$ about $\hat{Z}$ by 30 degrees. Here, $\hat{Z}$ is pointing out of the page.

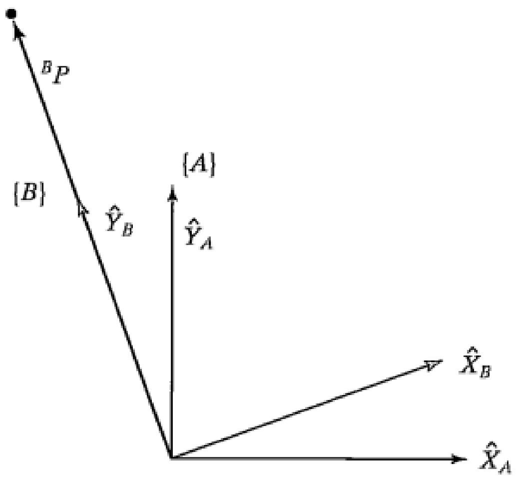


FIGURE 2.6: $\{B\}$ rotated 30 degrees about $\hat{Z}$.

Given, ${}^{B}P = \begin{bmatrix} 0.0 \\ 2.0 \\ 0.0 \end{bmatrix}$

we obtain

$$ {}^{A}R_{B} = \begin{bmatrix} \cos\theta & -sin\theta & 0 \\ sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.866 & -0.500 & 0.000 \\ 0.500 & 0.866 & 0.000 \\ 0.000 & 0.000 & 1.000 \end{bmatrix} $$

we calculate ${}^{A}P$ as

$$ {}^{A}P = {}^{A}R_{B}{}^{B}P = \begin{bmatrix} -1.000 \\ 1.732 \\ 0.000 \end{bmatrix} $$

- **Properties of a rotation matrix**

  orthonormal matrix(직교 행렬)

$$ {}^{A}_{B}R = {}^{B}_{A}R^{-1} = {}^{B}_{A}R^{T}. $$

$$ {}^{A}P = {}^{A}R_{B}{}^{B}P $$
$$ {}^{B}P = {}^{B}R_{A}{}^{A}P $$
$$ {}^{A}P = {}^{A}R_{B}{}^{B}R_{A}{}^{A}P $$
$$ {}^{A}R_{B}{}^{B}R_{A} = I $$

Indeed, from linear algebra, we know that the inverse of a matrix with orthonormal columns is equal to its transpose.

- Orthogonality conditions

$$\hat{x}\cdot\hat{y}=0, \hat{y}\cdot\hat{z}=0, \hat{z}\cdot\hat{x}=0$$

- Unit length conditions

$$|\hat{x}|=1, |\hat{y}|=1, |\hat{z}|=1$$

Only 3 components are independent among 9 components of a rotation matrix because 6 conditions are imposed.]

즉, 3개의 파라미터만 결정을 하면 orientation이 정해진다는 의미입니다.

1-2. Operators: Translation, Rotation, and Transformation

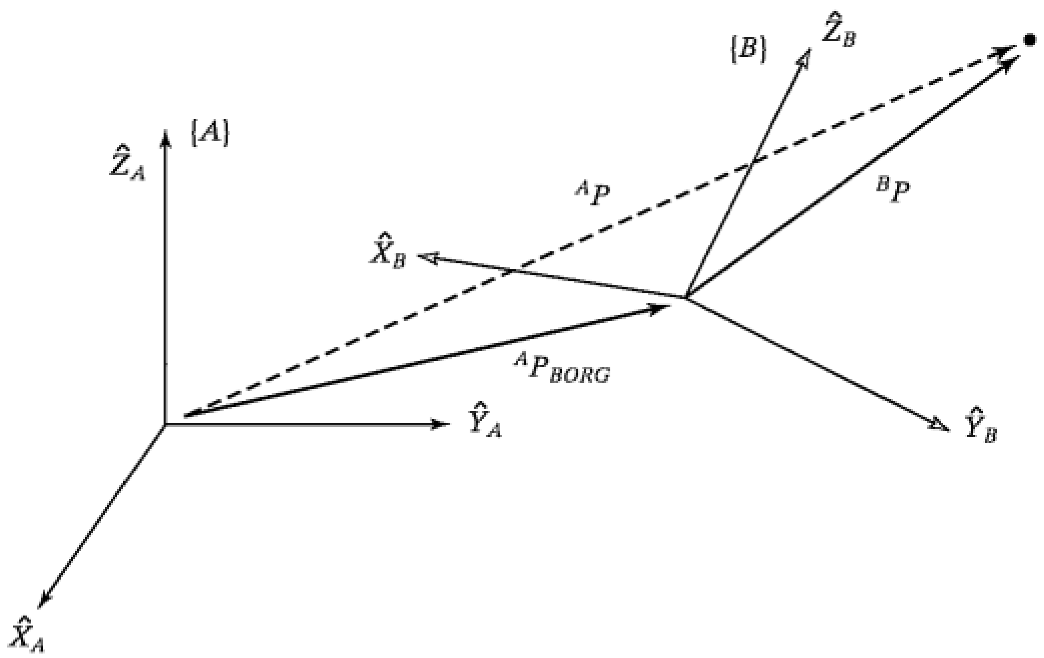


FIGURE 2.7: General transform of a vector.

$$^AP = {}^A_BR\,{}^BP + {}^AP_{BORG}.$$

$$\begin{bmatrix} ^AP \\ 1 \end{bmatrix} = \left[\begin{array}{ccc|c} & {}^A_BR & & ^AP_{BORG} \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \begin{bmatrix} ^BP \\ 1 \end{bmatrix}.$$

- Homogeneous transform (동차변환)

  4 x 4 matrix

$$H=\begin{bmatrix} R(3X3) & T(3X1)) \\ 0(1X3) & 1(1X1)) \end{bmatrix}$$

- Pure translation

$$Trans(Px,Py,Pz)=\begin{bmatrix} 1 & 0 & 0 & Px \\ 0 & 1 & 0 & Py \\ 0 & 0 & 1 & Pz \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Pure Rotation (Elementary rotation)
    - General rotation : a suitable sequence of three elementary rotations on condition that two successive rotations are not made about parallel axes.
    - Fixed angle representation (12 sets)

      Rotation with respect to the fixed axes 👉 Absolute transforms

      Typically, XYZ fixed angles (roll, pitch, yaw angles)
    - [Euler angle]() representation (12 sets)

      Rotation with respect to the current axes (or body-fixed axes) 👉 Relative transforms

      XYZ, ZXY, XZX, XYX, YXZ, YZX, YXY, YZY, ZXY, ZYZ, ZXZ, ZYX

      Typically, ZYX Euler angles and ZYZ Euler angles
    - 각도 계산 시 [arctan2]()를 이용

      arctan : range [-pi/2, pi/2]

      arctan2 : range [-pi, pi], arctan2(y, x) = arctan2(sin(theta), cos(theta))

```
result_arctan = np.arctan([-1/-1]) * 180 / np.pi
>>> [45.]
result_arctan2 = np.arctan2([-1], [-1]) * 180 / np.pi
>>> [-135.]
```

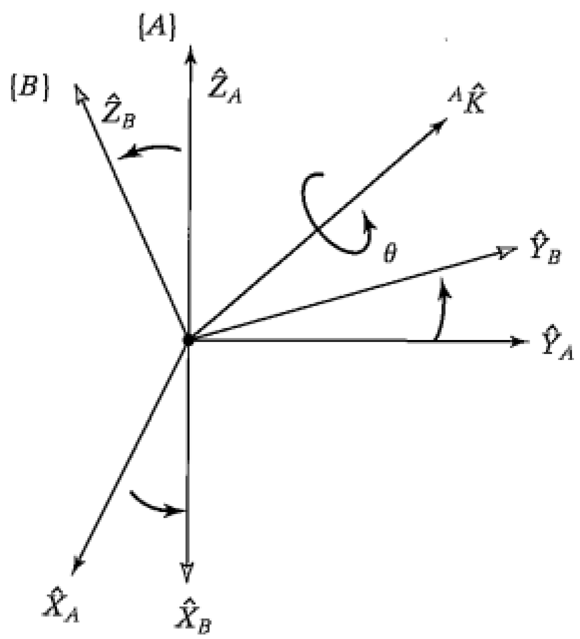


FIGURE 2.19: Equivalent angle–axis representation.

$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos(\theta) & -sin(\theta) \\ 0 & sin(\theta) & cos(\theta) \end{bmatrix}$	$Ry(\theta) = \begin{bmatrix} cos(\theta) & 0 & sin(\theta) \\ 0 & 1 & 0 \\ -sin(\theta) & 0 & cos(\theta) \end{bmatrix}$	$Rz(\theta) = \begin{bmatrix} cos(\theta) & -sin(\theta) & 0 \\ sin(\theta) & cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$

- Homogeneous transform

  H = Translation x Rotation,

  H = TR

$$H = \begin{bmatrix} R & T \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} I & T \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R & 0 \\ 0 & 1 \end{bmatrix}$$

${}^{A}T_{B}^{-1}$: Inverse transform matrix

$$
\begin{aligned}
&{}^{A}P = {}^{A}P_{BORG} + {}^{A}R_{B}{}^{B}P \\
&{}^{A}R_{B}^{T}{}^{A}P = {}^{A}R_{B}^{T}{}^{A}P_{BORG} + {}^{A}R_{B}^{T}{}^{A}R_{B}{}^{B}P = {}^{A}R_{B}^{T}{}^{A}P_{BORG} + {}^{B}P \\
&{}^{B}P = {}^{A}R_{B}^{T}{}^{A}P - {}^{A}R_{B}^{T}{}^{A}P_{BORG}
\end{aligned}
$$

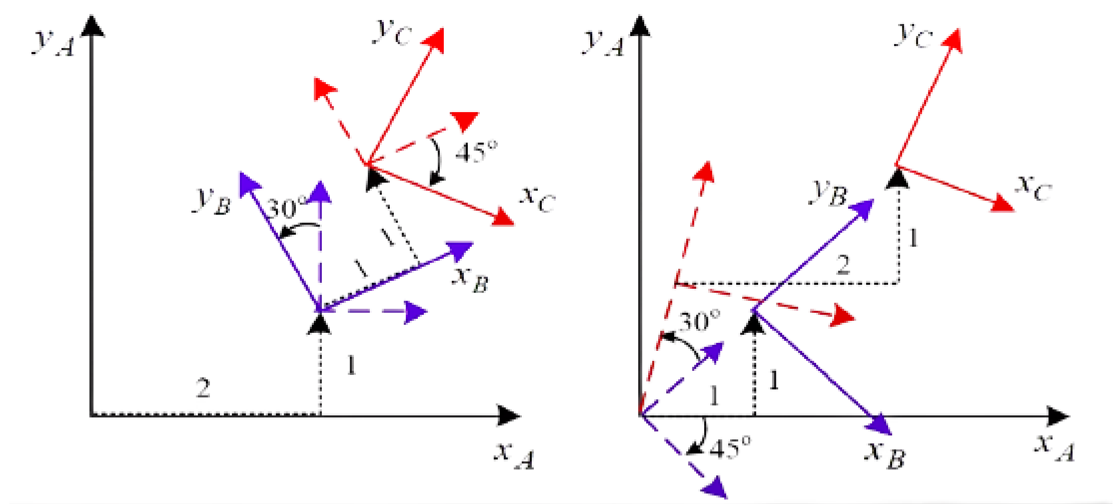
$${}^{B}_{A}T = \left[\begin{array}{ccc|c} & {}^{A}_{B}R^{T} & & -{}^{A}_{B}R^{T}\,{}^{A}P_{BORG} \\ \hline 0 & 0 & 0 & 1 \end{array}\right].$$

$${}^{A}T_{B}^{-1} = {}^{B}_{A}T$$

1-3. Relative transform and Absolute transform

- Relative transform
  - Translation relative to the current frame ==> Rotation relative to the current frame
  - Each transform is performed relative to the current frame.
  - The final frame is computed by **post-multiplying** (left -> right) the corresponding transformation matrices successively.
- Absolute transform
  - Rotation relative to the fixed frame -> Translation relative to the fixed frame
  - Each transform is performed relative to the fixed(reference) frame.
  - The final frame is computed by **pre-multiplying** (right -> left) the corresponding transformation matrices successively.

$$
{}^{A}T_{B} = \begin{bmatrix} \cos 30^{o} & -\sin 30^{o} & 0 & 2 \\ \sin 30^{o} & \cos 30^{o} & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad {}^{B}T_{C} = \begin{bmatrix} \cos(-45^{o}) & -\sin(-45^{o}) & 0 & 1 \\ \sin(-45^{o}) & \cos(-45^{o}) & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$



> H = T1 * R1 * T2 * R2
>
> Relative transform
>
> - (Translation of A -> B) ---> (Rotation of A -> B) ---> (Translation of B -> C) ---> (Rotation of B->C)
>
> Absolute transform
>
> - (Translation of A -> B) <--- (Rotation of A -> B) <--- (Translation of B -> C) <--- (Rotation of B->C)

1-4. Transform Equations

> Examples

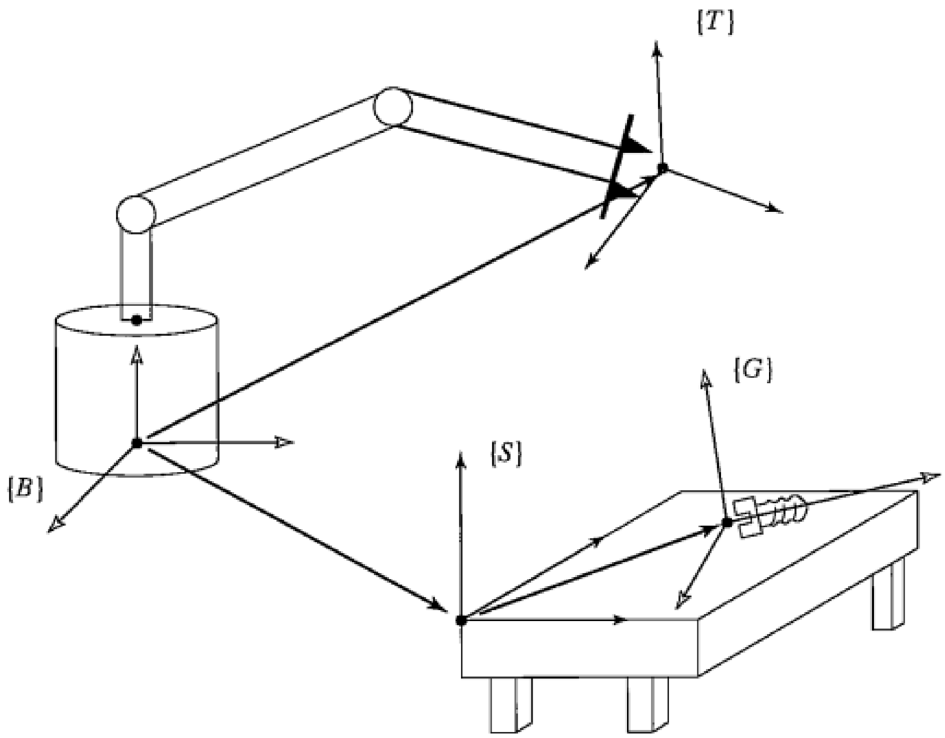


FIGURE 2.16: Manipulator reaching for a bolt.

EXAMPLE 2.6

Assume that we know the transform ${}^B_T T$ in Fig. 2.16, which describes the frame at the manipulator's fingertips $\{T\}$ relative to the base of the manipulator, $\{B\}$, that we know where the tabletop is located in space relative to the manipulator's base (because we have a description of the frame $\{S\}$ that is attached to the table as shown, ${}^B_S T$), and that we know the location of the frame attached to the bolt lying on the table relative to the table frame—that is, ${}^S_G T$. Calculate the position and orientation of the bolt relative to the manipulator's hand, ${}^T_G T$.

Guided by our notation (and, it is hoped, our understanding), we compute the bolt frame relative to the hand frame as

$$ {}^T_G T = {}^B_T T^{-1}\, {}^B_S T\, {}^S_G T. \tag{2.55}$$

1-5. More on representation of Orientation

- Inverse XYZ Fixed angles, ZYX Euler angles, ZYZ Euler angles problem

- **XYZ fixed angles**

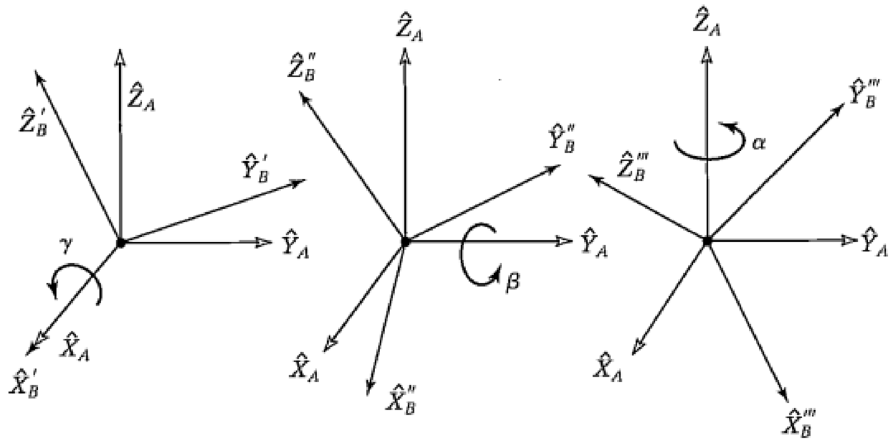


FIGURE 2.17: X–Y–Z fixed angles. Rotations are performed in the order $R_X(\gamma)$, $R_Y(\beta)$, $R_Z(\alpha)$.

$$
\begin{aligned}
{}^A_BR_{XYZ}(\gamma,\beta,\alpha) &= R_Z(\alpha)R_Y(\beta)R_X(\gamma) \\
&= \begin{bmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\gamma & -s\gamma \\ 0 & s\gamma & c\gamma \end{bmatrix}, \qquad (2.63)
\end{aligned}
$$

$$
{}^A_BR_{XYZ}(\gamma,\beta,\alpha) = \begin{bmatrix} c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma \\ s\alpha c\beta & s\alpha s\beta s\gamma + c\alpha c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma \\ -s\beta & c\beta s\gamma & c\beta c\gamma \end{bmatrix}. \qquad (2.64)
$$

$$
{}^A_BR_{XYZ}(\gamma,\beta,\alpha) = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}.
$$

- if $cos(\beta) \neq 0$

$$
\begin{aligned}
\beta &= \text{Atan2}(-r_{31}, \sqrt{r_{11}^2 + r_{21}^2}), \\
\alpha &= \text{Atan2}(r_{21}/c\beta, r_{11}/c\beta), \\
\gamma &= \text{Atan2}(r_{32}/c\beta, r_{33}/c\beta),
\end{aligned}
$$

we always compute the single solution for which $-90.0° \leq \beta \leq 90.0°$

- If $\beta = 90.0°$

$$\alpha - \gamma = atan2(-r_{12}, r_{22})$$

$$
\begin{aligned}
\beta &= 90.0°, \\
\alpha &= 0.0, \\
\gamma &= \text{Atan2}(r_{12}, r_{22}).
\end{aligned}
$$

- If $\beta = -90.0°$

$$\alpha + \gamma = atan2(-r_{12}, r_{22})$$

$$\beta = -90.0°,$$

$$\alpha = 0.0,$$

$$\gamma = -\mathrm{Atan2}(r_{12}, r_{22}).$$

  In those cases, only the sum or the difference of α and γ can be computed.

  -> **Infinitely many solution for α and γ**

  Not able to distinguish the rotation α and γ

  ->**Representation singularity** (1 자유도 상실)

- **ZYX Euler angles**

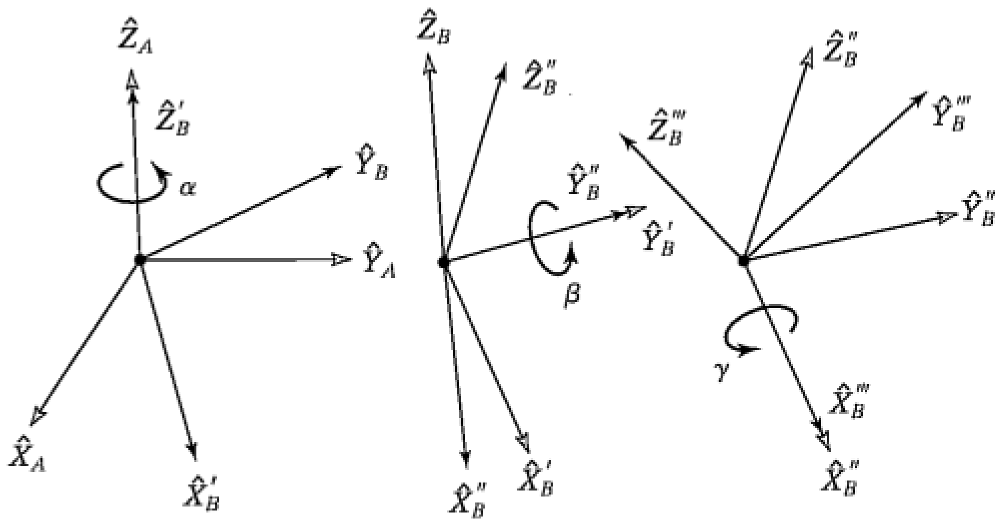


FIGURE 2.18: Z−Y−X Euler angles.

  three rotations taken about fixed axes yield the same final orientation as the same three rotations taken in opposite order about the axes of the moving frame.

- **ZYZ Euler angles**

  we always compute the single solution for which $0° \leq \beta \leq 180.0°$

  If $\beta = 0.0$or $\beta = 180.0$,

  In those cases, only the sum or the difference of α and γ can be computed.

  Not able to distinguish the rotation α and γ

  ->**Representation singularity** (1 자유도 상실)

1-6. Equivalent angle-axis representation

wikipedia

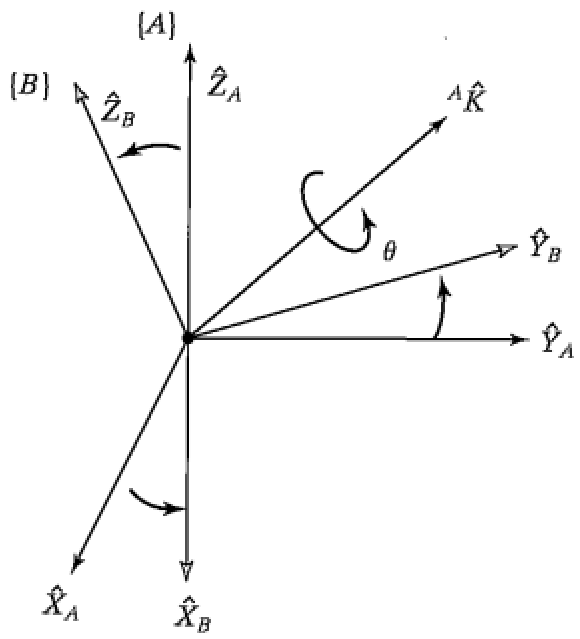


FIGURE 2.19: Equivalent angle–axis representation.

$$R_K(\theta) = \begin{bmatrix} k_x k_x v\theta + c\theta & k_x k_y v\theta - k_z s\theta & k_x k_z v\theta + k_y s\theta \\ k_x k_y v\theta + k_z s\theta & k_y k_y v\theta + c\theta & k_y k_z v\theta - k_x s\theta \\ k_x k_z v\theta - k_y s\theta & k_y k_z v\theta + k_x s\theta & k_z k_z v\theta + c\theta \end{bmatrix}$$

$$^A_B R_K(\theta) = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix},$$

$$\theta = \text{Acos}\left(\frac{r_{11} + r_{22} + r_{33} - 1}{2}\right)$$

$$\hat{K} = \frac{1}{2\sin\theta}\begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix}$$

Singularity : The solution given by fails if θ = 0° or θ = 180

1-7. Unit Quaternion

wikipedia

- **Euler parameters.**

$$\epsilon_1 = k_x \sin\frac{\theta}{2},$$
$$\epsilon_2 = k_y \sin\frac{\theta}{2},$$
$$\epsilon_3 = k_z \sin\frac{\theta}{2},$$
$$\epsilon_4 = \cos\frac{\theta}{2}.$$

$\epsilon_1^2 + \epsilon_2^2 + \epsilon_3^2 + \epsilon_4^2 = 1$, Only three parameters are independent.

$$R_\epsilon = \begin{bmatrix} 1 - 2\epsilon_2^2 - 2\epsilon_3^2 & 2(\epsilon_1\epsilon_2 - \epsilon_3\epsilon_4) & 2(\epsilon_1\epsilon_3 + \epsilon_2\epsilon_4) \\ 2(\epsilon_1\epsilon_2 + \epsilon_3\epsilon_4) & 1 - 2\epsilon_1^2 - 2\epsilon_3^2 & 2(\epsilon_2\epsilon_3 - \epsilon_1\epsilon_4) \\ 2(\epsilon_1\epsilon_3 - \epsilon_2\epsilon_4) & 2(\epsilon_2\epsilon_3 + \epsilon_1\epsilon_4) & 1 - 2\epsilon_1^2 - 2\epsilon_2^2 \end{bmatrix}.$$

$$\epsilon_1 = \frac{r_{32} - r_{23}}{4\epsilon_4},$$
$$\epsilon_2 = \frac{r_{13} - r_{31}}{4\epsilon_4},$$
$$\epsilon_3 = \frac{r_{21} - r_{12}}{4\epsilon_4},$$
$$\epsilon_4 = \frac{1}{2}\sqrt{1 + r_{11} + r_{22} + r_{33}}.$$

not Singularity

---

next 👉 forward kinematics