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Abstract. Robot arms with redundant degrees of freedom (DoF) are of major importance in advanced robotics where smooth trajectories (no brusque changes in the effector configuration) or obstacle avoidance in the working area are needed. Unfortunately, closed form solutions for the inverse kinematics problem of a more than 6 DoF are not available, and the problem is always solved by approximation using iterative numerical methods (i.e. Jacobian methods). The method that we describe in this paper allows the analytical description of the inverse kinematics of a robot arm with seven degrees of freedom. By formally substituting every two following joints with a spherical joint, it is possible to develop a kinematical model from which the joint angles can be extracted. The solution obtained has the advantage to be exact and to have a low computational cost. This model also allows to compute the redundancy manifold of the arm. This manifold can be described in the cartesian space as a circle and this analytical form opens a new perspective for the development of algorithms for obstacle avoidance, joint limits avoidance, or energy optimization.

1. Overview

One of the standard methods used for the control of robot arms is the so-called inverse kinematics, i.e. the transformation of the six cartesian coordinates of the end-effector (3 position, 3 orientation) into the n-dimensional angle space of the robot:

$$(\theta_1, \theta_2, \theta_3, \ldots, \theta_n)^T = \vec{F}(x_1, x_2, x_3, \ldots, x_6) \qquad (1)$$

In classical robotics up to six degrees of freedom can be handled in a closed form [5][1]. In the following case we consider seven joint angles. For the solution of this problem a first order Taylor expansion of the forward kinemaics is often used [5]:

$$\vec{x} = \vec{f}(\vec{\theta}) \quad \cong \quad \vec{f}(\vec{\theta}_0) + (\vec{\nabla}_\theta \vec{f}(\vec{\theta}_0))(\vec{\theta} - \vec{\theta}_0) \qquad (2)$$

$$\overrightarrow{\Delta x} = (\vec{\nabla}_\theta \vec{f}(\vec{\theta}_0))\ \overrightarrow{\Delta\theta} \qquad (3)$$

where $\vec{f}(\vec{\theta}_0) = \vec{F}^{-1}(\vec{\theta}_0) = \vec{x}_0$ is the current end-effector position and $\vec{\theta}_0$ the corresponding joint angle configuration of the robot. $\vec{\nabla}_\theta \vec{f}(\vec{\theta}_0)$ is also called the Jacobian matrix $\underline{J}$ at position $\vec{\theta}_0$ [6] and can easily be expressed by the current rotation axes of the arm.

Usually, equation (3) is devided by a time interval Δt in order to obtain a relation between the endeffector speed $\vec{v}$ and the anglular speed $\vec{\omega}$ relative to each robot joint:

$$\vec{v} = \underline{J}\big|_{\vec{\theta}_0} \vec{\omega} \qquad (4)$$

In order to calculate the inverse kinematics, one needs to invert $\underline{J}$:

$$\vec{\omega} = \underline{J}^{-1}\Big|_{\vec{\theta}_0} \vec{v} \qquad (5)$$

When $\underline{J}$ is singular or not rectangular (which is the case for 7 DoF), $\vec{\omega}$ can only be approximated since $\underline{J}$ is not invertible. It is clear, that $\vec{\omega}$ cannot be defined in a unique way because of the redundancy of the arm. To overcome this problem the pseudo-inverse [7] of $\underline{J}$ is often used [3]:

$$\vec{\omega} = \underline{J}^{\dagger}\vec{v} + (\underline{I} - \underline{J}^{\dagger}\underline{J})\vec{\varphi}, \qquad (6)$$

where $\underline{J}^{\dagger}$ is the pseudo inverse of $\underline{J}$ and $\underline{I}$ the identity matrix.

The first term of (6) represents a particular solution of the problem. It uses the minimal $\vec{\omega}$ which is necessary for the realization of $\vec{v}$ [10]. The second term corresponds to the homogeneous solution, i.e. the solution of $\vec{0} = \underline{J}\big|_{\theta_0}\vec{\omega}$. This is the set of $\vec{\omega}$ which leaves the position and the orientation of the endeffector unchanged (redundancy). $(\underline{I} - \underline{J}^{\dagger}\underline{J})$ is also called the null space projection matrix [4]. It transforms an arbitrarily choosen vector $\vec{\varphi}$ into the so-called null space [8] which describes all joint angular speeds $\vec{\omega}$ under the constraint $\vec{v} = \vec{0}$.

The previous considerations belong strongly to the standard robotics. Equation (6) does not allow to solve the problem of singular configurations as well as the use of the redundancy for joint limit or obstacle avoidance. (The treatment of singularities on none redundant robots is discussed in [9]. The introduction of optimization criteria can be found in [2] and [4]).

In the following, we will present a method for the closed form calculation of the inverse kinematics of an 7-DoF robot, which was up to now not available. This solution carries some advantages:

- no approximation[1] with Taylor expansion is needed. Therefore a direct control of each point in the working area of the robot is possible in contrast to equation (2) which only allows an iterative computation.
- no Jacobian has to be inverted. Therefore no singularity treatment is necessairy as far as the pure mapping is concerned.
- the use of the redundancy takes place in the cartesian space.

2. Closed form solution for the inverse kinematics

2.1. Kinematical model

The problem when calculating seven joint angles results from the underdetermination of the inverse kinematics. In the following part we develope a kinematical modell which allows to define the redundancy of the robot arm in the cartesian space. We show that the redundancy can be described as a simple parametrical curve. Through the definition of a point on this curve - for example by an optimization algorithm or by predefined control from the user - it is then possible to overcome the underdetermination.

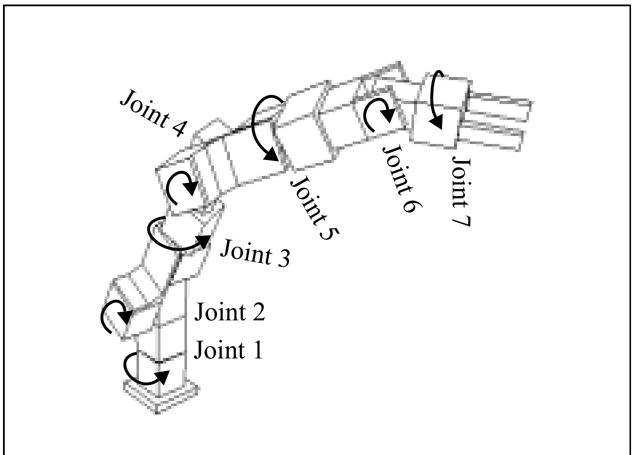

Figure 1. Seven DoF robot arm.

Figure 1 describes a typical robot arm with seven degrees of freedom. The arm is composed of a succession of roll and pitch joints. It is formally possible to fuse these two types of joints together in a spherical joint without any change in the kinematics (cf. fig. 2).

1. Of course, one obtains equation (4) by simply making use of the chain rule refering to the time derivation of equation (2), too. In this case (4), respectively (5), is an exact solution to the problem of mapping $\vec{v}$ on $\vec{\omega}$, however, it remains iterative according to the goal of reaching a certain endeffector position and orientation.

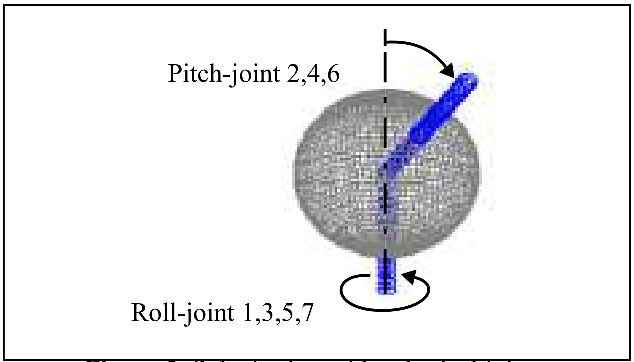

Figure 2. Substitution with spherical joints.

This formal transformation can be applied to joints 1/2, 3/4 and 5/6 at the joint positions 2, 4 and 6. Joint 7 remains unchanged. The previous substitution is not specific to the robot arm configuration we presented above. Nevertheless successive joint pairs need to be fusionable in spherical joints.

As the human arm also possesses seven degrees of freedom, it is possible to make a link with its anatomy (cf. fig. 3):

- Joint 1&2 at the position of the shoulder (origin of the reference coordinate system)
- Joint 3&4 at the position of the elbow: ($\vec{r}_U$)
- Joint 5&6 at the position of the wrist: ($\vec{r}_W$)
- Position of the endeffector: ($\vec{r}_{EEF}$)
- Upperarm: ($\vec{r}_U$)
- Forearm: ($\vec{r}_F$)
- Hand: ($\vec{r}_H$)
- Orientation of the gripper $\vec{r}_G$, with $\vec{r}_H \cdot \vec{r}_G = \vec{0}$.

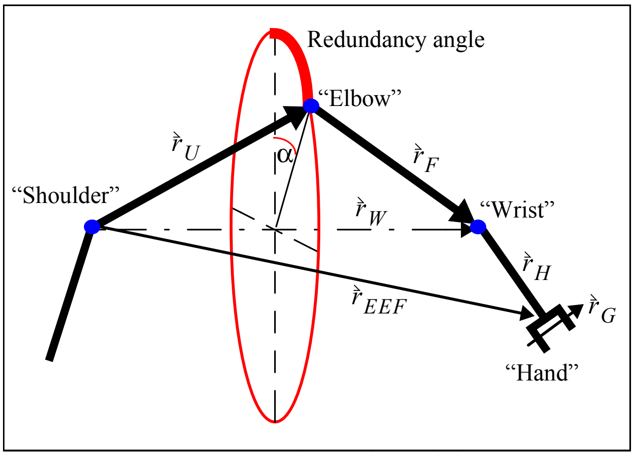

Figure 3. Kinematic model.

2.2. Redundancy

As far as the inverse kinematics are concerned, the position and orientation of the endeffector are given (i.e. $\vec{r}_{EEF}$, $\vec{r}_G$ and $\vec{r}_W$). When the formal representation of the spherical joints is used (shoulder joint, wrist joint), it is easy to see that the only remaining degree of freedom of the arm is definded by a circle. This circle represents the possible positions of the elbow around the shoulder-hand-axis (cf. fig. 3).

This result is easy to induce from the fact that the ending point of the upperarm describes a sphere centered on the shoulder joint, and that the starting point of the forearm describes a sphere centered on the wrist. As both of these points have to be the same, the redundancy curve is the result of the intersection of the two spheres: a circle. The mean point $\vec{r}_M$ and the radius R of this redundancy circle are easy to calculate from the arm construction:

$$R = \sqrt{\vec{r}_U^{\,2} - \left(\frac{\vec{r}_U^{\,2} - \vec{r}_F^{\,2} + \vec{r}_W^{\,2}}{2 \cdot |\vec{r}_W|}\right)^2} \tag{7}$$

$$\vec{r}_\mathrm{M} = \frac{\vec{r}_U^{\,2} - \vec{r}_F^{\,2} + \vec{r}_W^{\,2}}{2 \cdot |\vec{r}_W|^2}\vec{r}_W \tag{8}$$

If the position of the wrist is expressed in spherical coordinates ($\vec{r}_W = (\theta_W, \varphi_W, |\vec{r}_W|)^T$) the position of the elbow $\vec{r}_U$ can be written as:

$$\vec{r}_U = (\underline{R}_z^{\varphi_W} \cdot \underline{R}_y^{\theta_W} \cdot \underline{R}_z^{\alpha} \cdot \hat{e}_x) \cdot R + \vec{r}_M \tag{9}$$

with $\underline{R}_z$, $\underline{R}_y$ repectively the rotation matrix around z- and y-axis and α the redundancy angle. $\hat{e}_x$ is the basis vector in x-direction.

To begin with, let us consider the value of α as being given. The inverse kinematics are then easy to determine.

2.3. Calculation of the seven joint angles

Strictly spoken by the introduction of the redundancy angle α we make use of a constraint which allows to eliminate the underdetermination of the here discussed problem. Now it is possible to calculate $\vec{r}_U$, respectively $\vec{r}_F$ - and hence, all joint angles.

The latter ones correspond to sphere coordinates refering to their local coordinate system (cf. fig. 4)[1].

In the following we make use of a scheme which can be applied repeatedly to get all angles. First of all one has to transform the considered coordinate system, for instance the elbow system, into the reference system, simply by making use of the already known angles. The corresponding limb vector, for instance $\vec{r}_F$, has to be expressed in spherical coordinates.

Of course, $\vec{r}_U$ already belongs to the reference coordinate system. Therefore it does not need to be transformed. θ_1 and θ_2 can be easily obtained by the following relations:

$$\theta_1 = \mathrm{atan2}({}^{Ref}r_U^y, {}^{Ref}r_U^x) \tag{10}$$

$$\theta_2 = \mathrm{acos}\left(\frac{{}^{Ref}r_U^z}{|{}^{Ref}\vec{r}_U|}\right) \tag{11}$$

The upper left index describes the coordinate system we refer to.

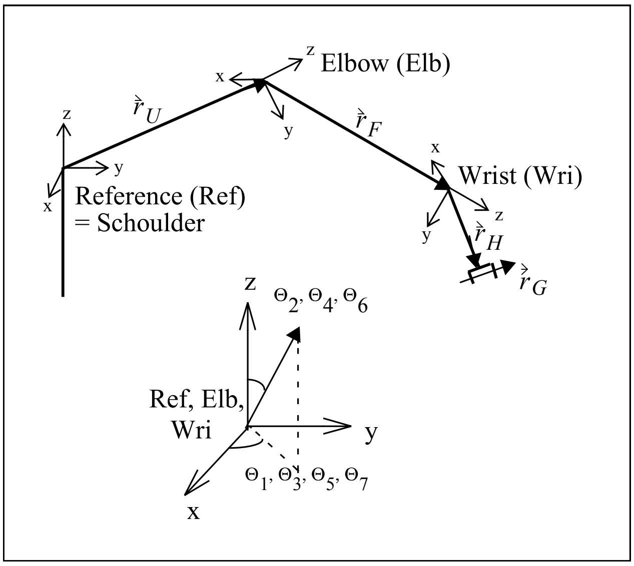

Figure 4. Reference coordinate system and joint spherical cordinates.

The next two joint angles belong to the elbow coordinate system. Hence it is necessary to perform the corresponding transformation first:

$${}^{Elb}\vec{r}_F = \underline{R}_y^{-\theta_2} \cdot \underline{R}_z^{-\theta_1} \cdot {}^{Ref}\vec{r}_F \tag{12}$$

Afterwards we can calculate θ_3 and θ_4 just as in equation (10) and (11):

$$\theta_3 = \mathrm{atan2}({}^{Elb}r_F^y, {}^{Elb}r_F^x) \tag{13}$$

$$\theta_4 = \mathrm{acos}\left(\frac{{}^{Elb}r_F^z}{|{}^{Elb}\vec{r}_F|}\right) \tag{14}$$

And in the same way we get θ_5 and θ_6:

$${}^{Wri}\vec{r}_H = \underline{R}_y^{-\theta_4} \cdot \underline{R}_z^{-\theta_3} \cdot \underline{R}_y^{-\theta_2} \cdot \underline{R}_z^{-\theta_1} \cdot {}^{Ref}\vec{r}_H \tag{15}$$

$$\theta_5 = \mathrm{atan2}({}^{Wri}r_H^y, {}^{Wri}r_H^x) \tag{16}$$

$$\theta_6 = \mathrm{acos}\left(\frac{{}^{Wri}r_H^z}{|{}^{Wri}\vec{r}_H|}\right) \tag{17}$$

1. In the case of other than the here treated joint sequences first of all one has to perform the corresponding transformation.

The last joint angle, namely θ_7, describes the gripper orientation. The corresponding vector $\vec{r}_G$ is given in reference coordinates (as well as all other $\vec{r}_{...}$). We know that it is always perpendicular to $\vec{r}_H$. Therefore - in addition to the transformation (15) - we have to rotate $\vec{r}_G$ into the x-y-plane of the wrist coordinate system:

$$^{Wri}\vec{r}_G = \underline{R}_y^{-\theta_6}\underline{R}_z^{-\theta_5}\underline{R}_y^{-\theta_4}\underline{R}_z^{-\theta_3}\underline{R}_y^{-\theta_2}\underline{R}_z^{-\theta_1} \cdot {}^{Ref}\vec{r}_G \tag{18}$$

$$\theta_7 = \mathrm{atan2}({}^{Wri}r_G^y, {}^{Wri}r_G^x) \tag{19}$$

With respect to the control of an robot arm through by the above calculated angles it is necessairy to take joint limits into consideration. If a joint limit is reached one has to switch between the two angle domains (θ,φ) and $(-\theta,\varphi\pm\pi)$. In the following section we give an example of how to avoid such switches, respectively configuration changes.

However, first of all we have to determine the redundancy angle which was so far assumed to be known.

2.4. The determination of the redundancy angle

In the absence of any constraints the redundancy angle α can be arbitrarily chosen within the interval $[0,2\pi]$. Of course, with respect to obstacle and joint limit avoidance one usually has to make use of the redundancy. For instance, in the presence of an obstacle - assuming that its position is well known and close enough to the arm - it is quite easy to calculate the remaining redundancy, simply by substracting the corresponding angle segment from the redundancy circle.

We would like to show that even more difficult problems can be treated in an analytical way: A typical joint limitation is that which occurs between the endeffector and the preceding limb. Let us assume that the minimal reachable angle is $\pi/2$. With reference to the proposed model this is equivalent to the following constraint (cf. fig. 5):

$$-\pi/2 \le \theta_6 \le \pi/2 \tag{20}$$

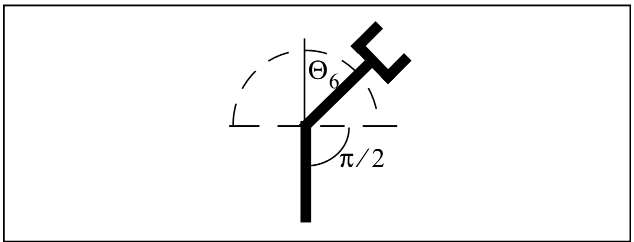

Figure 5. Joint angle constraint.

In order to obtain those states within the redundancy circle which do not violate equation [20] one can imagine a geometrical construction according to figure 6. It is easy to see that the intervall $[\kappa+\delta,\kappa-\delta]$ has to be avoided. Otherwise $\pi-\theta_6$ would be less than $\pi/2$, and therefore equation (20) not be fulfilled.

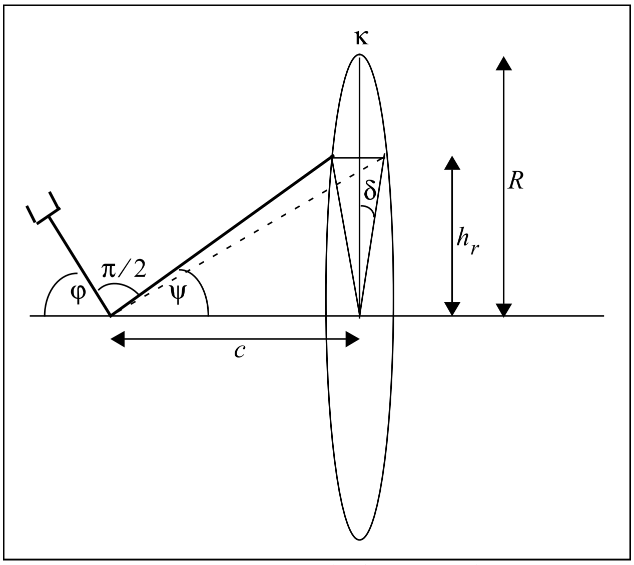

Figure 6. Geometrical construction

For calculating κ and δ one has to transform the shoulder-wrist-axis onto one axis of the reference coordinate system, for instance the z-axis. Then κ can be obtained by projecting the endeffector upon the x-y-plane. Using some quantities which can be extracted from figure 6 yields:

$$\delta = \mathrm{acos}\left(\frac{h_r}{R}\right) \tag{21}$$

$$\text{with}: \; \varphi = \mathrm{acos}\left(\frac{r_H^z}{|\vec{r}_H|}\right) \tag{22}$$

$$\psi = \pi/2 - \varphi \tag{23}$$

$$c = |\vec{r}_W| - |\vec{r}_M| \tag{24}$$

$$h_r = c \cdot \tan(\psi) \tag{25}$$

In a quite similar way one can make use of the redundancy in order to avoid configuration changes caused by other joint limits.

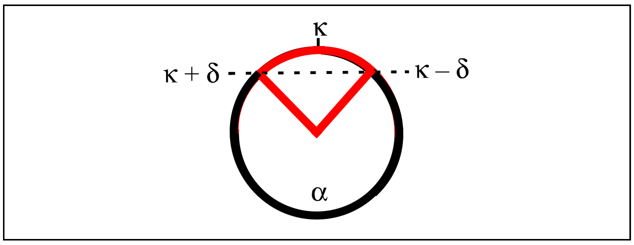

Figure 7. Limitation of the redundancy angle to avoid joint limits.

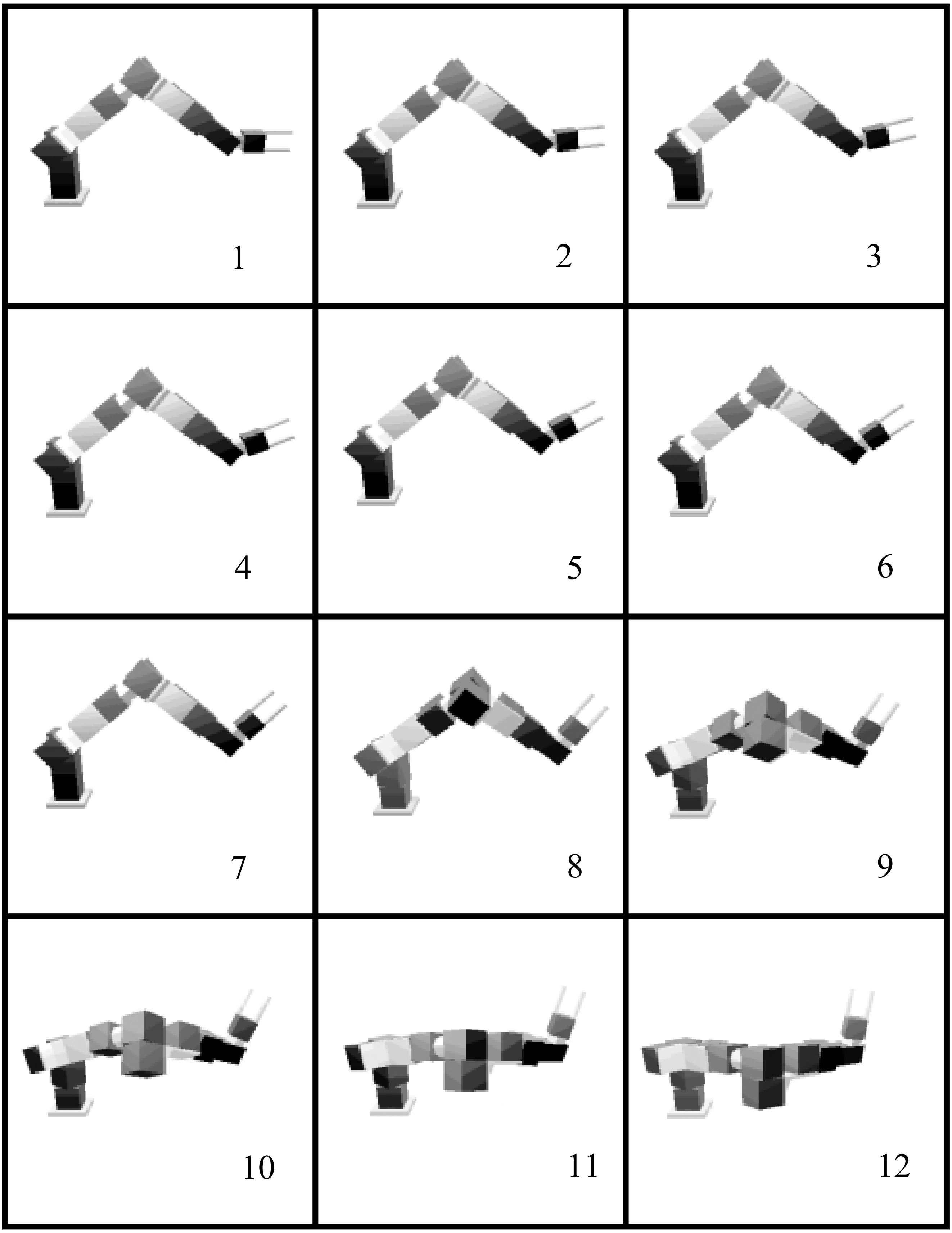

Figure 8. Joint limit avoidance application (see text)

Having calculated all the intervalls which are not allowed with respect to some certain constraints the redundancy angle α has to take place in those segments which remain after all the forbidden intervals have been substracted from the redundancy circle (cf. fig. 7).

3. Results

To validate the considerations above described we have implemented our model first within a simulation tool and then on a real robot arm.

In the image sequence figure 8, we show the result of our implementation of the joint limit avoidance described in section 2.4. While the end-effector moves towards a vertical orientation, the elbow follows the highest state which fulfills the above mentioned constraints.

Furthermore, we have implemented the inverse kinematics solution described in this paper on a real robot arm . This arm is part of a new project called NEUROS[1], which deals with a mobile service robot. Figure 9 serves for getting an impression of its complete appearence. It is easy to see that the joint sequence we refer to in this paper is not essential in order to make use of our closed form solution, as far as a transformation exists which maps the robot angles into spherical coordinates.

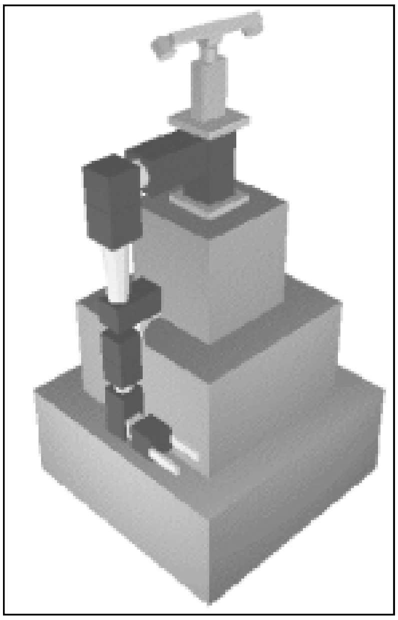

Figure 9. Our new robot called ARNOLD

4. Conclusions

In this paper we present a closed form solution for the inverse kinematics of a seven degree of freedom (7DoF) robot arm. It is based on the formal replacement of subsequent joints by spherical joints. In the resulting kinematical model the redundancy can be parameterized as a circle in the cartesian space.

We have shown how to make use of this *redundancy circle* in order to avoid, for example, joint limits.

In future work we will concentrate on fitting our model into a dynamical framework. It seems to be attractive to decompose the complex control problem for 7DoF into three single dynamics concerning the wrist (3DoF), the hand (3DoF) and the elbow (1DoF).

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