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Saturday, 30 May 2020

Important To Do List

- Formally prove the space X given by the norms in (4) is complete. **See Monday 1st June 2020**

- Try and understand the probabalistic smoothness calculations given in Lemma 7.4 of [8]. Once this is done, we can try adapting it to our more general situation. **See Friday, 4th September 2020**

Maybe Do

- Today I found an interesting survey on the Fourier Dimension [3]. If I find the time I should read through it more thoroughly to get some intuition. **See Sunday, 31st May 2020**

- I also found a survey on the application of the probabilistic method and the Baire category theorem in Harmonic analysis [7]. I feel this method is very exploitable in the types of problems I currently deal with, so if only for culture, this should be a useful read.

My main goals today were to finish up the slides, for my talk on Fourier dimension in pattern avoidance problems at the 2020 Ottawa Math Conference. I have essentially completed these slides; all that remains is to polish them up, and practice giving the presentation. My main goal during the presentation is to show that viewing avoiding sets Z geometrically leads to interesting questions, that the geometric quantities we consider lead to important consequences, and that the Fourier dimension question I am currently considering is interesting to study. I also thought about an idea which seemed to prevent adapting our result obtained by the 'queuing approach' in [5] to Thomas Körner's Baire category approach in [8], as well as finding a counterexample in a paper which removes the trivial case of [5] from the Fourier dimension case.

Ideas for Fourier Dimension Technique

Thomas Körner's paper [8] relies on Baire category arguments to construct generic measures μ supported on a subset of $\mathbf{T}$ avoiding solutions to m-term linear equations, such that for each $\xi \in \mathbf{Z}$,

$$|\widehat{\mu}(\xi)| \leq A(\xi).^1 \tag{1}$$

[1] Here $\{A(\xi)\}$ is a sequence given for each $\xi \in \mathbf{Z}$ by the formula

$$A(\xi) = B(\xi)|\xi|^{-\beta/2}\log(1+|\xi|)^{1/2},$$

where $\beta = (n-1)^{-1}$, and $\{B(\xi)\}$ is some fixed sequence of positive numbers such that $B(\xi) \to \infty$ as $|\xi| \to \infty$.

To obtain this bound generically, Körner works in the norm space X consisting of finite measures μ on $\mathbf{T}$ such that the quantity

$$\|\mu\|_X = \sup_{\xi \in \mathbf{Z}} \frac{|\widehat{\mu}(\xi)|}{A(\xi)}. \tag{2}$$

is finite. Then any measure in X satisfies (1) up to a multiplicative constant, and X is a Banach space, which enables one to use Baire-category techniques. A problem occurs in our Fourier dimension paper because I believe we can only construct finite measures μ such that for each $\varepsilon > 0$,

$$|\widehat{\mu}(\xi)| \lesssim_\varepsilon |\xi|^{\varepsilon-\beta/2}. \tag{3}$$

Such a measure does not satisfy quite as rigid an inequality as (1), instead having to satisfy infinitely many inequalities of the form (3), and as such I do not believe we can find a Banach space norm which encapsulates (3).

However, today I thought of an idea which might prove fruitful. For any measure μ satisfying (3) for each $\varepsilon > 0$, the quantities

$$\|\mu\|_\varepsilon = \sup_{\xi \in \mathbf{Z}} |\widehat{\mu}(\xi)||\xi|^{\beta/2-\varepsilon} \tag{4}$$

will be finite for all $\varepsilon > 0$. If we let X denote the family of all finite measures which satisfy (4) for all $\varepsilon > 0$, then the collection of seminorms $\{\|\cdot\|_\varepsilon : \varepsilon > 0\}$ might give X the structure of a Frechét space. Since Frechét spaces are complete metric spaces, we can still apply Baire category arguments here. It remains to check whether this really gives a complete metric space structure, however.

Finite Additivity of the Fourier Dimension

In [4], I found a result which constructs two disjoint, Borel sets $A, B \subset \mathbf{T}$, with $\dim_{\mathbf{F}}(A), \dim_{\mathbf{F}}(B) < 1$, but such that $A \cup B = \mathbf{T}$. The result of Theorem 1 of [5] is trivial when $\dim_{\mathbf{M}}(Z) < d$, for if $\pi : (\mathbf{T}^d)^n \to \mathbf{T}^d$ is given by projection onto the first d coordinates, then $\mathbf{T}^d - \pi(Z)$ has full Hausdorf dimension and avoids Z. If $Z = A \times \{0\}$, then $\mathbf{T} - \pi(Z) = B$, which is not full dimensional, so things are more complicated when dealing with Fourier dimension. I found this result in [3], which might be a useful survey to read through completely in order to get a better grasp on how the Fourier dimension behaves.

Sunday, 31st May 2020

Maybe Do

- Read Kahane's book [6] to obtain ideas on possible Fourier dimension constructions.
- Read Lyon's article [9] to obtain cultural background information about Fourier dimension in harmonic analysis.

Today I finished off my slides, prepared for the talk, and review Ekström and Schmeling's survey on Fourier dimension results [3]. The review is given below.

Ekström and Schmeling's "A Survey of Fourier Dimension"

This article gives a pretty diverse set of viewpoints about the Fourier dimension. I'll list off a couple of things that I hadn't heard about before:

- It is obvious that if μ is a finite Borel measure with Hausdorff dimension s and $\mu(E) > 0$, then $\dim_{\mathbf{H}}(E) \geq s$. Thus μ does not need to be supported on E completely. However, if we define the *modified Fourier dimension*

  $$\dim_{\mathbf{MF}}(E) = \sup\{\dim_{\mathbf{F}}(\mu) : \mu(E) > 0\}, \tag{5}$$

  we get a different dimension to the Fourier dimension

  $$\dim_{\mathbf{F}}(E) = \sup\{\dim_{\mathbf{F}}(\mu) : \operatorname{supp}(\mu) \subset E\}. \tag{6}$$

  The modified Fourier dimension *is* countably stable, i.e. for a countable collection of Borel sets $\{E_k\}$,

  $$\dim_{\mathbf{MF}}\left(\bigcup_k E_k\right) = \sup_k \left(\dim_{\mathbf{F}}(E_k)\right)$$

  The usual Fourier dimension is not even finitely stable, but is countably stable if we restrict ourselves to F_δ sets.

- It is interesting that we can define the Hausdorff dimension completely locally. Given a finite Borel measure μ on $\mathbf{R}^d$, and $x \in \mathbf{R}^d$, define the *local dimension* as

  $$\underline{\dim_{\mathbf{H}}}(\mu, x) = \liminf_{r \to 0} \log_r \mu(B_x(r)).$$

and

$$\overline{\dim_{\mathbf{H}}}(\mu, x) = \limsup_{r \to 0} \log_r \mu(B_x(r)).$$

Theorem 0.1. *If $\mu(E) > 0$ and $\underline{dim_{\mathbf{H}}}(\mu, x) \geq s$ for μ a.e. $x \in E$, then $dim_{\mathbf{H}}(E) \geq s$.*

Proof. For each $\varepsilon > 0$, there exists $r_0 > 0$ and $F \subset E$ with $\mu(F) > 0$ such that for each $x \in F$ and $r \leq r_0$, $\mu(B_x(r)) \geq r^{s-\varepsilon}$. This implies $\dim_{\mathbf{H}}(E) \geq \dim_{\mathbf{H}}(F) \geq s - \varepsilon$. We then take $\varepsilon \to 0$ to conclude $\dim_{\mathbf{H}}(E) \geq s$. □

Working locally might simplify calculations, rather than having to come up with uniform bounds on the dimension. There is no local dimension for the usual Fourier transform, since the Fourier transform is not monotone on sets. However, the modified Fourier dimension *is* monotone, so it might be possible to find a local charactierization of the Fourier dimension. For a measure μ, we define

$$\dim_{\mathbf{MF}}(\mu) = \sup\{\dim_{\mathbf{F}}(\eta) : \mu \ll \eta\},$$

so that $\dim_{\mathbf{MF}}(E) = \sup\{\dim_{\mathbf{MF}}(\mu) : \text{supp}(\mu) \subset E\}$. Then we can try and define the Fourier dimension locally.

- Given a family of measures M on a measure space X, it is interesting to find a family $\mathcal{U}$ of measurable subsets of X such that a measure μ on X is an element of M if and only if $\mu(E) = 0$ for all $E \in \mathcal{U}$. For instance, this is possible if a measure μ is fixed, and M is the family of all absolutely continuous measures with respect to μ. This is also possible for measures with Hausdorff dimension exceeding some value s - it suffices to consider the family of all sets with s Hausdorff measure zero. The family of *Rajchman measures* on $\mathbf{R}$ are the collection of all measures μ such that

$$\lim_{|\xi| \to \infty} \widehat{\mu}(\xi) = 0.$$

Suprisingly, one can find a family of sets $\mathcal{U}$ such that a measure μ is Rajchman if and only if it assigns mass zero to each set $E \in \mathcal{U}$. This is detailed in [9], which might also be a useful text to read for culture. If we let

$$M^{\perp} = \{E \subset X : \mu(E) = 0 \text{ for all } \mu \in M(X)\}$$

and

$$M^{\perp\perp} = \{\mu : \mu(E) = 0 \text{ for all } E \in M^{\perp}\}.$$

To show such a family exists for a given collection of measures M, it is necessary and sufficient to show $M^{\perp\perp} = M$. This is true for the family of measures with *modified Fourier dimension* greater than a given value, but not for the family of measures whose normal Fourier dimension is greater than a given value. It is an open question to explicitly identify the family of sets in these examples, however, which might be an interesting problem to work on.

- If E is a Borel subset of $\mathbf{R}$, then there exists a $C^{m+\alpha}$ diffeomorphism $f : \mathbf{R} \to \mathbf{R}$ such that

$$\dim_{\mathbf{F}}(f(E)) \geq \frac{\dim_{\mathbf{H}}(E)}{m+\alpha},$$

  In particular, if $m = 1$ and $\alpha = 0$, any Borel set is diffeomorphic to a Salem set. One can thus view the difference between Hausdorff and Fourier dimension as a measure of pertubation, in some sense. I should check whether it is possible to find a smooth faimly of deformations of E than smoothly deform the Fourier dimension up to the Hausdorff dimension. It is also an open question to construct an *explicit* diffeomorphism $f : \mathbf{R} \to \mathbf{R}$ between the Cantor set and a Salem set, since the construction above is obtained in a random fashion.

Looking at the references shows a list of books that might be useful to consult for intuition on the problems I'm working on now:

- J.P. Kahane's book [6] on random functions might be useful to obtain ideas on how to obtain Fourier dimension measures.
- R. Lyons article [9] might be useful for learning some cultural information about the history of Fourier dimension in harmonic analysis.

Monday 1st June 2020

Important To Do List

- Edit the proof in our existing draft on Fourier dimension to work completely rigorously for the new Frechét space. **See Tuesday 2nd June 2020**

I gave my presentation at the Ottawa Math Conference today. It went fairly well, I got a few questions so some people followed the talk. Later on in the day I wrote a formal proof that the space I thought up in Saturday, 30 May 2020 is actually a Frechét space. What remains is to modify the existing proof in the paper draft to completely rigorously prove that generic sets in the space are Salem and avoid patterns.

Tuesday 2nd June 2020

Today was mostly a writing day. I finished editing the proof in our paper to incorporate the new Frechét space idea, so now we can find sets with Fourier dimension

$$\frac{nd - \alpha}{n - 1/2}$$

avoiding patterns. All that remains to complete this project is to understand Körner's probabilistic analysis which enabled him to improve this bound in the special case of integer coefficient equations, see if we can generalize this to more general cases to yield a bound

$$\frac{nd - \alpha}{n - 1},$$

and then incorporate this into the paper.

Wednesday 3rd June 2020

Important To Do List

- Look back on decoupling argument and see if we can still utilize it to help obtain the probability bounds needed in our construction.
- Try and solve the tensorized special case of the argument.
- Prove that if we can prove the analogous problem in the discrete setting, then we automatically obtain the result in the general setting.

Today I cleaned up the proof in the Fourier dimension paper. Aside from expanding the background section, all that remains is to try and obtain the improved Fourier dimension by obtaining a square root cancellation in the final argument. I'll detail my thoughts on this in more detail in the next section.

Obtaining Square Root Cancellation

The only obstacle to obtaining the full result we desire in the Fourier dimension is understanding a fairly simple situation in probability; Consider a large integer K, let $W \subset \mathbf{T}^{dn}$ be a set, let $\varepsilon = K^{(1-n)/(dn-\alpha)}$, and suppose that $|W_\varepsilon| \leq K^{1-n}$, where

$$W_\varepsilon = \bigcup_{x \in W} B_\varepsilon(x).$$

Then take K independant and uniformly distributed random variables $X_1, \dots, X_K$ in $\mathbf{T}^d$. Let S be the set of indices $k_1 \in \{1, \dots, K\}$ such that there exists distinct indices $k_2, \dots, k_n \in \{1, \dots, K\}$ such that $(X_{k_1}, \dots, X_{k_n}) \in W_\varepsilon$. Standard expectation bounds imply that $\#(S) \leq K$ with high probability. For $\xi \in \mathbf{Z}^d$, we want to understand what conditions guarantee that with high probability square root cancellation occurs in the sum

$$\sum_{k \in S} e^{2\pi i \xi \cdot X_k},$$

i.e. under what conditions can we conclude that,

$$\left| \sum_{k \in S} e^{2\pi i \xi \cdot X_k} \right| \leq K^{1/2} \log(K)^{O(1)}.$$

If we write

$$Y_k = \begin{cases} e^{2\pi i \xi \cdot X_k} & : \text{if } k \in S, \\ 0 & : \text{if } k \notin S, \end{cases}$$

then our goal is to obtain square root cancellation in the sum $Y = Y_1 + \cdots + Y_K$. There are several reasons why we might expect this result to be the case:

- The variables $\{Y_k\}$ are identically distributed (but not independent) with $|Y_k| \leq 1$ for all $k \in \{1, \ldots, K\}$.
- The variables are only 'n coupled' with one another, so if K is large, the variables are not too coupled with one another. More precisely,

However, I am unable to find a general concentration result which can give us this result. To consider a simple analysis, let us suppose that there are sets $W_1, \ldots, W_n$ such that

$$W = W_1 \times \cdots \times W_n.$$

Is the problem at least solvable in this simple, tensorizable situation? I might also want to look back on the probabilistic decoupling techniques I was trying to utilize last year.

Sunday 4th June 2020

Today I discovered a useful probabilistic concentration result that reduces our square-root cancellation problem to concentration bounds on conditional expectations.

Reduction to Conditional Expectation Bound

Suppose we consider an independant family of random variables

$$\{X_{ij} : i \in \{1, \dots, n\}, j \in \{1, \dots, K\}\}$$

as well as a set W with $|W_\varepsilon| \leq K^{1-n}$, where $\varepsilon = K^{(1-n)/(dn-\alpha)}$. We let S be the random collection of all indices $k_1 \in \{1, \dots, K\}$ for which there is $k_2, \dots, k_n$ such that $(X_{1k_1}, \dots, X_{nk_n}) \in W_\varepsilon$. Our goal is to obtain a high probability bound of the form

$$\left| \sum_{k \in S} e^{2\pi i \xi \cdot X_{1k}} \right| \lesssim K^{1/2} \log(K)^{O(1)}.$$

To obtain an initial reduction, we consider an inequality known as McDiarmid's inequality.

Theorem 0.2. *Let Ω be a measurable space, and let $X_1, \dots, X_N$ be independant random variables taking values in Ω. Let $f : \Omega^N \to \mathbf{C}$ be any measurable function, and suppose there exists constants $A_1, \dots, A_N$ such that for any $i \in \{1, \dots, N\}$, $x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n \in \Omega$, and $x_i, x_i' \in \Omega$,*

$$|f(x_1, \dots, x_i, \dots, x_n) - f(x_1, \dots, x_i', \dots, x_n)| \leq A_i.$$

Then for any $t \geq 0$,

$$\mathbf{P}\left(|f(X_1, \dots, X_N) - \mathbf{E}(f(X_1, \dots, X_N))| \geq t\right) \leq 4 \exp\left(\frac{-2t^2}{A_1^2 + \dots + A_N^2}\right).$$

Suppose we fix particular values for the family of random variables $\{X_{21}, \dots, X_{2K}, \dots, X_{n1}, \dots, X_{nK}\}$. If we let

$$S_\xi(X_{11}, \dots, X_{1K}) = \sum_{k \in S} e^{2\pi i \xi \cdot X_{1k}},$$

then changing each coordinate changes the value of the overall conditional expectation by at most two. Thus McDiarmid's inequality implies that

$$\mathbf{P}\left(|S_\xi(X_{11}, \dots, X_{1K}) - \mathbf{E}(S_\xi(X_{11}, \dots, X_{1K}))| \geq t\right) \leq 4 \exp\left(\frac{-t^2}{2K}\right).$$

In particular, applying a union bound to this inequality shows that with high probability the difference between $S_\xi(X_{11},\dots,X_{1K})$ and $\mathbf{E}(S_\xi(X_{11},\dots,X_{1K}))$ for all $|\xi| \leq K^{-1/\beta}$ is at most $O(K^{1/2}\log(K)^{1/2})$, which is sufficient for our purposes. If we now introduce the randomness caused by the random variables, letting Σ be the σ algebra generated by $\{X_{21},\dots,X_{nK}\}$, then we see that we need only prove that square-root cancellation occurs for the conditional expectation

$$\mathbf{E}\left(\sum_{k\in S} e^{2\pi i\xi\cdot X_{1k}}\middle|\Sigma\right).$$

But this conditional expectation is equal to

$$K\cdot\mathbf{E}\left(e^{2\pi i\xi\cdot X}|\Sigma\right),$$

where X has the same distribution as $\{X_1,\dots,X_K\}$. For each $i \in \{1,\dots,K\}$, we let $E_i = \{X_{i1},\dots,X_{iK}\}$, and set $F = (\mathbf{T}^d \times E_2 \times \cdots \times E_n)\cap W_\varepsilon$. Then

$$\mathbf{E}\left(e^{2\pi i\xi\cdot X}|\Sigma\right) = \int_{\pi(F)} e^{-2\pi i\xi\cdot x}.$$

where $\pi : \mathbf{T}^{dn} \to \mathbf{T}^d$ is projection onto the first d coordinates. Thus we must obtain probability bounds on the fact that

$$\left|\int_{\pi(F)} e^{-2\pi i\xi\cdot x}\right| \leq K^{-1/2}.$$

As a final remark, if W is a surface specified by the equation $x_1 = f(x_2,\dots,x_n)$, it might be simpler to study random slices of the set

$$W'_\varepsilon = \{x\in\mathbf{T}^{dn} : |x_1 - f(x_2,\dots,x_n)| \leq \varepsilon\},$$

which contains W_δ for $\delta \lesssim \varepsilon$. It then follows that if we set $F' = (\mathbf{T}^d \times E_2\times\cdots\times E_n)\cap W'_\varepsilon$, then

$$\pi(F') = \bigcup\left\{B_\varepsilon(f(x_2,\dots,x_n)) : (x_2,\dots,x_n)\in E_2\times\cdots\times E_n\right\}.$$

This set seems quite tractable to study, and we should try and obtain the result for hyperplanes first to begin with. Unfortunately, I have to start studying for the Madison qualifying exams, so I will have to put the study of these sets off for some time.

Thursday 20th August 2020

Maybe Do

- Read Krantz's notes [15] to prepare for researching more general Harmonic analysis again.
- Look through Marco Vitturi's lecture notes to see which are worth working through.
- Read through Steven Krantz's book [14] to learn about real analytic functions.
- Read through William Ziemer's book [16] on Sobolev theory.
- Read through Evan and Gariepy's book [12] on the relation between Sobolev spaces and Hausdorff dimension.

It's been two months since my last research entry, given that I have spent the last two months preparing for the qualifying exam at Madison. I haven't returned back to the Fourier dimension questions I've been discussing, so I'll just spend this entry as a way to discuss references I found which might prove useful future reading.

- I found some notes [15] from a course in Harmonic analysis by Steven Krantz which might prove a useful refresher before I start researching Harmonic analysis at Wisconsin.
- I found a set of Lecture Notes by Marco Vitturi on various important topics on Harmonic analysis at this webpage.
- I also found other miscellanous books which might prove useful:
  - A book by Steven Krantz [14] systematically describing real analytic functions from the point of view of Harmonic analysis, which might prove useful.
  - A book [16] by William Ziemer systematically describing Sobolev theory.
  - A book [12] by Lawrence Evans and Ronald Gariepy on the relation between Sobolev spaces and Hausdorff dimension.

Friday, 4th September 2020

Important To Do List

- Look into uncertainty principles on Orlicz spaces in order to extend the smoothing bound as required.

In the past few days I've officially started my program at Madison. I also attended some interest talks:

- David Beltran gave an interesting lecture series on sparse domination. Consider an operator T whose input and output consists of scalar functions on a set X. The basic theory studies *pointwise sparse domination*, i.e. finding a family of subsets $\mathcal{S}$ of X and an exponent p such that for each $x \in X$,

  $$|Tf(x)| \lesssim \sum_{Q \in \mathcal{S}} \langle f \rangle_{Q,p} \cdot \mathbf{I}_Q(x)$$

  where

  $$\langle f \rangle_{Q,p} = \left(\fint_Q |f|^p \right)^{1/p}$$

  and such that there exists a disjoint family of sets $\{E_Q : Q \in \mathcal{S}\}$ such that $|E_Q| \geq \eta |Q|$ (such a family $\mathcal{S}$ is then called *η-sparse*). It then follows that we have a bound $\|Tf\|_{L^r(X)} \lesssim_r \|f\|_{L^r(X)}$. for any $r > p$.

  Bilinear sparse domination attempts to extend the family of operators we can sparsely dominate. Suppose that there exists $\eta \in (0,1]$ such that for any functions f_1 and f_2,

  $$\langle Tf_1, f_2 \rangle \lesssim \sup_{\eta-\text{sparse } \mathcal{S}} \sum_{Q \in \mathcal{S}} |Q| \cdot \langle f_1 \rangle_{Q,p} \langle f_2 \rangle_{Q,q^*}.$$

  Given this assumption, we have bounds

  $$\|Tf\|_{L^r(X)} \lesssim_r \|f\|_{L^r(X)}$$

  for any $r \in (p,q)$,

  $$\|Tf\|_{L^{p,\infty}(X)} \lesssim \|f\|_{L^p(X)}$$

  and

  $$\|Tf\|_{L^q(X)} \lesssim \|f\|_{L^{q,1}(X)}.$$

  This is proved in [2].

Sparse domination is easiest when working with operators at a single scale. Examples include a ball average operator on $\mathbf{R}^d$, i.e.

$$A_r f(x) = \fint_{B_r(x)} f(x)\, dx.$$

Then if $\text{supp}(f) \subset Q$, where $l(Q) = r$, $\text{supp}(A_r f) \subset 3Q$. Thus we think of A_r as being 'at scale r'. We have bounds

$$\|A_r f\|_{L^p(\mathbf{R}^d)} \leq \|f\|_{L^p(\mathbf{R}^d)} \quad \text{and} \quad \|A_r f\|_{L^\infty(\mathbf{R}^d)} \lesssim_d r^{-d} \|f\|_{L^1(\mathbf{R}^d)}.$$

Then for any f_1, f_2, applying Hölder's inequality shows

$$\begin{aligned}
\langle A_r(f_1 \mathbf{I}_Q), f_2 \rangle &= \langle A_r(f_1 \mathbf{I}_Q), f_2 \mathbf{I}_{3Q} \rangle \\
&\leq \|A_r(f_1 \mathbf{I}_Q)\|_{L^q(\mathbf{R}^d)} \|f_2 \mathbf{I}_{3Q}\|_{L^{q^*}(\mathbf{R}^d)} \\
&\lesssim_d r^{d(1/q - 1/p)} \|f_1 \mathbf{I}_Q\|_{L^p(\mathbf{R}^d)} \|f_2 \mathbf{I}_{3Q}\|_{L^{q^*}(\mathbf{R}^d)} \\
&= |Q| \langle f_1 \rangle_{Q,p} \langle f_2 \rangle_{Q,q^*}.
\end{aligned}$$

We can cover $\mathbf{R}^d$ by tiles of the form $3Q$ which gives the sparse domination required. The advantage of this argument is that it involves very little of the structure of the operator A_r, only that $\text{supp}(A_r f)$ is contained in an r-thickening of $\text{supp}(f)$, and that $\|A_r f\|_{L^q(\mathbf{R}^d)} \lesssim r^{d(1/q-1/p)} \|f\|_{L^p(\mathbf{R}^d)}$. This technique was first found in [11].

Dyadic decompositions also work for multiscale operators, if the operator can be decomposed into single scale operators dyadically and an exponential decay is involved. This method has been used to bound various psuedodifferential operators as in [1], and Bochner-Riesz multipliers as in [10].

In very recent work, David Beltran has managed to obtain bilinear sparse domination bounds that apply to multiscale operators. But I wasn't able to follow this section this clearly.

- Stefan Steinerberger gave a talk based on [13] on the relation between uncertainty principles and the 'smoothest average' operator. Given a fixed $\alpha > 0$ and $\lambda > 0$, we consider the set of all non-negative functions $u : \mathbf{R}^d \to \mathbf{R}$ such that

$$\int_{\mathbf{R}^d} u(x)\, dx = 1 \quad \text{and} \quad \int_{\mathbf{R}^d} |x|^\alpha \cdot u(x)\, dx = \lambda.$$

Out of these u, we try and and find such a u which minimizes the value

$$\sup_{f \in H^2(\mathbf{R}^d)} \frac{\|\nabla(f * u)\|_{L^2(\mathbf{R}^d)}}{\|f\|_{L^2(\mathbf{R}^d)}}.$$

Applying Plancherel, and the fact that the Fourier transform of $\nabla(u * f)$ is $2\pi i \cdot \widehat{u} \cdot \widehat{f} \xi$, this supremum is equal to

$$\|\xi \cdot \widehat{u}\|_{L^\infty(\mathbf{R}^d)},$$

which is the quantity we wish to minimize. Steinerberger obtains an uncertainty principle that for $\alpha > 0$ and $\beta > d/2$, and any $u \in L^1(\mathbf{R}^d)$,

$$\| |\xi|^\beta \cdot \widehat{u} \|^\alpha_{L^\infty(\mathbf{R}^d)} \cdot \| |x|^\alpha \cdot u \|^\beta_{L^1(\mathbf{R}^d)} \gtrsim_{\alpha,\beta,d} \|u\|^{\alpha+\beta}_{L^1(\mathbf{R}^d)}.$$

In this problem we only care about $d = 1$ and $\beta = 1$. Suprisingly, for certain parameters, the minimizer is *not smooth*, unlike in many problems in analysis. Indeed, for $\alpha \in \{2,3,4,5,6\}$, the indicator function of $[-1/2, 1/2]$ is a local minimizer for the appropriate value of λ.

I think an interesting problem is determining what happens if we consider tighter bounds on the moments. In particular, what happens if we consider the problem under averaging by a probability measure μ such that $\|\mu\|_{\psi_2} \leq \lambda$, where $\|\cdot\|_{\psi_2}$ is the sub-Gaussian norm of a random variable. This bound implies an asymptotic bound on all sufficiently large moments. Is the optimal measure here also uniformly distributed on an interval? The uncertainty principle of Steinerberger shows that if μ is a function, then for $\alpha < 2/d$,

$$\|\mu\|_{\psi_2} \gtrsim_\alpha \frac{1}{\| |\xi|^{1/\alpha} \cdot \widehat{\mu} \|_{L^\infty(\mathbf{R}^d)}^\alpha},$$

Thus if $\|\mu\|_{\psi_2} \leq \lambda$, then for each $\alpha < 2/d$, there exists a frequency $\xi_\alpha \in \mathbf{R}^d$ such that

$$|\widehat{\mu}(\xi_\alpha)| \gtrsim_\alpha |\xi_\alpha|^{-1/\alpha} \lambda^{-1/\alpha}.$$

Thus μ is somewhat singular in the sense of Fourier dimension with dimension smaller than $d/2$.

I also emailed Thomas Körner to try and get more info on the probabilistic calculations in his paper. Hopefully he'll email back (even though his paper is a decade old), and I'll be able to get more intuition about his technique.

Bibliography

[1] Laura Cladek David Beltran. Sparse bounds for psuedodifferential operators. 2017.

[2] Stefanie Petermichl Frédéric Bernicot, Dorothee Frey. Sharp weighted norm estimates beyond Caldeŕon-Zygmund theory. 2016.

[3] Jörg Schmeling Fredrik Ekström. A survey on the fourier dimension. 2017.

[4] Jörg Schmeling Fredrik Ekström, Tomas Persson. On the Fourier dimension and a modification. 2015.

[5] Joshua Zahl Jacob Denson, Malabika Pramanik. Large sets avoiding rough patterns. 2019.

[6] Jean-Pierre Kahane. *Some Random Series of Functions*. 1994.

[7] J.P. Kahane. Probabilities and Baire's theory in harmonic analysis. 2000.

[8] Thomas W. Körner. Fourier transforms of measures and algebraic relations on their supports. *Annales de L'Institut Fourier*, 2009.

[9] R. Lyons. Seventy years of Rajchman measures. 1995.

[10] Maria Carmen Reguera Michael T. Lacey, Darío Mena. Sparse bounds for bochner-riesz means. 2017.

[11] Scott Spencer Michael T. Lacey. Sparse bounds for oscillatory and random singular integrals. 2016.

[12] Lawrence C. Evans Ronald F. Gariepy. *Measure Theory and Fine Properties of Functions*. 1992.

[13] Stefan Steinerberger. Fourier uncertainty principles, scale space theory and the smoothest average.

[14] Harold R. Parks Steven G. Krantz. *A Primer on Real Analytic Functions*. 2002.

[15] Lina Lee Steven G. Krantz. *Explorations in Harmonic Analysis*. 2007.

[16] William P. Ziemer. *Weakly Differentiable Functions*. 1989.