JACOB DENSON

LABORATORY JOURNAL

Contents

TODO List

Saturday, 30 May 2020

Important To Do List

- Formally prove the space X given by the norms in (4) is complete. **See Monday 1st June 2020**

- Try and understand the probabalistic smoothness calculations given in Lemma 7.4 of [8]. Once this is done, we can try adapting it to our more general situation. **See Friday, 4th September 2020**

Maybe Do

- Today I found an interesting survey on the Fourier Dimension [3]. If I find the time I should read through it more thoroughly to get some intuition. **See Sunday, 31st May 2020**

- I also found a survey on the application of the probabilistic method and the Baire category theorem in Harmonic analysis [7]. I feel this method is very exploitable in the types of problems I currently deal with, so if only for culture, this should be a useful read.

My main goals today were to finish up the slides, for my talk on Fourier dimension in pattern avoidance problems at the 2020 Ottawa Math Conference. I have essentially completed these slides; all that remains is to polish them up, and practice giving the presentation. My main goal during the presentation is to show that viewing avoiding sets Z geometrically leads to interesting questions, that the geometric quantities we consider lead to important consequences, and that the Fourier dimension question I am currently considering is interesting to study. I also thought about an idea which enables us to allow us to use the type of bounds in [5] to Thomas Körner's Baire category method in [8]. I also found an example of how Fourier dimension behaves weirdly to show that the case of [5] when $s < d$ is nontrivial in the Fourier dimension scheme.

Ideas for Fourier Dimension Technique

Thomas Körner's paper [8] relies on Baire category arguments to construct generic measures μ supported on a subset of $\mathbf{T}$ avoiding solutions to m-term linear equations, such that for each $\xi \in \mathbf{Z}$,

$$|\widehat{\mu}(\xi)| \leq A(\xi).^1 \tag{1}$$

[1] Here $\{A(\xi)\}$ is a sequence given for each $\xi \in \mathbf{Z}$ by the formula

$$A(\xi) = B(\xi)|\xi|^{-\beta/2} \log(1+|\xi|)^{1/2},$$

where $\beta = (n-1)^{-1}$, and $\{B(\xi)\}$ is some fixed sequence of positive numbers such that $B(\xi) \to \infty$ as $|\xi| \to \infty$.

To obtain this bound generically, Körner works in the norm space X consisting of finite measures μ on $\mathbf{T}$ such that the quantity

$$\|\mu\|_X = \sup_{\xi \in \mathbf{Z}} \frac{|\widehat{\mu}(\xi)|}{A(\xi)}. \tag{2}$$

is finite. Then any measure in X satisfies (1) up to a multiplicative constant, and X is a Banach space, which enables one to use Baire-category techniques. A problem occurs in our Fourier dimension paper because I believe we can only construct finite measures μ such that for each $\varepsilon > 0$,

$$|\widehat{\mu}(\xi)| \lesssim_\varepsilon |\xi|^{\varepsilon-\beta/2}. \tag{3}$$

Such a measure does not satisfy quite as rigid an inequality as (1), instead having to satisfy infinitely many inequalities of the form (3), and as such I do not believe we can find a Banach space norm which encapsulates (3).

However, today I thought of an idea which might prove fruitful. For any measure μ satisfying (3) for each $\varepsilon > 0$, the quantities

$$\|\mu\|_\varepsilon = \sup_{\xi \in \mathbf{Z}} |\widehat{\mu}(\xi)||\xi|^{\beta/2-\varepsilon} \tag{4}$$

will be finite for all $\varepsilon > 0$. If we let X denote the family of all finite measures which satisfy (4) for all $\varepsilon > 0$, then the collection of seminorms $\{\|\cdot\|_\varepsilon : \varepsilon > 0\}$ might give X the structure of a Frechét space. Since Frechét spaces are complete metric spaces, we can still apply Baire category arguments here. It remains to check whether this really gives a complete metric space structure, however.

Finite Additivity of the Fourier Dimension

In [4], I found a result which constructs two disjoint, Borel sets $A, B \subset \mathbf{T}$, with $\dim_{\mathbf{F}}(A), \dim_{\mathbf{F}}(B) < 1$, but such that $A \cup B = \mathbf{T}$. The result of Theorem 1 of [5] is trivial when $\dim_{\mathbf{M}}(Z) < d$, for if $\pi : (\mathbf{T}^d)^n \to \mathbf{T}^d$ is given by projection onto the first d coordinates, then $\mathbf{T}^d - \pi(Z)$ has full Hausdorf dimension and avoids Z. If $Z = A \times \{0\}$, then $\mathbf{T} - \pi(Z) = B$, which is not full dimensional, so things are more complicated when dealing with Fourier dimension. I found this result in [3], which might be a useful survey to read through completely in order to get a better grasp on how the Fourier dimension behaves.

Sunday, 31st May 2020

Maybe Do

- Read Kahane's book [6] to obtain ideas on possible Fourier dimension constructions.
- Read Lyon's article [9] to obtain cultural background information about Fourier dimension in harmonic analysis.

Today I finished off my slides, prepared for the talk, and review Ekström and Schmeling's survey on Fourier dimension results [3]. The review is given below.

Ekström and Schmeling's "A Survey of Fourier Dimension"

This article gives a pretty diverse set of viewpoints about the Fourier dimension. I'll list off a couple of things that I hadn't heard about before:

- It is obvious that if μ is a finite Borel measure with Hausdorff dimension s and $\mu(E) > 0$, then $\dim_{\mathbf{H}}(E) \geq s$. Thus μ does not need to be supported on E completely. However, if we define the *modified Fourier dimension*

  $$\dim_{\mathbf{MF}}(E) = \sup\{\dim_{\mathbf{F}}(\mu) : \mu(E) > 0\}, \tag{5}$$

  we get a different dimension to the Fourier dimension

  $$\dim_{\mathbf{F}}(E) = \sup\{\dim_{\mathbf{F}}(\mu) : \mathrm{supp}(\mu) \subset E\}. \tag{6}$$

  The modified Fourier dimension *is* countably stable, i.e. for a countable collection of Borel sets $\{E_k\}$,

  $$\dim_{\mathbf{MF}}\left(\bigcup_k E_k\right) = \sup_k \left(\dim_{\mathbf{F}}(E_k)\right)$$

  The usual Fourier dimension is not even finitely stable, but is countably stable if we restrict ourselves to F_δ sets.

- It is interesting that we can define the Hausdorff dimension completely locally. Given a finite Borel measure μ on $\mathbf{R}^d$, and $x \in \mathbf{R}^d$, define the *local dimension* as

  $$\underline{\dim_{\mathbf{H}}}(\mu, x) = \liminf_{r \to 0} \log_r \mu(B_x(r)).$$

and

$$\overline{\dim_{\mathbf{H}}}(\mu, x) = \limsup_{r \to 0} \log_r \mu(B_x(r)).$$

Theorem 0.1. *If $\mu(E) > 0$ and $\underline{\dim_{\mathbf{H}}}(\mu, x) \geq s$ for μ a.e. $x \in E$, then $\dim_{\mathbf{H}}(E) \geq s$.*

Proof. For each $\varepsilon > 0$, there exists $r_0 > 0$ and $F \subset E$ with $\mu(F) > 0$ such that for each $x \in F$ and $r \leq r_0$, $\mu(B_x(r)) \geq r^{s-\varepsilon}$. This implies $\dim_{\mathbf{H}}(E) \geq \dim_{\mathbf{H}}(F) \geq s - \varepsilon$. We then take $\varepsilon \to 0$ to conclude $\dim_{\mathbf{H}}(E) \geq s$. □

Working locally might simplify calculations, rather than having to come up with uniform bounds on the dimension. There is no local dimension for the usual Fourier transform, since the Fourier transform is not monotone on sets. However, the modified Fourier dimension *is* monotone, so it might be possible to find a local charactierization of the Fourier dimension. For a measure μ, we define

$$\dim_{\mathbf{MF}}(\mu) = \sup\{\dim_{\mathbf{F}}(\eta) : \mu \ll \eta\},$$

so that $\dim_{\mathbf{MF}}(E) = \sup\{\dim_{\mathbf{MF}}(\mu) : \text{supp}(\mu) \subset E\}$. Then we can try and define the Fourier dimension locally.

- Given a family of measures M on a measure space X, it is interesting to find a family $\mathcal{U}$ of measurable subsets of X such that a measure μ on X is an element of M if and only if $\mu(E) = 0$ for all $E \in \mathcal{U}$. For instance, this is possible if a measure μ is fixed, and M is the family of all absolutely continuous measures with respect to μ. This is also possible for measures with Hausdorff dimension exceeding some value s - it suffices to consider the family of all sets with s Hausdorff measure zero. The family of *Rajchman measures* on $\mathbf{R}$ are the collection of all measures μ such that

  $$\lim_{|\xi| \to \infty} \widehat{\mu}(\xi) = 0.$$

  Suprisingly, one can find a family of sets $\mathcal{U}$ such that a measure μ is Rajchman if and only if it assigns mass zero to each set $E \in \mathcal{U}$. This is detailed in [9], which might also be a useful text to read for culture. If we let

  $$M^{\perp} = \{E \subset X : \mu(E) = 0 \text{ for all } \mu \in M(X)\}$$

  and

  $$M^{\perp\perp} = \{\mu : \mu(E) = 0 \text{ for all } E \in M^{\perp}\}.$$

  To show such a family exists for a given collection of measures M, it is necessary and sufficient to show $M^{\perp\perp} = M$. This is true for the family of measures with *modified Fourier dimension* greater than a given value, but not for the family of measures whose normal Fourier dimension is greater than a given value. It is an open question to explicitly identify the family of sets in these examples, however, which might be an interesting problem to work on.

- If E is a Borel subset of $\mathbf{R}$, then there exists a $C^{m+\alpha}$ diffeomorphism $f : \mathbf{R} \to \mathbf{R}$ such that

$$\dim_{\mathbf{F}}(f(E)) \geq \frac{\dim_{\mathbf{H}}(E)}{m+\alpha},$$

  In particular, if $m = 1$ and $\alpha = 0$, any Borel set is diffeomorphic to a Salem set. One can thus view the difference between Hausdorff and Fourier dimension as a measure of pertubation, in some sense. I should check whether it is possible to find a smooth faimly of deformations of E than smoothly deform the Fourier dimension up to the Hausdorff dimension. It is also an open question to construct an *explicit* diffeomorphism $f : \mathbf{R} \to \mathbf{R}$ between the Cantor set and a Salem set, since the construction above is obtained in a random fashion.

Looking at the references shows a list of books that might be useful to consult for intuition on the problems I'm working on now:

- J.P. Kahane's book [6] on random functions might be useful to obtain ideas on how to obtain Fourier dimension measures.
- R. Lyons article [9] might be useful for learning some cultural information about the history of Fourier dimension in harmonic analysis.

Monday 1st June 2020

Important To Do List

- Edit the proof in our existing draft on Fourier dimension to work completely rigorously for the new Frechét space. **See Tuesday 2nd June 2020**

I gave my presentation at the Ottawa Math Conference today. It went fairly well, I got a few questions so some people followed the talk. Later on in the day I wrote a formal proof that the space I thought up in Saturday, 30 May 2020 is actually a Frechét space. What remains is to modify the existing proof in the paper draft to completely rigorously prove that generic sets in the space are Salem and avoid patterns.

Tuesday 2nd June 2020

Today was mostly a writing day. I finished editing the proof in our paper to incorporate the new Frechét space idea, so now we can find sets with Fourier dimension

$$\frac{nd-\alpha}{n-1/2}$$

avoiding patterns. All that remains to complete this project is to understand Körner's probabilistic analysis which enabled him to improve this bound in the special case of integer coefficient equations, see if we can generalize this to more general cases to yield a bound

$$\frac{nd-\alpha}{n-1},$$

and then incorporate this into the paper.

Wednesday 3rd June 2020

Important To Do List

- Look back on decoupling argument and see if we can still utilize it to help obtain the probability bounds needed in our construction.
- Try and solve the tensorized special case of the argument.
- Prove that if we can prove the analogous problem in the discrete setting, then we automatically obtain the result in the general setting.

Today I cleaned up the proof in the Fourier dimension paper. Aside from expanding the background section, all that remains is to try and obtain the improved Fourier dimension by obtaining a square root cancellation in the final argument. I'll detail my thoughts on this in more detail in the next section.

Obtaining Square Root Cancellation

The only obstacle to obtaining the full result we desire in the Fourier dimension is understanding a fairly simple situation in probability; Consider a large integer K, let $W \subset \mathbf{T}^{dn}$ be a set, let $\varepsilon = K^{(1-n)/(dn-\alpha)}$, and suppose that $|W_\varepsilon| \leq K^{1-n}$, where

$$W_\varepsilon = \bigcup_{x \in W} B_\varepsilon(x).$$

Then take K independant and uniformly distributed random variables $X_1, \dots, X_K$ in $\mathbf{T}^d$. Let S be the set of indices $k_1 \in \{1, \dots, K\}$ such that there exists distinct indices $k_2, \dots, k_n \in \{1, \dots, K\}$ such that $(X_{k_1}, \dots, X_{k_n}) \in W_\varepsilon$. Standard expectation bounds imply that $\#(S) \leq K$ with high probability. For $\xi \in \mathbf{Z}^d$, we want to understand what conditions guarantee that with high probability square root cancellation occurs in the sum

$$\sum_{k \in S} e^{2\pi i \xi \cdot X_k},$$

i.e. under what conditions can we conclude that,

$$\left| \sum_{k \in S} e^{2\pi i \xi \cdot X_k} \right| \leq K^{1/2} \log(K)^{O(1)}.$$

If we write

$$Y_k = \begin{cases} e^{2\pi i \xi \cdot X_k} & : \text{if } k \in S, \\ 0 & : \text{if } k \notin S, \end{cases}$$

then our goal is to obtain square root cancellation in the sum $Y = Y_1 + \dots + Y_K$. There are several reasons why we might expect this result to be the case:

- The variables $\{Y_k\}$ are identically distributed (but not independent) with $|Y_k| \leq 1$ for all $k \in \{1, \dots, K\}$.
- The variables are only 'n coupled' with one another, so if K is large, the variables are not too coupled with one another.

However, I am unable to find a general concentration result which can give us this result. To consider a simple analysis, let us suppose that there are sets $W_1, \dots, W_n$ such that

$$W = W_1 \times \dots \times W_n.$$

Is the problem at least solvable in this simple, tensorizable situation? I might also want to look back on the probabilistic decoupling techniques I was trying to utilize last year.

Sunday 4th June 2020

Today I discovered a useful probabilistic concentration result that reduces our square-root cancellation problem to concentration bounds on conditional expectations.

Reduction to Conditional Expectation Bound

Suppose we consider an independant family of random variables

$$\{X_{ij} : i \in \{1, \dots, n\}, j \in \{1, \dots, K\}\}$$

as well as a set W with $|W_\varepsilon| \leq K^{1-n}$, where $\varepsilon = K^{(1-n)/(dn-\alpha)}$. We let S be the random collection of all indices $k_1 \in \{1, \dots, K\}$ for which there is $k_2, \dots, k_n$ such that $(X_{1k_1}, \dots, X_{nk_n}) \in W_\varepsilon$. Our goal is to obtain a high probability bound of the form

$$\left| \sum_{k \in S} e^{2\pi i \xi \cdot X_{1k}} \right| \lesssim K^{1/2} \log(K)^{O(1)}.$$

To obtain an initial reduction, we consider an inequality known as McDiarmid's inequality.

Theorem 0.2. *Let Ω be a measurable space, and let $X_1, \dots, X_N$ be independant random variables taking values in Ω. Let $f : \Omega^N \to \mathbf{C}$ be any measurable function, and suppose there exists constants $A_1, \dots, A_N$ such that for any $i \in \{1, \dots, N\}$, $x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n \in \Omega$, and $x_i, x_i' \in \Omega$,*

$$|f(x_1, \dots, x_i, \dots, x_n) - f(x_1, \dots, x_i', \dots, x_n)| \leq A_i.$$

Then for any $t \geq 0$,

$$\mathbf{P}\left(|f(X_1, \dots, X_N) - \mathbf{E}(f(X_1, \dots, X_N))| \geq t\right) \leq 4 \exp\left(\frac{-2t^2}{A_1^2 + \dots + A_N^2} \right).$$

Suppose we fix particular values for the family of random variables $\{X_{21}, \dots, X_{2K}, \dots, X_{n1}, \dots, X_{nK}\}$. If we let

$$S_\xi(X_{11}, \dots, X_{1K}) = \sum_{k \in S} e^{2\pi i \xi \cdot X_{1k}},$$

then changing each coordinate changes the value of the overall conditional expectation by at most two. Thus McDiarmid's inequality implies that

$$\mathbf{P}\left(|S_\xi(X_{11}, \dots, X_{1K}) - \mathbf{E}(S_\xi(X_{11}, \dots, X_{1K}))| \geq t\right) \leq 4 \exp\left(\frac{-t^2}{2K} \right).$$

In particular, applying a union bound to this inequality shows that with high probability the difference between $S_\xi(X_{11}, \dots, X_{1K})$ and $\mathbf{E}(S_\xi(X_{11}, \dots, X_{1K}))$ for all $|\xi| \leq K^{-1/\beta}$ is at most $O(K^{1/2} \log(K)^{1/2})$, which is sufficient for our purposes. If we now introduce the randomness caused by the random variables, letting Σ be the σ algebra generated by $\{X_{21}, \dots, X_{nK}\}$, then we see that we need only prove that square-root cancellation occurs for the conditional expectation

$$\mathbf{E}\left(\sum_{k \in S} e^{2\pi i \xi \cdot X_{1k}} \middle| \Sigma \right).$$

But this conditional expectation is equal to

$$K \cdot \mathbf{E}\left(e^{2\pi i \xi \cdot X} | \Sigma \right),$$

where X has the same distribution as $\{X_1, \dots, X_K\}$. For each $i \in \{1, \dots, K\}$, we let $E_i = \{X_{i1}, \dots, X_{iK}\}$, and set $F = (\mathbf{T}^d \times E_2 \times \cdots \times E_n) \cap W_\varepsilon$. Then

$$\mathbf{E}\left(e^{2\pi i \xi \cdot X} | \Sigma \right) = \int_{\pi(F)} e^{-2\pi i \xi \cdot x}.$$

where $\pi : \mathbf{T}^{dn} \to \mathbf{T}^d$ is projection onto the first d coordinates. Thus we must obtain probability bounds on the fact that

$$\left| \int_{\pi(F)} e^{-2\pi i \xi \cdot x} \right| \leq K^{-1/2}.$$

As a final remark, if W is a surface specified by the equation $x_1 = f(x_2, \dots, x_n)$, it might be simpler to study random slices of the set

$$W'_\varepsilon = \{x \in \mathbf{T}^{dn} : |x_1 - f(x_2, \dots, x_n)| \leq \varepsilon\},$$

which contains W_δ for $\delta \lesssim \varepsilon$. It then follows that if we set $F' = (\mathbf{T}^d \times E_2 \times \cdots \times E_n) \cap W'_\varepsilon$, then

$$\pi(F') = \bigcup \left\{ B_\varepsilon(f(x_2, \dots, x_n)) : (x_2, \dots, x_n) \in E_2 \times \cdots \times E_n \right\}.$$

This set seems quite tractable to study, and we should try and obtain the result for hyperplanes first to begin with. Unfortunately, I have to start studying for the Madison qualifying exams, so I will have to put the study of these sets off for some time.

Thursday 20th August 2020

Maybe Do

- Read Krantz's notes [15] to prepare for researching more general Harmonic analysis again.
- Look through Marco Vitturi's lecture notes to see which are worth working through.
- Read through Steven Krantz's book [14] to learn about real analytic functions.
- Read through William Ziemer's book [16] on Sobolev theory.
- Read through Evan and Gariepy's book [12] on the relation between Sobolev spaces and Hausdorff dimension.

It's been two months since my last research entry, given that I have spent the last two months preparing for the qualifying exam at Madison. I haven't returned back to the Fourier dimension questions I've been discussing, so I'll just spend this entry as a way to discuss references I found which might prove useful future reading.

- I found some notes [15] from a course in Harmonic analysis by Steven Krantz which might prove a useful refresher before I start researching Harmonic analysis at Wisconsin.
- I found a set of Lecture Notes by Marco Vitturi on various important topics on Harmonic analysis at this webpage.
- I also found other miscellanous books which might prove useful:
  - A book by Steven Krantz [14] systematically describing real analytic functions from the point of view of Harmonic analysis, which might prove useful.
  - A book [16] by William Ziemer systematically describing Sobolev theory.
  - A book [12] by Lawrence Evans and Ronald Gariepy on the relation between Sobolev spaces and Hausdorff dimension.

Friday, 4th September 2020

Important To Do List

- Look into uncertainty principles on Orlicz spaces in order to extend the smoothing bound as required.

In the past few days I've officially started my program at Madison. I also attended some interest talks:

- David Beltran gave an interesting lecture series on sparse domination. Consider an operator T whose input and output consists of scalar functions on a set X. The basic theory studies *pointwise sparse domination*, i.e. finding a family of subsets $\mathcal{S}$ of X and an exponent p such that for each $x \in X$,

  $$|Tf(x)| \lesssim \sum_{Q \in \mathcal{S}} \langle f \rangle_{Q,p} \cdot \mathbf{I}_Q(x)$$

  where

  $$\langle f \rangle_{Q,p} = \left(\fint_Q |f|^p \right)^{1/p}$$

  and such that there exists a disjoint family of sets $\{E_Q : Q \in \mathcal{S}\}$ such that $|E_Q| \geq \eta |Q|$ (such a family $\mathcal{S}$ is then called *η-sparse*). It then follows that we have a bound $\|Tf\|_{L^r(X)} \lesssim_r \|f\|_{L^r(X)}$. for any $r > p$.

  Bilinear sparse domination attempts to extend the family of operators we can sparsely dominate. Suppose that there exists $\eta \in (0, 1]$ such that for any functions f_1 and f_2,

  $$\langle Tf_1, f_2 \rangle \lesssim \sup_{\eta-\text{sparse } \mathcal{S}} \sum_{Q \in \mathcal{S}} |Q| \cdot \langle f_1 \rangle_{Q,p} \langle f_2 \rangle_{Q,q^*}.$$

  Given this assumption, we have bounds

  $$\|Tf\|_{L^r(X)} \lesssim_r \|f\|_{L^r(X)}$$

  for any $r \in (p, q)$,

  $$\|Tf\|_{L^{p,\infty}(X)} \lesssim \|f\|_{L^p(X)}$$

  and

  $$\|Tf\|_{L^q(X)} \lesssim \|f\|_{L^{q,1}(X)}.$$

  This is proved in [2].

Sparse domination is easiest when working with operators at a single scale. Examples include a ball average operator on $\mathbf{R}^d$, i.e.

$$A_r f(x) = \fint_{B_r(x)} f(x)\, dx.$$

Then if $\text{supp}(f) \subset Q$, where $l(Q) = r$, $\text{supp}(A_r f) \subset 3Q$. Thus we think of A_r as being 'at scale r'. We have bounds

$$\|A_r f\|_{L^p(\mathbf{R}^d)} \leq \|f\|_{L^p(\mathbf{R}^d)} \quad \text{and} \quad \|A_r f\|_{L^\infty(\mathbf{R}^d)} \lesssim_d r^{-d} \|f\|_{L^1(\mathbf{R}^d)}.$$

Then for any f_1, f_2, applying Hölder's inequality shows

$$\begin{aligned} \langle A_r(f_1 \mathbf{I}_Q), f_2 \rangle &= \langle A_r(f_1 \mathbf{I}_Q), f_2 \mathbf{I}_{3Q} \rangle \\ &\leq \|A_r(f_1 \mathbf{I}_Q)\|_{L^q(\mathbf{R}^d)} \|f_2 \mathbf{I}_{3Q}\|_{L^{q^*}(\mathbf{R}^d)} \\ &\lesssim_d r^{d(1/q-1/p)} \|f_1 \mathbf{I}_Q\|_{L^p(\mathbf{R}^d)} \|f_2 \mathbf{I}_{3Q}\|_{L^{q^*}(\mathbf{R}^d)} \\ &= |Q| \langle f_1 \rangle_{Q,p} \langle f_2 \rangle_{Q,q^*}. \end{aligned}$$

We can cover $\mathbf{R}^d$ by tiles of the form $3Q$ which gives the sparse domination required. The advantage of this argument is that it involves very little of the structure of the operator A_r, only that $\text{supp}(A_r f)$ is contained in an r-thickening of $\text{supp}(f)$, and that $\|A_r f\|_{L^q(\mathbf{R}^d)} \lesssim r^{d(1/q-1/p)} \|f\|_{L^p(\mathbf{R}^d)}$. This technique was first found in [11].

Dyadic decompositions also work for multiscale operators, if the operator can be decomposed into single scale operators dyadically and an exponential decay is involved. This method has been used to bound various psuedodifferential operators as in [1], and Bochner-Riesz multipliers as in [10].

In very recent work, David Beltran has managed to obtain bilinear sparse domination bounds that apply to multiscale operators. But I wasn't able to follow this section this clearly.

- Stefan Steinerberger gave a talk based on [13] on the relation between uncertainty principles and the 'smoothest average' operator. Given a fixed $\alpha > 0$ and $\lambda > 0$, we consider the set of all non-negative functions $u : \mathbf{R}^d \to \mathbf{R}$ such that

$$\int_{\mathbf{R}^d} u(x)\, dx = 1 \quad \text{and} \quad \int_{\mathbf{R}^d} |x|^\alpha \cdot u(x)\, dx = \lambda.$$

Out of these u, we try and and find such a u which minimizes the value

$$\sup_{f \in H^2(\mathbf{R}^d)} \frac{\|\nabla (f * u)\|_{L^2(\mathbf{R}^d)}}{\|f\|_{L^2(\mathbf{R}^d)}}.$$

Applying Plancherel, and the fact that the Fourier transform of $\nabla(u * f)$ is $2\pi i \cdot \widehat{u} \cdot \widehat{f} \xi$, this supremum is equal to

$$\|\xi \cdot \widehat{u}\|_{L^\infty(\mathbf{R}^d)},$$

which is the quantity we wish to minimize. Steinerberger obtains an uncertainty principle that for $\alpha > 0$ and $\beta > d/2$, and any $u \in L^1(\mathbf{R}^d)$,

$$\| |\xi|^\beta \cdot \widehat{u} \|_{L^\infty(\mathbf{R}^d)}^\alpha \cdot \| |x|^\alpha \cdot u \|_{L^1(\mathbf{R}^d)}^\beta \gtrsim_{\alpha,\beta,d} \|u\|_{L^1(\mathbf{R}^d)}^{\alpha+\beta}.$$

In this problem we only care about $d = 1$ and $\beta = 1$. Suprisingly, for certain parameters, the minimizer is *not smooth*, unlike in many problems in analysis. Indeed, for $\alpha \in \{2,3,4,5,6\}$, the indicator function of $[-1/2, 1/2]$ is a local minimizer for the appropriate value of λ.

I think an interesting problem is determining what happens if we consider tighter bounds on the moments. In particular, what happens if we consider the problem under averaging by a probability measure μ such that $\|\mu\|_{\psi_2} \leq \lambda$, where $\|\cdot\|_{\psi_2}$ is the sub-Gaussian norm of a random variable. This bound implies an asymptotic bound on all sufficiently large moments. Is the optimal measure here also uniformly distributed on an interval? The uncertainty principle of Steinerberger shows that if μ is a function, then for $\alpha < 2/d$,

$$\|\mu\|_{\psi_2} \gtrsim_\alpha \frac{1}{\||\xi|^{1/\alpha} \cdot \widehat{\mu}\|_{L^\infty(\mathbf{R}^d)}^\alpha},$$

Thus if $\|\mu\|_{\psi_2} \leq \lambda$, then for each $\alpha < 2/d$, there exists a frequency $\xi_\alpha \in \mathbf{R}^d$ such that

$$|\widehat{\mu}(\xi_\alpha)| \gtrsim_\alpha |\xi_\alpha|^{-1/\alpha} \lambda^{-1/\alpha}.$$

Thus μ is somewhat singular in the sense of Fourier dimension with dimension smaller than $d/2$.

I also emailed Thomas Körner to try and get more info on the probabilistic calculations in his paper. Hopefully he'll email back (even though his paper is a decade old), and I'll be able to get more intuition about his technique.

Tuesday 2nd February 2021

Maybe Do

- Andreas wants me to study a characterization of radial Fourier multipliers. Is it possible to study characterizations of multipliers which are symmetric with respect to other families of group actions, and how do things improve in this scenario?
- Can one apply an analytic interpolation argument to show an endpoint estimate holds for this technique?
- Are there Knapp examples which show this result is tight?

Today I read through Tobias Weth and Tolga Yesil's paper *Fourier-Extension Estimates for Symmetric Functions and Applications to Nonlinear Helmholtz Equations*. Here are some notes I took on this paper:

- The classical Stein-Tomas theorem, in it's dual form, says that if E is the extension operator, defined for each $f \in C(S^{d-1})$ by setting
$$Ef(x) = \int_{S^{d-1}} e^{2\pi i \xi \cdot x} f(\xi) d\sigma(\xi) = \mathcal{F}^{-1}(f\sigma)(x),$$
then $\|Ef\|_{L^q(\mathbf{R}^d)} \lesssim_{d,q} \|f\|_{L^2(S^{d-1})}$ holds for $d \geq 2$ and
$$q \geq \frac{2(d+1)}{(d-1)}.$$
The decay estimate $\mathcal{F}^{-1}\sigma(x) \lesssim \langle x \rangle^{-(d-1)/2}$ shows that $\mathcal{F}^{-1}\sigma \in L^q(\mathbf{R}^d)$ for $q \geq 2d/(d-1)$. Thus for *constant* functions f we have a bound
$$\|Ef\|_{L^q(\mathbf{R}^d)} \lesssim_{d,q} \|f\|_{L^2(S^{d-1})}$$
for
$$q > \frac{2d}{(d-1)}.$$
The goal of the present paper is to prove an intermediary between these two results. Given a subgroup G of isometries of S^{d-1}, i.e. a subgroup of $O(d)$, let $L^2_G(S^{d-1})$ be the space of all square integrable functions $f : S^{d-1} \to \mathbf{C}$ such that $f(Ax) = f(x)$ for all $A \in G$. A natural questions is to determine whether one can improve the range of values q for which E is bounded from $L^2_G(S^{d-1})$ to $L^q(\mathbf{R}^d)$. To further improve the range of values q, we can introduce a subset $F \subset \mathbf{R}^d$, and ask whether the extension map E is bounded from $L^2_G(S^{d-1})$ to $L^q(F)$.

- Since the Knapp example is axially symmetric, we cannot improve the bounds for q without restricting to a subset of Euclidean space for such a family of symmetries. On the other hand, introducing weights can eliminate some extremizers for the standard extension bound which make the Stein result type.
- In this paper, for each $k > 0$ the authors consider the group of isometries $G = O(d-k) \times O(k)$. For each $\alpha \geq \beta > 0$ the authors consider the sets
$$F_{\alpha\beta} = \{(x,y) \in \mathbf{R}^{n-k} \times \mathbf{R}^k : |x| \leq c \max(|y|^{-\alpha}, |y|^{-\beta})\}$$
for some $c > 0$, and prove bounds of the form
$$\|Ef\|_{L^q(F_{\alpha\beta})} \lesssim \|f\|_{L^2(S^{d-1})}.$$
This bound holds for the following parameters:
  - If $k = 1$, $\alpha, \beta > 1/(d-1)$, and
$$q > \frac{2(d-1-1/\alpha)}{d-2}.$$
Similarly, if $k = d-1$, $\alpha, \beta < (d-1)$ and
$$q > \frac{2(d-1)-2\beta}{d-2}.$$
  - If $k \in \{2, \dots, d-2\}$, and
$$q > \max\left\{ \frac{2k - 2\beta(d-k)}{k-1}, \frac{2(d-k) - 2/\alpha}{d-k-1} \right\}.$$

I am not sure if these results are tight. Can one come up with symmetric variants of the Knapp example which prove this tightness. Can one apply an analytic interpolation argument to give the endpoint result?
- If $f \in L^2_G(S^{d-1})$, then there exists a function $u : [0,1] \to \mathbf{C}$ such that $u(|x|) = f(x,y)$ for each $x \in \mathbf{R}^{d-k}$ and $y \in \mathbf{R}^k$ such that $|x|^2 + |y|^2 = 1$. A simple location applying the coarea formula implies that if σ_1 is the surface measure on S^{d-k-1} and σ_2 is the surface measure on S^{k-1}, then
$$Ef(x,y) \propto_{d,k} \int_0^1 u(r) r^{d-k-1} (1-r^2)^{k/2-1} \widehat{\sigma_1}(rx) \widehat{\sigma_2}((1-r^2)^{1/2} y)\, dr.$$
Taking in absolute values shows that
$$|Ef(x,y)| \lesssim_{d,k} \left(\int_0^1 r^{d-k-1} (1-r^2)^{k/2-1} |\widehat{\sigma_1}(rx)|^2 |\widehat{\sigma_2}((1-r^2)^{1/2} y)|^2\, dr \right)^{1/2} \cdot \|f\|_{L^2(S^{d-1})}.$$
We may assume, without loss of generality, that $q \geq 2$, so that
$$\begin{aligned} \|Ef\|_{L^q(F_{\alpha,\beta})} &\lesssim_{d,k} \left(\int_{F_{\alpha\beta}} \left(\int_0^1 r^{d-k-1} (1-r^2)^{k/2-1} |\widehat{\sigma_1}(rx)|^2 |\widehat{\sigma_2}((1-r^2)^{1/2} y)|^2\, dr \right)^{q/2} dx\, dy \right)^{1/q} \cdot \|f\|_{L^2(S^{d-1})} \\ &\leq \left(\int_0^1 r^{(d-k-1)(q/2)} (1-r^2)^{(k/2-1)(q/2)} \int_{F_{\alpha\beta}} |\widehat{\sigma_1}(rx)|^q |\widehat{\sigma_2}((1-r^2)^{1/2} y)|^q\, dr\, dx\, dy \right)^{1/q} \\ &= \left(\int_0^1 r^{(d-k-1)(q/2)-(d-k)} (1-r^2)^{(k/2-1)(q/2)-k/2} H_{\alpha\beta}(r)\, dr \right)^{1/q}. \end{aligned}$$

Here

$$H_{\alpha\beta}(r) = \int_{\mathbf{R}^d} \mathbf{I}_{F_{\alpha\beta}}(x,y) |\widehat{\sigma_1}(x)|^q |\widehat{\sigma_2}(y)|^q \; dx \; dy.$$

To bound the integral, one uses the standard stationary phase decay estimates for unit spheres, and then some tricky, case by case manipulations. It is assume that $q > 4$ in the argument, but I'm not sure why intuitively. This also occurs in Geoffrey Bentsen's argument in the paper *L^p Regularity Estimates for a Class of Integral Operators With Fold Blowdown Singularities*, which might be worth reading.

- Given $f \in \mathcal{S}_G(\mathbf{R}^d)$, let Rf be the solution to the Helmholtz equation $-\Delta u - u = f$ subject to Sommerfeld's outgoing radiation condition $\partial_r u(x) - iu(x) = o(|x|^{-(d-1)/2})$. Then there is a classical result (Kenig, Ruiz, Sogge) that says that

$$\|Rf\|_{L^{p'}(\mathbf{R}^d)} \lesssim \|f\|_{L^p(\mathbf{R}^d)}$$

if $2(d+1)/(d-1) \leq p \leq 2d/(d-2)$. The relation proved in this paper is that if we have a bound

$$\|w \cdot Ef\|_{L^q(\mathbf{R}^d)} \lesssim \|f\|_{L^2_G(S^{d-1})}$$

and suppose

$$1 - \frac{2}{d} \leq 1/p + 1/r < \frac{q+2}{2q}\frac{d-1}{d},$$

that if $q \geq 2$, then $\max(1/r, 1/p) < (d-1)/2d$, and if $q < 2$,

$$\begin{aligned} \frac{2q}{(d-1)(2-q)p} - \frac{(d-1)q - 2(d-3)}{2d(2-q)} &\\ < 1/r &\\ < \frac{(d-1)(2-q)}{2pq} + \frac{(d-1)[(d-1)q - 2(d-3)]}{4dq}. \end{aligned}$$

Then $\|wR(wf)\|_{L^r(\mathbf{R}^d)} \lesssim \|f\|_{L^{p'}(\mathbf{R}^d)}$ for any $f \in \mathcal{S}_G(\mathbf{R}^d)$.

Thursday 4th February 2021

Important To Do List

- Determine whether the problem about the image of sets has been studied in the literature before.

Fourier Dimension Under Curved Maps

One interesting application of the main result of my paper *Large Salem Sets Avoiding Configurations* is that if $\gamma : [0,1] \to \mathbf{R}^d$ is a Lipschitz curve, then there exists $E \subset [0,1]$ with $\dim_{\mathbf{F}}(E) \geq 4/9$ such that $\gamma(E)$ contains no isosceles triangles. It would be interesting if, for some families of curves, we could conclude from this that $\dim_{\mathbf{F}}(\gamma(E)) \geq 4/9$. Today I tried to do some computations to show this was true when $d = 2$ and γ had nonvanishing curvature, but I ran into trouble. The main technique I tried was that if μ is supported on E and $|\widehat{\mu}(\xi)| \lesssim |\xi|^{-2/9}$, then

$$\begin{aligned} \int e^{2\pi i \xi \cdot x} d\gamma_* \mu(x) &= \int_0^1 e^{2\pi i \xi \cdot \gamma(t)} d\mu(x) \\ &= \sum_{n=-\infty}^{\infty} \widehat{\mu}(n) \int_0^1 e^{2\pi i (\xi \cdot \gamma(t) - nt)}\; dt \\ &= \sum_{n=-\infty}^{\infty} \widehat{\mu}(n) I(\xi, n) \end{aligned}$$

There are many problems that we run into with this approach, however. Standard applications of stationary phase are reasonable when $|n| \ll |\xi|$ and ξ is aligned with γ', because we conclude that

$$|I(\xi, n)| \lesssim \frac{1}{|\xi|}$$

which shows that

$$\sum_{|n| \ll |\xi|} |\widehat{\mu}(n)| |I(\xi, n)| \lesssim |\xi|^{-2/9}.$$

If $|n| \gg |\xi|$, then stationary phase shows that

$$|I(\xi, n)| \lesssim 1/|n|$$

which gives that

$$\sum_{|n| \gg |\xi|} |\widehat{\mu}(n)||I(\xi, n)| \lesssim |\xi|^{-2/9}$$

In this case, the only problem that remains is when $|\xi| \approx |n|$, in which case there can be stationary points that cause problems. Furthermore, it is unclear how to use curvature to get decay if ξ is not aligned with γ', but with γ''. I couldn't seem to find papers discussing this kind of problem, so maybe I should ask Andreas / Malabika / Rob whether this kind of problem has been considered in the literature before.

Monday 15th February 2021

Important To Do List

- Plug in tight incidence example into radial multiplier problem. Does this give sharp examples here?

- In the Heisenberg group, are complex lines which are close to being coplanar, also close to being coincident?

Today I read through Josh Zahl's paper *Sphere tangencies, line incidences, and Lie's line-sphere correspondence*, which uses Lie sphere theory to bound the number of tangencies between a family of spheres in $\mathbf{R}^3$. I'm hoping that the techniques in this paper can help me understand the radial multiplier problem I'm currently working on.

Let S_1, S_2 be spheres of radii r_1 and r_2, and centers $x_1, x_2 \in \mathbf{R}^3$. These spheres have a tangential intersection if $|x_1 - x_2| = |r_1 + r_2|$, where we have an *external tangency*, or $|x_1 - x_2| = |r_1 - r_2|$, where we have an *internal tangency*. The goal of Josh's paper is to study the combinatorial properties of the contact points of a finite family of spheres $\mathcal{S}$ in $\mathbf{R}^3$.

Given a set $\mathcal{S}$ with $\#(\mathcal{S}) = n$, one might want to determine the maximum number of pairs $S_1, S_2 \in \mathcal{S}$ such that S_1 and S_2 are in contact with one another. However, it is possible for the number of contact pairs to be as high as n^2. In fact, there exist infinite families, known as *pencils of contracting spheres*, such that every sphere is in contact with every other sphere in the family. We may therefore modify the problem, and ask, for a fixed $2 \leq k \leq n$, how many pencils $\mathcal{P}$ there are such that $\#(\mathcal{S} \cap \mathcal{P}) \geq k$ (we say that $\mathcal{P}$ is *k rich*).

Theorem 0.3. *For $3 \leq k \leq n$, the set $\mathcal{S}$ determines $O(n^{3/2}k^{-5/2} + n/k)$ k-rich pencils.*

In this situation, things can go wrong for $k = 2$. Indeed, there exists pairs of infinite families $\mathcal{C}_1$ and $\mathcal{C}_2$, such that each sphere in $\mathcal{C}_1$ is in contact with every sphere in $\mathcal{C}_2$, but no sphere in $\mathcal{C}_1$ is in contact with any other sphere in $\mathcal{C}_1$. We call these a *pair of complimentary conic sections*. If we let $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$, where $\mathcal{S}_1 \subset \mathcal{C}_1$, $\mathcal{S}_2 \subset \mathcal{C}_2$, and $\#(\mathcal{S}_1) = \#(\mathcal{S}_2) = n/2$, then $\mathcal{S}$ determines $n^2/4$ distinct 2-rich pencils. Thus we do not get a $O(n^{3/2})$ bound as hinted at by the last theorem. This example is essentially the only thing that can go wrong.

Theorem 0.4. *Either there is a pair of conic sections $\mathcal{C}_1$ and $\mathcal{C}_2$ such that $\#(\mathcal{S} \cap \mathcal{C}_i) \geq \sqrt{n}$, or there are $O(n^{3/2})$ 2-rich pencils.*

A corollary of these two theorems is that, provided $\mathcal{S}$ does not contain many spheres in a single pencil, nor many spheres in a pair of conic sections, then we have an incidence bound.

Theorem 0.5. *If $\mathcal{S}$ does not depend any $\sqrt{n}$-rich pencils, and there do not exist any conic sections $\mathcal{C}_1$ and $\mathcal{C}_2$ such that $\#(\mathcal{S} \cap \mathcal{C}_i) \geq \sqrt{n}$, then there are at most $O(n^{3/2})$ pairs in $\mathcal{S}$ which are in contact with one another.*

The idea of the paper is to exploit the theory of 'Lie sphere theory'. The idea here is to parameterize the family of all 'oriented' spheres in $\mathbf{R}^3$ with the quadric surface

$$Q = \{\xi \in \mathbf{P}^5 : \xi_1^2 + \dots + \xi_4^2 - \xi_5^2 - \xi_0^2 = 0\},$$

such that if we define bilinear form

$$\langle \xi, \eta \rangle = \xi_1 \eta_1 + \dots + \xi_4 \eta_4 - \xi_5 \eta_5 - \xi_0 \eta_0,$$

then two spheres corresponding to points $\xi, \eta \in Q$ are in contact if and only if $\langle \xi, \eta \rangle = 0$. Another useful geometry is due to Plücker, which parameterizes the family of all lines in $\mathbf{R}^3$ by the quadric surface

$$W = \{\xi \in \mathbf{P}^5 : \xi_1 \xi_6 - \xi_2 \xi_5 + \xi_3 \xi_4 = 0\}.$$

If we associate the bilinear form

$$(\xi, \eta) = \xi_1 \eta_6 - \xi_2 \eta_5 + \xi_3 \eta_4 + \xi_4 \eta_3 - \xi_5 \eta_2 + \xi_6 \eta_1,$$

then two lines corresponding to points $\xi, \eta \in W$ are coplanar if and only if $(\xi, \eta) = 0$.

Both Q and W are quadric surfaces in $\mathbf{P}^5$ associated to bilinear forms. But the bilinear forms have different signatures over the real numbers, so they are projectively equivalent. However, all nondegenerate bilinear forms over the *complex numbers* are equivalent, so if we complexify, forming the quadric surfaces $Q^{\mathbf{C}}$ and $W^{\mathbf{C}}$ in $\mathbf{C}\,\mathbf{P}^5$, then there is a natural projective equivalence $\phi : Q^{\mathbf{C}} \to W^{\mathbf{C}}$. The elements of $Q^{\mathbf{C}}$ do not seem to parameterize anything relevant to this problem, but $W^{\mathbf{C}}$ parameterizes the family of all *complex lines* in $\mathbf{C}^3$. And it turns out that we can choose ϕ in such a way that the points in $\phi(Q^{\mathbf{C}})$ correspond to complex lines lying entirely in the *Heisenberg group*

$$\mathbf{H} = \{(x, y, z) \in \mathbf{C}^3 : \mathrm{Im}(z) = \mathrm{Im}(x\overline{y})\}.$$

Lines in $\mathbf{H}$ have the special property that, if they are coplanar, then they intersect (possibly at ∞). Since two spheres corresponding to $\xi, \eta \in Q$ are in contact if and only if $\phi(\xi)$ and $\phi(\eta)$ correspond to lines in coplanar, then if ξ and η are in contact, $\phi(\xi)$ and $\phi(\eta)$ correspond to lines that intersect in $\mathbf{R}^3$. Thus we have reduced the combinatorial problem of spheres in contact to a standard incidence problem in $\mathbf{R}^3$, where we can apply standard techniques of incidence geometry in $\mathbf{R}^3$, i.e. detecting ruled surfaces.

Unfortunately, I don't think these techniques will apply to the radial multiplier problem, since it seems very important for lines to be exactly incident for us to apply the correspondence - two spheres may be 'close' to being in contact, which would imply that the complex lines are 'close' to being coplanar, but this might not necessarily imply that these lines are 'close' to being coincident. Perhaps it might be a good idea to double check this is the case? Also, it might be good to try and plug in the pencils of contacting spheres, and conic sections, into the radial multiplier problem to see if they give tight estimates.

Tuesday 4th March, 2021

Today I attended the talk of Diogo Oliveira e Silva on the topic of sharp restriction theory, which focused on the problem of extremizers for the Tomas-Stein theorem. The main result is that if $d \in \{3,4,5,6,7\}$, and $q \geq 6$ is an even integer, then constant functions are the unique maximizers to the extension operator from $L^2(S^{d-1})$ to $L^q(\mathbf{R}^d)$. Here were the main techniques in the case where $q = 6$:

- (Calculus of Variations): If f maximizes the $L^2(S^{d-1})$ to $L^q(\mathbf{R}^d)$ bound, and $\|f\|_{L^2(S^{d-1})} = 1$, then

$$\begin{aligned}\|E\|^6 &= \|Ef\|_{L^6(\mathbf{R}^d)}^6 \\ &= \langle |Ef|^4 Ef, Ef \rangle \\ &= \langle R(|Ef|^4 Ef), f \rangle \\ &\leq \|E\| \| |Ef|^5 \|_{L^{6/5}(\mathbf{R}^d)} \\ &= \|E\| \|Ef\|_{L^6(\mathbf{R}^d)}^6 \\ &= \|E\|^6.\end{aligned}$$

  Thus all the inequalities in this chain are actually tight. Thus $R(|Ef|^4 Ef) = \lambda f$ for some scalar λ. Expanding this expression out, this means that

$$(f\sigma * f\sigma * f\sigma) * (f_* \sigma * f_* \sigma * f_* \sigma) = \lambda f$$

  for some scalar λ, where $f_*(x) = f(-x)$ is the antipodal reflection of f. One of the primary goals of this paper is to show that this equation implies that $f \in C^\infty(S^{d-1})$.

- (Symmetry): Since σ is a positive measure, $\|(f\sigma)^{(*3)}\|_{L^2(\mathbf{R}^d)} \leq \| |f| \sigma^{(*3)} \|$, so $f \geq 0$ almost everywhere.

*

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