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LABORATORY JOURNAL

Contents

Saturday, 30 May 2020

Important To Do List

- Formally prove the space X given by the norms in (4) is complete.
- Try and understand the probabistic smoothness calculations given in Lemma 7.4 of [5]. Once this is done, we can try adapting it to our more general situation.

Maybe Do

- Today I found an interesting survey on the Fourier Dimension [1]. If I find the time I should read through it more thoroughly to get some intuition.
- I also found a survey on the application of the probabilistic method and the Baire category theorem in Harmonic analysis [4]. I feel this method is very exploitable in the types of problems I currently deal with, so if only for culture, this should be a useful read.

My main goals today were to finish up the slides, for my talk on Fourier dimension in pattern avoidance problems at the 2020 Ottawa Math Conference. I have essentially completed these slides; all that remains is to polish them up, and practice giving the presentation. My main goal during the presentation is to show that viewing avoiding sets Z geometrically leads to interesting questions, that the geometric quantities we consider lead to important consequences, and that the Fourier dimension question I am currently considering is interesting to study. I also thought about an idea which seemed to prevent adapting our result obtained by the 'queuing approach' in [3] to Thomas Körner's Baire category approach in [5], as well as finding a counterexample in a paper which removes the trivial case of [3] from the Fourier dimension case.

Ideas for Fourier Dimension Technique

Thomas Körner's paper [5] relies on Baire category arguments to construct generic measures μ supported on a subset of $\mathbf{T}$ avoiding solutions to m-term linear equations, such that for each $\xi \in \mathbf{Z}$,

$$|\widehat{\mu}(\xi)| \leq A(\xi).^1 \tag{1}$$

To obtain this bound generically, Körner works in the norm space X consisting of finite measures μ on $\mathbf{T}$ such that the quantity

$$\|\mu\|_X = \sup_{\xi \in \mathbf{Z}} \frac{|\widehat{\mu}(\xi)|}{A(\xi)}. \tag{2}$$

is finite. Then any measure in X satisfies (1) up to a multiplicative constant, and X is a Banach space, which enables one to use Baire-category techniques. A problem occurs in our Fourier dimension paper because I believe we can only construct finite measures μ such that for each $\varepsilon > 0$,

$$|\widehat{\mu}(\xi)| \lesssim_\varepsilon |\xi|^{\varepsilon - \beta/2}. \tag{3}$$

Such a measure does not satisfy quite as rigid an inequality as (1), instead having to satisfy infinitely many inequalities, and as such I do not believe we can find a Banach space norm which encapsulates (3).

However, today I thought of an idea which might prove fruitful. For any measure μ satisfying (3) for each $\varepsilon > 0$, the quantities

$$\|\mu\|_\varepsilon = \sup_{\xi \in \mathbf{Z}} |\widehat{\mu}(\xi)||\xi|^{\beta/2-\varepsilon} \tag{4}$$

will be finite for all $\varepsilon > 0$. If we let X denote the family of all finite measures which satisfy (4) for all $\varepsilon > 0$, then the collection of seminorms $\{\|\cdot\|_\varepsilon : \varepsilon > 0\}$ might give X the structure of a Frechét space. Since Frechét spaces are complete metric spaces, we can still apply Baire category arguments here. I haven't formally thought this through, so I'll add this to my to-do list later.

[1] Here $\{A(\xi)\}$ is a sequence given for each $\xi \in \mathbf{Z}$ by the formula

$$A(\xi) = B(\xi)|\xi|^{-\beta/2} \log(1 + |\xi|)^{1/2},$$

where $\beta = (n-1)^{-1}$, and $\{B(\xi)\}$ is some fixed sequence of positive numbers such that $B(\xi) \to \infty$ as $|\xi| \to \infty$.

Fourier Dimension Counterexample

In [2], I found a result which construct two disjoint, Borel sets $A, B \subset \mathbf{T}$, with $\dim_{\mathbf{F}}(A), \dim_{\mathbf{F}}(B) < 1$, but for such that $A \cup B = \mathbf{T}$. The result of Theorem 1 of [3] is trivial when $\dim_{\mathbf{M}}(Z) < d$, for if $\pi : (\mathbf{T}^d)^n \to \mathbf{T}^d$ is given by projection onto the first d coordinates, then $\mathbf{T}^d - \pi(Z)$ has full Hausdorff dimension and avoids Z. If $Z = A \times \{0\}$, then $\mathbf{T} - \pi(Z) = B$, which is not full dimensional, so things are more complicated when dealing with Fourier dimension. I found this result in [1], which might be a useful survey to read through completely in order to get a better grasp on how the Fourier dimension behaves.

Bibliography

[1] Jörg Schmeling Fredrik Ekström. A survey on the fourier dimension. 2017.

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[3] Joshua Zahl Jacob Denson, Malabika Pramanik. Large sets avoiding rough patterns. 2019.

[4] J.P. Kahane. Probabilities and Baire's theory in harmonic analysis. 2000.

[5] Thomas W. Körner. Fourier transforms of measures and algebraic relations on their supports. *Annales de L'Institut Fourier*, 2009.