Cartesian Products Avoiding Patterns

by

Jacob Denson

A THESIS SUBMITTED IN PARTIAL FULFILLMENT
OF THE REQUIREMENTS FOR THE DEGREE OF

Master of Science

in

THE FACULTY OF SCIENCE
(Mathematics)

The University of British Columbia
(Vancouver)

April 2019

The following individuals certify that they have read, and recommend to the Faculty of Graduate and Postdoctoral Studies for acceptance, the thesis entitled:

Cartesian Products Avoiding Patterns

submitted by **Jacob Denson** in partial fulfillment of the requirements for the degree of **Master of Science** in **Mathematics**.

Examining Committee:

Malabika Pramanik, Mathematics
Supervisor

Joshua Zahl, Mathematics
Supervisor

Abstract

The pattern avoidance problem seeks to construct a set $X \subset \mathbf{R}^d$ with large dimension that avoids a prescribed pattern such as three term arithmetic progressions, or more general patterns such as avoiding points $x_1,\dots,x_n$ such that $f(x_1,\dots,x_n) = 0$ for a given function f. Previous work on the subject has considered patterns described by polynomials, or functions f satisfying certain regularity conditions. We provide an exposition of this work, as well as considering new strategies to avoid 'rough patterns. There are several problems that fit into the framework of rough pattern avoidance. As a first application, if $Y \subset [0,1]$ is a set with Minkowski dimension α, we construct a set $X \subset [0,1]$ with Hausdorff dimension $1-\alpha$ so that $X+X$ is disjoint from Y. As a second application, given a set Y with dimension close to one, we can construct a set $X \subset Y$ of dimension $1/2$ that avoids isosceles triangles.

Lay Summary

Imagine looking at a patch of carpet with a microscope. Zooming in, we see the carpet is really a collection of twines tied together. On closer inspection, those twines break off into smaller twines. The carpet is rough at all scales. Classical geometric shapes like polyhedra do not model objects with complexity at arbitrarily small scales. Instead, you need a fractal: shapes with structure at small scales. Such models are often useful in applied mathematics and small-scale physics.

It is easy to construct polyhedra with geometric properties at a single scale. But it is non-trivial to construct fractals with properties at many scales. For instance, how do we construct a fractal which intersects any line in at most two points? In this thesis, we begin with an exposition on previous construction, and then provide new random fractal construction techniques.

Preface

At UBC, a preface may be required. Be sure to check the GPS guidelines as they may have specific content to be included.

Contents

Abstract iii

Lay Summary v

Preface vii

Acknowledgments xi

1 Introduction 1

2 Background 3
- 2.1 Configuration Avoidance . . . 3
- 2.2 Minkowski Dimension . . . 6
- 2.3 Hausdorff Dimension . . . 6
- 2.4 Dyadic Scales . . . 8
- 2.5 Frostman Measures . . . 11
- 2.6 Dyadic Fractional Dimension . . . 17
- 2.7 Beyond Hyperdyadics . . . 22

3 Related Work 25
- 3.1 Keleti: A Translate Avoiding Set . . . 25
- 3.2 Generalizing Keleti's Argument . . . 27
- 3.3 Fraser/Pramanik: Smooth Configurations . . . 28

4 Avoiding Rough Sets 35
- 4.1 Frequently Used Notation and Terminology . . . 37
- 4.2 Avoidance at Discrete Scales . . . 39
- 4.3 Fractal Discretization . . . 43
- 4.4 Dimension Bounds . . . 46

4.5 Applications 48
4.5.1 Sum-sets avoiding specified sets 48
4.5.2 Subsets of Lipschitz curves avoiding isosceles triangles . . 49

5 Conclusions **53**

Bibliography **55**

Acknowledgments

Thank those people who helped you.

Don’t forget your parents or loved ones.

You may wish to acknowledge your funding sources.

Chapter 1

Introduction

In this thesis, we study a simple family of questions:

How large can Euclidean sets be not containing geometric patterns?.

For instance, what is the maximal size of a set $X \subset \mathbf{R}^d$ that contains no three collinear points? Or what is the maximal size of a set not containing the vertices of an isosceles triangle? Aside from pure geometric interest, these problems provide useful settings to test methods of ergodic theory, additive combinatorics and harmonic analysis.

Sets which avoid the patterns we consider are highly irregular, in many ways behaving like a fractal set. Our understanding of them requires techniques from geometric measure theory. In particular, we use the *fractal dimension* to measure the size of a set when determining how large a pattern avoiding set can be.

At present, many fundamental questions about geometric structure and it's relation to fractal dimension remain unsolved. One might expect sets with sufficiently large fractal dimension contain patterns. For example, Theorem 6.8 of [12] shows that any set $X \subset \mathbf{R}^d$ with Hausdorff dimension exceeding one must contain three collinear points. On the other hand, Theorem 2.3 of [10] constructs a set $X \subset \mathbf{R}^d$ with full Hausdorff dimension such that no four points in X form the vertices of a parallelogram. Thus the problem depends on the particular aspects of the configurations involved. For most geometric configurations, it remains unknown at what threshold patterns are guaranteed, or whether such a threshold exists at all. So finding sets with large Hausdorff dimension avoiding patterns is an important topic to determine for which classes of patterns our intuition remains true.

Our goal in this thesis is to derive new methods for constructing sets avoiding patterns. In particular, we expand on a number of general *pattern dissection meth-*

ods which have proven useful in the area, originally developed by Keleti but also studied notably by Mathé, Pramanik, and Fraser. The main contribution we give to pattern dissection methods is to use random dissections to avoid patterns which lack structural information. Thus we can expand the utility of interval dissection methods from regular families of patterns to a set of *fractal avoidance problems*, that previous methods were completely unavailable to address. Such problems include finding large X such that the angles formed by any three distinct points of X avoid a specified set Y of angles, where Y has a fixed fractal dimension.

Chapter 2

Background

2.1 Configuration Avoidance

We consider an ambient set $\mathbf{A}$. It's *n-point configuration space* is

$$\mathcal{C}^n(\mathbf{A}) = \{(a_1,\dots,a_n) \in \mathbf{A}^n : a_i \neq a_j \text{ if } i \neq j\}.$$

An *n point configuration*, or *n point pattern*, is a subset of $\mathcal{C}^n(\mathbf{A})$. More generally, we define the general *configuration space* of $\mathbf{A}$ as $\mathcal{C}(\mathbf{A}) = \bigcup_{n=1}^{\infty} \mathcal{C}^n(\mathbf{A})$, and a *pattern*, or *configuration*, on $\mathbf{A}$ as a subset of $\mathcal{C}(\mathbf{A})$.

Our main focus in this thesis is the *pattern avoidance problem*. For a fixed configuration $\mathcal{C}$ on $\mathbf{A}$, we say a set $X \subset \mathbf{A}$ *avoids* $\mathcal{C}$ if $\mathcal{C}(X)$ is disjoint from $\mathcal{C}$. The pattern avoidance problem asks to find sets X of maximal size avoiding a fixed configuration $\mathcal{C}$. Often $\mathcal{C}$ describes algebraic or geometric structure, and the pattern avoidance problem asks to find the maximal size of a set before it is guaranteed to have such structure.

Example (Isoceles Triangle Configuration)**.** *Let*

$$\mathcal{C} = \left\{(x_1,x_2,x_3) \in \mathcal{C}^3(\mathbf{R}^2) : |x_1 - x_2| = |x_1 - x_3|\right\}.$$

Then $\mathcal{C}$ is a 3-point configuration, and a set $X \subset \mathbf{R}^2$ avoids $\mathcal{C}$ if and only if it does not contain all three vertices of an isoceles triangle.

Example (Linear Independence Configuration)**.** *Let V be a vector space over a* *field K. We set*

$$\mathcal{C} = \bigcup_{n=1}^{\infty} \{(x_1, \dots, x_n) \in \mathcal{C}^n(V) : \textit{there is } a_1, \dots, a_n \in K \textit{ s.t. } a_1 x_1 + \dots + a_n x_n = 0\}.$$

A set $X \subset V$ avoids $\mathcal{C}$ if and only if X is a linearly independent subset of X. Note that if V is infinite dimensional, then $\mathcal{C}$ cannot be replaced by a n point configuration for any n; arbitrarily large tuples must be considered. We will be interested in the case where $K = \mathbf{Q}$, and $V = \mathbf{R}$. Our interest here is to find analytically large linearly independant subsets of V.

Even though our problem formulation assumes configurations are formed by distinct sets of points, one can still formulate avoidance problems involving repeated points by a simple trick.

Example (Sum Set Configuration)**.** *Let G be an abelian group, and fix $Y \subset G$. Set*

$$\mathcal{C}^1 = \{g \in \mathcal{C}^1(G) : g + g \in Y\} \quad \textit{and} \quad \mathcal{C}^2 = \{(g_1, g_2) \in \mathcal{C}^2(G) : g_1 + g_2 \in Y\}.$$

Then define $\mathcal{C} = \mathcal{C}^1 \cup \mathcal{C}^2$. A set $X \subset G$ avoids $\mathcal{C}$ if and only if $(X + X) \cap Y = \emptyset$.

Depending on the structure of the ambient space $\mathbf{A}$ and the configuration $\mathcal{C}$, there are various ways of measuring the size of sets $X \subset \mathbf{A}$ for the purpose of the pattern avoidance problem:

- If $\mathbf{A}$ is finite, we wish to find a set X with large cardinality.
- If $\{\mathbf{A}_n\}$ is an increasing family of finite sets with $\mathbf{A} = \bigcup_n \mathbf{A}_n$, we wish to find a set X such that $X \cap \mathbf{A}_n$ has large cardinality asymptotically as $n \to \infty$.
- If $\mathbf{A} = \mathbf{R}^d$, but $\mathcal{C}$ is a discrete configuration, then a satisfactory goal is to find a set X with large Lebesgue measure avoiding $\mathcal{C}$.

In this thesis, inspired by results in these three settings, we establish methods for avoiding non-discrete configurations $\mathcal{C}$ in $\mathbf{R}^d$. Here, Lebesgue measure completely fails to measure the size of pattern avoiding solutions, as the next theorem shows, under the often true assumption that $\mathcal{C}$ is *translation invariant*, i.e. that if $(a_1, \dots, a_n) \in \mathcal{C}$ and $b \in \mathbf{R}^d$, $(a_1 + b, \dots, a_n + b) \in \mathcal{C}$.

Theorem 1. *Let $\mathcal{C}$ be a n-point configuration on $\mathbf{R}^d$. Suppose*

(A) *$\mathcal{C}$ is translation invariant.*

(B) *For any $\varepsilon > 0$, there is $(a_1, \dots, a_n) \in \mathcal{C}$ with diam$\{a_1, \dots, a_n\} \le \varepsilon$.*

Then no set with positive Lebesgue measure avoids $\mathcal{C}$.

Proof. Let $X \subset \mathbf{R}^d$ have positive Lebesgue measure. The Lebesgue density theorem shows that there exists a point $x \in X$ such that

$$\lim_{l(Q) \to 0} \frac{|X \cap Q|}{|Q|} = 1, \tag{2.1}$$

where Q ranges over all cubes in $\mathbf{R}^d$ with $x \in Q$, and $l(Q)$ denotes the sidelength of Q. Fix $\varepsilon > 0$, to be specified later, and choose r small enough that $|X \cap Q| \ge (1-\varepsilon)|Q|$ for any cube Q with $x \in Q$ and $l(Q) \le r$. Now let Q_0 denote a cube centered at x with $l(Q_0) \le r$. Applying Property (B), we find $C = (a_1, \dots, a_n) \in \mathcal{C}$ such that

$$\operatorname{diam}\{a_1, \dots, a_n\} \le l(Q_0)/2. \tag{2.2}$$

For each $p \in Q_0$, let $C(p) = (a_1(p), \dots, a_n(p))$, where $a_i(p) = p + (a_i - a_1)$. Property (A) implies $C(p) \in \mathcal{C}$ for each $p \in \mathbf{R}^d$. A union bound shows

$$|\{p \in Q_0 : C(p) \notin \mathcal{C}(X)\}| \le \sum_{i=1}^{d} |\{p \in Q_0 : a_i(p) \notin X\}|. \tag{2.3}$$

We have $a_i(p) \notin X$ precisely when $p + (a_i - a_1) \notin X$, so

$$|\{p \in Q_0 : a_i(p) \notin X\}| = |(Q_0 + (a_i - a_1)) \cap X^c|. \tag{2.4}$$

Note $Q_0 + (a_i - a_1)$ is a cube with the same sidelength as Q_0. Equation (2.2) implies $|a_i - a_1| \le l(Q_0)/2$, so $x \in Q_0 + (a_i - a_1)$. Thus (2.1) shows

$$|Q_0 + (a_i - a_1)) \cap X^c| \le \varepsilon |Q_0|. \tag{2.5}$$

Combining (2.3), (2.4), and (2.5), we find

$$|\{p \in Q_0 : C(p) \notin \mathcal{C}(X)\}| \le \varepsilon d |Q_0|.$$

Provided $\varepsilon d < 1$, this means there is $p \in Q_0$ with $C(p) \in \mathcal{C}(X)$. But this means X does not avoid $\mathcal{C}$. $\square$

Since no set of positive Lebesgue measure can avoid non-discrete configurations, we cannot use the Lebesgue measure to quantify the size of pattern avoiding sets for many non-discrete configurations. Geometric measure theory provides us with a quantity which can distinguish between the size of sets of measure zero. This is the *fractional dimension* of a set.

There are many variants of fractional dimension. Here we choose to use the Minkowski dimension and the Hausdorff dimension. They assign the same dimension to any smooth manifold with non-vanishing curvature, but vary over more singular sets. One major difference is that Minkowski dimension measures relative density at a single scale, whereas Hausdorff dimension measures relative density at countably many scales.

2.2 Minkowski Dimension

We begin by discussing the Minkowski dimension, which is the simplest to define. Given $l > 0$, and a bounded set $E \subset \mathbf{R}^d$, we let $N(l,E)$ denote the *covering number* of E, i.e. the minimum number of sidelength l cubes required to cover E. We define the *lower* and *upper* Minkowski dimension as

$$\dim_{\underline{\mathbf{M}}}(E) = \liminf_{l \to 0} \frac{\log(N(l,E))}{\log(1/l)} \quad \text{and} \quad \dim_{\overline{\mathbf{M}}}(E) = \limsup_{l \to 0} \frac{\log(N(l,E))}{\log(1/l)}.$$

If $\dim_{\overline{\mathbf{M}}}(E) = \dim_{\underline{\mathbf{M}}}(E)$, then we refer to this common quantity as the *Minkowski dimension* of E, denoted $\dim_{\mathbf{M}}(E)$. Thus $\dim_{\underline{\mathbf{M}}}(E) < s$ if there exists a sequence of lengths $\{l_k\}$ converging to zero with $N(l_k,E) \leq (1/l_k)^s$, and $\dim_{\overline{\mathbf{M}}}(E) < s$ if $N(l,E) \leq (1/l)^s$ for *all* sufficiently small lengths l.

2.3 Hausdorff Dimension

For $E \subset \mathbf{R}^d$ and $\delta > 0$, we define the *Hausdorff content*

$$H^s_\delta(E) = \inf \left\{ \sum_{k=1}^\infty l(Q_k)^s : E \subset \bigcup_{k=1}^\infty Q_k, l(Q_k) \leq \delta \right\}.$$

The *s-dimensional Hausdorff measure* of E is

$$H^s(E) = \lim_{\delta \to 0} H^s_\delta(E) = \sup_{\delta > 0} H^s_\delta(E).$$

It is easy to see H^s is an exterior measure on $\mathbf{R}^d$, and $H^s(E \cup F) = H^s(E) + H^s(F)$ if the Hausdorff distance $d(E,F)$ between E and F is positive. So H^s is actually a metric exterior measure, and the Caratheodory extension theorem shows all Borel sets are measurable with respect to H^s. Sometimes, it is convenient to use the exterior measure

$$H^s_\infty(E) = \inf \left\{ \sum_{k=1}^\infty l(Q_k)^s : E \subset \bigcup_{k=1}^\infty Q_k \right\}.$$

The majority of Borel sets which occur in practice fail to be measurable with respect to H^s_∞, but the exterior measure H^s_∞ has the useful property that $H^s_\infty(E) = 0$ if and only if $H^s(E) = 0$.

Lemma 2. *Consider $t < s$, and $E \subset \mathbf{R}^d$.*

(i) If $H^t(E) < \infty$, then $H^s(E) = 0$.

(ii) If $H^s(E) \neq 0$, then $H^t(E) = \infty$.

Proof. Suppose that $H^t(E) = A < \infty$. Then for any $\delta > 0$, there is a cover of E by a collection of intervals $\{Q_k\}$, such that $l(Q_k) \leq \delta$ for each k, and

$$\sum l(Q_k)^t \leq A < \infty.$$

But then

$$H^s_\delta(E) \leq \sum l(Q_k)^s \leq \sum l(Q_k)^{s-t} l(Q_k)^t \leq \delta^{s-t} A.$$

As $\delta \to 0$, we conclude $H^s(E) = 0$, proving *(i)*. And *(ii)* is just the contrapositive of (i), and therefore immediately follows. □

Corollary 3. *If $s > d$, $H^s = 0$.*

Proof. The measure H^d is just the Lebesgue measure on $\mathbf{R}^d$, so

$$H^d[-N,N]^d = (2N)^d.$$

If $s > d$, Lemma 2 shows $H^s[-N,N]^d = 0$. By countable additivity, taking $N \to \infty$ shows $H^s(\mathbf{R}^d) = 0$. Since H^s is a positive measure, $H^s(E) = 0$ for all E. □

Given any Borel set E, Corollary 3, combined with Lemma 2, implies there is a unique value $s_0 \in [0,d]$ such that $H^s(E) = 0$ for $s > s_0$, and $H^s(E) = \infty$ for $0 \leq s < s_0$. We refer to s_0 as the *Hausdorff dimension* of E, denoted $\dim_{\mathbf{H}}(E)$.

Theorem 4. *For any bounded set E,* $\dim_{\mathbf{H}}(E) \leq \dim_{\underline{\mathbf{M}}}(E) \leq \dim_{\overline{\mathbf{M}}}(E)$.

Proof. Given $l > 0$, we have a simple bound $H^s_l(E) \leq N(l,E) \cdot l^s$. If $\dim_{\underline{\mathbf{M}}}(E) < s$, then there exists a sequence $\{l_k\}$ with $l_k \to 0$, and $N(l_k,E) \leq (1/l_k)^s$. Thus we conclude that

$$H^s(E) = \lim_{k\to\infty} H^s_{l_k}(E) \leq \lim_{k\to\infty} N(l_k,E) \cdot l_k^s \leq 1,$$

Thus $\dim_{\mathbf{H}}(E) \leq s$. Taking infima over all s, we find $\dim_{\mathbf{H}}(E) \leq \dim_{\underline{\mathbf{M}}}(E)$. $\square$

Remark. *If* $\dim_{\mathbf{H}}(E) < d$*, then* $|E| = H^d(E) = 0$*. Thus any set with fractional dimension less than d must have measure zero. This means we can use the dimension as a way of distinguishing between sets of measure zero, which is precisely what we need to study configuration avoidance problem for non-discrete configurations.*

The fact that Hausdorff dimension is defined over multiple scales simultaneously makes it more stable under analytical operations. In particular, for any family of at most countably many sets $\{E_k\}$,

$$\dim_{\mathbf{H}}\left\{\bigcup E_k\right\} = \sup\left\{\dim_{\mathbf{H}}(E_k)\right\}.$$

This need not be true for the Minkowski dimension; a single point has Minkowski dimension zero, but the rational numbers, which are a countable union of points, have Minkowski dimension one. An easy way to make Minkowski dimension countably stable is to define the *modified Minkowski dimensions*

$$\dim_{\underline{\mathbf{MM}}}(E) = \inf\left\{ s : E \subset \bigcup_{i=1}^{\infty} E_i, \dim_{\underline{\mathbf{M}}}(E_i) \leq s \right\}$$

and

$$\dim_{\overline{\mathbf{MM}}}(E) = \inf\left\{ s : E \subset \bigcup_{i=1}^{\infty} E_i, \dim_{\overline{\mathbf{M}}}(E_i) \leq s \right\}.$$

This notion of dimension, in a disguised form, appears in Chapter BLAH.

2.4 Dyadic Scales

It is now useful to introduce the dyadic notation we utilize throughout this thesis. At the cost of losing topological perspective about $\mathbf{R}^d$, applying dyadic techniques often allows us to elegantly discretize certain problems in Euclidean space.

Fix an integer N. The classic family of dyadic cubes with *branching factor N* is given by setting, for each integer $k \geq 0$,

$$\mathcal{D}_k^d = \left\{ \prod_{i=1}^{d} \left[\frac{n_i}{N^k}, \frac{n_i+1}{N^k} \right] : n \in \mathbf{Z}^d \right\},$$

and then setting $\mathcal{D} = \bigcup_{k \geq 0} \mathcal{D}_k^d$. Elements of $\mathcal{D}$ are known as *dyadic cubes*, and elements of $\mathcal{D}_k^d$ are known as *dyadic cubes of generation k*. The most important properties of the dyadic cubes is that for each k, $\mathcal{D}_k^d$ is a cover of $\mathbf{R}^d$ by cubes of sidelength $1/N^k$, and for any two cubes $Q, R \in \mathcal{D}$, either their interiors are disjoint, or one cube is nested in the other.

- For each cube $Q \in \mathcal{D}_{k+1}^d$, there is a unique cube $R \in \mathcal{D}_k^d$ such that $Q \subset R$. We refer to R as the *parent* of Q, and Q as a *child* of R. Each cube in $\mathcal{D}$ has exactly N^d children. For $Q \in \mathcal{D}_{k+1}^d$, we let $Q^* \in \mathcal{D}_k^d$ denote it's parent.

- We say a set $E \subset \mathbf{R}^d$ is $\mathcal{D}_k$ *discretized* if it is a union of cubes in $\mathcal{D}_k^d$. If E is $\mathcal{D}_k$ discretized, we define

  $$\mathcal{D}_k(E) = \{Q \in \mathcal{Q}_k^d : Q \subset E\}.$$

  Then $E = \bigcup \mathcal{D}_k(E)$.

- Given $k \geq 0$, and $E \subset \mathbf{R}^d$, we let

  $$E(1/N^k) = \bigcup \{Q \in \mathcal{D}_k^d : Q \cap E \neq \emptyset\}.$$

  Then $E(1/N^k)$ is the smallest $\mathcal{D}_k$ discretized set containing E in it's interior. Our choice of notation invites thinking of $E(1/N^k)$ as a discretized version of the classic $1/N^k$ thickening

  $$\{x \in \mathbf{R}^d : d(x,E) < 1/N^k\}.$$

  We have no need for the standard thickening in this thesis, so there is no notational conflict.

Since any cube is covered by at most $O_d(1)$ cubes in $\mathcal{D}$ of comparable sidelength, from the perspective of geometric measure theory, working with dyadic cubes is normally equivalent to working with the class of all cubes.

Our main purpose with working with dyadic cubes is to construct *fractal-type sets*. By this, we mean defining sets X as the intersection of a nested family of sets $\{X_k\}$, where each X_k is $\mathcal{D}_k$ discretized, and each succesive set X_{k+1} is obtained from X_k by application of a simple procedure applied recursively. Such a construction satisfies Property (i), (ii), and (v) of Falconer's definition of a fractal, justifying the name.

Example. *A classic fractal-type set is the Cantor set C. We form C from the family of dyadic cubes with branching factor $M = 3$. We initially set $C_0 = [0,1]$. Then, given the $\mathcal{D}_k$ discretized set C_k, we consider each $I \in \mathcal{D}_k^1(C_k)$, and let $\mathcal{D}_{k+1}^1(I) = \{I_1, I_2, I_3\}$, where I_1, I_2, I_3 are listed in order. We set*

$$C_{k+1} = \bigcup \{I_1 \cup I_3 : I \in \mathcal{D}_k^1(C_k)\}.$$

Then $C = \bigcap_{k \geq 0} C_k$ is the Cantor set.

Unfortunately, a *constant* branching factor is not sufficient to describe the types of recursive procedures we discuss in this thesis. This motivates a more general iterated family of cubes. Instead of a single branching factor N, we fix a sequence of positive integers $\{N_k : k \geq 1\}$, with $N_k \geq 2$ for all k, which give the branching factor at each stage of the dyadic cubes we define.

- For each $k \geq 0$, we define

$$\mathcal{Q}_k^d = \left\{ \prod_{i=1}^d \left[\frac{m_i}{N_1 \dots N_k}, \frac{m_i + 1}{N_1 \dots N_k} \right] : m \in \mathbf{Z}^d \right\}.$$

  These are the *dyadic cubes of generation k*. We let $\mathcal{Q}^d = \bigcup_{k \geq 0} \mathcal{Q}_k^d$. Note that any two cubes $\mathcal{Q}^d$ are either nested within one another, or disjoint from one another.

- We set $l_k = (N_1 \dots N_k)^{-1}$. Then l_k is the sidelength of the cubes in $\mathcal{Q}_k^d$.

- Given $Q \in \mathcal{Q}_{k+1}^d$, we let $Q^* \in \mathcal{Q}_k^d$ denote the *parent cube* of Q, i.e. the unique dyadic cube of generation k such that $Q \subset Q^*$.

- We say a set $E \subset \mathbf{R}^d$ is $\mathcal{Q}_k$ *discretized* if it is a union of cubes in $\mathcal{Q}_k^d$. In this case, we let

$$\mathcal{Q}_k(E) = \{Q \in \mathcal{Q}_k^d : Q \subset E\}$$

  denote the family of cubes whose union is E.

- For $E \subset \mathbf{R}^d$ and $k \geq 0$, we let $E(l_k) = \{Q \in \mathcal{Q}_k^d : Q \cap E = \emptyset\}$.

Sometimes, our recursive constructions need a family of 'intermediary' cubes that lie between the scales $\mathcal{Q}_k^d$ and $\mathcal{Q}_{k+1}^d$. In this case, we consider a supplementary sequence $\{M_k : k \geq 1\}$ with $M_k \,|\, N_k$ for each k.

- For $k \geq 1$, we define

$$\mathcal{R}_k^d = \left\{ \prod_{i=1}^d \left[\frac{m_i}{N_1 \dots N_{k-1} M_k}, \frac{m_i + 1}{N_1 \dots N_{k-1} M_k} \right] : m \in \mathbf{Z}^d \right\}.$$

- We set $r_k = (N_1 \dots N_{k-1} M_k)^{-1}$. Then r_k is the sidelength of a cube in $\mathcal{R}_k^d$.

- The notions of being $\mathcal{R}_k^d$ discretized, the collection of cubes $\mathcal{R}_k^d(E)$, and the sets $E(r_k)$, are defined as should be expected.

The cubes in $\mathcal{R}_k^d$ are coarser than those in $\mathcal{Q}_k^d$, but finer than those in $\mathcal{Q}_{k-1}^d$.

Remark. *We note that there is some notational conflict between the cubes $\mathcal{D}^d$ and the cubes $\mathcal{Q}^d$, but since we never use both families simultaneously in a single argument, it should be clear which notation we are using.*

2.5 Frostman Measures

It is often easy to upper bound Hausdorff dimension, but non-trivial to *lower bound* the Hausdorff dimension of a given set. A key technique to finding a lower bound is *Frostman's lemma*, which says that a set has large Hausdorff dimension if and only if it supports a probability measure which obeys a certain decay law on small sets. We say a Borel probability measure μ is a *Frostman measure* of dimension s if it is non-zero, compactly supported, and there exists $C > 0$ such that for any cube Q, $\mu(Q) \leq C \cdot l(Q)^s$. The proof of Frostman's lemma will utilize a technique often useful, known as the *mass distribution principle*. Such a principle was stated in BLAH.

Lemma 5. *For each $k \geq 0$, let $\mathcal{E}_k$ be a finite collection of non-empty, compact, disjoint subsets of $\mathbf{R}^d$, such that for each $A \in \mathcal{E}_{k+1}$, there is a unique $A^* \in \mathcal{E}_k$ such that $A \subset A^*$, and for any $A^* \in \mathcal{E}_k$, there is at least one $A \in \mathcal{E}_{k+1}$ with $A \subset A^*$. Let*

$E_k = \bigcup \mathcal{E}_k$, and let $E_\infty = \bigcap E_k$. Let $\mu : \bigcup_{k \geq 0} \mathcal{E}_k \to [0,\infty)$ be a function such that for any i, and for any $A \in \mathcal{E}_i$,

$$\mu(A) = \sum \{\mu(B) : B \in \mathcal{E}_{i+1}, B \subset A\}. \tag{2.6}$$

Then μ extends to a finite Borel measure on $\mathbf{R}^d$ supported on E_∞.

Proof. Let $\mathcal{E} = \bigcup_{k \geq 0} \mathcal{E}_k \cup \mathcal{P}(\mathbf{R}^d - E_k)$. Then $\mathcal{E}$ is a *semi-ring* of sets:

- $\mathcal{E}$ *is closed under intersections*: Let $A, B \in \mathcal{E}$. Then without loss of generality, swapping A and B if necessary, there exists $i \leq j$ such that $A \in \mathcal{E}_i$ or $A \subset \mathbf{R}^d - E_i$, and $B \in \mathcal{E}_j$ or $B \subset \mathbf{R}^d - E_j$.

  If $A \in \mathcal{E}_i$, and $B \in \mathcal{E}_j$, then either $A \cap B = \emptyset$, or $B \subset A$, and $A \cap B = B$. In either case, $A \cap B \in \mathcal{E}$.

  If $A \subset \mathbf{R}^d - E_i$, then $A \cap B \subset A \subset \mathbf{R}^d - E_i$, so $A \cap B \in \mathcal{E}$. Similarily, if $B \subset \mathbf{R}^d - E_j$, then $A \cap B \subset \mathbf{R}^d - E_j$, so $A \cap B \in \mathcal{E}$.

  This addresses all cases.

- $\mathcal{E}$ *is closed under relative complements*: Let $A, B \in \mathcal{E}$. Then there exists i, j such that $A \in \mathcal{E}_i$ or $A \subset \mathbf{R}^d - E_i$, and $B \in \mathcal{E}_j$ or $B \subset \mathbf{R}^d - E_j$.

  Suppose $A \in \mathcal{E}_i$ and $B \in \mathcal{E}_j$. If $i \geq j$, then either $A \cap B = \emptyset$, or $A \subset B$. In either case, $A - B = \emptyset \in \mathcal{E}$. If $i \leq j$, then either $A \cap B = \emptyset$, or $B \subset A$. In the latter case,

  $$A - B = \left(\bigcup \{C : C \in \mathcal{E}_j, C \neq B, C \subset A\} \right) \cup \{A - E_j\}.$$

  This gives $A - B$ as a disjoint union of elements of $\mathcal{E}$.

  Suppose $A \in \mathcal{E}_i$, and $B \subset \mathbf{R}^d - E_j$. If $i \geq j$, then $A \subset E_i \subset E_j$, so $A - B = A$. If $i \leq j$, then

  $$A - B = (A \cap E_j) \cup (A \cap (\mathbf{R}^d - B)).$$

  The former set can be written as $\bigcup \{A' \in \mathcal{E}_j : A' \subset A\}$, and the latter set is a subset of $\mathbf{R}^d - E_j$. This gives $A - B$ as a disjoint union of elements of $\mathcal{E}$.

  If $A \subset \mathbf{R}^d - E_i$, then $A - B \subset \mathbf{R}^d - E_i$, so $A - B \in \mathcal{E}$.

  This addresses all cases.

The function μ extends to a function on $\mathcal{E}$ by setting $\mu(E) = 0$ if $E \subset \mathbf{R}^d - E_k$ for some k. This does not conflict with our previous definition, since if $A \in \mathcal{E}_i$, then $A \cap E_j \neq \emptyset$ for all $j \geq 0$. We claim that μ is then a *pre-measure* on the semi-ring:

- It is certainly true that $\mu(\emptyset) = 0$.

- Let $A \in \mathcal{E}$, and suppose $A = \bigcup_i A_i$, where $\{A_i\}$ is an at most countable subcollection of disjoint sets from $\mathcal{E}$. We fix k such that $A \in \mathcal{E}_k$, or $A \subset \mathbf{R}^d - E_k$, and for each i, fix k_i such that $A_i \in \mathcal{E}_{k_i}$, or $A \subset \mathbf{R}^d - E_{k_i}$.

  If $A \in \mathbf{R}^d - E_k$ for some k, then $A_i \in \mathbf{R}^d - E_k$ for each i, and thus $\mu(A_i) = 0$ for all i. Since $\mu(A) = 0$, this means $\mu(A) = \sum \mu(A_i)$.

  Suppose $A \in \mathcal{E}_k$, and let $A_\infty = A \cap E_\infty$. If $B \in \mathcal{E}_k$ for any k, then $B \cap A_\infty$ is *relatively open* in A_∞ because the sets in $\mathcal{E}_k$ cover A_∞ and are separated. Note that $A_\infty \cap (\mathbf{R}^d - E_k) = \emptyset$ for all k. Thus the family of sets $\{A_i : A_i \in \mathcal{E}_{k_i}\}$ is a relatively open cover of A_∞. Since A_∞ is compact, it must have a finite subcover. But applying compactness, $A_i \cap A_\infty \neq \emptyset$ for all i with $A_i \in \mathcal{E}_{k_i}$, so we conclude the set $\{A_i : A_i \in \mathcal{E}_{k_i}\}$ is finite.

  Since the family of A_i with $A_i \in \mathcal{E}_{k_i}$ is finite, we can repeatedly apply (2.6) so that, without loss of generality, we may assume there is a common integer N such that $A_i \in \mathcal{E}_N$ for all i with $\mu(A_i) > 0$. To prove that $\mu(A) = \sum \mu(A_i)$, it now suffices to show that for each $B \in \mathcal{E}_N$ with $B \subset A$, $B = A_i$ for some i. But this is obvious, since $\{A_i : A_i \in \mathcal{E}_N\}$ covers A_∞, and every such B satisfies $B \cap A_\infty \neq \emptyset$.

The Caratheodory extension theorem guarantees that μ extends to a measure on the σ algebra generated by $\mathcal{E}$, denoted $\sigma(\mathcal{E})$. We claim this σ algebra contains all Borel sets. Let F be a closed set. Then

$$F \cap E_\infty = \bigcap_k \left[\bigcup \{A : A \in \mathcal{E}_k, A \cap F \neq \emptyset\} \right] = \bigcap_k [G_k].$$

Each of the sets G_k is a finite union of elements in $\mathcal{E}_k$, hence $G_k \in \sigma(\mathcal{E})$, and so $F \cap E_\infty = \bigcap_k G_k \in \sigma(\mathcal{E})$. We also find

$$F \cap E_\infty^c = \bigcup_k F \cap (\mathbf{R}^d - E_k) = \bigcup_k G'_k.$$

Since $G'_k \in \mathcal{E}$ for all k, we conclude $F \cap E_\infty^c \in \sigma(\mathcal{E})$. Thus

$$F = (F \cap E_\infty) \cup (F \cap E_\infty^c) \in \sigma(\mathcal{E}).$$

Since F was an arbitrary closed set, this implies all Borel sets are measurable with respect to $\sigma(\mathcal{E})$. $\square$

An alternative approach, leading to a more general theorem, is given by employing weak convergence.

Lemma 6. *A Cauchy sequence of non-negative Borel measures on $\mathbf{R}^d$ is convergent with respect to the weak $*$ topology.*

Proof. Let $\{\mu_i\}$ be a Cauchy sequence of non-negative, regular Borel measures. This means that for each $f \in C_c(\mathbf{R}^d)$, the sequence of numbers

$$\left\{ \int f d\mu_i \right\}$$

is Cauchy. Fix a compact set K. then we can find a function $\varphi \in C_c(\mathbf{R}^d)$ such that $\mathbf{I}_K \leq \varphi$. This means that

$$\mu_i(K) = \int \mathbf{I}_K d\mu_i \leq \int \varphi d\mu_i.$$

Since $\{\int \varphi d\mu_i\}$ is Cauchy, it is uniformly bounded in i. In particular, $\mu_i(K)$ is uniformly bounded in i. Thus the measures $\mu_i|_K$ are uniformly finite, and we can apply the Banach Alaoglu theorem to find a regular Borel measure μ^K such that $\mu_i|_K \to \mu^K$ weakly. Note that if $f \in C_c(\mathbf{R}^d)$ is supported on $K_1 \cap K_2$ for two compact sets K_1 and K_2, then

$$\int f d\mu^{K_1} = \lim_{i\to\infty} \int_{K_1} f d\mu_i = \lim_{i\to\infty} \int_{K_2} f d\mu_i = \int f d\mu^{K_2}.$$

Thus we can define a measure μ such that

$$\int f d\mu = \lim_{K\to\infty} \int f d\mu^K.$$

For any $f \in C_c(\mathbf{R}^d)$ supported on a compact set K,

$$\int f d\mu = \int f d\mu^K = \lim_{i\to\infty} \int_K f d\mu_i = \lim_{i\to\infty} \int f d\mu_i.$$

Since f was arbitrary, $\mu_i \to \mu$ weakly. $\square$

Lemma 7 (Mass Distribution Principle)**.** *Let $f : \mathcal{Q}^d \to [0,\infty)$ be a function such that for any $Q_0 \in \mathcal{Q}^d$,*

$$\sum_{Q^*=Q_0} f(Q) = f(Q_0), \tag{2.7}$$

Then there exists a regular Borel measure μ supported on

$$\bigcap_{k=1}^{\infty} \left[\bigcup \{ Q \in \mathcal{Q}_k^d : f(Q) > 0 \} \right]$$

such that for each $Q \in \mathcal{Q}^d$,

$$\mu(Q) \geq f(Q), \tag{2.8}$$

and for any set E and $k \geq 0$,

$$\mu(E) \leq \sum f(Q), \tag{2.9}$$

where Q ranges over all cubes in $\mathcal{Q}_k^d(E(l_k))$.

Proof. For each i, let μ_i be a regular Borel measure such that for each $j \leq i$, and $Q \in \mathcal{Q}_j^d$, $\mu_i(Q) = f(Q)$. One such choice is given, for each $f \in C_c(\mathbf{R}^d)$, by the equation

$$\int f d\mu_i = \sum_{Q \in \mathcal{Q}_i^d} \frac{f(Q)}{|Q|} \int_Q f \, dx.$$

We claim that $\{\mu_i\}$ is a Cauchy sequence in the space of all regular Borel measures on $\mathbf{R}^d$, viewing the space as a locally convex space under the weak topology. Fix $\varphi \in C_c(\mathbf{R}^d)$, and choose some N such that φ is supported on $[-N,N]^d$. Set $Q_0 = [-N,N]^d$. Since φ is compactly supported, φ is *uniformly continuous*, so for each $\varepsilon > 0$, if i is suitably large, there is a sequence of values $\{a_Q : Q \in \mathcal{Q}_i^d(Q_0)\}$ such that if $x \in Q$, $|\varphi(x) - a_Q| \leq \varepsilon$. But this means that

$$\sum (a_Q - \varepsilon) \mathbf{I}_Q \leq \varphi \leq \sum (a_Q + \varepsilon) \mathbf{I}_Q,$$

where Q ranges over cubes in $\mathcal{Q}_i^d(Q_0)$. Thus for any $j \geq i$,

$$\int \varphi \, d\mu_j \leq \sum (a_Q + \varepsilon) \mu_j(Q) = \sum (a_Q + \varepsilon) f(Q)$$

and

$$\int \varphi \, d\mu_j \geq \sum (a_Q - \varepsilon) \mu_j(Q) = \sum (a_Q - \varepsilon) f(Q).$$

In particular, if $j, j' \geq i$,

$$\left| \int \varphi d\mu_j - \int \varphi d\mu_{j'} \right| \leq 2\varepsilon \sum_{Q \in \mathcal{Q}_i(Q_0)} f(Q) = 2\varepsilon \sum_{Q \in \mathcal{Q}_0(Q_0)} f(Q).$$

Since ε and φ were arbitrary, this shows $\{\mu_i\}$ is Cauchy.

Applying Lemma 6, there exists a regular Borel measure μ such that $\mu_i \to \mu$ weakly. For any $Q \in \mathcal{Q}^d$, since Q is a closed set,

$$\mu(Q) \geq \limsup_{i \to \infty} \mu_i(Q) = f(Q).$$

Conversely, if E is an arbitrary set, then $E \subset E(l_i)^\circ$ for any i, so

$$\mu(E) \leq \liminf_{i \to \infty} \mu_i(E(l_i)^\circ) \leq \liminf_{i \to \infty} \mu_i(E(l_i)) = \sum f(Q),$$

where Q ranges over the cubes in $\mathcal{Q}_i^d(E(l_k))$. $\square$

Lemma 8 (Frostman's Lemma)**.** *If E is a Borel set, $H^s(E) > 0$ if and only if there exists an s dimensional Frostman measure supported on E.*

Proof. Suppose that μ is s dimensional and supported on E. If $H^s(F) = 0$, then for each $\varepsilon > 0$ there is a sequence of cubes $\{Q_k\}$ with $\sum_{k=1}^\infty l(Q_k)^s \leq \varepsilon$. But then

$$\mu(F) \leq \sum_{k=1}^\infty \mu(Q_k) \lesssim \sum_{k=1}^\infty l(Q_k)^s \leq \varepsilon.$$

Taking $\varepsilon \to 0$, we conclude $\mu(F) = 0$. Thus μ is absolutely continuous with respect to H^s. Since $\mu(E) > 0$, this means that $H^s(E) > 0$.

To prove the converse, we will suppose for simplicity that E is compact. We work dyadically, with the classical family of dyadic cubes $\mathcal{D}$. By translating, we may assume that $H^s(E \cap [0,1]^d) > 0$, and so without loss of generality we may assume $E \subset [0,1]^d$. For each $Q \in \mathcal{D}_k^d$, define $f^+(Q) = H^s_\infty(E \cap Q)$. Then $f^+(Q) \leq 1/2^{ks}$, and f^+ is subadditive. We try and define a function f to which we can apply Lemma 7. We initially define f by setting $f([0,1]^d) = f^+([0,1]^d)$. Given $Q \in \mathcal{D}_k^d$, we enumerate it's children as $Q_1, \dots, Q_M \in \mathcal{D}_{k+1}^d$. We then consider any values $A_1, \dots, A_M \geq 0$ such that

$$A_1 + \cdots + A_M = f(Q), \tag{2.10}$$

and for each k,

$$A_k \leq f^+(Q_k). \tag{2.11}$$

This is feasible to do because $f^+(Q_1) + \cdots + f^+(Q_M) \geq f^+(Q)$. We then define $f(Q_k) = A_k$ for each k. Equation (2.10) implies the recursive constraint is satisfied, so f is a well defined function. Furthermore, (2.10) implies (2.7) of Lemma 7, and so the mass distribution principle implies that there exists a measure μ supported on E, satisfying (2.9) and (2.8). In particular, (2.9) implies μ is non-zero. For each $Q \in \mathcal{D}_k^d$, $\#[\mathcal{D}_k^d(Q(l_k))] = 3^d = O_d(1)$, so we can apply (2.8) to conclude

$$\mu(Q) \lesssim_d \max f(Q') \leq 1/2^{ks} = l(Q)^s.$$

where Q' ranges over the cubes in $\mathcal{D}_k^d(Q(l_k))$. Given any cube Q, we find k with $1/2^{k-1} \leq l(Q) \leq 1/2^k$. Then Q is covered by $O_d(1)$ dyadic cubes in $\mathcal{D}_k^d$, and so $\mu(Q) \lesssim l(Q)^s$. Thus μ is a Frostman measure of dimension s. $\square$

Frostman's lemma implies that to study the Hausdorff dimension of the set, it suffices to understand the class of measures which can be supported on that set. The Fourier dimension of a set also studies this perspective, by slightly refining the measure bound required by the Frostman dimension in frequency space.

2.6 Dyadic Fractional Dimension

We wish to establish results about fractional dimension 'dyadically', working with the family of cubes $\mathcal{Q}^d$ specified in terms of a sequence $\{N_k : k \geq 1\}$. We begin with Minkowski dimension. For each m, let $N_{\mathcal{Q}}(m, E)$ denote the minimal number of cubes in $\mathcal{Q}_m^d$ required to cover E. This is often easy to calculate, up to a multiplicative constant, by greedily selecting cubes which intersect E.

Lemma 9. *For any set E,*

$$N_{\mathcal{Q}}(m,E) \approx_d \#\{Q \in \mathcal{Q}_m^d : Q \cap E \neq \emptyset\} \approx_d N(l_m, E).$$

Proof. Let $\mathcal{E} = \{Q \in \mathcal{Q}_m^d : Q \cap E \neq \emptyset\}$. Then $N(l_m, E) \leq N_{\mathcal{Q}}(m,E) \leq \#(E)$. Conversely, let $\{Q_k\}$ be a minimal cover of E by cubes. Then each cube Q_k intersects at most 3^d cubes in $\mathcal{Q}_k^d$, so $\#(\mathcal{E}) \leq 3^d \cdot N(l_m, E) \leq 3^d \cdot N_{\mathcal{Q}}(m,E)$. $\square$

Thus it is natural to ask whether it is true that for any set E,

$$\begin{gathered} \dim_{\underline{\mathbf{M}}}(E) = \liminf_{k\to\infty} \frac{\log[N_{\mathcal{Q}}(k,E)]}{\log[1/l_k]} \\ \text{and} \\ \dim_{\overline{\mathbf{M}}}(E) = \limsup_{k\to\infty} \frac{\log[N_{\mathcal{Q}}(k,E)]}{\log[1/l_k]}. \end{gathered} \tag{2.12}$$

The answer depends on the choice of $\{N_k\}$. In particular, we find that a sufficient condition is that

$$N_{k+1} \lesssim_\varepsilon (N_1 \dots N_k)^\varepsilon \quad \text{for any } \varepsilon > 0. \tag{2.13}$$

We will see that this condition allows us to work dyadically in many scenarios when it comes to fractal dimension.

Lemma 10. *If* (2.13) *holds, then* (2.12) *holds.*

Proof. Fix a length l, and find k with $l_{k+1} \leq l \leq l_k$. Applying Lemma 9 shows

$$N(l,E) \leq N(l_{k+1},E) \lesssim_d N_{\mathcal{Q}}(k+1,E)$$

and

$$N(l,E) \geq N(l_k,E) \gtrsim_d N_{\mathcal{Q}}(k,E).$$

Thus

$$\frac{\log[N(l,E)]}{\log[1/l]} \leq \left[\frac{\log(1/l_{k+1})}{\log(1/l_k)}\right] \frac{\log[N_{\mathcal{Q}}(k+1,E)]}{\log[1/l_{k+1}]} + O_d(1/k)$$

and

$$\frac{\log[N(l,E)]}{\log[1/l]} \geq \left[\frac{\log(1/l_k)}{\log(1/l_{k+1})}\right] \frac{\log[N_{\mathcal{Q}}(k,E)]}{\log[1/l_k]} + O_d(1/k).$$

Thus, provided that

$$\frac{\log(1/l_{k+1})}{\log(1/l_k)} \to 1, \tag{2.14}$$

the conclusion of the theorem is true. But (2.14) is equivalent to the condition that

$$\frac{\log(N_{k+1})}{\log(N_1) + \dots + \log(N_k)} \to 0,$$

and this is equivalent to (2.13). $\square$

Any constant branching factor satisfies 2.13 for the Minkowski dimension. In particular, we can work fairly freely with the classical dyadic cubes without any problems occuring. But more importantly for our work, we can let the sequence $\{N_k\}$ increase rapidly.

Lemma 11. *If $N_k = 2^{\lfloor 2^{k\psi(k)} \rfloor}$, where $\psi(k)$ is any decreasing sequence of positive numbers tending to zero, such that $\psi(k) \gtrsim \log(k)/k$, then* (2.13) *holds.*

Proof. We note that $\log(N_k) = 2^{k\psi(k)} + O(1)$. Thus

$$\begin{aligned}
\frac{\log(N_{k+1})}{\log(N_1) + \cdots + \log(N_k)} &= \frac{2^{(k+1)\psi(k+1)} + O(1)}{2^{\psi(1)} + 2^{2\psi(2)} + \cdots + 2^{k\psi(k)} + O(k)} \\
&\lesssim \frac{2^{(k+1)\psi(k+1)}}{2^{\psi(k)} + 2^{2\psi(k)} + \cdots + 2^{k\psi(k)}} \\
&\lesssim \frac{2^{(k+1)\psi(k+1)}}{2^{(k+1)\psi(k)}}(2^{\psi(k)} - 1) \\
&\leq (2^{\psi(k)} - 1) \to 0.
\end{aligned}$$

This is equivalent to the fact that $N_{k+1} \lesssim_\varepsilon (N_1 \dots N_k)^\varepsilon$ for any $\varepsilon > 0$. $\square$

We refer to any sequence $\{l_k\}$ constructed by $\{N_k\}$ satisfying the conditions of Lemma 11 as a *subhyperdyadic* sequence. If a sequence is generated by a sequence $\{N_k = 2^{\lfloor 2^{ck} \rfloor}\}$, for some fixed $c > 0$, the values $\{l_k\}$ are referred to as *hyperdyadic*. The next (counter) example shows that hyperdyadic sequences are essentially the 'boundary' for sequences that can be used to measure the Minkowski dimension.

Example. *We consider a multi-scale dyadic construction, utilizing the two families $\mathcal{Q}^d$ and $\mathcal{R}^d$. Fix $0 \leq c < 1$, and define $N_k = 2^{\lfloor 2^{ck} \rfloor}$, and $M_k = 2^{\lfloor c2^{ck} \rfloor}$. Then $M_k \,|\, N_k$ for each k. We recursively define a nested family of sets $\{E_k\}$, with each E_k a $\mathcal{Q}_k^d$ discretized set, and set $E = \bigcap E_k$. We define $E_0 = [0,1]$. Then, given E_k, we divide each sidelength l_k dyadic interval in E_k into M_{k+1} intervals, and then keep the first sidelength l_k/M_{k+1} interval from this set, which is formed from N_{k+1}/M_{k+1} sidelength l_{k+1} dyadic intervals. Thus $\#(\mathcal{Q}_0(E_0)) = 1$, and*

$$\#(\mathcal{Q}_{k+1}(E_{k+1})) = (N_{k+1}/M_{k+1})\#(\mathcal{Q}_k(E_k))$$

so

$$\#(\mathcal{Q}_k(E_k)) = \frac{N_1 \dots N_k}{M_1 \dots M_k} \tag{2.15}$$

Noting that $\log(N_i) = 2^{ci} + O(1)$*, and* $\log(M_i) = c2^{ci} + O(1)$*, we conclude that*

$$\frac{\log \#(\mathcal{Q}_k(E_k))}{\log(1/l_k)} = \frac{(1-c)(2^c + \cdots + 2^{ck}) + O(k)}{(2^c + \cdots + 2^{ck}) + O(k)} = 1 - c + O\left(\frac{k}{2^{ck}}\right) \to 1 - c.$$

On the other hand, if $r_{k+1} = l_k/M_{k+1}$*, then for each* k*,*

$$\#(\mathcal{R}^d_{k+1}(E_k)) = \#(\mathcal{Q}_k(E_k)) = \frac{N_1 \dots N_k}{M_1 \dots M_k},$$

and so

$$\begin{aligned}
\frac{\log \#(\mathcal{R}^d_{k+1}(E_k))}{\log(1/r_{k+1})} &= \frac{(1-c)(2^c + \cdots + 2^{ck}) + O(k)}{(2^c + \cdots + 2^{ck}) + c2^{c(k+1)} + O(k)} \\
&= \frac{\left(\frac{(1-c)2^c}{2^c-1}\right) 2^{ck} + O(k)}{\left(\frac{2^c}{2^c-1} + c2^c\right) 2^{ck} + O(k)} \\
&= \frac{1-c}{1-c+c2^c} + O\left(\frac{k}{2^{ck}}\right) \to \frac{1-c}{1-c+c2^c} < 1 - c.
\end{aligned}$$

In particular,

$$\dim_{\underline{\mathbf{M}}}(E) \neq \liminf_{k \to \infty} \frac{\log [N(l_k, E)]}{\log(1/l_k)},$$

so measurements at hyperdyadic scales fail to establish general results about the Minkowski dimension.

We now move on to calculating Hausdorff dimension dyadically. The natural quantity to consider is the measure defined for any E as $H^s_{\mathcal{Q}}(E) = \lim_{m\to\infty} H^s_{\mathcal{Q},m}(E)$, where

$$H^s_{\mathcal{Q},m}(E) = \inf\left\{ \sum_k l(Q_k)^s : E \subset \bigcup_k^\infty Q_k,\ Q_k \in \bigcup_{i \geq m} \mathcal{Q}^d_i \text{ for each } k \right\}.$$

A similar argument to the standard Hausdorff measures shows there is a unique s_0 such that $H^s_{\mathcal{Q}}(E) = \infty$ for $s < s_0$, and $H^s_{\mathcal{Q}}(E) = 0$ for $s > s_0$. It is obvious that $H^s_{\mathcal{Q}}(E) \geq H^s(E)$ for any set E, so we certainly have $s_0 \geq \dim_{\mathbf{H}}(E)$. The next lemma guarantees that $s_0 = \dim_{\mathbf{H}}(E)$, under the same conditions on the sequence $\{N_k\}$ as found in Lemma 10.

Lemma 12. *If* (2.13) *holds, then for any* $\varepsilon > 0$*,* $H^s_{\mathcal{Q}}(E) \lesssim_{s,\varepsilon} H^{s-\varepsilon}(E)$*.*

Proof. Fix $\varepsilon > 0$ and m. Let $E \subset \bigcup Q_k$, where $l(Q_k) \leq l_m$ for each k. Then for each k, we can find i_k such that $l_{i_k+1} \leq l(Q_k) \leq l_{i_k}$. Then Q_k is covered by $O_d(1)$ elements of $\mathcal{Q}^d_{i_k}$, and

$$H^s_{\mathcal{Q},m}(E) \lesssim_d \sum l^s_{i_k} \leq \sum (l_{i_k}/l_{i_k+1})^s l(Q_k)^s \leq \sum \left(l_{i_k}/l_{i_{k+1}}\right)^s l^{\varepsilon}_{i_k} l(Q_k)^{s-\varepsilon}. \quad (2.16)$$

By assumption,

$$l_{i_k+1} = \frac{1}{N_1 \dots N_{i_k} N_{i_k+1}} \gtrsim_{s,\varepsilon} (N_1 \dots N_{i_k})^{1+\varepsilon/s} = l^{1+\varepsilon/s}_{i_k}. \quad (2.17)$$

Putting (2.16) and (2.17) together, we conclude that $H^s_{\mathcal{Q},m}(E) \lesssim_{d,s,\varepsilon} \sum l(Q_k)^{s-\varepsilon}$. Since $\{Q_k\}$ was an arbitrary cover of E, we conclude $H^s_{\mathcal{Q},m}(E) \lesssim H^{s-\varepsilon}(E)$, and since m was arbitrary, that $H^s_{\mathcal{Q}}(E) \lesssim H^{s-\varepsilon}(E)$. $\square$

Finally, we consider computing whether we can establish that a measure is a Frostman measure dyadically.

Theorem 13. *If* (2.13) *holds, and if μ is a Borel measure such that $\mu(Q) \lesssim l(Q)^s$ for each $Q \in \mathcal{Q}^d_k$, then μ is a Frostman measure of dimension $s - \varepsilon$ for each $\varepsilon > 0$.*

Proof. Given a cube Q, find k such that $l_{k+1} \leq l(Q) \leq l_k$. Then Q is covered by $O_d(1)$ cubes in $\mathcal{Q}^d_k$, which shows

$$\mu(Q) \lesssim_d l^s_k = [(l_k/l)^s l^{\varepsilon}] l^{s-\varepsilon} \leq [l^{s+\varepsilon}_k / l^s_{k+1}] l^{s-\varepsilon} = \left[\frac{N^s_{k+1}}{(N_1 \dots N_k)^{\varepsilon}}\right] l^{s-\varepsilon} \lesssim_{\varepsilon} l^{s-\varepsilon}. \ \square$$

Remark. *The dyadic construction showing that Minkowski dimension cannot be measured only at hyperdyadic scales also shows that a bound on a measure μ on hyperdyadic cubes does not imply the correct bound at all scales. It is easy to show from* (2.15) *that the canonical measure μ for this example satisfies $\mu(Q) \lesssim l(Q)^{1-c}$ for all $Q \in \mathcal{Q}_k(E_k)$, yet we know that for the set constructed in that example,*

$$\dim_{\mathbf{H}}(E) \leq \dim_{\underline{\mathbf{M}}}(E) < 1 - c.$$

Frostman's lemma implies we cannot possibly have $\mu(Q) \lesssim_{\varepsilon} l(Q)^{1-c-\varepsilon}$ for all $\varepsilon > 0$ and all *cubes Q.*

Let's recognize the utility of this approach from the perspective of a dyadic construction. Suppose we have a sequence of nested sets $\{E_k\}$, where E_k is a $\mathcal{Q}_k$ discretized subset of $[0,1]^d$, and for each $Q_0 \in \mathcal{Q}_k(E_k)$, there is at least one cube

$Q \in \mathcal{Q}_{k+1}(E_{k+1})$ with $Q^* = Q_0$. Then we can set $E = \bigcap E_k$ as a 'limit' of the discretizations E_k. We can associate with this construction a finite measure μ. It is defined by setting $f([0,1]^d) = 1$, and setting, for each $Q \in \mathcal{Q}_{k+1}(E_{k+1})$,

$$f(Q) = \frac{f(Q^*)}{\#\{Q' \in \mathcal{Q}_{k+1}(E_{k+1}) : (Q')^* = Q^*\}}.$$

The mass distribution principle shows that there exists a Borel measure μ, which we refer to as the *canonical measure* associated with this construction. For this measure, it is often easy to show from a combinatorial argument that $f(Q) \lesssim l(Q)^s$ if $Q \in \mathcal{Q}^d$, which leads to bounds on the measure μ. This makes Theorem 13 useful.

2.7 Beyond Hyperdyadics

If we are to use a faster increasing sequence of branching factors than the last section guarantees, we therefore must exploit some extra property of our construction, which is not always present in general sets. Here, we rely on a *uniform mass distribution* between scales. Given the uniformity assumption, the lengths can decrease as fast as desired. We utilize a multi-scale set of dyadic cubes.

Lemma 14. *Let μ be a measure supported on a set E. Suppose that*

(A) *For any $Q \in \mathcal{Q}_k^d$, $\mu(Q) \lesssim l_k^s$.*

(B) *For each $Q \in \mathcal{R}_k^d$, $\#[\mathcal{Q}_k^d(E(l_k) \cap Q)] \lesssim 1$.*

(C) *For any $Q \in \mathcal{R}_{k+1}^d$ with parent cube $Q^* \in \mathcal{Q}_k^d$, $\mu(Q) \lesssim (r_{k+1}/l_k)^d \mu(Q^*)$.*

Then μ is a Frostman measure of dimension s.

Proof. We establish the general bound $\mu(Q) \lesssim l(Q)^s$ for all cubes Q by separating into two different cases:

- Suppose there is k with $r_{k+1} \leq l(Q) \leq l_k$. Then $\#(\mathcal{R}_{k+1}^d(Q(r_{k+1}))) \lesssim (l/r_{k+1})^d$. Properties (A) and (C) imply each of these cubes has measure at most $O((r_{k+1}/l_k)^d l_k^s)$, so we obtain that

$$\mu(Q) \lesssim (l/r_{k+1})^d (r_{k+1}/l_k)^d l_k^s = l^d / l_k^{d-s} \lesssim l^s.$$

- Suppose there exists k with $l_k \leq l \leq r_k$. Then $\#[\mathcal{R}_k^d(Q(r_k))] \lesssim 1$. Combining this with Property (C) gives $\#[\mathcal{Q}_k^d(Q(r_k) \cap E(l_k))] \lesssim 1$. This and Property (A) then shows
$$\mu(Q) \lesssim l_k^s \leq l^s.$$

This addresses all cases, so μ is a Frostman measure of dimension s. $\square$

TODO: ADDRESS OTHER FRACTIONAL DIMENSION CASES?

Chapter 3

Related Work

Here, we discuss the main papers which influenced our results. In particular, the work of Keleti on translate avoiding sets, Fraser and Pramanik's work on sets avoiding smooth configurations, Mathé's result on sets avoiding algebraic varieties, and Schmerkin's result on sets with large Fourier dimension avoiding smooth configurations.

3.1 Keleti: A Translate Avoiding Set

Keleti's two page paper constructs a full dimensional set $X \subset [0,1]$ such that for each $t \neq 0$, X intersects $t+X$ in at most one place. The set X is then said to *avoid translates*. This paper contains the core idea behind the *interval dissection* method adapted in Fraser and Pramanik's paper. We also adapt this technique in our paper, which makes the result of interest.

Lemma 15. *Let X be a set. Then X avoids translates if and only if there do not exists values $x_1 < x_2 \leq x_3 < x_4$ in X with $x_2 - x_1 = x_4 - x_3$. In particular, a set X avoids translates if and only if it avoids the four point configuration*

$$\mathcal{C} = \{(x_1,x_2,x_3,x_4) : x_1 < x_2 \leq x_3 < x_4,\ x_2 - x_1 = x_4 - x_3\}.$$

Proof. Suppose $(t+X) \cap X$ contains two points $a < b$. Without loss of generality, we may assume that $t > 0$. If $a \leq b - t$, then the equation

$$a - (a-t) = t = b - (b-t)$$

satisfies the constraints, since $a - t < a \leq b - t < b$ are all elements of X. We also have

$$(b-t)-(a-t)=b-a,$$

which satisfies the constraints if $a-t<b-t\le a<b$. This covers all possible cases. Conversely, if there are $x_1<x_2\le x_3<x_4$ in X with

$$x_2-x_1=t=x_4-x_3,$$

then $X+t$ contains $x_2=x_1+(x_2-x_1)$ and $x_4=x_3+(x_4-x_3)$. □

The basic, but fundamental idea of the interval dissection technique is to introduce memory into Cantor set constructions. Keleti constructs a nested family of discrete sets $\{X_k\}$, with X_k a $\mathcal{Q}_k$ discretized set, and with $X=\bigcap X_k$. The sequence $\{N_k\}$ will be specified later, but each N_k will be a multiple of 10. We initialize $X_0=[0,1]$. The novel feature of the argument is to add a queue of intervals to the construction, initially just containing $[0,1]$. To construct the sequence $\{X_k\}$, Keleti iteratively performs the following procedure:

Algorithm 1 Construction of the Sets $\{X_k\}$:

Set $k=0$.
Repeat
 Take off an interval I from the front of the queue.
 For all $J\in\mathcal{Q}_k(X_k)$:
  Order the intervals in $\mathcal{Q}_{k+1}(J)$ as $J_1,\dots,J_N$.
  If $J\subset I$, add all intervals J_i to X_{k+1} with $i\equiv 0$ modulo 10.
  Else add all J_i with $i\equiv 5$ modulo 10.
 Add all intervals in $\mathcal{Q}^d_{k+1}$ to the end of the queue.
 Increase k by 1.

Each iteration of the algorithm produces a new set X_k, and so leaving the algorithm to repeat infinitely produces a sequence $\{X_k\}$ whose intersection is X.

Lemma 16. *The set X is translate avoiding.*

Proof. If X is not translate avoiding, there is $x_1<x_2\le x_3<x_4$ with $x_2-x_1=x_4-x_3$. Since $l_k\to 0$, there is a suitably large integer N such that x_1 is contained in an interval $I\in\mathcal{Q}_N$ not containing x_2,x_3, or x_4. At stage N of the algorithm, the interval I is added to the end of the queue, and at a much later stage M, the interval I is retrieved. Find the startpoints x_1°,x_2°, $x_3^\circ,x_4^\circ\in l_M\mathbf{Z}$ to the intervals in

$\mathcal{Q}_M$ containing x_1, x_2, x_3, and x_4. Then we can find n and m such that $x_4^\circ - x_3^\circ = (10n)l_M$, and $x_2^\circ - x_1^\circ = (10m+5)l_M$. In particular, this means that $|(x_4^\circ - x_3^\circ) - (x_2^\circ - x_1^\circ)| \geq 5L_M$. But

$$\begin{aligned}|(x_4^\circ - x_3^\circ) - (x_2^\circ - x_1^\circ)| &= |[(x_4^\circ - x_3^\circ) - (x_2^\circ - x_1^\circ)] - [(x_4 - x_3) - (x_2 - x_1)]| \\ &\leq |x_1^\circ - x_1| + \cdots + |x_4^\circ - x_4| \leq 4L_M\end{aligned}$$

which gives a contradiction. □

It is easy to see from the algorithm that

$$\#(\mathcal{Q}_k(X_k)) = (l_{k-1}/10l_k) \cdot \#(\mathcal{Q}_{k-1}(X_{k-1})).$$

Closing the recursive definition shows

$$\#(\mathcal{Q}_k(X_k)) = \frac{1}{10^k l_k}.$$

In particular, this means $|X_k| = 1/10^k$, so X has measure zero irrespective of our parameters. Nonetheless, the canonical measure μ on X defined with respect to the decomposition $\{X_k\}$ satisfies $\mu(I) = 10^k l_k$ for all $I \in \mathcal{B}(l_k, X)$. If $10^k l_k^\varepsilon \lesssim_\varepsilon 1$ for all ε, then we can establish the bounds $\mu(I) \lesssim_\varepsilon l_k^{1-\varepsilon}$ for all ε. In particular, this is true if $N_k = 10^{\lfloor \log(k+2) \rfloor}$. And because this sequence does not rapidly decrease too fast, we can apply Lemma 13 to show μ is a Frostman measure of dimension $1-\varepsilon$ for each $\varepsilon > 0$, so X has full Hausdorff dimension.

3.2 Generalizing Keleti's Argument

Before we move onto other methods which developed Keleti's argument work, it is useful to dwell on what general properties this argument has:

- *Simplification to a Discrete Problem*: A major part of Keleti's argument is solving a discrete version of the configuration argument. We could summarize the result of Keleti's discrete argument in a lemma.

  Lemma 17. *Let T_1, T_2 be disjoint, $\mathcal{Q}_k$ discretized sets. Then we can find $S_1 \subset T_1$ and $S_2 \subset T_2$ such that*

  (i) For each k, S_k is a $\mathcal{Q}_{k+1}$ discretized subset of T_k.

(ii) If $x_1 \in S_1$ and $x_2, x_3, x_4 \in S_2$, then $x_2 - x_1 \neq x_4 - x_3$.

(iii) For each cube $Q \in \mathcal{Q}_k$ with $Q \subset T_k$,

$$\#(\mathcal{Q}_{k+1}(Q \cap S_k)) \geq (1/10) \cdot \#(\mathcal{Q}_{k+1}(Q)).$$

- *Iterative Application of Discrete Solution*: Keleti then repeatedly applies his argument iteratively. In particular, Property (i) of Lemma 17 allows him to apply his argument iteratively. The reason why Keleti obtains a configuration avoiding set in the limit is because of Property (ii). Most importantly the reason why Keleti obtains a set with full Hausdorff dimension, if the sequence $\{N_k\}$ decreases rapidly enough, is because of Property (iii).

Of course, it is not possible to extend the discrete solution of the configuration argument to general configurations; this part of Keleti's method strongly depends on the arithmetic structure of the configuration. The iterative application of a discrete solution, however, can be applied in generality. In various senses, this technique is applied in all the results we review in this chapter, and in the new results we provide later on in this thesis.

3.3 Fraser/Pramanik: Smooth Configurations

Inspired by Keleti's result, Pramanik and Fraser obtained a generalization of the queue method which allows one to find sets avoiding $n+1$ point configurations given by the zero sets of smooth functions, i.e.

$$\mathcal{C} = \{(x_0, \dots, x_n) \in \mathbf{R}^{dn} : f(x_0, \dots, x_n) = 0\},$$

under mild regularity conditions on the function $f : [0,1]^{dn} \to [0,1]^m$. Here, a simple pidgeonholing strategy suffices.

Theorem 18 (Pramanik and Fraser). *Fix $m \leq d(n-1)$. Consider a countable family of functions $\{f_k : [0,1]^{dn} \to [0,1]^m\}$ each of which being C^2, and such that for each k, Df_k has full rank at any $(x_1, \dots, x_n)$ with $f(x_1, \dots, x_n) = 0$, where the $(x_1, \dots, x_n)$ are distinct. Then there exists a set $X \subset \mathbf{R}^d$ with Hausdorff dimension $m/(n-1)$ such that X avoids the configuration*

$$\mathcal{C} = \bigcup_k \{(x_1, \dots, x_n) \in \mathcal{C}^n(\mathbf{R}^d) : f(x_1, \dots, x_n) = 0\}.$$

Remark. *For simplicity, we only prove the result for a single function, rather than a countable family of functions. The only major difference between the two approaches is the choice of scales we must choose later on in the argument.*

Pramanik and Fraser also simplify to a discrete version of their problem, which they then iteratively apply. In the discrete setting, rather than making a linear shift in one of the intervals we avoid as in Keleti's approach, one must use the smoothness properties of the function to find large segments of an interval avoiding. Corollary 21 gives the discrete solution that Pramanik and Fraser utilizes.

Lemma 19. *Fix $n > 1$. Let $T \subset [0,1]^d$ be $\mathcal{Q}_k$ discretized, and $T' \subset [0,1]^{(n-1)d}$ be $\mathcal{Q}_k$ discretized. Let $B \subset T \times T'$ be $\mathcal{Q}_{k+1}$ discretized. Then there exists a $\mathcal{Q}_{k+1}$ discretized set $S \subset T$, and a $\mathcal{Q}_{k+1}$ discretized set $B' \subset T'$, such that*

(A) $(S \times T') \cap B \subset S \times B'$.

(B) *For every $Q \in \mathcal{Q}_k^d$, there exists $\mathcal{R}(Q) \subset \mathcal{R}_{k+1}^d(Q)$, such that*

$$\#(\mathcal{R}(Q)) \geq (1/2) \cdot \#(\mathcal{R}_{k+1}^d(Q)).$$

and for each $R \in \mathcal{R}_{k+1}^d(Q)$,

$$\#(\mathcal{Q}_{k+1}(R)) = \begin{cases} 1 & : R \in \mathcal{R}(Q,) \\ 0 & : R \notin \mathcal{R}(Q). \end{cases}$$

(C) $\#(\mathcal{Q}_{k+1}(B')) \leq 2(N_1 \dots N_k)^d \,(M_{k+1}/N_{k+1})^d \cdot \#(\mathcal{Q}_{k+1}(B))$.

Proof. Fix $Q_0 \in \mathcal{Q}_k(T)$. For each $R \in \mathcal{R}_{k+1}(Q_0)$, define a *slab* $S[R] = R \times T'$, and for each $Q \in \mathcal{Q}_{k+1}(Q_0)$, define a *wafer* $W[Q] = Q \times T'$. We say a wafer $W[Q]$ is *good* if

$$\#(\mathcal{Q}_{k+1}(W[Q] \cap B)) \leq (2/N_{k+1}^d) \cdot \#(\mathcal{Q}_{k+1}(B)), \tag{3.1}$$

Then at most $N_{k+1}^d/2$ wafers are bad. We call a slab *good* if it contains a wafer which is good. Since a slab is the union of $(N_{k+1}/M_{k+1})^d$ wafers, at most $M_{k+1}^d/2 = (1/2) \cdot \#(\mathcal{R}_{k+1}(Q_0))$ slabs are bad. Thus if we set

$$\mathcal{R}(Q_0) = \{R \in \mathcal{R}_{k+1}(Q_0) : S[R] \text{ is good}\},$$

then

$$\#(\mathcal{R}(Q_0)) \geq (1/2) \cdot \#(\mathcal{R}_{k+1}(Q_0)). \tag{3.2}$$

For each $R \in \mathcal{R}(Q_0)$, we pick $Q_R \in \mathcal{Q}_{k+1}(R)$ such that $W[Q_R]$ is good, and define

$$S = \bigcup\{Q_R : R \in \mathcal{R}(Q_0)\}.$$

Equation (3.2) implies S satisfies Property (B).

Let B' be the union of all cubes $Q' \in \mathcal{Q}_{k+1}(T')$ such that there is $Q \in \mathcal{Q}_{k+1}(S)$ with $Q \times Q' \in \mathcal{Q}_{k+1}(B)$. By definition, Property (A) is then satisfied. For each $Q \in \mathcal{Q}_{k+1}(S)$, $W[Q]$ is good, so (3.1) implies

$$\#\{Q' : Q \times Q' \in \mathcal{Q}_{k+1}(B)\} \leq (2/N_{k+1}^d) \cdot \#(\mathcal{Q}_{k+1}(B)).$$

But $\#(\mathcal{Q}_{k+1}(S)) \leq \#(\mathcal{R}_{k+1}(T)) \leq (1/r_{k+1})^d = (N_1 \dots N_k)^d M_{k+1}^d$, so

$$\begin{aligned}\#(\mathcal{Q}_{k+1}(B')) &\leq \#(\mathcal{Q}_{k+1}(S))[(2/N_{k+1}^d) \cdot \#(\mathcal{Q}_{k+1}(B))] \\ &\leq 2(N_1 \dots N_k)^d (M_{k+1}/N_{k+1})^d \#(\mathcal{Q}_{k+1}(B)),\end{aligned}$$

which establishes Property (C). □

We apply the lemma recursively $n-1$ times to continually reduce the dimensionality of the avoidance problem we are considering. Eventually, we obtain the case where $n = 0$, and we are in need of a final technique.

Lemma 20. *Fix $n > 1$. Let $T \subset [0,1]^d$ be $\mathcal{Q}_k$ discretized, and let $B \subset T$ be $\mathcal{Q}_{k+1}$ discretized. Suppose*

$$\#\,\mathcal{Q}_{k+1}(B) \leq \left[C \cdot 2^{n-1} (N_1 \dots N_k)^{d(n-1)} (M_{k+1}/N_{k+1})^{d(n-1)}\right] (1/l_{k+1})^{dn-m}.$$

and

$$N_{k+1} > \left[C \cdot 2^n (N_1 \dots N_k)^{2dn}\right]^{1/m} M_{k+1}^{d(n-1)/m} \tag{3.3}$$

Then there exists a $\mathcal{Q}_{k+1}$ discretized set $S \subset T$ such that

(A) $S \cap B = \emptyset$.

(B) *For each $Q_0 \in \mathcal{Q}_k(T)$, there is $\mathcal{R}(Q_0) \subset \mathcal{R}_{k+1}(Q_0)$ with*

$$\#(\mathcal{R}(Q_0)) \geq (1/2)\#(\mathcal{R}_{k+1}(Q_0)),$$

such that for each $R \in \mathcal{R}_{k+1}(Q_0)$,

$$\#(\mathcal{Q}_{k+1}(R \cap S)) = \begin{cases} 1 & : R \in \mathcal{R}(Q_0), \\ 0 & : R \notin \mathcal{R}(Q_0). \end{cases}$$

Proof. For each $Q_0 \in \mathcal{Q}_k(T)$, we set

$$\mathcal{R}(Q_0) = \{R \in \mathcal{R}_{k+1}(Q_0) : \#(\mathcal{Q}_{k+1}(R \cap B)) \leq (2/M_{k+1}^d) \cdot \#(\mathcal{Q}_{k+1}(B))\}.$$

Since $\mathcal{R}_{k+1}(Q_0) = M_{k+1}^d$,

$$\#(\mathcal{R}(Q_0)) \geq \#(\mathcal{R}_{k+1}(Q_0)) - (M_{k+1}^d/2) \geq (1/2) \cdot \#(\mathcal{R}_{k+1}(T)).$$

Now (3.3) implies that for each $R \in \mathcal{R}(Q_0)$,

$$\begin{aligned} \#(\mathcal{Q}_{k+1}(R \cap B)) &\leq (2/M_{k+1}^d) \cdot \#(\mathcal{Q}_{k+1}(B)) \\ &\leq (2/M_{k+1}^d) \left(C \cdot 2^{n-1} (N_1 \dots N_k)^{2dn} (M_{k+1}/N_{k+1})^{d(n-1)} \right). \\ &= \left[2^n C (N_1 \dots N_k)^{2dn} \right] \left(M_{k+1}^{d(n-2)} / N_{k+1}^{m-d} \right) \\ &< (N_{k+1}/M_{k+1})^d \\ &= \mathcal{Q}_{k+1}(R) \end{aligned}$$

Thus for each $R \in \mathcal{R}(Q_0)$, we can find $Q_R \in \mathcal{Q}_{k+1}(R)$ such that $Q_R \cap B = \emptyset$. And so if we set

$$S = \bigcup \{Q_R : R \in \mathcal{R}(Q_0), Q_0 \in \mathcal{Q}_k(T)\},$$

then (A) and (B) are satisfied. $\square$

Corollary 21. *Let $f : [0,1]^{dn} \to [0,1]^m$ be C^2, and have full rank at every point $(x_1, \dots, x_n)$ with all $x_1, \dots, x_n$ distinct, and $f(x_1, \dots, x_n) = 0$. Then there exists a universal constant C depending only on f such that, if* (3.3) *is satisfied, then for any disjoint, $\mathcal{Q}_k$ discretized sets $T_1, \dots, T_n \subset [0,1]^d$, we can find $\mathcal{Q}_{k+1}$ discretized sets $S_1 \subset T_1, \dots, S_n \subset T_n$ such that*

(A) *If $x_1 \in S_1, \dots, x_n \in S_n$, then $f(x_1, \dots, x_n) \neq 0$.*

(B) *For each k, and for each $Q_0 \in \mathcal{Q}_k(T_k)$, there is $\mathcal{R}(Q_0) \subset \mathcal{R}_{k+1}(Q_0)$ with*

$$\#(\mathcal{R}(Q_0)) \geq (1/2) \cdot \#(\mathcal{R}_{k+1}(Q_0)),$$

and for each $R \in \mathcal{R}_{k+1}(Q_0)$,

$$\#(\mathcal{Q}_{k+1}(R \cap S)) = \begin{cases} 1 & : R \in \mathcal{R}(Q_0), \\ 0 & : R \notin \mathcal{R}(Q_0). \end{cases}$$

Proof. Since f is C^2 and has full rank on the set

$$V(f) = \{(x_1, \dots, x_n) \in \mathcal{C}^n(\mathbf{R}^d) : f(x_1, \dots, x_n) = 0\},$$

the implicit function theorem implies $V(f)$ is a smooth manifold of dimension $nd - m$ in $\mathbf{R}^{dn}$, and the coarea formula implies the existence of a constant C such that for each k,

$$\#\{Q \in \mathcal{Q}_k^{dn} : Q \cap V(f) \neq \emptyset\} \leq C/l_k^{dn-m}.$$

To apply Lemma 19 and 20, we set

$$B = \#\{Q \in \mathcal{Q}_k^{dn} : Q \cap V(f) \neq \emptyset\}.$$

Applying Lemma 19 iteratively $n-1$ times, then finishing with Lemma 20, constructs the sets $S_1, \dots, S_n$. $\square$

Remark. *The values of $N_1, \dots, N_k$, exponentials 2^k, and constants, are roughly insignificant in comparison to M_{k+1} and N_{k+1}, when these numbers are hyperdyadic for a suitably large constant, or grow faster than hyperdyadic. In particular, we can roughly read off the Hausdorff dimension of the set $m/(n-1)$ we construct from the inverse power of M_{k+1} in* (3.3)*, i.e. that $N_{k+1} \gtrapprox M_{k+1}^{d(n-1)/m}$.*

Just like in Keleti's proof, Pramanik and Fraser's technique applies a discrete result, Corollary 21, iteratively at many scales to obtain a high dimensional set avoiding the zeroes of a function. We construct a nested family $\{X_k : k \geq 0\}$ of $\mathcal{Q}_k$ discretized sets, converging to a set X, which we will show is translate avoiding. We intiailize $X_0 = [0,1]$. Our queue shall consist of n tuples of disjoint intervals $(T_1, \dots, T_n)$, all of the same length, which initially consists of all possible tuples of intervals in $\mathcal{Q}_1^d([0,1]^d)$. To construct the sequence $\{X_k\}$, we perform the following iterative procedure:

Algorithm 2 Construction of the Sets $\{X_k\}$

Set $k = 0$

Repeat

Take off an n tuple $(T_1', \dots, T_n')$ from the front of the queue

Set $T_i = T_i' \cap X_k$ for each i

Apply Corollary 21 to the sets $T_1, \dots, T_d$, obtaining $\mathcal{Q}_{k+1}$ discretized sets $S_1, \dots, S_n$ satisfying Properties (A), (B), and (C) of that Lemma.

Set $X_{k+1} = X_k - \bigcup_{i=1}^n T_i - S_i$.

Add all n tuples of disjoint cubes $(T_1', \dots, T_n')$ in $\mathcal{Q}_{k+1}^d(X_{k+1})$ to the back of the queue.

Increase k by 1.

Lemma 22. *The set X constructed by the procedure avoids the configuration*

$$\mathcal{C} = \{(x_1, \dots, x_n) \in \mathcal{C}^n(\mathbf{R}^d) : f(x_1, \dots, x_n) = 0\}.$$

Proof. Suppose $x_1, \dots, x_n \in X$ are distinct. Then at some stage k, $x_1, \dots, x_n$ lie in disjoint cubes $T_1', \dots, T_n' \in \mathcal{Q}_k^d(X_k)$, for some large k. At this stage, $(T_1', \dots, T_n')$ is added to the back of the queue, and therefore, at some much later stage N, the tuple $(T_1', \dots, T_n')$ is taken off the front. Sets $S_1 \subset T_1', \dots, S_n \subset T_n'$ are constructed satisfying Property (A) of Corollary 21. Since $x_1, \dots, x_n \in X$, we must have $x_i \in S_i$ for each i, so $f(x_1, \dots, x_n) \neq 0$. □

What remains is to bound the Hausdorff dimension of X. For this, one must use a sequence of factors $\{N_k\}$ which is *just* faster than a hyperdyadic sequence of factors.

Theorem 23. *If $M_k = 2^{2^{k \log k}}$, X has Hausdorff dimension exceeding $m/(n-1)$.*

Proof. First, we construct the canonical measure μ on X. Property (B) of Corollary 21 implies that for each $Q \in \mathcal{Q}_{k+1}^d$, $\mu(Q) \leq 2/M_{k+1}^d \mu(Q^*)$, which implies that for $Q \in \mathcal{Q}_k^d$, $\mu(Q) \leq 2^k/(M_k \dots M_1)^d$. By Lemma 14, it suffices to show that for each $\varepsilon > 0$,

$$\frac{2^k}{(M_{k+1} \dots M_1)^d} \lesssim_\varepsilon \frac{1}{(N_1 \dots N_{k+1})^{m/(n-1)(1-\varepsilon)}} \tag{3.4}$$

Set

$$N_{k+1} = A(N_1 \dots N_k)^{2dn/m} M_{k+1}^{d(n-1)/m},$$

where A is an arbitrary constant not depending on k so that (3.3) holds. Then inequality (3.4) is then implied if

$$M_{k+1} \gtrsim_\varepsilon \left((N_1 \dots N_k)^{6dn}[2^k A^d]\right)^{1/\varepsilon}.$$

This equation is satisfied if $M_k = 2^{2^{k \log k}}$. $\square$

Chapter 4

Avoiding Rough Sets

In the last chapter, we saw that many authors have considered the pattern avoidance problem for configurations $\mathcal{C}$ which take the form of many general classes of smooth shapes; in Mathé's work, $\mathcal{C}$ can take the form of an algebraic variety of low degree, and in Pramanik and Fraser's work, $\mathcal{C}$ can take the form of a smooth manifold. In this chapter, we consider the pattern avoidance problem for an even more general class of 'rough' patterns, that are the countable union of sets with controlled lower Minkowski dimension.

Theorem 24. *Let $\alpha \geq d$, and suppose $\mathcal{C} \subset \mathcal{C}^n(\mathbf{R}^d)$ is the countable union of precompact sets, each with lower Minkowski dimension at most α. Then there exists a set $X \subset [0,1]^d$ with Hausdorff dimension at least $(nd-\alpha)/(n-1)$ avoiding $\mathcal{C}$.*

A major question in modern geometric measure theory is whether sufficiently large sets are forced to contain copies of certain patterns. Intuitively, one expects the answer to be yes, and many results in the literature support this intuition. For example, the Lebesgue density theorem implies that a set of positive Lebesgue measure contains an affine copy of every finite set. And any set $X \subset \mathbf{R}^2$ with Hausdorff dimension exceeding one must contain three collinear points. On the other hand, there is a distinct genre of results that challenges this intuition. Keleti [9] constructs a set $X \subset \mathbf{R}$ that avoids all solutions of the equation $x_2 - x_1 = x_4 - x_3$ with $x_1 < x_2 \leq x_3 < x_4$, and which consequently does not contain any nontrivial arithmetic progression. Given any triangle, Falconer [**?**] constructs a full dimensional planar set that does not contain any similar copy of the vertex set of the triangle. Maga [10] provides a set $X \subset \mathbf{R}^2$ of full Hausdorff dimension such that no four points in X form the vertices of a parallelogram. The pattern avoidance problem (informally stated) asks: for a given pattern, how large can the dimension of a set $X \subset \mathbf{R}^d$ be before it is forced to contain a copy of this pattern?

One way to formalize the notion of a pattern is as follows. If $d \geq 1$ and $n \geq 2$ are integers, we define a pattern to be a set $Z \subset \mathbf{R}^{dn}$. We say that a set $X \subset \mathbf{R}^d$ avoids the pattern Z if for every n-tuple of distinct points $x_1, \dots, x_n \in X$, we have $(x_1, \dots, x_n) \notin Z$. For example, a set $X \subset \mathbf{R}^2$ does not contain three collinear points if and only if it avoids the pattern

$$Z = \{(x_1, x_2, x_3) \in \mathbf{R}^6 : \ |(x_1 - x_2) \wedge (x_1 - x_3)| = 0\}.$$

Similarly, a set $X \subset \mathbf{R}^2$ avoids the pattern

$$Z = \{(x_1, x_2, x_3, x_4) \in \mathbf{R}^8 : \ x_1 + x_4 = x_2 + x_3\}$$

if and only if no four points in X form the vertices of a (possibly degenerate) parallelogram.

A number of recent articles have established pattern avoidance results for increasingly general patterns. In [11], Máthé constructs a set $X \subset \mathbf{R}^d$ that avoids a pattern specified by a countable union of algebraic varieties of controlled degree. In [6], Fraser and the second author consider the pattern avoidance problem for countable unions of C^1 manifolds. In this paper, we consider the pattern avoidance problem for an even more general class of 'rough' patterns $Z \subset \mathbf{R}^{dn}$, that are the countable union of sets with controlled lower Minkowski dimension.

Theorem 25. *Let $\alpha \geq d$, and let $Z \subset \mathbf{R}^{dn}$ be a countable union of compact sets, each with lower Minkowski dimension at most α. Then there exists a set $X \subset [0,1]^d$ with Hausdorff dimension at least $(nd - \alpha)/(n-1)$ such that whenever $x_1, \dots, x_n \in X$ are distinct, we have $(x_1, \dots, x_n) \notin Z$.*

Remarks.

1. *When $\alpha < d$, the pattern avoidance problem is trivial, since $X = [0,1]^d - \pi(Z)$ is full dimensional and solves the pattern avoidance problem, where $\pi(x_1, \dots, x_n) = x_1$ is a projection map from $\mathbf{R}^{dn}$ to $\mathbf{R}^d$. We will therefore assume that $\alpha \geq d$ in our proof of the theorem. Note that obtaining a full dimensional set in the case $\alpha = d$, however, is still interesting.*

2. *Theorem 25 is trivial when $\alpha = dn$, since we can set $X = \emptyset$. We will therefore assume that $\alpha < dn$ in our proof of the theorem.*

3. *When Z is a countable union of smooth manifolds in $\mathbf{R}^{nd}$ of co-dimension m, we have $\alpha = nd - m$. In this case Theorem 25 yields a set in $\mathbf{R}^d$ with Hausdorff dimension at least $(nd-\alpha)/(n-1) = m/(n-1)$. This recovers Theorem 1.1 and 1.2 from [6], making Theorem 25 a generalization of these results.*

4. *Since Theorem 25 does not require any regularity assumptions on the set Z, it can be applied in contexts that cannot be addressed using previous methods. Two such applications, new to the best of our knowledge, have been recorded in Section 4.5; see Theorems 33 and 34 there.*

The set X in Theorem 25 is obtained by constructing a sequence of approximations to X, each of which avoids the pattern Z at different scales. For a sequence of lengths $l_k \searrow 0$, we construct a nested family of sets $\{X_k\}$, where X_k is a union of cubes of sidelength l_k that avoids Z at scales close to l_k. The set $X = \bigcap X_k$ avoids Z at all scales. While this proof strategy is not new, our method for constructing the sets $\{X_k\}$ has several innovations that simplify the analysis of the resulting set $X = \bigcap X_k$. In particular, through a probabilistic selection process we are able to avoid the complicated queuing techniques used in [9] and [6], that required storage of data from each step of the iterated construction, to be retrieved at a much later stage of the construction process.

At the same time, our construction continues to share certain features with [6]. For example, between each pair of scales l_{k-1} and l_k, we carefully select an intermediate scale r_k. The set $X_k \subset X_{k-1}$ avoids Z at scale l_k, and it is 'evenly distributed' at scale r_k; the set X_k is a union of cubes of length l_k whose midpoints resemble (a large subset of) generalized arithmetic progression of step size r_k. The details of a single step of this construction are described in Section 4.2. In Section 4.3, we explain how the length scales l_k and r_k for X are chosen, and prove its avoidance property. In Section 4.4 we analyze the size of X and show that it satisfies the conclusions of Theorem 25.

4.1 Frequently Used Notation and Terminology

(A) A *dyadic length* is a number l of the form 2^{-k} for some non-negative integer k.

(B) Given a length $l > 0$, we let $\mathcal{B}_l^d$ denote the set of all closed cubes in $\mathbf{R}^d$ with sidelength l and corners on the lattice $(l \cdot \mathbf{Z})^d$, i.e.

$$\mathcal{B}_l^d = \{[a_1, a_1 + l] \times \cdots \times [a_d, a_d + l] : \ a_k \in l \cdot \mathbf{Z}\}.$$

(C) A set $E \subset \mathbf{R}^d$ is *l discretized* if it is a union of cubes in $\mathcal{B}_l^d$. For any set $E \subset \mathbf{R}^d$, and any length $l > 0$, we let

$$E(l) = \bigcup \{I \in \mathcal{B}_l^d : \ I \cap E \neq \emptyset\}.$$

Then $E(l)$ is the smallest l discretized set with $E \subset E(l)^\circ$. Given an l discretized set E, we let

$$\mathcal{B}_l^d(E) = \{I \in \mathcal{B}_l^d : \ I \subset E\}.$$

Then $E = \bigcup \mathcal{B}_l^d(E)$.

(D) The *lower Minkowski dimension* of a bounded set $Z \subset \mathbf{R}^d$ is defined as

$$\dim_{\underline{\mathbf{M}}}(Z) = \liminf_{l \to 0} \frac{\log \left[\# \mathcal{B}_l^d(Z(l))\right]}{\log[1/l]}.$$

(E) If $\alpha \geq 0$ and $\delta > 0$, we define the dyadic Hausdorff content of a set $E \subset \mathbf{R}^d$ as

$$H_\delta^\alpha(E) = \inf \left\{ \sum_{k=1}^\infty l_k^\alpha : \ E \subset \bigcup_{k=1}^\infty I_k \right\},$$

where the infinum is taken over all families of cubes $\{I_k\}$ such that for all k, $I_k \in \mathcal{B}_{l_k}^d$, and l_k is a dyadic length with $l_k \leq \delta$. The α-dimensional dyadic Hausdorff measure H^α on $\mathbf{R}^d$ is $H^\alpha(E) = \lim_{\delta \to 0} H_\delta^\alpha(E)$, and the *Hausdorff dimension* of a set E is $\dim_{\mathbf{H}}(E) = \inf\{\alpha \geq 0 : \ H^\alpha(E) = 0\}$.

(F) Given $K \in \mathcal{B}_l^{dn}$, we can decompose K as $K_1 \times \cdots \times K_n$ for unique cubes $K_1, \dots, K_n \in \mathcal{B}_l^d$. We say K is *strongly non-diagonal* if the cubes $K_1, \dots, K_n$ are distinct. Strongly non-diagonal cubes will play an important role in Section 4.2, when we solve a discrete version of Theorem 25.

(G) Adopting the terminology of [8], we say a collection of sets $\{U_k\}$ is a *strong cover* of a set E if $E \subset \limsup U_k$, which means every element of E is contained in infinitely many of the sets U_k. This idea will be useful in Section

4.3.

(H) A *Frostman measure* of dimension α is a non-zero compactly supported probability measure μ on $\mathbf{R}^d$ such that for every dyadic cube I of sidelength l, $\mu(I) \lesssim l^\alpha$. Note that a measure μ satisfies this inequality for every cube I if and only if it satisfies the inequality for cubes whose sidelengths are dyadic lengths. *Frostman's lemma* says that

$$\dim_{\mathbf{H}}(E) = \sup \left\{ \alpha : \begin{array}{c} \text{there is a Frostman measure of} \\ \text{dimension } \alpha \text{ supported on } E \end{array} \right\}.$$

4.2 Avoidance at Discrete Scales

In this section we describe a method for avoiding Z at a single scale. We apply this technique in Section 4.3 at many scales to construct a set X avoiding Z at all scales. This single scale avoidance technique is the core building block of our construction, and the efficiency with which we can avoid Z at a single scale has direct consequences on the Hausdorff dimension of the set X obtained in Theorem 25.

At a single scale, we solve a discretized version of the problem, where all sets are unions of cubes at two dyadic lengths $l > s$. In this discrete setting, Z is replaced by a discretized version of itself, a union of cubes in $\mathcal{B}_s^{dn}$ denoted by G. Given a set E, which is a union of cubes in $\mathcal{B}_l^d$, our goal is to construct a set $F \subset E$ that is a union of cubes in $\mathcal{B}_s^d$, such that F^n is disjoint from strongly non-diagonal cubes (see Definition (F)) in $\mathcal{B}_s^{dn}(G)$. Using the setup introduced at the end of the introduction, we will later choose $l = l_k$, $s = l_{k+1}$, and $E = X_k$. The set X_{k+1} will be defined as the set F constructed.

In order to ensure the final set X obtained in Theorem 25 has large Hausdorff dimension regardless of the rapid decay of scales used in the construction of X, it is crucial that F is uniformly distributed at intermediate scales between l and s. We achieve this by decomposing E into sub-cubes in $\mathcal{B}_r^d$ for some intermediate scale $r \in [s, l]$, and distributing F as evenly among these intermediate sub-cubes as possible. This is possible assuming a mild regularity condition on the number of cubes in G, i.e. Equation (4.1).

Lemma 26. *Fix two distinct dyadic lengths l and s, with $l > s$. Let $E \subseteq [0,1]^d$ be a nonempty, l discretized set, and let $G \subset \mathbf{R}^{dn}$ be a nonempty s discretized set* *such that*

$$(l/s)^d \leq \#\mathcal{B}_s^{dn}(G) \leq \frac{1}{2}(l/s)^{dn}. \tag{4.1}$$

Then there exists a dyadic length $r \in [s,l]$ of size

$$r \sim \left(l^{-d} s^{dn} \#\mathcal{B}_s^{dn}(G)\right)^{\frac{1}{d(n-1)}}, \tag{4.2}$$

and an s discretized set $F \subset E$ satisfying the following three properties:

(A) Avoidance*: For any n distinct cubes $I_1, \ldots, I_n \in \mathcal{B}_s^d(F)$, $I_1 \times \cdots \times I_n \notin \mathcal{B}_s^d(G)$.*

(B) Non-Concentration*: For any $I \in \mathcal{B}_r^d(E)$, $\#(\mathcal{B}_s^d(F \cap I)) \leq 1$.*

(C) Large Size*: For every $I \in \mathcal{B}_l^d(E)$, $\#(\mathcal{B}_s^d(F \cap I)) \geq \#(\mathcal{B}_r^d(E \cap I))/2$.*

Remark. *Property (A) says that F avoids strongly non-diagonal cubes in $\mathcal{B}_s^{dn}(G)$. Properties (B) and (C) together imply that for every $I \in \mathcal{B}_l^d(E)$, at least half of the cubes in $\mathcal{B}_r^d(I)$ contribute a single sub-cube of sidelength s to F; the rest contribute none.*

Proof. Let r be the smallest dyadic length at least as large as R, where

$$R = \left(2l^{-d} s^{dn} \#\mathcal{B}_s^{dn}(G)\right)^{\frac{1}{d(n-1)}}. \tag{4.3}$$

This choice of r satisfies (4.2). The inequalities in (4.1) ensure that $r \in [s,l]$; more precisely, the left inequality in (4.1) implies R is bounded from below by s, and the right inequality implies R is bounded from above by l. The minimality of r ensures $s \leq r \leq l$.

For each $I \in \mathcal{B}_r^d(E)$, let J_I be an element of $\mathcal{B}_s^d(I)$ chosen uniformly at random; these choices are independent as I ranges over the elements of $\mathcal{B}_r^d(E)$. Define

$$U = \bigcup \left\{ J_I : I \in \mathcal{B}_r^d(E) \right\},$$

and

$$\mathcal{K}(U) = \{K \in \mathcal{B}_s^{dn}(G) : K \in U^n, K \text{ strongly non-diagonal}\}.$$

Note that the sets U and $\mathcal{K}(U)$ are random sets, in the sense that they depend on the random variables $\{J_I\}$. Define

$$F(U) = \bigcup \left(\mathcal{B}_s^d(U) - \{\pi(K) : K \in \mathcal{K}(U)\} \right), \tag{4.4}$$

where $\pi\colon \mathbf{R}^{dn} \to \mathbf{R}^d$ is the projection map $(x_1,\dots,x_n) \mapsto x_1$, for $x_i \in \mathbf{R}^d$. Given any strongly non-diagonal cube $K = J_1 \times \cdots \times J_n \in \mathcal{B}_s^{dn}(G)$, either $K \notin \mathcal{B}_s^{dn}(U^n)$, or $K \in \mathcal{B}_s^{dn}(U^n)$. If the former occurs then $K \notin \mathcal{B}_s^{dn}(F(U)^n)$ since $F(U) \subset U$, so $\mathcal{B}_s^{dn}(F(U)^n) \subset \mathcal{B}_s^{dn}(U^n)$. If the latter occurs then $K \in \mathcal{K}(U)$, and since $\pi(K) = J_1$, $J_1 \notin \mathcal{B}_s^d(F(U))$. In either case, $K \notin \mathcal{B}_s^{dn}(F(U)^n)$, so $F(U)$ satisfies Property (A). By construction, U contains at most one sub cube $J \in \mathcal{B}_s^{dn}$ for each $I \in \mathcal{B}_l^{dn}(E)$. Since $F(U) \subset U$, $F(U)$ satisfies Property (B). Thus the set $F(U)$ satisfies Properties (A) and (B) regardless of which values are assumed by the random variables $\{J_I\}$. Next we will show that with non-zero probability, the set $F(U)$ satisfies Property (C).

For each cube $J \in \mathcal{B}_s^d(E)$, there is a unique 'parent' cube $J^* \in \mathcal{B}_r^d(E)$ such that $J \subset J^*$. Since $\#\mathcal{B}_s^d(J^*) = (r/s)^d$, and J_{J^*} is chosen uniformly at random from $\mathcal{B}_s^d(J^*)$,

$$\mathbf{P}(J \subset U) = \mathbf{P}(J_{J^*} = J) = (s/r)^d.$$

The cubes $\{J_I\}$ are chosen independently, so if $J_1,\dots,J_n$ are distinct cubes in $\mathcal{B}_s^d(E)$, then

$$\mathbf{P}(J_1,\dots,J_n \in U) = \begin{cases} (s/r)^{dn} & \text{if } J_1,\dots,J_n \text{ have distinct parents,} \\ 0 & \text{otherwise.} \end{cases} \tag{4.5}$$

Let $K = J_1 \times \cdots \times J_n \in \mathcal{B}_s^{dn}(G)$. If the cubes $J_1,\dots,J_n$ are distinct, we deduce from (4.5) that

$$\mathbf{P}(K \subset U^n) = \mathbf{P}(J_1,\dots,J_n \in U) \le (s/r)^{dn}. \tag{4.6}$$

By (4.6), linearity of expectation, and (4.3),

$$\mathbf{E}(\#\mathcal{K}(U)) = \sum_{K \in \mathcal{B}_s^{dn}(G)} \mathbf{P}(K \subset U^n) \le \#\mathcal{B}_s^{dn}(G) \cdot (s/r)^{dn} \le 0.5 \cdot (l/r)^d.$$

In particular, there exists at least one (non-random) set U_0 such that

$$\#\mathcal{K}(U_0) \le \mathbf{E}(\#\mathcal{K}(U)) \le 0.5 \cdot (l/r)^d. \tag{4.7}$$

In other words, $F(U_0) \subset U_0$ is obtained by removing at most $0.5 \cdot (l/r)^d$ cubes in $\mathcal{B}_s^d$ from U_0. For each $I \in \mathcal{B}_l^d(E)$, we know that $\#\mathcal{B}_s^d(I \cap U_0) = (l/r)^d$. Combining this with (4.7), we arrive at the estimate

$$\begin{aligned}
\#\mathcal{B}_s^d(I \cap F(U_0)) &= \mathcal{B}_s^d(I \cap U_0) - \#\{\pi(K) : K \in \mathcal{K}(U_0), \pi(K) \in F(U_0)\} \\
&\geq \mathcal{B}_s^d(I \cap U_0) - \#(\mathcal{K}(U_0)) \\
&\geq (l/r)^d - 0.5 \cdot (l/r)^d \geq 0.5 \cdot (l/r)^d
\end{aligned}$$

In other words, $F(U_0)$ satisfies Property (C). Setting $F = F(U_0)$ completes the proof. □

Remarks.

1. *While Lemma 26 uses probabilistic arguments, the conclusion of the lemma is not a probabilistic statement. In particular, one can find a suitable F constructively by checking every possible choice of U (there are finitely many) to find one particular choice U_0 which satisfies* (4.7)*, and then defining F by* (4.4)*. Thus the set we obtain in Theorem 25 exists by purely constructive means.*

2. *At this point, it is possible to motivate the numerology behind the dimension bound* $\dim(X) \geq (dn-\alpha)/(n-1)$ *from Theorem 25, albeit in the context of Minkowski dimension. We will pause to do so here before returning to the proof of Theorem 25. For simplicity, let $\alpha > d$, and suppose that $Z \subset \mathbf{R}^{dn}$ satisfies*
$$\#\mathcal{B}_s^{dn}(Z(s)) \sim s^{-\alpha} \quad \textit{for every } s \in (0,1]. \tag{4.8}$$
Let $l = 1$, $E = [0,1]^d$, and let $s > 0$ be a small parameter. If s is chosen sufficiently small compared to d, n, and α, then (4.1) *is satisfied with $G = \bigcup \mathcal{B}_s^{dn}(Z(s))$. We can then apply Lemma 26 to find a dyadic scale $r \sim s^{(dn-\alpha)/d(n-1)}$ and a set F that avoids the strongly non-diagonal cubes of $\mathcal{B}_s^{dn}(Z(s))$. The set F is a union of approximately $r^{-d} \sim s^{-(dn-\alpha)/(n-1)}$ cubes of sidelength s. Thus informally, the set F resembles a set with Minkowski dimension α when viewed at scale s.*

   The set X constructed in Theorem 25 will be obtained by applying Lemma 26 iteratively at many scales. At each of these scales, X will resemble a set of Minkowski dimension $(dn-\alpha)/(n-1)$. A careful analysis of the construction (performed in Section 4.4) shows that X actually has Hausdorff dimension at least $(dn-\alpha)/(n-1)$.

3. *Lemma 26 is the core method in our avoidance technique. The remaining argument is fairly modular. If, for a special case of Z, one can improve*

the result of Lemma 26 so that r is chosen on the order of $s^{\beta/d}$, then the remaining parts of our paper can be applied near verbatim to yield a set X with Hausdorff dimension β, as in Theorem 25.

4.3 Fractal Discretization

In this section we construct the set X from Theorem 25 by applying Lemma 26 at many scales. Let us start by fixing a strong cover Z that we will work with in the sequel.

Lemma 27. *Let $Z \subset \mathbf{R}^{dn}$ be a countable union of bounded sets with Minkowski dimension at most α, and let $\varepsilon_k \searrow 0$ with $2\varepsilon_k < dn - \alpha$ for all k. Then there exists a sequence of dyadic lengths $\{l_k\}$ and a sequence of sets $\{Z_k\}$, such that*

(A) Strong Cover*: The interiors $\{Z_k^\circ\}$ of the sets $\{Z_k\}$ form a strong cover of Z.*

(B) Discreteness*: For all $k \geq 0$, Z_k is an l_k discretized subset of $\mathbf{R}^{dn}$.*

(C) Sparsity*: For all $k \geq 0$, $l_k^{-d} \leq \#\mathcal{B}_{l_k}^{dn}(Z_k) \leq l_k^{-\alpha-\varepsilon_k}$.*

(D) Rapid Decay*: For all $k > 1$,*

$$l_k^{dn-\alpha-\varepsilon_k} \leq 0.5 \cdot l_{k-1}^{dn}, \tag{4.9}$$

$$l_k^{\varepsilon_k} \leq l_{k-1}^{2d}. \tag{4.10}$$

Proof. We can write $Z \subset \bigcup_{i=1}^{\infty} Y_i$, with $\dim_{\underline{\mathbf{M}}}(Y_i) \leq \alpha$ for each i. Without loss of generality, we may assume that for any l,

$$\#\mathcal{B}_l^{nd}(Y_i(l)) \geq l^{-d}. \tag{4.11}$$

If (4.11) fails to be satisfied for some set Y_i, we consider the d dimensional hyperplane

$$H = \{(x_1, \dots, x_1) : x_1 \in [0,1]^d\}.$$

and replace Y_i with $Y_i \cup H$. Since $\dim_{\underline{\mathbf{M}}}(Y_i) \leq \alpha$ for each index i. Let $\{i_k\}$ be a sequence of integers that repeats each integer infinitely often.

The lengths $\{l_k\}$ and sets $\{Z_k\}$ are defined inductively. As a base case, set $l_0 = 1$ and $Z_0 = [0,1]^{nd}$. Suppose that the lengths $l_0, \dots, l_{k-1}$ have been chosen. Since

$\dim_{\underline{\mathbf{M}}}(Y_{i_k}) \leq \alpha$, and $\varepsilon_k > 0$, Definition (D) implies that there exists arbitrarily small lengths l that satisfy

$$\#\mathcal{B}_l^{dn}(Y_{i_k}(l)) \leq l^{-\alpha-\varepsilon_k}. \tag{4.12}$$

In particular, we can choose a dyadic length $l = l_k$ small enough to satisfy (4.9), (4.10), and (4.12). With this choice of l_k, Property (D) is satisfied. Define $Z_k = Y_{i_k}(l_k)$. This choice of Z_k clearly satisfies Property (B), and Property (C) is implied by (4.11) and (4.12).

It remains to verify that the sets $\{Z_k^\circ\}$ strongly cover Z. Fix a point $z \in Z$. Then there exists an index i such that $z \in Y_i$, and there is a subsequence $k_1, k_2, \dots$ such that $i_{k_j} = i$ for each j. But then $z \in Y_i \subset Z_{i_{k_j}}^\circ$, so z is contained in each of the sets $Z_{i_{k_j}}^\circ$, and thus $z \in \limsup Z_i^\circ$. □

To construct X, we consider a nested, decreasing family of sets $\{X_k\}$, where each X_k is an l_k discretized subset of $\mathbf{R}^d$. We then set $X = \bigcap X_k$. The goal is to choose X_k such that X_k^n does not contain any *strongly non diagonal* cubes in Z_k.

Lemma 28. *Let $\{l_k\}$ be a sequence of positive numbers converging to zero, and let $Z \subset \mathbf{R}^{dn}$. Let $\{Z_k\}$ be a sequence of sets, with each Z_k an l_k discretized subset of $\mathbf{R}^{dn}$, such that the interiors $\{Z_k^\circ\}$ strongly cover Z. For each index k, let X_k be an l_k discretized subset of $\mathbf{R}^d$. Suppose that for each k, $\mathcal{B}_{l_k}^{dn}(X_k^n) \cap \mathcal{B}_{l_k}^{dn}(Z_k)$ contains no strongly non diagonal cubes. If $X = \bigcap X_k$, then for any distinct $x_1, \dots, x_n \in X$, we have $(x_1, \dots, x_n) \notin Z$.*

Proof. Let $z \in Z$ be a point with distinct coordinates $z_1, \dots, z_n$. Define

$$\Delta = \{(w_1, \dots, w_n) \in \mathbf{R}^{dn} : \text{ there exists } i \neq j \text{ such that } w_i = w_j\}.$$

Then $d(\Delta, z) > 0$, where d is the Hausdorff distance between Δ and z. Since $\{Z_k\}$ strongly covers Z, there is a subsequence $\{k_m\}$ such that $z \in Z_{k_m}^\circ$ for every index m. Since l_k converges to 0 and thus l_{k_m} converges to 0, if m is sufficiently large then $\sqrt{dn} \cdot l_{k_m} < d(\Delta, z)$. Note that $\sqrt{dn} \cdot l_{km}$ is the diameter of a cube in $\mathcal{B}_{l_{k_m}}^{dn}$. For such a choice of m, any cube $I \in \mathcal{B}_{l_{k_m}}^d$ which contains z is strongly non-diagonal. Furthermore, $z \in Z_{k_m}^\circ$. Since X_{k_m} and Z_{k_m} share no cube which contains z, this implies $z \notin X_{k_m}$. In particular, this means $z \notin X^n$. □

All that remains is to apply the discrete lemma to choose the sets X_k.

Lemma 29. *Given a sequence of dyadic length scales $\{l_k\}$ obeying,* (4.9), (4.10), *and* (4.12) *as above, there exists a sequence of sets $\{X_k\}$ and a sequence of dyadic intermediate scales $\{r_k\}$ with $l_k \le r_k \le l_{k-1}$ for each $k \ge 1$, such that each set X_k is an l_k discretized subset of $[0,1]^d$ such that $\mathcal{B}^{dn}_{l_k}(X_k^d) \cap \mathcal{B}^{dn}_{l_k}(Z_k)$ contains no strongly non diagonal cubes. Furthermore, for each index $k \ge 1$ we have*

$$r_k \lesssim l_k^{(dn-\alpha-\varepsilon_k)/d(n-1)}, \tag{4.13}$$

$$\#\mathcal{B}^d_{l_k}(X_k \cap I) \ge 0.5 \cdot (l_{k-1}/r_k)^d \quad \text{for each } I \in \mathcal{B}^d_{l_{k-1}}(X_{k-1}), \tag{4.14}$$

$$\#\mathcal{B}^d_{l_k}(X_k \cap I) \le 1 \quad \text{for each } I \in \mathcal{B}^d_{r_k}(X_{k-1}). \tag{4.15}$$

Proof. We construct X_k by induction, using Lemma 26 at each step. Set $X_0 = [0,1]^d$. Next, suppose that the sets $X_0, \dots, X_{k-1}$ have been defined. Our goal is to apply Lemma 26 to $E = X_{k-1}$ and $G = Z_k$ with $l = l_{k-1}$ and $s = l_k$. This will be possible once we verify the hypothesis (4.1), which in this case takes the form

$$(l_{k-1}/l_k)^d \le \#\mathcal{B}^{dn}_{l_k}(Z_k) \le 0.5 \cdot (l_{k-1}/l_k)^{dn}. \tag{4.16}$$

The right hand side follows from Property (C) of Lemma 27 and (4.10). On the other hand, Property (C) and the fact that $l_{k-1} \le 1$ implies that

$$(l_{k-1}/l_k)^d \le l_k^{-d} \le \#\mathcal{B}^{dn}_{l_k}(Z_k),$$

establishing the left inequality in (4.16). Applying Lemma 26 as described above now produces a dyadic length

$$r \sim \left(l_{k-1}^{-d} l_k^{dn} \#\mathcal{B}^{dn}_{l_k}(Z_k)\right)^{\frac{1}{d(n-1)}} \tag{4.17}$$

and an l_k discretized set $F \subset X_{k-1}$. The set F satisfies Properties (A), (B), and (C) from the statement of Lemma 26. Define $r_k = r$ and $X_k = F$. The estimate (4.13) on r_k follows from (4.17) using the known bounds (4.10) and (4.12):

$$r_k \lesssim \left(l_{k-1}^{-d} l_k^{dn-\alpha-0.5\varepsilon_k}\right)^{\frac{1}{d(n-1)}} = \left(l_{k-1}^{-d} l_k^{0.5\varepsilon_k} l_k^{dn-\alpha-\varepsilon_k}\right)^{\frac{1}{d(n-1)}} = \left(l_{k-1}^{-2d} l_k^{\varepsilon_k}\right)^{\frac{1}{2d(n-1)}} l_k^{\frac{dn-\alpha-\varepsilon_k}{d(n-1)}} \lesssim l_k^{\frac{dn-\alpha-\varepsilon_k}{d(n-1)}}.$$

The requirements (4.14) and (4.15) follow from Properties (B) and (C) of Lemma 26 respectively. □

Now we have defined the sets $\{X_k\}$, we set $X = \bigcap X_k$. Since X_k avoids strongly non-diagonal cubes in Z_k, Lemma 28 implies that if $x_1, \dots, x_n \in X$ are distinct, then $(x_1,\dots,x_n) \notin Z$. To finish the proof of Theorem 25, we must show that $\dim_{\mathbf{H}}(X) \geq (dn-\alpha)/(n-1)$. This will be done in the next section.

4.4 Dimension Bounds

To complete the proof of Theorem 25, we must show that $\dim_{\mathbf{H}}(X) \geq (dn-\alpha)/(n-1)$. In view of Definition (H), we will do this by constructing a Frostman measure of appropriate dimension supported on X. We start by recursively defining a positive function $f : \bigcup_{i=1}^{\infty} \mathcal{B}^d_{l_i} \to [0,1]$. Set $f([0,1]^d) = 1$, and $f(I) = 0$ for every other $I \in \mathcal{B}^d_{l_0}$. Suppose now that $f(I)$ has been defined for all cubes $I \in \mathcal{B}^d_{l_{k-1}}$, and let $I \in \mathcal{B}^d_{l_k}$. Consider the unique 'parent cube' $I^* \in \mathcal{B}^d_{l_{k-1}}$ for which $I \subset I^*$. Define

$$f(I) = \begin{cases} f(I^*)/\#\mathcal{B}^d_{l_k}(X_k \cap I^*) & \text{if } I \subset X_k, \\ 0 & \text{otherwise.} \end{cases} \tag{4.18}$$

Observe that $f(I) > 0$ if and only if $I \cap X \neq \emptyset$, and for each index $k \geq 1$, if $I^* \in \mathcal{B}^d_{l_{k-1}}$,

$$\sum \{f(I) : \, I \in \mathcal{B}^d_{l_k}(I^*)\} = \sum \{f(I) : \, I \in \mathcal{B}^d_{l_k}(X_k \cap I^*)\} = f(I^*). \tag{4.19}$$

In particular, for each index k we have

$$\sum_{I \in \mathcal{B}^d_{l_k}} f(I) = 1.$$

By a standard argument involving weak convergence (CITE PRAMANIK FRASER HERE?), there exists a Borel probability measure μ supported on $\bigcap X_k = X$, such that for each $I \in \bigcup_{i=1}^{\infty} \mathcal{B}^d_{l_i}$, $\mu(I) \geq f(I)$, and for any Borel set E, and any $k \geq 0$,

$$\mu(E) \leq \sum \{f(I) : \, I \in \mathcal{B}_{l_k}(E(l_k))\}. \tag{4.20}$$

To complete the proof of Theorem 25 we will show that μ is a Frostman measure of dimension $(dn-\alpha)/(n-1) - \varepsilon$ for every $\varepsilon > 0$.

Lemma 30. *For each $k \geq 1$, if $I \in \mathcal{B}^d_{l_k}$,*

$$\mu(I) \lesssim l_k^{\frac{dn-\alpha}{n-1} - \eta_k}, \quad \textit{where} \quad \eta_k = \frac{n+1}{2(n-1)} \cdot \varepsilon_k \searrow 0 \textit{ as } k \to \infty.$$

Proof. Let $I \in \mathcal{B}^d_{l_k}$ and let $I^* \in \mathcal{B}^d_{l_{k-1}}$ be the parent cube of I. If $f(I) > 0$, then $I \subset X_k$, and combining (4.18), (4.14), and (4.13) with the fact that $f(I^*) \leq 1$, we obtain

$$f(I) = \frac{f(I^*)}{\#\mathcal{B}^d_{l_k}(X_k \cap I^*)} \leq 2\left(\frac{r_k}{l_{k-1}}\right)^d \lesssim \frac{l_k^{\frac{dn-\alpha-\varepsilon_k}{n-1}}}{l_{k-1}^d} \leq l_k^{\frac{dn-\alpha}{n-1}-\eta_k}(l_k^{\varepsilon_k/2}/l_{k-1}^d) \leq l_k^{\frac{dn-\alpha}{n-1}-\eta_k}. \tag{4.21}$$

Given any cube $I_0 \in \mathcal{B}^d_{l_k}$, we have $\#\mathcal{B}^d_{l_k}(I_0(l_k)) \lesssim 1$, so we can apply (4.20) to conclude that

$$\mu(I_0) \leq \sum\{f(I) : I \in \mathcal{B}^d_{l_k}(I(l_k))\} \lesssim l_k^{\frac{dn-\alpha}{n-1}-\eta_k}. \qquad \square$$

Corollary 31. *For each $k \geq 1$ and each $I \in \mathcal{B}^d_{r_k}$, $\mu(I) \lesssim (r_k/l_{k-1})^d l_{k-1}^{\frac{dn-\alpha}{n-1}-\eta_{k-1}}$.*

Proof. We begin by noting that $\mathcal{B}^d_{r_k}(I(r_k)) \lesssim 1$. If we combine this with (4.15), we find that

$$\#(\mathcal{B}^d_{l_k}(I(r_k) \cap X_k)) \lesssim 1. \tag{4.22}$$

For each $J \in \mathcal{B}^d_{l_k}(I(r_k) \cap X_k)$, let $J^* \in \mathcal{B}^d_{l_{k-1}}$ denote the parent cube of J. Equation (4.21) of Lemma 30 combined with (4.18) and (4.14) implies

$$f(J) = \frac{f(J^*)}{\#\mathcal{B}^d_{l_k}(X_k \cap J^*)} \lesssim (r_k/l_{k-1})^d l_{k-1}^{\frac{dn-\alpha}{n-1}-\eta_k}.$$

Thus (4.20) and (4.22) show that

$$\mu(I) \lesssim (r_k/l_{k-1})^d l_{k-1}^{\frac{dn-\alpha}{n-1}-\eta_k}. \qquad \square$$

Lemma 30 and Corollary 31 allow us to control the behavior of μ at all scales.

Lemma 32. *For every $\alpha \in [d, dn)$, and for each $\varepsilon > 0$, there is a constant C_ε so that for all dyadic lengths $l \in (0,1]$ and all $I \in \mathcal{B}^d_l$, we have*

$$\mu(I) \leq C_\varepsilon l^{\frac{dn-\alpha}{n-1}-\varepsilon}. \tag{4.23}$$

Proof. Fix $\varepsilon > 0$. Since $\eta_k \searrow 0$ as $k \to \infty$, there is a constant C_ε so that $l_k^{-\eta_k} \leq C_\varepsilon l_k^{-\varepsilon}$ for each $k \geq 1$. For instance, if ε_k is decreasing, we could choose $C_\varepsilon = l_{k_0}^{-\eta_{k_0}}$, where k_0 is the largest integer for which $\eta_{k_0} \geq \varepsilon$. Next, let k be the (unique) index so that $l_{k+1} \leq l < l_k$. We will split the proof of (4.23) into two cases, depending on the position of l within $[l_{k+1}, l_k]$.

Case 1: If $r_{k+1} \leq l \leq l_k$, we can cover I by $(l/r_{k+1})^d$ cubes in $\mathcal{B}^d_{r_{k+1}}$. A union bound combined with Corollary 31 gives

$$\begin{aligned}
\mu(I) &\lesssim (l/r_{k+1})^d (r_{k+1}/l_k)^d l_k^{\frac{dn-\alpha}{n-1}-\eta_k} \\
&= (l/l_k)^d l_k^{\frac{dn-\alpha}{n-1}-\eta_k} \\
&= l^{\frac{dn-\alpha}{n-1}} (l/l_k)^{\frac{\alpha-d}{n-1}} l_k^{-\eta_k} \\
&\leq l^{\frac{dn-\alpha}{n-1}-\eta_k} \\
&\leq C_\varepsilon l^{\frac{dn-\alpha}{n-1}-\varepsilon}.
\end{aligned} \tag{4.24}$$

The penultimate inequality is a consequence of our assumption $\alpha \geq d$.

Case 2: If $l_{k+1} \leq l \leq r_{k+1}$, we can cover I by a single cube in $\mathcal{B}^d_{r_{k+1}}$. By (4.15), each cube in $\mathcal{B}^d_{r_{k+1}}$ contains at most one cube $I_0 \in \mathcal{B}^d_{l_{k+1}}(X_{k+1})$, so by Lemma 30,

$$\mu(I) \leq \mu(I_0) \lesssim l_{k+1}^{\frac{dn-\alpha}{n-1}-\eta_{k+1}} \leq C_\varepsilon l_{k+1}^{\frac{dn-\alpha}{n-1}-\varepsilon} \leq C_\varepsilon l^{\frac{dn-\alpha}{n-1}-\varepsilon}. \qquad \square$$

Applying Frostman's lemma to Lemma 32 gives $\dim_{\mathbf{H}}(X) \geq \frac{dn-\alpha}{n-1} - \varepsilon$ for every $\varepsilon > 0$, which concludes the proof of Theorem 25.

4.5 Applications

As discussed in the introduction, Theorem 25 generalizes Theorems 1.1 and 1.2 from [6]. In this section, we present two applications of Theorem 25 in settings where previous methods do not yield any results.

4.5.1 Sum-sets avoiding specified sets

Theorem 33. *Let $Y \subset \mathbf{R}^d$ be a countable union of sets of Minkowski dimension at most $\beta < d$. Then there exists a set $X \subset \mathbf{R}^d$ with Hausdorff dimension at least $d - \beta$ such that $X + X$ is disjoint from Y.*

Proof. Define $Z = Z_1 \cup Z_2$, where

$$Z_1 = \{(x,y) : x+y \in Y\} \quad \text{and} \quad Z_2 = \{(x,y) : y \in Y/2\}.$$

Since Y is a countable union of sets of Minkowski dimension at most β, Z is a countable union of sets with lower Minkowski dimension at most $d+\beta$. Applying Theorem 25 with $n = 2$ and $\alpha = d+\beta$ produces a set $X \subset \mathbf{R}^d$ with Hausdorff dimension $2d-(d+\beta) = d-\beta$ such that $(x,y) \notin Z$ for all $x,y \in X$ with $x \neq y$. We claim that $X+X$ is disjoint from Y. To see this, first suppose $x,y \in X$, $x \neq y$. Since X avoids Z_1, we conclude that $x+y \notin Y$. Suppose now that $x = y \in X$. Since X avoids Z_2, we deduce that $X \cap (Y/2) = \emptyset$, and thus for any $x \in X$, $x+x = 2x \notin Y$. This completes the proof. □

4.5.2 Subsets of Lipschitz curves avoiding isosceles triangles

In [6], Fraser and the second author prove that there exists a set $S \subset [0,1]$ with dimension $\log_3 2$ such that for any simple C^2 curve $\gamma\colon [0,1] \to \mathbf{R}^n$ with bounded non-vanishing curvature, $\gamma(S)$ does not contain the vertices of an isosceles triangle. Our method enables us to obtain a result that works for Lipschitz curves with small Lipschitz constants. The dimensional bound that we provide is slightly worse than [6] ($1/2$ instead of $\log_3 2$), and the set we obtain only works for a single Lipschitz curve, not for many curves simultaneously.

Theorem 34. *Let $f\colon [0,1] \to \mathbf{R}^{n-1}$ be Lipschitz with*

$$\|f\|_{Lip} := \sup\big\{|f(x)-f(y)|/|x-y| : x,y \in [0,1], x \neq y\big\} < 1.$$

Then there is a set $X \subset [0,1]$ of Hausdorff dimension $1/2$ so that the set $\{(t,f(t)) : t \in X\}$ does not contain the vertices of an isosceles triangle.

Corollary 35. *Let $f\colon [0,1] \to \mathbf{R}^{n-1}$ be C^1. Then there is a set $X \subset [0,1]$ of Hausdorff dimension $1/2$ so that the set $\{(t,f(t)) : t \in X\}$ does not contain the vertices of an isosceles triangle.*

Proof of Corollary 35. The graph of any C^1 function can be locally expressed, after possibly a translation and rotation, as the graph of a Lipschitz function with small Lipschitz constant. In particular, there exists an interval $I \subset [0,1]$ of positive length so that the graph of f restricted to I, after being suitably translated, rotated, and dilated, is the graph of a Lipschitz function $g\colon [0,1] \to \mathbf{R}^{n-1}$ with

Lipschitz constant at most $1/2$. Since isosceles triangles remain invariant under these transformations, the corollary is a consequence of Theorem 34. $\square$

Proof of Theorem 34. Set

$$Z = \left\{ (x_1,x_2,x_3) \in [0,1]^3 : \begin{array}{c} \text{The three points } p_j = (x_j, f(x_j)), 1 \le j \le 3 \\ \text{form the vertices of an isosceles triangle} \end{array} \right\}. \tag{4.25}$$

In the next lemma, we show Z has lower Minkowski dimension at most two. By Theorem 25, there is a set $X \subset [0,1]$ of Hausdorff dimension $1/2$ so that for each distinct $x_1,x_2,x_3 \in X$, we have $(x_1,x_2,x_3) \notin Z$. This is precisely the statement that for each $x_1,x_2,x_3 \in X$, the points $(x_1, f(x_1))$, $(x_2, f(x_2))$, and $(x_3, f(x_3))$ do not form the vertices of an isosceles triangle. $\square$

Lemma 36. *Let $f\colon [0,1] \to \mathbf{R}^{n-1}$ be Lipschitz with $\|f\|_{Lip} < 1$. Then the set Z given by* (4.25) *has Minkowski dimension at most two.*

Proof. First, notice that three points $p_1, p_2, p_3 \in \mathbf{R}^n$ form an isosceles triangle, with p_3 as the apex, if and only if $p_3 \in H_{p_1,p_2}$, where

$$H_{p_1,p_2} = \left\{ x \in \mathbf{R}^n : \left(x - \frac{p_1+p_2}{2} \right) \cdot (p_2 - p_1) = 0 \right\}. \tag{4.26}$$

To prove Z has Minkowski has dimension at most two, it suffices to show that the set

$$W = \left\{ x \in [0,1]^3 : \ p_3 = (x_3, f(x_3)) \in H_{p_1,p_2} \right\}$$

has upper Minkowski dimension at most 2. This is because Z is the union of three copies of W, obtained by permuting coordinates. To bound the upper Minkowski dimension of W, we prove the estimate

$$\#\big(\mathcal{B}^3_\delta(W(\delta))\big) \le C\delta^{-2}\log(1/\delta) \quad \text{for all } 0 < \delta < 1, \tag{4.27}$$

where C is a constant independent of δ, and δ is a dyadic scale.

Fix $0 < \delta < 1$. Note that

$$\#\big(\mathcal{B}^3_\delta(W)\big) = \sum_{k=0}^{1/\delta} \sum_{\substack{I_1,I_2 \in \mathcal{B}^1_\delta[0,1] \\ d(I_1,I_2)=k\delta}} \#\mathcal{B}^3_\delta(W(\delta) \cap I_1 \times I_2 \times [0,1]). \tag{4.28}$$

Our next task is to bound each of the summands in (4.28). Let $I_1, I_2 \in \mathcal{B}^1_\delta[0,1]$, and let $k = \delta^{-1} d(I_1, I_2)$. Let x_1 be the midpoint of I_1, and x_2 the midpoint of I_2. Let $(y_1, y_2, y_3) \in W \cap I_1 \times I_2 \times [0,1]$. Then it follows from (4.26) that

$$\left(y_3 - \frac{y_1+y_2}{2}\right) \cdot (y_2 - y_1) + \left(f(y_3) - \frac{f(y_2)+f(y_1)}{2}\right) \cdot (f(y_2) - f(y_1)) = 0.$$

We know $|x_1 - y_1|, |x_2 - y_2| \le \delta/2$, so

$$\begin{aligned}
&\left|\left(y_3 - \frac{y_1+y_2}{2}\right)(y_2-y_1) - \left(y_3 - \frac{x_1+x_2}{2}\right)(x_2-x_1)\right| \\
&\quad \le \frac{|y_1-x_1|+|y_2-x_2|}{2}|y_2-y_1| + \Big(|y_1-x_1|+|y_2-x_2|\Big)\left|y_3 - \frac{x_1+x_2}{2}\right| \\
&\quad \le (\delta/2)\cdot 1 + \delta \cdot 1 \le 3\delta/2.
\end{aligned} \tag{4.29}$$

Conversely, we know $|f(x_1) - f(y_1)|, |f(x_2) - f(y_2)| \le \delta/2$ because $\|f\|_{\mathrm{Lip}} < 1$, and a similar calculation yields

$$\begin{aligned}
&\Bigg|\left(f(y_3) - \frac{f(y_1)+f(y_2)}{2}\right)\cdot(f(y_2)-f(y_1)) \\
&\quad - \left(f(y_3) - \frac{f(x_1)+f(x_2)}{2}\right)\cdot(f(x_2)-f(x_1))\Bigg| \le 3\delta/2.
\end{aligned} \tag{4.30}$$

Putting (4.29) and (4.30) together, we conclude that

$$\left|\left(y_3 - \frac{x_1+x_2}{2}\right)(x_2-x_1) + \left(f(y_3) - \frac{f(x_2)+f(x_1)}{2}\right)\cdot(f(x_2)-f(x_1))\right| \le 3\delta. \tag{4.31}$$

Since $|(x_2 - x_1, f(x_2) - f(x_1))| \ge |x_2 - x_1| \ge k\delta$, we can interpret (4.31) as saying the point $(y_3, f(y_3))$ is contained in a $3/k$ thickening of the hyperplane $H_{(x_1,f(x_1)),(x_2,f(x_2))}$. Given another value $y' \in W \cap I_1 \cap I_2 \cap [0,1]$, it satisfies a variant of the inequality (4.31), and we can subtract the difference between the two inequalities to conclude

$$\left|\left(y_3 - y_3'\right)(x_2-x_1) + (f(y_3) - f(y_3'))\cdot(f(x_2)-f(x_1))\right| \le 6\delta. \tag{4.32}$$

The triangle difference inequality applied with (4.32) implies

$$\begin{aligned}(f(y_3)-f(y_3'))\cdot(f(x_2)-f(x_1)) &\geq |y_3-y_3'||x_2-x_1|-6\delta \\ &= (k+1)\delta\cdot|y_3-y_3'|-6\delta.\end{aligned} \tag{4.33}$$

Conversely,

$$\begin{aligned}(f(y_3)-f(y_3'))\cdot(f(x_2)-f(x_1)) &\leq \|f\|_{\mathrm{Lip}}^2|y_3-y_3'||x_2-x_1| \\ &= \|f\|_{\mathrm{Lip}}^2\cdot(k+1)\delta\cdot|y_3-y_3'|.\end{aligned} \tag{4.34}$$

Combining (4.33) and (4.34) and rearranging, we see that

$$|y_3-y_3'| \leq \frac{6}{(k+1)(1-\|f\|_{\mathrm{Lip}}^2)} \lesssim \frac{1}{k+1}, \tag{4.35}$$

where the implicit constant depends only on $\|f\|_{\mathrm{Lip}}$ (and blows up as $\|f\|_{\mathrm{Lip}}$ approaches 1). We conclude that

$$\#\mathcal{B}_\delta^3(W(\delta)\cap I_1\times I_2\times[0,1]) \lesssim \frac{1}{k+1}, \tag{4.36}$$

which holds uniformly over any value of k.

We are now ready to bound the sum from (4.28). Note that for each value of k, there are at most $2/\delta$ pairs (I_1,I_2) with $d(I_1,I_2)=k\delta$. Indeed, there are $1/\delta$ choices for I_1 and then at most two choices for I_2. Equation (4.36) shows

$$\#\mathcal{B}_\delta^3(W(\delta)) = \sum_{k=0}^{1/\delta}\sum_{\substack{I_1,I_2\in\mathcal{B}_\delta^1[0,1]\\ d(I_1,I_2)=k\delta}}\#\mathcal{B}_\delta^3(W(\delta)\cap I_1\times I_2\times[0,1]) \lesssim \delta^{-2}\sum_{k=0}^{1/\delta}\frac{1}{k+1} \lesssim \delta^{-2}\log(1/\delta).$$

In the above inequalities, the implicit constants depend on $\|f\|_{\mathrm{Lip}}$, but they are independent of δ. This establishes (4.27) and completes the proof. $\square$

Chapter 5

Conclusions

Bibliography

[1] K. Athreya. Large deviation rates for branching processes–i. single type case. *Ann. Appl. Probab.*, 4:779–790, 1994.

[2] J. Balogh, R. Morris, and W. Samotij. Independent sets in hypergraphs. *J. Amer. Math. Soc.*, 28:669–709, 2015.

[3] J. Biggins and N. Bingham. Large deviations in the supercritical branching process. *Adv. in Appl. Probab.*, 25:757–772.

[4] K. Dickman. On the frequency of numbers containing prime factors of a certain relative magnitude.

[5] K. Falconer. *Fractal Geometry: Mathematical Foundations and Applications*. Wiley, 2003.

[6] R. Fraser and M. Pramanik. Large sets avoiding patterns. *Anal. PDE*, 11: 1083–1111, 2018. → pages 36, 37, 48, 49

[7] T. Harris. *The Theory of Branching Processes*. Springer-Verlag, 1963.

[8] N. H. Katz and T. Tao. Some connections between Falconer's distance set conjecture, and sets of Furstenburg type. *New York J. Math.*, 7:149–187, 2001. → page 38

[9] T. Keleti. A 1-dimensional subset of the reals that intersects each of its translates in at most a single point. *Real Anal. Exchange*, 24:843–845, 1999. → pages 35, 37

[10] P. Maga. Full dimensional sets without given patterns. *Real Anal. Exchange*, 36:79–90, 2010. → pages 1, 35

[11] A. Máthé. Sets of large dimension not containing polynomial configurations. *Adv. Math.*, 316:691–709, 2017. → page 36

[12] P. Mattila. *Fourier analysis and Hausdorff dimension*. Cambridge University Press, 2015. → page 1

[13] I. Z. Ruzsa. Difference sets without squares.

[14] B. Sudakov, E. Szemerédi, and V. Vu. On a question of erdös and moser.

[15] T. Tao. 245 C, Notes 5: Hausdorff dimension, 2009. URL https://terrytao.wordpress.com/2009/05/19/245c-notes-5-hausdorff-dimension-optional/.