Cartesian Products Avoiding Patterns

by

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Abstract

The pattern avoidance problem seeks to construct a set $X \subset \mathbf{R}^d$ with large dimension that avoids a prescribed pattern such as three term arithmetic progressions, or more general patterns such as avoiding points $x_1, \dots, x_n$ such that $f(x_1, \dots, x_n) = 0$ (three term arithmetic progressions are specified by the pattern $x_1 - 2x_2 + x_3 = 0$). Previous work on the subject has considered patterns described by polynomials, or by functions f satisfying certain regularity conditions. We consider the case of 'rough patterns. There are several problems that fit into the framework of rough pattern avoidance. As a first application, if $Y \subset [0,1]$ is a set with Minkowski dimension α, we construct a set $X \subset [0,1]$ with Hausdorff dimension $1 - \alpha$ so that $X + X$ is disjoint from Y. As a second application, given a set Y of dimension close to one, we can construct a subset $X \subset Y$ of dimension $1/2$ that avoids isosceles triangles.

Lay Summary

The lay or public summary explains the key goals and contributions of the research/scholarly work in terms that can be understood by the general public. It must not exceed 150 words in length.

Preface

At **UBC!** (**UBC!**), a preface may be required. Be sure to check the **GPS!** (**GPS!**) guidelines as they may have specific content to be included.

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Acknowledgments

Thank those people who helped you.

Don't forget your parents or loved ones.

You may wish to acknowledge your funding sources.

Chapter 1

Introduction

In this thesis, we study a simple question:

How large can Euclidean sets be not containing geometric patterns?.

The patterns manifest as certain configurations of point tuples, often affine invariant. For instance, we might like to find large subsets X of $\mathbf{R}^d$ that contain no three collinear points, or do not contain the vertices of any isoceles triangle. Aside from purely geometric interest, these problems provide useful scenarios in which to test the methods of ergodic theory, additive combinatorics and harmonic analysis.

Sets which avoid the configurations we consider are highly irregular, and so also require techniques from geometric measure theory to be understood. In particular, we use the *fractal dimension* as a measure of a sets size when comparing how large two sets which avoid patterns are.

At the time of the writing of this thesis, many fundamental questions about the relation between the geometric arrangement of a set and it's fractal dimension remain unsolved. It might be expected that sets with sufficiently large Hausdorff dimension contain patterns. For example, Theorem 6.8 of [12] shows that any set $X \subset \mathbf{R}^d$ with Hausdorff dimension exceeding one must contain three colinear points. On the other hand, [10] constructs a set $X \subset \mathbf{R}^d$ with full Hausdorff dimension such that no four points in X form the vertices of a parallelogram. Thus the problem must depend on the particular aspects of the configurations involved. For most geometric configurations, it remains unknown at what threshold patterns are guaranteed, or whether such a threshold exists at all. So finding sets with large Hausdorff dimension avoiding patterns is an important topic to determine for which classes of patterns our intuition remains true.

Our goal in this thesis is to derive new methods for constructing sets avoiding patterns. And rather than studying particular instances of the configuration avoidance problem, we choose to study general methods for finding large subsets of space avoiding any pattern. In particular, we expand on a number of general *pattern dissection methods* which have proven useful in the area, originally developed by Keleti but also studied notably by Mathé, and Pramanik/Fraser.

The main contribution we give to pattern dissection methods is using random dissections to avoid patterns without much structural information. Thus we can expand the utility of interval dissection methods from regular patterns to a set of *fractal avoidance problems*, that previous methods were completely unavailable to address. Such problems include finding large X such that the angles formed by any three distinct points of X avoid a specified set Y of angles, where Y has a fixed fractal dimension.

Chapter 2

Background

2.1 Configuration Avoidance

We consider an ambient set $\mathbf{A}$. It's *n-point configuration space* is

$$\mathcal{C}^n(\mathbf{A}) = \{(x_1,\dots,x_n) \in X^n : x_i \neq x_j \text{ if } i \neq j\}.$$

An *n point configuration*, or *pattern*, is a subset of $\mathcal{C}^n(\mathbf{A})$. More generally, we define the general *configuration space* of $\mathbf{A}$ as $\mathcal{C}(\mathbf{A}) = \bigcup_{n=1}^\infty \mathcal{C}^n(\mathbf{A})$, and a *pattern*, or *configuration*, on $\mathbf{A}$ is a subset of $\mathcal{C}(\mathbf{A})$.

Our main focus in this thesis is the *pattern avoidance problem*. For a fixed configuration $\mathcal{C}$ on $\mathbf{A}$, we say a set $X \subset \mathbf{A}$ *avoids* $\mathcal{C}$ if $\mathcal{C}(X)$ is disjoint from $\mathcal{C}$. The pattern avoidance problem asks to find sets X of maximal size avoiding a fixed configuration $\mathcal{C}$. The set $\mathcal{C}$ often describes the presence of algebraic or geometric structure, and so we are trying to find large sets which do not possess any instance of such structure.

Example (Isoceles Triangle Configuration)**.** *Let*

$$\mathcal{C} = \left\{ (x_1,x_2,x_3) \in \mathcal{C}^3(\mathbf{R}^2) : |x_1 - x_2| = |x_1 - x_3| \right\}.$$

Then $\mathcal{C}$ is a 3-point configuration, and a set $X \subset \mathbf{R}^2$ avoids $\mathcal{C}$ if and only if it does not contain all three vertices of an isoceles triangle.

Example (Linear Independence Configuration)**.** *Let V be a vector space over a field K. We set*

$$\mathcal{C} = \bigcup_{n=1}^{\infty} \{(x_1,\dots,x_n) \in \mathcal{C}^n(V) : \textit{for any } a_1,\dots,a_n \in K,\ a_1x_1 + \dots + a_nx_n \neq 0\}.$$

A subset $X \subset V$ avoids $\mathcal{C}$ if and only if X is a linearly independent subset of X. Note that if V is infinite dimensional, then $\mathcal{C}$ cannot be replaced by a n point configuration for any n; arbitrarily large tuples must be considered. We will be interested in the case where $K = \mathbf{Q}$, and $V = \mathbf{R}$, where we will be looking for analytically large, linearly independant subsets of V.

Even though our problem formulation assumes configurations are formed by distinct sets of points, one can still formulate avoidance problems involving repeated points by a simple trick.

Example (Sum Set Configuration)**.** *Let G be an abelian group, and fix $Y \subset G$. Set*

$$\mathcal{C}^1 = \{g \in \mathcal{C}^1(G) : g + g \in Y\} \quad \textit{and} \quad \mathcal{C}^2 = \{(g_1,g_2) \in \mathcal{C}^2(G) : g_1 + g_2 \in Y\}.$$

Then set $\mathcal{C} = \mathcal{C}^1 \cup \mathcal{C}^2$. A set $X \subset G$ avoids $\mathcal{C}$ if and only if $(X + X) \cap Y = \emptyset$.

Depending on the structure of the ambient space $\mathbf{A}$ and the configuration $\mathcal{C}$, there are various ways of measuring the size of sets $X \subset \mathbf{A}$ for the purpose of the pattern avoidance problem:

- If $\mathbf{A}$ is finite, the goal is to find a set X with large cardinality.
- If $\{\mathbf{A}_n\}$ is an increasing family of finite sets with $\mathbf{A} = \lim \mathbf{A}_n$, the goal is to find a set X such that $X \cap \mathbf{A}_n$ has large cardinality asymptotically in n.
- If $\mathbf{A} = \mathbf{R}^d$, but $\mathcal{C}$ is a discrete configuration, then a satisfactory goal is to find a set X with large Lebesgue measure avoiding $\mathcal{C}$.

In this thesis, inspired by results in these three settings, we establish methods for avoiding non-discrete configurations $\mathcal{C}$ in $\mathbf{R}^d$. Here, Lebesgue measure completely fails to measure the size of pattern avoiding solutions, as the next theorem shows, under the often true assumption that $\mathcal{C}$ is *translation invariant*, i.e. that if $(a_1,\dots,a_n) \in \mathcal{C}$ and $b \in \mathbf{R}^d$, $(a_1+b,\dots,a_n+b) \in \mathcal{C}$.

Theorem 1. *Let $\mathcal{C}$ be a n-point configuration on $\mathbf{R}^d$. Suppose*

(A) *$\mathcal{C}$ is translation invariant.*

(B) *For any $\varepsilon > 0$, there is $(a_1,\dots,a_n) \in \mathcal{C}$ with $diam\{a_1,\dots,a_n\} \leq \varepsilon$.*

Then no set with positive Lebesgue measure avoids $\mathcal{C}$.

Proof. Let $X \subset \mathbf{R}^d$ have positive Lebesgue measure. The Lebesgue density theorem shows that there exists a point $x \in X$ such that

$$\lim_{l(Q)\to 0} \frac{|X \cap Q|}{|Q|} = 1, \tag{2.1}$$

where Q ranges over all cubes in $\mathbf{R}^d$ with $x \in Q$, and $l(Q)$ denotes the sidelength of Q. Fix $\varepsilon > 0$, to be specified later, and choose r small enough that $|X \cap Q| \geq (1-\varepsilon)|Q|$ for any cube Q with $x \in Q$ and $l(Q) \leq r$. Now let Q_0 denote the cube centered at x with $l(Q_0) \leq r$. Applying Property (B), we find $C = (a_1,\dots,a_n) \in \mathcal{C}$ such that

$$\operatorname{diam}\{a_1,\dots,a_n\} \leq l(Q_0)/2. \tag{2.2}$$

For each $p \in Q_0$, let $C(p) = (a_1(p),\dots,a_n(p))$, where $a_i(p) = p + (a_i - a_1)$. Property (A) implies $C(p) \in \mathcal{C}$ for each $p \in \mathbf{R}^d$. A union bound shows

$$|\{p \in Q_0 : C(p) \notin \mathcal{C}(X)\}| \leq \sum_{i=1}^d |\{p \in Q_0 : a_i(p) \notin X\}|. \tag{2.3}$$

We have $a_i(p) \notin X$ precisely when $p + (a_i - a_1) \notin X$, so

$$|\{p \in Q_0 : a_i(p) \notin X\}| = |(Q_0 + (a_i - a_1)) \cap X^c|. \tag{2.4}$$

Note $Q_0 + (a_i - a_1)$ is a cube with the same sidelength as Q_0. Equation (2.2) implies $|a_i - a_1| \leq l(Q_0)/2$, so $x \in Q_0 + (a_i - a_1)$. Thus (2.1) shows

$$|Q_0 + (a_i - a_1)) \cap X^c| \leq \varepsilon |Q_0|. \tag{2.5}$$

Combining (2.3), (2.4), and (2.5), we find

$$|\{p \in Q_0 : C(p) \notin \mathcal{C}(X)\}| \leq \varepsilon d |Q_0|.$$

Provided $\varepsilon d < 1$, this means there is $p \in Q_0$ with $C(p) \in \mathcal{C}(X)$. $\square$

Since no set of positive Lebesgue measure can avoid non-discrete configurations, we cannot use the Lebesgue measure to quantify the size of pattern avoiding sets. Fortunately, there is a quantity which can distinguish between the size of sets of measure zero. This is the *fractional dimension* of a set.

There are many variants of fractional dimension. Here we choose to use the Minkowski dimension, the Hausdorff dimension, and the Fourier dimension. They assign the same dimension to any smooth manifold[1], but vary over more singular sets. One major difference is that Minkowski dimension measures relative density at a single scale, whereas Hausdorff dimension measures relative density at countably many scales. The Fourier dimension is a refinement of the Hausdorff dimension which gives greater control on the set in the frequency domain, and therefore gives additional structural information on sets.

2.2 Minkowski Dimension

We begin by discussing the Minkowski dimension, which is the simplest to define. Given $l > 0$, and a bounded set $E \subset \mathbf{R}^d$, we let $N(l, E)$ denote the *covering number* of E, the minimum number of sidelength l cubes required to cover E. We define

[1]For Fourier dimension, the smooth manifold also needs to have nonvanishing curvature.

the *lower* and *upper* Minkowski dimension as

$$\dim_{\underline{\mathbf{M}}}(E) = \liminf_{l \to 0} \frac{\log(N(l,E))}{\log(1/l)} \quad \text{and} \quad \dim_{\overline{\mathbf{M}}}(E) = \limsup_{l \to 0} \frac{\log(N(l,E))}{\log(1/l)}.$$

If $\dim_{\overline{\mathbf{M}}}(E) = \dim_{\underline{\mathbf{M}}}(E)$, then we refer to this common quantity as the *Minkowski dimension* of E, denoted $\dim_{\mathbf{M}}(E)$. Thus $\dim_{\underline{\mathbf{M}}}(E) < s$ if there exists a sequence of lengths $\{l_k\}$ converging to zero with $N(l_k,E) \leq (1/l_k)^s$, and $\dim_{\overline{\mathbf{M}}}(E) < s$ if $N(l,E) \leq (1/l)^s$ for *all* sufficiently small lengths l.

2.3 Hausdorff Dimension

For $E \subset \mathbf{R}^d$ and $\delta > 0$, we define the *Hausdorff content*

$$H^s_\delta(E) = \inf \left\{ \sum_{k=1}^{\infty} l(Q_k)^s : E \subset \bigcup_{k=1}^{\infty} Q_k, l(Q_k) \leq \delta \right\}.$$

The *s-dimensional Hausdorff measure* of E is

$$H^s(E) = \lim_{\delta \to 0} H^s_\delta(E).$$

It is easy to see H^s is an exterior measure on $\mathbf{R}^d$, and $H^s(E \cup F) = H^s(E) + H^s(F)$ if the Hausdorff distance $d(E,F)$ between E and F is positive. So H^s is actually a metric exterior measure, and the Caratheodory extension theorem shows all Borel sets are measurable with respect to H^s.

Lemma 2. *Consider $t < s$, and $E \subset \mathbf{R}^d$.*

(i) If $H^t(E) < \infty$, then $H^s(E) = 0$.

(ii) If $H^s(E) \neq 0$, then $H^t(E) = \infty$.

Proof. Suppose that $H^t(E) = A < \infty$. Then for any $\delta > 0$, there is a cover of E by a collection of intervals $\{Q_k\}$, such that $l(Q_k) \leq \delta$ for each k, and

$$\sum l(Q_k)^t \leq A < \infty.$$

But then

$$H^s_\delta(E) \le \sum l(Q_k)^s \le \sum l(Q_k)^{s-t} l(Q_k)^t \le \delta^{s-t} A.$$

As $\delta \to 0$, we conclude $H^s(E) = 0$, proving *(i)*. And *(ii)* is just the contrapositive of (i), and therefore immediately follows. □

Corollary 3. *If $s > d$, $H^s = 0$.*

Proof. The measure H^d is just the Lebesgue measure on $\mathbf{R}^d$, so

$$H^d[-N,N]^d = (2N)^d.$$

If $s > d$, Lemma 2 shows $H^s[-N,N]^d = 0$. By countable additivity, taking $N \to \infty$ shows $H^s(\mathbf{R}^d) = 0$. Thus $H^s(E) = 0$ for all E if $s > d$. □

Given any Borel set E, Corollary 3, combined with Lemma 2, implies there is a unique value $s_0 \in [0,d]$ such that $H^s(E) = 0$ for $s > s_0$, and $H^s(E) = \infty$ for $0 \le s < s_0$. We refer to s_0 as the *Hausdorff dimension* of E, denoted $\dim_{\mathbf{H}}(E)$.

Theorem 4. *For any bounded set E,* $\dim_{\mathbf{H}}(E) \le \dim_{\underline{\mathbf{M}}}(E) \le \dim_{\overline{\mathbf{M}}}(E)$.

Proof. Given $l > 0$, we have a simple bound $H^s_l(E) \le N(l,E) \cdot l^s$. If $\dim_{\underline{\mathbf{M}}}(E) < s$, then there exists a sequence $\{l_k\}$ with $l_k \to 0$, and $N(l_k,E) \le (1/l_k)^s$. Thus we conclude that

$$H^s(E) = \lim_{k \to \infty} H^s_{l_k}(E) \le \lim_{k \to \infty} N(l_k,E) \cdot l_k^s \le 1,$$

This $\dim_{\mathbf{H}}(E) \le s$. Taking infima over all s, we find $\dim_{\mathbf{H}}(E) \le \dim_{\underline{\mathbf{M}}}(E)$. □

Remark. *If* $\dim_{\mathbf{H}}(E) < d$, *then* $|E| = H^d(E) = 0$. *Thus any set with fractional dimension less than d must have measure zero, and so the dimension is a way of distinguishing between sets of measure zero, which is precisely what we need to study the configuration avoidance problem for non-discrete configurations.*

The fact that Hausdorff dimension is defined with respect to multiple scales makes it more stable under analytical operations. In particular,

$$\dim_{\mathbf{H}} \left\{ \bigcup E_k \right\} = \sup \{ \dim_{\mathbf{H}}(E_k).\}$$

This need not be true for the Minkowski dimension; a single point has Minkowski dimension zero, but the rational numbers, which are a countable union of points, have Minkowski dimension one. An easy way to make Minkowski dimension countably stable is to define the *modified Minkowski dimensions*

$$\dim_{\underline{\mathbf{MB}}}(E) = \inf\left\{ s : E \subset \bigcup_{i=1}^{\infty} E_i, \dim_{\underline{\mathbf{M}}}(E_i) \leq s \right\}$$

and

$$\dim_{\overline{\mathbf{MB}}}(E) = \inf\left\{ s : E \subset \bigcup_{i=1}^{\infty} E_i, \dim_{\overline{\mathbf{M}}}(E_i) \leq s \right\}.$$

This notion of dimension, in a disguised form, appears in Chapter BLAH.

2.4 Dyadic Scales

It is now useful to introduce the dyadic notation we utilize throughout this thesis. At the cost of losing topological perspective about $\mathbf{R}^d$, applying dyadic techniques often allows us to elegantly discretize certain problems in Euclidean space. We introduce fractional dimension through a dyadic framework, and all our constructions will be done dyadically. This section introduces some notation we will use throughout the remainder of the thesis. We fix a sequence of positive integers $\{N_k : k \geq 1\}$, which will change over each argument in the thesis, but will remain constant throughout each argument.

- For each $k \geq 0$, we define

  $$\mathcal{Q}_k^d = \left\{ \prod_{k=1}^{d} \left[\frac{m_k}{N_1 \dots N_k}, \frac{m_k + 1}{N_1 \dots N_k} \right] : m \in \mathbf{Z}^d \right\}.$$

  These are the *dyadic cubes of generation k*. We let $\mathcal{Q}^d = \bigcup_{k \geq 0} \mathcal{Q}_k^d$. Note that any two cubes $\mathcal{Q}^d$ are either nested within one another, or their interiors do not intersect.

- We set
$$l_k = \frac{1}{N_1 \dots N_k}.$$
Then l_k is the sidelength of the cubes in $\mathcal{Q}_k^d$.

- Given $Q \in \mathcal{Q}_{k+1}^d$, we let $Q^* \in \mathcal{Q}_k^d$ denote the *parent cube* of Q, i.e. the unique dyadic cube of generation k such that $Q \subset Q^*$.

- We say a set $E \subset \mathbf{R}^d$ is $\mathcal{Q}_k^d$ *discretized* if it is a union of cubes in $\mathcal{Q}_k^d$. In this case, we let
$$\mathcal{Q}_k(E) = \{Q \in \mathcal{Q}_k^d : Q \subset E\}$$
denote the family of cubes whose union is E.

It is most common to set $N_k = 2$ for all k, which gives the standard families of dyadic cubes. But in our methods, it is necessary to allow more general sequences $\{N_k\}$, which are not necessarily bounded.

Sometimes, we need to rely on certain 'intermediary' cubes that lie between the scales $\mathcal{Q}_k^d$ and $\mathcal{Q}_{k+1}^d$. In this case, we consider a supplementary sequence $\{M_k : k \geq 1\}$ with $M_k | N_k$ for each k.

- For $k \geq 1$, we define
$$\mathcal{R}_k^d = \left\{ \prod_{k=1}^d \left[\frac{m_k}{N_1 \dots N_{k-1} M_k}, \frac{m_k + 1}{N_1 \dots N_{k-1} M_k} \right] : m \in \mathbf{Z}^d \right\}.$$

- We set
$$r_k = \frac{1}{N_1 \dots N_{k-1} M_k}$$
to be the sidelength of a cube in $\mathcal{R}_k^d$.

- For $E \subset \mathbf{R}^d$, the notions of being $\mathcal{R}_k^d$ discretized, and the collection of cubes $\mathcal{R}_k^d(E)$, are defined as should be expected.

Thus the cubes in $\mathcal{R}_k^d$ are coarser than those in $\mathcal{Q}_k^d$, but finer than those in $\mathcal{Q}_{k-1}^d$.

2.5 Frostman Measures

It is often easy to upper bound Hausdorff dimension, but non-trivial to *lower bound* the Hausdorff dimension of a given set. A key technique to finding a lower bound is *Frostman's lemma*, which says that a set has large Hausdorff dimension if and only if it supports a probability measure which obeys a certain decay law on small sets. We say a Borel measure μ is a *Frostman measure* of dimension s if it is non-zero, compactly supported, and for any cube Q, $\mu(Q) \lesssim l(Q)^s$. The proof of Frostman's lemma will utilize a technique often useful, known as the *mass distribution principle*.

Lemma 5 (Mass Distribution Principle)**.** *Let $\mu : \mathcal{Q}^d \to [0,\infty)$ be a function such that for any k, and for any $Q_0 \in \mathcal{Q}^d_k$,*

$$\sum \left\{ \mu(Q) : Q \in \mathcal{Q}^d_{k+1}, Q^* = Q_0 \right\} = \mu(Q_0). \tag{2.6}$$

Then μ extends uniquely to a regular Borel measure on $\mathbf{R}^d$.

Proof. We begin by defining a sequence of regular Borel measures $\{\mu_k\}$ by setting, for each $f \in C_c(\mathbf{R}^d)$,

$$\int f d\mu_k = \sum_Q \mu(Q) \int_Q f,$$

where Q ranges over all cubes in $\mathcal{Q}^d_k$. Similarily, define a family of operators $\{E_k\}$ on regular Borel measures by the formula

$$\int f(x) dE_k(\nu) = \sum_Q \nu(Q) \int_Q f,$$

where Q ranges over all cubes in $\mathcal{Q}^d_k$. Equation (2.6) then says that $E_j(\mu_k) = \mu_j$ for each $j \leq k$.

If $\nu_i \to \nu$ weakly, then $\nu_i(Q) \to \nu(Q)$ for each fixed $Q \in \mathcal{Q}^d_k$. Thus if $f \in C_c(\mathbf{R}^d)$, then the support of f intersects only finitely many cubes in $\mathcal{Q}^d_k$, and this implies

$$\int f(x)dE_k(\nu_i) = \sum \nu_i(Q)\int_Q f\,dx \to \sum \nu(Q)\int_Q f\,dx = \int f dE_k(\nu).$$

Thus the operators $\{E_k\}$ are continuous with respect to the weak topology.

Now fix an interval $Q \in \mathcal{Q}_0^d$. The measures $\{\mu_k|_Q\}$, restricted to Q, all have total mass $\mu(Q)$. Thus the Banach-Alaoglu theorem implies that is a subsequence $\mu_{k_i}|_Q$ converging weakly on Q to some measure μ_Q. By continuity,

$$E_j(\mu_Q) = \lim_{i\to\infty} E_j(\mu_{k_i}|_Q) = \lim_{i\to\infty} E_j(\mu_{k_i})|_Q = \mu_j|_Q.$$

This means precisely that μ_Q is the extension of μ to a Borel measure on Q. If we patch together the measures μ_Q over all choices of $Q \in \mathcal{Q}_0^d$, i.e. setting

$$\mu(E) = \sum_{n\in\mathbf{Z}^d} \mu_Q\left(E \cap [n_1, n_1+1) \times \cdots \times [n_d, n_d+1)\right),$$

then this extends the function μ to a regular Borel measure. The uniqueness of the extension is guaranteed, because the intervals in $\mathcal{Q}^d$ generate the entire Borel sigma algebra. □

Lemma 6 (Frostman's Lemma)**.** *If E is Borel, $H^s(E) > 0$ if and only if there exists an s dimensional Frostman measure supported on E.*

Proof. Suppose that μ is s dimensional and supported on E. If $H^s(F) = 0$, then for each $\varepsilon > 0$ there is a sequence of cubes $\{Q_k\}$ with $\sum_{k=1}^\infty l(Q_k)^s \leq \varepsilon$. But then

$$\mu(F) \leq \sum_{k=1}^\infty \mu(Q_k) \lesssim \sum_{k=1}^\infty l(Q_k)^s \leq \varepsilon.$$

Taking $\varepsilon \to 0$, we conclude $\mu(F) = 0$. Thus μ is absolutely continuous with respect to H^s. But since $\mu(E) > 0$, this means that $H^s(E) > 0$.

Conversely, suppose $H^s(E) > 0$. We work with the classical family of dyadic cubes, i.e. with the sequence $\{N_k = 2 : k \geq 1\}$. By translating, we may assume that

$H^s(E\cap[0,1]^d) > 0$, and so without loss of generality we may assume $E \subset [0,1]^d$. Fix m, and for each $Q \in \mathcal{Q}_k^d$, define $\mu^+(Q) = H^s_{1/2^m}(E\cap Q)$. Then $\mu^+(Q) \leq 1/2^{ks}$, and μ^+ is subadditive. We use it to recursively define a Frostman measure μ, such that $\mu(Q) \leq \mu^+(Q)$ for each $Q \in \mathcal{Q}_k^d$. We initially define μ by setting $\mu([0,1]^d) = \mu^+([0,1]^d)$. Given $Q \in \mathcal{Q}_k^d$, we enumerate it's children as $Q_1,\dots,Q_M \in \mathcal{Q}_{k+1}^d$. We then consider any values $A_1,\dots,A_M \geq 0$ such that

$$A_1 + \dots + A_M = \mu(Q), \tag{2.7}$$

and for each k,

$$A_k \leq \mu^+(Q_k). \tag{2.8}$$

This is feasible to do because $\mu^+(Q_1) + \dots + \mu^+(Q_M) \geq \mu^+(Q)$. We then define $\mu(Q_k) = A_k$ for each k. Equation (2.7) implies the recursive constraint is satisfied, so μ is a well defined function. Equation (2.8) implies (2.6) of Lemma 5, and so the mass distribution principle implies μ extends to a Borel measure which satisfies $\mu(Q) \leq \mu^+(Q) \leq 1/2^{ks}$ for each $Q \in \mathcal{Q}_k^d$. Given any cube Q, we find k with $1/2^{k-1} \leq l(Q) \leq 1/2^k$. Then Q is covered by $O_d(1)$ dyadic cubes in $\mathcal{Q}_k^d$, and so $\mu(Q) \lesssim l(Q)^s$ for any cube Q. And this means that we have shown directly that μ is a Frostman measure of dimension s. □

Frostman's lemma implies that to study the Hausdorff dimension of the set, it suffices to understand the class of measures which can be supported on that set. The Fourier dimension of a set also studies this perspective, by a slightly refinement of the measure bound required on the Frostman dimension of a measure.

2.6 Fourier Dimension

The applicability of Fourier analysis to the analysis of dimension begins by converting the measure bound required on the Frostman dimension onto a condition on the Fourier transform of the measure. For a Borel measure μ, we define the s

energy of μ as

$$I_s(\mu) = \int\int \frac{d\mu(x)d\mu(y)}{|x-y|^s}.$$

A simple rearrangement shows that for each y,

$$\int\int \frac{d\mu(x)d\mu(y)}{|x-y|^s} = s\int_0^\infty \frac{\mu(B(x,r))}{r^{s+1}}\,dr,$$

where $B(x,r)$ is the open ball of radius r about the point x. This enables us to relate the Frostman bound with energy integrals.

Theorem 7. *For any set E,*

$$\dim_{\mathbf{H}}(E) = \sup\{s : \textit{there is } \mu \textit{ supported on } E \textit{ with } I_s(\mu) < \infty\}.$$

Proof. For each $x \in \mathbf{R}^d$ and $r > 0$, let $B(x,r)$ denote the open ball of radius r centered at x. We note μ is an s dimensional Frostman measure if and only if $\mu(B(x,r)) \lesssim r^s$, since every dyadic cube with sidelength r is contained in $O_d(1)$ balls of radius r, and every ball of radius r is contained in $O_d(1)$ dyadic cubes with sidelength r. Thus if μ is a Frostman measure with dimension less than s, then $I_s(\mu) < \infty$. Conversely, if $I_s(\mu) < \infty$, then for μ almost every y,

$$\int \frac{d\mu(x)}{|x-y|^s} < \infty$$

In particular, there is $M < \infty$, and E_0 with $\mu(E_0) > 0$ such that for any $y \in E_0$,

$$\int \frac{d\mu(x)}{|x-y|^s} \leq M.$$

If we let $\nu(E) = \mu(E \cap E_0)$, then for any $y \in \mathbf{R}^d$, if $\nu(B(y,r)) > 0$, there is $y_0 \in E \cap B(y,r)$, so $B(y,r) \subset B(y_0,2r)$, and this implies

$$\nu(B(y,r)) \leq \nu(B(y_0,2r)) \leq \int_{B(y_0,2r)} d\nu(x) \leq 2^s r^s \int_{B(y_0,2r)} \frac{d\nu(x)}{|x-y|^s} \leq (2^s M) r^s.$$

Thus ν is a Frostman measure of dimension s supported on E. $\square$

Lemma 8. *There exists a constant $C(d,s)$ such that*

$$I_s(\mu) = C(d,s) \int k_{d-s}(\xi) |\widehat{\mu}(\xi)|^2 \, d\xi.$$

Proof. We can convert the energy integral into a condition on the Fourier transform. If we define the *Riesz kernels* $k_s(x) = 1/|x|^s$, for $0 < s < d$, then

$$I_s(\mu) = \int (k_s * \mu) d\mu.$$

Naively applying the multiplication formula for the Fourier transform, we find

$$\begin{aligned} \int (k_s * \mu)(x) d\mu(x) &= \int \widehat{k_s * \mu}(\xi) \overline{\widehat{\mu}(\xi)} \, dx \\ &= \int \widehat{k_s}(\xi) |\widehat{\mu}(\xi)|^2 \, dx \\ &= C(d,s) \int k_{d-s}(\xi) |\widehat{\mu}(\xi)|^2 \, dx. \end{aligned}$$

where we have used the fact that the Fourier transform of k_s is equal to $C(d,s)k_{d-s}$ for some constant $C(d,s)$. This is not a rigorous argument, because the kernels k_s and the measure μ do not lie in $L^1(\mathbf{R}^d)$. But one can interpret this idea in a distributional sense, and therefore obtain the formula by a technical argument involving approximations to the identity, which we leave out of this thesis. A formal proof can be found in BLAH. $\square$

Thus the application of energy integrals translates the problem of finding a Frostman measure bound to frequency space. In particular, if μ is a Frostman measure of dimension s, then

$$\int \frac{|\widehat{\mu}(\xi)|^2}{|\xi|^{d-s}} \, d\xi < \infty.$$

A weak type bound implies that we should have

$$|\widehat{\mu}(\xi)|^2 \lesssim |\xi|^{s/2} \tag{2.9}$$

for *most* values ξ. Thus we obtain a stronger condition if we require that (2.9) holds for *all* values ξ. In particular, we say a measure μ is a measure with *Fourier dimension s* if (2.9) holds for *all* values ξ. If this is true, then $I_t(\mu) < \infty$ for all $t < s$. Thus if we define the *Fourier dimension* of a set E as

$$\dim_{\mathbf{F}}(E) = \sup \left\{ s : \begin{array}{c} \text{there is } \mu \text{ supported on } E \text{ with} \\ |\widehat{\mu}(\xi)| \lesssim |\xi|^{s/2} \text{ for all } \xi \in \mathbf{R}^d \end{array} \right\},$$

then $\dim_{\mathbf{F}}(E) \leq \dim_{\mathbf{H}}(E)$. We say a set E is *Salem* if $\dim_{\mathbf{F}}(E) = \dim_{\mathbf{H}}(E)$.

Most classical, self-similar fractals (for instance, the classical Cantor set) have Fourier dimension zero. Nonetheless, most *random s* dimensional subsets of $\mathbf{R}^d$ are almost surely Salem. As of now, except in very particular cases, the only way to find sets with large Fourier dimension is to input randomness into their construction. Using randomness, in Chapter BLAH we are able to find Salem sets with large Fourier dimension.

2.7 Dyadic Fractional Dimension

We wish to establish results about fractional dimension 'dyadically'. We begin with Minkowski dimension. For each m, let $N_{\mathcal{Q}}(m,E)$ denote the minimal number of cubes in $\mathcal{Q}_m^d$ required to cover E. This is often easy to calculate, up to a multiplicative constant, by greedily selecting cubes which intersect E.

Lemma 9. *For any set E,*

$$N_{\mathcal{Q}}(m,E) \sim_d \#\{Q \in \mathcal{Q}_m^d : Q \cap E \neq \emptyset\} \sim_d N(l_m,E).$$

Proof. Let $\mathcal{E} = \{Q \in \mathcal{Q}_m^d : Q \cap E \neq \emptyset\}$. Then $\mathcal{E}$ is certainly a cover of E by dyadic cubes, which shows $N_{\mathcal{Q}}(m,E) \leq \#(E)$. Conversely, let $\{Q_k\}$ be a minimal cover of E. Then $Q_k \cap E \neq \emptyset$ for each m, so $\{Q_k\} \subset \mathcal{E}$. But if $Q \in \mathcal{E}$, then it intersects a cube in $\{Q_k\}$, and since each cube Q_k intersects at most 2^d other cubes in $\mathcal{Q}_k^d$, we conclude that $\#(\mathcal{E}) \leq (2^d+1)N_{\mathcal{Q}}(k,E)$. The bound $N(l_m,E) \leq N_{\mathcal{Q}}(m,E)$ is obvious, whereas for each Q with $l(Q) = l_m$, Q is covered by 2^d+1 cubes in $\mathcal{Q}_m^d$, so $N_{\mathcal{Q}}(m,E) \leq (2^d+1)N(l_m,E)$. □

Thus it is natural to ask whether it is true that for any set E,

$$\dim_{\underline{\mathbf{M}}}(E) = \liminf_{k\to\infty} \frac{\log[N_{\mathcal{Q}}(k,E)]}{\log[1/l_k]}$$
$$\text{and} \tag{2.10}$$
$$\dim_{\overline{\mathbf{M}}}(E) = \limsup_{k\to\infty} \frac{\log[N_{\mathcal{Q}}(k,E)]}{\log[1/l_k]}.$$

The answer depends on the choice of $\{N_k\}$.

Lemma 10. *If, for all* $\varepsilon > 0$, $N_{k+1} \lesssim_\varepsilon (N_1 \dots N_k)^\varepsilon$, *then* (2.10) *holds.*

Proof. Fix a length l, and find k with $l_{k+1} \leq l \leq l_k$. Applying Lemma 9 shows

$$N(l,E) \leq N(l_{k+1},E) \lesssim_d N_{\mathcal{Q}}(k+1,E)$$

and

$$N(l,E) \geq N(l_k,E) \gtrsim_d N_{\mathcal{Q}}(k,E).$$

Thus

$$\frac{\log[N(l,E)]}{\log[1/l]} \leq \left[\frac{\log(1/l_{k+1})}{\log(1/l_k)}\right] \frac{\log[N_{\mathcal{Q}}(k+1,E)]}{\log[1/l_{k+1}]} + O_d(1/k)$$

and

$$\frac{\log[N(l,E)]}{\log[1/l]} \geq \left[\frac{\log(1/l_k)}{\log(1/l_{k+1})}\right] \frac{\log[N_{\mathcal{Q}}(k,E)]}{\log[1/l_k]} + O_d(1/k).$$

Thus, provided that

$$\frac{\log(1/l_{k+1})}{\log(1/l_k)} \to 1, \tag{2.11}$$

the conclusion of the theorem is true. But (2.11) is equivalent to the condition that

$$\frac{\log(N_{k+1})}{\log(N_1)+\cdots+\log(N_k)} \to 0,$$

and this is equivalent to the assumption of the theorem. $\square$

Any constant branching factor satisfies the hypothesis of Lemma 10 for the Minkowski dimension. In particular, we can construct sets over classical dyadic cubes without any problems occuring. But more importantly for our work, we can let the sequence $\{N_k\}$ increase rapidly.

Lemma 11. *If $N_k = 2^{\lfloor 2^{k\psi(k)} \rfloor}$, where $\psi(k)$ is any decreasing sequence of positive numbers tending to zero, but for which $k\psi(k) \to \infty$, then $N_{k+1} \lesssim_\varepsilon (N_1 \dots N_k)^\varepsilon$ for any $\varepsilon > 0$.*

Proof. We note that $\log(N_k) = 2^{k\psi(k)} + O(1)$. Thus

$$\begin{aligned}\frac{\log(N_{k+1})}{\log(N_1)+\cdots+\log(N_k)} &= \frac{2^{(k+1)\psi(k+1)} + O(1)}{2^{\psi(1)} + 2^{2\psi(2)} + \cdots + 2^{k\psi(k)} + O(k)} \\ &\lesssim \frac{2^{(k+1)\psi(k+1)}}{2^{\psi(k)} + 2^{2\psi(k)} + \dots 2^{k\psi(k-1)}} \\ &\lesssim \frac{2^{(k+1)\psi(k+1)}}{2^{(k+1)\psi(k)}}(2^{\psi(k)} - 1) = 2^{\psi(k)} - 1 \to 0.\end{aligned}$$

This is equivalent to the fact that $N_{k+1} \lesssim_\varepsilon (N_1 \dots N_k)^\varepsilon$ for any $\varepsilon > 0$. $\square$

We refer to any sequence $\{l_k\}$ constructed by $\{N_k\}$ satisfying the conditions of Lemma 11 as a *subhyperdyadic* sequence. If a sequence is generated by a sequence $\{N_k = 2^{\lfloor 2^{ck} \rfloor}\}$, for some fixed $c > 0$, the lengths are referred to as *hyperdyadic*. The next (counter) example shows that hyperdyadic sequences are essentially the 'boundary' for sequences that can be used to measure the Minkowski dimension.

Example. *We consider a multi-scale dyadic construction. Fix $0 \leq c < 1$, and define $N_k = 2^{\lfloor 2^{ck} \rfloor}$, and $M_k = 2^{\lfloor c2^{ck} \rfloor}$. Then $M_k \,|\, N_k$ for each k. We recursively define* *a nested family of sets* $\{E_k\}$*, with each* E_k *a* $\mathcal{Q}_k^d$ *discretized set, and set* $E = \bigcap E_k$*. We define* $E_0 = [0,1]$*. Then, given* E_k*, we divide each sidelength* l_k *dyadic interval in* E_k *into* M_{k+1} *intervals, and then keep the first sidelength* l_k/M_{k+1} *interval from this set, which is formed from* N_{k+1}/M_{k+1} *sidelength* l_{k+1} *dyadic intervals. Thus* $\#(\mathcal{Q}_0(E_0)) = 1$*, and*

$$\#(\mathcal{Q}_{k+1}(E_{k+1})) = (N_{k+1}/M_{k+1})\#(\mathcal{Q}_k(E_k))$$

so

$$\#(\mathcal{Q}_k(E_k)) = \frac{N_1 \dots N_k}{M_1 \dots M_k} \tag{2.12}$$

Noting that $\log(N_i) = 2^{ci} + O(1)$*, and* $\log(M_i) = c2^{ci} + O(1)$*, we conclude that*

$$\frac{\log \#(\mathcal{Q}_k(E_k))}{\log(1/l_k)} = \frac{(1-c)(2^c + \dots + 2^{ck}) + O(k)}{(2^c + \dots + 2^{ck}) + O(k)} = 1 - c + O\left(\frac{k}{2^{ck}}\right) \to 1 - c.$$

On the other hand, if $r_{k+1} = l_k/M_{k+1}$*, then for each* k*,*

$$\#(\mathcal{R}_{k+1}^d(E_k)) = \#(\mathcal{Q}_k(E_k)) = \frac{N_1 \dots N_k}{M_1 \dots M_k},$$

and so

$$\begin{aligned}
\frac{\log \#(\mathcal{R}_{k+1}^d(E_k))}{\log(1/r_{k+1})} &= \frac{(1-c)(2^c + \dots + 2^{ck}) + O(k)}{(2^c + \dots + 2^{ck}) + c2^{c(k+1)} + O(k)} \\
&= \frac{\left(\frac{(1-c)2^c}{2^c - 1}\right) 2^{ck} + O(k)}{\left(\frac{2^c}{2^c - 1} + c2^c\right) 2^{ck} + O(k)} \\
&= \frac{1-c}{1-c+c2^c} + O\left(\frac{k}{2^{ck}}\right) \to \frac{1-c}{1-c+c2^c} < 1 - c.
\end{aligned}$$

In particular,

$$\dim_{\underline{\mathbf{M}}}(E) \neq \liminf_{k \to \infty} \frac{\log[N(l_k, E)]}{\log(1/l_k)} = \liminf_{k \to \infty} \frac{\log[N_{\mathcal{Q}}(k, E)]}{\log(1/l_k)},$$

so measurements at hyperdyadic scales fail to establish general results about the Minkowski dimension.

We now move on to calculating Hausdorff dimension dyadically. The natural quantity to consider is the measure defined for any E as $H^s_{\mathcal{Q}}(E) = \lim H^s_{\mathcal{Q},m}(E)$, where

$$H^s_{\mathcal{Q},m}(E) = \inf\left\{\sum_k l(Q_k)^s : E \subset \bigcup_k^\infty Q_k,\ Q_k \in \bigcup_{i\geq m} \mathcal{Q}^d_i \text{ for each } k\right\}.$$

A similar argument to the standard Hausdorff measures shows there is a unique s_0 such that $H^s_{\mathcal{Q}}(E) = \infty$ for $s < s_0$, and $H^s_{\mathcal{Q}}(E) = 0$ for $s > s_0$. It is obvious that $H^s_{\mathcal{Q}}(E) \geq H^s(E)$ for any set E, so we certainly have $s_0 \geq \dim_{\mathbf{H}}(E)$. The next lemma guarantees that $s_0 = \dim_{\mathbf{H}}(E)$, under the same conditions on the sequence $\{N_k\}$ as found in Lemma 10.

Lemma 12. *If, for any $\varepsilon > 0$, $N_{k+1} \lesssim_\varepsilon (N_1 \dots N_k)^\varepsilon$, then for any $\varepsilon > 0$,*

$$H^s_{\mathcal{Q}}(E) \lesssim_\varepsilon H^{s-\varepsilon}(E).$$

Proof. Fix $\varepsilon > 0$ and m. Let $E \subset \bigcup Q_k$, where $l(Q_k) \leq l_m$ for each k. Then for each k, we can find i_k such that $l_{i_k+1} \leq l(Q_k) \leq l_{i_k}$. Then Q_k is covered by $O_d(1)$ elements of $\mathcal{Q}^d_{i_k}$, and

$$H^s_{\mathcal{Q},m}(E) \lesssim_d \sum l^s_{i_k} \leq \sum (l_{i_k}/l_{i_k+1})^s l(Q_k)^s \leq \sum \left(l_{i_k}/l_{i_{k+1}}\right)^s l^\varepsilon_{i_k} l(Q_k)^{s-\varepsilon} \tag{2.13}$$

By assumption,

$$l_{i_k+1} = \frac{1}{N_1 \dots N_{i_k} N_{i_k+1}} \gtrsim_{s,\varepsilon} (N_1 \dots N_{i_k})^{1+\varepsilon/s} = l^{1+\varepsilon/s}_{i_k}. \tag{2.14}$$

Putting (2.13) and (2.14) together, we conclude that $H^s_{\mathcal{Q},m}(E) \lesssim_{d,s,\varepsilon} \sum l(Q_k)^{s-\varepsilon}$. Since $\{Q_k\}$ was an arbitrary cover of E, we conclude $H^s_{\mathcal{Q},m}(E) \lesssim H^{s-\varepsilon}(E)$, and since m was arbitrary, that $H^s_{\mathcal{Q}}(E) \lesssim H^{s-\varepsilon}(E)$. □

Finally, we consider computing whether we can establish that a measure is a Frostman measure dyadically. First, we recognize the utility of this approach from the perspective of a dyadic construction. Suppose we have a sequence $\{\mathcal{E}_k\}$, where $\mathcal{E}_k \subset \mathcal{Q}_k^d$, and $\mathcal{E}_{k+1}^* = \mathcal{E}_k$. Then the sets $\{E_k = \bigcup \mathcal{E}_k\}$ are nested, and we can set $E = \bigcap E_k$ as a 'limit' of the discretizations E_k. We can associate with this construction a finite measure μ supported on E. It is defined by setting $\mu([0,1]^d) = 1$, and for each $Q \in \mathcal{E}_{k+1}$, setting

$$\mu(Q) = \frac{\mu(Q^*)}{\#\{Q_0 \in \mathcal{E}_{k+1} : Q_0^* = Q^*\}}.$$

The mass distribution principle extends μ to a Borel measure, and we refer to it as the *canonical* measure associated with this construction. For this measure, it is often easy to show that $\mu(Q) \lesssim l(Q)^s$ if $Q \in \mathcal{Q}^d$. The conditions of Lemma 10 are then sufficient to infer that μ is a Frostman measure of dimension $s - \varepsilon$ for all $\varepsilon > 0$.

Theorem 13. *If $N_{k+1} \lesssim_\varepsilon (N_1 \dots N_k)^\varepsilon$ for all $\varepsilon > 0$, and if μ is a Borel measure such that $\mu(Q) \lesssim l(Q)^s$ for each $Q \in \mathcal{Q}_k^d$, then μ is a Frostman measure of dimension $s - \varepsilon$ for each $\varepsilon > 0$.*

Proof. Given a cube Q, find k such that $l_{k+1} \leq l(Q) \leq l_k$. Then Q is covered by $O_d(1)$ cubes in $\mathcal{Q}_k^d$, which shows

$$\mu(Q) \lesssim_d l_k^s = [(l_k/l)^s l^\varepsilon] l^{s-\varepsilon} \leq [l_k^{s+\varepsilon}/l_{k+1}^s] l^{s-\varepsilon} = \left[\frac{N_{k+1}^s}{(N_1 \dots N_k)^\varepsilon}\right] l^{s-\varepsilon} \lesssim_\varepsilon l^{s-\varepsilon}. \quad \square$$

Remark. *The dyadic construction showing that Minkowski dimension cannot be measured only at hyperdyadic scales also shows that a bound on a measure μ on hyperdyadic cubes does not imply the correct bound at all scales. It is easy to show from* (2.12) *that the canonical measure μ for this example satisfies $\mu(Q) \lesssim l(Q)^{1-c}$ when Q is hyperdyadic, yet we know that for the set constructed in that example,*

$$\dim_{\mathbf{H}}(E) \leq \dim_{\underline{\mathbf{M}}}(E) < 1 - c.$$

Frostman's lemma implies we cannot possibly have $\mu(Q) \lesssim_\varepsilon l(Q)^{1-c-\varepsilon}$ *for all* $\varepsilon > 0$ *and* all *cubes* Q*.*

2.8 Beyond Hyperdyadics

If we are to use a faster increasing sequence of branching factors than the last section guarantees, we therefore must exploit some extra property of our construction, which is not always present in general sets. Here, we rely on a *uniform mass distribution* between scales. Given the uniformity assumption, the lengths can decrease as fast as desired. We utilize a multi-scale set of dyadic cubes.

Lemma 14. *Let* μ *be a measure supported on a set* E*. Suppose that*

(A) *For any* $Q \in \mathcal{Q}_k^d$*,* $\mu(Q) \lesssim l_k^s$*.*

(B) *For each* $Q \in \mathcal{R}_k^d$*,* $\#\{Q \in \mathcal{Q}_k^d : E \cap Q \neq \emptyset\} = O(1)$*.*

(C) *For any* $Q \in \mathcal{R}_{k+1}^d$ *with parent cube* $Q^* \in \mathcal{Q}_k^d$*,* $\mu(Q) \lesssim (r_{k+1}/l_k)^d \mu(Q^*)$*.*

Then μ *is a Frostman measure of dimension* s*.*

Proof. We establish the general bound $\mu(Q) \lesssim l(Q)^s$ for all cubes Q in two different cases:

- Suppose there is k with $r_{k+1} \leq l \leq l_k$. Then we can cover Q by at most $O_d((1/r_{k+1})^d)$ cubes in $\mathcal{R}_k^d$. By Properties (A) and (C), each of these cubes has measure at most $O((r_{k+1}/l_k)^d l_k^s)$, so we obtain that

$$\mu(Q) \lesssim (l/r_{k+1})^d (r_{k+1}/l_k)^d l_k^s = l^d / l_k^{d-s} \lesssim l^s.$$

- Suppose there exists k with $l_k \leq l \leq r_k$. Then we can cover Q by $O_d(1)$ cubes in $\mathcal{R}_k^d$, and for each such cube, Property (C) shows there are $O(1)$ cubes in $\mathcal{Q}_k^d$ which have mass. Thus

$$\mu(Q) \lesssim l_k^s \leq l^s.$$

This addresses all cases, so μ is a Frostman measure of dimension s. $\square$

TODO: ADDRESS OTHER FRACTIONAL DIMENSION CASES?

2.9 Extras: Hyperdyadic Covers

Nonetheless, it will be useful for us to know that we can 'decompose' a set with a prescribed Hausdorff dimension hyperdyadically. We say a sequence of sets $\{E_k\}$ is a *strong cover* of a set E if $E \subset \limsup E_k$, or equivalently, if every $x \in E$ lies in infinitely many of the sets E_k. In this section, we inclusively treat hyperdyadic cubes, i.e. we assume $l_k = 2^{-\lfloor (1+\varepsilon)^k \rfloor}$ for some fixed $0 < \varepsilon \leq 1$.

Theorem 15. *Suppose $E \subset [0,1]^d$ is a set with $\dim(E) \leq s$. Fix $\varepsilon > 0$, and write $l_k = 2^{-\lfloor (1+\varepsilon)^k \rfloor}$. Then there exists a strong cover of E by sets $\{E_k\}$, where E_k is a union of $O((1+\varepsilon)^{2k} l_k^{-s})$ cubes in $\mathcal{Q}_k^d$.*

Proof. For each hyperdyadic number l_k, we can find a collection of cubes $\{Q_{k,i}\}$ covering E with $l(Q_{k,i}) \leq l_k$ for all i, and

$$\sum_{i=1}^{\infty} l(Q_{k,i})^{s+C\varepsilon} \lesssim 1. \tag{2.15}$$

For each k and i, find $j_{k,i}$ such that $l_{j_{k,i}+1} \leq l(Q_{k,i}) \leq l_{j_{k,i}}$. Note $l_{j_{k,i}+1} \lesssim l_{j_{k,i}}^{1+\varepsilon}$, so

$$\begin{aligned}\sum_{i=1}^{\infty} l_{j_{k,i}}^{s+(C+s)\varepsilon} &= \sum_{i=1}^{\infty} l(Q_{k,i})^{s+(C+s)\varepsilon} (l_{j_{k,i}}/l(Q_{k,i}))^{s+(C+s)\varepsilon} \\ &\lesssim \sum_{i=1}^{\infty} l(Q_{k,i})^{s+(C+s)\varepsilon} l_{j_{k,i}}^{-s\varepsilon} \lesssim \sum_{i=1}^{\infty} l(Q_{k,i})^{s+C\varepsilon} \lesssim 1.\end{aligned}$$

Thus, replacing C with $C+s$, and replacing $Q_{k,i}$ with the $O_d(1)$ cubes in $\mathcal{Q}_{j_{k,i}}(Q_{k,i})$, we may assume without loss of generality that all cubes in the decomposition corresponding to (2.15) are hyperdyadic.

For $k_2 \geq k_1$, we let

$$Y_{k_1,k_2} = \bigcup \{Q_{k_1,i} : l(Q_{k_1,i}) = l_{k_2}.\}$$

Note that Y_{k_1,k_2} is the union of $O((1/l_{k_2})^{s+C\varepsilon})$ cubes in $\mathcal{Q}_{k_2}$. We let Z_{k_1,k_2} be the collection of hyperdyadic cubes covering Y_{k_1,k_2} which minimize

$$\sum \left\{ l(Q)^s : Q \in Z_{k_1,k_2} \right\}$$

and such that $l(Q) \geq l_{k_2}$ for each Q. Then clearly

$$\sum_{Q \in Z_{k_1,k_2}} l(Q)^s \lesssim l_{k_2}^{-C\varepsilon}.$$

In particular, this means $l(Q) \lesssim l_{k_2}^{-C\varepsilon/s}$ for each $Q \in Z_{k_1,k_2}$. Moreover, for each hyperdyadic Q_0 with $l(Q_0) \geq l_{k_2}$,

$$\sum_{Q \subset Q_0} l(Q)^s \leq l(Q_0)^s.$$

Now we define $E_k = \bigcup \left\{ \bigcup_{k_1,k_2} Z_{k_1,k_2} \cap \mathcal{Q}_k \right\}$. Since E_k only contains cubes from Z_{k_1,k_2} where $k_1 \leq k_2$, and $l_k \lesssim l_{k_2}^{-\varepsilon/\alpha}$ TODO: THERE IS SOMETHING WRONG HERE THIS DOESN'T IMPLY $\log(1/l_k)^2$?. But this means that there are only $O(\log(1/l_k)^2) = O((1+\varepsilon)^{2k})$ such choices of (k_1,k_2). But this means that

$$|E_k| = \sum l(Q)^d = l_k^{d-s} \sum l(Q)^s \lesssim (1+\varepsilon)^{2k} l_k^{d-s},$$

which implies that E_k is the union of at most $O((1+\varepsilon)^{2k} l_k^{-s})$ cubes in $\mathcal{Q}_k$. $\square$

Chapter 3

Related Work

Here, we discuss the main papers which influenced our results. In particular, the work of Keleti on translate avoiding sets, Fraser and Pramanik's work on sets avoiding smooth configurations, Mathé's result on sets avoiding algebraic varieties, and Schmerkin's result on sets with large Fourier dimension avoiding smooth configurations.

3.1 Keleti: A Translate Avoiding Set

Keleti's two page paper constructs a full dimensional set $X \subset [0,1]$ such that for each $t \neq 0$, X intersects $t+X$ in at most one place. The set X is then said to *avoid translates*. This paper contains the core idea behind the *interval dissection* method adapted in Fraser and Pramanik's paper. We also adapt this technique in our paper, which makes the result of interest.

It is often convenient to avoid certain configurations when they are expressed in terms of an equation, which is also exploited in Fraser and Pramanik's work, and in Mathé's.

Lemma 16. *Let X be a set. Then X avoids translates if and only if there do not exists values $x_1 < x_2 \leq x_3 < x_4$ in X with $x_2 - x_1 = x_4 - x_3$.*

Proof. Suppose $(t+X) \cap X$ contains two points $a < b$. Without loss of generality, we may assume that $t > 0$. If $a \leq b - t$, then the equation

$$a - (a - t) = t = b - (b - t)$$

satisfies the constraints, since $a - t < a \leq b - t < b$ are all elements of X. We also have

$$(b - t) - (a - t) = b - a,$$

which satisfies the constraints if $a - t < b - t \leq a < b$. This covers all possible cases. Conversely, if there are $x_1 < x_2 \leq x_3 < x_4$ in X with

$$x_2 - x_1 = t = x_4 - x_3,$$

then $X + t$ contains $x_2 = x_1 + (x_2 - x_1)$ and $x_4 = x_3 + (x_4 - x_3)$. $\square$

The basic, but fundamental idea to Keleti's technique is to introduce memory into Cantor set constructions. Keleti constructs a nested family of discrete sets $X_0 \supset X_1 \supset \dots$ converging to X, with each X_k a union of disjoint intervals in $\mathcal{Q}_k^d$, where the sequence $\{N_k\}$ will be chosen later, but a multiple of 10. We initialize $X_0 = [0,1]$, and $l_0 = 1$. Furthermore, we consider a queue of intervals, initially just containing $[0,1]$. To construct the sequence $\{X_k\}$, Keleti iteratively performs the following procedure:

Algorithm 1 Construction of the Sets $\{X_k\}$:

Set $k = 0$.

Repeat

Take off an interval I from the front of the queue.

For all $J \in \mathcal{Q}_k(X_k)$:

Order the intervals in $\mathcal{Q}_{k+1}(J)$ as $J_0, J_1, \dots, J_N$.

If $J \subset I$, add all intervals J_i to X_{k+1} with $i \equiv 0$ modulo 10.

Else add all J_i with $i \equiv 5$ modulo 10.

Add all intervals in $\mathcal{Q}^d_{k+1}$ to the end of the queue.

Increase k by 1.

Each iteration of the algorithm produces a new set X_k, and so leaving the algorithm to repeat infinitely produces a sequence $\{X_k\}$ whose intersection is X.

Lemma 17. *The set X is translate avoiding.*

Proof. If X is not translate avoiding, there is $x_1 < x_2 \leq x_3 < x_4$ with $x_2 - x_1 = x_4 - x_3$. Since $l_k \to 0$, there is a suitably large integer N such that x_1 is contained in an interval $I \in \mathcal{Q}_N$ not containing x_2, x_3, or x_4. At stage N of the algorithm, the interval I is added to the end of the queue, and at a much later stage M, the interval I is retrieved. Find the startpoints x_1°, x_2°, $x_3^\circ, x_4^\circ \in l_M\mathbf{Z}$ to the intervals in $\mathcal{Q}_M$ containing x_1, x_2, x_3, and x_4. Then we can find n and m such that $x_4^\circ - x_3^\circ = (10n)l_M$, and $x_2^\circ - x_1^\circ = (10m+5)l_M$. In particular, this means that $|(x_4^\circ - x_3^\circ) - (x_2^\circ - x_1^\circ)| \geq 5L_M$. But

$$\begin{aligned} |(x_4^\circ - x_3^\circ) - (x_2^\circ - x_1^\circ)| &= |[(x_4^\circ - x_3^\circ) - (x_2^\circ - x_1^\circ)] - [(x_4 - x_3) - (x_2 - x_1)]| \\ &\leq |x_1^\circ - x_1| + \cdots + |x_4^\circ - x_4| \leq 4L_M \end{aligned}$$

which gives a contradiction. □

It is easy to see from the definition of the algorithm that

$$\#(\mathcal{Q}_k(X_k)) = (l_{k-1}/10l_k)\#(\mathcal{Q}_{k-1}(X_{k-1})).$$

Thus closing the recursive definition shows

$$\#(\mathcal{Q}_k(X_k)) = \frac{1}{10^k l_k}.$$

In particular, this means $|X_k| = 1/10^k$, so X has measure zero irrespective of our parameters. Nonetheless, the canonical measure μ on X defined with respect to the decomposition $\{X_k\}$ satisfies $\mu(I) = 10^k l_k$ for all $I \in \mathcal{B}(l_k, X)$. If $10^k l_k^\varepsilon \lesssim_\varepsilon 1$ for all ε, then we can establish the bounds $\mu(I) \lesssim_\varepsilon l_k^{1-\varepsilon}$ for all ε. In particular, this is true if $N_k = 10^{\lfloor \log(k+2) \rfloor}$. And because this sequence does not rapidly decrease too fast, we can apply Lemma 13 to show μ is a Frostman measure of dimension $1-\varepsilon$ for each $\varepsilon > 0$, so X has full Hausdorff dimension.

3.2 Generalizing Keleti's Argument

Before we move onto other methods which developed Keleti's argument work, it is useful to dwell on what general properties this argument has:

- *Simplification to a Discrete Problem*: A major part of Keleti's argument is solving a discrete version of the configuration argument. We could summarize the result of Keleti's discrete argument in a lemma.

  Lemma 18. *Let T_1, T_2 be disjoint, $\mathcal{Q}_k$ discretized sets. Then we can find $S_1 \subset T_1$ and $S_2 \subset T_2$ such that*

  (i) For each k, S_k is a $\mathcal{Q}_{k+1}$ discretized subset of T_k.

  (ii) If $x_1 \in S_1$ and $x_2, x_3, x_4 \in S_2$, then $x_2 - x_1 \neq x_4 - x_3$.

  (iii) For each cube $Q \in \mathcal{Q}_k$ with $Q \subset T_k$,

  $$\#(\mathcal{Q}_{k+1}(Q \cap S_k)) \geq (1/10) \cdot \#(\mathcal{Q}_{k+1}(Q)).$$

- *Iterative Application of Discrete Solution*: Keleti then repeatedly applies his argument iteratively. In particular, Property (i) of Lemma 18 allows him to apply his argument iteratively. The reason why Keleti obtains a configuration avoiding set in the limit is because of Property (ii). Most importantly the reason why Keleti obtains a set with full Hausdorff dimension, if the sequence $\{N_k\}$ decreases rapidly enough, is because of Property (iii).

Of course, it is not possible to extend the discrete solution of the configuration argument to general configurations; this part of Keleti's method strongly depends on the arithmetic structure of the configuration. The iterative application of a discrete solution, however, can be applied in generality. In various senses, this technique is applied in all the results we review in this chapter.

3.3 Fraser/Pramanik: Smooth Configurations

Inspired by Keleti's result, Pramanik and Fraser obtained a generalization of the queue method which allows one to find sets avoiding $n+1$ point configurations given by the zero sets of smooth functions, i.e.

$$\mathcal{C} = \{(x_0, \dots, x_n) \in \mathbf{R}^{dn} : f(x_0, \dots, x_n) = 0\},$$

under mild regularity conditions on the function $f : [0,1]^{dn} \to [0,1]^m$. Here, a simple pidgeonholing strategy suffices.

Theorem 19 (Pramanik and Fraser)**.** *Fix $m \leq d(n-1)$. Consider a countable family of functions $\{f_k : [0,1]^{dn} \to [0,1]^m\}$ each of which being C^2, and such that for each k, Df_k has full rank at any $(x_1, \dots, x_n)$ with $f(x_1, \dots, x_n) = 0$, where the $(x_1, \dots, x_n)$ are distinct. Then there exists a set $X \subset \mathbf{R}^d$ with Hausdorff dimension $m/(n-1)$ such that X avoids the configuration*

$$\mathcal{C} = \bigcup_k \{(x_1, \dots, x_n) \in \mathcal{C}^n(\mathbf{R}^d) : f(x_1, \dots, x_n) = 0\}.$$

Remark. *For simplicity, we only prove the result for a single function, rather than a countable family of functions. The only major difference between the two approaches is the choice of scales we must choose later on in the argument.*

Pramanik and Fraser also simplify to a discrete version of their problem, which they then iteratively apply. In the discrete setting, rather than making a linear shift in one of the intervals we avoid as in Keleti's approach, one must use the smoothness properties of the function to find large segments of an interval avoiding. Corollary 22 gives the discrete solution that Pramanik and Fraser utilizes.

Lemma 20. *Fix $n > 1$. Let $T \subset [0,1]^d$ be $\mathcal{Q}_k$ discretized, and $T' \subset [0,1]^{(n-1)d}$ be $\mathcal{Q}_k$ discretized. Let $B \subset T \times T'$ be $\mathcal{Q}_{k+1}$ discretized. Then there exists a $\mathcal{Q}_{k+1}$ discretized set $S \subset T$, and a $\mathcal{Q}_{k+1}$ discretized set $B' \subset T'$, such that*

(A) $(S \times T') \cap B \subset S \times B'$.

(B) *For every $Q \in \mathcal{Q}_k^d$, there exists $\mathcal{R}(Q) \subset \mathcal{R}_{k+1}^d(Q)$, such that*

$$\#(\mathcal{R}(Q)) \geq (1/2) \cdot \#(\mathcal{R}_{k+1}^d(Q)).$$

and for each $R \in \mathcal{R}_{k+1}^d(Q)$,

$$\#(\mathcal{Q}_{k+1}(R)) = \begin{cases} 1 & : R \in \mathcal{R}(Q,) \\ 0 & : R \notin \mathcal{R}(Q). \end{cases}$$

(C) $\#(\mathcal{Q}_{k+1}(B')) \leq 2(N_1 \dots N_k)^d \left(M_{k+1}/N_{k+1}\right)^d \cdot \#(\mathcal{Q}_{k+1}(B))$.

Proof. Fix $Q_0 \in \mathcal{Q}_k(T)$. For each $R \in \mathcal{R}_{k+1}(Q_0)$, define a *slab* $S[R] = R \times T'$, and for each $Q \in \mathcal{Q}_{k+1}(Q_0)$, define a *wafer* $W[Q] = Q \times T'$. We say a wafer $W[Q]$ is *good* if

$$\#(\mathcal{Q}_{k+1}(W[Q] \cap B)) \leq (2/N_{k+1}^d) \cdot \#(\mathcal{Q}_{k+1}(B)), \tag{3.1}$$

Then at most $N_{k+1}^d/2$ wafers are bad. We call a slab *good* if it contains a wafer which is good. Since a slab is the union of $(N_{k+1}/M_{k+1})^d$ wafers, at most $M_{k+1}^d/2 = (1/2) \cdot \#(\mathcal{R}_{k+1}(Q_0))$ slabs are bad. Thus if we set

$$\mathcal{R}(Q_0) = \{R \in \mathcal{R}_{k+1}(Q_0) : S[R] \text{ is good}\},$$

then

$$\#(\mathcal{R}(Q_0)) \geq (1/2) \cdot \#(\mathcal{R}_{k+1}(Q_0)). \tag{3.2}$$

For each $R \in \mathcal{R}(Q_0)$, we pick $Q_R \in \mathcal{Q}_{k+1}(R)$ such that $W[Q_R]$ is good, and define

$$S = \bigcup \{Q_R : R \in \mathcal{R}(Q_0)\}.$$

Equation (3.2) implies S satisfies Property (B).

Let B' be the union of all cubes $Q' \in \mathcal{Q}_{k+1}(T')$ such that there is $Q \in \mathcal{Q}_{k+1}(S)$ with $Q \times Q' \in \mathcal{Q}_{k+1}(B)$. By definition, Property (A) is then satisfied. For each $Q \in \mathcal{Q}_{k+1}(S)$, $W[Q]$ is good, so (3.1) implies

$$\#\{Q' : Q \times Q' \in \mathcal{Q}_{k+1}(B)\} \leq (2/N_{k+1}^d) \cdot \#(\mathcal{Q}_{k+1}(B)).$$

But $\#(\mathcal{Q}_{k+1}(S)) \leq \#(\mathcal{R}_{k+1}(T)) \leq (1/r_{k+1})^d = (N_1 \dots N_k)^d M_{k+1}^d$, so

$$\begin{aligned}\#(\mathcal{Q}_{k+1}(B')) &\leq \#(\mathcal{Q}_{k+1}(S))[(2/N_{k+1}^d) \cdot \#(\mathcal{Q}_{k+1}(B))] \\ &\leq 2(N_1 \dots N_k)^d (M_{k+1}/N_{k+1})^d \#(\mathcal{Q}_{k+1}(B)),\end{aligned}$$

which establishes Property (C). □

We apply the lemma recursively $n-1$ times to continually reduce the dimensionality of the avoidance problem we are considering. Eventually, we obtain the case where $n = 0$, and we are in need of a final technique.

Lemma 21. *Fix $n > 1$. Let $T \subset [0,1]^d$ be $\mathcal{Q}_k$ discretized, and let $B \subset T$ be $\mathcal{Q}_{k+1}$ discretized. Suppose*

$$\# \mathcal{Q}_{k+1}(B) \leq \left[C \cdot 2^{n-1} (N_1 \dots N_k)^{d(n-1)} (M_{k+1}/N_{k+1})^{d(n-1)} \right] (1/l_{k+1})^{dn-m}.$$

and

$$N_{k+1} > \left[C \cdot (1+2^d) 2^n (N_1 \dots N_k)^{2dn} \right]^{1/m} M_{k+1}^{d(n-1)/m} \tag{3.3}$$

Then there exists a $\mathcal{Q}_{k+1}$ discretized set $S \subset T$ such that

(A) $S \cap B = \emptyset$.

(B) *For each $Q_0 \in \mathcal{Q}_k(T)$, there is $\mathcal{R}(Q_0) \subset \mathcal{R}_{k+1}(Q_0)$ with*

$$\#(\mathcal{R}(Q_0)) \geq (1/2)\#(\mathcal{R}_{k+1}(Q_0)),$$

such that for each $R \in \mathcal{R}_{k+1}(Q_0)$,

$$\#(\mathcal{Q}_{k+1}(R \cap S)) = \begin{cases} 1 & : R \in \mathcal{R}(Q_0), \\ 0 & : R \notin \mathcal{R}(Q_0). \end{cases}$$

Proof. For each $Q_0 \in \mathcal{Q}_k(T)$, we set

$$\mathcal{R}(Q_0) = \{R \in \mathcal{R}_{k+1}(Q_0) : \#(\mathcal{Q}_{k+1}(R \cap B)) \leq (2/M_{k+1}^d) \cdot \#(\mathcal{Q}_{k+1}(B))\}.$$

Since $\mathcal{R}_{k+1}(Q_0) = M_{k+1}^d$,

$$\#(\mathcal{R}(Q_0)) \geq \#(\mathcal{R}_{k+1}(Q_0)) - (M_{k+1}^d/2) \geq (1/2) \cdot \#(\mathcal{R}_{k+1}(T)).$$

Now (3.3) implies that for each $R \in \mathcal{R}(Q_0)$,

$$\begin{aligned}
\#(\mathcal{Q}_{k+1}(R \cap B)) &\leq (2/M_{k+1}^d) \cdot \#(\mathcal{Q}_{k+1}(B)) \\
&\leq (2/M_{k+1}^d) \left(C \cdot 2^{n-1} (N_1 \dots N_k)^{2dn} (M_{k+1}/N_{k+1})^{d(n-1)} \right). \\
&= \left[2^n C (N_1 \dots N_k)^{2dn} \right] \left(M_{k+1}^{d(n-2)} / N_{k+1}^{m-d} \right) \\
&< \left(\frac{1}{1+2^d} \right) (N_{k+1}/M_{k+1})^d \\
&= \left(\frac{1}{1+2^d} \right) \mathcal{Q}_{k+1}(R)
\end{aligned}$$

Thus for each $R \in \mathcal{R}(Q_0)$, we can find $Q_R \in \mathcal{Q}_{k+1}(R)$ such that $Q_R \cap B = \emptyset$. And so if we set

$$S = \bigcup \{Q_R : R \in \mathcal{R}(Q_0), Q_0 \in \mathcal{Q}_k(T)\},$$

Then (A) and (B) are satisfied. $\square$

Corollary 22. *Let $f : [0,1]^{dn} \to [0,1]^m$ be C^2, and have full rank at every point $(x_1, \dots, x_n)$ with all $x_1, \dots, x_n$ distinct, and $f(x_1, \dots, x_n) = 0$. Then there exists a universal constant C depending only on f such that, if* (3.3) *is satisfied, then for any disjoint, $\mathcal{Q}_k$ discretized sets $T_1, \dots, T_n \subset [0,1]^d$, we can find $\mathcal{Q}_{k+1}$ discretized sets $S_1 \subset T_1, \dots, S_n \subset T_n$ such that*

(A) *If $x_1 \in S_1, \dots, x_n \in S_n$, then $f(x_1, \dots, x_n) \neq 0$.*

(B) *For each k, and for each $Q_0 \in \mathcal{Q}_k(T_k)$, there is $\mathcal{R}(Q_0) \subset \mathcal{R}_{k+1}(Q_0)$ with*

$$\#(\mathcal{R}(Q_0)) \geq (1/2) \cdot \#(\mathcal{R}_{k+1}(Q_0)),$$

and for each $R \in \mathcal{R}_{k+1}(Q_0)$,

$$\#(\mathcal{Q}_{k+1}(R \cap S)) = \begin{cases} 1 & : R \in \mathcal{R}(Q_0), \\ 0 & : R \notin \mathcal{R}(Q_0). \end{cases}$$

Proof. Since f is C^2 and has full rank on the set

$$V(f) = \{(x_1, \dots, x_n) \in \mathcal{C}^n(\mathbf{R}^d) : f(x_1, \dots, x_n) = 0\},$$

the implicit function theorem implies $V(f)$ is a smooth manifold of dimension $nd - m$ in $\mathbf{R}^{dn}$, and the coarea formula implies the existence of a constant C such that for each k,

$$\#\{Q \in \mathcal{Q}_k^{dn} : Q \cap V(f) \neq \emptyset\} \leq C/l_k^{dn-m}.$$

To apply Lemma 20 and 21, we set

$$B = \#\{Q \in \mathcal{Q}_k^{dn} : Q \cap V(f) \neq \emptyset\}.$$

Applying Lemma 20 iteratively $n-1$ times, then finishing with Lemma 21, constructs the sets $S_1, \dots, S_n$. $\square$

Remark. *The values of $N_1, \dots, N_k$, exponentials 2^k, and constants, are roughly insignificant in comparison to M_{k+1} and N_{k+1}, when these numbers are hyperdyadic for a suitably large constant, or grow faster than hyperdyadic. In particular, we can roughly read off the Hausdorff dimension of the set $m/(n-1)$ we construct from the inverse power of M_{k+1} in* (3.3)*, i.e. that $N_{k+1} \gtrapprox M_{k+1}^{d(n-1)/m}$.*

Just like in Keleti's proof, Pramanik and Fraser's technique applies a discrete result, Corollary 22, iteratively at many scales to obtain a high dimensional set avoiding the zeroes of a function. We construct a nested family $\{X_k : k \geq 0\}$ of $\mathcal{Q}_k$ discretized sets, converging to a set X, which we will show is translate avoiding. We intiailize $X_0 = [0,1]$. Our queue shall consist of n tuples of disjoint intervals $(T_1, \dots, T_n)$, all of the same length, which initially consists of all possible tuples of intervals in $\mathcal{Q}_1^d([0,1]^d)$. To construct the sequence $\{X_k\}$, we perform the following iterative procedure:

Algorithm 2 Construction of the Sets $\{X_k\}$

Set $k = 0$

Repeat

Take off an n tuple $(T_1', \dots, T_n')$ from the front of the queue

Set $T_i = T_i' \cap X_k$ for each i

Apply Corollary 22 to the sets $T_0, \dots, T_d$, obtaining $\mathcal{Q}_{k+1}$ discretized sets $S_1, \dots, S_n$ satisfying Properties (A), (B), and (C) of that Lemma.

Set $X_{k+1} = X_k - \bigcup_{i=1}^n T_i - S_i$.

Add all n tuples of disjoint cubes $(T_1', \dots, T_n')$ in $\mathcal{Q}_{k+1}^d(X_{k+1})$ to the back of the queue.

Increase k by 1.

Lemma 23. *The set X constructed by the procedure avoids the configuration*

$$\mathcal{C} = \{(x_1, \dots, x_n) \in \mathcal{C}^n(\mathbf{R}^d) : f(x_1, \dots, x_n) = 0\}.$$

Proof. Suppose $x_1, \dots, x_n \in X$ are distinct. Then at some stage k, $x_1, \dots, x_n$ lie in disjoint cubes $T_1', \dots, T_n' \in \mathcal{Q}_k^d(X_k)$, for some large k. At this stage, $(T_1', \dots, T_n')$ is added to the back of the queue, and therefore, at some much later stage N, the tuple $(T_1', \dots, T_n')$ is taken off the front. Sets $S_1 \subset T_1', \dots, S_n \subset T_n'$ are constructed satisfying Property (A) of Corollary 22. Since $x_1, \dots, x_n \in X$, we must have $x_i \in S_i$ for each i, so $f(x_1, \dots, x_n) \neq 0$. $\square$

What remains is to bound the Hausdorff dimension of X.

Theorem 24. *If $M_k = 2^{2^{k^2}}$, X has Hausdorff dimension exceeding $m/(n-1)$.*

Proof. First, we construct the canonical measure μ on X. Property (B) of Corollary 22 implies that for each $Q \in \mathcal{Q}_{k+1}^d$, $\mu(Q) \leq 2/M_{k+1}^d \mu(Q^*)$, which implies that for $Q \in \mathcal{Q}_k^d$, $\mu(Q) \leq 2^k/(M_k \dots M_1)^d$. By Lemma 14, it suffices to show that for each $\varepsilon > 0$,

$$\frac{2^k}{(M_{k+1} \dots M_1)^d} \lesssim_\varepsilon \frac{1}{(N_1 \dots N_{k+1})^{m/(n-1)(1-\varepsilon)}} \tag{3.4}$$

Set

$$N_{k+1} = A(N_1 \dots N_k)^{2dn/m} M_{k+1}^{d(n-1)/m},$$

where A is an arbitrary constant not depending on k so that (3.3) holds. Then inequality (3.4) is then implied if

$$M_{k+1} \gtrsim_\varepsilon \left((N_1 \dots N_k)^{6dn} [2^k A^d] \right)^{1/\varepsilon}.$$

This equation is satisfied if $M_k = 2^{2^{k^2}}$. $\square$

3.4 Mathé: Polynomial Configurations

Mathé's result constructs sets avoiding algebraic varieties.

Theorem 25 (Mathé). *Let $\{f_k : \mathbf{R}^{nd} \to \mathbf{R}\}$ be a countable family of rational coefficient polynomials with degree at most m. Then there exists a set $X \subset [0,1]^d$ with*

Hausdorff dimension d/m which avoids the configurations

$$\mathcal{C} = \bigcup_k \{(x_1, \dots, x_n) \in \mathbf{R}^{nd} : f(x_1, \dots, x_n) = 0\}.$$

Originally, Mathé's result does not explicitly use a discretization method analogous to Keleti and Pramanik and Fraser, but his proof strategy can be reconfigured to work in this setting. For the purpose of brevity, we do not carry out the complete argument, merely giving the discretization method below. By first trying to avoid the zero sets of the derivatives of the function f, one can reduce to the case where a partial derivative of f is non-vanishing on the $\mathcal{Q}_k$ discretized sets we start with. The key technique is that f maps discrete lattices of points to a discrete, 'one dimensional lattice' in $\mathbf{R}^d$, and the degree of the polynomial gives us the difference in lengths between the two lattices. We revisit this idea in Chapter BLAH.

Theorem 26. *Let $f : [0,1]^{dn} \to \mathbf{R}$ be a rational coefficient polynomial of degree m, and disjoint, $\mathcal{Q}_k$ discretized sets $T_1, \dots, T_n \subset [0,1]^d$, such that $|\partial_1 f| \neq 0$ on $T_1 \times \dots \times T_n$. Then there exists constants C_0 and C_1, depending only on f and $T_1, \dots, T_n$, such that if*

$$N_{k+1} \geq \max(4C_1\sqrt{nd}, 2C_0) \cdot (N_1 \dots N_k)^{m-1} \cdot M_{k+1}^m. \tag{3.5}$$

Then there exists $\mathcal{Q}_{k+1}$ discretized sets $S_1 \subset T_1, \dots, S_n \subset T_n$ such that

(A) *$f(x) \neq 0$ for $x \in S_1 \times \dots \times S_n$.*

(B) *For each i, and for each $R \in \mathcal{R}_{k+1}^d(T_i)$, $\#(\mathcal{Q}_{k+1}^d(R \cap S_i)) = 1$.*

Proof. Without loss of generality, by considering an appropriate integer multiple of f, we may assume f has integer coefficients. Let $\mathbf{A} \subset (r_{k+1} \cdot \mathbf{Z})^d$. Since f has degree m, $f(\mathbf{A}) \subset r_{k+1}^m \cdot \mathbf{Z}$. Suppose that $|\partial_1 f| \geq C_0$ on $T_1 \times \dots \times T_n$, and $\|\nabla f\|_{L^\infty[0,1]^{nd}} \leq C_1$. Then the mean value theorem guarantees that if $\delta \leq r_{k+1}$, then

$$|f(a + \delta e_1) - f(a)| \geq C_0 \delta.$$

If $C_0\delta \leq r_{k+1}^m/2$, then this implies

$$d(f(a+\delta e_1),(r_{k+1}^m \cdot \mathbf{Z})) \geq C_0\delta.$$

In particular, we set $\delta = r_{k+1}^m/2C_0$, then

$$d(f(a+\delta e_1), r_{k+1}^m \cdot \mathbf{Z}) \geq r_{k+1}^m/2.$$

Define

$$S_i = \begin{cases} \bigcup [a_1, a_1 + l_{k+1}] \times \cdots \times [a_d, a_d + l_{k+1}] & : i > 1 \\ \bigcup [a_1 + \delta e_1, a_1 + \delta e_1 + s] \times [a_2, a_2 + l_{k+1}] \times \cdots \times [a_d, a_d + l_{k+1}] & : i = 1. \end{cases}$$

where a ranges over all startpoints to intervals $[a_1, b_1] \times \cdots \times [a_d, b_d] \in \mathcal{R}_{k+1}^d(T_i)$. Then (3.5) implies that

$$d(f(S_1 \times \cdots \times S_n), r_{k+1}^m \cdot \mathbf{Z}) \geq r_{k+1}^m/2 - C_1\sqrt{nd}\, l_{k+1} \geq r_{k+1}^m/4.$$

In particular, this implies Property (A). $\square$

3.5 Schmerkin: Salem Sets Avoiding APs

Schmerkin is the first of the papers to obtain a result involving Fourier dimension.

Theorem 27. *There exists a set $X \subset [0,1]$ with $\dim_{\mathbf{F}}(X) = 1$, which avoids*

$$\mathcal{C} = \{(x,y,z) \in \mathcal{C}^3 : x < y < z, z - y = y - x\}.$$

In other words, X contains no nontrivial three term arithmetic progressions.

Schmerkin implements two main ideas. First, working modulo m, Behrend's construction of subsets of $\mathbf{Z}/m\mathbf{Z}$ of cardinality $m^{1-\varepsilon}$ avoiding arithmetic progressions is exploited to find subsets of $\mathbf{Z}$.

Chapter 4

Avoiding Rough Sets

In the last chapter, we saw that many authors have considered the pattern avoidance problem for configurations $\mathcal{C}$ which take the form of many general classes of smooth shapes; in Mathé's work, $\mathcal{C}$ can take the form of an algebraic variety of low degree, and in Pramanik and Fraser's work, $\mathcal{C}$ can take the form of a smooth manifold. In this chapter, we consider the pattern avoidance problem for an even more general class of 'rough' patterns, that are the countable union of sets with controlled lower Minkowski dimension.

Theorem 28. *Let $\alpha \geq d$, and suppose $\mathcal{C} \subset \mathcal{C}^n(\mathbf{R}^d)$ is the countable union of precompact sets, each with lower Minkowski dimension at most α. Then there exists a set $X \subset [0,1]^d$ with Hausdorff dimension at least $(nd-\alpha)/(n-1)$ avoiding $\mathcal{C}$.*

Remarks.

(1) When $\alpha < d$, the pattern avoidance problem is trivial, since $X = [0,1)^d - \pi(Z)$ is full dimensional and solves the pattern avoidance problem, where $\pi(x_1,\dots,x_n) = x_1$ is a projection map from $\mathbf{R}^{dn}$ to $\mathbf{R}^d$. We will therefore assume that $\alpha \geq d$ in our proof of the theorem. Note that obtaining a full dimensional set in the case $\alpha = d$, however, is still interesting.

(2) Theorem 28 is trivial when $\alpha = dn$, since we can set $X = \emptyset$. We will therefore assume that $\alpha < dn$ in our proof of the theorem.

(3) When Z is a countable union of smooth manifolds in $\mathbf{R}^{nd}$ of co-dimension m, we have $\alpha = nd - m$. In this case Theorem 28 yields a set in $\mathbf{R}^d$ with Hausdorff dimension at least $(nd-\alpha)/(n-1) = m/(n-1)$. This recovers Theorem 1.1 and 1.2 from [6], making Theorem 28 a generalization of these results.

(4) Since Theorem 28 does not require any regularity assumptions on the set Z, it can be applied in contexts that cannot be addressed using previous methods. Two such applications, new to the best of our knowledge, have been recorded in Section 4.4; see Theorems 36 and 37 there.

The set X in Theorem 28 is obtained as a limit of discretized sets $\{X_k\}$, each of which avoids a certain discretization of the configuration $\mathcal{C}$ at the scale l_k. While this proof strategy is not new, our method for constructing the sets $\{X_k\}$ has several innovations that simplify the analysis of the resulting set $X = \bigcap X_k$. In particular, through a probabilistic selection process we are able to avoid the complicated queuing techniques used in [9] and [6], that required storage of data from each step of the iterated construction to be retrieved at a much later stage of the construction process.

At the same time, our construction shares certain features with [6], in particular, the strategy of iterative discretization. The details of a single step of this construction are described in Section 4.1. In Section 4.2, we explain how the length scales l_k and r_k for X are chosen, and prove its avoidance property. In Section 4.3 we analyze the size of X and show that it satisfies the conclusions of Theorem 28.

4.1 Avoidance at Discrete Scales

In this section we describe a method for avoiding Z at a single scale. We apply this technique in Section 4.2 at many scales to construct a set X avoiding Z at all scales. This single scale avoidance technique is the core building block of our construction, and the efficiency with which we can avoid Z at a single scale has direct consequences on the Hausdorff dimension of the set X constructed obtained in Theorem 28.

At a single scale, we solve a discretized version of the problem, where all sets are unions of cubes at two dyadic lengths $l > s$. In this discrete setting, Z is replaced by a discretized version of itself, a union of cubes in $\mathcal{B}_s^{dn}$ denoted by G. Given a set E, which is a union of cubes in $\mathcal{B}_l^d$, our goal is to construct a set $F \subset E$ that is a union of cubes in $\mathcal{B}_s^d$, such that F^n is disjoint from strongly non-diagonal cubes (see Definition **??**) in $\mathcal{B}_s^{dn}(G)$. Using the setup introduced at the end of the introduction, we will later choose $l = l_k$, $s = l_{k+1}$, and $E = X_k$. The set X_{k+1} will be defined as the set F constructed.

In order to ensure the final set X obtained in Theorem 28 has large Hausdorff dimension regardless of the rapid decay of scales used in the construction of X, it is crucial that F is uniformly distributed at intermediate scales between l and s. We achieve this by decomposing E into sub-cubes in $\mathcal{B}_r^d$ for some intermediate scale $r \in [s,l]$, and distributing F as evenly among these intermediate sub-cubes as possible. This is possible assuming a mild regularity condition on the number of cubes in G, i.e. Equation (4.1).

Lemma 29. *Fix two distinct dyadic lengths l and s, with $l > s$. Let $E \subseteq [0,1)^d$ be a nonempty union of cubes in $\mathcal{B}_l^d$, and let $G \subset \mathbf{R}^{dn}$ be a nonempty union of cubes in $\mathcal{B}_s^{dn}$ such that*

$$(l/s)^d \leq \#\mathcal{B}_s^{dn}(G) \leq \frac{1}{2}(l/s)^{dn}. \tag{4.1}$$

Then there exists a dyadic length $r \in [s,l]$ of size

$$r \sim \left(l^{-d} s^{dn} \#\mathcal{B}_s^{dn}(G)\right)^{\frac{1}{d(n-1)}}, \tag{4.2}$$

and a set $F \subset E$ that is a nonempty union of cubes in $\mathcal{B}_s^d(E)$ satisfying the following three properties:

(A) Avoidance*: For any choice of distinct cubes $J_1, \dots, J_n \in \mathcal{B}_s^d(F)$, $J_1 \times \cdots \times J_n \notin \mathcal{B}_s^{dn}(G)$.*

(B) Non-Concentration*: For every $I' \in \mathcal{B}_r^d(E)$, there is at most one $J \in \mathcal{B}_s^d(F)$ with $J \subset I'$.*

(C) Large Size*: For every $I \in \mathcal{B}_l^d(E)$, $\#\mathcal{B}_s^d(F \cap I) \geq \#\mathcal{B}_r^d(I)/2 = (l/r)^d/2$.*

Remark. *Property (A) says that F avoids strongly non-diagonal cubes in $\mathcal{B}_s^{dn}(G)$. Properties (B) and (C) together imply that for every $I \in \mathcal{B}_l^d(E)$, at least half the cubes $I' \in \mathcal{B}_r^d(I)$ contribute a single sub-cube of sidelength s to F; the rest contribute none.*

Proof. Let r be the smallest dyadic length at least as large as R, where

$$R = \left(2l^{-d} s^{dn} \# \mathcal{B}_s^{dn}(G)\right)^{\frac{1}{d(n-1)}}. \tag{4.3}$$

This choice of r satisfies (4.2). The inequalities in (4.1) ensure that $r \in [s,l]$; more precisely, the left inequality in (4.1) implies R is bounded from below by s, and the right inequality implies R is bounded from above by l. The minimality of r ensures $s \leq r \leq l$.

For each $I' \in \mathcal{B}_r^d(E)$, let $J_{I'}$ be an element of $\mathcal{B}_s^d(I)$ chosen uniformly at random; these choices are independent as I' ranges over the elements of $\mathcal{B}_r^d(E)$. Define

$$U = \bigcup \left\{ J_{I'} : \, I' \in \mathcal{B}_r^d(E) \right\},$$

and

$$\mathcal{K}(U) = \{K \in \mathcal{B}_s^{dn}(G) : \, K \in U^n, K \text{ strongly non-diagonal}\}.$$

Note that the sets U and $\mathcal{K}(U)$ are random sets, in the sense that they depend on the random variables $\{J_{I'}\}$. Define

$$F(U) = U - \{\pi(K) : \, K \in \mathcal{K}(U)\}, \tag{4.4}$$

where $\pi \colon \mathbf{R}^{dn} \to \mathbf{R}^d$ is the projection map $(x_1, \dots, x_n) \mapsto x_1$, for $x_i \in \mathbf{R}^d$. Thus π sends the cube $J_1 \times \cdots \times J_n \in \mathcal{B}_s^{dn}$ to the cube $J_1 \in \mathcal{B}_s^d$. Given any strongly non-diagonal cube $K = J_1 \times \cdots \times J_n \in \mathcal{B}_s^{dn}(G)$, either $K \notin \mathcal{B}_s^{dn}(U^n)$, or $K \in \mathcal{B}_s^{dn}(U^n)$.

If the former occurs then $K \notin \mathcal{B}_s^{dn}(F(U)^n)$ since $F(U) \subset U$, so $\mathcal{B}_s^{dn}(F(U)^n) \subset \mathcal{B}_s^{dn}(U^n)$. If the latter occurs then $K \in \mathcal{K}(U)$, and since $\pi(K) = J_1$, $J_1 \notin \mathcal{B}_s^d(F(U))$. In either case, $K \notin \mathcal{B}_s^{dn}(F(U)^n)$, so $F(U)$ satisfies Property (A). By construction, U contains at most one subcube $J \in \mathcal{B}_s^{dn}$ for each $I \in \mathcal{B}_l^{dn}(E)$. Since $F(U) \subset U$, $F(U)$ satisfies Property (B). Thus the set $F(U)$ satisfies Properties (A) and (B) regardless of which values are assumed by the random variables $\{J_{I'}\}$. Next we will show that with non-zero probability, the set $F(U)$ satisfies Property (C).

For each cube $J \in \mathcal{B}_s^d(E)$, there is a unique 'parent' cube $I' \in \mathcal{B}_r^d(E)$ such that $J \subset I'$. Since I' contains $(r/s)^d$ cubes of sidelength s, and $J_{I'}$ is chosen uniformly at random from $\mathcal{B}_s^d(I')$,

$$\mathbf{P}(J \subset U) = \mathbf{P}(J_{I'} = J) = (s/r)^d.$$

The cubes $J_{I'}$ are chosen independently, so if $J_1, \dots, J_n$ are distinct cubes in $\mathcal{B}_s^d(E)$, then

$$\mathbf{P}(J_1, \dots, J_n \in U) = \begin{cases} (s/r)^{dn} & \text{if } J_1, \dots, J_n \text{ have distinct parents,} \\ 0 & \text{otherwise.} \end{cases} \tag{4.5}$$

Let $K = J_1 \times \cdots \times J_n \in \mathcal{B}_s^{dn}(G)$. If the cubes $J_1, \dots, J_n$ are distinct, we deduce from (4.5) that

$$\mathbf{P}(K \subset U^n) = \mathbf{P}(J_1, \dots, J_n \in U) \leq (s/r)^{dn}. \tag{4.6}$$

By (4.6), linearity of expectation, and (4.3),

$$\mathbf{E}(\#\mathcal{K}(U)) = \sum_{K \in \mathcal{B}_s^{dn}(G)} \mathbf{P}(K \subset U^n) \leq \#\mathcal{B}_s^{dn}(G) \cdot (s/r)^{dn} \leq 0.5 \cdot (l/r)^d.$$

In particular, there exists at least one (non-random) set U_0 such that

$$\#\mathcal{K}(U_0) \leq \mathbf{E}(\#\mathcal{K}(U)) \leq 0.5 \cdot (l/r)^d. \tag{4.7}$$

In other words, $F(U_0) \subset U_0$ is obtained by removing at most $0.5 \cdot (l/r)^d$ cubes in $\mathcal{B}_s^d$ from U_0. For each $I \in \mathcal{B}_l^d(E)$, we know that $\#\mathcal{B}_s^d(I \cap U_0) = (l/r)^d$. Combining this with (4.7), we arrive at the estimate

$$\begin{aligned}\#\mathcal{B}_s^d(I \cap F(U_0)) &= \mathcal{B}_s^d(I \cap U_0) - \#\{\pi(K) : K \in \mathcal{K}(U_0), \pi(K) \in F(U_0)\} \\ &\geq \mathcal{B}_s^d(I \cap U_0) - \#(\mathcal{K}(U_0)) \\ &\geq (l/r)^d - 0.5 \cdot (l/r)^d \geq 0.5 \cdot (l/r)^d\end{aligned}$$

In other words, $F(U_0)$ satisfies Property (C). Setting $F = F(U_0)$ completes the proof. $\square$

Remarks.

(1) While Lemma 29 uses probabilistic arguments, the conclusion of the lemma is not a probabilistic statement. In particular, one can find a suitable F constructively by checking every possible choice of U (there are finitely many) to find one particular choice U_0 which satisfies (4.7)*, and then defining F by* (4.4)*. Thus the set we obtain in Theorem 28 exists by purely constructive means.*

(2) At this point, it is possible to motivate the numerology behind the dimension bound $\dim(X) \geq (dn - \alpha)/(n-1)$ *from Theorem 28, albeit in the context of Minkowski dimension. We will pause to do so here before returning to the proof of Theorem 28. For simplicity, let $\alpha > d$, and suppose that $Z \subset \mathbf{R}^{dn}$ satisfies*

$$\#\mathcal{B}_s^{dn}(Z) \sim s^{-\alpha} \quad \textit{for every } s \in (0,1]. \tag{4.8}$$

Let $l = 1$, $E = [0,1)^d$, and let $s > 0$ be a small parameter. If s is chosen sufficiently small compared to d, n, and α, then (4.1) *is satisfied with $G = \bigcup \mathcal{B}_s^{dn}(Z)$. We can then apply Lemma 29 to find a dyadic scale $r \sim s^{(dn-\alpha)/d(n-1)}$ and a set F that avoids the strongly non-diagonal cubes of $\mathcal{B}_s^{dn}(Z)$. The set F is a union of approximately $r^{-d} \sim s^{-(dn-\alpha)/(n-1)}$ cubes* *of sidelength s. Thus informally, the set F resembles a set with Minkowski dimension α when viewed at scale s.*

The set X constructed in Theorem 28 will be obtained by applying Lemma 29 iteratively at many scales. At each of these scales, X will resemble a set of Minkowski dimension $(dn-\alpha)/(n-1)$. A careful analysis of the construction (performed in Section 4.3) shows that X actually has Hausdorff dimension at least $(dn-\alpha)/(n-1)$.

(3) *Lemma 29 is the core method in our avoidance technique. The remaining argument is fairly modular. If, for a special case of Z, one can improve the result of Lemma 29 so that r is chosen on the order of $s^{\beta/d}$, then the remaining parts of our paper can be applied near verbatim to yield a set X with Hausdorff dimension β, as in Theorem 28.*

4.2 Fractal Discretization

In this section we construct the set X from Theorem 28 by applying Lemma 29 at many scales. Let us start by fixing a strong cover Z that we will work with in the sequel.

Lemma 30. *Let $Z \subset \mathbf{R}^{dn}$ be a countable union of bounded sets with Minkowski dimension at most α, and let $\varepsilon_k \searrow 0$ with $2\varepsilon_k < dn-\alpha$ for all k. Then there exists a sequence of dyadic lengths $\{l_k\}$ and a strong cover of Z by a sequence of sets $\{Z_k\}$, such that*

(A) Discreteness*: For all $k \geq 0$, Z_k is a union of cubes in $\mathcal{B}^{dn}_{l_k}$.*

(B) Sparsity*: For all $k \geq 0$, $l_k^{-d} \leq \#\mathcal{B}^{dn}_{l_k}(Z_k) \leq l_k^{-\alpha-\varepsilon_k}$.*

(C) Rapid Decay*: For all $k > 1$,*

$$l_k^{dn-\alpha-\varepsilon_k} \leq 0.5 \cdot l_{k-1}^{dn}, \tag{4.9}$$

$$l_k^{\varepsilon_k} \leq l_{k-1}^{2d}. \tag{4.10}$$

Proof. We can write $Z = \bigcup_{i=1}^{\infty} Y_i$, with $\dim_{\underline{\mathbf{M}}}(Y_i) \le \alpha$ for each i. Consider the d dimensional hyperplane

$$H = \{(x_1, \ldots, x_1) : \; x_1 \in [0,1)^d\}.$$

Let $Y_i' = Y_i \cup H$. In particular, this means for any l,

$$\#\mathcal{B}_l^{nd}(Y_i') \ge \#\mathcal{B}_l^{nd}(H) = l^{-d}. \tag{4.11}$$

Note that $\dim_{\underline{\mathbf{M}}}(Y_i') \le \alpha$ for each index i. Let $\{i_k\}$ be a sequence of integers that repeats each integer infinitely often.

The lengths $\{l_k\}$ and sets $\{Z_k\}$ are defined inductively. As a base case, set $l_0 = 1$ and $Z_0 = [0,1)^d$. Suppose that the lengths $l_0, \ldots, l_{k-1}$ have been chosen. Since $\dim_{\underline{\mathbf{M}}}(Y_{i_k}) \le \alpha$, Definition **??** implies that there exists arbitrarily small lengths l that satisfy

$$\#\mathcal{B}_l^{dn}(Y_{i_k}') \le l^{-\alpha - \frac{\varepsilon_k}{4}}.$$

Since $\varepsilon_k > 0$, this means that there exist arbitrarily small dyadic lengths l that satisfy

$$\#\mathcal{B}_l^{dn}(Y_{i_k}') \le l^{-\alpha - \frac{\varepsilon_k}{2}}. \tag{4.12}$$

In particular, we can choose a dyadic length $l = l_k$ small enough to satisfy (4.9), (4.10), and (4.12). With this choice of l_k, we have that Property (C) is satisfied. Define Z_k to be the union of the cubes in $\mathcal{B}_{l_k}^{dn}(Y_{i_k}')$. This choice of Z_k clearly satisfies Property (A), and Property (B) is implied by (4.11) and (4.12).

It remains to verify that the sets $\{Z_k\}$ strongly cover Z. Fix a point $z \in Z$. Then there exists an index i such that $z \in Y_i$, and there is a subsequence $k_1, k_2, \ldots$ such that $i_{k_j} = i$ for each j. But then $z \in Y_i \subset Y_i' \subset Z_{i_{k_j}}$, so z is contained in each of the sets $Z_{i_{k_j}}$, and thus $z \in \limsup Z_i$. □

To construct X, we consider a nested, decreasing family of discretized sets $\{X_k\}$, where X_k is a union of cubes in $\mathcal{B}_{l_k}^d(X_k)$. We then set $X = \bigcap X_k$. The goal is to choose X_k such that X_k^n is disjoint from *strongly non diagonal* cubes in Z_k.

Lemma 31. *Let $Z \subset \mathbf{R}^{dn}$, let $\{Z_k\}$ be a sequence of sets that strongly cover Z, and let $\{l_k\}$ be a sequence of lengths converging to zero. For each index k, let X_k be a union of cubes in $\mathcal{B}_{l_k}^d$. Suppose that for each k, X_k^n avoids strongly non-diagonal cubes in $\mathcal{B}_{l_k}^{dn}(Z_k)$. If $X = \bigcap X_k$, then for any distinct $x_1, \dots, x_n \in X$, we have $(x_1, \dots, x_n) \notin Z$.*

Proof. Let $z \in Z$ be a point with distinct coordinates $z_1, \dots, z_n$. Define

$$\Delta = \{(w_1, \dots, w_n) \in \mathbf{R}^{dn} : \text{ there exists } i \neq j \text{ such that } w_i = w_j\}.$$

Then $d(\Delta, z) > 0$, where d is the Hausdorff distance between Δ and z. Since $\{Z_k\}$ strongly covers Z, there is a subsequence $\{k_m\}$ such that $z \in Z_{k_m}$ for every index m. Since l_k converges to 0 and thus l_{k_m} converges to 0, if m is sufficiently large then $\sqrt{dn} \cdot l_{k_m} < d(\Delta, z)$. Note that $\sqrt{dn} \cdot l_{km}$ is the diameter of a cube in $\mathcal{B}_{l_{km}}^{dn}$. For such a choice of m, if $I \in \mathcal{B}_{l_{km}}^{dn}(Z_{k_m})$ is the (unique) cube in $\mathcal{B}_{l_{km}}^{dn}$ containing z, then $I \cap \Delta = \emptyset$. But this means I is strongly non-diagonal. Since X_{k_m} avoids the strongly non-diagonal cubes of Z_{k_m}, we conclude that $z \notin X_{k_m}^n$. In particular, this means $z \notin X^n$. □

All that remains is to apply the discrete lemma to choose the sets X_k.

Lemma 32. *Given a sequence of dyadic length scales $\{l_k\}$ obeying,* (4.9), (4.10), *and* (4.12) *as above, there exists a sequence of sets $\{X_k\}$ and a sequence of dyadic intermediate scales $\{r_k\}$ with $l_k \leq r_k \leq l_{k-1}$ for each $k \geq 1$, such that each set X_k is a union of cubes in $\mathcal{B}_{l_k}^d(X_{k-1})$ that avoids the strongly non-diagonal cubes of $\mathcal{B}_{l_k}^{dn}(Z_k)$. Furthermore, for each index $k \geq 1$ we have*

$$r_k \lesssim l_k^{(dn-\alpha-\varepsilon_k)/d(n-1)}, \tag{4.13}$$

$$\#\mathcal{B}_{l_k}^d(X_k \cap I) \geq 0.5 \cdot (l_{k-1}/r_k)^d \quad \textit{for each } I \in \mathcal{B}_{l_{k-1}}^d(X_{k-1}), \tag{4.14}$$

$$\#\mathcal{B}_{l_k}^d(X_k \cap I') \leq 1 \quad \textit{for each } I' \in \mathcal{B}_{r_k}^d(X_{k-1}). \tag{4.15}$$

Proof. We construct X_k by induction, using Lemma 29 at each step. Set $X_0 = [0,1)^d$. Next, suppose that the sets $X_0, \dots, X_{k-1}$ have been defined. Our goal is to apply Lemma 29 to $E = X_{k-1}$ and $G = Z_k$ with $l = l_{k-1}$ and $s = l_k$. This will be possible once we verify the hypothesis (4.1), which in this case takes the form

$$(l_{k-1}/l_k)^d \leq \#\mathcal{B}^{dn}_{l_k}(Z_k) \leq 0.5 \cdot (l_{k-1}/l_k)^{dn}. \tag{4.16}$$

The right hand side follows from Property (B) of Lemma 30 and (4.10). On the other hand, Property (B) and the fact that $l_{k-1} \leq 1$ implies that

$$(l_{k-1}/l_k)^d \leq l_k^{-d} \leq \#\mathcal{B}^{dn}_{l_k}(Z_k),$$

establishing the left inequality in (4.16). Applying Lemma 29 as described above now produces a dyadic length

$$r \sim \left(l_{k-1}^{-d} l_k^{dn} \#\mathcal{B}^{dn}_{l_k}(Z_k)\right)^{\frac{1}{d(n-1)}} \tag{4.17}$$

and a set $F \subset X_{k-1}$ that is a union of cubes in $\mathcal{B}^d_{l_k}$. The set F satisfies Properties (A), (B), and (C) from the statement of Lemma 29. Define $r_k = r$ and $X_k = F$. The estimate (4.13) on r_k follows from (4.17) using the known bounds (4.10) and (4.12):

$$r_k \lesssim \left(l_{k-1}^{-d} l_k^{dn-\alpha-0.5\varepsilon_k}\right)^{\frac{1}{d(n-1)}} = \left(l_{k-1}^{-d} l_k^{0.5\varepsilon_k} l_k^{dn-\alpha-\varepsilon_k}\right)^{\frac{1}{d(n-1)}} = \left(l_{k-1}^{-2d} l_k^{\varepsilon_k}\right)^{\frac{1}{2d(n-1)}} l_k^{\frac{dn-\alpha-\varepsilon_k}{d(n-1)}} \lesssim l_k^{\frac{dn-\alpha-\varepsilon_k}{d(n-1)}}.$$

The requirements (4.14) and (4.15) follow from Properties (B) and (C) of Lemma 29 respectively. □

Now we have defined the sets $\{X_k\}$, we set $X = \bigcap X_k$. Since X_k avoids strongly non-diagonal cubes in Z_k, Lemma 31 implies that if $x_1, \dots, x_n \in X$ are distinct, then $(x_1, \dots, x_n) \notin Z$. To finish the proof of Theorem 28, we must show that $\dim_{\mathbf{H}}(X) \geq (dn - \alpha)/(n-1)$. This will be done in the next section.

4.3 Dimension Bounds

To complete the proof of Theorem 28, we must show that $\dim_{\mathbf{H}}(X) \geq (dn - \alpha)/(n-1)$. In view of Definition **??**, we will do this by constructing a Frostman measure of appropriate dimension supported on X.

We start by defining a premeasure on $\bigcup_{i=1}^{\infty} \mathcal{B}_{l_i}^d[0,1)^d$. Set $\mu([0,1)^d) = 1$. Suppose now that $\mu(I)$ has been defined for all cubes in $\mathcal{B}_{l_{k-1}}^d[0,1)^d$, and let $J \in \mathcal{B}_{l_k}^d$. Consider the unique 'parent cube' $I \in \mathcal{B}_{l_{k-1}}^d$ for which $J \subset I$. Define

$$\mu(J) = \begin{cases} \mu(I)/\#\mathcal{B}_{l_k}^d(X_k \cap I) & \text{if } J \subset X_k, \\ 0 & \text{otherwise.} \end{cases} \tag{4.18}$$

Observe that for each index $k \geq 1$ and each $I \in \mathcal{B}_{l_{k-1}}^d$,

$$\sum_{J \in \mathcal{B}_{l_k}^d(I)} \mu(J) = \sum_{J \in \mathcal{B}_{l_k}^d(X_k \cap I)} \mu(J) = \mu(I). \tag{4.19}$$

In particular, for each index k we have

$$\sum_{I \in \mathcal{B}_{l_k}} \mu(I) = 1.$$

By a standard argument involving the Caratheodory extension theorem [5, Proposition 1.7], the premeasure μ extends to a measure on the Borel subsets of $[0,1)^d$. Note that for each $k \geq 1$, μ is supported on X_k. Thus μ is supported on $\bigcap X_k = X$. To complete the proof of Theorem 28 we will show that μ is a Frostman measure of dimension $(dn-\alpha)/(n-1) - \varepsilon$ for every $\varepsilon > 0$.

Lemma 33. *For each $k \geq 1$ and each $J \in \mathcal{B}_{l_k}^d(X)$,*

$$\mu(J) \lesssim l_k^{\frac{dn-\alpha}{n-1} - \eta_k}, \quad \textit{where} \quad \eta_k = \frac{n+1}{2(n-1)} \cdot \varepsilon_k \searrow 0 \textit{ as } k \to \infty.$$

Proof. Let $J \in \mathcal{B}_{l_k}^d$ and let $I \in \mathcal{B}_{l_{k-1}}^d$ be the parent cube of J. Since μ is a probability measure, we have $\mu(I) \leq 1$. Combining (4.18), (4.14), (4.13), and (4.10) we obtain

$$\mu(J) \leq \frac{2r_k^d}{l_{k-1}^d}\mu(I) \leq \frac{2r_k^d}{l_{k-1}^d} \lesssim \frac{l_k^{\frac{dn-\alpha-\varepsilon_k}{n-1}}}{l_{k-1}^d} = l_k^{\frac{dn-\alpha}{n-1}-\eta_k}\left(l_k^{0.5\varepsilon_k}/l_{k-1}^d\right) \leq l_k^{\frac{dn-\alpha}{n-1}-\eta_k}. \quad \square$$

Corollary 34. *For each $k \geq 1$ and each $I' \in \mathcal{B}_{r_k}^d(X_{k-1})$,*

$$\mu(I') \lesssim (r_k/l_{k-1})^d l_{k-1}^{\frac{dn-\alpha}{n-1}-\eta_{k-1}}. \tag{4.20}$$

Proof. Let us fix a cube $I' \in \mathcal{B}_{r_k}^d(X_{k-1})$, and let I denote its unique parent cube in $\mathcal{B}_{l_{k-1}}^d(X_{k-1})$. According to (4.15), I' contains at most one cube in $\mathcal{B}_{l_k}^d(I)$; let us denote this cube by J if it exists. Then the mass distribution rule given by (4.18) dictates that zero

$$\mu(I') = \mu(X_k \cap I') = \begin{cases} \mu(J) = \mu(I)/\#\mathcal{B}_{l_k}^d(X_k \cap I) & \text{if } \#\mathcal{B}_{l_k}^d(X_k \cap I') = 1, \\ 0 & \text{if } \#\mathcal{B}_{l_k}^d(X_k \cap I') = 0. \end{cases}$$

Using the estimate (4.14) and applying Lemma 33 to $I \in \mathcal{B}_{l_{k-1}}^d(X)$, we arrive at the claimed bound (4.20). $\square$

Lemma 33 and Corollary 34 allow us to control the behavior of μ at all scales.

Lemma 35. *For every $\alpha \in [d, dn)$, and for each $\varepsilon > 0$, there is a constant C_ε so that for all dyadic lengths $l \in (0,1]$ and all $I \in \mathcal{B}_l^d$, we have*

$$\mu(I) \leq C_\varepsilon l^{\frac{dn-\alpha}{n-1}-\varepsilon}. \tag{4.21}$$

Proof. Fix $\varepsilon > 0$. Since $\eta_k \searrow 0$ as $k \to \infty$, there is a constant C_ε so that $l_k^{-\eta_k} \leq C_\varepsilon l_k^{-\varepsilon}$ for each $k \geq 1$. For instance, if ε_k is decreasing, we could choose $C_\varepsilon = l_{k_0}^{-\eta_{k_0}}$, where k_0 is the largest integer for which $\eta_{k_0} \geq \varepsilon$. Next, let k be the (unique) index so that $l_{k+1} \leq l < l_k$. We will split the proof of (4.21) into two cases, depending on the position of l within $[l_{k+1}, l_k]$.

Case 1: If $r_{k+1} \le l \le l_k$, we can cover I by $(l/r_{k+1})^d$ cubes in $\mathcal{B}^d_{r_{k+1}}$. By Corollary 34,

$$
\begin{aligned}
\mu(I) &\lesssim (l/r_{k+1})^d (r_{k+1}/l_k)^d l_k^{\frac{dn-\alpha}{n-1}-\eta_k} \\
&= (l/l_k)^d l_k^{\frac{dn-\alpha}{n-1}-\eta_k} \\
&= l^{\frac{dn-\alpha}{n-1}} (l/l_k)^{\frac{\alpha-d}{n-1}} l_k^{-\eta_k} \\
&\le l^{\frac{dn-\alpha}{n-1}-\eta_k} \\
&\le C_\varepsilon l^{\frac{dn-\alpha}{n-1}-\varepsilon}.
\end{aligned}
\tag{4.22}
$$

The penultimate inequality is a consequence of our assumption $\alpha \ge d$.

Case 2: If $l_{k+1} \le l \le r_{k+1}$, we can cover I by a single cube in $\mathcal{B}^d_{r_{k+1}}$. By (4.15), each cube in $\mathcal{B}^d_{r_{k+1}}$ contains at most one cube $I_0 \in \mathcal{B}^d_{l_{k+1}}(X_{k+1})$, so by Lemma 33,

$$
\mu(I) \le \mu(I_0) \lesssim l_{k+1}^{\frac{dn-\alpha}{n-1}-\eta_{k+1}} \le C_\varepsilon l_{k+1}^{\frac{dn-\alpha}{n-1}-\varepsilon} \le C_\varepsilon l^{\frac{dn-\alpha}{n-1}-\varepsilon}. \qquad \square
$$

Applying Frostman's lemma to Lemma 35 gives $\dim_{\mathbf{H}}(X) \ge \frac{dn-\alpha}{n-1} - \varepsilon$ for every $\varepsilon > 0$, which concludes the proof of Theorem 28.

4.4 Applications

As discussed in the introduction, Theorem 28 generalizes Theorems 1.1 and 1.2 from [6]. In this section, we present two applications of Theorem 28 in settings where previous methods do not yield any results.

4.4.1 Sum-sets avoiding specified sets

Theorem 36. *Let $Y \subset \mathbf{R}^d$ be a countable union of sets of Minkowski dimension at most $\beta < d$. Then there exists a set $X \subset \mathbf{R}^d$ with Hausdorff dimension at least $d - \beta$ such that $X + X$ is disjoint from Y.*

Proof. Define $Z = Z_1 \cup Z_2$, where

$$Z_1 = \{(x,y) : x+y \in Y\} \quad \text{and} \quad Z_2 = \{(x,y) : y \in Y/2\}.$$

Since Y is a countable union of sets of Minkowski dimension at most β, Z is a countable union of sets with lower Minkowski dimension at most $d+\beta$. Applying Theorem 28 with $n=2$ and $\alpha = d+\beta$ produces a set $X \subset \mathbf{R}^d$ with Hausdorff dimension $2d-(d+\beta) = d-\beta$ such that $(x,y) \notin Z$ for all $x,y \in X$ with $x \neq y$. We claim that $X+X$ is disjoint from Y. To see this, first suppose $x,y \in X$, $x \neq y$. Since X avoids Z_1, we conclude that $x+y \notin Y$. Suppose now that $x = y \in X$. Since X avoids Z_2, we deduce that $X \cap (Y/2) = \emptyset$, and thus for any $x \in X$, $x+x = 2x \notin Y$. This completes the proof. $\square$

4.4.2 Subsets of Lipschitz curves avoiding isosceles triangles

4.4.3 Subsets of Lipschitz curves avoiding isosceles triangles

In [6], Fraser and the second author prove that there exists a set $S \subset [0,1]$ with dimension $\log_3 2$ such that for any simple C^2 curve $\gamma\colon [0,1] \to \mathbf{R}^n$ with bounded non-vanishing curvature, $\gamma(S)$ does not contain the vertices of an isosceles triangle. Our method enables us to obtain a result that works for a Lipschitz curve. Currently, we are able to provide a slightly worse dimensional bound ($1/2$ instead of $\log_3 2$), and are unable to ensure uniformity across all Lipschitz curves with a given Lipschitz constant.

Theorem 37. *Let $f\colon [0,1] \to \mathbf{R}^{n-1}$ be Lipschitz with $\|f\|_{Lip} < 1$. Then there is a set $X \subset [0,1]$ of Hausdorff dimension $1/2$ so that the set $\{(t, f(t)) : t \in X\}$ does not contain the vertices of an isosceles triangle.*

Proof. Set

$$Z = \left\{ (x_1,x_2,x_3) \in [0,1]^3 : \begin{array}{c} (x_1, f(x_1)), (x_2, f(x_2)), (x_3, f(x_3)) \\ \text{form the vertices of an isosceles triangle} \end{array} \right\}.$$

In the next lemma, we show Z has lower Minkowski dimension at most two. By Theorem 28, there is a set $X_1 \subset [0,1]$ of Hausdorff dimension $1/2$ so that for each distinct $x_1, x_2, x_3 \in X_1$, we have $(x_1, x_2, x_3) \notin Z$. This is precisely the statement that for each $x_1, x_2, x_3 \in X$, the points $(x_1, f(x_1))$, $(x_2, f(x_2))$, and $(x_3, f(x_3))$ do not form the vertices of an isosceles triangle. To complete the proof, let $X = \{(x, f(x)) : x \in X_1\}$. $\square$

Lemma 38. *Let $f\colon [0,1] \to \mathbf{R}^{n-1}$ be Lipschitz with $\|f\|_{Lip} < 1$. Then the set*

$$Z = \left\{ (x_1, x_2, x_3) \in [0,1]^3 : \begin{array}{c} (x_1, f(x_1)), (x_2, f(x_2)), (x_3, f(x_3)) \\ \textit{form the vertices of an isosceles triangle} \end{array} \right\}$$

has Minkowski dimension at most two.

Proof. First, notice that three points $p, q, r \in \mathbf{R}^n$ form an isosceles triangle, with r as the apex, if and only if $r \in H_{p,q}$, where

$$H_{p,q} = \left\{ x \in \mathbf{R}^n : \left(x - \frac{p+q}{2} \right) \cdot (q-p) = 0 \right\}.$$

To prove Z has Minkowski has dimension at most two, it suffices to show that the set

$$W = \left\{ x \in [0,1]^3 : (x_3, f(x_3)) \in H_{(x_1, f(x_1)),(x_2, f(x_2))} \right\}$$

has Minkowski dimension at most 2, because Z is the union of three copies of W, obtained by permuting coordinates.

Fix $0 < \delta < 1$, and consider $I_1, I_2 \in \mathcal{B}^1_\delta[0,1]$, together with an integer $k > 0$ such that $d(I_1, I_2) = k \cdot \delta$. Let x_1 be the midpoint of I_1, and x_2 the midpoint of I_2. Suppose $(y_1, y_2, y_3) \in W \cap I_1 \times I_2 \times [0,1]$. Then

$$\left(y_3 - \frac{y_1 + y_2}{2} \right) \cdot (y_2 - y_1) + \left(f(y_3) - \frac{f(y_2) + f(y_1)}{2} \right) \cdot (f(y_2) - f(y_1)) = 0.$$

We know $|x_1 - y_1|, |x_2 - y_2| \leq \delta/2$, so

$$\begin{aligned}
&\left|\left(y_3 - \frac{y_1+y_2}{2}\right)(y_2 - y_1) - \left(y_3 - \frac{x_1+x_2}{2}\right)(x_2 - x_1)\right| \\
&\quad \leq \frac{|y_1 - x_1| + |y_2 - x_2|}{2}|y_2 - y_1| + \Big(|y_1 - x_1| + |y_2 - x_2|\Big)\left|y_3 - \frac{x_1+x_2}{2}\right| \\
&\quad \leq (\delta/2)\cdot 1 + \delta \cdot 1 \leq 3\delta/2.
\end{aligned} \tag{4.23}$$

Conversely, we know $|f(x_1) - f(y_1)|, |f(x_2) - f(y_2)| \leq \delta/2$ because $\|f\|_{\mathrm{Lip}} < 1$, and a similar calculation yields

$$\begin{aligned}
&\left|\left(f(y_3) - \frac{f(y_1)+f(y_2)}{2}\right)\cdot(f(y_2) - f(y_1))\right. \\
&\qquad \left. - \left(f(y_3) - \frac{f(x_1)+f(x_2)}{2}\right)\cdot(f(x_2) - f(x_1))\right| \leq 3\delta/2.
\end{aligned} \tag{4.24}$$

Putting (4.23) and (4.24) together, we conclude that

$$\left|\left(y_3 - \frac{x_1+x_2}{2}\right)(x_2 - x_1) + \left(f(y_3) - \frac{f(x_2)+f(x_1)}{2}\right)\cdot(f(x_2) - f(x_1))\right| \leq 3\delta. \tag{4.25}$$

Since $|(x_2 - x_1, f(x_2) - f(x_1))| \geq k\delta$, we can interpret (4.25) as saying (y_1, y_2, y_3) is contained in a $3/k$ thickening of the hyperplane $H_{(x_1, f(x_1)),(x_2, f(x_2))}$. Given another y' contained in a $3/k$ thickening of the hyperplane, it satisfies a variant of the inequality (4.25), we can subtract the difference between the two inequalities to conclude

$$\left|\left(y_3 - y_3'\right)(x_2 - x_1) + (f(y_3) - f(y_3'))\cdot(f(x_2) - f(x_1))\right| \leq 6\delta. \tag{4.26}$$

The triangle difference inequality applied with (4.26) implies

$$(f(y_3)-f(y_3'))\cdot(f(x_2)-f(x_1)) \geq |y_3-y_3'||x_2-x_1|-6\delta \geq k\delta\cdot|y_3-y_3'|-6\delta. \tag{4.27}$$

Conversely,

$$(f(y_3)-f(y_3'))\cdot(f(x_2)-f(x_1)) \leq \|f\|_{\mathrm{Lip}}^2|y_3-y_3'||x_2-x_1| \leq \|f\|_{\mathrm{Lip}}^2(k+1)\delta|y_3-y_3'|. \tag{4.28}$$

Combining (4.27) and (4.28) and rearranging gives that if k is sufficiently large, depending only on $\|f\|_{\mathrm{Lip}}$,

$$|y_3-y_3'| \leq \frac{6}{k-(k+1)\|f\|_{\mathrm{Lip}}^2} \lesssim 1/k \tag{4.29}$$

so $\#\mathcal{B}_\delta^3(W\cap I_1\times I_2\times[0,1])\lesssim 1/k$. If k is too small to use this bound, we just conclude the trivial bound that $\#\mathcal{B}_\delta^3(W\cap I_1\times I_2\times[0,1])\lesssim 1/\delta$.

For each value of k, there are at most $O(1/\delta)$ pairs (I_1,I_2) with $d(I_1,I_2)=k\delta$. And the maximum value of k for any pairs I_1,I_2 is $O(1/\delta)$. Thus

$$\begin{aligned}
\#\mathcal{B}_\delta^3(W) &= \sum_{k=0}^{O(1)}\sum_{d(I_1,I_2)=k\delta}\#\mathcal{B}_\delta^3(W\cap I_1\times I_2\times[0,1]) \\
&\quad+\sum_{k=O(1)}^{O(1/\delta)}\sum_{d(I_1,I_2)=k\delta}\#\mathcal{B}_\delta^3(W\cap I_1\cap I_2\times[0,1]) \\
&= O(1)\cdot O(1/\delta)\cdot O(1/\delta)+O\left(\sum_{k=O(1)}^{O(1/\delta)}1/k\right)\cdot O(1/\delta)\cdot O(1/\delta) \\
&= O(\log(1/\delta)\cdot(1/\delta^2)).
\end{aligned}$$

This shows W has upper Minkowski dimension at most 2. $\square$

Chapter 5

Extensions to Low Rank Configurations

5.1 Boosting the Dimension of Pattern Avoiding Sets by Low Rank Coordinate Changes

We now consider finding subsets of $[0,1]$ avoiding solutions to the equation $y = f(Tx)$, where T is a rank k linear transformation with integer coefficients with respect to standard coordinates, and f is real-valued and Lipschitz continuous. Fix a constant A bounding the operator norm of T, in the sense that $|Tx| \leq A|x|$ for all $x \in \mathbf{R}^n$, and a constant B such that $|f(x+y) - f(x)| \leq B|y|$ for all x and y for which the equation makes sense (if f is C^1, this is equivalent to a bound $\|\nabla f\|_\infty \leq B$). Consider sets $J_0, J_1, \dots, J_n, \subset [0,1]$, which are unions of intervals of length $1/M$, with startpoints lying on integer multiples of $1/M$. The next theorem works as a 'building block lemma' used in our algorithm for constructing a set avoiding solutions to the equation with Hausdorff dimension k and full Minkowski dimension.

Theorem 39. *For infinitely many integers N, there exists $S_i \subset J_i$ avoiding solutions to $y = f(Tx)$ with $y \in S_0$ and $x_n \in S_n$, such that*

- *For $n \neq 0$, if we decompose each J_i into length $1/N$ consecutive intervals, S_i contains an initial portion $\Omega(1/N^k)$ of each length $1/N$ interval. This part of the decomposition gives the Hausdorff dimension $1/k$ bound for the set we will construct.*

- *If we decompose J_i into length $1/N$ intervals, and then subdivide these intervals into length $\Omega(1/N^k)$ intervals, then S_0 contains a subcollection of these $1/N^k$ intervals which contains a total length $\Omega(1/N)$ of a fraction $1-1/M$ of the length $1/N$ intervals. This property gives that our resultant set will have full Minkowski dimension.*

The implicit constants in these bounds depend only on A, B, n, and k.

Proof. Split each interval of J_a into length $1/N$ intervals, and then set

$$\mathbf{A} = \{x : x_a \text{ is a startpoint of a } 1/N \text{ interval in } J_a\}$$

Since the startpoints of the intervals are integer multiples of $1/N$, $T(\mathbf{A})$ is contained with a rank k sublattice of $(\mathbf{Z}/N)^m$. The operator norm also guarantees $T(\mathbf{A})$ is contained within the ball B_A of radius A in $\mathbf{R}^m$. Because of the lattice structure of the image, $|x-y| \gtrsim_n 1/N$ for each distinct pair $x, y \in T(\mathbf{A})$. For any R, we can cover $\Sigma \cap B_A$ by $O_{n,k}((A/R)^k)$ balls of radius R. If $R \gtrsim_n 1/N$, then each ball can contain only a single element of $T(\mathbf{A})$, so we conclude that $|T(\mathbf{A})| \lesssim_{n,k} (AN)^k$. If we define the set of 'bad points' to be

$$\mathbf{B} = \{y \in [0,1] : \text{there is } x \in \mathbf{A} \text{ such that } y = f(T(x))\}$$

Then

$$|\mathbf{B}| = |f(T(\mathbf{A}))| \leq |T(\mathbf{A})| = O_{A,n,k}(N^k)$$

For simplicity, we now introduce an integer constant $C_0 = C_0(A,n,k,M)$ such that $|\mathbf{B}| \leq (C_0/M^2)N^k$. We now split each length $1/M$ interval in J_0 into length $1/N$ intervals, and filter out those intervals containing more than $C_0 N^{k-1}$ elements of

B. Because of the cardinality bound we have on **B** there can be at most N/M^2 such intervals, so we discard at most a fraction $1/M$ of any particular length $1/M$ interval in J_0. If we now dissect the remaining intervals into $4C_0N^{k-1}$ intervals of length $1/4C_0N^k$, and discard any intervals containing an element of **B**, or adjacent to such an interval, then the remaining such intervals I satisfy $d(I,\mathbf{B}) \geq 1/4C_0N^k \gtrsim_{A,n,M,k} (1/N^k)$, and because of our bound on the number of elements of **B** in these intervals, there are at least C_0N^{k-1} intervals remaining, with total length exceeding $C_0N^{k-1}/4C_0N^k = \Omega(1/N)$. If f is C^1 with $\|\nabla f\|_\infty \leq B$, or more generally, if f is Lipschitz continuous of magnitude B, then

$$|f(Tx) - f(Tx')| \leq AB|x - x'|$$

and so we may choose $S_i \subset J_i$ by thickening each startpoint $x \in J_i$ to a length $O(1/N^k)$ interval while still avoiding solutions to the equation $y = f(T(x))$. $\square$

Remark. *If T is a rank k linear transformation with rational coefficients, then there is some number a such that aT has integer coefficients, and then the equation $y = f(Tx)$ is the same as the equation $y = f_0((aT)(x))$, where $f_0(x) = f(x)/a$. Since f_0 is also Lipschitz continuous, we conclude that we still get the dimension $1/k$ bound if T has rational rather than integral coefficients. More generally, this trick shows the result applies unperturbed if all coefficients of T are integer multiples of some fixed real number. More generally, by varying the lengths of our length $1/N$ decomposition by a constant amount, we can further generalize this to the case where each column of T are integers multiples of some fixed real number.*

Remark. *To form* **A**, *we take startpoints lying at equal spaced $1/N$ points. However, by instead taking startpoints at varying points in the length $1/N$ intervals, we might be able to make points cluster more than in the original algorithm. Maybe the probabalistic method would be able to guarantee the existence of a choice of startpoints whose images are tightly clustered together.*

Remark. *Since the condition $y = f(Tx)$ automatically assumes a kind of 'non-vanishing derivative' condition on our solutions, we do not need to assume the* *regularity of f, and so the theorem extends naturally to a more general class of functions than Rob's result, i.e. the Lipschitz continuous functions.*

Using essentially the same approach as the last argument shows that we can avoid solutions to $y = f(Tx)$, where y and x are now vectors in some $\mathbf{R}^m$, and T has rank k. If we consider unions of $1/M$ cubes $J_0, \dots, J_n$. If we fix startpoints of each x_k forming lattice spaced apart by $\Omega(1/N)$, and consider the space $\mathbf{A}$ of products, then there are $O(N^k)$ points in $T(\mathbf{A})$, and so there are $O(N^k)$ elements in $\mathbf{B}$. We now split each $1/M$ cube in J_0 into length $1/N$ cubes, and discard those cubes which contain more than $O(N^{k-m})$ bad points, then we discard at most $1 - 1/M$ of all such cubes. We can dissect the remaining length $1/N$ cubes into $O(N^{k-m})$ length $\Omega(1/N^{k/m})$ cubes, and as in the previous argument, the cubes not containing elements of $\mathbf{B}$ nor adjacent to an element have total volume $\Omega(1/N^m)$, which we keep. The startpoints in the other intervals T_i may then be thickened to a length $\Omega(1/N^{m/k})$ portion while still avoiding solutions. This gives a set with full Minkowski dimension and Hausdorff dimension m/k avoiding solutions to $y = f(Tx)$. (I don't yet understand Minkowski dimension enough to understand this, but the techniques of the appendix make proving the Hausdorff dimension $1/k$ bound easy)

5.2 Extension to Well Approximable Numbers

If the coefficients of the linear transformation T in the equation $y = f(Tx)$ are non-rational, then the images of startpoints under the action of T do not form a lattice, and so points may not overlap so easily when avoiding solutions to the equations $y = f(Tx)$. However, if T is 'very close' to a family of rational coefficient linear transformations, then we can show the images of the startpoints are 'very close' to a lattice, which will still enable us to find points avoiding solutions by replacing the direct combinatorial approach in the argument for integer matrices with a covering argument.

Suppose that T is a real-coefficient linear transformation with the property that for each coefficient x there are infinitely many rational numbers p/q with

$|x - p/q| \leq 1/q^\alpha$, for some fixed α. For infinitely many K, we can therefore find a linear transformation S with coefficients in $\mathbf{Z}/K$ with each coefficient of T differing from the corresponding coefficient in S by at most $1/K^\alpha$. Then for each x, we find

$$\|(T-S)(x)\|_\infty \leq (n/K^\alpha)\|x\|_\infty$$

If we now consider $T_0, \dots, T_n$, splitting $T_1, \dots, T_n$ into length $1/N$ intervals, and considering $\mathbf{A}$ as in the last section, then $S(\mathbf{A})$ lie in a k dimensional sublattice of $(\mathbf{Z}/KN)^m$, hence containing at most $(2A)^k(KN)^k = O_{T,n}((KN)^k)$ points. By our error term calculation of $T-S$, the elements of $T(\mathbf{A})$ are contained in cubes centered at these lattice points with side-lengths $2n/K^\alpha$, or balls centered at these points with radius $n^{3/2}/K^\alpha$. If $\|\nabla f\| \leq B$, then the images of the radius $n^{3/2}/K^\alpha$ balls under the action of f are contained in length $Bn^{3/2}/K^\alpha$ intervals. Thus the total length of the image of all these balls under f is $(Bn^{3/2}/K^\alpha)(2A)^k(KN)^k = (2A)^k Bn^{3/2}K^{k-\alpha}N^k$. If $k < \alpha$, then we can take K arbitrarily large, so that there exists intervals with $\text{dist}(I, \mathbf{B}) = \Omega_{A,k,M}(1/N^k)$. But I believe that, after adding the explicit constants in, we cannot let $k = \alpha$.

Remark. *One problem is that, if T has rank k, we might not be able to choose S to be rank k as well. Is this a problem? If T has full rank, then the set of all such matrices is open so if T and S are close enough, S also has rank k, but this need not be true if T does not have full rank.*

Example. *If T has rank 1, then Dirichlet's theorem says that every irrational number x can be approximated by infinitely many p/q with $|x - p/q| < 1/q^2$, so every real-valued rank 1 linear transformation can be avoided with a dimension one bound.*

5.3 Applications of Low Rank Coordinate Changes

Example. *Our initial exploration of low rank coordinate changes was inspired by trying to find solutions to the equation*

$$y - x = (u - w)^2$$

Our algorithm gives a Hausdorff dimension $1/2$ *set avoiding solutions to this equation. This equals Mathé's result. But this dimension for us now depends on the shifts involved in the equation, not on the exponent, so we can actually avoid solutions to the equation*

$$y - x = (u - w)^n$$

for any n, in a set of Hausdorff dimension $1/2$. *More generally, if* X *is a set, then given a smooth function* f *of* n *variables, we can find a set* X *of Hausdorff dimension* $1/n$ *such that there is no* $x \in X$, *and* $y_1, \dots, y_n \in X - X$ *such that* $x = f(y_1, \dots, y_n)$. *This is better than the* $1/2n$ *bound that is obtained by Malabika and Fraser's result.*

Example. *For any fixed m, we can find a set* $X \subset \mathbf{R}^n$ *of full Hausdorff dimension which contains no solutions to*

$$a_1 x_1 + \dots + a_n x_n = 0$$

for any rational numbers a_n *which are not all zero. Since Malabika/Fraser's technique's solutions are bounded by the number of variables, they cannot let* $n \to \infty$ *to obtain a linearly independant set over the rational numbers. But since the Hausdorff dimension of our sets now only depends on the rank of* T, *rather than the total number of variables in* T, *we can let* $n \to \infty$ *to obtain full sets linearly independant over the rationals. More generally, for any Lipschitz continuous function* $f : \mathbf{R} \to \mathbf{R}$, *we can find a full Hausdorff dimensional set such that there are no solutions*

$$f(a_1 x_1 + \dots + a_n x_n, y)$$

for any n, and for any rational numbers a_n that are not all zero.

Example. *The easiest applications of the low rank coordinate change method are probably involving configuration problems involving pairwise distances between m points in $\mathbf{R}^n$, where $m \ll n$, since this can best take advantage of our rank condition. Perhaps one way to encompass this is to avoid m vertex polyhedra in n dimensional space, where $m \ll n$. In order to distinguish this problem from something that can be solved from Mathé's approach, we can probably find a high dimensional set avoiding m vertex polyhedra on a parameterized n dimensional manifold, where $m \ll n$. There is a result in projective geometry which says that every projectively invariant property of m points in $\mathbf{RP}^d$ is expressible as a function in the $\binom{m}{d}$ bracket polynomials with respect to these m points. In particular, our result says that we can avoid a countable collection of such invariants in a dimension $1/\binom{m}{d}$ set. This is a better choice of coordinates than Euclidean coordinates if $\binom{m}{d} \leq m$. Update: I don't think this is ever the case.*

Remark. *Because of how we construct our set X, we can find a dimension $1/k$ set avoiding solutions to $y = f(Tx)$ for all rank k rational matrices T, without losing any Hausdorff dimension. Maybe this will help us avoid solutions to more general problems?*

Example. *Given a smooth curve Γ in $\mathbf{R}^n$, can we find a subset E with high Hausdorff dimension avoiding isoceles triangles. That is, if the curve is parameterized by $\gamma : [0,1] \to \mathbf{R}^n$, can we find $E \subset [0,1]$ such that for any t_1, t_2, t_3, $\gamma(t_1)$, $\gamma(t_2)$, and $\gamma(t_3)$ do not form the vertices of an isoceles triangle. This is, in a sense, a non-linear generalization of sets avoiding arithmetic progressions, since if Γ is a line, an isoceles triangle is given by arithmetic progressions. Assuming our curve is simple, we must avoid zeroes of the function*

$$|\gamma(t_1) - \gamma(t_2)|^2 = |\gamma(t_2) - \gamma(t_3)|^2$$

If we take a sufficiently small segment of this curve, and we assume the curve has non-zero curvature on this curve, we can assume that $t_1 < t_2 < t_3$ in our dissection *method.*

If the coordinates of γ are given by polynomials with maximum degree d, then the equation

$$|\gamma(t_1) - \gamma(t_2)|^2 - |\gamma(t_2) - \gamma(t_3)|^2$$

is a polynomial of degree $2d$, and so Mathé's result gives a set of dimension $1/2d$ avoiding isoceles triangles. In the case where Γ is a line, then the function $f(t_1,t_2,t_3) = \gamma(t_1) + \gamma(t_3) - 2\gamma(t_2)$ avoids arithmetic progresions, and Mathé's result gives a dimension one set avoiding such progressions. Rob and Malabika's algorithm easily gives a set with dimension $1/2$ for any curve Γ. Our algorithm doesn't seem to be able to do much better here.

Example. *What is the largest dimension of a set in Euclidean space such that for any value λ, there is at most one pair of points x,y in the set such that $|x-y| = \lambda$.*

Example. *What is the largest dimension of a set which avoids certain angles, i.e. for which a triplet x,y,z avoids certain planar configurations.*

Example. *A set of points $x_0,\dots,x_d \in \mathbf{R}^d$ lie in a hyperplane if and only if the determinant formed by the vectors $x_n - x_0$, for $n \in \{1,\dots,d\}$, is zero. This is a degree d polynomial, hence Mathé's result gives a dimension one set with no set of $d+1$ points lying in a hyperplane. On the other hand, a theorem of Mattila shows that every analytic set E with dimension exceeding one contains $d+1$ points in a hyperplane. Can we generalize this to a more general example avoiding points on a rotational, translation invariant family of manifolds using our results?*

Example. *Given a set F not containing the origin, what is the largest Hausdorff dimension of a set E such that for any for any distinct rational $a_1,\dots,a_N$, the sum $a_1E + \dots + a_NE$ does not contain any elements of F. Thus the vector space over the rationals generated by E does not contain any elements of F. We can also take the non-linear values $f(a_1E + \dots + a_NE)$ avoiding elements of F. F must have non-empty interior for the problem to be interesting. Then can we find a smooth function f with non-nanishing derivative which vanishes over F, or a family of smooth functions with non-vanishing derivative around F.*

Chapter 6

Fourier Dimension and Fractal Avoidance

There is much interplay between geometric measure theory and Fourier analysis. In particular, the Frostman dimension of a measure μ can be related to a integrability condition on it's Fourier transform. If we replace this integrability condition with a uniform bound on the decay of the Fourier transform, we obtain a stricter requirement. This leads to the *Fourier dimension* of a measure. A natural question then arises; do sets supporting measures with large Fourier dimension naturally contain patterns?

6.1 Basic Technique

Theorem 40. *Let K be a collection of at most $A \cdot N^s$, strongly non-diagonal sidelength $1/N$ dyadic cubes in $[0,1]^{nd}$. Then there exists a random set $X \subset [0,1]^d$, which is a union of sidelength $1/N$ dyadic cubes in $[0,1]^d$, and a constant C, depending only on A, d, and n, such that with probability $1 - C/\log N$,*

- $X^d \cap K = \emptyset$,
- *If μ is the probability measure equal to a constant multiple of the Lebesgue*

measure on X, and supported on X, then for any $m \in \{-N,\dots,N\}^d$,

$$|\widehat{\mu}(m) - \widehat{dx}(m)| \leq \frac{C(\log N)^{1-1/n}}{N^{d-s/n}}.$$

Proof. Fix $p = 1/(AN^s \log N)^{1/n}$. Let $\{X_Q\}$ be a family of independant and identically distributed $\{0,1\}$ valued Bernoulli random variable, for each sidelength $1/N$ dyadic cube, such that $\mathbf{P}(X_Q = 1) = p$ for each Q, and define $X = \bigcup\{Q : X_Q = 1\}$. Then Chernoff's inequality implies there exists a universal constant c such that if $S = \sum_Q X_Q$, then

$$\mathbf{P}\left(S \geq (1/2)(N^d p)\right) \geq 1 - 2e^{-cN^d p}.$$

Substituting in parameters, we conclude

$$\mathbf{P}\left(S \geq \left[1/2A^{1/n}(\log N)^{1/n}\right] \cdot N^{d-s/n}\right) \geq 1 - 2\exp\left(\frac{-N^{d-s/n}}{A^{1/n}(\log N)^{1/n}}\right) \quad (6.1)$$

Thus X is the union of a large number of cubes with high probability.

Given any cube $Q_1 \times \cdots \times Q_n \in K$, we have

$$\mathbf{P}(Q_1 \times \dots Q_n \subset X) = \mathbf{P}(Q_1 \in X, \dots, Q_n \in X) = p^n.$$

Thus

$$\mathbf{E}(\#\{Q_1 \times \cdots \times Q_n \in X^n\}) \leq AN^s p^n = 1/\log N.$$

In particular, Markov's inequality implies

$$\begin{aligned} \mathbf{P}(\#\{Q_1 \times \cdots \times Q_n \in X^n\} = 0) &= \mathbf{P}(\#\{Q_1 \times \cdots \times Q_n \in X^n\} < 1) \\ &\geq 1 - 1/\log N. \end{aligned} \quad (6.2)$$

Thus X^d avoids K completely with high probability.

Now we analyze the Fourier decay of the set X, conditioning on the value of

S. For each cube Q, define

$$f_Q = \begin{cases} \left(\frac{N^d}{S} - 1\right) \cdot \mathbf{I}_Q & : X_Q = 1, \\ (-1) \cdot \mathbf{I}_Q & : X_Q = 0. \end{cases}$$

If $f = \sum_Q f_Q$, then $f\,dx = d\mu - dx$. For each x, $\mathbf{E}[f_Q(x)|S] = 0$. If $x \notin Q$, this is obvious, since $f_Q(x) = 0$ always holds, and if $x \in Q$, we find

$$\begin{aligned} \mathbf{E}[f_Q(x)|S] &= \mathbf{P}(X_Q = 1|S)(N^d/S - 1) + \mathbf{P}(X_Q = 0|S)(-1) \\ &= (S/N^d)(N^d/S - 1) + (1 - S/N^d)(-1) = 0, \end{aligned}$$

This implies that for each $m \in \mathbf{Z}^d$,

$$\mathbf{E}\left[\widehat{f_Q}(m)|S\right] = \int \mathbf{E}[f_Q(x)|S] e^{-2\pi i x \cdot m}\, dx = 0.$$

Now $|\widehat{f_Q}(m)| \leq 1/S$ for all Q, and since the family $\left\{\widehat{f_Q}(m)\right\}$ are independent random variables as Q varies over all dyadic cubes, we can apply Hoeffding's inequality to conclude that there exists a universal constant c such that for the random variable $\widehat{f}(m) = \sum \widehat{f_Q}(m)$,

$$\mathbf{P}\left(|\widehat{f}(m)| \geq \log N/S\right) \leq 2/N^{c \log N}$$

If we now take a union bound over all $m \in \{-N, \dots, N\}^d$, we can guarantee that

$$\mathbf{P}\left(|\widehat{f}(m)| \leq \log N/S \text{ for all } m \in \{-N, \dots, N\}^d\right) \geq 1 - 2^{d+1}/N^{c \log N - d}. \quad (6.3)$$

Thus $\widehat{f}$ has good decay with high probability.

Combining (6.1), (6.2), and (6.3), we conclude that there exists a constant C such that with probability at least

$$1 - 2\exp\left(\frac{-N^{d-s/n}}{A^{1/n}(\log N)^{1/n}}\right) - 1/\log N - \frac{2^{d+1}}{N^{c \log N - d}} \geq 1 - C/\log N,$$

the set X avoids K, and for all $m \in \{-N,\dots,N\}^d$,

$$|\widehat{f}(m)| \leq \frac{C(\log N)^{1-1/n}}{N^{d-s/n}}. \qquad \square$$

6.2 Another Try

In the last few chapters, we have discovered that the presence of many configurations is not guaranteed by large Hausdorff dimension. We now turn to an analysis of a different dimensional quantity which in many cases gives much more structure than Hausdorff dimension. We construct a set E

Proof. Let $S_0 = \{0\}$. Then, given S_k, let N^s

$$S_{k+1} = \left\{(j, mN_{k+1}^{1-\alpha} + X_{jk}) : j \in S_k, 0 \leq m < N_{k+1}^{\alpha}\right\}$$

where X_{jk} is chosen uniformly randomly from all integers in $[0, N_k^{\alpha})$. Write $E_k = \{\bigcup Q_j : j \in S_k\}$. Then $\#(S_{k+1}) = N_{k+1}^{\alpha}\#(S_k)$, and since $\#(S_0) = 1$, we find $\#(S_k) = (N_1 \dots N_k)^{\alpha}$. Now define $E_k = \{\bigcup Q_j : j \in S_k\}$, and probability measures

$$f_k = (N_1 \dots N_k)^{1-\alpha}\mathbf{I}_{E_k}.$$

Then $\widehat{f_{k+1}}(m) - \widehat{f_k}(m) = \sum_{j \in S_k} Y_j(m)$, where

$$Y_j(m) = \int_{I_j} (f_{k+1}(x) - f_k(x))e(mx)\, dx$$

Note that $\mathbf{E}[f_{k+1}(x)] = \mathbf{E}[f_k(x)]$ for each x, so that $\mathbf{E}[Y_j(m)] = 0$. Note also that

$$|Y_j(m)| \leq 2l_k/l_{k+1}^{1-\alpha} = \frac{2N_{k+1}^{1-\alpha}}{(N_1 \dots N_k)^{\alpha}}.$$

Thus Hoeffding's inequality implies

$$\mathbf{P}\left(|\widehat{f_{k+1}}(m)-\widehat{f}_k(m)|\geq t\right)\leq 4\exp\left(-\left(t\frac{(N_1\dots N_k)^{\alpha}}{4N_{k+1}^{1-\alpha}\#(S_k)^{1/2}}\right)^2\right)$$
$$\leq 4\exp\left(-\left(t\frac{(N_1\dots N_k)^{\alpha/2}}{2N_{k+1}^{1-\alpha}}\right)^2\right)$$

Thus we can guarantee with high probability that for all m,

$$|\widehat{f_{k+1}}(m)-\widehat{f}_k(m)|\lessapprox\frac{N_{k+1}^{1-\alpha}}{(N_1\dots N_k)^{\alpha/2}}$$

So if $N_{k+1}\lesssim(N_1\dots N_k)^{\varepsilon}$ for all $\varepsilon>0$, we obtain the right bound. But we need something deeper.

Take 2: Write

$$f_{k+1}(x)=(N_1\dots N_{k+1})^{1-\alpha}\sum_{j\in S_k}\sum_{m=0}^{N_{k+1}^{1-\alpha}}Y_{jm}$$

where Y_{jm} is the indicator function of

$$\left[\frac{j}{N_1\dots N_k}+\frac{m}{N_1\dots N_{k+1}^{\alpha}}+\frac{X_{jk}}{N_1\dots N_{k+1}},\frac{j}{N_1\dots N_k}+\frac{m}{N_1\dots N_{k+1}^{\alpha}}+\frac{X_{jk}+1}{N_1\dots N_{k+1}}\right]$$

Note that

$$\widehat{Y_{jm}}(n)=\frac{1}{2\pi in}e\left(\frac{jn}{N_1\dots N_k}+\frac{mn}{N_1\dots N_{k+1}^{\alpha}}+\frac{X_{jk}n}{N_1\dots N_{k+1}}\right)\left(e\left(\frac{n}{N_1\dots N_{k+1}}\right)-1\right),$$

and

$$\mathbf{E}\left(\widehat{Y_{jm}}(n)\right)=\frac{1}{2\pi inN_{k+1}^{\alpha}}e\left(\frac{jn}{N_1\dots N_k}+\frac{mn}{N_1\dots N_{k+1}^{\alpha}}\right)\left(e\left(\frac{N_{k+1}^{\alpha}n}{N_1\dots N_{k+1}}\right)-1\right).$$

and $|\widehat{Y_{jm}}| \leq 1/(N_1 \dots N_{k+1})$. Thus Hoeffding's inequality implies, if we ignore the expectation,

$$\begin{aligned}\mathbf{P}\left(|\widehat{f_{k+1}}(n)| \geq t(N_1 \dots N_{k+1})^{1-\alpha}\right) &\leq 4\exp\left(\frac{-t^2(N_1 \dots N_{k+1})^2}{4(N_1 \dots N_{k+1})^{\alpha}}\right) \\ &\leq 4\exp\left(-\left(t(N_1 \dots N_{k+1})^{1-\alpha/2}/2\right)^2\right).\end{aligned}$$

Thus with high probability, for all $|n| \leq N_1 \dots N_{k+1}$,

$$|\widehat{f_{k+1}}(n)| \lessapprox \frac{1}{(N_1 \dots N_{k+1})^{\alpha/2}}.$$

□

6.3 Schmerkin's Method

We now try and adapt Schmerkin's Method to our situation, introducing translations into the avoiding set. Let $K \subset \mathbf{Z}_N^n$. Our goal is to find $X \subset \mathbf{Z}_N$ such that if $x_1, \dots, x_n$ are distinct, then $(x_1, \dots, x_n) \notin K$, such that if $f : \mathbf{Z}_N \to \mathbf{R}$ is the probability distribution which uniformly randomly chooses a point on X, such that the Fourier transform of f has small L^∞ norm.

We consider an intermediary scale M, with $M\,|\,N$. Then we consider the set

$$K' = \{x \in \mathbf{Z}_N^n : \text{There is } y \in K \text{ with } x - y \in M\mathbf{Z}_N^n\}.$$

We construct X such that if $x_1, \dots, x_n \in X$ are distinct, then $(x_1, \dots, x_n) \notin Y$. In particular, this means that if we define, for each $x \in X$, a uniformly random selected integer $n_x \in M\mathbf{Z}_N$, then we can define $x' = x + n_x$, and $X' = \{x' : x \in X\}$. Now if $I_x : \mathbf{Z}_N \to \mathbf{R}$ denotes the delta function at x', then

$$\widehat{I_x}(k) = (1/N)e(-kx'/N) = (1/N)e(-kx/N)e(-n_x).$$

And so

$$\mathbf{E}(\widehat{I_x}(k)) = (M/N)e(-kx/N)\sum_{m=1}^{N/M} e(-(kM/N)m).$$

If $kM/N \notin \mathbf{Z}$, $\mathbf{E}(\widehat{I_x}(k)) = 0$. If $kM/N \in \mathbf{Z}$, then $\mathbf{E}(\widehat{I_x}(k)) = e(-kx/N)$. And

$$|\widehat{I_x}(k)| \leq 1.$$

Since I_x is independant, we can apply Hoeffding's inequality to conclude that

$$\mathbf{P}\left(\widehat{f}(k) \geq t\right) \leq \exp\left(\frac{-t^2}{4|X|}\right)$$

so $\widehat{f}(k) \lesssim |X|^{1/2}$ with high probability. This should give the Fourier dimension bound we want? Except we do lose some dimension by introducing the parameter M. If we pick N to be very large, but M very small, then this shouldn't introduce many problems? What about if we pick M randomly to prevent problems on the linear frequencies?

6.4 Tail Bounds on Cube Selection

The aim of this section is to obtain tail bounds on the total number of intersections between the random set we select in the discrete selection schema, with the goal of obtaining a Fourier dimension bound on the set we construct. Consider the random process generating our interval selection scheme. We have three fixed lengths l, r, and s. We start with a set E, which is a non-empty union of cubes in $\mathcal{B}_l^d$, and a set G, which is a union of cubes in $\mathcal{B}_s^{dn}$. For each cube $J \in \mathcal{B}_r^d(E)$, we select a single random element of $\mathcal{B}_s^d(J)$, denoted J_I. We let $U = \bigcup J_I$. The aim of this section is to obtain tail bounds on the size of the random set

$$\mathcal{K}(U) = \{K \in \mathcal{B}_s^{dn}(G) : K \in U^n, K \text{ strongly non-diagonal}\}.$$

Here, this is obtained by applying the statistical decoupling techniques of Bourgain and Tzafriri.

To understand the distribution of this random variable, it will be convenient to simplify notation. We write $\mathbf{I}(I_J = I_0)$ as X_{J,I_0}. We then set

$$Z = \#\mathcal{K}(U) = \sum_{J_1,\dots,J_n} \sum_K \left(\prod_{i=1}^n X_{J_i,K_i} \right),$$

where $J_1,\dots,J_n$ ranges over all distinct choices of n elements from $\mathcal{B}_l^d(E)$, and $K = K_1 \times \cdots \times K_n$ ranges over all elements of $\mathcal{B}_l^{dn}(G) \cap \mathcal{B}_s^{dn}(J_1 \times \cdots \times J_n)$. The random cubes $\{I_J\}$ form an independant family, which should make the distribution easier to bound. But Z is a multiplicative linear combination of this independant family. Random variables of this form as known as a *chaos*.

The idea behind statistical decoupling is to obtain tail bounds on the chaos Z by studying the tail bounds on the modified random variable

$$W = \sum_{J_1,\dots,J_n} \sum_K \left(\prod_{i=1}^n X^i_{J_i,K_i} \right),$$

where $X^1,\dots,X^n$ are independant and identically distributed copies of X.

Lemma 41. *Let $F : [0,\infty) \to [0,\infty)$ be a monotone, convex function. Then*

$$\mathbf{E}(F(Z)) \leq \mathbf{E}(F(n^n W)).$$

Proof. Consider a partition $\mathcal{J}_1,\dots,\mathcal{J}_n$ selected at random from all possible partitions of $\mathcal{B}_s^d(E)$ into n classes. Then, for any distinct set of intervals $J_1,\dots,J_n \in \mathcal{B}_s^d(E)$,

$$\mathbf{P}(J_1 \in \mathcal{J}_1,\dots,J_n \in \mathcal{J}_n) = \mathbf{P}(J_1 \in \mathcal{J}_1)\dots\mathbf{P}(J_n \in \mathcal{J}_n) = 1/n^n. \tag{6.4}$$

Thus if we consider

$$Z' = \sum_{J_1 \in \mathcal{J}_1, \dots, J_n \in \mathcal{J}_n} \sum_K \left(\prod_{i=1}^n X_{J_i, K_i} \right),$$

where J_i ranges over elements of $\mathcal{J}_i$ for each $i \in [n]$. If we let $\mathbf{E}_{\mathcal{J}}$ denote conditioning with respect to all random variables except those depending on $\mathcal{J}_1, \dots, \mathcal{J}_n$, then (6.4) implies

$$\mathbf{E}_{\mathcal{J}}(Z') = \sum_{J_1, \dots, J_n} \sum_K \mathbf{P}(J_1 \in \mathcal{J}_1, \dots, J_n \in \mathcal{J}_n) \left(\prod_{i=1}^n X_{J_i, K_i} \right) = Z/n^n.$$

Thus Jensen's inequality implies that

$$F(Z) = F(n^n \mathbf{E}_{\mathcal{J}}(Z')) \leq \mathbf{E}_{\mathcal{J}} F(n^n Z').$$

In particular, this means we can select a single, non-random partition $\mathcal{J}_1, \dots, \mathcal{J}_n$ of $\mathcal{B}_s^d(E)$ such that $F(Z) \leq F(n^n Z')$. Because $\mathcal{J}_1, \dots, \mathcal{J}_n$ are disjoint sets, Z' is identically distributed to

$$W' = \sum_{J_1 \in \mathcal{J}_1, \dots, J_n \in \mathcal{J}_n} \sum_K \left(\prod_{i=1}^n X^i_{J_i, K_i} \right).$$

But $W' \leq W$, and so by monotonicity,

$$\mathbf{E}(F(Z)) \leq \mathbf{E}(F(n^n W')) \leq \mathbf{E}(F(n^n W)). \qquad \square$$

Lemma 42. *MOMENT BOUND THEOREM.*

Proof. We now bound the moments of W, which by Lemma 41 bounds the moments of Z. We write

$$W^m = \sum_{J_{j,i}} \sum_{K_1, \dots, K_m} \left(\prod_{i=1}^n \prod_{j=1}^m X^i_{J_{j,i}, K_{j,i}} \right).$$

Thus

$$\mathbf{E}(W^m) = \sum_{J_{j,i}} \sum_{K_1,\ldots,K_m} \prod_{i=1}^{n} \mathbf{E}\left(\prod_{j=1}^{m} X^i_{J_{j,i},K_{j,i}}\right)$$

The inner expectation is zero except if $\pi(K_{j,i}) = J_{j,i}$ for all j and i, and also, if $\pi(K_{j_0,i}) = \pi(K_{j_1,i})$, then $K_{j_0,i} = K_{j_1,i}$. And then the expectation is $\varepsilon^{\#\{K_{1,i},\ldots,K_{m,i}\}}$.

□

6.5 Singular Value Decomposition of Tensors

An *n tensor* with respect to $\mathbf{R}^{k_1},\ldots,\mathbf{R}^{k_n}$, is a multi-linear function

$$A : \mathbf{R}^{k_1} \times \cdots \times \mathbf{R}^{k_n} \to \mathbf{R}.$$

The family of all n tensors forms a vector space, denoted $\mathbf{R}^{k_1} \otimes \cdots \otimes \mathbf{R}^{k_n}$, or $\bigotimes \mathbf{R}^{k_i}$. Given a family of vectors v_i, with $v_i \in \mathbf{R}^{k_i}$, we can consider a tensor $v_1 \otimes \cdots \otimes v_n$, such that

$$(v_1 \otimes \cdots \otimes v_n)(w_1,\ldots,w_n) = (v_1 \cdot w_1)\ldots(v_n \cdot w_n).$$

It is easily verified that a basis for $\mathbf{R}^{k_1} \otimes \cdots \otimes \mathbf{R}^{k_n}$ is given by

$$\{e_{j_1} \otimes \cdots \otimes e_{j_n} : j_i \in [k_i]\}.$$

Thus a tensor A can be identified with a family of coefficients

$$\{A_{j_1\ldots j_n} : 1 \le j_i \le k_i\}$$

such that

$$A(v_1,\ldots,v_n) = \sum A_{j_1\ldots j_n} \cdot e_{j_1} \otimes \cdots \otimes e_{j_n}.$$

From this perspective, a tensor is a multi-dimensional array given by n indices. This basis induces a natural inner product for which the basis is orthonormal. The norm with respect to this inner product is called the *Frobenius norm*, denoted $\|A\|_F$.

Recall that if A is a rank r, $m \times n$ matrix, then the *singular value decomposition* says that there are orthonormal families of vectors $\{u_1, \dots, u_r\}$ and $\{v_1, \dots, v_r\}$ in $\mathbf{R}^m$ and $\mathbf{R}^n$ respectively, and values $s_1, \dots, s_r > 0$ such that $A = \sum s_i (u_i^T v_i)$. We may view A as a 2 tensor in $\mathbf{R}^m \otimes \mathbf{R}^n$ by setting $A(v,w) = v^T A w$. Then the singular value decomposition says that $A = \sum s_i (u_i \otimes v_i)$. Thus, viewing A as a bilinear function, for any vectors $x \in \mathbf{R}^m$ and y in $\mathbf{R}^n$, $A(x,y) = \sum s_i (u_i \cdot x)(v_i \cdot y)$. We want to try and generalize this result to n tensors in $\mathbf{R}^{k_1} \otimes \dots \otimes \mathbf{R}^{k_n}$.

One can reduce certain quantities over tensors to quantities defined with respect to linear maps. For instance, for each index $i_0 \in [n]$, we can define a linear map

$$A_i : \bigotimes_{i' \neq i} \mathbf{R}^{k_{i'}} \to \mathbf{R}^{k_i}$$

by letting $A_{i_0}(v_1, \dots, \widehat{v_i}, \dots, v_n)$ to be the unique vector $w \in \mathbf{R}^{k_i}$ such that for any vector $v_i \in \mathbf{R}^{k_i}$,

$$v_i \cdot w = A(v_1, \dots, v_n).$$

The matrix A_i is known as the *i'th unfolding of A*. We define the i rank of A to be $\text{rank}_i(A) = \text{rank}(A_i)$. This generalizes the column and row rank of matrices. Unlike in the case of matrices, these quantities can differ for different indices i. One can also obtain the i rank

The natural definition of the rank of a tensor in the study of the higher order singular value decomposition is defined in analogy with the fact that a rank r matrix can be written as the sum of r rank one matrices. We say a tensor is rank one if it can be written in the form $v_1 \otimes \dots \otimes v_n$ for some vectors $v_i \in \mathbf{R}^{k_i}$. The *rank* of a tensor A, denoted $\text{rank}(A)$, is the minimal number of rank one tensors which can be written in a linear combination to form A. We have $\text{rank}(A_i) \leq \text{rank}(A)$ for each i, but there may be a gap in these ranks.

To obtain an SVD decomposition, fix an index i. Then we can perform a singular value decomposition on A_i, obtaining orthonormal basis

6.6 Decoupling Concentration Bounds

We require some methods in non-asymptotic probability theory in order to understand the concentration structure of the random quantities we study. Most novel of these is relying on a probabilistic decoupling bound of Bourgain.

In this section, we fix a single underlying probability space, assumed large enough for us to produce any additional random quantities used in this section. Given a scalar valued random variable X, we define its *Orlicz norm* as

$$\|X\|_{\psi_2} = \inf\left\{\lambda \geq 0 : \mathbf{E}\left(\exp\left(|X|^2/\lambda^2\right)\right) \leq 2\right\}.$$

The family of random variables with finite Orlicz norm forms a Banach space, known as the space of *subgaussian random variables*.

Theorem 43. *Let X be a sub-gaussian random variable.*

- *For all $t \geq 0$,*
$$\mathbf{P}(|X| \geq t) \leq 2\exp\left(\frac{-t^2}{\|X\|_{\psi_2}^2}\right).$$
- *Conversely, if there is a constant $K > 0$ such that for all $t \geq 0$,*
$$\mathbf{P}(|X| \geq t) \leq 2\exp\left(\frac{-t^2}{K^2}\right),$$
then $\|X\|_{\psi_2} \leq 2K$.
- *For any bounded random variable X, $\|X\|_{\psi_2} \lesssim \|X\|_\infty$.*
- $\|X - \mathbf{E}X\|_{\psi_2} \lesssim \|X\|$.
- *If $X_1, \dots, X_n$ are independent, then*
$$\|X_1 + \cdots + X_n\|_{\psi_2} \lesssim \left(\|X_1\|_{\psi_2}^2 + \cdots + \|X_n\|_{\psi_2}^2\right)^{1/2}.$$

Proof. See [16]. □

Given a random vector X, taking values in $\mathbf{R}^n$, we say it is *uniformly subgaussian* if there is a constant A such that for any $x \in \mathbf{R}^n$, $\|X \cdot x\|_{\psi_2} \leq A \cdot |x|$. We define

$$\|X\|_{\psi_2} = \sup_{|x|=1} \|X \cdot x\|_{\psi_2},$$

so that $\|X \cdot x\|_{\psi_2} \leq \|X\|_{\psi_2} \cdot |x|$ for any random vector X.

The classic Hanson-Wright inequality gives a concentration bound for the magnitude of a quadratic form $X^T AX$ in terms of the matrix norms of A, assuming X is a random vector with independant, mean zero, subgaussian coordinates.

Theorem 44 (Hanson-Wright)**.** *Let $X = (X_1, \dots, X_k)$ be a random vector with independant, mean zero, subgaussian coordinates, with $\|X_i\|_{\psi_2} \leq K$ for each i, and let A be a diagonal-free $k \times k$ matrix. Then there is a universal constant c such that for every $t \geq 0$,*

$$\mathbf{P}\left(|X^T AX| \geq t\right) \leq 2\exp\left(-c \cdot \min\left(\frac{t^2}{K^4\|A\|_F^2}, \frac{t}{K^2\|A\|}\right)\right).$$

Here $\|A\|$ is the operator norm of A, and $\|A\|_F$ it's Frobenius norm.

In our work, we require a modified version of the Hanson Wright inequality where we replace the independant random variables $X_1, \dots, X_k$ with independant random vectors taking values in $\mathbf{R}^d$, and where A is replaced with a n tensor on $\mathbf{R}^{k \times d}$. To make things more tractable, we assume that $A_I = 0$ for each multi-index $I = ((i_1, j_1), \dots, (i_n, j_n))$ if there is $r \neq r'$ with $i_r = i_{r'}$. If this is satisfied, we say A is a *non-diagonal tensor*.

Theorem 45. *Let $X = (X_1, \dots, X_k)$, where $X_1, \dots, X_k$ are independant, mean zero random vectors taking values in $\mathbf{R}^d$, with $\|X_i\|_{\psi_2} \leq K$ for each i. Let A be a non-diagonal n-tensor in $\mathbf{R}^{k \times d}$. Then for $t \geq 0$,*

$$\mathbf{P}\left(|A(X, \dots, X)| \geq t\right) \leq 2 \cdot \exp\left(-c(n) \cdot \min\left(\frac{t^2}{K^4\|A\|_F^2}, \dots, \frac{t^{2/n}}{K^{4/n}\|A\|_F^{2/n}}\right)\right).$$

The main technique involved is a statistical decoupling result due to J. Bourgain and L. Tzafriri. We follow the proof of the Hanson-Wright inequality from [16], altering various calculations to incorporate the vector-valued coordinates $X_1,\dots,X_n$, and incorporating more robust calculations of the moments of higher order Gaussian chaos.

Lemma 46 (Decoupling)**.** *If $F : \mathbf{R} \to \mathbf{R}$ is convex, and $X = (X_1,\dots,X_k)$, where $X_1,\dots,X_k$ are independant, mean zero random vectors taking values in $\mathbf{R}^d$. If A is a non-diagonal n-tensor in $\mathbf{R}^{d\times k}$, then*

$$\mathbf{E}(F(A(X,\dots,X))) \leq \mathbf{E}(F(n^n \cdot A(X^1,X^2,\dots,X^n))),$$

where $X^1,\dots,X^n$ are independant copies of X.

Proof. Given a set $I \subset [k]$, let $\pi_I : \mathbf{R}^{k\times d} \to \mathbf{R}^{k\times d}$ be defined by

$$\pi_I(X)_{ij} = \begin{cases} X_{ij} & : \text{if } i \in I, \\ 0 & : \text{if } i \notin I \end{cases}.$$

Thus given $X = (X_1,\dots,X_k)$, the vector $\pi_I(X)$ is obtained by zeroeing out all vectors X_i with $i \notin I$. Consider a random partition of $[k]$ into n classes $I_1,\dots,I_n$, chosen uniformly at random. Thus given any $x \in \mathbf{R}^{d\times k}$, we have

$$\mathbf{E}(A(\pi_{I_1}(x),\dots,\pi_{I_n}(x))) = \sum_{I=((i_1,j_1),\dots,(i_n,j_n))} A_I \cdot x_{i_1 j_1}\dots x_{i_n j_n}\mathbf{P}(i_1 \in I_1,\dots,i_n \in I_n).$$

If $A_I \neq 0$, the indices $\{i_r\}$ are distinct from one another. Since $(I_1,\dots,I_n)$ is selected uniformly at random, this means the positions of the indices i_r are independent of one another. Thus

$$\mathbf{P}(i_1 \in I_1,\dots,i_n \in I_n) = \mathbf{P}(i_1 \in I_1)\dots\mathbf{P}(i_n \in I_n) = 1/n^n.$$

Thus we have shown

$$A(x,\ldots,x) = n^n \cdot \mathbf{E}(A(\pi_{I_1}(x),\ldots,\pi_{I_n}(x))).$$

In particular, substituting X for x, we have

$$A(X,\ldots,X) = n^n \cdot \mathbf{E}_I(A(\pi_{I_1}(X),\ldots,\pi_{I_n}(X)))$$

Applying Jensen's inequality, we find

$$\begin{aligned}
\mathbf{E}_X(F(A(X,\ldots,X))) &= \mathbf{E}_X(F(n^n \cdot \mathbf{E}_I(A(\pi_{I_1}(X),\ldots,\pi_{I_n}(X))))) \\
&\le \mathbf{E}_X\,\mathbf{E}_I\,F(n^n \cdot A(\pi_{I_1}(X),\ldots,\pi_{I_n}(X))) \\
&= \mathbf{E}_I\,\mathbf{E}_X\,F(n^n \cdot A(\pi_{I_1}(X),\ldots,\pi_{I_n}(X))).
\end{aligned}$$

Thus we may select a *deterministic* partition I for which

$$\mathbf{E}(F(A(X,\ldots,X))) \le \mathbf{E}(F(n^n \cdot A(\pi_{I_1}(X),\ldots,\pi_{I_n}(X)))).$$

Since the classes $I_1,\ldots,I_n$ are disjoint from one another, $(\pi_{I_1}(X),\ldots,\pi_{I_n}(X))$ is identically distributed to $(\pi_{I_1}(X^1),\ldots,\pi_{I_n}(X^n))$, so

$$\mathbf{E}(F(n^n \cdot A(\pi_{I_1}(X),\ldots,\pi_{I_n}(X)))) = \mathbf{E}(F(n^n \cdot A(\pi_{I_1}(X^1),\ldots,\pi_{I_n}(X^n)))).$$

Let $Z = A(X^1,\ldots,X^n) - A(\pi_{I_1}(X^1),\ldots,\pi_{I_n}(X^n))$. If $\mathbf{E}'$ denotes conditioning with respect to $\pi_{I_1}(X^1),\ldots,\pi_{I_n}(X^n)$, then this expectation fixes $A(\pi_{I_1}(X_1),\ldots,\pi_{I_n}(X_n))$, but $\mathbf{E}'(Z) = 0$. Thus we can apply Jensen's inequality again to conclude

$$\begin{aligned}
\mathbf{E}(F(n^n \cdot A(\pi_{I_1}(X^1),\ldots,\pi_{I_n}(X^n)))) &= \mathbf{E}(F(n^n \cdot \mathbf{E}'(A(\pi_{I_1}(X^1),\ldots,\pi_{I_n}(X^n)) + Z))) \\
&= \mathbf{E}(F(n^n \cdot \mathbf{E}'(A(X^1,\ldots,X^n)))) \\
&\le \mathbf{E}(\mathbf{E}'(F(n^n A(X^1,\ldots,X^n)))) \\
&= \mathbf{E}(F(n^n A(X^1,\ldots,X^n))). \qquad \square
\end{aligned}$$

Remark. *Note this theorem also implies that for each $\lambda \in \mathbf{R}$,*

$$\mathbf{E}(F((\lambda/n^n)\cdot A(X,\ldots,X))) \le \mathbf{E}(F(\lambda \cdot A(X^1,\ldots,X^n))).$$

Lemma 47 (Comparison)**.** *Let $X^1,\ldots,X^n$ be independant random vectors taking values in $\mathbf{R}^{k\times d}$*

To make things easy for ourselves, we assume that A vanishes along it's diagonal. Reinterpreting the quantities in the Hanson-Wright inequality in this setting gives the result. For each i and j, we define A_{ij} be an $d\times d$ matrix, then we set $X^TAX = \sum X_i^T A_{ij} X_j$,

$$\|A\|_F = \sqrt{\sum \|A_{ij}\|_F^2}, \quad \text{and} \quad \|A\| = \sup\left\{ |\sum x_i^T A_{ij} y_j| : \sum |x_i|^2 = \sum |y_j|^2 = 1 \right\}.$$

These are precisely the definitions of the operator norm and Frobenius norm if we interpret A as a $dn\times dn$ matrix by expanding out the entries of each A_{ij}, and X as a random vector taking values in $\mathbf{R}^{dn}$ by expanding out the entries of each X_i.

Chapter 7

Conclusions

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