Cartesian Products Avoiding Patterns

by

Jacob Denson

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Cartesian Products Avoiding Patterns

submitted by **Jacob Denson** in partial fulfillment of the requirements for the degree of **Master of Science** in **Mathematics**.

Examining Committee:

Malabika Pramanik, Mathematics
Supervisor

Joshua Zahl, Mathematics
Supervisor

Abstract

The pattern avoidance problem seeks to construct a set $X \subset \mathbf{R}^d$ with large dimension that avoids a prescribed pattern such as three term arithmetic progressions, or more general patterns such as avoiding points $x_1, \dots, x_n$ such that $f(x_1, \dots, x_n) = 0$ for a given function f. Previous work on the subject has considered patterns described by polynomials, or functions f satisfying certain regularity conditions. We provide an exposition of this work, as well as considering new strategies to avoid 'rough patterns. There are several problems that fit into the framework of rough pattern avoidance. As a first application, if $Y \subset [0,1]$ is a set with Minkowski dimension s, we construct a set $X \subset [0,1]$ with Hausdorff dimension $1-s$ so that $X+X$ is disjoint from Y. As a second application, given a set Y with dimension close to one, we can construct a set $X \subset Y$ of dimension $1/2$ that avoids isosceles triangles.

Lay Summary

In geometry, we are often interested in constructing shapes with certain properties. For instance, given three points, can one find a circle connecting these three points? Most questions of this type involving classical shapes like circles and polygons have been answered. But many open questions remain about more modern families of shapes. Here, we focus on *fractals*; Examples of fractals include the Koch snowflake and Sierpinski triangle. This class of shapes often occurs in small scale physics and computer graphics.

In this thesis, we focus on constructing large fractals which avoid the existence of certain configurations. For example, can one construct a large fractal so that one cannot form an equilateral triangle from three points lying on the fractal? We begin with an exposition on constructions in the literature for avoiding configurations, and then provide new construction techniques utilizing randomness.

Preface

This thesis gives an exposition by the author, of the pattern avoidance problem and the geometric measure theory required to understand the pattern avoidance problem in the non-discrete setting. In Chapter 4, the author presents details of joint work with his supervisors Dr. Joshua Zahl and Dr. Malabika Pramanik. The results of that section have been submitted for publication.

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Chapter 1

Introduction

In this thesis, we study a simple family of questions:

How large can Euclidean sets be not containing geometric patterns?.

For instance, what is the maximal size of a set $X \subset \mathbf{R}^d$ that contains no three collinear points? Or what is the maximal size of a set not containing the vertices of an isosceles triangle? Aside from pure geometric interest, these problems provide useful settings to test methods of ergodic theory, additive combinatorics and harmonic analysis.

Sets which avoid the patterns we consider are highly irregular, in many ways behaving like a fractal set. Our understanding of their structure requires techniques from geometric measure theory. We use the *fractal dimension* to measure the size of a set when determining how large a pattern avoiding set can be. In particular, we use the Hausdorff dimension and Minkowski dimension, as well as various other more modern variants of fractal dimension which give more structural information about a set.

At present, many fundamental questions about geometric structure and it's relation to fractal dimension remain unsolved. One might expect sets with sufficiently large fractal dimension contain patterns. For example, Theorem 6.8 of [12] shows that any set $X \subset \mathbf{R}^d$ with Hausdorff dimension exceeding one must contain three collinear points. On the other hand, Theorem 2.3 of [10] constructs a set $X \subset \mathbf{R}^d$ with full Hausdorff dimension such that no four points in X form the vertices of a parallelogram. Thus the problem depends on the particular aspects of the configurations involved. For most geometric configurations, it remains unknown at what threshold patterns are guaranteed, or whether such a threshold exists at all. So finding sets with large Hausdorff dimension avoiding patterns is an important topic to determine for which classes of patterns our intuition remains true.

Our goal in this thesis is to derive new methods for constructing sets avoiding patterns. In particular, we expand on a number of general *pattern dissection methods* which have proven useful in the area, originally developed by Keleti but also studied notably by Mathé, Pramanik, and Fraser. After a background to the techniques we use is established in chapter 2, we give a full exposition of pattern dissection methods in chapter 3. The main contribution we give to pattern dissection methods is to use random dissections to avoid patterns which lack structural information. Thus we can expand the utility of interval dissection methods from regular families of patterns to a set of *fractal avoidance problems*, that previous methods were completely unavailable to address. Such problems include finding large X such that the angles formed by any three distinct points of X avoid a specified set Y of angles, where Y has a fixed fractal dimension. This method is described in chapter 4.

Chapter 2

Background

2.1 Configuration Avoidance

We consider an ambient set $\mathbf{A}$. It's *n-point configuration space* is

$$\mathcal{C}^n(\mathbf{A}) = \{(a_1,\dots,a_n) \in \mathbf{A}^n : a_i \neq a_j \text{ if } i \neq j\}.$$

An *n point configuration*, or *n point pattern*, is a subset of $\mathcal{C}^n(\mathbf{A})$. More generally, we define the general *configuration space* of $\mathbf{A}$ as $\mathcal{C}(\mathbf{A}) = \bigcup_{n=1}^\infty \mathcal{C}^n(\mathbf{A})$, and a *pattern*, or *configuration*, on $\mathbf{A}$ as a subset of $\mathcal{C}(\mathbf{A})$.

Our main focus in this thesis is the *pattern avoidance problem*. For a fixed configuration $\mathcal{C}$ on $\mathbf{A}$, we say a set $X \subset \mathbf{A}$ *avoids* $\mathcal{C}$ if $\mathcal{C}(X)$ is disjoint from $\mathcal{C}$. The pattern avoidance problem asks to find sets X of maximal size avoiding a fixed configuration $\mathcal{C}$. Often $\mathcal{C}$ describes algebraic or geometric structure, and the pattern avoidance problem asks to find the maximal size of a set before it is guaranteed to have such structure.

Example (Isoceles Triangle Configuration)**.** Let

$$\mathcal{C} = \left\{ (x_1,x_2,x_3) \in \mathcal{C}^3(\mathbf{R}^2) : |x_1 - x_2| = |x_1 - x_3| \right\}.$$

Then $\mathcal{C}$ is a 3-point configuration, and a set $X \subset \mathbf{R}^2$ avoids $\mathcal{C}$ if and only if it does not contain all three vertices of an isoceles triangle.

Example (Linear Independence Configuration)**.** Let V be a vector space over a field K. We set

$$\mathcal{C} = \bigcup_{n=1}^{\infty} \{(x_1,\dots,x_n) \in \mathcal{C}^n(V) : \text{there is } a_1,\dots,a_n \in K \text{ s.t. } a_1x_1 + \cdots + a_nx_n = 0\}.$$

A set $X \subset V$ avoids $\mathcal{C}$ if and only if X is a linearly independent subset of V. Note that if V is infinite dimensional, then $\mathcal{C}$ cannot be replaced by a n point configuration for any n; arbitrarily large tuples must be considered. We will be interested in the case where $K = \mathbf{Q}$, and $V = \mathbf{R}$. Our interest here is to find analytically large linearly independant subsets of V.

Even though our problem formulation assumes configurations are formed by distinct sets of points, one can still formulate avoidance problems involving repeated points by a simple trick, exemplified by the next example.

Example (Sum Set Configuration)**.** Let G be an abelian group, and fix $Y \subset G$. Set

$$\mathcal{C}^1 = \{g \in \mathcal{C}^1(G) : g + g \in Y\} \quad \text{and} \quad \mathcal{C}^2 = \{(g_1,g_2) \in \mathcal{C}^2(G) : g_1 + g_2 \in Y\}.$$

Then define $\mathcal{C} = \mathcal{C}^1 \cup \mathcal{C}^2$. A set $X \subset G$ avoids $\mathcal{C}$ if and only if $(X+X) \cap Y = \emptyset$.

Depending on the structure of the ambient space $\mathbf{A}$ and the configuration $\mathcal{C}$, there are various ways of measuring the size of sets $X \subset \mathbf{A}$ for the purpose of the pattern avoidance problem:

- If $\mathbf{A}$ is finite, we wish to find a set X with large cardinality.
- If $\{\mathbf{A}_n\}$ is an increasing family of finite sets with $\mathbf{A} = \bigcup_n \mathbf{A}_n$, we wish to find a set X such that $X \cap \mathbf{A}_n$ has large cardinality asymptotically as $n \to \infty$.
- If $\mathbf{A} = \mathbf{R}^d$, but $\mathcal{C}$ is a discrete configuration, then a satisfactory goal is to find a set X with large Lebesgue measure avoiding $\mathcal{C}$.

In this thesis, inspired by results in these three settings, we establish methods for avoiding non-discrete configurations $\mathcal{C}$ in $\mathbf{R}^d$. Here, Lebesgue measure completely fails to measure the size of pattern avoiding solutions, as the next theorem shows, under the often true assumption that $\mathcal{C}$ is *translation invariant*, i.e. that if $(a_1,\dots,a_n) \in \mathcal{C}$ and $b \in \mathbf{R}^d$, $(a_1+b,\dots,a_n+b) \in \mathcal{C}$.

Theorem 1. *Let $\mathcal{C}$ be a n-point configuration on $\mathbf{R}^d$. Suppose*

(A) *$\mathcal{C}$ is translation invariant.*

(B) *For any $\varepsilon > 0$, there is $(a_1,\dots,a_n) \in \mathcal{C}$ with diam$\{a_1,\dots,a_n\} \leq \varepsilon$.*

Then no set with positive Lebesgue measure avoids $\mathcal{C}$.

Proof. Let $X \subset \mathbf{R}^d$ have positive Lebesgue measure. The Lebesgue density theorem shows that there exists a point $x \in X$ such that

$$\lim_{l(Q)\to 0} \frac{|X \cap Q|}{|Q|} = 1, \tag{2.1}$$

where Q ranges over all axis-oriented cubes in $\mathbf{R}^d$ with $x \in Q$, and $l(Q)$ denotes the sidelength of Q. Fix $\varepsilon > 0$, to be specified later, and choose r small enough that $|X \cap Q| \geq (1-\varepsilon)|Q|$ for any cube Q with $x \in Q$ and $l(Q) \leq r$. Now let Q_0 denote a cube centered at x with $l(Q_0) \leq r$. Applying Property (B), we find $C = (a_1,\dots,a_n) \in \mathcal{C}$ such that

$$\operatorname{diam}\{a_1,\dots,a_n\} \leq l(Q_0)/2. \tag{2.2}$$

For each $p \in Q_0$, let $C(p) = (a_1(p),\dots,a_n(p))$, where $a_i(p) = p + (a_i - a_1)$. A union bound shows

$$|\{p \in Q_0 : C(p) \not\in \mathcal{C}(X)\}| \leq \sum_{i=1}^{d} |\{p \in Q_0 : a_i(p) \not\in X\}|. \tag{2.3}$$

We have $a_i(p) \notin X$ precisely when $p + (a_i - a_1) \notin X$, so

$$|\{p \in Q_0 : a_i(p) \notin X\}| = |(Q_0 + (a_i - a_1)) \cap X^c|. \tag{2.4}$$

Note $Q_0 + (a_i - a_1)$ is a cube with the same sidelength as Q_0. Equation (2.2) implies $|a_i - a_1| \leq l(Q_0)/2$, so $x \in Q_0 + (a_i - a_1)$. Thus (2.1) shows

$$|Q_0 + (a_i - a_1)) \cap X^c| \leq \varepsilon |Q_0|. \tag{2.5}$$

Combining (2.3), (2.4), and (2.5), we find

$$|\{p \in Q_0 : C(p) \notin \mathcal{C}(X)\}| \leq \varepsilon d |Q_0|.$$

Provided $\varepsilon d < 1$, this means there is $p \in Q_0$ with $C(p) \in \mathcal{C}(X)$. Property (A) implies $C(p) \in \mathcal{C}$, so X does not avoid $\mathcal{C}$. $\square$

Since no set of positive Lebesgue measure can avoid non-discrete configurations, we cannot use the Lebesgue measure to quantify the size of pattern avoiding sets for many non-discrete configurations. Geometric measure theory provides us with a quantity which can distinguish between the size of sets of measure zero. This is the *fractional dimension* of a set.

There are many variants of fractional dimension. Here we choose to use the Minkowski dimension and the Hausdorff dimension. They assign the same dimension to any smooth manifold, but vary over more singular sets. One major difference is that Minkowski dimension measures relative density at a single scale, whereas Hausdorff dimension measures relative density at countably many scales.

2.2 Minkowski Dimension

Given $l > 0$, and a bounded set $E \subset \mathbf{R}^d$, we let $N(l, E)$ denote the *covering number* of E, i.e. the minimum number of sidelength l cubes required to cover E. We define the *lower* and *upper* Minkowski dimension as

$$\dim_{\underline{\mathbf{M}}}(E) = \liminf_{l \to 0} \left[\frac{\log(N(l,E))}{\log(1/l)} \right] \quad \text{and} \quad \dim_{\overline{\mathbf{M}}}(E) = \limsup_{l \to 0} \left[\frac{\log(N(l,E))}{\log(1/l)} \right].$$

If $\dim_{\overline{\mathbf{M}}}(E) = \dim_{\underline{\mathbf{M}}}(E)$, then we refer to this common quantity as the *Minkowski dimension* of E, denoted $\dim_{\mathbf{M}}(E)$. Thus $\dim_{\underline{\mathbf{M}}}(E) < s$ if there exists a sequence of lengths $\{l_k\}$ converging to zero such that for each k, E is covered by fewer than $(1/l_k)^s$ sidelength l_k cubes, and $\dim_{\overline{\mathbf{M}}}(E) < s$ if E is covered by fewer than $(1/l)^s$ sidelength l cubes for *any* suitably small l.

Remark. Any cube with sidelength r is covered in $O_d(1)$ balls of radius r. Conversely, any ball of radius r is covered by $O_d(1)$ cubes of sidelength r. Thus for any $r > 0$, if we temporarily define $N_B(r,E)$ to be the optimal number of radius r *balls* it takes to cover E, then $N(r,E) \sim_d N_B(r,E)$. As $r \to 0$, this means

$$\frac{\log(N(r,E))}{\log(1/r)} = \frac{\log(N_B(r,E))}{\log(1/r)} + o(1).$$

In particular, $\dim_{\underline{\mathbf{M}}}(E) < s$ if and only if there exists a sequence of lengths $\{r_k\}$ such that E is covered by $(1/r_k)^s$ radius r_k *balls*, and $\dim_{\overline{\mathbf{M}}}(E) < s$ if and only if E is covered by $(1/r)^s$ radius r *balls*, for any suitably small $r > 0$.

It is often easy to upper bound the Minkowski dimension of a set, simply by providing a cover of the set and counting the number of cubes that cover it. We now provide an example. We say a set $S \subset \mathbf{R}^d$ is an s dimensional *Lipschitz manifold* if there exists a family of bounded, open subsets $\{U_\alpha\}$ of $\mathbf{R}^s$, together with a family of bi-Lipschitz maps $\{f_\alpha : U_\alpha \to S\}$, such that the sets $\{f_\alpha(U_\alpha)\}$ form a relatively open cover of S. Every C^1 manifold in $\mathbf{R}^d$ is a Lipschitz manifold. This example proves useful in Chapter 4.

Theorem 2. *Let $S \subset \mathbf{R}^d$ be a* Lipschitz manifold *of dimension s, and let K be a compact subset of S. Then* $\dim_{\overline{\mathbf{M}}}(K) \leq s$.

Proof. Since K is compact, we can find finitely many bi-Lipschitz maps $f_1, \dots, f_N$

such that the family $\{f_i(U_i) : 1 \leq i \leq N\}$ covers K. Then there is C such that for each i, if $x, y \in U_i$,

$$|f_i(x) - f_i(y)| \leq C \cdot |x - y|.$$

Since each U_i is bounded, for any $r > 0$, we can find a family of balls $B_{i,1}, \dots, B_{i,M_i}$ of radius (r/C) covering U_i, such that the centres $x_{i,1}, \dots, x_{i,M_i}$ of the balls also lie in U_i, and $M_i \lesssim (C/r)^s$. But then the balls of radius r with centres lying in

$$\{f_i(x_{i,j}) : 1 \leq i \leq N, 1 \leq j \leq M_i\}$$

cover K, and there are $\sum_{i=1}^N M_i \lesssim N/(Cr)^s = (N/C^s)/r^s$ such balls. Since C and N are independent of r, and $r > 0$ was arbitrary, this shows $\dim_{\overline{\mathbf{M}}}(K) \leq s$. $\square$

2.3 Hausdorff Dimension

For $E \subset \mathbf{R}^d$ and $\delta > 0$, we define the *Hausdorff content*

$$H^s_\delta(E) = \inf \left\{ \sum_{k=1}^\infty l(Q_k)^s : E \subset \bigcup_{k=1}^\infty Q_k, l(Q_k) \leq \delta \right\}.$$

The *s-dimensional Hausdorff measure* of E is

$$H^s(E) = \lim_{\delta \to 0} H^s_\delta(E) = \sup \left\{ H^s_\delta(E) : \delta > 0 \right\}.$$

It is easy to see H^s is an exterior measure on $\mathbf{R}^d$, and $H^s(E \cup F) = H^s(E) + H^s(F)$ if the Hausdorff distance $d(E,F)$ between E and F is positive. So H^s is actually a metric exterior measure, and the Caratheodory extension theorem shows all Borel sets are measurable with respect to H^s. Sometimes, it is convenient to use the exterior measure

$$H^s_\infty(E) = \inf \left\{ \sum_{k=1}^\infty l(Q_k)^s : E \subset \bigcup_{k=1}^\infty Q_k \right\}.$$

The majority of Borel sets which occur in practice fail to be measurable with respect to H^s_∞, but the exterior measure H^s_∞ has the useful property that $H^s_\infty(E) = 0$ if and only if $H^s(E) = 0$.

Lemma 3. *Consider $t < s$, and $E \subset \mathbf{R}^d$.*

(i) If $H^t(E) < \infty$, then $H^s(E) = 0$.

(ii) If $H^s(E) \neq 0$, then $H^t(E) = \infty$.

Proof. Suppose that $H^t(E) = A < \infty$. Then for any $\delta > 0$, there is a cover of E by a collection of intervals $\{Q_k\}$, such that $l(Q_k) \leq \delta$ for each k, and

$$\sum l(Q_k)^t \leq A < \infty.$$

But then

$$H^s_\delta(E) \leq \sum l(Q_k)^s \leq \sum l(Q_k)^{s-t} l(Q_k)^t \leq \delta^{s-t} A.$$

As $\delta \to 0$, we conclude $H^s(E) = 0$, proving *(i)*. And *(ii)* is just the contrapositive of (i), and therefore immediately follows. □

Corollary 4. *If $s > d$, $H^s = 0$.*

Proof. The measure H^d is just the Lebesgue measure on $\mathbf{R}^d$, so

$$H^d[-N,N]^d = (2N)^d.$$

If $s > d$, Lemma 3 shows $H^s[-N,N]^d = 0$. By countable additivity, taking $N \to \infty$ shows $H^s(\mathbf{R}^d) = 0$. Since H^s is a positive measure, $H^s(E) = 0$ for all E. □

Given any Borel set E, Corollary 4, combined with Lemma 3, implies there is a unique value $s_0 \in [0,d]$ such that $H^s(E) = 0$ for $s > s_0$, and $H^s(E) = \infty$ for $0 \leq s < s_0$. We refer to s_0 as the *Hausdorff dimension* of E, denoted $\dim_{\mathbf{H}}(E)$.

Theorem 5. *For any bounded set E,* $\dim_{\mathbf{H}}(E) \leq \dim_{\underline{\mathbf{M}}}(E) \leq \dim_{\overline{\mathbf{M}}}(E)$.

Proof. Given $l > 0$, we have a simple bound $H_l^s(E) \leq N(l,E) \cdot l^s$. If $\dim_{\underline{\mathbf{M}}}(E) < s$, then there exists a sequence $\{l_k\}$ with $l_k \to 0$, and $N(l_k,E) \leq (1/l_k)^s$. Thus we conclude that

$$H^s(E) = \lim_{k\to\infty} H_{l_k}^s(E) \leq \lim_{k\to\infty} N(l_k,E) \cdot l_k^s \leq 1,$$

Thus $\dim_{\mathbf{H}}(E) \leq s$. Taking infima over all s, we find $\dim_{\mathbf{H}}(E) \leq \dim_{\underline{\mathbf{M}}}(E)$. □

Remark. If $\dim_{\mathbf{H}}(E) < d$, then $|E| = H^d(E) = 0$. Thus any set with fractional dimension less than d must have measure zero. This means we can use the dimension as a way of distinguishing between sets of measure zero, which is precisely what we need to study the configuration avoidance problem for non-discrete configurations.

The fact that Hausdorff dimension is defined over multiple scales simultaneously makes it more stable under analytical operations. In particular, for any family of at most countably many sets $\{E_k\}$,

$$\dim_{\mathbf{H}} \left\{ \bigcup E_k \right\} = \sup \{\dim_{\mathbf{H}}(E_k)\}.$$

This need not be true for the Minkowski dimension; a single point has Minkowski dimension zero, but the rational numbers, which are a countable union of points, have Minkowski dimension one. An easy way to make Minkowski dimension countably stable is to define the *modified Minkowski dimensions*

$$\dim_{\underline{\mathbf{MM}}}(E) = \inf \left\{ s : E \subset \bigcup_{i=1}^{\infty} E_i, \dim_{\underline{\mathbf{M}}}(E_i) \leq s \right\}$$

and

$$\dim_{\overline{\mathbf{MM}}}(E) = \inf \left\{ s : E \subset \bigcup_{i=1}^{\infty} E_i, \dim_{\overline{\mathbf{M}}}(E_i) \leq s \right\}.$$

This notion of dimension, in a disguised form, appears in Chapter 4.

2.4 Dyadic Scales

It is now useful to introduce the dyadic notation we utilize throughout this thesis. At the cost of losing topological perspective about $\mathbf{R}^d$, applying dyadic techniques often allows us to elegantly discretize certain problems in Euclidean space.

Fix an integer N. The classic family of dyadic cubes with *branching factor N* is given by setting, for each integer $k \geq 0$,

$$\mathcal{D}_k^d = \left\{ \prod_{i=1}^d \left[\frac{n_i}{N^k}, \frac{n_i+1}{N^k} \right] : n \in \mathbf{Z}^d \right\},$$

and then setting $\mathcal{D}^d = \bigcup_{k \geq 0} \mathcal{D}_k^d$. Elements of $\mathcal{D}^d$ are known as *dyadic cubes*, and elements of $\mathcal{D}_k^d$ are known as *dyadic cubes of generation k*. The most important properties of the dyadic cubes is that for each k, $\mathcal{D}_k^d$ is a cover of $\mathbf{R}^d$ by cubes of sidelength $1/N^k$, and for any two cubes $Q_1, Q_2 \in \mathcal{D}^d$, either their interiors are disjoint, or one cube is nested in the other.

- For each cube $Q_1 \in \mathcal{D}_{k+1}^d$, there is a unique cube $Q_2 \in \mathcal{D}_k^d$ such that $Q_1 \subset Q_2$. We refer to Q_2 as the *parent* of Q_1, and Q_1 as a *child* of Q_2. Each cube in $\mathcal{D}$ has exactly N^d children. For $Q \in \mathcal{D}_{k+1}^d$, we let $Q^* \in \mathcal{D}_k^d$ denote it's parent.

- We say a set $E \subset \mathbf{R}^d$ is $\mathcal{D}_k$ *discretized* if it is a union of cubes in $\mathcal{D}_k^d$. If E is $\mathcal{D}_k$ discretized, we define

$$\mathcal{D}_k(E) = \{Q \in \mathcal{Q}_k^d : Q \subset E\}.$$

  Then $E = \bigcup \mathcal{D}_k(E)$.

- Given $k \geq 0$, and $E \subset \mathbf{R}^d$, we let

$$E(1/N^k) = \bigcup \{Q \in \mathcal{D}_k^d : Q \cap E \neq \emptyset\}.$$

  Then $E(1/N^k)$ is the smallest $\mathcal{D}_k$ discretized set containing E in it's interior. Our choice of notation invites thinking of $E(1/N^k)$ as a discretized version of the classic $1/N^k$ thickening

$$\{x \in \mathbf{R}^d : d(x,E) < 1/N^k\}.$$

We have no need for the standard thickening in this thesis, so there is no notational conflict.

Since any cube is covered by at most $O_d(1)$ cubes in $\mathcal{D}$ of comparable sidelength, from the perspective of geometric measure theory, working with dyadic cubes is normally equivalent to working with the class of all cubes.

Our main purpose with working with dyadic cubes is to construct *fractal-type sets*. By this, we mean defining sets X as the intersection of a nested family of sets $\{X_k\}$, where each X_k is $\mathcal{D}_k$ discretized, and each succesive set X_{k+1} is obtained from X_k by application of a simple, recursive procedurew. Such a construction satisfies Property (i), (ii), and (v) of Falconer's definition of a fractal, justifying the name.

Example. A classic fractal-type set is the Cantor set C. We form C from the family of dyadic cubes with branching factor $N = 3$. We initially set $C_0 = [0,1]$. Then, given the $\mathcal{D}_k$ discretized set C_k, we consider each $I \in \mathcal{D}^1_k(C_k)$, and let $\mathcal{D}^1_{k+1}(I) = \{I_1, I_2, I_3\}$, where I_1, I_2, I_3 are given in increasing order with respect to their appearance in I. We set

$$C_{k+1} = \bigcup \{I_1 \cup I_3 : I \in \mathcal{D}^1_k(C_k)\}.$$

Then $C = \bigcap_{k \geq 0} C_k$ is the Cantor set.

Unfortunately, a *constant* branching factor is not sufficient to describe the types of recursive procedures we discuss in this thesis. This motivates a more general iterated family of cubes. Instead of a single branching factor N, we fix a sequence of positive integers $\{N_k : k \geq 1\}$, with $N_k \geq 2$ for all k, which gives the branching factor at each stage of the class of cubes we define.

- For each $k \geq 0$, we define

$$\mathcal{Q}_k^d = \left\{ \prod_{i=1}^d \left[\frac{m_i}{N_1 \dots N_k}, \frac{m_i + 1}{N_1 \dots N_k} \right] : m \in \mathbf{Z}^d \right\}.$$

These are the *dyadic cubes of generation k*. We let $\mathcal{Q}^d = \bigcup_{k \geq 0} \mathcal{Q}_k^d$. Note that any two cubes $\mathcal{Q}^d$ are either nested within one another, or their interiors are disjoint.

- We set $l_k = (N_1 \dots N_k)^{-1}$. Then l_k is the sidelength of the cubes in $\mathcal{Q}_k^d$.

- Given $Q \in \mathcal{Q}_{k+1}^d$, we let $Q^* \in \mathcal{Q}_k^d$ denote the *parent cube* of Q, i.e. the unique dyadic cube of generation k such that $Q \subset Q^*$.

- We say a set $E \subset \mathbf{R}^d$ is $\mathcal{Q}_k$ *discretized* if it is a union of cubes in $\mathcal{Q}_k^d$. In this case, we let

$$\mathcal{Q}_k(E) = \{Q \in \mathcal{Q}_k^d : Q \subset E\}$$

denote the family of cubes whose union is E.

- For $E \subset \mathbf{R}^d$ and $k \geq 0$, we let $E(l_k) = \{Q \in \mathcal{Q}_k^d : Q \cap E = \emptyset\}$.

Sometimes, our recursive constructions need a family of 'intermediary' cubes that lie between the scales $\mathcal{Q}_k^d$ and $\mathcal{Q}_{k+1}^d$. In this case, we consider a supplementary sequence $\{M_k : k \geq 1\}$ with $M_k \,|\, N_k$ for each k.

- For $k \geq 1$, we define

$$\mathcal{R}_k^d = \left\{ \prod_{i=1}^d \left[\frac{m_i}{N_1 \dots N_{k-1} M_k}, \frac{m_i + 1}{N_1 \dots N_{k-1} M_k} \right] : m \in \mathbf{Z}^d \right\}.$$

- We set $r_k = (N_1 \dots N_{k-1} M_k)^{-1}$. Then r_k is the sidelength of a cube in $\mathcal{R}_k^d$.

- The notions of being $\mathcal{R}_k^d$ discretized, the collection of cubes $\mathcal{R}_k^d(E)$, and the sets $E(r_k)$, are defined as should be expected.

The cubes in $\mathcal{R}_k^d$ are coarser than those in $\mathcal{Q}_k^d$, but finer than those in $\mathcal{Q}_{k-1}^d$.

Remark. We note that there is some notational conflict between the cubes $\mathcal{D}^d$ and the cubes $\mathcal{Q}^d$, but since we never use both families simultaneously in a single argument, it should be clear which notation we are using.

2.5 Frostman Measures

It is often easy to upper bound Hausdorff dimension, but non-trivial to *lower bound* the Hausdorff dimension of a given set. A key technique to finding a lower bound is *Frostman's lemma*, which says that a set has large Hausdorff dimension if and only if it supports a Borel measure obeying a decay law on small sets. We say a Borel probability measure μ is a *Frostman measure* of dimension s if it is non-zero, compactly supported, and there exists $C > 0$ such that for any cube Q, $\mu(Q) \leq C \cdot l(Q)^s$. The proof of Frostman's lemma will utilize a technique often useful, known as the *mass distribution principle*. To prove the mass distribution principle, we apply weak convergence.

Lemma 6. *Suppose $\{\mu_i\}$ is a Cauchy sequence of non-negative Borel measures on $\mathbf{R}^d$, in the sense that for any $f \in C_c(\mathbf{R}^d)$, the sequence*

$$\left\{ \int f d\mu_i \right\}$$

is Cauchy. Then there is a regular Borel measure μ such that $\mu_i \to \mu$ weakly.

Proof. Fix a compact set K. then we can find a function $\varphi \in C_c(\mathbf{R}^d)$ such that $\mathbf{I}_K \leq \varphi$. This means that

$$\mu_i(K) = \int \mathbf{I}_K d\mu_i \leq \int \varphi d\mu_i.$$

Since $\{\int \varphi d\mu_i\}$ is Cauchy, it is uniformly bounded in i. In particular, $\mu_i(K)$ is uniformly bounded in i. Thus the measures $\mu_i|_K$ are uniformly finite, and we can apply the Banach Alaoglu theorem to find a regular Borel measure μ^K such that $\mu_i|_K \to \mu^K$ weakly. Note that if $f \in C_c(\mathbf{R}^d)$ is supported on $K_1 \cap K_2$ for two compact sets K_1 and K_2, then

$$\int f d\mu^{K_1} = \lim_{i\to\infty} \int_{K_1} f d\mu_i = \lim_{i\to\infty} \int_{K_2} f d\mu_i = \int f d\mu^{K_2}.$$

Thus we can define a measure μ such that for each $f \in C_c(\mathbf{R}^d)$,

$$\int f d\mu = \lim_{K\to\infty} \int f d\mu^K.$$

For any $f \in C_c(\mathbf{R}^d)$ supported on a compact set K,

$$\int f d\mu = \int f d\mu^K = \lim_{i\to\infty} \int_K f d\mu_i = \lim_{i\to\infty} \int f d\mu_i.$$

Since f was arbitrary, $\mu_i \to \mu$ weakly. $\square$

Theorem 7 (Mass Distribution Principle)**.** *Let $f : \mathcal{Q}^d \to [0,\infty)$ be a function such that for any $Q_0 \in \mathcal{Q}^d$,*

$$\sum_{Q^*=Q_0} f(Q) = f(Q_0), \tag{2.6}$$

Then there exists a regular Borel measure μ supported on

$$\bigcap_{k=1}^{\infty} \left[\bigcup \{Q \in \mathcal{Q}_k^d : f(Q) > 0\} \right]$$

such that for each $Q \in \mathcal{Q}^d$,

$$\mu(Q^\circ) \leq f(Q) \leq \mu(Q), \tag{2.7}$$

and for any set E and $k \geq 0$,

$$\mu(E) \leq \sum \left\{ f(Q) : Q \in \mathcal{Q}_k^d(E(l_k)) \right\}. \tag{2.8}$$

Proof. For each i, let μ_i be a regular Borel measure such that for each $j \leq i$, and

$Q \in \mathcal{Q}_j^d$, $\mu_i(Q) = \mu_i(Q^\circ) = f(Q)$. One such choice is given, for each $f \in C_c(\mathbf{R}^d)$, by the equation

$$\int f d\mu_i = \sum_{Q \in \mathcal{Q}_i^d} \frac{f(Q)}{|Q|} \int_Q f\, dx.$$

We claim that $\{\mu_i\}$ is a Cauchy sequence in the space of all regular Borel measures on $\mathbf{R}^d$, viewing the space as a locally convex space under the weak topology. Fix $\varphi \in C_c(\mathbf{R}^d)$, and choose some N such that φ is supported on $[-N,N]^d$. Set $Q_0 = [-N,N]^d$. Since φ is compactly supported, φ is *uniformly continuous*, so for each $\varepsilon > 0$, if i is suitably large, there is a sequence of values $\{a_Q : Q \in \mathcal{Q}_i^d(Q_0)\}$ such that if $x \in Q$, $|\varphi(x) - a_Q| \leq \varepsilon$. But this means that

$$\sum (a_Q - \varepsilon)\mathbf{I}_Q \leq \varphi \leq \sum (a_Q + \varepsilon)\mathbf{I}_Q,$$

where Q ranges over cubes in $\mathcal{Q}_i^d(Q_0)$. Thus for any $j \geq i$,

$$\int \varphi\, d\mu_j \leq \sum (a_Q + \varepsilon)\mu_j(Q) = \sum (a_Q + \varepsilon) f(Q)$$

and

$$\int \varphi\, d\mu_j \geq \sum (a_Q - \varepsilon)\mu_j(Q) = \sum (a_Q - \varepsilon) f(Q).$$

In particular, if $j, j' \geq i$,

$$\left| \int \varphi d\mu_j - \int \varphi d\mu_{j'} \right| \leq 2\varepsilon \sum_{Q \in \mathcal{Q}_i(Q_0)} f(Q) = 2\varepsilon \sum_{Q \in \mathcal{Q}_0(Q_0)} f(Q).$$

Since ε and φ were arbitrary, this shows $\{\mu_i\}$ is Cauchy.

Applying Lemma 6, there exists a regular Borel measure μ such that $\mu_i \to \mu$ weakly. For any $Q \in \mathcal{Q}^d$, since Q is a closed set,

$$\mu(Q) \geq \limsup_{i \to \infty} \mu_i(Q) = f(Q).$$

Conversely, if E is an arbitrary set, then $E \subset E(l_i)^\circ$ for any i, so

$$\mu(E) \leq \liminf_{i \to \infty} \mu_i(E(l_i)^\circ) \leq \liminf_{i \to \infty} \mu_i(E(l_i)) = \sum f(Q),$$

where Q ranges over the cubes in $\mathcal{Q}_i^d(E(l_i))$. $\square$

Remark. The reason why we cannot necessarily find a function μ which *precisely* extends f is that in the weak limit, mass from cubes can 'leak' into the mass of adjacent cubes. This is why we must use the weaker 'extension bounds' (2.7) and (2.8). If, for each $Q_0 \in \mathcal{Q}^d$,

$$\lim_{k \to \infty} \sum \{ f(Q) : Q \in \mathcal{Q}_k^d(Q_0(l_k)) - \mathcal{Q}_k^d(Q_0) \} = 0,$$

then (2.7) and (2.8) actually satisfies $\mu(Q) = f(Q)$ for all $Q \in \mathcal{Q}$, so μ gives an extension of f. One such condition that guarantees this is that $f(Q) \lesssim l(Q)^s$ for some $s > d-1$. Another is that for any k, and for any two cubes $Q_0, Q_1 \in \mathcal{Q}_k^d$ with $Q_0 \cap Q_1 \neq \emptyset$, either $f(Q_0) = 0$ or $f(Q_1) = 0$. If f does not satisfy such a bound, however, then mass from cubes can 'leak' into the mass of adjacent cubes if the measure is suitably concentrated at the boundary. This doesn't cause us to 'gain' or 'lose' any mass in the weak limit, as (2.7) shows, so the proof still is a 'mass distribution', it just means that the mass may be shared by adjacent cubes.

Theorem 8 (Frostman's Lemma)**.** *If E is a Borel set, $H^s(E) > 0$ if and only if there exists an s dimensional Frostman measure supported on E.*

Proof. Suppose that μ is s dimensional and supported on E. If $H^s(F) = 0$, then for each $\varepsilon > 0$ there is a sequence of cubes $\{Q_k\}$ with $\sum_{k=1}^\infty l(Q_k)^s \leq \varepsilon$. But then

$$\mu(F) \leq \sum_{k=1}^\infty \mu(Q_k) \lesssim \sum_{k=1}^\infty l(Q_k)^s \leq \varepsilon.$$

Taking $\varepsilon \to 0$, we conclude $\mu(F) = 0$. Thus μ is absolutely continuous with respect to H^s. Taking the contrapositive of what we just proved, since $\mu(E) > 0$, this means that $H^s(E) > 0$.

To prove the converse, we will suppose for simplicity that E is compact. We work dyadically with the classical family of dyadic cubes $\mathcal{D}$. By translating, we may assume that $H^s(E \cap [0,1]^d) > 0$, and so without loss of generality we may assume $E \subset [0,1]^d$. For each $Q \in \mathcal{D}^d_k$, define $f^+(Q) = H^s_\infty(E \cap Q)$. Then $f^+(Q) \leq 1/2^{ks}$, and f^+ is subadditive. We try and define a function f to which we can apply Theorem 7. We initially define f by setting $f([0,1]^d) = f^+([0,1]^d)$. Given $Q \in \mathcal{D}^d_k$, we enumerate it's children as $Q_1, \dots, Q_M \in \mathcal{D}^d_{k+1}$. We then consider any values $A_1, \dots, A_M \geq 0$ such that

$$A_1 + \dots + A_M = f(Q), \tag{2.9}$$

and for each k,

$$A_k \leq f^+(Q_k). \tag{2.10}$$

This is feasible to do because $f^+(Q_1) + \dots + f^+(Q_M) \geq f^+(Q)$. We then define $f(Q_k) = A_k$ for each k. Equation (2.9) implies the recursive constraint is satisfied, so f is a well defined function. Furthermore, (2.9) implies (2.6) of Lemma 7, and so the mass distribution principle implies that there exists a measure μ supported on E, satisfying (2.8) and (2.7). In particular, (2.8) implies μ is non-zero. For each $Q \in \mathcal{D}^d_k$, $\#[\mathcal{D}^d_k(Q(l_k))] = 3^d = O_d(1)$, so we can apply (2.7) to conclude

$$\mu(Q) \lesssim_d \max f(Q') \leq 1/2^{ks} = l(Q)^s.$$

where Q' ranges over the cubes in $\mathcal{D}^d_k(Q(l_k))$. Given any cube Q, we find k with $1/2^{k-1} \leq l(Q) \leq 1/2^k$. Then Q is covered by $O_d(1)$ dyadic cubes in $\mathcal{D}^d_k$, and so $\mu(Q) \lesssim l(Q)^s$. Thus μ is a Frostman measure of dimension s. □

2.6 Dyadic Fractional Dimension

It is often natural for us to establish results about fractional dimension 'dyadically', working with the family of cubes $\mathcal{Q}^d$. $\{N_k : k \geq 1\}$. We begin with Minkowski dimension. For each m, let $N_{\mathcal{Q}}(m,E)$ denote the minimal number of cubes in $\mathcal{Q}_m^d$ required to cover E. This is often easy to calculate, up to a multiplicative constant, by greedily selecting cubes which intersect E.

Lemma 9. *For any set E,*

$$N_{\mathcal{Q}}(m,E) \approx_d \#\{Q \in \mathcal{Q}_m^d : Q \cap E \neq \emptyset\} \approx_d N(l_m, E).$$

Proof. Let $\mathcal{E} = \{Q \in \mathcal{Q}_m^d : Q \cap E \neq \emptyset\}$. Then $N(l_m,E) \leq N_{\mathcal{Q}}(m,E) \leq \#(E)$. Conversely, let $\{Q_k\}$ be a minimal cover of E by cubes. Then each cube Q_k intersects at most 3^d cubes in $\mathcal{Q}_k^d$, so $\#(\mathcal{E}) \leq 3^d \cdot N(l_m,E) \leq 3^d \cdot N_{\mathcal{Q}}(m,E)$. □

Thus it is natural to ask whether it is true that for any set E,

$$\begin{aligned} \dim_{\underline{\mathbf{M}}}(E) &= \liminf_{k\to\infty} \frac{\log[N_{\mathcal{Q}}(k,E)]}{\log[1/l_k]} \\ &\text{and} \\ \dim_{\overline{\mathbf{M}}}(E) &= \limsup_{k\to\infty} \frac{\log[N_{\mathcal{Q}}(k,E)]}{\log[1/l_k]}. \end{aligned} \tag{2.11}$$

The answer depends on the choice of $\{N_k\}$. In particular, a sufficient condition (and as we see later, essentially necessary) is that

$$N_{k+1} \lesssim_\varepsilon (N_1 \dots N_k)^\varepsilon \quad \text{for any } \varepsilon > 0. \tag{2.12}$$

We will see that this condition allows us to work dyadically in many scenarios when it comes to fractal dimension.

Theorem 10. *If* (2.12) *holds, then* (2.11) *holds.*

Proof. Fix a length l, and find k with $l_{k+1} \leq l \leq l_k$. Applying Lemma 9 shows

$$N(l,E) \leq N(l_{k+1},E) \lesssim_d N_{\mathcal{Q}}(k+1,E)$$

and

$$N(l,E) \geq N(l_k,E) \gtrsim_d N_{\mathcal{Q}}(k,E).$$

Thus
$$\frac{\log[N(l,E)]}{\log[1/l]} \le \left[\frac{\log(1/l_{k+1})}{\log(1/l_k)}\right] \frac{\log[N_{\mathcal{Q}}(k+1,E)]}{\log[1/l_{k+1}]} + O_d(1/k)$$
and
$$\frac{\log[N(l,E)]}{\log[1/l]} \ge \left[\frac{\log(1/l_k)}{\log(1/l_{k+1})}\right] \frac{\log[N_{\mathcal{Q}}(k,E)]}{\log[1/l_k]} + O_d(1/k).$$
Provided that
$$\frac{\log(1/l_{k+1})}{\log(1/l_k)} \to 1, \tag{2.13}$$
the conclusion of the theorem is true. But (2.13) is equivalent to the condition that
$$\frac{\log(N_{k+1})}{\log(N_1)+\cdots+\log(N_k)} \to 0,$$
and this is equivalent to (2.12). □

Any constant branching factor satisfies (2.12) for the Minkowski dimension. In particular, we can work fairly freely with the classical dyadic cubes without any problems occuring. But more importantly for our work, we can let the sequence $\{N_k\}$ increase rapidly.

Theorem 11. *If $N_k = 2^{\lfloor 2^{k\psi(k)} \rfloor}$, where $\psi(k)$ is any decreasing sequence of positive numbers tending to zero, such that*
$$\psi(k) \ge \log_2(k)/k, \tag{2.14}$$
then (2.12) *holds.*

Proof. We note that $\log(N_k) = 2^{k\psi(k)} + O(1)$, and that (2.14) implies that
$$2^{\psi(1)} + \cdots + 2^{k\psi(k)} \ge k.$$

Putting these two facts together, we conclude that

$$\begin{aligned}\frac{\log(N_{k+1})}{\log(N_1)+\cdots+\log(N_k)} &= \frac{2^{(k+1)\psi(k+1)}+O(1)}{2^{\psi(1)}+2^{2\psi(2)}+\cdots+2^{k\psi(k)}+O(k)} \\ &\lesssim \frac{2^{(k+1)\psi(k+1)}}{2^{\psi(k)}+2^{2\psi(k)}+\cdots+2^{k\psi(k)}} \\ &\lesssim \frac{2^{(k+1)\psi(k+1)}}{2^{(k+1)\psi(k)}}(2^{\psi(k)}-1) \\ &\leq (2^{\psi(k)}-1) \to 0.\end{aligned}$$

This is equivalent to (2.12). $\square$

We refer to any sequence $\{l_k\}$ constructed by $\{N_k\}$ satisfying (2.14) for some function ψ as a *subhyperdyadic* sequence. If a sequence $\{l_k\}$ is generated by a sequence $\{N_k\}$ such that for some fixed $c > 0$,

$$N_k = 2^{\lfloor 2^{ck} \rfloor},$$

then the values $\{l_k\}$ are referred to as *hyperdyadic*. The next (counter) example shows that hyperdyadic sequences are essentially the 'boundary' for sequences that can be used to measure the Minkowski dimension.

Example. We consider a multi-scale dyadic construction, utilizing the two families $\mathcal{Q}^d$ and $\mathcal{R}^d$. Fix $0 \leq c < 1$, and define $N_k = 2^{\lfloor 2^{ck} \rfloor}$, and $M_k = 2^{\lfloor c2^{ck} \rfloor}$. Then $M_k \,|\, N_k$ for each k. We recursively define a nested family of sets $\{E_k\}$, with each E_k a $\mathcal{Q}_k^d$ discretized set, and set $E = \bigcap E_k$. We define $E_0 = [0,1]$. Then, given E_k, for each $Q \in \mathcal{Q}_k(E_k)$, we select a *single* cube $R_Q \in \mathcal{R}_{k+1}(E_k)$, and define $E_{k+1} = \bigcup R_Q$. Then $\#(\mathcal{Q}_0(E_0)) = 1$, and

$$\#(\mathcal{Q}_{k+1}(E_{k+1})) = (N_{k+1}/M_{k+1})\#(\mathcal{Q}_k(E_k)),$$

which we can simplify to read

$$\#(\mathcal{Q}_k(E_k)) = \frac{N_1 \dots N_k}{M_1 \dots M_k} \tag{2.15}$$

Noting that $\log(N_i) = 2^{ci} + O(1)$, and $\log(M_i) = c2^{ci} + O(1)$, we conclude that

$$\frac{\log \#(\mathcal{Q}_k(E_k))}{\log(1/l_k)} = \frac{(1-c)(2^c + \dots + 2^{ck}) + O(k)}{(2^c + \dots + 2^{ck}) + O(k)} \to 1 - c.$$

On the other hand, for each k,

$$\#(\mathcal{R}^d_{k+1}(E_k)) = \#(\mathcal{Q}_k(E_k)) = \frac{N_1 \dots N_k}{M_1 \dots M_k},$$

and so

$$\begin{aligned}
\frac{\log \#(\mathcal{R}^d_{k+1}(E_k))}{\log(1/r_{k+1})} &= \frac{(1-c)(2^c + \dots + 2^{ck}) + O(k)}{(2^c + \dots + 2^{ck}) + c2^{c(k+1)} + O(k)} \\
&= \frac{(1-c) \cdot 2^{c(k+1)} + O(k)}{(1 - c + c2^c) \cdot 2^{c(k+1)} + O(k)} \\
&\to \frac{1-c}{1-c+c2^c} < 1 - c.
\end{aligned}$$

In particular,

$$\dim_{\underline{\mathbf{M}}}(E) \neq \liminf_{k \to \infty} \frac{\log [N(l_k, E)]}{\log(1/l_k)},$$

so measurements at hyperdyadic scales fail to establish general results about the Minkowski dimension.

We now move on to calculating Hausdorff dimension dyadically. The natural quantity to consider is the measure defined for any set E as $H^s_{\mathcal{Q}}(E) = \lim_{m \to \infty} H^s_{\mathcal{Q},m}(E)$, where

$$H^s_{\mathcal{Q},m}(E) = \inf \left\{ \sum_k l(Q_k)^s : E \subset \bigcup_k^\infty Q_k,\ Q_k \in \bigcup_{i \geq m} \mathcal{Q}^d_i \text{ for each } k \right\}.$$

A similar argument to the standard Hausdorff measures shows there is a unique s_0 such that $H^s_{\mathcal{Q}}(E) = \infty$ for $s < s_0$, and $H^s_{\mathcal{Q}}(E) = 0$ for $s > s_0$. It is obvious that $H^s_{\mathcal{Q}}(E) \geq H^s(E)$ for any set E, so we certainly have $s_0 \geq \dim_{\mathbf{H}}(E)$. The next lemma guarantees that $s_0 = \dim_{\mathbf{H}}(E)$, under the same conditions on the sequence $\{N_k\}$ as found in Theorem 10.

Lemma 12. *If* (2.12) *holds, then for any* $\varepsilon > 0$, $H^s_{\mathcal{Q}}(E) \lesssim_{s,\varepsilon} H^{s-\varepsilon}(E)$.

Proof. Fix $\varepsilon > 0$ and m. Let $E \subset \bigcup Q_k$, where $l(Q_k) \leq l_m$ for each k. Then for each k, we can find i_k such that $l_{i_k+1} \leq l(Q_k) \leq l_{i_k}$. Then Q_k is covered by $O_d(1)$ elements of $\mathcal{Q}^d_{i_k}$, and

$$H^s_{\mathcal{Q},m}(E) \lesssim_d \sum l^s_{i_k} \leq \sum (l_{i_k}/l_{i_k+1})^s l(Q_k)^s \leq \sum \left(l_{i_k}/l_{i_{k+1}}\right)^s l^{\varepsilon}_{i_k} l(Q_k)^{s-\varepsilon}. \quad (2.16)$$

By assumption,

$$l_{i_k+1} = \frac{1}{N_1 \dots N_{i_k} N_{i_k+1}} \gtrsim_{s,\varepsilon} (N_1 \dots N_{i_k})^{1+\varepsilon/s} = l^{1+\varepsilon/s}_{i_k}. \quad (2.17)$$

Putting (2.16) and (2.17) together, we conclude that $H^s_{\mathcal{Q},m}(E) \lesssim_{d,s,\varepsilon} \sum l(Q_k)^{s-\varepsilon}$. Since $\{Q_k\}$ was an arbitrary cover of E, we conclude $H^s_{\mathcal{Q},m}(E) \lesssim H^{s-\varepsilon}(E)$, and since m was arbitrary, that $H^s_{\mathcal{Q}}(E) \lesssim H^{s-\varepsilon}(E)$. $\square$

Finally, we consider computing whether we can establish that a measure is a Frostman measure dyadically.

Theorem 13. *If* (2.12) *holds, and if* μ *is a Borel measure such that* $\mu(Q) \lesssim l(Q)^s$ *for each* $Q \in \mathcal{Q}^d_k$, *then* μ *is a Frostman measure of dimension* $s - \varepsilon$ *for each* $\varepsilon > 0$.

Proof. Given a cube Q, find k such that $l_{k+1} \leq l(Q) \leq l_k$. Then Q is covered by $O_d(1)$ cubes in $\mathcal{Q}^d_k$, which shows

$$\mu(Q) \lesssim_d l^s_k = [(l_k/l)^s l^{\varepsilon}] l^{s-\varepsilon} \leq [l^{s+\varepsilon}_k / l^s_{k+1}] l^{s-\varepsilon} = \left[\frac{N^s_{k+1}}{(N_1 \dots N_k)^{\varepsilon}}\right] l^{s-\varepsilon} \lesssim_{\varepsilon} l^{s-\varepsilon}. \quad \square$$

Let's recognize the utility of this approach from the perspective of a dyadic construction. Suppose we have a sequence of nested sets $\{E_k\}$, where E_k is a $\mathcal{Q}_k$ discretized subset of $[0,1]^d$, and for each $Q_0 \in \mathcal{Q}_k(E_k)$, there is at least one cube $Q \in \mathcal{Q}_{k+1}(E_{k+1})$ with $Q^* = Q_0$. Then we can set $E = \bigcap E_k$ as a 'limit' of the discretizations E_k. We can associate with this construction a finite measure μ. It is defined by setting $f([0,1]^d) = 1$, and setting, for each $Q \in \mathcal{Q}_{k+1}(E_{k+1})$,

$$f(Q) = \frac{f(Q^*)}{\#\{Q' \in \mathcal{Q}_{k+1}(E_{k+1}) : (Q')^* = Q^*\}}.$$

The mass distribution principle then gives a Borel measure μ, which we refer to as the *canonical measure* associated with this construction. For this measure, it is often easy to show from a combinatorial argument that $f(Q) \lesssim l(Q)^s$ if $Q \in \mathcal{Q}^d$, which also implies $\mu(Q) \lesssim l(Q)^s$ for $Q \in \mathcal{Q}^d$. This makes Theorem 13 useful.

Remark. The dyadic construction showing that Minkowski dimension cannot be measured only at hyperdyadic scales also shows that a bound on a measure μ on hyperdyadic cubes does not imply the correct bound at all scales. It is easy to show from (2.15) that the canonical measure μ for this example satisfies, for each k, each $Q \in \mathcal{Q}_k(E_k)$, and each $\varepsilon > 0$,

$$\mu(Q) = (\#(\mathcal{Q}_k(E_k)))^{-1} \lesssim_\varepsilon l(Q)^{1-c-\varepsilon}$$

for all $Q \in \mathcal{Q}_k(E_k)$, yet we know that for the set constructed in that example,

$$\dim_{\mathbf{H}}(E) \leq \dim_{\underline{\mathbf{M}}}(E) < 1 - c.$$

Frostman's lemma implies we cannot possibly have $\mu(Q) \lesssim_\varepsilon l(Q)^{1-c-\varepsilon}$ for all $\varepsilon > 0$ and *all* cubes Q.

2.7 Beyond Hyperdyadics

If we use a faster increasing sequence of branching factors than that satisfying (2.12), we must exploit some extra property of our construction, which is not always present in general sets. Here, we rely on a *uniform mass distribution* between scales. Given the uniformity assumption, the lengths can decrease as fast as desired. We utilize the multi-scale set of dyadic cubes $\mathcal{Q}^d$ and $\mathcal{R}^d$ introduced in Section 2.4.

Lemma 14. *Let μ be a measure supported on a set E. Suppose that*

(A) *For any $Q \in \mathcal{Q}_k^d$, $\mu(Q) \lesssim l_k^s$.*

(B) *For each $R \in \mathcal{R}_{k+1}^d$, $\#[\mathcal{Q}_{k+1}^d(E(l_k) \cap R)] \lesssim 1$.*

(C) *For any $R \in \mathcal{R}_{k+1}^d$ with parent cube $Q \in \mathcal{Q}_k^d$, $\mu(R) \lesssim (1/M_{k+1})^d \cdot \mu(Q)$.*

Then μ is a Frostman measure of dimension s.

Proof. We establish the general bound $\mu(Q) \lesssim l(Q)^s$ for all cubes Q by separating into two different cases:

- Suppose there is k with $r_{k+1} \leq l(Q) \leq l_k$. Then $\#(\mathcal{R}_{k+1}^d(Q(r_{k+1}))) \lesssim (l/r_{k+1})^d$. Properties (A) and (C) imply each of these cubes has measure at most $O((r_{k+1}/l_k)^d l_k^s)$, so we obtain that

$$\mu(Q) \lesssim (l/r_{k+1})^d (r_{k+1}/l_k)^d l_k^s = l^d / l_k^{d-s} \lesssim l^s.$$

- Suppose there exists k with $l_k \leq l \leq r_k$. Then $\#[\mathcal{R}_k^d(Q(r_k))] \lesssim 1$. Combining this with Property (C) gives $\#[\mathcal{Q}_k^d(Q(r_k) \cap E(l_k))] \lesssim 1$. This and Property (A) then shows

$$\mu(Q) \lesssim l_k^s \leq l^s.$$

This addresses all cases, so μ is a Frostman measure of dimension s. □

This theorem is commonly used here when dealing with multi-scale dyadic fractal constructions, whose character is summarized in the following Theorem.

Theorem 15. *let $E = \bigcap E_k$, where $\{E_k\}$ is a nested family of subsets of $\mathbf{R}^d$, such that E_k is $\mathcal{Q}_k$ discretized for each k. Suppose that*

(A) *For each $Q \in \mathcal{Q}_k(E_k)$, there exists a set $\mathcal{R}_Q \subset \mathcal{R}_{k+1}(Q)$ such that $\#(\mathcal{R}_Q) \geq (1/2) \cdot \#(\mathcal{R}_{k+1}(Q))$ such that*

$$\#\left(\mathcal{Q}_{k+1}(R \cap E_{k+1})\right) = \begin{cases} 1 & : R \in \mathcal{R}_Q, \\ 0 & : R \notin \mathcal{R}_Q. \end{cases}$$

(B) *For any $\varepsilon > 0$, $N_1 \dots N_k \lesssim_\varepsilon N_{k+1}^\varepsilon$.*

(C) *There is $0 < s \leq 1$ such that for any $\varepsilon > 0$, $N_{k+1} \lesssim_\varepsilon M_{k+1}^{(1+\varepsilon)/s}$.*

Then E has Hausdorff dimension sd.

Proof. Consider the function $f : \mathcal{Q}^d \to [0,\infty)$ defined with respect to the sequence $\{E_k\}$, which generates the canonical measure μ supported on E using the mass distribution principle. For each $R \in \mathcal{R}_{k+1}(E_k)$ with parent cube $Q \in \mathcal{Q}_k(E_k)$, Property (A) shows

$$f(R) \leq (2/M_{k+1}^d) f(Q). \tag{2.18}$$

Properties (B) and (C) imply that for each $\varepsilon > 0$,

$$1/M_{k+1} \lesssim_\varepsilon r_{k+1}^{1-\varepsilon} \lesssim_\varepsilon l_{k+1}^{s(1-2\varepsilon)}. \tag{2.19}$$

If $Q \in \mathcal{Q}_{k+1}(E_{k+1})$ has parent cubes $R \in \mathcal{R}_{k+1}(E_k)$ and $Q^* \in \mathcal{Q}_k(E_k)$, then by (2.18), (2.19), and the fact that $f(Q^*) \leq 1$,

$$f(Q) = f(R) \leq (2/M_{k+1}^d) f(Q^*) \leq (2/M_{k+1}^d) \lesssim_\varepsilon l_{k+1}^{ds(1-2\varepsilon)}. \tag{2.20}$$

If Q is a cube with length l_k, then $\#(\mathcal{Q}_k^d(Q(l_k))) = O_d(1)$, which combined with

(2.8), shows that

$$\mu(Q) \lesssim_\varepsilon l_{k+1}^{ds(1-2\varepsilon)}. \tag{2.21}$$

Property (B) of Lemma 14 is implied by Property (A). Equation (2.21) is a form of Property (A) in Lemma 14. Together with (2.8), (2.18) implies that the canonical measure μ satisfies Property (C) of Lemma 14. Thus all assumptions of Lemma 14 are satisfied, and so we conclude μ is a Frostman measure of dimension $ds(1-2\varepsilon)$ for each $\varepsilon > 0$. Taking $\varepsilon \to 0$, and applying Frostman's lemma, we conclude that the Hausdorff dimension of the support of μ, which is a subset of E, is greater than or equal to ds. For each k, we find that for each $\varepsilon > 0$,

$$\#(\mathcal{Q}_{k+1}(E_{k+1})) \leq \#(\mathcal{R}_{k+1}(E_k)) \leq (1/M_{k+1})^d \lesssim_\varepsilon l_{k+1}^{ds-\varepsilon}.$$

Taking $k \to \infty$, and then $\varepsilon \to 0$, we conclude that

$$\dim_{\mathbf{H}}(E) \leq \dim_{\underline{\mathbf{M}}}(E) \leq ds.$$

Since we already know $\dim_{\mathbf{H}}(E) \geq ds$, this completes the proof. □

Chapter 3

Related Work

Here, we discuss the main papers which influenced our results. In particular, the work of Keleti on translate avoiding sets, Fraser and Pramanik's work on sets avoiding smooth configurations, and Mathé's result on sets avoiding algebraic varieties.

3.1 Keleti: A Translate Avoiding Set

Keleti's two page paper constructs a full dimensional set $X \subset [0,1]$ such that for each $t \neq 0$, X intersects $t+X$ in at most one place. The set X is then said to *avoid translates*. This paper contains the core idea behind the *discretization* method adapted in Fraser and Pramanik's paper. We also adapt this technique in our paper, which makes the result of interest.

Lemma 16. *Let X be a set. Then X avoids translates if and only if there do not exists values $x_1 < x_2 \leq x_3 < x_4$ in X with $x_2 - x_1 = x_4 - x_3$. In particular, a set X avoids translates if and only if it avoids the four point configuration*

$$\mathcal{C} = \{(x_1,x_2,x_3,x_4) : x_1 < x_2 \leq x_3 < x_4,\ x_2 - x_1 = x_4 - x_3\}.$$

Proof. Suppose $(t+X) \cap X$ contains two points $a < b$. Without loss of generality, we may assume that $t > 0$. If $a \leq b - t$, then the equation

$$a - (a - t) = t = b - (b - t)$$

satisfies the constraints, since $a - t < a \leq b - t < b$ are all elements of X. We also have

$$(b - t) - (a - t) = b - a,$$

which satisfies the constraints if $a - t < b - t \leq a < b$. This covers all possible cases. Conversely, if there are $x_1 < x_2 \leq x_3 < x_4$ in X with

$$x_2 - x_1 = t = x_4 - x_3,$$

then $X + t$ contains $x_2 = x_1 + (x_2 - x_1)$ and $x_4 = x_3 + (x_4 - x_3)$. $\square$

The basic, but fundamental idea of the interval dissection technique is to introduce memory into Cantor set constructions. Keleti constructs a nested family of discrete sets $\{X_k\}$, with X_k a $\mathcal{Q}_k$ discretized set, and with $X = \bigcap X_k$. The sequence $\{N_k\}$ will be specified later, but each N_k will be a multiple of 10. We initialize $X_0 = [0,1]$. The novel feature of the argument is to add a queue of intervals to the construction, initially just containing $[0,1]$. To construct the sequence $\{X_k\}$, Keleti iteratively performs the following procedure:

Algorithm 1 Construction of the Sets $\{X_k\}$:

Set $k = 0$.

Repeat

Take off an interval I from the front of the queue.

For all $J \in \mathcal{Q}_k(X_k)$:

Order the intervals in $\mathcal{Q}_{k+1}(J)$ as $J_1, \dots, J_N$.

If $J \subset I$, add all intervals J_i to X_{k+1} with $i \equiv 0$ modulo 10.

Else add all J_i with $i \equiv 5$ modulo 10.

Add all intervals in $\mathcal{Q}^d_{k+1}$ to the end of the queue.

Increase k by 1.

Each iteration of the algorithm produces a new set X_k, and so leaving the algorithm to repeat infinitely produces a sequence $\{X_k\}$ whose intersection is X.

Lemma 17. *The set X is translate avoiding.*

Proof. If X is not translate avoiding, there is $x_1 < x_2 \leq x_3 < x_4$ with $x_2 - x_1 = x_4 - x_3$. Since $l_k \to 0$, there is a suitably large integer N such that x_1 is contained in an interval $I \in \mathcal{Q}_N$ not containing x_2, x_3, or x_4. At stage N of the algorithm, the interval I is added to the end of the queue, and at a much later stage M, the interval I is retrieved. Find the startpoints x_1°, x_2°, $x_3^\circ, x_4^\circ \in l_M\mathbf{Z}$ to the intervals in $\mathcal{Q}_M$ containing x_1, x_2, x_3, and x_4. Then we can find n and m such that $x_4^\circ - x_3^\circ = (10n)l_M$, and $x_2^\circ - x_1^\circ = (10m+5)l_M$. In particular, this means that $|(x_4^\circ - x_3^\circ) - (x_2^\circ - x_1^\circ)| \geq 5L_M$. But

$$\begin{aligned}|(x_4^\circ - x_3^\circ) - (x_2^\circ - x_1^\circ)| &= |[(x_4^\circ - x_3^\circ) - (x_2^\circ - x_1^\circ)] - [(x_4 - x_3) - (x_2 - x_1)]| \\ &\leq |x_1^\circ - x_1| + \cdots + |x_4^\circ - x_4| \leq 4L_M\end{aligned}$$

which gives a contradiction. □

It is easy to see from the algorithm that

$$\#(\mathcal{Q}_k(X_k)) = (N_k/10) \cdot \#(\mathcal{Q}_{k-1}(X_{k-1})).$$

Closing the recursive definition shows

$$\#(\mathcal{Q}_k(X_k)) = \frac{1}{10^k l_k}.$$

In particular, this means $|X_k| = 1/10^k$, so $|X|$ has Lebesgue measure zero regardless of how we choose the parameters $\{N_k\}$.

Theorem 18. *For some sequence $\{N_k\}$, the set X has full Hausdorff dimension.*

Proof. Set $N_k = 10M_k$ for each $k \geq 1$, then one sees that for each $R \in \mathcal{R}_k(X_k)$, $\#(\mathcal{Q}_{k+1}(R \cap X_{k+1})) = 1$. Thus Property (A) of Theorem 15 is satisfied. Furthermore, Property (C) of Theorem 15 is satisfied with $s = 1$. We conclude that if $N_1 \dots N_k \lesssim_\varepsilon N_{k+1}^\varepsilon$, then all assumptions of Theorem 15 is satisfied, and we conclude X is a set with full Hausdorff dimension. This is true, for instance, if we set $N_k = 2^{2^{\lfloor k \log k \rfloor}}$. $\square$

The most important feature of Keleti's argument is his reduction of a non-discrete configuration avoidance problem to a sequence of discrete avoidance problems on cubes. Let us summarize the result of Keleti's discrete argument in a lemma.

Lemma 19. *Let $T_1, T_2 \subset \mathbf{R}$ be disjoint, $\mathcal{Q}_k$ discretized sets. Then we can find $S_1 \subset T_1$ and $S_2 \subset T_2$ such that*

(i) For each k, S_k is a $\mathcal{Q}_{k+1}$ discretized subset of T_k.

(ii) If $x_1 \in S_1$ and $x_2, x_3, x_4 \in S_2$, then $x_2 - x_1 \neq x_4 - x_3$.

(iii) For each cube $Q \in \mathcal{Q}_k$ with $Q \subset T_k$,

$$\#(\mathcal{Q}_{k+1}(Q \cap S_k)) \geq (1/10) \cdot \#(\mathcal{Q}_{k+1}(Q)).$$

Property (i) of Lemma 19 then allows Keleti to apply his argument iteratively at each Dyadic scale. Property (ii) implies Keleti obtains a configuration avoiding set in the limit of these iterations. Most importantly the reason why Keleti obtains a set with full Hausdorff dimension, if the sequence $\{N_k\}$ grows suitably fast in k, is because of Property (iii).

Of course, it is not possible to extend the discrete solution of the configuration argument to general configurations; this part of Keleti's method strongly depends on the arithmetic structure of the configuration. The iterative application of a discrete solution, however, can be applied in generality. Combined with Theorem 15, this technique gives a powerful method to reduce configuration avoidance problems about Hausdorff dimension to discrete avoidance problems on cubes.

3.2 Fraser/Pramanik: Smooth Configurations

Inspired by Keleti's result, Pramanik and Fraser obtained a generalization of the queue method which allows one to find sets avoiding $n+1$ point configurations given by the zero sets of smooth functions, i.e.

$$\mathcal{C} = \{(x_0, \dots, x_n) \in \mathbf{R}^{dn} : f(x_0, \dots, x_n) = 0\},$$

under mild regularity conditions on the function $f : [0,1]^{dn} \to [0,1]^m$. Here, a simple pidgeonholing strategy suffices.

Theorem 20 (Pramanik and Fraser)**.** *Fix $m \leq d(n-1)$. Consider a countable family of functions $\{f_k : [0,1]^{dn} \to [0,1]^m\}$ each of which being C^2, and such that for each k, Df_k has full rank at any $(x_1, \dots, x_n)$ with $f(x_1, \dots, x_n) = 0$, where the $(x_1, \dots, x_n)$ are distinct. Then there exists a set $X \subset \mathbf{R}^d$ with Hausdorff dimension $m/(n-1)$ such that X avoids the configuration*

$$\mathcal{C} = \bigcup_k \{(x_1, \dots, x_n) \in \mathcal{C}^n(\mathbf{R}^d) : f(x_1, \dots, x_n) = 0\}.$$

Remark. For simplicity, we only prove the result for a single function, rather than a countable family of functions. The only major difference between the two approaches is the choice of scales we must choose later on in the argument.

Pramanik and Fraser also simplify to a discrete version of their problem, which they then iteratively apply. In the discrete setting, rather than making a linear shift in one of the intervals we avoid as in Keleti's approach, one must use the smoothness properties of the function to find large segments of an interval avoiding. Corollary 23 gives the discrete solution that Pramanik and Fraser utilizes.

Lemma 21. *Fix $n > 1$. Let $T \subset [0,1]^d$ be $\mathcal{Q}_k$ discretized, and $T' \subset [0,1]^{(n-1)d}$ be $\mathcal{Q}_k$ discretized. Let $B \subset T \times T'$ be $\mathcal{Q}_{k+1}$ discretized. Then there exists a $\mathcal{Q}_{k+1}$ discretized set $S \subset T$, and a $\mathcal{Q}_{k+1}$ discretized set $B' \subset T'$, such that*

(A) $(S \times T') \cap B \subset S \times B'$.

(B) *For every $Q \in \mathcal{Q}_k^d$, there exists $\mathcal{R}(Q) \subset \mathcal{R}_{k+1}^d(Q)$, such that*

$$\#(\mathcal{R}(Q)) \geq (1/2) \cdot \#(\mathcal{R}_{k+1}^d(Q)).$$

and for each $R \in \mathcal{R}_{k+1}^d(Q)$,

$$\#(\mathcal{Q}_{k+1}(R)) = \begin{cases} 1 & : R \in \mathcal{R}(Q,) \\ 0 & : R \notin \mathcal{R}(Q). \end{cases}$$

(C) $\#(\mathcal{Q}_{k+1}(B')) \leq 2(N_1 \dots N_k)^d \, (M_{k+1}/N_{k+1})^d \cdot \#(\mathcal{Q}_{k+1}(B))$.

Proof. Fix $Q_0 \in \mathcal{Q}_k(T)$. For each $R \in \mathcal{R}_{k+1}(Q_0)$, define a *slab* $S[R] = R \times T'$, and for each $Q \in \mathcal{Q}_{k+1}(Q_0)$, define a *wafer* $W[Q] = Q \times T'$. We say a wafer $W[Q]$ is *good* if

$$\#(\mathcal{Q}_{k+1}(W[Q] \cap B)) \leq (2/N_{k+1}^d) \cdot \#(\mathcal{Q}_{k+1}(B)), \tag{3.1}$$

Then at most $N_{k+1}^d/2$ wafers are bad. We call a slab *good* if it contains a wafer which is good. Since a slab is the union of $(N_{k+1}/M_{k+1})^d$ wafers, at most $M_{k+1}^d/2 = (1/2) \cdot \#(\mathcal{R}_{k+1}(Q_0))$ slabs are bad. Thus if we set

$$\mathcal{R}(Q_0) = \{R \in \mathcal{R}_{k+1}(Q_0) : S[R] \text{ is good}\},$$

then

$$\#(\mathcal{R}(Q_0)) \geq (1/2) \cdot \#(\mathcal{R}_{k+1}(Q_0)). \tag{3.2}$$

For each $R \in \mathcal{R}(Q_0)$, we pick $Q_R \in \mathcal{Q}_{k+1}(R)$ such that $W[Q_R]$ is good, and define

$$S = \bigcup \{Q_R : R \in \mathcal{R}(Q_0)\}.$$

Equation (3.2) implies S satisfies Property (B).

Let B' be the union of all cubes $Q' \in \mathcal{Q}_{k+1}(T')$ such that there is $Q \in \mathcal{Q}_{k+1}(S)$ with $Q \times Q' \in \mathcal{Q}_{k+1}(B)$. By definition, Property (A) is then satisfied. For each $Q \in \mathcal{Q}_{k+1}(S)$, $W[Q]$ is good, so (3.1) implies

$$\#\{Q' : Q \times Q' \in \mathcal{Q}_{k+1}(B)\} \leq (2/N_{k+1}^d) \cdot \#(\mathcal{Q}_{k+1}(B)).$$

But $\#(\mathcal{Q}_{k+1}(S)) \leq \#(\mathcal{R}_{k+1}(T)) \leq (1/r_{k+1})^d = (N_1 \dots N_k)^d M_{k+1}^d$, so

$$\begin{aligned}
\#(\mathcal{Q}_{k+1}(B')) &\leq \#(\mathcal{Q}_{k+1}(S))[(2/N_{k+1}^d) \cdot \#(\mathcal{Q}_{k+1}(B))] \\
&\leq 2(N_1 \dots N_k)^d (M_{k+1}/N_{k+1})^d \#(\mathcal{Q}_{k+1}(B)),
\end{aligned}$$

which establishes Property (C). □

We apply the lemma recursively $n-1$ times to continually reduce the dimensionality of the avoidance problem we are considering. Eventually, we obtain the case where $n = 0$, and then avoiding the configuration is easy.

Lemma 22. *Fix $n > 1$. Let $T \subset [0,1]^d$ be $\mathcal{Q}_k$ discretized, and let $B \subset T$ be $\mathcal{Q}_{k+1}$ discretized. Suppose*

$$\# \mathcal{Q}_{k+1}(B) \leq \left[C \cdot 2^{n-1} (N_1 \dots N_k)^{d(n-1)} (M_{k+1}/N_{k+1})^{d(n-1)} \right] (1/l_{k+1})^{dn-m}.$$

and

$$N_{k+1} \geq \left[C \cdot 2^n (N_1 \dots N_k)^{2dn} \right]^{1/m} M_{k+1}^{d(n-1)/m} \tag{3.3}$$

Then there exists a $\mathcal{Q}_{k+1}$ discretized set $S \subset T$ such that

(A) $S \cap B = \emptyset$.

(B) *For each* $Q_0 \in \mathcal{Q}_k(T)$*, there is* $\mathcal{R}(Q_0) \subset \mathcal{R}_{k+1}(Q_0)$ *with*

$$\#(\mathcal{R}(Q_0)) \geq (1/2)\#(\mathcal{R}_{k+1}(Q_0)),$$

such that for each $R \in \mathcal{R}_{k+1}(Q_0)$*,*

$$\#(\mathcal{Q}_{k+1}(R \cap S)) = \begin{cases} 1 & : R \in \mathcal{R}(Q_0), \\ 0 & : R \notin \mathcal{R}(Q_0). \end{cases}$$

Proof. For each $Q_0 \in \mathcal{Q}_k(T)$, we set

$$\mathcal{R}(Q_0) = \{R \in \mathcal{R}_{k+1}(Q_0) : \#(\mathcal{Q}_{k+1}(R \cap B)) \leq (2/M_{k+1}^d) \cdot \#(\mathcal{Q}_{k+1}(B))\}.$$

Since $\mathcal{R}_{k+1}(Q_0) = M_{k+1}^d$,

$$\#(\mathcal{R}(Q_0)) \geq \#(\mathcal{R}_{k+1}(Q_0)) - (M_{k+1}^d/2) \geq (1/2) \cdot \#(\mathcal{R}_{k+1}(T)).$$

Now (3.3) implies that for each $R \in \mathcal{R}(Q_0)$,

$$\begin{aligned} \#(\mathcal{Q}_{k+1}(R \cap B)) &\leq (2/M_{k+1}^d) \cdot \#(\mathcal{Q}_{k+1}(B)) \\ &\leq (2/M_{k+1}^d) \left(C \cdot 2^{n-1} (N_1 \dots N_k)^{2dn} (M_{k+1}/N_{k+1})^{d(n-1)} \right). \\ &= \left[2^n C (N_1 \dots N_k)^{2dn} \right] \left(M_{k+1}^{d(n-2)} / N_{k+1}^{m-d} \right) \\ &< (N_{k+1}/M_{k+1})^d \\ &= \mathcal{Q}_{k+1}(R) \end{aligned}$$

Thus for each $R \in \mathcal{R}(Q_0)$, we can find $Q_R \in \mathcal{Q}_{k+1}(R)$ such that $Q_R \cap B = \emptyset$. And so if we set

$$S = \bigcup \{Q_R : R \in \mathcal{R}(Q_0), Q_0 \in \mathcal{Q}_k(T)\},$$

then (A) and (B) are satisfied. $\square$

Corollary 23. *Let $f : [0,1]^{dn} \to [0,1]^m$ be C^2, and have full rank at every point $(x_1,\dots,x_n)$ with all $x_1,\dots,x_n$ distinct, and $f(x_1,\dots,x_n) = 0$. Then there exists a universal constant C depending only on f such that, if* (3.3) *is satisfied, then for any disjoint, $\mathcal{Q}_k$ discretized sets $T_1,\dots,T_n \subset [0,1]^d$, we can find $\mathcal{Q}_{k+1}$ discretized sets $S_1 \subset T_1,\dots,S_n \subset T_n$ such that*

(A) *If $x_1 \in S_1,\dots,x_n \in S_n$, then $f(x_1,\dots,x_n) \neq 0$.*

(B) *For each k, and for each $Q_0 \in \mathcal{Q}_k(T_k)$, there is $\mathcal{R}(Q_0) \subset \mathcal{R}_{k+1}(Q_0)$ with*

$$\#(\mathcal{R}(Q_0)) \geq (1/2) \cdot \#(\mathcal{R}_{k+1}(Q_0)),$$

and for each $R \in \mathcal{R}_{k+1}(Q_0)$,

$$\#(\mathcal{Q}_{k+1}(R \cap S)) = \begin{cases} 1 & : R \in \mathcal{R}(Q_0), \\ 0 & : R \notin \mathcal{R}(Q_0). \end{cases}$$

Proof. Since f is C^2 and has full rank on the set

$$V(f) = \{(x_1,\dots,x_n) \in \mathcal{C}^n(\mathbf{R}^d) : f(x_1,\dots,x_n) = 0\},$$

the implicit function theorem implies $V(f)$ is a smooth manifold of dimension $nd - m$ in $\mathbf{R}^{dn}$, and the coarea formula implies the existence of a constant C such that for each k,

$$\#\{Q \in \mathcal{Q}_k^{dn} : Q \cap V(f) \neq \emptyset\} \leq C/l_k^{dn-m}.$$

To apply Lemma 21 and 22, we set

$$B = \#\{Q \in \mathcal{Q}_k^{dn} : Q \cap V(f) \neq \emptyset\}.$$

Applying Lemma 21 iteratively $n-1$ times, then finishing with Lemma 22, constructs the sets $S_1,\dots,S_n$. □

Just like in Keleti's proof, Pramanik and Fraser's technique applies a discrete result, Corollary 23, iteratively at many scales, with the help of a queueing process, to obtain a high dimensional set avoiding the zeroes of a function. We construct a nested family $\{X_k : k \geq 0\}$ of $\mathcal{Q}_k$ discretized sets, converging to a set X, which we will show is translate avoiding. We initialize $X_0 = [0,1]$. Our queue shall consist of n tuples of disjoint intervals $(T_1,\dots,T_n)$, all of the same length, which initially consists of all possible tuples of intervals in $\mathcal{Q}_1^d([0,1]^d)$. To construct the sequence $\{X_k\}$, we perform the following iterative procedure:

Algorithm 2 Construction of the Sets $\{X_k\}$

Set $k = 0$

Repeat

Take off an n tuple $(T_1',\dots,T_n')$ from the front of the queue

Set $T_i = T_i' \cap X_k$ for each i

Apply Corollary 23 to the sets $T_1,\dots,T_d$, obtaining $\mathcal{Q}_{k+1}$ discretized sets $S_1,\dots,S_n$ satisfying Properties (A), (B), and (C) of that Lemma.

Set $X_{k+1} = X_k - \bigcup_{i=1}^n T_i - S_i$.

Add all n tuples of disjoint cubes $(T_1',\dots,T_n')$ in $\mathcal{Q}_{k+1}^d(X_{k+1})$ to the back of the queue.

Increase k by 1.

Lemma 24. *The set X constructed by the procedure avoids the configuration*

$$\mathcal{C} = \{(x_1,\dots,x_n) \in \mathcal{C}^n(\mathbf{R}^d) : f(x_1,\dots,x_n) = 0\}.$$

Proof. Suppose $x_1,\dots,x_n \in X$ are distinct. Then at some stage k, $x_1,\dots,x_n$ lie in disjoint cubes $T_1',\dots,T_n' \in \mathcal{Q}_k^d(X_k)$, for some large k. At this stage, $(T_1',\dots,T_n')$ is added to the back of the queue, and therefore, at some much later stage N, the tuple $(T_1',\dots,T_n')$ is taken off the front. Sets $S_1 \subset T_1',\dots,S_n \subset T_n'$ are constructed satisfying Property (A) of Corollary 23. Since $x_1,\dots,x_n \in X$, we must have $x_i \in S_i$ for each i, so $f(x_1,\dots,x_n) \neq 0$. $\square$

What remains is to show that for some sequence of parameters $\{N_k\}$ and $\{M_k\}$

satisfying (3.3), we can apply Theorem 15. The conclusion of Corollary 23 shows Property (A) of Theorem 15 is always satisfied. If we set

$$N_{k+1} = \left\lceil \left[C \cdot 2^n (N_1 \dots N_k)^{2dn} \right]^{1/m} M_{k+1}^{d(n-1)/m} \right\rceil,$$

so that (3.3), then both Properties (B) and (B) of Theorem 15 are satisfied with $s = m/d(n-1)$ if $N_1 \dots N_k \lesssim_\varepsilon M_{k+1}^\varepsilon$ for each $\varepsilon > 0$. This is true, for instance, if we set $M_k = 2^{\lfloor 2^{k \log k} \rfloor}$, and Theorem 15 then shows the resultant set X has Hausdorff dimension $m/(n-1)$.

3.3 Mathé: Polynomial Configurations

Mathé's result constructs sets avoiding algebraic varieties.

Theorem 25 (Mathé)**.** *Let $\{f_k : \mathbf{R}^{nd} \to \mathbf{R}\}$ be a countable family of rational coefficient polynomials with degree at most m. Then there exists a set $X \subset [0,1]^d$ with Hausdorff dimension d/m which avoids the configuration*

$$\mathcal{C} = \bigcup_k \{(x_1, \dots, x_n) \in \mathcal{C}^n(\mathbf{R}^d) : f_k(x_1, \dots, x_n) = 0\}.$$

Originally, Mathé's result does not explicitly use a discretization method analogous to Keleti and Pramanik and Fraser, but his proof strategy can be reconfigured to work in this setting. For the purpose of brevity, we do not carry out the complete argument, merely giving the discretization method below. By first trying to avoid the zero sets of the partial derivatives of the function f, one can reduce to the case where a partial derivative of f is non-vanishing on the $\mathcal{Q}_k$ discretized sets we start with. The key technique is that f maps discrete lattices of points to a discrete, 'one dimensional lattice' in $\mathbf{R}^d$, and the degree of the polynomial gives us the difference in lengths between the two lattices. This idea was present in Keleti's work, and one can view Mathé's result as a generalization along these lines. We revisit this idea in Chapter BLAH.

Theorem 26. *Let $f : [0,1]^{dn} \to \mathbf{R}$ be a rational coefficient polynomial of degree m, and disjoint, $\mathcal{Q}_k$ discretized sets $T_1, \dots, T_n \subset [0,1]^d$, such that* $\inf|\partial_1 f| \neq 0$ *on $T_1 \times \cdots \times T_n$. Then there exists a constant C, depending only on f, such that if*

$$N_{k+1} \geq C \cdot (N_1 \dots N_k)^{m-1} \cdot M_{k+1}^m, \tag{3.4}$$

then there exists $\mathcal{Q}_{k+1}$ discretized sets $S_1 \subset T_1, \dots, S_n \subset T_n$ such that

(A) *$f(x) \neq 0$ for $x \in S_1 \times \cdots \times S_n$.*

(B) *For each i, and for each $R \in \mathcal{R}_{k+1}^d(T_i)$, $\#(\mathcal{Q}_{k+1}^d(R \cap S_i)) = 1$.*

Proof. Without loss of generality, by considering an appropriate integer multiple of f, we may assume f has integer coefficients. Let $\mathbf{A} \subset (r_{k+1} \cdot \mathbf{Z})^d$. Since f has degree m, $f(\mathbf{A}) \subset r_{k+1}^m \cdot \mathbf{Z}$. Suppose that $c_0 \leq |\partial_1 f| \leq C_0$ on $T_1 \times \cdots \times T_n$. Then the mean value theorem guarantees that if $\delta \leq r_{k+1}$, then

$$c_0 \delta \leq |f(a + \delta e_1) - f(a)| \leq C_0 \delta.$$

Pick $\varepsilon \in (0,1)$ small enough that $\varepsilon / c_0 < (1-\varepsilon)/C_0$. If

$$l_{k+1} \leq [(1-\varepsilon)/C_0 - \varepsilon/c_0] r_{k+1}^m \tag{3.5}$$

we can find $\delta \in l_{k+1} \cdot \mathbf{Z}$ is such that

$$[\varepsilon/c_0] r_{k+1}^m \leq \delta \leq [(1-\varepsilon)/C_0] r_{k+1}^m.$$

Equation (3.5) is guaranteed by (3.4) for a sufficiently large constant C. We then find $d(f(\mathbf{A} + \delta e_1), r_{k+1}^m \cdot \mathbf{Z}) \geq \varepsilon r_{k+1}^m$. For $i > 1$, define

$$S_i = \bigcup [a_1, a_1 + l_{k+1}] \times \cdots \times [a_d, a_d + l_{k+1}],$$

where $a = (a_1, \ldots, a_d)$ ranges over all startpoints to intervals

$$[a_1, a_1 + r_{k+1}] \times \cdots \times [a_d, a_d + r_{k+1}] \in \mathcal{R}^d_{k+1}(T_i).$$

Similarily, define

$$S_1 = \bigcup [a_1 + \delta e_1, a_1 + \delta e_1 + s] \times [a_2, a_2 + l_{k+1}] \times \cdots \times [a_d, a_d + l_{k+1}],$$

where a ranges over all startpoints to intervals

$$[a_1, a_1 + r_{k+1}] \times \cdots \times [a_d, a_d + r_{k+1}] \in \mathcal{R}^d_{k+1}(T_1).$$

Because f is C^1, we can find C_2 such that if $x, y \in \mathbf{R}^d$ satisfy $|x_i - y_i| \leq t$ for all i, then $|f(x) - f(y)| \leq C_2 t$. But then (3.4), for a sufficiently large constant C, implies that

$$d(f(S_1 \times \cdots \times S_n), r^m_{k+1} \cdot \mathbf{Z}) \geq \varepsilon r^m_{k+1} - C_2 l_{k+1} \geq (\varepsilon/2) r^m_{k+1}.$$

In particular, this implies Property (A). $\square$

Chapter 4

Avoiding Rough Sets

In the last chapter, we saw that many authors have considered the pattern avoidance problem for configurations $\mathcal{C}$ which take the form of many general classes of smooth shapes; in Mathé's work, $\mathcal{C}$ can take the form of an algebraic variety of low degree, and in Pramanik and Fraser's work, $\mathcal{C}$ can take the form of a smooth manifold. In this chapter, we consider the pattern avoidance problem for an even more general class of 'rough' patterns, that are the countable union of sets with controlled lower Minkowski dimension.

Theorem 27. *Let $s \geq d$, and suppose $\mathcal{C} \subset \mathcal{C}^n(\mathbf{R}^d)$ is the countable union of precompact sets, each with lower Minkowski dimension at most s. Then there exists a set $X \subset [0,1]^d$ with Hausdorff dimension at least $(nd-s)/(n-1)$ avoiding $\mathcal{C}$.*

Remarks.

1. When $s < d$, avoiding the configuration $\mathcal{C}$ is trivial. If we define $\pi : \mathcal{C}^n(\mathbf{R}^d) \to \mathbf{R}^d$ by $\pi(x_1, \dots, x_n) = x_1$, then the set $X = [0,1]^d - \pi(\mathcal{C})$ is full dimension and avoids $\mathcal{C}$. Note that obtaining a full dimensional set in the case $s = d$, however, is still interesting.

2. Theorem 27 is trivial when $s = dn$, since we can set $X = \emptyset$. We will therefore assume that $s < dn$ in our proof of the theorem.

3. Let $f : \mathbf{R}^{dn} \to \mathbf{R}^m$ be a C^1 map such that f has full rank at any point $(x_1, \dots, x_n) \in \mathcal{C}^n(\mathbf{R}^d)$ with $f(x_1, \dots, x_n) = 0$. If we set

$$\mathcal{C} = \{x \in \mathcal{C}^n(\mathbf{R}^d) : f(x) = 0\},$$

Then $\mathcal{C}$ is a C^1 submanifold of $\mathcal{C}^n(\mathbf{R}^d)$ of dimension $nd - m$. A submanifold of Euclidean space is σ compact, so we can write $\mathcal{C} = \bigcup K_i$, where each K_i is a compact set. Applying Theorem 2 shows $\dim_{\underline{\mathbf{M}}}(K_i) \leq nd - m$ for each i, so we can apply Theorem 27 with $s = nd - m$ to yield a set in $\mathbf{R}^d$ with Hausdorff dimension at least

$$\frac{nd - s}{n - 1} = \frac{m}{n - 1}.$$

This recovers Theorem 20, making Theorem 27 a generalization of Pramanik and Fraser's result.

4. Since Theorem 27 does not require any regularity assumptions on the set $\mathcal{C}$, it can be applied in contexts that cannot be addressed using previous methods in the literature. Two such applications, new to the best of our knowledge, have been recorded in Chapter 5; see Theorems 31 and 32 there.

Like with the results considered in the last chapter, we construct the set X in Theorem 27 by repeatedly applying a discrete avoidance result at the scales corresponding to cubes $\mathcal{Q}^d$, constructing a Frostman measure with equal mass at intermediary scales, and applying Lemma 14. However, our method has several innovations that simplify the analysis of the resulting set $X = \bigcap X_k$ than from previous results. In particular, through a probabilistic selection process we are able to use a simplified queueing technique then that used in [9] and [6], that required storage of data from each step of the iterated construction to be retrieved at a much later stage of the construction process.

4.1 Avoidance at Discrete Scales

In this section we describe a method for avoiding a discretized version of $\mathcal{C}$ at a single scale. We apply this technique in Section 4.2 at many scales to construct a set X avoiding $\mathcal{C}$ at all scales. In the discrete setting, $\mathcal{C}$ is replaced by a union of cubes in $\mathcal{Q}_{k+1}^{dn}$ denoted by B. We say a cube $Q = Q_1 \times \cdots \times Q_n \in \mathcal{Q}_{k+1}^{dn}$ is *strongly non-diagonal* if the n cubes $Q_1, \dots, Q_n$ are distinct. Given a $\mathcal{Q}_k$ discretized set $T \subset \mathbf{R}^d$, our goal is to construct a $\mathcal{Q}_{k+1}$ discretized set $S \subset T$, such that $\mathcal{Q}_{k+1}^{dn}(F^n)$ does not contain any strongly non-diagonal cubes of $\mathcal{Q}_{k+1}^{dn}(B)$.

Lemma 28. *Fix k, $s \in [1, dn)$, and $\varepsilon \in [0, (dn-s)/2)$. Let $T \subset \mathbf{R}^d$ be a nonempty, $\mathcal{Q}_k$ discretized set, and let $B \subset \mathbf{R}^{dn}$ be a nonempty $\mathcal{Q}_{k+1}$ discretized set such that*

$$\#(\mathcal{Q}_{k+1}(B)) \leq N_{k+1}^{s+\varepsilon}.$$

Then there exists a constant $C(s,d,n) > 0$, depending only on s, d, and n, such that, provided

$$N_{k+1} \geq C(s,d,n) \cdot M_{k+1}^{\frac{d(n-1)}{dn-s-\varepsilon}}, \tag{4.1}$$

then there is a $\mathcal{Q}_{k+1}$ discretized set $S \subset T$ satisfying the following three properties:

(A) *For any collection of n distinct cubes $Q_1, \dots, Q_n \in \mathcal{Q}_{k+1}(S)$,*

$$Q_1 \times \cdots \times Q_n \notin \mathcal{Q}_{k+1}(B).$$

(B) *For each $Q \in \mathcal{Q}_k(T)$, there exists $\mathcal{R}_Q \subset \mathcal{R}_{k+1}(Q)$ such that*

$$\#(\mathcal{R}_Q) \geq \#(\mathcal{R}_{k+1}(Q)),$$

and if $R \in \mathcal{R}_{k+1}(Q)$,

$$\#(\mathcal{Q}_{k+1}(R \cap S)) = \begin{cases} 1 & : R \in \mathcal{R}_Q \\ 0 & : R \notin \mathcal{R}_Q. \end{cases}$$

Proof. For each $R \in \mathcal{R}_{k+1}(T)$, pick Q_R uniformly at random from $\mathcal{Q}_{k+1}(R)$; these choices are independent as R ranges over $\mathcal{R}_{k+1}(T)$. Define

$$A = \bigcup \{Q_R : \ R \in \mathcal{R}_{k+1}(T)\},$$

and

$$\mathcal{K}(A) = \{K \in \mathcal{Q}_{k+1}(B) \cap \mathcal{Q}_{k+1}(A^n) : \ K \text{ strongly non-diagonal}\}.$$

The sets A and $\mathcal{K}(A)$ are random, in the sense that they depend on the random variables $\{Q_R\}$. Define

$$S(A) = \bigcup \Big[\mathcal{Q}_{k+1}(A) - \{\pi(K) : \ K \in \mathcal{K}(A)\} \Big], \tag{4.2}$$

where $\pi \colon \mathbf{R}^{dn} \to \mathbf{R}^d$ is the projection map $(x_1, \dots, x_n) \mapsto x_1$, for $x_i \in \mathbf{R}^d$.

Given any strongly non-diagonal cube $K = K_1 \times \cdots \times K_n \in \mathcal{Q}_{k+1}(B)$, either $K \notin \mathcal{Q}_{k+1}(A^n)$, or $K \in \mathcal{Q}_{k+1}(A^n)$. If the former occurs then $K \notin \mathcal{Q}_{k+1}(S(A))$ since $S(A) \subset A$, so $\mathcal{Q}_{k+1}(S(A)^n) \subset \mathcal{Q}_{k+1}(A^n)$. If the latter occurs then $K \in \mathcal{K}(A)$, and since $\pi(K) = K_1$, $K_1 \notin \mathcal{Q}_{k+1}(S(A))$. In either case, $K \notin \mathcal{Q}_{k+1}(S(A)^n)$, so $S(A)$ satisfies Property (A). By construction, we know that for each $R \in \mathcal{R}_{k+1}(T)$,

$$\#(\mathcal{Q}_{k+1}(S(A) \cap R)) \leq \#(\mathcal{Q}_{k+1}(A \cap R)) = 1. \tag{4.3}$$

To conclude that $S(A)$ satisfies Property (B), it therefore suffices to show that

$$\#(\mathcal{K}(A)) \leq (1/2) \cdot M_{k+1}^d.$$

We now show this is true with non-zero probability.

For each cube $Q \in \mathcal{Q}_{k+1}(T)$, there is a unique 'parent' cube $R \in \mathcal{R}_{k+1}(T)$ such that $Q \subset R$. Note that (4.1) implies, if $C(s,d,n) \geq 4d$, that $N_{k+1} \geq 4d \cdot M_{k+1}$.

Since Q_R is chosen uniformly from $\mathcal{Q}_{k+1}(R)$, for any $Q \in \mathcal{Q}_{k+1}(R)$,

$$\mathbf{P}(Q \subset A) = \mathbf{P}(Q_R = Q) = (\mathcal{Q}_{k+1}(R))^{-1} = (M_{k+1}/N_{k+1})^d.$$

Suppose $K_1, \dots, K_n \in \mathcal{Q}_{k+1}(T)$ are distinct cubes. If the cubes have distinct parents in $\mathcal{R}^d_{k+1}$, we can apply the independence of the random cubes $\{Q_R\}$ to conclude that

$$\mathbf{P}(K_1, \dots, K_n \subset A) = (M_{k+1}/N_{k+1})^{dn}.$$

If the cubes $K_1, \dots, K_n$ do not have distinct parents, (4.3) shows

$$\mathbf{P}(K_1, \dots, K_n \subset A) = 0.$$

In either case, we conclude that

$$\mathbf{P}(K_1, \dots, K_n \subset A) \leq (M_{k+1}/N_{k+1})^{dn}. \tag{4.4}$$

Let $K = K_1 \times \cdots \times K_n$ be a strongly non-diagonal cube in $\mathcal{Q}_{k+1}(B)$. We deduce from (4.4) that

$$\mathbf{P}(K \subset A^n) = \mathbf{P}(K_1, \dots, K_n \subset A) \leq (M_{k+1}/N_{k+1})^{dn}. \tag{4.5}$$

If

$$C(s,d,n) \geq 4^{\frac{1}{dn-s}} \geq 2^{\frac{1}{dn-s-\varepsilon}},$$

by (4.5), linearity of expectation, and (4.1), we conclude

$$\begin{aligned}
\mathbf{E}(\#(\mathcal{K}(A))) &= \sum_{K \in \mathcal{Q}_{k+1}(B)} \mathbf{P}(K \subset A^n) \\
&\leq \#(\mathcal{Q}_{k+1}(B)) \cdot (M_{k+1}/N_{k+1})^{dn} \\
&\leq M_{k+1}^{dn}/N_{k+1}^{dn-s-\varepsilon} \\
&\leq (1/2) \cdot M_{k+1}^d.
\end{aligned}$$

In particular, there exists at least one (non-random) set A_0 such that

$$\#(\mathcal{K}(A_0)) \leq \mathbf{E}(\#(\mathcal{K}(A))) \leq (1/2) \cdot M_{k+1}^d. \tag{4.6}$$

In other words, $S(A_0) \subset A_0$ is obtained by removing at most $(1/2) \cdot M_{k+1}^d$ cubes in $\mathcal{B}_s^d$ from A_0. For each $Q \in \mathcal{B}_l^d(T)$, we know that $\#\,\mathcal{Q}_{k+1}(Q \cap A_0) = M_{k+1}^d$. Combining this with (4.6), we arrive at the estimate

$$\begin{aligned}
\#\,\mathcal{Q}_{k+1}(Q \cap S(A_0)) &= \mathcal{Q}_{k+1}(Q \cap A_0) - \#\{\pi(K) :\ K \in \mathcal{K}(A_0), \pi(K) \in A_0\} \\
&\geq \mathcal{Q}_{k+1}(Q \cap A_0) - \#(\mathcal{K}(A_0)) \\
&\geq M_{k+1}^d - (1/2) \cdot M_{k+1}^d \geq (1/2) \cdot M_{k+1}^d
\end{aligned}$$

Thus, $S(A_0)$ satisfies Property (B). Setting $S = S(A_0)$ completes the proof. $\square$

Remarks.

1. While Lemma 28 uses probabilistic arguments, the proof of the lemma is still constructive. In particular, one can find a suitable S constructively by checking every possible choice of A (there are finitely many) to find one particular choice A_0 which satisfies (4.6), and then defining S by (4.2). Thus the set we obtain in Theorem 27 exists by purely constructive means.

2. As with the proofs in Chapter 3, the fact that we can choose

$$N_{k+1} \sim M_{k+1}^{(1-\varepsilon)/t}$$

where $t = (dn - s)/d(n-1)$, implies we should be able to iteratively apply Lemma 28 to obtain a set with Hausdorff dimension $(dn - s)/(n-1)$.

4.2 Fractal Discretization

In this section we construct the set X from Theorem 27 by applying Lemma 28 at many scales. Let us start by fixing a strong cover of $\mathcal{C}$, as well as the branching factors $\{N_k\}$, that we will work with in the sequel.

Lemma 29. *Let $\mathcal{C} \subset \mathcal{C}^n(\mathbf{R}^d)$ be a countable union of bounded sets with Minkowski dimension at most s, and let $\varepsilon_k \searrow 0$ with $\varepsilon_k < (dn-s)/2$ for all k. Then there exists a choice of branching factors $\{N_k : k \geq 1\}$, and a sequence of sets $\{B_k : k \geq 1\}$, with $B_k \subset \mathbf{R}^{dn}$ for all k, such that*

(A) Strong Cover*: The interiors $\{B_k^\circ\}$ of the sets $\{B_k\}$ form a strong cover of $\mathcal{C}$.*

(B) Discreteness*: For all $k \geq 1$, B_k is a $\mathcal{Q}_k$ discretized subset of $\mathbf{R}^{dn}$.*

(C) Sparsity*: For all $k \geq 1$, $\mathcal{Q}_k(B_k) \leq N_k^{s+\varepsilon_k}$.*

(D) Rapid Decay*: For all $\varepsilon > 0$ and $k \geq 1$, N_k is a power of two such that*

$$N_1 \dots N_{k-1} \lesssim_\varepsilon N_k^\varepsilon,$$

and for all $k \geq 1$,

$$N_k \geq C(s,d,n).$$

Proof. We can write $\mathcal{C} = \bigcup_{i=1}^\infty Y_i$, with $\dim_{\underline{\mathbf{M}}}(Y_i) \leq s$ for each i. Let $\{i_k\}$ be a sequence of integers that repeats each integer infinitely often. The branching factors $\{N_k\}$ and sets $\{B_k\}$ are defined inductively. Suppose that the lengths $N_1, \dots, N_{k-1}$ have been chosen. Since $\dim_{\underline{\mathbf{M}}}(Y_{i_k}) < s + \varepsilon_k$, there are arbitrarily small lengths $l \leq l_{k-1}$ such that if $N_k = l_{k-1}/l$,

$$\#(\mathcal{Q}_k(Y_{i_k}(l))) \leq (1/l)^{s+(\varepsilon_k/2)}. \tag{4.7}$$

In particular, we can choose l small enough that $l \leq l_{k-1}^{2s/\varepsilon+2}$, which together with (4.7) implies

$$\#(\mathcal{Q}_k(Y_{i_k})(l)) \leq N_k^{s+\varepsilon_k}. \tag{4.8}$$

We can certainly also choose l small enough that

$$N_k \geq \max(C(s,d,n), (N_1 \dots N_{k-1})^{1/\varepsilon_k}).$$

We know that l_{k-1} is a power of two, so we can ensure N_k is a power of two. We then set $l_k = l$. With this choice, Property (D) is satisfied. And if $B_k = \bigcup \mathcal{Q}_k(Y_{i_k}(l_k))$, then this choice of B_k clearly satisfies Property (B). And Property (C) is precisely Equation (4.8).

It remains to verify that the sets $\{B_k^\circ\}$ strongly cover $\mathcal{C}$. Fix a point $z \in \mathcal{C}$. Then there exists an index i such that $z \in Y_i$, and there is a subsequence $k_1, k_2, \dots$ such that $i_{k_j} = i$ for each j. But then $z \in Y_i \subset B^\circ_{i_{k_j}}$, so z is contained in each of the sets $B^\circ_{i_{k_j}}$, and thus $z \in \limsup B_i^\circ$. Thus Property (A) is proved. □

To construct X, we consider a nested, decreasing family of sets $\{X_k\}$, where each X_k is an l_k discretized subset of $\mathbf{R}^d$. We then set $X = \bigcap X_k$. The goal is to choose X_k such that X_k^n does not contain any *strongly non diagonal* cubes in B_k.

Lemma 30. *For each k, let $B_k \subset \mathbf{R}^{dn}$ be a $\mathcal{Q}_k$ discretized set, such that the interiors $\{B_k^\circ\}$ strongly cover $\mathcal{C}$. For each index k, let X_k be a $\mathcal{Q}_k$ discretized set such that $\mathcal{Q}_k(X_k^n) \cap \mathcal{Q}_k(B_k)$ contains no strongly non diagonal cubes. If $X = \bigcap X_k$, then X avoids B.*

Proof. Let $x = (x_1, \dots, x_n) \in \mathcal{C}$ be a point such that $x_1, \dots, x_n$ are distinct. Define

$$\Delta = \{(y_1, \dots, y_n) \in \mathbf{R}^{dn} : \text{ there exists } i \neq j \text{ such that } y_i = y_j\}.$$

Then $d(\Delta, x) > 0$, where d is the Hausdorff distance between Δ and x. Since $\{B_k\}$ strongly covers $\mathcal{C}$, there is a subsequence $\{k_m\}$ such that $x \in B^\circ_{k_m}$ for every index m. Since l_k converges to 0 and thus l_{k_m} converges to 0, if m is sufficiently large then $\sqrt{dn} \cdot l_{k_m} < d(\Delta, x)$. Note that $\sqrt{dn} \cdot l_{k_m}$ is the diameter of a cube in $\mathcal{Q}_{k_m}$. For such a choice of m, any cube $Q \in \mathcal{Q}^d_{k_m}$ which contains x is strongly non-diagonal. Thus $x \in B^\circ_{k_m}$. Since X_{k_m} and B_{k_m} share no cube which contains x, this implies $x \notin X_{k_m}$. In particular, this means $x \notin X^n$. □

All that remains is to apply the discrete lemma to choose the sequence $\{X_k\}$. Given $\mathcal{C}$, apply Lemma 29 to choose a sequence $\{N_k\}$ and a sequence $\{B_k\}$. We recursively define the sequence $\{X_k\}$. Set $X_0 = [0,1]^d$. Then, Property (D) of Lemma 29 implies

$$\#(\mathcal{Q}_{k+1}(B_{k+1})) \leq N_{k+1}^{s+\varepsilon_k}$$

Let M_{k+1} be the largest power of two smaller than

$$\left(\frac{N_{k+1}}{C(s,d,n)}\right)^{\frac{dn-s-\varepsilon_k}{d(n-1)}} \tag{4.9}$$

Then $1 \leq M_{k+1} \leq N_{k+1}$, and (4.1) is satisfied. Set $B = B_{k+1}$, and $T = X_k$. Then we can apply Lemma 28 to find a set S satisfying all Properties of that Lemma, and set $X_{k+1} = S$.

Property (A) of Lemma 28 together with Lemma 30 implies that the set $X = \bigcap X_k$ avoids $\mathcal{C}$. Property (B) of Lemma 28 shows that the sequence $\{X_k\}$ satisfies Property (A) of Theorem 15. Property (D) of Lemma 29 implies Property (B) of Theorem 15 is satisfied. And by definition, i.e. the choice of $\{M_k\}$ as given by (4.9),

$$N_k \leq M_k^{\frac{d(n-1)}{dn-s-\varepsilon_k}} \lesssim_\varepsilon M_k^{\frac{1+\varepsilon}{t}} \tag{4.10}$$

where $t = (nd-s)/d(n-1)$. (4.10) is a form of Property (C) of Theorem 15. Thus all assumptions of Theorem 15 are satisfied, and so we find X has Hausdorff dimension $(nd-s)/(n-1)$.

Chapter 5

Applications

As discussed in the introduction, Theorem 27 generalizes Theorems 1.1 and 1.2 from [6]. In this chapter, we present two applications of Theorem 27 in settings where previous methods do not yield any results.

5.1 Sum-sets avoiding specified sets

Theorem 31. *Let $Y \subset \mathbf{R}^d$ be a countable union of sets with lower Minkowski dimension at most t. Then there exists a set $X \subset \mathbf{R}^d$ with Hausdorff dimension at least $d - t$ such that $X + X$ is disjoint from Y.*

Proof. Define $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$, where

$$\mathcal{C}_1 = \{(x,y) : x + y \in Y\} \quad \text{and} \quad \mathcal{C}_2 = \{(x,y) : y \in Y/2\}.$$

Since Y is a countable union of sets with lower Minkowski dimension at most t, $\mathcal{C}$ is a countable union of sets with lower Minkowski dimension at most $d + t$. Applying Theorem 27 with $n = 2$ and $s = d + t$ produces a set $X \subset \mathbf{R}^d$ with Hausdorff dimension $2d - (d + t) = d - t$ such that $(x,y) \notin \mathcal{C}$ for all $x, y \in X$ with $x \neq y$. We claim that $X + X$ is disjoint from Y. To see this, first suppose $x, y \in X$, $x \neq y$. Since X avoids Z_1, we conclude that $x + y \notin Y$. Suppose now that

$x = y \in X$. Since X avoids Z_2, we deduce that $X \cap (Y/2) = \emptyset$, and thus for any $x \in X$, $x + x = 2x \notin Y$. This completes the proof. $\square$

5.2 Subsets of Lipschitz curves avoiding isosceles triangles

In [6], Fraser and the second author prove that there exists a set $S \subset [0,1]$ with dimension $\log_3 2$ such that for any simple C^2 curve $\gamma\colon [0,1] \to \mathbf{R}^n$ with bounded non-vanishing curvature, $\gamma(S)$ does not contain the vertices of an isosceles triangle. Our method enables us to obtain a result that works for Lipschitz curves with small Lipschitz constants. The dimensional bound that we provide is slightly worse than [6] ($1/2$ instead of $\log_3 2$), and the set we obtain only works for a single Lipschitz curve, not for many curves simultaneously.

Theorem 32. *Let $f\colon [0,1] \to \mathbf{R}^{n-1}$ be Lipschitz with*

$$\|f\|_{Lip} := \sup\big\{|f(x) - f(y)|/|x-y| : x, y \in [0,1], x \neq y\big\} < 1.$$

Then there is a set $X \subset [0,1]$ of Hausdorff dimension $1/2$ so that the set

$$\big\{(t, f(t)) :\ t \in X\big\}$$

does not contain the vertices of an isosceles triangle.

Corollary 33. *Let $f\colon [0,1] \to \mathbf{R}^{n-1}$ be C^1. Then there is a set $X \subset [0,1]$ of Hausdorff dimension $1/2$ so that the set*

$$\big\{(t, f(t)) :\ t \in X\big\}$$

does not contain the vertices of an isosceles triangle.

Proof of Corollary 33. The graph of any C^1 function can be locally expressed, after possibly a translation and rotation, as the graph of a Lipschitz function with small Lipschitz constant. In particular, there exists an interval $I \subset [0,1]$ of positive length so that the graph of f restricted to I, after being suitably translated and rotated, is the graph of a Lipschitz function $g\colon [0,1] \to \mathbf{R}^{n-1}$ with Lipschitz constant at most $1/2$. Since isosceles triangles remain invariant under these transformations, the corollary is a consequence of Theorem 32. $\square$

Proof of Theorem 32. Set

$$\mathcal{C} = \left\{ (x_1,x_2,x_3) \in \mathcal{C}^3[0,1] : \begin{array}{c} \text{The points } p_j = (x_j, f(x_j)) \text{ form} \\ \text{the vertices of an isosceles triangle} \end{array} \right\}. \tag{5.1}$$

In the next lemma, we show $\mathcal{C}$ has lower Minkowski dimension at most two. By Theorem 27, there is a set $X \subset [0,1]$ of Hausdorff dimension $1/2$ so that X avoids $\mathcal{C}$. This is precisely the statement that for each $x_1,x_2,x_3 \in X$, the points $(x_1, f(x_1))$, $(x_2, f(x_2))$, and $(x_3, f(x_3))$ do not form the vertices of an isosceles triangle. $\square$

Lemma 34. *Let $f\colon [0,1] \to \mathbf{R}^{n-1}$ be Lipschitz with $\|f\|_{Lip} < 1$. Then the set $\mathcal{C}$ given by* (5.1) *satisfies* $\dim_{\overline{\mathbf{M}}}(\mathcal{C}) \leq 2$.

Proof. First, notice that three points $p_1, p_2, p_3 \in \mathbf{R}^n$ form an isosceles triangle, with p_3 as the apex, if and only if $p_3 \in H_{p_1,p_2}$, where

$$H_{p_1,p_2} = \left\{ x \in \mathbf{R}^n : \left(x - \frac{p_1 + p_2}{2} \right) \cdot (p_2 - p_1) = 0 \right\}. \tag{5.2}$$

To prove $\mathcal{C}$ has Minkowski has dimension at most two, it suffices to show the set

$$W = \left\{ x \in [0,1]^3 : \; p_3 = (x_3, f(x_3)) \in H_{p_1,p_2} \right\}$$

has upper Minkowski dimension at most 2. This is because $\mathcal{C}$ is covered by three copies of W, obtained by permuting coordinates. We work with the family of dyadic cubes $\mathcal{D}^n$. To bound the upper Minkowski dimension of W, we prove the estimate

$$\#\big(\mathcal{D}_k(W(1/2^k))\big) \le Ck4^k \quad \text{for all } k \ge 1, \tag{5.3}$$

where C is a constant independent of k. Then for any $\varepsilon > 0$, (5.3) implies that for suitably large k,

$$\#\big(\mathcal{D}_k(W(1/2^k))\big) \le 2^{(2+\varepsilon)k}.$$

Since ε was arbitrary, this shows $\dim_{\overline{\mathbf{M}}}(W) \le 2$.

To establish (5.3), we write

$$\#\big(\mathcal{D}_k(W(1/2^k))\big) = \sum_{m=0}^{2^k} \sum_{\substack{I_1,I_2\in\mathcal{D}_k[0,1] \\ d(I_1,I_2)=m/2^k}} \#\Big(\mathcal{D}_k(W(1/2^k)\cap(I_1\times I_2\times[0,1])\Big). \tag{5.4}$$

Our next task is to bound each of the summands in (5.4). Let $I_1, I_2 \in \mathcal{D}_k[0,1]$, and let $m = 2^k \cdot d(I_1,I_2)$. Let x_1 be the midpoint of I_1, and x_2 the midpoint of I_2. Let $(y_1,y_2,y_3) \in W\cap(I_1\times I_2\times[0,1])$. Then it follows from (5.2) that

$$\left(y_3-\frac{y_1+y_2}{2}\right)\cdot(y_2-y_1)+\left(f(y_3)-\frac{f(y_2)+f(y_1)}{2}\right)\cdot(f(y_2)-f(y_1))=0.$$

We know $|x_1-y_1|, |x_2-y_2| \le 1/2^{k+1}$, so

$$\begin{aligned}
&\left|\left(y_3-\frac{y_1+y_2}{2}\right)(y_2-y_1)-\left(y_3-\frac{x_1+x_2}{2}\right)(x_2-x_1)\right| \\
&\quad\le \frac{|y_1-x_1|+|y_2-x_2|}{2}|y_2-y_1|+\Big(|y_1-x_1|+|y_2-x_2|\Big)\left|y_3-\frac{x_1+x_2}{2}\right| \\
&\quad\le (1/2^{k+1})\cdot 1+(1/2^k)\cdot 1 \le 3/2^{k+1}.
\end{aligned} \tag{5.5}$$

Conversely, $|f(x_1)-f(y_1)|, |f(x_2)-f(y_2)| \le 1/2^{k+1}$ because $\|f\|_{\mathrm{Lip}} \le 1$, and a similar calculation yields

$$\begin{aligned}\Big|\left(f(y_3)-\frac{f(y_1)+f(y_2)}{2}\right)\cdot(f(y_2)-f(y_1))\\ -\left(f(y_3)-\frac{f(x_1)+f(x_2)}{2}\right)\cdot(f(x_2)-f(x_1))\Big|\leq 3/2^{k+1}.\end{aligned}\tag{5.6}$$

Putting (5.5) and (5.6) together, we conclude that

$$\begin{aligned}\Big|\left(y_3-\frac{x_1+x_2}{2}\right)(x_2-x_1)\\ +\left(f(y_3)-\frac{f(x_2)+f(x_1)}{2}\right)\cdot(f(x_2)-f(x_1))\Big|\leq 3/2^{k}.\end{aligned}\tag{5.7}$$

Since $|(x_2-x_1, f(x_2)-f(x_1))| \geq |x_2-x_1| \geq m/2^k$, we can interpret (5.7) as saying the point $(y_3, f(y_3))$ is contained in a $3/k$ thickening of the hyperplane $H_{(x_1,f(x_1)),(x_2,f(x_2))}$. Given another value $y' \in W \cap (I_1 \cap I_2 \cap [0,1])$, it satisfies a variant of the inequality (5.7), and we can subtract the difference between the two inequalities to conclude

$$\left|(y_3-y_3')(x_2-x_1)+(f(y_3)-f(y_3'))\cdot(f(x_2)-f(x_1))\right|\leq 6/2^k.\tag{5.8}$$

The triangle difference inequality applied with (5.8) implies

$$\begin{aligned}(f(y_3)-f(y_3'))\cdot(f(x_2)-f(x_1))&\geq |y_3-y_3'||x_2-x_1|-6/2^k\\ &=\frac{(m+1)\cdot|y_3-y_3'|-6}{2^k}.\end{aligned}\tag{5.9}$$

Conversely,

$$\begin{aligned}(f(y_3)-f(y_3'))\cdot(f(x_2)-f(x_1))&\leq \|f\|_{\mathrm{Lip}}^2\cdot|y_3-y_3'||x_2-x_1|\\ &=\|f\|_{\mathrm{Lip}}^2\cdot(m+1)/2^k\cdot|y_3-y_3'|.\end{aligned}\tag{5.10}$$

Combining (5.9) and (5.10) and rearranging, we see that

$$|y_3 - y_3'| \leq \frac{6}{(m+1)(1-\|f\|_{\mathrm{Lip}}^2)} \lesssim \frac{1}{m+1}, \tag{5.11}$$

where the implicit constant depends only on $\|f\|_{\mathrm{Lip}}$. We conclude that

$$\#\,\mathcal{Q}_k(W(1/2^k) \cap (I_1 \times I_2 \times [0,1])) \lesssim \frac{2^k}{m+1}, \tag{5.12}$$

which holds uniformly over any value of m.

We are now ready to bound the sum from (5.4). Note that for each value of m, there are at most 2^{k+1} pairs (I_1, I_2) with $d(I_1, I_2) = m/2^k$. Indeed, there are 2^k choices for I_1 and then at most two choices for I_2. Equation (5.12) shows

$$\begin{aligned}
\#\,\mathcal{Q}_k(W(1/2^k)) &= \sum_{m=0}^{2^k} \sum_{\substack{I_1, I_2 \in \mathcal{Q}_k[0,1] \\ d(I_1,I_2)=m/2^k}} \#\,\mathcal{Q}_k(W(1/2^k) \cap (I_1 \times I_2 \times [0,1])) \\
&\lesssim 4^k \sum_{m=0}^{2^k} \frac{1}{m+1} \lesssim k/4^k.
\end{aligned}$$

In the above inequalities, the implicit constants depend on $\|f\|_{\mathrm{Lip}}$, but they are independent of k. This establishes (5.3) and completes the proof. $\square$

Chapter 6

Conclusions

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