Cartesian Products Avoiding Patterns

by

Jacob Denson

BSc. Computing Science, University of Alberta, 2017

A THESIS SUBMITTED IN PARTIAL FULFILLMENT
OF THE REQUIREMENTS FOR THE DEGREE OF

Master of Science

in

THE FACULTY OF SCIENCE
(Mathematics)

The University of British Columbia
(Vancouver)

April 2019

The following individuals certify that they have read, and recommend to the Faculty of Graduate and Postdoctoral Studies for acceptance, the thesis entitled:

Cartesian Products Avoiding Patterns

submitted by **Jacob Denson** in partial fulfillment of the requirements for the degree of **Master of Science** in **Mathematics**.

Examining Committee:

Malabika Pramanik, Mathematics
Supervisor

Joshua Zahl, Mathematics
Supervisor

Abstract

The pattern avoidance problem seeks to construct a set $X \subset \mathbf{R}^d$ with large dimension that avoids a prescribed pattern such as three term arithmetic progressions, or more general patterns such as avoiding points $x_1, \dots, x_n$ such that $f(x_1, \dots, x_n) = 0$ (three term arithmetic progressions are specified by the pattern $x_1 - 2x_2 + x_3 = 0$). Previous work on the subject has considered patterns described by polynomials, or by functions f satisfying certain regularity conditions. We consider the case of 'rough patterns. There are several problems that fit into the framework of rough pattern avoidance. As a first application, if $Y \subset [0,1]$ is a set with Minkowski dimension α, we construct a set $X \subset [0,1]$ with Hausdorff dimension $1 - \alpha$ so that $X + X$ is disjoint from Y. As a second application, given a set Y of dimension close to one, we can construct a subset $X \subset Y$ of dimension $1/2$ that avoids isosceles triangles.

Lay Summary

The lay or public summary explains the key goals and contributions of the research/scholarly work in terms that can be understood by the general public. It must not exceed 150 words in length.

Preface

At University of British Columbia (UBC), a preface may be required. Be sure to check the Graduate and Postdoctoral Studies (GPS) guidelines as they may have specific content to be included.

Table of Contents

List of Figures

Glossary

This glossary uses the handy `acroynym` package to automatically maintain the glossary. It uses the package's `printonlyused` option to include only those acronyms explicitly referenced in the LaTeX source.

GPS Graduate and Postdoctoral Studies

Acknowledgments

Thank those people who helped you.

Don’t forget your parents or loved ones.

You may wish to acknowledge your funding sources.

Chapter 1

Introduction

In this thesis, we study a simple question:

How large can Euclidean sets be not containing geometric patterns?.

The patterns manifest as certain configurations of point tuples, often affine invariant. For instance, we might like to find large subsets X of $\mathbf{R}^d$ that contain no three collinear points, or do not contain the vertices of any isoceles triangle. Aside from purely geometric interest, these problems provide useful scenarios in which to test the methods of ergodic theory, additive combinatorics and harmonic analysis. Sets which avoid the configurations we consider are highly irregular, and so also require techniques from geometric measure theory to be understood. In particular, we use the *fractal dimension* as a measure of a sets size when comparing how large two sets which avoid patterns are.

At the time of the writing of this thesis, many fundamental questions about the relation between the geometric arrangement of a set and it's fractal dimension remain unsolved. It might be expected that sets with sufficiently large Hausdorff dimension contain patterns. For example, Theorem 6.8 of [12] shows that any set $X \subset \mathbf{R}^d$ with Hausdorff dimension exceeding one must contain three colinear points. On the other hand, [10] constructs a set $X \subset \mathbf{R}^d$ with full Hausdorff dimension such that no four points in X form the vertices of a parallelogram. Thus the problem must depend on the particular aspects of the configurations involved. For most geometric configurations, it remains unknown at what threshold patterns are guaranteed, or whether such a threshold exists at all. So finding sets with large Hausdorff dimension avoiding patterns is an important topic to determine for which classes of patterns our intuition remains true.

Our goal in this thesis is to derive new methods for constructing sets avoiding patterns. And rather than studying particular instances of the configuration avoidance problem, we choose to study general methods for finding large subsets of space avoiding any pattern. In particular, we expand on a number of general *pattern dissection methods* which have proven useful in the area, originally developed by Keleti but also studied notably by Mathé, and Pramanik/Fraser.

The main contribution we give to pattern dissection methods is using random dissections to avoid patterns without much structural information. Thus we can expand the utility of interval dissection methods from regular patterns to a set of *fractal avoidance problems*, that previous methods were completely unavailable to address. Such problems include finding large X such that the angles formed by any three distinct points of X avoid a specified set Y of angles, where Y has a fixed fractal dimension.

Chapter 2

Background

2.1 Configuration Avoidance

We begin by describing the problem we wish to address in this thesis. We consider an ambient set $\mathbf{A}$. It's *n-point configuration space* is

$$\mathcal{C}^n(\mathbf{A}) = \{(x_1, \dots, x_n) \in X^n : x_i \neq x_j \text{ if } i \neq j\}.$$

An *n point configuration*, or *pattern*, is a subset of $\mathcal{C}^n(\mathbf{A})$. More generally, we define the general *configuration space* of $\mathbf{A}$ as $\mathcal{C}(\mathbf{A}) = \bigcup_{n=1}^{\infty} \mathcal{C}^n(\mathbf{A})$, and a *pattern*, or *configuration*, on $\mathbf{A}$ is a subset of $\mathcal{C}(\mathbf{A})$.

Our main focus on the thesis is the *pattern avoidance problem*. For a fixed configuration $\mathcal{C}$ on $\mathbf{A}$, we say a set $X \subset \mathbf{A}$ *avoids* $\mathcal{C}$ if $\mathcal{C}(X)$ is disjoint from $\mathcal{C}$. The pattern avoidance problem asks to find sets X of maximal size avoiding a fixed configuration $\mathcal{C}$. The set $\mathcal{C}$ often describes the presence of algebraic or geometric structure, and so we are trying to find large sets which do not possess this structure.

Example (Isoceles Triangle Configuration)**.** *Let*

$$\mathcal{C} = \left\{ (x_1, x_2, x_3) \in \mathcal{C}^3(\mathbf{R}^2) : |x_1 - x_2| = |x_1 - x_3| \right\}.$$

Then $\mathcal{C}$ is a 3-point configuration, and a set $X \subset \mathbf{R}^2$ avoids $\mathcal{C}$ if and only if it does not contain all vertices of an isoceles triangle.

Example (Linear Independence Configuration)**.** *Let V be a vector space over a field K. We set*

$$\mathcal{C} = \bigcup_{n=1}^{\infty} \{(x_1, \dots, x_n) \in \mathcal{C}^n(V) : \textit{for any } a_1, \dots, a_n \in K,\ a_1 x_1 + \dots + a_n x_n \neq 0\}.$$

A subset $X \subset V$ avoids $\mathcal{C}$ if and only if X is a linearly independent subset of X. Note that if V is infinite dimensional, then $\mathcal{C}$ cannot be replaced by a n point configuration for any n; arbitrarily large tuples must be considered. We will be interested in the case where $K = \mathbf{Q}$, and $V = \mathbf{R}$, where we will be looking for analytically large, linearly independant subsets of V.

Even though our problem formulation assumes configurations are formed by distinct sets of points, one can still formulate avoidance problems involving repeated points in our framework by a simple trick.

Example (Sum Set Configuration)**.** *Let G be an abelian group, and fix $Y \subset G$. Set*

$$\mathcal{C}^1 = \{g \in \mathcal{C}^1(G) : g + g \in Y\} \quad \textit{and} \quad \mathcal{C}^2 = \{(g_1, g_2) \in \mathcal{C}^2(G) : g_1 + g_2 \in Y\}.$$

Then set $\mathcal{C} = \mathcal{C}^1 \cup \mathcal{C}^2$. A set $X \subset G$ avoids $\mathcal{C}$ if and only if $(X + X) \cap Y = \emptyset$.

Depending on the structure of the ambient space $\mathbf{A}$ and the configuration $\mathcal{C}$, there are various ways of measuring the size of sets $X \subset \mathbf{A}$:

- If $\mathbf{A}$ is finite, the goal is to find a set X with large cardinality.
- If $\{\mathbf{A}_n\}$ is an increasing family of finite sets with $\mathbf{A} = \lim \mathbf{A}_n$, the goal is to find a set X such that $X \cap \mathbf{A}_n$ has large cardinality asymptotically in n.
- If $\mathbf{A} = \mathbf{R}^d$, but $\mathcal{C}$ is a discrete configuration, then a satisfactory goal is to find a set X with large Lebesgue measure avoiding $\mathcal{C}$.

In this thesis, inspired by results in these three settings, we establish methods for avoiding non-discrete configurations $\mathcal{C}$ in $\mathbf{R}^d$. Here, Lebesgue measure completely fails to measure the size of pattern avoiding solutions, as the next theorem shows, under the often true assumption that $\mathcal{C}$ is *translation invariant*, i.e. that if $(a_1,\dots,a_n) \in \mathcal{C}$ and $b \in \mathbf{R}^d$, $(a_1+b,\dots,a_n+b) \in \mathcal{C}$.

Theorem 1. *Let $\mathcal{C}$ be a n-point configuration on $\mathbf{R}^d$. Suppose*

(A) *$\mathcal{C}$ is translation invariant.*

(B) *For any $\varepsilon > 0$, there is $(a_1,\dots,a_n) \in \mathcal{C}$ with diam$\{a_1,\dots,a_n\} \leq \varepsilon$.*

Then no set with positive Lebesgue measure avoids $\mathcal{C}$.

Proof. Let $X \subset \mathbf{R}^d$ have positive Lebesgue measure. The Lebesgue density theorem shows that there exists a point $x \in X$ such that

$$\lim_{l(Q)\to 0} \frac{|X \cap Q|}{|Q|} = 1, \tag{2.1}$$

where Q ranges over all cubes in $\mathbf{R}^d$, and $l(Q)$ denotes the sidelength of Q. Fix $\varepsilon > 0$, to be specified later, and choose r small enough that $|X \cap Q| \geq (1-\varepsilon)|Q|$ for any cube Q with $x \in Q$ and $l(Q) \leq r$. Now let Q_0 denote the cube centered at x with $l(Q_0) \leq r$. Applying Property (B), we find $C = (a_1,\dots,a_n) \in \mathcal{C}$ such that diam$\{a_1,\dots,a_n\} \leq l(Q)/2$. For each $p \in Q_0$, let $C(p) = (a_1(p),\dots,a_n(p))$, where $a_i(p) = p + (a_i - a_1)$. Property (A) implies $C(p) \in \mathcal{C}$ for each $p \in \mathbf{R}^d$. A union bound shows

$$|\{p \in Q_0 : C(p) \not\in \mathcal{C}(X)\}| \leq \sum_{i=1}^{d} |\{p \in Q_0 : a_i(p) \not\in X\}|. \tag{2.2}$$

Note that $a_i(p) \not\in X$ precisely when $p + (a_i - a_1) \not\in X$, so

$$|\{p \in Q_0 : a_i(p) \not\in X\}| = |(Q_0 + (a_i - a_1)) \cap X^c|. \tag{2.3}$$

Note $Q_0 + (a_i - a_1)$ is a cube with the same sidelength as Q_0. Since $|a_i - a_1| \leq r/2$, $x \in Q_0 + (a_i - a_1)$. Thus (2.1) shows

$$|Q_0 + (a_i - a_1)) \cap X^c| \leq \varepsilon |Q_0|. \tag{2.4}$$

Combining (2.2), (2.3), and (2.4), we find

$$|\{p \in Q_0 : C(p) \notin \mathcal{C}(X)\}| \leq \varepsilon d |Q_0|.$$

Provided $\varepsilon d < 1$, this means there is $p \in Q_0$ with $C(p) \in \mathcal{C}(X)$. $\square$

Since no set of positive Lebesgue measure can avoid non-discrete configurations, we cannot use the Lebesgue measure to quantify the size of pattern avoiding sets. Fortunately, there is a quantity which can distinguish between the size of sets of measure zero. This is the *fractional dimension* of a set.

2.2 Fractional Dimension

There are many variants of fractional dimension. Here we choose to use the Minkowski dimension, and the Hausdorff dimension. They assign the same dimension to any smooth manifold, but vary over more singular sets. The biggest difference is that Minkowski dimension measures relative density at a single scale, whereas Hausdorff dimension measures relative density at countably many scales. This makes Hausdorff dimension more stable under analytical operations.

We begin by discussing the Minkowski dimension, which is the simplest to define. Given a length l, and a bounded set $E \subset \mathbf{R}^d$, we let $N(l,E)$ denote the length l covering number of E, i.e. the minimum number of sidelength l cubes required to cover E. We define the *lower* and *upper* Minkowski dimension as

$$\dim_{\underline{\mathbf{M}}}(E) = \liminf_{l \to 0} \frac{\log(N(l,E))}{\log(1/l)} \quad \dim_{\overline{\mathbf{M}}}(E) = \limsup_{l \to 0} \frac{\log(N(l,E))}{\log(1/l)}.$$

If $\dim_{\overline{\mathbf{M}}}(E) = \dim_{\underline{\mathbf{M}}}(E)$, then we refer to this common quantity as the *Minkowski* *dimension* of E, denoted $\dim_{\mathbf{M}}(E)$. Thus $\dim_{\underline{\mathbf{M}}}(E) < s$ if there *exists* a sequence of lengths $\{l_k\}$ converging to zero with $N(l_k, E) \leq (1/l_k)^s$, and $\dim_{\overline{\mathbf{M}}}(E) < s$ if $N(l, E) \leq (1/l)^s$ for *all* sufficiently small lengths l.

Hausdorff dimension is slightly more technical to define. For $E \subset \mathbf{R}^d$ and $\delta > 0$, we define the *Hausdorff content*

$$H^s_\delta(E) = \inf \left\{ \sum_{k=1}^{\infty} l(Q_k)^s : E \subset \bigcup_{k=1}^{\infty} Q_k, l(Q_k) \leq \delta \right\}.$$

We then set $H^s(E) = \lim_{\delta \to 0} H^s_\delta(E)$. It is easy to see H^s is an exterior measure, and $H^s(E \cup F) = H^s(E) + H^s(F)$ if the Hausdorff distance $d(E, F)$ between E and F is positive. So H^s is actually a metric exterior measure, and the Caratheodory extension theorem shows all Borel sets are measurable with respect to H^s. We then define the dimension of E to be the supremum of the values s such that $H^s(E) < \infty$.

Lemma 2. *Consider $t < s$, and a set E.*

- *If $H^t(E) < \infty$, then $H^s(E) = 0$.*
- *If $H^s(E) \neq 0$, then $H^t(E) = \infty$.*

Proof. If, for any cover of E by a collection of intervals $\{Q_k\}$, $\sum l(Q_k)^t \leq A$, and $l(Q_k) \leq \delta$, then

$$\sum l(Q_k)^s \leq \sum l(Q_k)^{s-t} l(Q_k)^t \leq \delta^{s-t} A.$$

Thus $H^s_\delta(E) \leq \delta^{s-t} A$, and taking $\delta \to 0$, we conclude $H^s(E) = 0$. The latter point is just proved by taking contrapositives. □

Remark. *It is easy to see directly from the definition that H^d is the Lebesgue measure on $\mathbf{R}^d$. Thus $H^d[-N, N]^d = (2N)^d$. If $s > d$, Lemma 2 shows $H^s[-N, N]^d = 0$, and so by countable additivity, taking $N \to \infty$ shows $H^s(\mathbf{R}^d) = 0$. Thus $H^s(E) = 0$ for all E if $s > d$.*

Given any Borel set E, the last remark, combined with Lemma 2, implies there is a unique value $s_0 \in [0, d]$ such that $H^s(E) = 0$ for $s > s_0$, and $H^s(E) = \infty$ for $0 \leq s < s_0$. We refer to s_0 as the *Hausdorff dimension* of E, denoted $\dim_{\mathbf{H}}(E)$.

Theorem 3. *For any bounded set E,* $\dim_{\mathbf{H}}(E) \leq \dim_{\underline{\mathbf{M}}}(E) \leq \dim_{\overline{\mathbf{M}}}(E)$.

Proof. Suppose there is a sequence $\{l_k\}$ with $l_k \to 0$ such that for all $\varepsilon > 0$, $N(l_k, E) \leq (1/l_k)^{s-\varepsilon}$ for sufficiently large k. We then consider the simple bound $H^s_{l_k}(E) \leq N(l_k, E) \cdot l_k^s \leq l_k^{\varepsilon}$, and taking $l_k \to 0$ shows $H^s(E) = 0$. Thus $\dim_{\mathbf{H}}(E) \leq s$, and since $\{l_k\}$ was arbitrary, we conclude $\dim_{\mathbf{H}}(E) \leq \dim_{\underline{\mathbf{M}}}(E)$. $\square$

The fact that Hausdorff dimension is defined with respect to multiple scales makes it more stable under analytical operations. In particular,

$$\dim_{\mathbf{H}} \left\{ \bigcup E_k \right\} = \sup \{ \dim_{\mathbf{H}}(E_k).\}$$

This need not be true for the Minkowski dimension; a single point has Minkowski dimension zero, but the rational numbers, which are a countable union of points, have Minkowski dimension one. An easy way to make Minkowski dimension countably stable is to define the *modified Minkowski dimensions*

$$\dim_{\underline{\mathbf{MB}}}(E) = \inf \left\{ s : E \subset \bigcup_{i=1}^{\infty} E_i, \dim_{\underline{\mathbf{M}}}(E_i) \leq s \right\}$$

and

$$\dim_{\overline{\mathbf{MB}}}(E) = \inf \left\{ s : E \subset \bigcup_{i=1}^{\infty} E_i, \dim_{\overline{\mathbf{M}}}(E_i) \leq s \right\}.$$

This notion of dimension, in a disguised form, appears later on in this thesis.

Remark. *If* $\dim_{\mathbf{H}}(E) < d$, *then* $|E| = H^d(E) = 0$. *Thus any set with fractional dimension less than d must have measure zero, and so the dimension is a way of distinguishing between sets of measure zero, which is precisely what we needed in the configuration avoidance problem.*

2.3 Dyadic Scales

It is now useful to introduce the dyadic notation we utilize throughout this thesis. We focus on $[0,1]^d$, since all the sets we construct will be a subset of the unit cube. At the cost of losing certain topological information about $[0,1]^d$, applying dyadic techniques often allows us to elegantly discretize certain problems in Euclidean space. We introduce fractional dimension through a dyadic framework, and all our constructions will be done dyadically.

We fix a sequence of positive integers $\{N_k : k \geq 1\}$, referred to as *branching factors*. Using these integers, we define a sequence of positive rational numbers $\{l_k : k \geq 0\}$, recursively such that $l_0 = 1$, and $l_{k+1} = l_k / N_{k+1}$. For each $k \geq 0$, set

$$\mathcal{Q}_k^d = \left\{ [a_1 l_k, (a_1+1) l_k] \times \cdots \times [a_d l_k, (a_d+1) l_k] : a \in \mathbf{Z}^d \right\},$$

and $\mathcal{Q}^d = \bigcup_{k \geq 0} \mathcal{Q}_k^d$. We call the set $\mathcal{Q}_k^d$ the *dyadic cubes of generation k*. For any set $E \subset [0,1]^d$, we let $\mathcal{Q}_k^d(E) = \{Q \in \mathcal{Q}_k^d : Q \cap E \neq \emptyset\}$.

The most common class of dyadic cubes in analysis is obtained from setting $N_k = 2$ for each k. We reserve a special notation for this class of dyadic cubes: the class of all such cubes is denoted by $\mathcal{D}^d$, and the generation k cubes by $\mathcal{D}_k^d$. These dyadic cubes have a constant branching factor, which makes them easy to analyze. But it is necessary in our methods to let the branching factor to tend to ∞. This is why we have to introduce the more general family of dyadic cubes given above.

Sometimes, we need to rely on certain 'intermediary' cubes that lie between the scales $\mathcal{Q}_k^d$ and $\mathcal{Q}_{k+1}^d$. This is the case in an iterative construction such that each step can be split down into two different scales. In this case, we write $N_k = K_k M_k$, for two integers K_k and M_k, write $r_k = l_{k-1}/K_k = M_k l_k$, and

$$\mathcal{R}_k^d = \left\{ [a_1 r_k, (a_1+1) r_k] \times \cdots \times [a_d r_k, (a_d+1) r_k] : a \in \mathbf{Z}^d \right\}.$$

Thus the classes of dyadic cubes we consider, in order from coarsest scale to finest scale, can be listed as $\{\mathcal{Q}_0, \mathcal{R}_1, \mathcal{Q}_1, \mathcal{R}_2, \mathcal{Q}_2, \dots\}$

2.4 Frostman Measures

It is often easy to upper bound Hausdorff dimension, but non-trivial to *lower bound* the Hausdorff dimension of a given set. A key technique to finding a lower bound is *Frostman's lemma*, which says that a set has large Hausdorff dimension if and only if it supports a probability measure which obeys a certain decay law on small sets. We say a measure μ is a *Frostman measure* of dimension s if it is non-zero, compactly supported, and for any cube Q, $\mu(Q) \lesssim l(Q)^s$. The proof of Frostman's lemma will utilize a technique often useful, known as the *mass distribution principle*.

Lemma 4 (Mass Distribution Principle)**.** *Let $\mu : \mathcal{Q}^d \to [0,\infty)$ be a function such that for any k, and for any $Q_0 \in \mathcal{Q}^d_k$,*

$$\sum \left\{ \mu(Q) : Q \in \mathcal{Q}^d_{k+1}, Q^* = Q_0 \right\} = \mu(Q_0).$$

Then μ extends uniquely to a regular Borel measure on $\mathbf{R}^d$.

Proof. We can define a sequence of regular Borel measures $\{\mu_k\}$ by setting, for each $f \in C_c(\mathbf{R}^d)$,

$$\int f d\mu_k = \sum \left\{ \mu(Q) \int_Q f : Q \in \mathcal{Q}^d_k \right\}$$

Similarily, define a family of operators $\{E_k\}$ on regular Borel measures by the formula

$$\int f(x) dE_k(\nu) = \sum \left\{ \nu(Q) \int_Q f : Q \in \mathcal{Q}^d_k \right\}.$$

The main condition of the theorem then says that $E_j(\mu_k) = \mu_j$ if $j \leq k$. Note that the operators $\{E_k\}$ are each continuous with respect to the weak topology; If $\nu_i \to \nu$ weakly, then $\nu_i(Q) \to \nu(Q)$ for each fixed $Q \in \mathcal{Q}^d_k$. Thus if $f \in C_c(\mathbf{R}^d)$, then the support of f intersects only finitely many cubes in $\mathcal{Q}_k^d$, and this implies

$$\int f(x) dE_k(\nu_i) = \sum \nu_i(Q) \int_Q f\, dx \to \sum \nu(Q) \int_Q f\, dx = \int f dE_k(\nu).$$

Now fix an interval $Q \in \mathcal{Q}_1^d$. The measures $\{\mu_k|_Q\}$, restricted to Q, all have total mass $\mu(Q)$. Thus the Banach-Alaoglu theorem implies that is a subsequence $\mu_{k_i}|_Q$ converging weakly on Q to some measure μ_Q. By continuity,

$$E_j(\mu_Q) = \lim E_j(\mu_{k_i}|_Q) = \lim E_j(\mu_{k_i})|_Q = \mu_j|_Q.$$

This means precisely that μ_Q is the extension of μ to a Borel measure on Q. If we patch together the measures μ_Q over all choices of $Q \in \mathcal{Q}_0^d$, we obtain a measure extending μ on all of $\mathbf{R}^d$. The uniqueness of the extension is guaranteed by the fact that the intervals upon which μ are defined generate the entire Borel sigma algebra. $\square$

Lemma 5 (Frostman's Lemma)**.** *If E is Borel, $H^s(E) > 0$ if and only if there exists an s dimensional Frostman measure supported on E.*

Proof. Suppose that μ is s dimensional and supported on E. If $H^s(F) = 0$, then for $\varepsilon > 0$ there is a sequence of cubes $\{Q_k\}$ with $\sum_{k=1}^\infty l(Q_k)^s \leq \varepsilon$. But then

$$\mu(F) \leq \mu\left(\bigcup_{k=1}^\infty Q_k\right) \leq \sum_{k=1}^\infty \mu(Q_k) \lesssim \sum_{k=1}^\infty l(Q_k)^s \leq \varepsilon.$$

Taking $\varepsilon \to 0$, we conclude $\mu(F) = 0$. Thus μ is absolutely continuous with respect to H^s. But since $\mu(E) > 0$, this means that $H^s(E) > 0$.

Conversely, suppose $H^s(E) > 0$. Then by translating, we may assume that $H^s(E \cap [0,1]^d) > 0$, and so without loss of generality we may assume $E \subset [0,1]^d$. Fix m, and for each $Q \in \mathcal{D}_k^d$, define $\mu^+(Q) = H_m^s(E \cap Q)$. Then $\mu^+(Q) \leq 1/2^{ks}$, and μ^+ is subadditive. We use it to recursively define a Frostman measure μ, such that $\mu(Q) \leq \mu^+(Q)$ for each $Q \in \mathcal{D}_k^d$. We initially define μ by setting $\mu([0,1]^d) =$

$\mu^+([0,1]^d)$. Given $Q \in \mathcal{D}_k^d$, we enumerate it's children as $Q_1, \dots, Q_M \in \mathcal{D}_{k+1}^d$. We then consider any values $A_1, \dots, A_M \geq 0$ such that

$$A_1 + \cdots + A_M = \mu(Q), \quad \text{and } A_k \leq \mu^+(Q_k) \text{ for each } k.$$

This is feasible to do because $\mu^+(Q_1) + \cdots + \mu^+(Q_M) \geq \mu^+(Q)$. We then define $\mu(Q_k) = A_k$ for each k. The recursive constraint is satisfied, so μ is well defined. The mass distribution principle then implies that μ extends to a full measure, which satisfies $\mu(Q) \leq \mu^+(Q) \leq 1/2^{ks}$ for each $Q \in \mathcal{D}_k^d$. Given any cube Q, we can cover it by $O_d(1)$ dyadic cubes with length at most twice that of Q. This shows $\mu(Q) \lesssim_d l(Q)^s$ for any cube Q. And this means that we have shown directly that μ is a Frostman measure of dimension s. □

Frostman's lemma implies that to study the Hausdorff dimension of the set, it suffices to understand the class of measures which can be supported on that set.

2.5 Dyadic Fractional Dimension

We wish to establish results about fractional dimension 'dyadically'. We begin with Minkowski dimension. If we construct a set $E = \bigcap E_k$ such that E_k is the union of cubes in $\mathcal{Q}_k^d$, it is often easy to establish estimates on $N(l,E)$ for $l = l_k$ simply by counting the cardinality of $\mathcal{Q}_k^d(E)$, as the next lemma indicates.

Lemma 6. *For any set E, $\#(\mathcal{Q}_k^d(E)) \sim_d N(l_k,E)$.*

Proof. Since $\mathcal{Q}_k^d(E)$ is a cover of E by sidelength l_k cubes, minimality shows $N(l_k,E) \leq \#(\mathcal{Q}_k^d(E))$. On the other hand, given *any* cover of E by sidelength l_k cubes, we can replace each cube in the cover by at most 2^d cubes in $\mathcal{Q}_k^d$, so $N(l_k,E) \leq 2^d \#(\mathcal{Q}_k^d(E))$. □

Thus it is natural to ask whether it is true that for any set E,

$$\dim_{\underline{\mathbf{M}}}(E) = \liminf_{k\to\infty} \frac{\log\left[\#(\mathcal{Q}_k^d(E))\right]}{\log(1/l_k)}$$
$$\text{and} \tag{2.5}$$
$$\dim_{\overline{\mathbf{M}}}(E) = \limsup_{k\to\infty} \frac{\log\left[\#(\mathcal{Q}_k^d(E))\right]}{\log(1/l_k)}.$$

The answer depends on the choice of branching factors $\{N_k\}$.

Lemma 7. *If, for all $\varepsilon > 0$, $N_k \leq (N_1 \dots N_{k-1})^\varepsilon$ for large k, then* (2.5) *holds.*

Proof. Fix a length l, and find k with $l_{k+1} \leq l \leq l_k$. Then applying Lemma 6 shows

$$N(l,E) \leq N(l_{k+1},E) \lesssim_d \#(\mathcal{Q}_{k+1}^d(E))$$

and

$$N(l,E) \geq N(l_k,E) \gtrsim_d \#(\mathcal{Q}_k^d(E)).$$

Thus

$$\frac{\log[N(l,E)]}{\log(1/l)} \leq \left[\frac{\log(1/l_{k+1})}{\log(1/l_k)}\right] \frac{\log\left[\#(\mathcal{Q}_{k+1}^d(E))\right]}{\log(1/l_{k+1})} + O_d(1/k)$$

and

$$\frac{\log[N(l,E)]}{\log(1/l)} \geq \left[\frac{\log(1/l_k)}{\log(1/l_{k+1})}\right] \frac{\log\left[\#(\mathcal{Q}_k^d(E))\right]}{\log(1/l_k)} + O_d(1/k).$$

Thus, provided that

$$\frac{\log(1/l_{k+1})}{\log(1/l_k)} \to 1, \tag{2.6}$$

The conclusion of the theorem is true. But $l_{k+1} = l_k/N_k$, and $l_k = 1/N_1 \dots N_{k-1}$, so (2.6) is equivalent to the condition that

$$\frac{\log(N_k)}{\log(N_1) + \dots + \log(N_{k-1})} \to 0,$$

and this is equivalent to the assumption of the theorem. $\square$

Any constant branching factor satisfies the hypothesis of Lemma 7 for the Minkowski dimension. In particular, we can construct sets in $\mathcal{D}^d$ without any problems occuring. But more importantly for our work, we can let the branching factors increase rapidly.

Lemma 8. *If $N_k = 2^{\lfloor 2^{k\psi(k)} \rfloor}$, where $\psi(k)$ is any decreasing sequence of positive numbers tending to zero, but for which $k\psi(k) \to \infty$, then $N_{k+1} \leq (N_1 \dots N_k)^{\varepsilon}$ for any $\varepsilon > 0$, when k is sufficiently large.*

Proof. We note that $\log N_k = 2^{k\psi(k)} + O(1)$. Thus

$$\begin{aligned}
\frac{\log(N_k)}{\log(N_1) + \cdots + \log(N_{k-1})} &\lesssim \frac{2^{k\psi(k)}}{2^{\psi(1)} + 2^{2\psi(2)} + \cdots + 2^{(k-1)\psi(k-1)}} \\
&\leq \frac{2^{k\psi(k)}}{2^{\psi(k-1)} + 2^{2\psi(k-1)} + \dots 2^{(k-1)\psi(k-1)}} \\
&= \frac{2^{k\psi(k)}(2^{\psi(k-1)} - 1)}{2^{k\psi(k-1)} - 2^{\psi(k-1)}} \\
&\lesssim 2^{k[\psi(k) - \psi(k-1)]}(2^{\psi(k-1)} - 1) \\
&\leq 2^{\psi(k-1)} - 1 \to 0.
\end{aligned}$$

This is equivalent to the fact that $N_{k+1} \leq (N_1 \dots N_k)^{\varepsilon}$ for any $\varepsilon > 0$, where k is sufficiently large. □

We refer to any sequence $\{l_k\}$ constructed by $\{N_k\}$ satisfying the conditions of Lemma 8 as a *subhyperdyadic* sequence. If a sequence is generated by branching factors of the form $2^{\lfloor 2^{ck} \rfloor}$ for a fixed $c > 0$, the lengths are referred to as *hyperdyadic*. The next example shows that hyperdyadic sequences are essentially the 'boundary' for sequences that can be used to measure the Minkowski dimension.

Example. *Construct a subset of* $\mathbf{R}$ *as follows. Let $[N_k] = [K_k] \times [M_k]$, where N_k, K_k, and M_k are parameters to be specified later. Define $E_0 = [0,1]$. Given E_k, define E_{k+1} by dividing each sidelength l_k dyadic interval in E_k into K_{k+1} intervals, and then keeping only the first interval. Then $\#(\mathcal{Q}^d_{k+1}(E_{k+1})) = M_k \cdot \#(\mathcal{Q}^d_k(E_k))$,* *and since* $\#(\mathcal{Q}_0^d(E_0)) = 1$, $\#(\mathcal{Q}_k^d(E_k)) = M_1 \dots M_k$. *Thus*

$$\begin{aligned}
\frac{\log\left[N(l_k,E)\right]}{\log(1/l_k)} &\sim \frac{\log\left[\#(\mathcal{Q}_k^d(E_k))\right]}{\log(1/l_k)} \\
&= \frac{\log(M_1)+\cdots+\log(M_k)}{\log(N_1)+\cdots+\log(N_k)} \\
&= 1 - \frac{\log(K_1)+\cdots+\log(K_k)}{\log(N_1)+\cdots+\log(N_k)}.
\end{aligned}$$

On the other hand, if $r_k = l_k/K_{k+1}$, $N(r_k,E) \sim_d \#(S_k) = M_1 \dots M_k$, *so*

$$\begin{aligned}
\frac{\log\left[N(r_k,E)\right]}{\log(1/r_k)} &\sim \frac{\log\left[\#(\mathcal{Q}_k^d(E_k))\right]}{\log(1/r_k)} \\
&= \frac{\log(M_1)+\cdots+\log(M_k)}{\log(N_1)+\cdots+\log(N_k)+\log(K_{k+1})} \\
&= 1 - \frac{\log(K_1)+\cdots+\log(K_{k+1})}{\log(N_1)+\cdots+\log(N_k)+\log(K_{k+1})}.
\end{aligned}$$

Set $N_k = 2^{\lfloor 2^{ck} \rfloor}$, *and* $K_k = 2^{\lfloor c2^{ck} \rfloor}$. *Then*

$$\log(K_1)+\cdots+\log(K_k) = O(k) + c\sum_{i=1}^{k} 2^{ck} = O(k) + c\frac{2^{c(k+1)}-2^c}{2^c-1}$$

and

$$\log(N_1)+\cdots+\log(N_k) = O(k) + \sum_{i=1}^{k} 2^{ck} = O(k) + \frac{2^{c(k+1)}-2^c}{2^c-1}.$$

Thus

$$\frac{\log\left[N(l_k,E)\right]}{\log(1/l_k)} \to 1-c \quad \textit{and} \quad \frac{\log\left[N(r_k,E)\right]}{\log(1/l_k)} \to \frac{1-c}{1-c+c2^c}.$$

In particular, for any $0 < c \leq 1$,

$$\dim_{\underline{\mathbf{M}}}(E) \neq \liminf_{k\to\infty} \frac{\log\left[N(l_k,E)\right]}{\log(1/l_k)},$$

so measurements at hyperdyadic scales fail to establish general results about the

Minkowski dimension. This example can also be extended to show that measurements at hyperdyadic scales also fail to establish results about Hausdorff dimension, as discussed later in this section.

We now move on to calculating Hausdorff dimension dyadically. The natural quantity to consider here is the measure $H^s_{\mathcal{Q}}(E) = \lim H^s_{\mathcal{Q},m}(E)$, where

$$H^s_{\mathcal{Q},m}(E) = \inf\left\{\sum_k l(Q_k)^s : E \subset \bigcup_k^\infty Q_k,\ Q_k \in \bigcup_{i \geq m} \mathcal{Q}_i \text{ for each } k\right\}.$$

A similar argument to the standard Hausdorff measures shows there is a unique s_0 such that $H^s(E) = \infty$ for $s < s_0$, and $H^s(E) = 0$ for $s > s_0$. It is obvious that $H^s_{\mathcal{Q}}(E) \geq H^s(E)$ for any set E, so we certainly have $s_0 \geq \dim_{\mathbf{H}}(E)$. The next lemma gurantees that $s_0 = \dim_{\mathbf{H}}(E)$, under the same conditions on $\{N_k\}$ as found in Lemma 7.

Lemma 9. *If, for any $\varepsilon > 0$, $N_{k+1} \leq (N_1 \dots N_k)^\varepsilon$ for sufficiently large k, then $H^s_{\mathcal{Q}}(E) \lesssim_d H^{s-\varepsilon}(E)$ for each $\varepsilon > 0$.*

Proof. Fix $\varepsilon > 0$. Let $E \subset Q_k$, where $l(Q_k) \leq l_m$ for each k. Then for each k, we can find i_k such that $l_{i_k+1} \leq l(Q_k) \leq l_{i_k}$. Then Q_k is covered by $O_d(1)$ elements of $\mathcal{Q}^d_{i_k}$, and

$$H^s_{\mathcal{Q},m}(E) \lesssim_d \sum l^s_{i_k} \leq \sum (l_{i_k}/l_{i_k+1})^s l(Q_k)^s \leq \sum \left(l_{i_k}/l_{i_{k+1}}\right)^s l^\varepsilon_{i_k} l(Q_k)^{s-\varepsilon} \qquad (2.7)$$

By assumption, if m is large enough, then

$$l_{i_k+1} = l_{i_k}/N_{i_k+1} \gtrsim l_{i_k}^{1+\varepsilon/s}. \qquad (2.8)$$

Putting (2.7) and (2.8) together, we conclude that $H^s_{\mathcal{Q},m}(E) \lesssim_d \sum l(Q_k)^{s-\varepsilon}$. $\square$

Finally, we consider computing whether we can establish that a measure is a Frostman measure dyadically. First, we recognize the utility of this approach from the perspective of a dyadic construction. Suppose we construct $E = \bigcap E_k$, where $\{E_k\}$ is a decreasing sequence of sets, with E_k the union of cubes in $\mathcal{Q}_k^d$. Then we can define a finite measure supported on E by setting $\mu([0,1]^d) = 1$, and for each $Q \in \mathcal{Q}_{k+1}^d$, if $Q \subset E_{k+1}$, and Q^* has M children in E_{k+1}, we define $\mu(Q) = \mu(Q^*)/M$, and if $Q \not\subset E_{k+1}$, we set $\mu(Q) = 0$. We refer to this as the *canonical measure* associated with the construction. For this measure, it is often easy to show that $\mu(Q) \lesssim l(Q)^s$ if $Q \in \mathcal{Q}^d$. The conditions of Lemma 7 also are sufficient to infer from this estimate that μ is a Frostman measure of dimension $s - \varepsilon$ for all $\varepsilon > 0$.

Theorem 10. *If, for all $\varepsilon > 0$, $N_k \leq (N_1 \dots N_{k-1})^\varepsilon$ for large enough k, and if μ is a Borel measure such that $\mu(Q) \lesssim l(Q)^s$ for each $Q \in \mathcal{Q}_k^d$, then μ is a Frostman measure of dimension $s - \varepsilon$ for each $\varepsilon > 0$.*

Proof. Given $Q \in \mathcal{D}_m^d$, find k such that $l_{k+1} \leq 1/2^m \leq l_k$. Then Q is covered by $O_d(1)$ cubes in $\mathcal{Q}_k^d$, which shows

$$\mu(Q) \lesssim_d l_k^s = [(l_k/l)^s l^\varepsilon] l^{s-\varepsilon} \leq [l_k^{s+\varepsilon}/l_{k+1}^s] l^{s-\varepsilon} = [N_k^s l_k^\varepsilon] l^{s-\varepsilon}.$$

But by assumption, $N_k^s \lesssim_\varepsilon 1/l_k^\varepsilon$ for all $\varepsilon > 0$, so the proof is complete. □

2.6 Beyond Hyperdyadics

If we are to use a faster increasing sequence of branching factors than the last section guarantees, we therefore must exploit some extra property of our construction, which is not always present in general sets. Here, we rely on a *uniform mass distribution* between scales. Given the uniformity assumption, the lengths can decrease as fast as desired. This decomposition requires an intermediary scale, so we write $N_k = K_k M_k$, and define $r_k = M_k l_k = l_{k-1}/K_k$, with $\mathcal{R}_k^d$ are the dyadic cubes of length r_k.

TODO: ADD OTHER SITUATIONS WITH THE UNIFORM MASS DISTRIBUTION.

Lemma 11. *Let μ be a measure supported on a set E. Suppose that*

(A) *For any* $Q \in \mathcal{Q}_k^d$, $\mu(Q) \lesssim l_k^s$.

(B) *For each* $Q \in \mathcal{R}_k^d$, $\#(\mathcal{Q}_k^d(E \cap Q)) = O(1)$.

(C) *For any* $Q \in \mathcal{R}_{k+1}^d$ *with parent cube* $Q^* \in \mathcal{Q}_k^d$, $\mu(Q) \lesssim (r_{k+1}/l_k)^d \mu(Q^*)$.

Then μ *is a Frostman measure of dimension* s.

Proof. Suppose Q has length l, and that we can find k with $r_{k+1} \leq l \leq l_k$. Then we can cover Q by at most $O_d((l/r_{k+1})^d)$ cubes in $\mathcal{R}_k^d$. By Properties (A) and (C), each of these cubes has measure at most $O((r_{k+1}/l_k)^d l_k^s)$, so we obtain that

$$\mu(Q) \lesssim (l/r_{k+1})^d (r_{k+1}/l_k)^d l_k^s = l^d / l_k^{d-s} \lesssim l^s.$$

On the other hand, suppose there exists k with $l_k \leq l \leq r_k$. Then we can cover Q by $O_d(1)$ cubes in $\mathcal{R}_k^d$, and for each such cube, Property (C) shows there are $O(1)$ cubes in $\mathcal{Q}_k^d$ which have mass. Thus

$$\mu(Q) \lesssim l_k^s \leq l^s.$$

This addresses all cases, so μ is a Frostman measure of dimension s. □

2.7 Extras: Should we Hyperdyadic Covers?

Nonetheless, it will be useful for us to know that we can 'decompose' a set with a prescribed Hausdorff dimension hyperdyadically. We say a sequence of sets $\{E_k\}$ is a *strong cover* of a set E if $E \subset \limsup E_k$, or equivalently, if every $x \in E$ lies in infinitely many of the sets E_k.

Theorem 12. *Suppose* $E \subset [0,1]^d$ *is a set with* $\dim(E) \leq \alpha$. *Fix* $\varepsilon > 0$*, and write* $l_k = 2^{-\lfloor (1+\varepsilon)^k \rfloor}$. *Then there exists a strong cover of* E *by sets* $\{E_k\}$*, where* E_k *is a union of* $O_\varepsilon(l_k^{-\alpha})$ *hyperdyadic cubes of sidelength* l_k.

Proof. For each hyperdyadic number l_k, we can find a collection of cubes $\{Q_{k,i}\}$ covering E with $l(Q_{k,i}) \leq l_k$ for all i, and

$$\sum_{i=1}^{\infty} l(Q_{k,i})^{\alpha+C\varepsilon} \lesssim 1.$$

For each k and i, find $j_{k,i}$ such that $l_{j_{k,i}+1} \leq l(Q_{k,i}) \leq l_{j_{k,i}}$. Note $l_{j_{k,i}+1} \lesssim l_{j_{k,i}}^{1+\varepsilon}$, so

$$\begin{aligned}\sum_{i=1}^{\infty} l_{j_{k,i}}^{\alpha+C\varepsilon} &= \sum_{i=1}^{\infty} l(Q_{k,i})^{\alpha+C\varepsilon} (l_{j_{k,i}}/l(Q_{k,i}))^{\alpha+C\varepsilon} \\ &\lesssim \sum_{i=1}^{\infty} l(Q_{k,i})^{\alpha+C\varepsilon} l_{j_{k,i}}^{-\varepsilon(\alpha+C\varepsilon)} \lesssim \sum_{i=1}^{\infty} l(Q_{k,i})^{\alpha+(C-\alpha+\varepsilon)\varepsilon}.\end{aligned}$$

Thus, replacing C with a slightly smaller constant, and replacing $Q_{k,i}$ with the $O_d(1)$ cubes in $\mathcal{Q}^d_{j_{k,i}}(Q_{k,i})$, we may assume all cubes in the decomposition are hyperdyadic. We let Y_{k_1,k_2} to be the union of all cubes in the decomposition $\{Q_{k_1,i}\}$ with hyperdyadic length l_{k_2}. Note that Y_{k_1,k_2} is the union of $O((1/l_{k_2})^{\alpha+C\varepsilon})$ cubes. We let $\mathbf{Q}_{k_1,k_2}$ be the collection of hyperdyadic cubes covering Y_{k_1,k_2} which minimize the quantity

$$\sum_{Q \in \mathbf{Q}_{k_1,k_2}} l(Q)^{\alpha}$$

and such that $l(Q) \geq l_{k_2}$ for each Q. Then clearly

$$\sum_{Q \in \mathbf{Q}_{k_1,k_2}} l(Q)^{\alpha} \lesssim l_{k_2}^{-C\varepsilon}.$$

In particular, this means $l(Q) \lesssim l_{k_2}^{-C\varepsilon/\alpha}$ for each $Q \in \mathbf{Q}_{k_1,k_2}$. Furthermore, for each hyperdyadic Q_0 with $l(Q_0) \geq l_{k_2}$,

$$\sum_{Q \subset Q_0} l(Q)^{\alpha} \leq l(Q_0)^{\alpha}.$$

Now we define E_k as the union of all cubes in the sets $\mathbf{Q}_{k_1,k_2}$ which have sidelength

l_k. Then clearly E_k is l_k discretized. Since E_k only contains cubes from $\mathbf{Q}_{k_1,k_2}$ where $k_2 \geq k_1$, and $l_{k_1} \lesssim l_{k_2}^{-C\varepsilon/\alpha}$, this means that there are only $O(\log(1/l_k)^2) = O(1+\varepsilon)$ such choices of (k_1,k_2). But this means that for $Q \in \mathbf{Q}_{k_1,k_2}$, and any hyperdyadic cube Q_0,

$$\sum_{Q \subset Q_0} l(Q)^\alpha \lesssim (1+\varepsilon)^2 l(Q_0)^\alpha$$

In particular,

$$|E_k| = \sum l(Q)^d = l_k^{d-\alpha} \sum l(Q)^\alpha \lesssim (1+\varepsilon)^2 l_k^{d-\alpha},$$

which implies that E_k is the union of at most $(1+\varepsilon)^2 l_k^{-\alpha}$ cubes. $\square$

Chapter 3

Related Work

3.1 Keleti: A Translate Avoiding Set

Keleti's two page paper constructs a full dimensional subset X of $[0,1]$ such that X intersects $t+X$ in at most one place for each nonzero real number t. If this is true, we say that X *avoids translates*. This paper contains the core idea behind the interval dissection method we adapt in our proof, which makes this paper of interest. In this section, and in the sequel, we shall find it is most convenient to avoid certain configurations by expressing them in terms of an equation, whose properties we can then exploit. One feature of translation avoidance is that the problem is specified in terms of a linear equation.

Lemma 13. *Let X be a set. Then X avoids translates if and only if there do not exists values $x_1 < x_2 \leq x_3 < x_4$ in X with $x_2 - x_1 = x_4 - x_3$.*

Proof. Suppose $(t+X) \cap X$ contains two points $a < b$. Without loss of generality, we may assume that $t > 0$. If $a \leq b - t$, then the equation

$$a - (a-t) = t = b - (b-t)$$

satisfies the constraints, since $a - t < a \leq b - t < b$ are all elements of X. We also have

$$(b-t)-(a-t)=b-a,$$

which satisfies the constraints if $a-t<b-t\leq a<b$. This covers all possible cases. Conversely, if there are $x_1<x_2\leq x_3<x_4$ in X with

$$x_2-x_1=t=x_4-x_3,$$

then $X+t$ contains $x_2=x_1+(x_2-x_1)$ and $x_4=x_3+(x_4-x_3)$. □

The basic, but fundamental idea to Keleti's technique is to introduce memory into Cantor set constructions. Keleti constructs a nested family of discrete sets $X_0\supset X_1\supset\dots$ converging to X, with each X_k a union of disjoint intervals in $\mathcal{Q}_k^d$, where the branching factors $\{N_k\}$ will be chosen later, but a multiple of 10. We initialize $X_0=[0,1]$, and $l_0=1$. Furthermore, we consider a queue of intervals, initially just containining $[0,1]$. To construct the sequence $\{X_k\}$, Keleti iteratively performs the following procedure:

Algorithm 1 Construction of the Sets $\{X_k\}$:

Set $k=0$.
Repeat
  Take off an interval I from the front of the queue.
  For all $J\in\mathcal{Q}_k^1$ contained in X_k:
    Order the intervals in $\mathcal{Q}_k^1(J)$ as $J_0,J_1,\dots,J_N$.
    If $J\subset I$, add all intervals J_i to X_{k+1} with $i\equiv 0$ modulo 10.
    Else add all J_i with $i\equiv 5$ modulo 10.
  Add all intervals in $\mathcal{Q}_{k+1}^d$ to the end of the queue.
  Increase k by 1.

Each iteration of the algorithm produces a new set X_k, and so leaving the algorithm to repeat infinitely produces a decreasing sequence $\{X_k\}$ whose intersection is X.

Lemma 14. *The set X is translate avoiding.*

Proof. If X is not translate avoiding, there is $x_1 < x_2 \leq x_3 < x_4$ with $x_2 - x_1 = x_4 - x_3$. Since $l_k \to 0$, there is a suitably large integer N such that x_1 is contained in an interval $I \in \mathcal{Q}_N^1$ not containing x_2, x_3, or x_4. At stage N of the algorithm, the interval I is added to the end of the queue, and at a much later stage M, the interval I is retrieved. Find the startpoints x_1°, x_2°, $x_3^\circ, x_4^\circ \in l_M\mathbf{Z}$ to the intervals in $\mathcal{Q}_M^1$ containing x_1, x_2, x_3, and x_4. Then we can find n and m such that $x_4^\circ - x_3^\circ = (10n)l_M$, and $x_2^\circ - x_1^\circ = (10m+5)l_M$. In particular, this means that $|(x_4^\circ - x_3^\circ) - (x_2^\circ - x_1^\circ)| \geq 5L_M$. But

$$\begin{aligned} |(x_4^\circ - x_3^\circ) - (x_2^\circ - x_1^\circ)| &= |[(x_4^\circ - x_3^\circ) - (x_2^\circ - x_1^\circ)] - [(x_4 - x_3) - (x_2 - x_1)]| \\ &\leq |x_1^\circ - x_1| + \dots + |x_4^\circ - x_4| \leq 4L_M \end{aligned}$$

which gives a contradiction. □

It is easy to see from the definition of the algorithm that

$$\#(\mathcal{Q}_k^1(X_k)) = (l_{k-1}/10l_k)\#(\mathcal{Q}_{k-1}^1(X_{k-1})).$$

Thus closing the recursive definition shows

$$\#(\mathcal{Q}_k^1(X_k)) = \frac{1}{10^k l_k}.$$

In particular, this means $|X_k| = 1/10^k$, so X has measure zero irrespective of our parameters. Nonetheless, the canonical measure μ on X defined with respect to the decomposition $\{X_k\}$ satisfies $\mu(I) = 10^k l_k$ for all $I \in \mathcal{B}(l_k, X)$. If $10^k l_k^\varepsilon \lesssim_\varepsilon 1$ for all ε, then we can establish the bounds $\mu(I) \lesssim_\varepsilon l_k^{1-\varepsilon}$ for all ε. In particular this is true if we set $l_k = 1/10^{\lfloor k \log k \rfloor}$. And because this sequence does not rapidly decrease too fast, we can apply Lemma 10 to show μ is a Frostman measure of dimension $1-\varepsilon$ for each $\varepsilon > 0$, so X has full Hausdorff dimension.

3.2 Fraser/Pramanik: Smooth Configurations

Inspired by Keleti's result, Pramanik and Fraser obtained a generalization of the queue method which allows one to find sets avoiding $d+1$ point configurations given by the zero sets of smooth functions, i.e.

$$\mathcal{C} = \{(x_0,\dots,x_n) : f(x_0,\dots,x_n) = 0\},$$

under mild regularity conditions on the function f. To obtain this result, rather than making a linear shift in one of the intervals we avoid as in Keleti's approach, one must use the smoothness properties of the function to find large segments of an interval avoiding solutions to another interval.

Lemma 15. *Let $T \subset [0,1]^d$, and $T' \subset [0,1]^{kd}$ be unions of cubes in $\mathcal{B}(l,\mathbf{R}^d)$ and $\mathcal{B}(l,\mathbf{R}^{kd})$. Fix three lengths, l, r, and s. Fix $B \subset T \times T'$, which is a union of cubes in $\mathcal{B}(s,\mathbf{R}^d)$. Then there exists a set $S \subset T$, which is a union of cubes in $\mathcal{B}(s,\mathbf{R}^d)$, and a set $B' \subset T'$, which is a union of cubes in $\mathcal{B}(s,\mathbf{R}^{kd})$ such that*

(A) *For every $I \in \mathcal{B}(l,T)$, and for at least a fraction $1-l$ of the cubes J in $\mathcal{B}(r,I)$, S contains a single cube in $\mathcal{B}(s,J)$, whereas $J \cap S = \emptyset$ otherwise.*

(B) $\#(\mathcal{B}(s,B')) \leq (1/l)^{d+1}(1/r)^d s^d \#(\mathcal{B}(s,B))$.

(C) $(S \times T') \cap B \subset S \cap B'$.

Proof. Fix $I \in \mathcal{B}(l,T)$. For each $J \in \mathcal{B}(r,I)$, define a *slab* to be

$$S[J] = \bigcup J \times \mathcal{B}(r,T').$$

Given any $K \in \mathcal{B}(s,\mathbf{R}^d)$, we define a *wafer* to be

$$W[K] = \bigcup K \times \mathcal{B}(s,T').$$

Note that a slab is the union of at most $(r/s)^d$ wafers, and I supports $(l/s)^d$ wafers.

In particular, if we say a wafer $W[K]$ is *good* if

$$\mathcal{B}(s, W[K] \cap B) \leq (s^d / l^{d+1})\, \mathcal{B}(s, B),$$

then at most l^{d+1}/s^d wafers are bad, which is a fraction l of all wafers in I. We call a slab good if it contains a good wafer. Then at most l of all slabs are bad. For each J such that $S[J]$ is good, we select a single $K_J \in \mathcal{B}(s, J)$ such that $W[K]$ is good. We then define

$$S = \bigcup_{I \in \mathcal{B}(l,T)} \left(\bigcup \{K_J : S[J] \text{ is good}\} \right).$$

Then S satisfies Property (A).

Let B' be the union of all cubes $Q \in \mathcal{B}(s, T')$ such that there is $K \in \mathcal{B}(s, S)$ with $K \times Q \in \mathcal{B}(s, B)$. Then by definition, Property (C) is satisfied. The choice of S implies that for each $K \in \mathcal{B}(s, S)$,

$$\#\{Q : K \times Q \in \mathcal{B}(s, B)\} \leq (s^d / l^{d+1})\, \mathcal{B}(s, B)$$

But $\#\mathcal{B}(s, S) \leq \mathcal{B}(r, T) \leq (1/r)^d$, so

$$\#\mathcal{B}(s, B') \leq \#\mathcal{B}(s, S)(s^d / l^{d+1})\, \mathcal{B}(s, B) \leq s^d (1/r)^d (1/l^{d+1})\, \mathcal{B}(s, B).$$

which establishes Property (B). □

We will use this lemma to continually reduce the dimensionality of the avoidance problem we are considering. Once we reach one dimension, we are in need of another lemma to complete the avoidance process.

Lemma 16. *Let T be a union of cubes in $\mathcal{B}(l, \mathbf{R}^d)$. Let $B \subset T$ be a union of cubes in $\mathcal{B}(s, T)$. Suppose*

$$\#\mathcal{B}(s, B) \leq C (1/l)^{(d+1)(k-1)} (1/r)^{d(k-1)} s^{m-n}$$

and
$$s \leq C^{-1/m} l^{(1/m)((d+1)k+1)} r^{n(k-1)/m}.$$

Then there exists $S \subset T$, which is a union of cubes in $\mathcal{B}(s,T)$ such that

(A) $S \cap B = \emptyset$.

(B) *For all but a fraction l of the cubes $I \in \mathcal{B}(r,T)$, $|S \cap I| \geq (1-l)s^n$.*

Proof. By pidgeonholing, it suffices to focus on the cubes $J \in \mathcal{B}(r,\mathbf{R}^d)$ for which

$$\#\mathcal{B}(s,J \cap B) \leq l^{d+1} r^{-d} \#\mathcal{B}(r,B)$$

We then define J to be equal to the union of all $J - B$, over the cubes J satisfying the inequality above. □

We again use a queuing process to a Cantor set construction. We construct a nested family of discrete sets $X_0 \supset X_1 \supset \dots$ converging to X, with each X_k a union of disjoint intervals in $\mathcal{Q}_k^d$, for a decreasing sequence of lengths $\{l_k\}$ converging to zero, to be chosen during the algorithm. We initialize $X_0 = [0,1]$, and $l_0 = 1/(d+1)$. Our queue shall consist of $d+1$ tuples of disjoint intervals $(T_0,\dots,T_d)$, all of the same length, which initially consists of all possible rearrangements of the intervals $\{[0,1/(d+1)),[1/(d+1),2/(d+1)),\dots,[d/(d+1,1)\}$. To construct the sequence $\{X_k\}$, we perform the following iterative procedure:

Algorithm 2 Construction of the Sets $\{X_k\}$

Set $k = 0$

Repeat

Take off a $d+1$ tuple $(T_0', \dots, T_d')$ from the front of the queue

Set $T_i = T_i' \cap X_k$ for each i

Apply Theorem **??** to the sets $T_0, \dots, T_d$, with $M = 1/l_k$ and with an appropriately large integer $N_k \in M\mathbf{Z}$, to obtain a rational constant C_k, not depending on N_k, and sets $S_0, \dots, S_d$, each unions of length C_k/N_k^d intervals, satisfying Property **??**, **??**, and **??**.

Set $l_{k+1} = C_k/N_k^d$.

Set $X_{k+1} = X_k - \bigcup_{i=0}^d (T_i - S_i)$.

Add all $d+1$ tuples of disjoint intervals $(T_0', \dots, T_d')$ in $\mathcal{B}(l_{k+1}, X_{k+1})$ to the back of the queue.

Increase k by 1.

Suppose that $x_0, \dots, x_d \in X$ are distinct. Then at some stage k, $x_0, \dots, x_d$ lie in distinct intervals $T_0', \dots, T_d' \in \mathcal{B}(l_k, X_k)$. At this stage, $(T_0', \dots, T_d')$ is added to the back of the queue, and therefore, at some much later stage N, is taken off the front. Sets $S_0 \subset T_0', \dots, S_d \subset T_d'$ are constructed satisfying Property **??**, and since $x_i \in S_i$ for each i, $f(x_0, \dots, x_d) \neq 0$. Thus X avoids the configuration $\mathcal{C}$.

What remains is to bound the Hausdorff dimension of X. First, we construct the canonical measure μ on X with respect to the Cantor-type decomposition $\{X_k\}$. We have $l_k = 1/M_k$, and $l_{k+1} = C_k/N_k^d$. We let $r_{k+1} = 1/N_k$. Let $K \in \mathcal{B}(l_{k+1}, X)$, $J \in \mathcal{B}(r_k, X)$, and $I \in \mathcal{B}(l_k, X)$. To calculate a bound on the mass of K, we consider the various decompositions considered in the algorithm:

- If I is subdivided in the non-specialized manner, then every length r_k interval recieves the same mass, which is allocated to a single length l_{k+1} interval it contains. Thus
$$\mu(K) = \mu(J) \leq (r_k/l_k)\mu(I).$$

- In I is subdivided in the specialized manner, at least a fraction $1-1/M_k$ of the length $1/N_k$ intervals are assigned mass. Since $M_k \geq 2$,

$$\mu(J) \leq (r_k/l_k)(1-1/M_k)^{-1} \leq (2r_k/l_k)\mu(I).$$

The set X_{k+1} contains more than C_k/N_k of the interval J, so

$$\mu(K) \leq \left(\frac{N_k}{C_k} l_{k+1}\right)\mu(J) \leq \frac{2l_{k+1}}{l_k C_k}\mu(I).$$

$(2/C_k)l_{k+1}$ vs. $l_{k+1}^{1/d}$

Thus in both cases, we find

- Carrying out the recursion based on the relationship

$$\mu(K) \leq (2l_{k+1}/l_k C_k)\mu(I),$$

we find that if $I \in \mathcal{B}(l_k, X)$, then

$$\mu(I) \leq \frac{2^k}{C_1 \dots C_{k-1}} l_k$$

Thus if there are $\psi(k) \to \infty$ such that

$$l_k \leq \left(\frac{C_1 \dots C_{k-1}}{2^k}\right)^{\psi(k)},$$

we conclude $\mu(I) \lesssim_\varepsilon l_k^{1-\varepsilon}$ for all $\varepsilon > 0$. Thus Property (A) is satisfied.

- $\log(1/l_{k+1})/\log(1/r_{k+1}) = d - \log(C_k)/\log(N_k) \to d$.

- $\mu(J) \lesssim (r_k/l_k)\mu(I)$, so Property (C) of Theorem **??** is satisfied.

Thus Theorem **??** shows that μ is a Frostman measure of dimension $1/(d-1)-\varepsilon$ for each $\varepsilon > 0$, and so X is a set with Hausdorff dimension at least $1/(d-1)$.

3.3 Mathés Result

Mathé's result can be reconfigured in terms of a building block strategy for implementation in our algorithm.

Theorem 17. *Let f be a polynomial of degree m, and consider unions of length $1/M$ intervals $T_0, \dots, T_d \subset [0,1]$, with rational start-points. If $\partial_0 f$ is non-vanishing on $T_0 \times \cdots \times T_d$, then there exists arbitrarily large integers N and a constant C not depending on N and sets $S_n \subset T_n$ such that*

- *$f(x) \neq 0$ for $x \in S_0 \times \cdots \times S_d$.*
- *If $T_0, \dots, T_d$ are split into length $1/N$ intervals, then S_n contains a length C/N^d region of each interval.*

Proof. Without loss of generality (by subdividing the initial intervals), let M be the greatest common divisor of all of the startpoints of the intervals in T_n. Divide each interval T_n into length $1/N$ intervals, and let $\mathbf{A} \subset (\mathbf{Z}/N)^d$ be the cartesian product of all startpoints of these length $1/N$ intervals. Since f has degree m, $f(\mathbf{A}) \subset \mathbf{Z}/N^m$. If $A_0 \leq |\partial_0 f| \leq A_1$ on $T_0 \times \cdots \times T_d$, then for any $a \in \mathbf{A}$, and δ_0, there exists δ_1 between 0 and δ_0 for which

$$|f(a+\delta_0 e_0)| - f(a)| = \delta_0 |(\partial_0 f)(a+\delta_1)|$$

If K is fixed such that $A_1 \leq (K-1)A_0$, so that we can choose

$$\frac{1/K}{A_0 N^m} \leq \delta_0 \leq \frac{(1-1/K)}{A_1 N^m}$$

Then

$$\frac{1/K}{N^m} \leq |f(a+\delta_0 e_0) - f(a)| \leq \frac{1-1/K}{N^m}$$

Thus $d(f(\mathbf{A}+\delta_0 e_0), \mathbf{Z}/N^m) \geq 1/KN^m$. Thus if we thicken the coordinates of $\mathbf{A}+\delta_0$ to intervals of length $O(1/N^m)$, then we obtain sets $S_0, \dots, S_n$ avoiding solutions. $\square$

Chapter 4

Avoiding Rough Sets

A number of recent articles have established pattern avoidance results for increasingly general patterns. In [11], Máthé constructs a set $X \subset \mathbf{R}^d$ that avoids a pattern specified by a countable union of algebraic varieties of controlled degree. In [6], Fraser and the second author consider the pattern avoidance problem for countable unions of C^1 manifolds. In this section, we consider the pattern avoidance problem for an even more general class of 'rough' patterns $Z \subset \mathbf{R}^{dn}$, that are the countable union of sets with controlled lower Minkowski dimension.

Theorem 18. *Suppose $\alpha \geq d$, and let $Z \subset \mathbf{R}^{dn}$ be a countable union of compact sets, each with lower Minkowski dimension at most α. Then there exists a set $X \subset [0,1)^d$ with Hausdorff dimension at least $(nd-\alpha)/(n-1)$ such that whenever $x_1,\dots,x_n \in X$ are distinct, we have $(x_1,\dots,x_n) \notin Z$.*

Remarks.

(1) When $\alpha < d$, the pattern avoidance problem is trivial, since $X = [0,1)^d - \pi(Z)$ is full dimensional and solves the pattern avoidance problem, where $\pi(x_1,\dots,x_n) = x_1$ is a projection map from $\mathbf{R}^{dn}$ to $\mathbf{R}^d$. Obtaining a full dimensional set in the case $\alpha = d$, however, is still interesting.

(2) Theorem 18 is trivial when $\alpha = dn$, since we can set $X = \emptyset$. We will therefore assume that $\alpha < dn$ in our proof of the theorem, without loss of generality.

(3) *When Z is a countable union of smooth manifolds in $\mathbf{R}^{nd}$ of co-dimension m, we have $\alpha = nd - m$. In this case Theorem 18 yields a set in $\mathbf{R}^d$ with Hausdorff dimension at least $(nd - \alpha)/(n-1) = m/(n-1)$. This recovers Theorem 1.1 and 1.2 from [6], making Theorem 18 a generalization of these results.*

(4) *Since Theorem 18 does not require any regularity assumptions on the set Z, it can be applied in contexts that cannot be addressed using previous methods. Two such applications, new to the best of our knowledge, have been recorded in Section 4.4; see Theorems 26 and 27 there.*

The set X in Theorem 18 is obtained by constructing a sequence of approximations to X, each of which avoids the pattern Z at different scales. For a sequence of lengths $l_k \searrow 0$, we construct a nested family of sets $\{X_k\}$, where X_k is a union of cubes of sidelength l_k that avoids Z at scales close to l_n. The set $X = \bigcap X_k$ avoids Z at all scales. While this proof strategy is not new, our method for constructing the sets $\{X_k\}$ has several innovations that simplify the analysis of the resulting set $X = \bigcap X_k$. In particular, through a probabilistic selection process we are able to avoid the complicated queuing techniques used in [9] and [6], that required storage of data from each step of the iterated construction, to be retrieved at a much later stage of the construction process.

At the same time, our construction continues to share certain features with [6]. For example, between each pair of scales l_{k-1} and l_k, we carefully select an intermediate scale r_k. The set $X_k \subset X_{k-1}$ avoids Z at scale l_k, and it is 'evenly distributed' at scale r_k: the set X_k is a union of intervals of length l_k whose midpoints resemble (a large subset of) an arithmetic progression of step size r_k. The details of a single step of this construction are described in Section 4.1. In Section 4.2, we explain how the length scales l_k and r_k for X are chosen, and prove its avoidance property. In Section 4.3 we analyze the size of X and show that it satisfies the conclusions of Theorem 18.

4.1 Avoidance at Discrete Scales

In this section we describe a method for avoiding Z at a single scale. We apply this technique in Section 4.2 at many scales to construct a set X avoiding Z at all scales. This single scale avoidance technique is the core building block of our construction, and the efficiency with which we can avoid Z at a single scale has direct consequences on the Hausdorff dimension of the set X obtained in Theorem 18.

At a single scale, we solve a discretized version of the problem, where all sets are unions of cubes at two dyadic lengths $l > s$. In this discrete setting, Z is replaced by a discretized version of itself, a union of cubes in $\mathcal{B}_s^{dn}$ denoted by G. Given a set E, which is a union of cubes in $\mathcal{B}_l^d$, our goal is to construct a set $F \subset E$ that is a union of cubes in $\mathcal{B}_s^d$, such that F^n is disjoint from strongly non-diagonal cubes (see Definition **??**) in $\mathcal{B}_s^{dn}(G)$. Using the setup introduced at the end of the introduction, we will later choose $l = l_k$, $s = l_{k+1}$, and $E = X_k$. The set X_{k+1} will be defined as the set F constructed.

In order to ensure the final set X obtained in Theorem 18 has large Hausdorff dimension regardless of the rapid decay of scales used in the construction of X, it is crucial that F is uniformly distributed at intermediate scales between l and s. We achieve this by decomposing E into sub-cubes in $\mathcal{B}_r^d$ for some intermediate scale $r \in [s,l]$, and distributing F as evenly among these intermediate sub-cubes as possible. This is possible assuming a mild regularity condition on the number of cubes in G, i.e. Equation (4.1).

Lemma 19. *Fix two distinct dyadic lengths l and s, with $l > s$. Let $E \subseteq [0,1)^d$ be a nonempty union of cubes in $\mathcal{B}_l^d$, and let $G \subset \mathbf{R}^{dn}$ be a nonempty union of cubes in $\mathcal{B}_s^{dn}$ such that*

$$(l/s)^d \leq \#\mathcal{B}_s^{dn}(G) \leq \frac{1}{2}(l/s)^{dn}. \tag{4.1}$$

Then there exists a dyadic length $r \in [s,l]$ of size

$$r \sim \left(l^{-d} s^{dn} \#\mathcal{B}_s^{dn}(G)\right)^{\frac{1}{d(n-1)}}, \tag{4.2}$$

and a set $F \subset E$ that is a nonempty union of cubes in $\mathcal{B}_s^d(E)$ satisfying the following three properties:

(A) Avoidance*: For any choice of distinct cubes $J_1, \dots, J_n \in \mathcal{B}_s^d(F)$, $J_1 \times \cdots \times J_n \notin \mathcal{B}_s^{dn}(G)$.*

(B) Non-Concentration*: For every $I' \in \mathcal{B}_r^d(E)$, there is at most one $J \in \mathcal{B}_s^d(F)$ with $J \subset I'$.*

(C) Large Size*: For every $I \in \mathcal{B}_l^d(E)$, $\#\mathcal{B}_s^d(F \cap I) \geq \#\mathcal{B}_r^d(I)/2 = (l/r)^d/2$.*

Remark. *Property (A) says that F avoids strongly non-diagonal cubes in $\mathcal{B}_s^{dn}(G)$. Properties (B) and (C) together imply that for every $I \in \mathcal{B}_l^d(E)$, at least half the cubes $I' \in \mathcal{B}_r^d(I)$ contribute a single sub-cube of sidelength s to F; the rest contribute none.*

Proof. Let r be the smallest dyadic length at least as large as R, where

$$R = \left(2l^{-d} s^{dn} \# \mathcal{B}_s^{dn}(G)\right)^{\frac{1}{d(n-1)}}. \tag{4.3}$$

This choice of r satisfies (4.2). The inequalities in (4.1) ensure that $r \in [s,l]$; more precisely, the left inequality in (4.1) implies R is bounded from below by s, and the right inequality implies R is bounded from above by l. The minimality of r ensures $s \leq r \leq l$.

For each $I' \in \mathcal{B}_r^d(E)$, let $J_{I'}$ be an element of $\mathcal{B}_s^d(I)$ chosen uniformly at random; these choices are independent as I' ranges over the elements of $\mathcal{B}_r^d(E)$. Define

$$U = \bigcup \left\{ J_{I'} : I' \in \mathcal{B}_r^d(E) \right\},$$

and

$$\mathcal{K}(U) = \{K \in \mathcal{B}_s^{dn}(G) : K \in U^n, K \text{ strongly non-diagonal}\}.$$

Note that the sets U and $\mathcal{K}(U)$ are random sets, in the sense that they depend on the random variables $\{J_{I'}\}$. Define

$$F(U) = U - \{\pi(K) : K \in \mathcal{K}(U)\}, \tag{4.4}$$

where $\pi : \mathbf{R}^{dn} \to \mathbf{R}^d$ is the projection map $(x_1, \ldots, x_n) \mapsto x_1$, for $x_i \in \mathbf{R}^d$. Thus π sends the cube $J_1 \times \cdots \times J_n \in \mathcal{B}_s^{dn}$ to the cube $J_1 \in \mathcal{B}_s^d$. Given any strongly non-diagonal cube $K = J_1 \times \cdots \times J_n \in \mathcal{B}_s^{dn}(G)$, either $K \notin \mathcal{B}_s^{dn}(U^n)$, or $K \in \mathcal{B}_s^{dn}(U^n)$. If the former occurs then $K \notin \mathcal{B}_s^{dn}(F(U)^n)$ since $F(U) \subset U$, so $\mathcal{B}_s^{dn}(F(U)^n) \subset \mathcal{B}_s^{dn}(U^n)$. If the latter occurs then $K \in \mathcal{K}(U)$, and since $\pi(K) = J_1$, $J_1 \notin \mathcal{B}_s^d(F(U))$. In either case, $K \notin \mathcal{B}_s^{dn}(F(U)^n)$, so $F(U)$ satisfies Property (A). By construction, U contains at most one subcube $J \in \mathcal{B}_s^{dn}$ for each $I \in \mathcal{B}_l^{dn}(E)$. Since $F(U) \subset U$, $F(U)$ satisfies Property (B). Thus the set $F(U)$ satisfies Properties (A) and (B) regardless of which values are assumed by the random variables $\{J_{I'}\}$. Next we will show that with non-zero probability, the set $F(U)$ satisfies Property (C).

For each cube $J \in \mathcal{B}_s^d(E)$, there is a unique 'parent' cube $I' \in \mathcal{B}_r^d(E)$ such that $J \subset I'$. Since I' contains $(r/s)^d$ elements of $\mathcal{B}_s^d(E)$, and $J_{I'}$ is chosen uniformly at random from $\mathcal{B}_s^d(I)$,

$$\mathbf{P}(J \subset U) = \mathbf{P}(J_{I'} = J) = (s/r)^d.$$

The cubes $J_{I'}$ are chosen independently, so if $J_1, \ldots, J_k$ are distinct cubes in $\mathcal{B}_s^d(E)$, then

$$\mathbf{P}(J_1, \ldots, J_k \in U) = \begin{cases} (s/r)^{dk} & \text{if } J_1, \ldots, J_k \text{ have distinct parents,} \\ 0 & \text{otherwise.} \end{cases} \tag{4.5}$$

Let $K = J_1 \times \cdots \times J_n \in \mathcal{B}_s^{dn}(G)$. If the cubes $J_1, \ldots, J_n$ are distinct, we deduce from (4.5) that

$$\mathbf{P}(K \subset U^n) = \mathbf{P}(J_1, \ldots, J_n \in U) \leq (s/r)^{dn}. \tag{4.6}$$

By (4.6) linearity of expectation, and (4.3),

$$\mathbf{E}(\#\mathcal{K}(U)) = \sum_{K \in \mathcal{B}_s^{dn}(G)} \mathbf{P}(K \subset U^n) \leq \#\mathcal{B}_s^{dn}(G) \cdot (s/r)^{dn} \leq 0.5 \cdot (l/r)^d.$$

In particular, there exists at least one (non-random) set U_0 such that

$$\#\mathcal{K}(U_0) \leq \mathbf{E}(\#\mathcal{K}(U)) \leq 0.5 \cdot (l/r)^d. \tag{4.7}$$

In other words, $F(U_0) \subset U_0$ is obtained by removing at most $0.5 \cdot (l/r)^d$ cubes in $\mathcal{B}_s^d$ from U_0. For each $I \in \mathcal{B}_l^d(E)$, we know that $\#\mathcal{B}_s^d(I \cap U_0) = (l/r)^d$. Combining this with (4.7), we arrive at the estimate

$$\begin{aligned} \#\mathcal{B}_s^d(I \cap F(U_0)) &= \mathcal{B}_s^d(I \cap U_0) - \#\{\pi(K) : K \in \mathcal{K}(U_0), \pi(K) \in F(U_0)\} \\ &\geq \mathcal{B}_s^d(I \cap U_0) - \#(\mathcal{K}(U_0)) \\ &\geq (l/r)^d - 0.5 \cdot (l/r)^d \geq 0.5 \cdot (l/r)^d \end{aligned}$$

In other words, $F(U_0)$ satisfies Property (C). Setting $F = F(U_0)$ completes the proof. □

Remarks.

(1) While Lemma 19 uses probabilistic arguments, the conclusion of the lemma is not a probabilistic statement. In particular, one can find a suitable F constructively by checking every possible choice of U (there are finitely many) to find one particular choice U_0 which satisfies (4.7)*, and then defining F by* (4.4)*. Thus the set we obtain in Theorem 18 exists by purely constructive means.*

(2) At this point, it is possible to motivate the numerology behind the dimension bound $\dim(X) \geq (dn - \alpha)/(n-1)$ *from Theorem 18, albeit in the context of Minkowski dimension. We will pause to do so here before returning to the proof of Theorem 18. For simplicity, let $\alpha > d$, and suppose that $Z \subset \mathbf{R}^{dn}$*

satisfies

$$\#\mathcal{B}_s^{dn}(Z) \sim s^{-\alpha} \quad \textit{for every } s \in (0,1]. \tag{4.8}$$

Let $l = 1$, $E = [0,1)^d$, and let $s > 0$ be a small parameter. If s is chosen sufficiently small compared to d,n, and α, then (4.1) *is satisfied with $G = \bigcup \mathcal{B}_s^{dn}(Z)$. We can then apply Lemma 19 to find a dyadic scale $r \sim s^{(dn-\alpha)/d(n-1)}$ and a set F that avoids the strongly non-diagonal cubes of $\mathcal{B}_s^{dn}(Z)$. The set F is a union of approximately $r^{-d} \sim s^{-(dn-\alpha)/(n-1)}$ cubes of sidelength s. Thus informally, the set F resembles a set with Minkowski dimension α when viewed at scale s.*

The set X constructed in Theorem 18 will be obtained by applying Lemma 19 iteratively at many scales. At each of these scales, X will resemble a set of Minkowski dimension $(dn-\alpha)/(n-1)$. A careful analysis of the construction (performed in Section 4.3) shows that X actually has Hausdorff dimension at least $(dn-\alpha)/(n-1)$.

(3) *Lemma 19 is the core method in our avoidance technique. The remaining argument is fairly modular. If, for a special case of Z, one can improve the result of Lemma 19 so that r is chosen on the order of $s^{\beta/d}$, then the remaining parts of our paper can be applied near verbatim to yield a set X with Hausdorff dimension β, as in Theorem 18.*

4.2 Fractal Discretization

In this section we construct the set X from Theorem 18 by applying Lemma 19 at many scales. Since Z is a countable union of bounded sets with Minkowski dimension at most α, there exists a strong cover (see Definition **??**) of Z by cubes restricted to a sequence of dyadic lengths $\{l_k\}$, with a quantitative bound on the number of cubes at each scale. We fix a cover so that the scales l_k converge to 0 very quickly.

Lemma 20. *Let $Z \subset \mathbf{R}^{dn}$ be a countable union of bounded sets with Minkowski dimension at most α, and let $\varepsilon_k \searrow 0$ with $2\varepsilon_k < dn - \alpha$ for all k. Then there exists a sequence of lengths $\{l_k\}$ and a strong cover of Z by a sequence of sets $\{Z_k\}$, such that*

(A) Discreteness*: For all $k \geq 0$, Z_k is a union of cubes in $\mathcal{B}^{dn}_{l_k}$.*

(B) Sparsity*: For all $k \geq 0$, $l_k^{-d} \leq \mathcal{B}^{dn}_{l_k}(Z_k) \leq l_k^{-\alpha-\varepsilon_k}$.*

(C) Rapid Decay*: For all $k > 1$,*

$$l_k^{dn-\alpha-\varepsilon_k} \leq 0.5 \cdot l_{k-1}^{dn} \quad \textit{and} \tag{4.9}$$

$$l_k^{\varepsilon_k} \leq l_{k-1}^{2d}. \tag{4.10}$$

Proof. We can write

$$Z = \bigcup_{i=1}^{\infty} Y_i \quad \text{with } \dim_{\underline{\mathbf{M}}}(Y_i) \leq \alpha \text{ for each } i.$$

Consider the d dimensional hyperplane

$$H = \{(x_1, \dots, x_1) : x_1 \in [0,1)^d\}$$

We then set $Y_i' = Y_i \cup H$. In particular, this means for any l,

$$\#\mathcal{B}^{nd}_l(Y_i') \geq \#\mathcal{B}^{nd}_l(H) = l^{-d}, \tag{4.11}$$

yet we still know $\dim_{\underline{\mathbf{M}}}(Y_i') \leq \alpha$ for each index i. Let $\{i_k\}$ be a sequence of integers that repeats each integer infinitely often.

The lengths $\{l_k\}$ and sets $\{Z_k\}$ are defined inductively. As a base case, set $l_0 = 1$ and $Z_0 = [0,1)^d$. Suppose that the lengths $l_0, \dots, l_{k-1}$ have been chosen. Since $\dim_{\underline{\mathbf{M}}}(Y_{i_k}) \leq \alpha$, Definition **??** implies that there exists arbitrarily small lengths l which saitsfy

$$\#\mathcal{B}^{dn}_l(Y_{i_k}') \leq l^{-\alpha-\frac{\varepsilon_k}{2}}. \tag{4.12}$$

In particular, we can choose $l = l_k$ small enough to satisfy (4.9), (4.10), and (4.12), so Property (C) is satisfied. We then set $Z_k = \bigcup \mathcal{B}^{dn}_{l_k}(Y'_{i_k})$. For any $z \in Z$, there is some i such that $z \in Y_i$, and there is a subsequence of integers $k_1, k_2, \dots$ such that $i_{k_j} = i$ for all j. But then $z \in Y_i \subset Y'_i \subset Z_{i_{k_j}}$, so z is contained in the infinite sequence of sets $Z_{i_{k_j}}$. In particular, $z \in \limsup Z_i$, and since z was arbitrary, $Z \subset \limsup Z_i$, so $\{Z_i\}$ is a strong cover of Z. Now Property (A) is satisfied by construction of the $\{Z_i\}$. And Property (B) is implied by (4.11) and (4.12). $\square$

To construct X, we consider a nested, decreasing family of discretized sets $\{X_k\}$, where X_k is a union of cubes in $\mathcal{B}^d_{l_k}(X_k)$. We then set $X = \bigcap X_k$. The goal is to choose X_k such that X_k^n is disjoint from *strongly non diagonal* cubes in Z_k.

Lemma 21. *Let $Z \subset \mathbf{R}^{dn}$, let $\{Z_k\}$ be a sequence of sets that strongly cover Z, and let $\{l_k\}$ be a sequence of lengths converging to zero. For each index k, let X_k be a union of cubes in $\mathcal{B}^d_{l_k}$. Suppose that for each k, X_k^n avoids strongly non-diagonal cubes in $\mathcal{B}^{dn}_{l_k}(Z_k)$. If $X = \bigcap X_k$, then for any distinct $x_1, \dots, x_n \in X$, we have $(x_1, \dots, x_n) \notin Z$.*

Proof. Let $z \in Z$ be a point with distinct coordinates $z_1, \dots, z_n$. Define

$$\Delta = \{(w_1, \dots, w_n) \in \mathbf{R}^{dn} : \text{there exists } i \neq j \text{ such that } w_i = w_j\}.$$

Then $d(\Delta, z) > 0$, where d is the Hausdorff distance between Δ and z. Since $\{Z_k\}$ strongly covers Z, there is a subsequence $\{k_m\}$ such that $z \in Z_{k_m}$ for every index m. Since $l_k \searrow 0$ and thus $l_{k_m} \searrow 0$, if m is sufficiently large then $\sqrt{dn} \cdot l_{k_m} < d(\Delta, z)$. Note that $\sqrt{dn} \cdot l_{km}$ is the diameter of a cube in $\mathcal{B}^{dn}_{l_{km}}$. For such a choice of m, if $I \in \mathcal{B}^{dn}_{l_{km}}(Z_{k_m})$ is the (unique) cube in $\mathcal{B}^{dn}_{l_{km}}$ containing z, then $I \cap \Delta = \emptyset$. But this means I is strongly non-diagonal. Since X_{k_m} avoids the strongly non-diagonal cubes of Z_{k_m}, we conclude that $z \notin X^n_{k_m}$. In particular, this means $z \notin X^n$. $\square$

All that remains is to apply the discrete lemma to choose the sets X_k.

Lemma 22. *Given a sequence of dyadic length scales l_k obeying,* (4.9)*,* (4.10)*, and* (4.12) *as above, there exists a sequence of sets $\{X_k\}$ and a sequence of dyadic* *intermediate scales $\{r_k\}$ with $l_k \leq r_k \leq l_{k-1}$ for each $k \geq 1$, such that each set X_k is a union of cubes in $\mathcal{B}^d_{l_k}(X_{k-1})$ that avoids the strongly non-diagonal cubes of $\mathcal{B}^{dn}_{l_k}(Z_k)$. Furthermore, for each index $k \geq 1$ we have*

$$r_k \lesssim l_k^{(dn-\alpha-\varepsilon_k)/d(n-1)}, \tag{4.13}$$

$$\#\mathcal{B}^d_{l_k}(X_k \cap I) \geq 0.5 \cdot (l_{k-1}/r_k)^d \quad \textit{for each } I \in \mathcal{B}^d_{l_{k-1}}(X_{k-1}), \tag{4.14}$$

$$\#\mathcal{B}^d_{l_k}(X_k \cap I') \leq 1 \quad \textit{for each } I' \in \mathcal{B}^d_{r_k}(X_{k-1}). \tag{4.15}$$

Proof. We construct X_k by induction, using Lemma 19 as building block. Set $X_0 = [0,1)^d$. Next, suppose that the sets $X_0, \ldots, X_{k-1}$ have been defined. Our goal is to apply Lemma 19 to $E = X_{k-1}$ and $G = Z_k$ with $l = l_{k-1}$ and $s = l_k$. This will be possible once we verify the hypothesis (4.1), which in this case takes the form

$$(l_{k-1}/l_k)^d \leq \#\mathcal{B}^{dn}_{l_k}(Z_k) \leq 0.5 \cdot (l_{k-1}/l_k)^{dn}. \tag{4.16}$$

The right hand side follows from Property (B) of Lemma 20 and (4.10). On the other hand, Property (B) and the fact that $l_{k-1} \leq 1$ implies that

$$(l_{k-1}/l_k)^d \leq l_k^{-d} \leq \#\mathcal{B}^{dn}_{l_k}(Z_k),$$

establishing the left inequality in (4.16). Applying Lemma 19 as described above now produces a dyadic length

$$r \sim \left(l_{k-1}^{-d} l_k^{dn} \#\mathcal{B}^{dn}_{l_k}(Z_k)\right)^{\frac{1}{d(n-1)}} \tag{4.17}$$

and a set $F \subset X_{k-1}$ that is a union of cubes in $\mathcal{B}^d_{l_k}$. The set F satisfies Properties (A), (B), and (C) from the statement of Lemma 19. Define $r_k = r$ and $X_k = F$. The estimate (4.13) on r_k follows from (4.17) using the known bounds (4.10) and (4.12):

$$r_k \lesssim \left(l_{k-1}^{-d} l_k^{dn-\alpha-0.5\varepsilon_k}\right)^{\frac{1}{d(n-1)}} = \left(l_{k-1}^{-d} l_k^{0.5\varepsilon_k} l_k^{dn-\alpha-\varepsilon_k}\right)^{\frac{1}{d(n-1)}} = \left(l_{k-1}^{-2d} l_k^{\varepsilon_k}\right)^{\frac{1}{2d(n-1)}} l_k^{\frac{dn-\alpha-\varepsilon_k}{d(n-1)}} \lesssim l_k^{\frac{dn-\alpha-\varepsilon_k}{d(n-1)}}.$$

The requirements (4.14) and (4.15) follow from Properties (B) and (C) of Lemma 19 respectively. $\square$

Now we have defined the sets $\{X_k\}$, we set $X = \bigcap X_k$. Since X_k avoids strongly non-diagonal cubes in Z_k, Lemma 21 implies that if $x_1, \dots, x_n \in X$ are distinct, then $(x_1, \dots, x_n) \notin Z$. To finish the proof of Theorem 18, we must show that $\dim_{\mathbf{H}}(X) \geq (dn - \alpha)/(n-1)$. This will be done in the next section.

4.3 Dimension Bounds

To complete the proof of Theorem 18, we must show that $\dim_{\mathbf{H}}(X) \geq (dn - \alpha)/(n-1)$. In view of Definition **??**, we will do this by constructing a Frostman measure of appropriate dimension supported on X.

We start by defining a premeasure on $\bigcup_{i=1}^{\infty} \mathcal{B}_{l_i}^d [0,1)^d$. Set $\mu([0,1)^d) = 1$. Suppose now that $\mu(I)$ has been defined for all cubes in $\mathcal{B}_{l_{k-1}}^d [0,1)^d$, and let $J \in \mathcal{B}_{l_k}^d$. Consider the unique 'parent cube' $I \in \mathcal{B}_{l_{k-1}}^d$ for which $J \subset I$. Define

$$\mu(J) = \begin{cases} \mu(I)/\#\mathcal{B}_{l_k}^d(X_k \cap I) & \text{if } J \subset X_k, \\ 0 & \text{otherwise.} \end{cases} \tag{4.18}$$

Observe that for each index $k \geq 1$ and each $I \in \mathcal{B}_{l_{k-1}}^d$,

$$\sum_{J \in \mathcal{B}_{l_k}^d(I)} \mu(J) = \sum_{J \in \mathcal{B}_{l_k}^d(X_k \cap I)} \mu(J) = \mu(I). \tag{4.19}$$

In particular, for each index k we have

$$\sum_{I \in \mathcal{B}_{l_k}} \mu(I) = 1.$$

By a standard argument involving the Caratheodory extension theorem [5, Proposition 1.7], the premeasure μ extends to a measure on the Borel subsets of $[0,1)^d$. Note that for each $k \geq 1$, μ is supported on X_k. Thus μ is supported on $\bigcap X_k = X$.

To complete the proof of Theorem 18 we will show that μ is a Frostman measure of dimension $(dn-\alpha)/(n-1)-\varepsilon$ for every $\varepsilon > 0$.

Lemma 23. *For each $k \geq 1$ and each $J \in \mathcal{B}^d_{l_k}(X)$,*

$$\mu(J) \lesssim l_k^{\frac{dn-\alpha}{n-1}-\eta_k}, \quad \text{where} \quad \eta_k = \frac{n+1}{2(n-1)} \cdot \varepsilon_k \searrow 0 \text{ as } k \to \infty.$$

Proof. Let $J \in \mathcal{B}^d_{l_k}$ and let $I \in \mathcal{B}^d_{l_{k-1}}$ be the parent cube of J. Since μ is a probability measure, we have $\mu(I) \leq 1$. Combining (4.18), (4.14), (4.13), and (4.10) we obtain

$$\mu(J) \leq \frac{2r_k^d}{l_{k-1}^d}\mu(I) \leq \frac{2r_k^d}{l_{k-1}^d} \lesssim \frac{l_k^{\frac{dn-\alpha-\varepsilon_k}{n-1}}}{l_{k-1}^d} = l_k^{\frac{dn-\alpha}{n-1}-\eta_k}\left(l_k^{0.5\varepsilon_k}/l_{k-1}^d\right) \leq l_k^{\frac{dn-\alpha}{n-1}-\eta_k}. \quad \square$$

Corollary 24. *For each $k \geq 1$ and each $I' \in \mathcal{B}^d_{r_k}(X_{k-1})$,*

$$\mu(I') \lesssim (r_k/l_{k-1})^d \cdot l_{k-1}^{\frac{dn-\alpha}{n-1}-\eta_{k-1}}. \tag{4.20}$$

Proof. Let us fix a cube $I' \in \mathcal{B}^d_{r_k}(X_{k-1})$, and let I denote its unique parent cube in $\mathcal{B}^d_{l_{k-1}}(X_{k-1})$. According to (4.15), I' contains at most one cube in $\mathcal{B}^d_{l_k}(I)$; let us denote this cube by J if it exists. Then the mass distribution rule given by (4.18) dictates that

$$\mu(I') = \mu(X_k \cap I') = \begin{cases} \mu(J) = \mu(I)/\#\mathcal{B}^d_{l_k}(X_k \cap I) & \text{if } \#\mathcal{B}^d_{l_k}(X_k \cap I') = 1, \\ 0 & \text{if } \#\mathcal{B}^d_{l_k}(X_k \cap I') = 0. \end{cases}$$

Using the estimate (4.14) and applying Lemma 23 to $I \in \mathcal{B}^d_{l_{k-1}}(X)$, we arrive at the claimed bound (4.20). $\square$

Lemma 23 and Corollary 24 allow us to control the behavior of μ at all scales.

Lemma 25. *For every $\alpha \in [d, dn)$, and for each $\varepsilon > 0$, there is a constant C_ε so* *that for all dyadic lengths $l \in (0,1]$ and all $I \in \mathcal{B}_l^d$, we have*

$$\mu(I) \le C_\varepsilon l^{\frac{dn-\alpha}{n-1} - \varepsilon}. \tag{4.21}$$

Proof. Fix $\varepsilon > 0$. Since $\eta_k \searrow 0$ as $k \to \infty$, there is a constant C_ε so that $l_k^{-\eta_k} \le C_\varepsilon l_k^{-\varepsilon}$ for each $k \ge 1$. For instance, if ε_k is decreasing, we could choose $C_\varepsilon = l_{k_0}^{-\eta_{k_0}}$, where k_0 is the largest integer for which $\eta_{k_0} \ge \varepsilon$. Next, let k be the (unique) index so that $l_{k+1} \le l < l_k$. We will split the proof of (4.21) into two cases, depending on the position of l within $[l_{k+1}, l_k]$.

Case 1: If $r_{k+1} \le l < l_k$, we can cover I by $(l/r_{k+1})^d$ cubes in $\mathcal{B}_{r_{k+1}}^d$. By Corollary 24,

$$\begin{aligned} \mu(I) &\lesssim (l/r_{k+1})^d (r_{k+1}/l_k)^d l_k^{\frac{dn-\alpha}{n-1} - \eta_k} \\ &= (l/l_k)^d l_k^{\frac{dn-\alpha}{n-1} - \eta_k} \\ &= l^{\frac{dn-\alpha}{n-1}} (l/l_k)^{\frac{\alpha-d}{n-1}} l_k^{-\eta_k} \\ &\le l^{\frac{dn-\alpha}{n-1} - \eta_k} \\ &\le C_\varepsilon l^{\frac{dn-\alpha}{n-1} - \varepsilon}. \end{aligned} \tag{4.22}$$

The penultimate inequality is a consequence of our assumption $\alpha \ge d$.

Case 2: If $l_{k+1} \le l \le r_{k+1}$, we can cover I by a single cube in $\mathcal{B}_{r_{k+1}}^d$. By (4.15), each cube in $\mathcal{B}_{r_{k+1}}^d$ contains at most one cube $I_0 \in \mathcal{B}_{l_{k+1}}^d(X_{k+1})$, so by Lemma 23,

$$\mu(I) \le \mu(I_0) \lesssim l_{k+1}^{\frac{dn-\alpha}{n-1} - \eta_{k+1}} \le C_\varepsilon l_{k+1}^{\frac{dn-\alpha}{n-1} - \varepsilon} \le C_\varepsilon l^{\frac{dn-\alpha}{n-1} - \varepsilon} \qquad \square$$

Applying Frostman's lemma to Lemma 25 gives $\dim_{\mathbf{H}}(X) \ge \frac{dn-\alpha}{n-1} - \varepsilon$ for every $\varepsilon > 0$, which concludes the proof of Theorem 18.

4.4 Applications

As discussed in the introduction, Theorem 18 generalizes Theorems 1.1 and 1.2 from [6]. In this section, we present two applications of Theorem 18 in settings where previous methods do not yield any results.

4.4.1 Sum-sets avoiding specified sets

Theorem 26. *Let $Y \subset \mathbf{R}^d$ be a countable union of sets of Minkowski dimension at most $\beta < d$. Then there exists a set $X \subset \mathbf{R}^d$ with Hausdorff dimension at least $d-\beta$ such that $X+X$ is disjoint from Y.*

Proof. Define $Z = Z_1 \cup Z_2$, where

$$Z_1 = \{(x,y) : x+y \in Y\} \quad \text{and} \quad Z_2 = \{(x,y) : y \in Y/2\}.$$

Since Y is a countable union of sets of Minkowski dimension at most β, Z is a countable union of sets with lower Minkowski dimension at most $d+\beta$. Applying Theorem 18 with $n=2$ and $\alpha = d+\beta$ produces a set $X \subset \mathbf{R}^d$ with Hausdorff dimension $2d-(d+\beta) = d-\beta$ such that $(x,y) \notin Z$ for any $x,y \in X$ with $x \neq y$. We claim that $X+X$ is disjoint from Y. To see this, first suppose $x,y \in X$, $x \neq y$. Since X avoids Z_1, we conclude that $x+y \notin Y$. Suppose now that $x = y \in X$. Since $(x,y) \notin Z_2$ for any $x,y \in X$, $x \notin Y/2$ for any $x \in X$. Thus for any $x \in X$, $x+x = 2x \notin Y$. This completes the proof. $\square$

4.4.2 Subsets of Lipschitz curves avoiding isosceles triangles

In [6], Fraser and the second author prove that if $\gamma \subset \mathbf{R}^n$ is a simple C^2 curve with non-vanishing curvature, then there exists a set $S \subset \gamma$ of Hausdorff dimension $1/2$ that does not contain the vertices of an isosceles triangle. Using Theorem 18, we generalize this result to Lipschitz curves.

Theorem 27. *Let $f\colon [0,1) \to \mathbf{R}^{n-1}$ be Lipschitz. Then there is a set $X \subset [0,1)$ of Hausdorff dimension $1/2$ so that the set $\{(t, f(t)) : t \in X\}$ does not contain the vertices of an isosceles triangle.*

Proof. Set

$$Z = \left\{ (x_1,x_2,x_3) \in [0,1]^3 : \begin{array}{c} (x_1, f(x_1)), (x_2, f(x_2)), (x_3, f(x_3)) \\ \text{form the vertices of an isosceles triangle} \end{array} \right\}.$$

In the next lemma, we show Z has lower Minkowski dimension at most two. By Theorem 18, there is a set $X_1 \subset [0,1]$ of Hausdorff dimension $1/2$ so that for each distinct $x_1,x_2,x_3 \in X_1$, we have $(x_1,x_2,x_3) \notin Z$. This is precisely the statement that for each $x_1,x_2,x_3 \in X$, the points $(x_1, f(x_1))$, $(x_2, f(x_2))$, and $(x_3, f(x_3))$ do not form the vertices of an isosceles triangle. To complete the proof, let $X = \{(x, f(x)) : x \in X_1\}$. $\square$

Lemma 28. *Let $f\colon [0,1) \to \mathbf{R}^{n-1}$ be Lipschitz. Then the set*

$$Z = \left\{ (x_1,x_2,x_3) \in [0,1]^3 : \begin{array}{c} (x_1, f(x_1)), (x_2, f(x_2)), (x_3, f(x_3)) \\ \textit{form the vertices of an isosceles triangle} \end{array} \right\}.$$

has Minkowski dimension at most two.

Proof. By translating and rescaling the range of f, which does not change the Minkowski dimension of the graph, we may assume withous loss of generality that f is $1/10$ Lipschitz, and that it's graph is contained in $[0,1]^n$. Fix $0 < \delta < 1$. It suffices to show that

$$\#\mathcal{B}^3_\delta(Z) \lesssim \delta^{-2} \log(1/\delta). \tag{4.23}$$

We have

$$|\mathcal{B}^3_\delta(Z)| = \sum_{I_1 \in \mathcal{B}^1_\delta([0,1])} \sum_{k=0}^{\log_2(1/\delta)} \sum_{\substack{I_2 \in \mathcal{B}^1_\delta([0,1]) \\ \operatorname{dist}(I_1,I_2) \sim \delta 2^k}} \#\{I_3 \in \mathcal{B}^1_\delta([0,1]) : I_1 \times I_2 \times I_3 \in \mathcal{B}^3_\delta(Z)\}.$$

In the above expression we abuse notation slightly and say that $\mathrm{dist}(I_1, I_2) \sim \delta$ if $I_1 = I_2$; this will not affect our estimates.

Note that for each $I_1 \in \mathcal{B}^1_\delta([0,1])$, there are roughly $(\delta 2^k)/\delta = 2^k$ intervals $I_2 \in \mathcal{B}^1_\delta([0,1])$ with $\mathrm{dist}(I_1, I_2) \sim \delta 2^k$. Thus to establish (4.23), it suffices to prove that for each $I_1 \in \mathcal{B}^1_\delta([0,1])$ and each $I_2 \in \mathcal{B}^1_\delta([0,1])$ with $\mathrm{dist}(I_1, I_2) \sim \delta 2^k$, we have

$$\#\{I_3 \in \mathcal{B}^1_\delta([0,1]) : I_1 \times I_2 \times I_3 \in \mathcal{B}^3_\delta(Z)\} \lesssim 2^{-k}/\delta. \tag{4.24}$$

For each distinct $p, q \in [0,1]^n$, define

$$H_{p,q} = \left\{ z \in \mathbf{R}^n : \left(z - \frac{p+q}{2} \right) \cdot (p - q) = 0 \right\}.$$

This is the hyperplane passing through the midpoint of p and q that is perpendicular to the line passing through p and q. We will call $H_{p,q}$ the perpendicular bisector of p and q.

Fix a choice of intervals I_1 and I_2 with $\mathrm{dist}(I_1, I_2) \sim \delta 2^k$. Let $\tilde{I}_1$ and $\tilde{I}_2$ denote the twofold dilates of I_1 and I_2, respectively. Note that if $I_3 \in \mathcal{B}^1_\delta([0,1])$ with $I_1 \times I_2 \times I_3 \in \mathcal{B}^3_\delta(Z)$, then there are points $x_j \in \tilde{I}_j,\ i = 1,2,3$ so that

$$(x_3, f(x_3)) \in H_{(x_1, f(x_1)),(x_2, f(x_2))}.$$

Consider the set

$$S_{I_1,I_2} = [0,1]^n \cap \bigcup_{\substack{x_1 \in \tilde{I}_1 \\ x_2 \in \tilde{I}_2}} H_{(x_1, f(x_1)),(x_2, f(x_2))}.$$

For each $x_1 \in \tilde{I}_1$ and $x_2 \in \tilde{I}_2$, the line passing through $(x_1, f(x_1))$ and $(x_2, f(x_2))$ makes an angle $\leq 1/10$ with the e_1 direction. Thus the hyperplane $H_{(x_1, f(x_1)),(x_2, f(x_2))}$ makes an angle $\leq 1/10$ with the hyperplane spanned by the $e_2, \ldots, e_n$ directions. Since $\tilde{I}_1$ and $\tilde{I}_2$ are intervals of length $\leq 3\delta$ that are $\sim \delta 2^k$ separated, S_{I_1,I_2} is contained in the $\sim 2^{-k}$ neighborhood of a hyperplane that makes an angle $\leq 1/10$

with the $e_2, \dots, e_n$ directions.

Suppose that $x_3, x_3' \in [0,1]$ satisfy

$$(x_3, f(x_3)) \in S_{I_1,I_2} \quad \text{and} \quad (x_3', f(x_3')) \in S_{I_1,I_2}. \tag{4.25}$$

Since f is $1/10$-Lipschitz, we must have

$$|f(x_3) - f(x_3')| \leq \frac{1}{10}|x_3 - x_3'|.$$

On the other hand, by (4.25) and the fact that S_{I_1,I_2} is contained in the $\sim 2^{-k}$ neighborhood of a hyperplane that makes an angle $\leq 1/10$ with the $e_2, \dots, e_n$ directions, we have

$$|f(x_3) - f(x_3')| \geq 10|x_3 - x_3'| - O(2^{-k}).$$

We conclude that $|x_3 - x_3'| \lesssim 2^{-k}$. This establishes (4.24). We conclude that (4.23) holds, so Z has lower Minkowski dimension at most 2. $\square$

Chapter 5

Extensions to Low Rank Configurations

5.1 Boosting the Dimension of Pattern Avoiding Sets by Low Rank Coordinate Changes

We now consider finding subsets of $[0,1]$ avoiding solutions to the equation $y = f(Tx)$, where T is a rank k linear transformation with integer coefficients with respect to standard coordinates, and f is real-valued and Lipschitz continuous. Fix a constant A bounding the operator norm of T, in the sense that $|Tx| \leq A|x|$ for all $x \in \mathbf{R}^n$, and a constant B such that $|f(x+y) - f(x)| \leq B|y|$ for all x and y for which the equation makes sense (if f is C^1, this is equivalent to a bound $\|\nabla f\|_\infty \leq B$). Consider sets $J_0, J_1, \dots, J_n, \subset [0,1]$, which are unions of intervals of length $1/M$, with startpoints lying on integer multiples of $1/M$. The next theorem works as a 'building block lemma' used in our algorithm for constructing a set avoiding solutions to the equation with Hausdorff dimension k and full Minkowski dimension.

Theorem 29. *For infinitely many integers N, there exists $S_i \subset J_i$ avoiding solutions to $y = f(Tx)$ with $y \in S_0$ and $x_n \in S_n$, such that*

- *For $n \neq 0$, if we decompose each J_i into length $1/N$ consecutive intervals, S_i contains an initial portion $\Omega(1/N^k)$ of each length $1/N$ interval. This part of the decomposition gives the Hausdorff dimension $1/k$ bound for the set we will construct.*

- *If we decompose J_i into length $1/N$ intervals, and then subdivide these intervals into length $\Omega(1/N^k)$ intervals, then S_0 contains a subcollection of these $1/N^k$ intervals which contains a total length $\Omega(1/N)$ of a fraction $1-1/M$ of the length $1/N$ intervals. This property gives that our resultant set will have full Minkowski dimension.*

The implicit constants in these bounds depend only on A, B, n, and k.

Proof. Split each interval of J_a into length $1/N$ intervals, and then set

$$\mathbf{A} = \{x : x_a \text{ is a startpoint of a } 1/N \text{ interval in } J_a\}$$

Since the startpoints of the intervals are integer multiples of $1/N$, $T(\mathbf{A})$ is contained with a rank k sublattice of $(\mathbf{Z}/N)^m$. The operator norm also guarantees $T(\mathbf{A})$ is contained within the ball B_A of radius A in $\mathbf{R}^m$. Because of the lattice structure of the image, $|x-y| \gtrsim_n 1/N$ for each distinct pair $x, y \in T(\mathbf{A})$. For any R, we can cover $\Sigma \cap B_A$ by $O_{n,k}((A/R)^k)$ balls of radius R. If $R \gtrsim_n 1/N$, then each ball can contain only a single element of $T(\mathbf{A})$, so we conclude that $|T(\mathbf{A})| \lesssim_{n,k} (AN)^k$. If we define the set of 'bad points' to be

$$\mathbf{B} = \{y \in [0,1] : \text{there is } x \in \mathbf{A} \text{ such that } y = f(T(x))\}$$

Then

$$|\mathbf{B}| = |f(T(\mathbf{A}))| \leq |T(\mathbf{A})| = O_{A,n,k}(N^k)$$

For simplicity, we now introduce an integer constant $C_0 = C_0(A,n,k,M)$ such that $|\mathbf{B}| \leq (C_0/M^2)N^k$. We now split each length $1/M$ interval in J_0 into length $1/N$ intervals, and filter out those intervals containing more than $C_0 N^{k-1}$ elements of

B. Because of the cardinality bound we have on **B** there can be at most N/M^2 such intervals, so we discard at most a fraction $1/M$ of any particular length $1/M$ interval in J_0. If we now dissect the remaining intervals into $4C_0N^{k-1}$ intervals of length $1/4C_0N^k$, and discard any intervals containing an element of **B**, or adjacent to such an interval, then the remaining such intervals I satisfy $d(I, \mathbf{B}) \geq 1/4C_0N^k \gtrsim_{A,n,M,k} (1/N^k)$, and because of our bound on the number of elements of **B** in these intervals, there are at least C_0N^{k-1} intervals remaining, with total length exceeding $C_0N^{k-1}/4C_0N^k = \Omega(1/N)$. If f is C^1 with $\|\nabla f\|_\infty \leq B$, or more generally, if f is Lipschitz continuous of magnitude B, then

$$|f(Tx) - f(Tx')| \leq AB|x - x'|$$

and so we may choose $S_i \subset J_i$ by thickening each startpoint $x \in J_i$ to a length $O(1/N^k)$ interval while still avoiding solutions to the equation $y = f(T(x))$. $\square$

Remark. *If T is a rank k linear transformation with rational coefficients, then there is some number a such that aT has integer coefficients, and then the equation $y = f(Tx)$ is the same as the equation $y = f_0((aT)(x))$, where $f_0(x) = f(x)/a$. Since f_0 is also Lipschitz continuous, we conclude that we still get the dimension $1/k$ bound if T has rational rather than integral coefficients. More generally, this trick shows the result applies unperturbed if all coefficients of T are integer multiples of some fixed real number. More generally, by varying the lengths of our length $1/N$ decomposition by a constant amount, we can further generalize this to the case where each column of T are integers multiples of some fixed real number.*

Remark. *To form* **A***, we take startpoints lying at equal spaced $1/N$ points. However, by instead taking startpoints at varying points in the length $1/N$ intervals, we might be able to make points cluster more than in the original algorithm. Maybe the probabalistic method would be able to guarantee the existence of a choice of startpoints whose images are tightly clustered together.*

Remark. *Since the condition $y = f(Tx)$ automatically assumes a kind of 'non-vanishing derivative' condition on our solutions, we do not need to assume the* *regularity of f, and so the theorem extends naturally to a more general class of functions than Rob's result, i.e. the Lipschitz continuous functions.*

Using essentially the same approach as the last argument shows that we can avoid solutions to $y = f(Tx)$, where y and x are now vectors in some $\mathbf{R}^m$, and T has rank k. If we consider unions of $1/M$ cubes $J_0, \dots, J_n$. If we fix startpoints of each x_k forming lattice spaced apart by $\Omega(1/N)$, and consider the space $\mathbf{A}$ of products, then there are $O(N^k)$ points in $T(\mathbf{A})$, and so there are $O(N^k)$ elements in $\mathbf{B}$. We now split each $1/M$ cube in J_0 into length $1/N$ cubes, and discard those cubes which contain more than $O(N^{k-m})$ bad points, then we discard at most $1 - 1/M$ of all such cubes. We can dissect the remaining length $1/N$ cubes into $O(N^{k-m})$ length $\Omega(1/N^{k/m})$ cubes, and as in the previous argument, the cubes not containing elements of $\mathbf{B}$ nor adjacent to an element have total volume $\Omega(1/N^m)$, which we keep. The startpoints in the other intervals T_i may then be thickened to a length $\Omega(1/N^{m/k})$ portion while still avoiding solutions. This gives a set with full Minkowski dimension and Hausdorff dimension m/k avoiding solutions to $y = f(Tx)$. (I don't yet understand Minkowski dimension enough to understand this, but the techniques of the appendix make proving the Hausdorff dimension $1/k$ bound easy)

5.2 Extension to Well Approximable Numbers

If the coefficients of the linear transformation T in the equation $y = f(Tx)$ are non-rational, then the images of startpoints under the action of T do not form a lattice, and so points may not overlap so easily when avoiding solutions to the equations $y = f(Tx)$. However, if T is 'very close' to a family of rational coefficient linear transformations, then we can show the images of the startpoints are 'very close' to a lattice, which will still enable us to find points avoiding solutions by replacing the direct combinatorial approach in the argument for integer matrices with a covering argument.

Suppose that T is a real-coefficient linear transformation with the property that for each coefficient x there are infinitely many rational numbers p/q with

$|x - p/q| \leq 1/q^{\alpha}$, for some fixed α. For infinitely many K, we can therefore find a linear transformation S with coefficients in $\mathbf{Z}/K$ with each coefficient of T differing from the corresponding coefficient in S by at most $1/K^{\alpha}$. Then for each x, we find

$$\|(T-S)(x)\|_{\infty} \leq (n/K^{\alpha})\|x\|_{\infty}$$

If we now consider $T_0, \dots, T_n$, splitting $T_1, \dots, T_n$ into length $1/N$ intervals, and considering $\mathbf{A}$ as in the last section, then $S(\mathbf{A})$ lie in a k dimensional sublattice of $(\mathbf{Z}/KN)^m$, hence containing at most $(2A)^k (KN)^k = O_{T,n}((KN)^k)$ points. By our error term calculation of $T - S$, the elements of $T(\mathbf{A})$ are contained in cubes centered at these lattice points with side-lengths $2n/K^{\alpha}$, or balls centered at these points with radius $n^{3/2}/K^{\alpha}$. If $\|\nabla f\| \leq B$, then the images of the radius $n^{3/2}/K^{\alpha}$ balls under the action of f are contained in length $Bn^{3/2}/K^{\alpha}$ intervals. Thus the total length of the image of all these balls under f is $(Bn^{3/2}/K^{\alpha})(2A)^k(KN)^k = (2A)^k Bn^{3/2} K^{k-\alpha} N^k$. If $k < \alpha$, then we can take K arbitrarily large, so that there exists intervals with $\text{dist}(I, \mathbf{B}) = \Omega_{A,k,M}(1/N^k)$. But I believe that, after adding the explicit constants in, we cannot let $k = \alpha$.

Remark. *One problem is that, if T has rank k, we might not be able to choose S to be rank k as well. Is this a problem? If T has full rank, then the set of all such matrices is open so if T and S are close enough, S also has rank k, but this need not be true if T does not have full rank.*

Example. *If T has rank 1, then Dirichlet's theorem says that every irrational number x can be approximated by infinitely many p/q with $|x - p/q| < 1/q^2$, so every real-valued rank 1 linear transformation can be avoided with a dimension one bound.*

5.3 Applications of Low Rank Coordinate Changes

Example. *Our initial exploration of low rank coordinate changes was inspired by trying to find solutions to the equation*

$$y - x = (u - w)^2$$

Our algorithm gives a Hausdorff dimension $1/2$ *set avoiding solutions to this equation. This equals Mathé's result. But this dimension for us now depends on the shifts involved in the equation, not on the exponent, so we can actually avoid solutions to the equation*

$$y - x = (u - w)^n$$

for any n, in a set of Hausdorff dimension $1/2$*. More generally, if* X *is a set, then given a smooth function* f *of n variables, we can find a set* X *of Hausdorff dimension* $1/n$ *such that there is no* $x \in X$*, and* $y_1, \dots, y_n \in X - X$ *such that* $x = f(y_1, \dots, y_n)$*. This is better than the* $1/2n$ *bound that is obtained by Malabika and Fraser's result.*

Example. *For any fixed m, we can find a set* $X \subset \mathbf{R}^n$ *of full Hausdorff dimension which contains no solutions to*

$$a_1 x_1 + \dots + a_n x_n = 0$$

for any rational numbers a_n *which are not all zero. Since Malabika/Fraser's technique's solutions are bounded by the number of variables, they cannot let* $n \to \infty$ *to obtain a linearly independant set over the rational numbers. But since the Hausdorff dimension of our sets now only depends on the rank of* T*, rather than the total number of variables in* T*, we can let* $n \to \infty$ *to obtain full sets linearly independant over the rationals. More generally, for any Lipschitz continuous function* $f : \mathbf{R} \to \mathbf{R}$*, we can find a full Hausdorff dimensional set such that there are no solutions*

$$f(a_1 x_1 + \dots + a_n x_n, y)$$

for any n, and for any rational numbers a_n that are not all zero.

Example. *The easiest applications of the low rank coordinate change method are probably involving configuration problems involving pairwise distances between m points in $\mathbf{R}^n$, where $m \ll n$, since this can best take advantage of our rank condition. Perhaps one way to encompass this is to avoid m vertex polyhedra in n dimensional space, where $m \ll n$. In order to distinguish this problem from something that can be solved from Mathé's approach, we can probably find a high dimensional set avoiding m vertex polyhedra on a parameterized n dimensional manifold, where $m \ll n$. There is a result in projective geometry which says that every projectively invariant property of m points in $\mathbf{RP}^d$ is expressible as a function in the $\binom{m}{d}$ bracket polynomials with respect to these m points. In particular, our result says that we can avoid a countable collection of such invariants in a dimension $1/\binom{m}{d}$ set. This is a better choice of coordinates than Euclidean coordinates if $\binom{m}{d} \leq m$. Update: I don't think this is ever the case.*

Remark. *Because of how we construct our set X, we can find a dimension $1/k$ set avoiding solutions to $y = f(Tx)$ for all rank k rational matrices T, without losing any Hausdorff dimension. Maybe this will help us avoid solutions to more general problems?*

Example. *Given a smooth curve Γ in $\mathbf{R}^n$, can we find a subset E with high Hausdorff dimension avoiding isoceles triangles. That is, if the curve is parameterized by $\gamma : [0,1] \to \mathbf{R}^n$, can we find $E \subset [0,1]$ such that for any t_1, t_2, t_3, $\gamma(t_1)$, $\gamma(t_2)$, and $\gamma(t_3)$ do not form the vertices of an isoceles triangle. This is, in a sense, a non-linear generalization of sets avoiding arithmetic progressions, since if Γ is a line, an isoceles triangle is given by arithmetic progressions. Assuming our curve is simple, we must avoid zeroes of the function*

$$|\gamma(t_1) - \gamma(t_2)|^2 = |\gamma(t_2) - \gamma(t_3)|^2$$

If we take a sufficiently small segment of this curve, and we assume the curve has non-zero curvature on this curve, we can assume that $t_1 < t_2 < t_3$ in our dissection *method.*

If the coordinates of γ are given by polynomials with maximum degree d, then the equation

$$|\gamma(t_1) - \gamma(t_2)|^2 - |\gamma(t_2) - \gamma(t_3)|^2$$

is a polynomial of degree $2d$*, and so Mathé's result gives a set of dimension* $1/2d$ *avoiding isoceles triangles. In the case where* Γ *is a line, then the function* $f(t_1,t_2,t_3) = \gamma(t_1) + \gamma(t_3) - 2\gamma(t_2)$ *avoids arithmetic progresions, and Mathé's result gives a dimension one set avoiding such progressions. Rob and Malabika's algorithm easily gives a set with dimension* $1/2$ *for any curve* Γ*. Our algorithm doesn't seem to be able to do much better here.*

Example. *What is the largest dimension of a set in Euclidean space such that for any value* λ*, there is at most one pair of points* x,y *in the set such that* $|x-y| = \lambda$*.*

Example. *What is the largest dimension of a set which avoids certain angles, i.e. for which a triplet* x,y,z *avoids certain planar configurations.*

Example. *A set of points* $x_0,\dots,x_d \in \mathbf{R}^d$ *lie in a hyperplane if and only if the determinant formed by the vectors* $x_n - x_0$*, for* $n \in \{1,\dots,d\}$*, is zero. This is a degree d polynomial, hence Mathé's result gives a dimension one set with no set of* $d+1$ *points lying in a hyperplane. On the other hand, a theorem of Mattila shows that every analytic set E with dimension exceeding one contains* $d+1$ *points in a hyperplane. Can we generalize this to a more general example avoiding points on a rotational, translation invariant family of manifolds using our results?*

Example. *Given a set F not containing the origin, what is the largest Hausdorff dimension of a set E such that for any for any distinct rational* $a_1,\dots,a_N$*, the sum* $a_1E + \dots + a_NE$ *does not contain any elements of F. Thus the vector space over the rationals generated by E does not contain any elements of F. We can also take the non-linear values* $f(a_1E + \dots + a_NE)$ *avoiding elements of F. F must have non-empty interior for the problem to be interesting. Then can we find a smooth function f with non-nanishing derivative which vanishes over F, or a family of smooth functions with non-vanishing derivative around F.*

Chapter 6

Fourier Dimension and Fractal Avoidance

In the last few chapters, we have discovered that the presence of many configurations is not guaranteed by large Hausdorff dimension. We now turn to an analysis of a different dimensional quantity which in many cases gives much more structure than Hausdorff dimension. We construct a set E

Proof. Let $S_0 = \{0\}$. Then, given S_k, let N^s

$$S_{k+1} = \left\{ (j, mN_{k+1}^{1-\alpha} + X_{jk}) : j \in S_k, 0 \leq m < N_{k+1}^{\alpha} \right\}$$

where X_{jk} is chosen uniformly randomly from all integers in $[0, N_k^{\alpha})$. Write $E_k = \{\bigcup Q_j : j \in S_k\}$. Then $\#(S_{k+1}) = N_{k+1}^{\alpha} \#(S_k)$, and since $\#(S_0) = 1$, we find $\#(S_k) = (N_1 \dots N_k)^{\alpha}$. Now define $E_k = \{\bigcup Q_j : j \in S_k\}$, and probability measures

$$f_k = (N_1 \dots N_k)^{1-\alpha} \mathbf{I}_{E_k}.$$

Then $\widehat{f_{k+1}}(m) - \widehat{f_k}(m) = \sum_{j \in S_k} Y_j(m)$, where

$$Y_j(m) = \int_{I_j} (f_{k+1}(x) - f_k(x)) e(mx)\, dx$$

Note that $\mathbf{E}[f_{k+1}(x)] = \mathbf{E}[f_k(x)]$ for each x, so that $\mathbf{E}[Y_j(m)] = 0$. Note also that

$$|Y_j(m)| \leq 2l_k/l_{k+1}^{1-\alpha} = \frac{2N_{k+1}^{1-\alpha}}{(N_1 \dots N_k)^{\alpha}}.$$

Thus Hoeffding's inequality implies

$$\begin{aligned}\mathbf{P}\left(|\widehat{f_{k+1}}(m) - \widehat{f}_k(m)| \geq t\right) &\leq 4\exp\left(-\left(t\frac{(N_1\dots N_k)^{\alpha}}{4N_{k+1}^{1-\alpha}\#(S_k)^{1/2}}\right)^2\right) \\ &\leq 4\exp\left(-\left(t\frac{(N_1\dots N_k)^{\alpha/2}}{2N_{k+1}^{1-\alpha}}\right)^2\right)\end{aligned}$$

Thus we can guarantee with high probability that for all m,

$$|\widehat{f_{k+1}}(m) - \widehat{f}_k(m)| \lessapprox \frac{N_{k+1}^{1-\alpha}}{(N_1 \dots N_k)^{\alpha/2}}$$

So if $N_{k+1} \lesssim (N_1 \dots N_k)^{\varepsilon}$ for all $\varepsilon > 0$, we obtain the right bound. But we need something deeper.

Take 2: Write

$$f_{k+1}(x) = (N_1 \dots N_{k+1})^{1-\alpha} \sum_{j \in S_k} \sum_{m=0}^{N_{k+1}^{1-\alpha}} Y_{jm}$$

where Y_{jm} is the indicator function of

$$\left[\frac{j}{N_1 \dots N_k} + \frac{m}{N_1 \dots N_{k+1}^{\alpha}} + \frac{X_{jk}}{N_1 \dots N_{k+1}}, \frac{j}{N_1 \dots N_k} + \frac{m}{N_1 \dots N_{k+1}^{\alpha}} + \frac{X_{jk}+1}{N_1 \dots N_{k+1}}\right]$$

Note that

$$\widehat{Y_{jm}}(n) = \frac{1}{2\pi in} e\left(\frac{jn}{N_1 \dots N_k} + \frac{mn}{N_1 \dots N_{k+1}^{\alpha}} + \frac{X_{jk}n}{N_1 \dots N_{k+1}}\right)\left(e\left(\frac{n}{N_1 \dots N_{k+1}}\right) - 1\right),$$

and

$$\mathbf{E}\left(\widehat{Y_{jm}}(n)\right) = \frac{1}{2\pi i n N_{k+1}^{\alpha}} e\left(\frac{jn}{N_1 \dots N_k} + \frac{mn}{N_1 \dots N_{k+1}^{\alpha}}\right)\left(e\left(\frac{N_{k+1}^{\alpha} n}{N_1 \dots N_{k+1}}\right) - 1\right).$$

and $|\widehat{Y_{jm}}| \leq 1/(N_1 \dots N_{k+1})$. Thus Hoeffding's inequality implies, if we ignore the expectation,

$$\begin{aligned}\mathbf{P}\left(|\widehat{f_{k+1}}(n)| \geq t(N_1 \dots N_{k+1})^{1-\alpha}\right) &\leq 4\exp\left(\frac{-t^2(N_1 \dots N_{k+1})^2}{4(N_1 \dots N_{k+1})^{\alpha}}\right) \\ &\leq 4\exp\left(-\left(t(N_1 \dots N_{k+1})^{1-\alpha/2}/2\right)^2\right).\end{aligned}$$

Thus with high probability, for all $|n| \leq N_1 \dots N_{k+1}$,

$$|\widehat{f_{k+1}}(n)| \lessapprox \frac{1}{(N_1 \dots N_{k+1})^{\alpha/2}}.$$

$\square$

6.1 Schmerkin's Method

We now try and adapt Schmerkin's Method to our situation, introducing translations into the avoiding set. Let $K \subset \mathbf{Z}_N^n$. Our goal is to find $X \subset \mathbf{Z}_N$ such that if $x_1, \dots, x_n$ are distinct, then $(x_1, \dots, x_n) \notin K$, such that if $f : \mathbf{Z}_N \to \mathbf{R}$ is the probability distribution which uniformly randomly chooses a point on X, such that the Fourier transform of f has small L^∞ norm.

We consider an intermediary scale M, with $M \mid N$. Then we consider the set

$$K' = \{x \in \mathbf{Z}_N^n : \text{There is } y \in K \text{ with } x - y \in M\mathbf{Z}_N^n\}.$$

We construct X such that if $x_1, \dots, x_n \in X$ are distinct, then $(x_1, \dots, x_n) \notin Y$. In particular, this means that if we define, for each $x \in X$, a uniformly random selected integer $n_x \in M\mathbf{Z}_N$, then we can define $x' = x + n_x$, and $X' = \{x' : x \in X\}$.

Now if $I_x : \mathbf{Z}_N \to \mathbf{R}$ denotes the delta function at x', then

$$\widehat{I_x}(k) = (1/N)e(-kx'/N) = (1/N)e(-kx/N)e(-n_x).$$

And so

$$\mathbf{E}(\widehat{I_x}(k)) = (M/N)e(-kx/N)\sum_{m=1}^{N/M} e(-(kM/N)m).$$

If $kM/N \not\in \mathbf{Z}$, $\mathbf{E}(\widehat{I_x}(k)) = 0$. If $kM/N \in \mathbf{Z}$, then $\mathbf{E}(\widehat{I_x}(k)) = e(-kx/N)$. And

$$|\widehat{I_x}(k)| \leq 1.$$

Since I_x is independant, we can apply Hoeffding's inequality to conclude that

$$\mathbf{P}\left(\widehat{f}(k) \geq t\right) \leq \exp\left(\frac{-t^2}{4|X|}\right)$$

so $\widehat{f}(k) \lesssim |X|^{1/2}$ with high probability. This should give the Fourier dimension bound we want? Except we do lose some dimension by introducing the parameter M. If we pick N to be very large, but M very small, then this shouldn't introduce many problems? What about if we pick M randomly to prevent problems on the linear frequencies?

6.2 Tail Bounds on Cube Selection

The aim of this section is to obtain tail bounds on the total number of intersections between the random set we select in the discrete selection schema, with the goal of obtaining a Fourier dimension bound on the set we construct. Consider the random process generating our interval selection scheme. We have three fixed lengths l, r, and s. We start with a set E, which is a non-empty union of cubes in $\mathcal{B}_l^d$, and a set G, which is a union of cubes in $\mathcal{B}_s^{dn}$. For each cube $J \in \mathcal{B}_r^d(E)$, we select a single random element of $\mathcal{B}_s^d(J)$, denoted J_I. We let $U = \bigcup J_I$. The aim of this section is to obtain tail bounds on the size of the random set

$$\mathcal{K}(U) = \{K \in \mathcal{B}_s^{dn}(G) : K \in U^n, K \text{ strongly non-diagonal}\}.$$

Here, this is obtained by applying the statistical decoupling techniques of Bourgain and Tzafriri.

To understand the distribution of this random variable, it will be convenient to simplify notation. We write $\mathbf{I}(I_J = I_0)$ as X_{J,I_0}. We then set

$$Z = \#\mathcal{K}(U) = \sum_{J_1,\dots,J_n} \sum_K \left(\prod_{i=1}^n X_{J_i,K_i} \right),$$

where $J_1,\dots,J_n$ ranges over all distinct choices of n elements from $\mathcal{B}_l^d(E)$, and $K = K_1 \times \cdots \times K_n$ ranges over all elements of $\mathcal{B}_l^{dn}(G) \cap \mathcal{B}_s^{dn}(J_1 \times \cdots \times J_n)$. The random cubes $\{I_J\}$ form an independant family, which should make the distribution easier to bound. But Z is a multiplicative linear combination of this independant family. Random variables of this form as known as a *chaos*.

The idea behind statistical decoupling is to obtain tail bounds on the chaos Z by studying the tail bounds on the modified random variable

$$W = \sum_{J_1,\dots,J_n} \sum_K \left(\prod_{i=1}^n X^i_{J_i,K_i} \right),$$

where $X^1,\dots,X^n$ are independant and identically distributed copies of X.

Lemma 30. *Let $F : [0,\infty) \to [0,\infty)$ be a monotone, convex function. Then*

$$\mathbf{E}(F(Z)) \leq \mathbf{E}(F(n^n W)).$$

Proof. Consider a partition $\mathcal{J}_1,\dots,\mathcal{J}_n$ selected at random from all possible partitions of $\mathcal{B}_s^d(E)$ into n classes. Then, for any distinct set of intervals $J_1,\dots,J_n \in$

$\mathcal{B}_s^d(E)$,

$$\mathbf{P}(J_1 \in \mathcal{J}_1, \ldots, J_n \in \mathcal{J}_n) = \mathbf{P}(J_1 \in \mathcal{J}_1) \ldots \mathbf{P}(J_n \in \mathcal{J}_n) = 1/n^n. \tag{6.1}$$

Thus if we consider

$$Z' = \sum_{J_1 \in \mathcal{J}_1, \ldots, J_n \in \mathcal{J}_n} \sum_K \left(\prod_{i=1}^n X_{J_i, K_i} \right),$$

where J_i ranges over elements of $\mathcal{J}_i$ for each $i \in [n]$. If we let $\mathbf{E}_{\mathcal{J}}$ denote conditioning with respect to all random variables except those depending on $\mathcal{J}_1, \ldots, \mathcal{J}_n$, then (6.1) implies

$$\mathbf{E}_{\mathcal{J}}(Z') = \sum_{J_1, \ldots, J_n} \sum_K \mathbf{P}(J_1 \in \mathcal{J}_1, \ldots, J_n \in \mathcal{J}_n) \left(\prod_{i=1}^n X_{J_i, K_i} \right) = Z/n^n.$$

Thus Jensen's inequality implies that

$$F(Z) = F(n^n \mathbf{E}_{\mathcal{J}}(Z')) \leq \mathbf{E}_{\mathcal{J}} F(n^n Z').$$

In particular, this means we can select a single, non-random partition $\mathcal{J}_1, \ldots, \mathcal{J}_n$ of $\mathcal{B}_s^d(E)$ such that $F(Z) \leq F(n^n Z')$. Because $\mathcal{J}_1, \ldots, \mathcal{J}_n$ are disjoint sets, Z' is identically distributed to

$$W' = \sum_{J_1 \in \mathcal{J}_1, \ldots, J_n \in \mathcal{J}_n} \sum_K \left(\prod_{i=1}^n X^i_{J_i, K_i} \right).$$

But $W' \leq W$, and so by monotonicity,

$$\mathbf{E}(F(Z)) \leq \mathbf{E}(F(n^n W')) \leq \mathbf{E}(F(n^n W)). \qquad \square$$

Lemma 31. *MOMENT BOUND THEOREM.*

Proof. We now bound the moments of W, which by Lemma 30 bounds the moments of Z. We write

$$W^m = \sum_{J_{j,i}} \sum_{K_1,\dots,K_m} \left(\prod_{i=1}^{n} \prod_{j=1}^{m} X^i_{J_{j,i},K_{j,i}} \right).$$

Thus

$$\mathbf{E}(W^m) = \sum_{J_{j,i}} \sum_{K_1,\dots,K_m} \prod_{i=1}^{n} \mathbf{E} \left(\prod_{j=1}^{m} X^i_{J_{j,i},K_{j,i}} \right)$$

The inner expectation is zero except if $\pi(K_{j,i}) = J_{j,i}$ for all j and i, and also, if $\pi(K_{j_0,i}) = \pi(K_{j_1,i})$, then $K_{j_0,i} = K_{j_1,i}$. And then the expectation is $\varepsilon^{\#\{K_{1,i},\dots,K_{m,i}\}}$.

$\square$

6.3 Singular Value Decomposition of Tensors

An *n tensor* with respect to $\mathbf{R}^{k_1},\dots,\mathbf{R}^{k_n}$, is a multi-linear function

$$A : \mathbf{R}^{k_1} \times \cdots \times \mathbf{R}^{k_n} \to \mathbf{R}.$$

The family of all n tensors forms a vector space, denoted $\mathbf{R}^{k_1} \otimes \cdots \otimes \mathbf{R}^{k_n}$, or $\bigotimes \mathbf{R}^{k_i}$. Given a family of vectors v_i, with $v_i \in \mathbf{R}^{k_i}$, we can consider a tensor $v_1 \otimes \cdots \otimes v_n$, such that

$$(v_1 \otimes \cdots \otimes v_n)(w_1,\dots,w_n) = (v_1 \cdot w_1)\dots(v_n \cdot w_n).$$

It is easily verified that a basis for $\mathbf{R}^{k_1} \otimes \cdots \otimes \mathbf{R}^{k_n}$ is given by

$$\{e_{j_1} \otimes \cdots \otimes e_{j_n} : j_i \in [k_i]\}.$$

Thus a tensor A can be identified with a family of coefficients

$$\{A_{j_1\dots j_n} : 1 \le j_i \le k_i\}$$

such that

$$A(v_1,\dots,v_n) = \sum A_{j_1\dots j_n} \cdot e_{j_1} \otimes \cdots \otimes e_{j_n}.$$

From this perspective, a tensor is a multi-dimensional array given by n indices. This basis induces a natural inner product for which the basis is orthonormal. The norm with respect to this inner product is called the *Frobenius norm*, denoted $\|A\|_F$.

Recall that if A is a rank r, $m \times n$ matrix, then the *singular value decomposition* says that there are orthonormal families of vectors $\{u_1, \dots, u_r\}$ and $\{v_1, \dots, v_r\}$ in $\mathbf{R}^m$ and $\mathbf{R}^n$ respectively, and values $s_1, \dots, s_r > 0$ such that $A = \sum s_i (u_i^T v_i)$. We may view A as a 2 tensor in $\mathbf{R}^m \otimes \mathbf{R}^n$ by setting $A(v,w) = v^T A w$. Then the singular value decomposition says that $A = \sum s_i (u_i \otimes v_i)$. Thus, viewing A as a bilinear function, for any vectors $x \in \mathbf{R}^m$ and y in $\mathbf{R}^n$, $A(x,y) = \sum s_i (u_i \cdot x)(v_i \cdot y)$. We want to try and generalize this result to n tensors in $\mathbf{R}^{k_1} \otimes \cdots \otimes \mathbf{R}^{k_n}$.

One can reduce certain quantities over tensors to quantities defined with respect to linear maps. For instance, for each index $i_0 \in [n]$, we can define a linear map

$$A_i : \bigotimes_{i' \neq i} \mathbf{R}^{k_{i'}} \to \mathbf{R}^{k_i}$$

by letting $A_{i_0}(v_1, \dots, \widehat{v_i}, \dots, v_n)$ to be the unique vector $w \in \mathbf{R}^{k_i}$ such that for any vector $v_i \in \mathbf{R}^{k_i}$,

$$v_i \cdot w = A(v_1, \dots, v_n).$$

The matrix A_i is known as the *i'th unfolding of* A. We define the i rank of A to be $\text{rank}_i(A) = \text{rank}(A_i)$. This generalizes the column and row rank of matrices. Unlike in the case of matrices, these quantities can differ for different indices i. One can also obtain the i rank

The natural definition of the rank of a tensor in the study of the higher order singular value decomposition is defined in analogy with the fact that a rank r matrix can be written as the sum of r rank one matrices. We say a tensor is rank one if it can be written in the form $v_1 \otimes \cdots \otimes v_n$ for some vectors $v_i \in \mathbf{R}^{k_i}$. The *rank* of a tensor A, denoted $\text{rank}(A)$, is the minimal number of rank one tensors which can be written in a linear combination to form A. We have $\text{rank}(A_i) \leq \text{rank}(A)$ for each i, but there may be a gap in these ranks.

To obtain an SVD decomposition, fix an index i. Then we can perform a singular value decomposition on A_i, obtaining orthonormal basis

6.4 Decoupling Concentration Bounds

We require some methods in non-asymptotic probability theory in order to understand the concentration structure of the random quantities we study. Most novel of these is relying on a probabilistic decoupling bound of Bourgain.

In this section, we fix a single underlying probability space, assumed large enough for us to produce any additional random quantities used in this section. Given a scalar valued random variable X, we define its *Orlicz norm* as

$$\|X\|_{\psi_2} = \inf\left\{\lambda \geq 0 : \mathbf{E}\left(\exp\left(|X|^2/\lambda^2\right)\right) \leq 2\right\}.$$

The family of random variables with finite Orlicz norm forms a Banach space, known as the space of *subgaussian random variables*.

Theorem 32. *Let X be a sub-gaussian random variable.*

- *For all $t \geq 0$,*
$$\mathbf{P}\left(|X| \geq t\right) \leq 2\exp\left(\frac{-t^2}{\|X\|_{\psi_2}^2}\right).$$

- *Conversely, if there is a constant $K > 0$ such that for all $t \geq 0$,*
$$\mathbf{P}\left(|X| \geq t\right) \leq 2\exp\left(\frac{-t^2}{K^2}\right),$$
then $\|X\|_{\psi_2} \leq 2K$.

- *For any bounded random variable X, $\|X\|_{\psi_2} \lesssim \|X\|_\infty$.*

- $\|X - \mathbf{E}X\|_{\psi_2} \lesssim \|X\|$.

- *If $X_1, \dots, X_n$ are independent, then*

$$\|X_1 + \cdots + X_n\|_{\psi_2} \lesssim \left(\|X_1\|_{\psi_2}^2 + \cdots + \|X_n\|_{\psi_2}^2 \right)^{1/2}.$$

Proof. See [16]. $\square$

Given a random vector X, taking values in $\mathbf{R}^n$, we say it is *uniformly subgaussian* if there is a constant A such that for any $x \in \mathbf{R}^n$, $\|X \cdot x\|_{\psi_2} \leq A \cdot |x|$. We define

$$\|X\|_{\psi_2} = \sup_{|x|=1} \|X \cdot x\|_{\psi_2},$$

so that $\|X \cdot x\|_{\psi_2} \leq \|X\|_{\psi_2} \cdot |x|$ for any random vector X.

The classic Hanson-Wright inequality gives a concentration bound for the magnitude of a quadratic form $X^T A X$ in terms of the matrix norms of A, assuming X is a random vector with independant, mean zero, subgaussian coordinates.

Theorem 33 (Hanson-Wright)**.** *Let $X = (X_1, \dots, X_k)$ be a random vector with independant, mean zero, subgaussian coordinates, with $\|X_i\|_{\psi_2} \leq K$ for each i, and let A be a diagonal-free $k \times k$ matrix. Then there is a universal constant c such that for every $t \geq 0$,*

$$\mathbf{P}\left(|X^T A X| \geq t\right) \leq 2 \exp\left(-c \cdot \min\left(\frac{t^2}{K^4 \|A\|_F^2}, \frac{t}{K^2 \|A\|} \right) \right).$$

Here $\|A\|$ is the operator norm of A, and $\|A\|_F$ it's Frobenius norm.

In our work, we require a modified version of the Hanson Wright inequality where we replace the independant random variables $X_1, \dots, X_k$ with independant random vectors taking values in $\mathbf{R}^d$, and where A is replaced with a n tensor on $\mathbf{R}^{k \times d}$. To make things more tractable, we assume that $A_I = 0$ for each multi-index $I = ((i_1, j_1), \dots, (i_n, j_n))$ if there is $r \neq r'$ with $i_r = i_{r'}$. If this is satisfied, we say A is a *non-diagonal tensor*.

Theorem 34. *Let $X = (X_1, \dots, X_k)$, where $X_1, \dots, X_k$ are independant, mean zero random vectors taking values in $\mathbf{R}^d$, with $\|X_i\|_{\psi_2} \leq K$ for each i. Let A be a non-diagonal n-tensor in $\mathbf{R}^{k\times d}$. Then for $t \geq 0$,*

$$\mathbf{P}\left(|A(X,\dots,X)| \geq t\right) \leq 2\cdot\exp\left(-c(n)\cdot\min\left(\frac{t^2}{K^4\|A\|_F^2},\dots,\frac{t^{2/n}}{K^{4/n}\|A\|_F^{2/n}}\right)\right).$$

The main technique involved is a statistical decoupling result due to J. Bourgain and L. Tzafriri. We follow the proof of the Hanson-Wright inequality from [16], altering various calculations to incorporate the vector-valued coordinates $X_1, \dots, X_n$, and incorporating more robust calculations of the moments of higher order Gaussian chaos.

Lemma 35 (Decoupling)**.** *If $F : \mathbf{R} \to \mathbf{R}$ is convex, and $X = (X_1, \dots, X_k)$, where $X_1, \dots, X_k$ are independant, mean zero random vectors taking values in $\mathbf{R}^d$. If A is a non-diagonal n-tensor in $\mathbf{R}^{d\times k}$, then*

$$\mathbf{E}(F(A(X,\dots,X))) \leq \mathbf{E}(F(n^n\cdot A(X^1,X^2,\dots,X^n))),$$

where $X^1, \dots, X^n$ are independant copies of X.

Proof. Given a set $I \subset [k]$, let $\pi_I : \mathbf{R}^{k\times d} \to \mathbf{R}^{k\times d}$ be defined by

$$\pi_I(X)_{ij} = \begin{cases} X_{ij} & : \text{if } i \in I, \\ 0 & : \text{if } i \notin I \end{cases}.$$

Thus given $X = (X_1, \dots, X_k)$, the vector $\pi_I(X)$ is obtained by zeroeing out all vectors X_i with $i \notin I$. Consider a random partition of $[k]$ into n classes $I_1, \dots, I_n$, chosen uniformly at random. Thus given any $x \in \mathbf{R}^{d\times k}$, we have

$$\mathbf{E}(A(\pi_{I_1}(x),\dots,\pi_{I_n}(x))) = \sum_{I=((i_1,j_1),\dots,(i_n,j_n))} A_I\cdot x_{i_1 j_1}\dots x_{i_n j_n}\mathbf{P}(i_1 \in I_1,\dots,i_n \in I_n).$$

If $A_I \neq 0$, the indices $\{i_r\}$ are distinct from one another. Since $(I_1, \dots, I_n)$ is selected uniformly at random, this means the positions of the indices i_r are independent of one another. Thus

$$\mathbf{P}(i_1 \in I_1, \ldots, i_n \in I_n) = \mathbf{P}(i_1 \in I_1) \ldots \mathbf{P}(i_n \in I_n) = 1/n^n.$$

Thus we have shown

$$A(x, \ldots, x) = n^n \cdot \mathbf{E}(A(\pi_{I_1}(x), \ldots, \pi_{I_n}(x))).$$

In particular, substituting X for x, we have

$$A(X, \ldots, X) = n^n \cdot \mathbf{E}_I(A(\pi_{I_1}(X), \ldots, \pi_{I_n}(X)))$$

Applying Jensen's inequality, we find

$$\begin{aligned}
\mathbf{E}_X(F(A(X, \ldots, X))) &= \mathbf{E}_X(F(n^n \cdot \mathbf{E}_I(A(\pi_{I_1}(X), \ldots, \pi_{I_n}(X))))) \\
&\leq \mathbf{E}_X \, \mathbf{E}_I \, F(n^n \cdot A(\pi_{I_1}(X), \ldots, \pi_{I_n}(X))) \\
&= \mathbf{E}_I \, \mathbf{E}_X \, F(n^n \cdot A(\pi_{I_1}(X), \ldots, \pi_{I_n}(X))).
\end{aligned}$$

Thus we may select a *deterministic* partition I for which

$$\mathbf{E}(F(A(X, \ldots, X))) \leq \mathbf{E}(F(n^n \cdot A(\pi_{I_1}(X), \ldots, \pi_{I_n}(X)))).$$

Since the classes $I_1, \ldots, I_n$ are disjoint from one another, $(\pi_{I_1}(X), \ldots, \pi_{I_n}(X))$ is identically distributed to $(\pi_{I_1}(X^1), \ldots, \pi_{I_n}(X^n))$, so

$$\mathbf{E}(F(n^n \cdot A(\pi_{I_1}(X), \ldots, \pi_{I_n}(X)))) = \mathbf{E}(F(n^n \cdot A(\pi_{I_1}(X^1), \ldots, \pi_{I_n}(X^n)))).$$

Let $Z = A(X^1, \ldots, X^n) - A(\pi_{I_1}(X^1), \ldots, \pi_{I_n}(X^n))$. If $\mathbf{E}'$ denotes conditioning with respect to $\pi_{I_1}(X^1), \ldots, \pi_{I_n}(X^n)$, then this expectation fixes $A(\pi_{I_1}(X_1), \ldots, \pi_{I_n}(X_n))$, but $\mathbf{E}'(Z) = 0$. Thus we can apply Jensen's inequality again to conclude

$$\begin{aligned}
\mathbf{E}(F(n^n \cdot A(\pi_{I_1}(X^1), \ldots, \pi_{I_n}(X^n)))) &= \mathbf{E}(F(n^n \cdot \mathbf{E}'(A(\pi_{I_1}(X^1), \ldots, \pi_{I_n}(X^n)) + Z))) \\
&= \mathbf{E}(F(n^n \cdot \mathbf{E}'(A(X^1, \ldots, X^n)))) \\
&\leq \mathbf{E}(\mathbf{E}'(F(n^n A(X^1, \ldots, X^n)))) \\
&= \mathbf{E}(F(n^n A(X^1, \ldots, X^n))). \qquad \square
\end{aligned}$$

Remark. *Note this theorem also implies that for each $\lambda \in \mathbf{R}$,*

$$\mathbf{E}(F((\lambda/n^n) \cdot A(X, \ldots, X))) \leq \mathbf{E}(F(\lambda \cdot A(X^1, \ldots, X^n))).$$

Lemma 36 (Comparison)**.** *Let $X^1, \ldots, X^n$ be independant random vectors taking values in $\mathbf{R}^{k \times d}$*

To make things easy for ourselves, we assume that A vanishes along it's diagonal. Reinterpreting the quantities in the Hanson-Wright inequality in this setting gives the result. For each i and j, we define A_{ij} be an $d \times d$ matrix, then we set $X^T A X = \sum X_i^T A_{ij} X_j$,

$$\|A\|_F = \sqrt{\sum \|A_{ij}\|_F^2}, \quad \text{and} \quad \|A\| = \sup\left\{ |\sum x_i^T A_{ij} y_j| : \sum |x_i|^2 = \sum |y_j|^2 = 1 \right\}.$$

These are precisely the definitions of the operator norm and Frobenius norm if we interpret A as a $dn \times dn$ matrix by expanding out the entries of each A_{ij}, and X as a random vector taking values in $\mathbf{R}^{dn}$ by expanding out the entries of each X_i.

Chapter 7

Conclusions

Bibliography

[1] K. Athreya. Large deviation rates for branching processes–i. single type case. *Ann. Appl. Probab.*, 4:779–790, 1994.

[2] J. Balogh, R. Morris, and W. Samotij. Independent sets in hypergraphs. *J. Amer. Math. Soc.*, 28:669–709, 2015.

[3] J. Biggins and N. Bingham. Large deviations in the supercritical branching process. *Adv. in Appl. Probab.*, 25:757–772.

[4] K. Dickman. On the frequency of numbers containing prime factors of a certain relative magnitude.

[5] K. Falconer. *Fractal Geometry: Mathematical Foundations and Applications*. Wiley, 2003. → page 40

[6] R. Fraser and M. Pramanik. Large sets avoiding patterns. *Anal. PDE*, 11: 1083–1111, 2018. → pages 30, 31, 43

[7] T. Harris. *The Theory of Branching Processes*. Springer-Verlag, 1963.

[8] N. H. Katz and T. Tao. Some connections between Falconer's distance set conjecture, and sets of Furstenburg type. *New York J. Math.*, 7:149–187, 2001.

[9] T. Keleti. A 1-dimensional subset of the reals that intersects each of its translates in at most a single point. *Real Anal. Exchange*, 24:843–845, 1999. → page 31

[10] P. Maga. Full dimensional sets without given patterns. *Real Anal. Exchange*, 36:79–90, 2010. → page 1

[11] A. Máthé. Sets of large dimension not containing polynomial configurations. *Adv. Math.*, 316:691–709, 2017. → page 30

[12] P. Mattila. *Fourier analysis and Hausdorff dimension*. Cambridge University Press, 2015. → page 1

[13] I. Z. Ruzsa. Difference sets without squares.

[14] B. Sudakov, E. Szemerédi, and V. Vu. On a question of erdös and moser.

[15] T. Tao. 245 C, Notes 5: Hausdorff dimension, 2009. URL https://terrytao.wordpress.com/2009/05/19/245c-notes-5-hausdorff-dimension-optional/.

[16] Vershynin. → pages 64, 65