Lemma 1. *For each $k \geq 0$, let $\mathcal{E}_k$ be a finite collection of non-empty, bounded Borel subsets of $\mathbf{R}^d$, such that*

- Separation*: For each k, the collection*

$$\left\{ \overline{A} : A \in \mathcal{E}_k \right\} isdisjoint.$$

- Nesting*: For each $A \in \mathcal{E}_{k+1}$, there is a unique $A^* \in \mathcal{E}_k$ such that $A \subset A^*$, and for any $A^* \in \mathcal{E}_k$, there is at least one $A \in \mathcal{E}_{k+1}$ with $A \subset A^*$.*

Let $E_k = \bigcup \mathcal{E}_k$, and let $E_\infty = \bigcap E_k$. Recursively define a function $\mu : \bigcup_{k \geq 0} \mathcal{E}_k \to [0, \infty)$ by setting $\mu(A) = 1$ if $A \in \mathcal{E}_0$, and for each $A \in \mathcal{E}_{k+1}$, setting

$$\mu(A) = \frac{\mu(A^*)}{\#\{A' \in \mathcal{E}_{k+1} : A' \subset A^*\}}.$$

Then μ extends to a finite Borel measure on $\mathbf{R}^d$ supported on E_∞.

Proof. Let $\mathcal{E} = \bigcup_{k \geq 0} \mathcal{E}_k \cup 2^{\mathbf{R}^d - E_k}$. Then $\mathcal{E}$ is a *semi-ring* of sets:

- *$\mathcal{E}$ is closed under intersections*: Let $A, B \in \mathcal{E}$. Then without loss of generality, swapping A and B if necessary, there exists $i \leq j$ such that $A \in \mathcal{E}_i$ or $A \subset \mathbf{R}^d - E_i$, and $B \in \mathcal{E}_j$ or $B \subset \mathbf{R}^d - E_j$.

  If $A \in \mathcal{E}_i$, and $B \in \mathcal{E}_j$, then either $A \cap B = \emptyset$, or $B \subset A$, and $A \cap B = B$. In either case, $A \cap B \in \mathcal{E}$.

  If $A \subset \mathbf{R}^d - E_i$, then $A \cap B \subset A \subset \mathbf{R}^d - E_i$, so $A \cap B \in \mathcal{E}$. Similarily, if $B \subset \mathbf{R}^d - E_j$, then $A \cap B \subset \mathbf{R}^d - E_j$, so $A \cap B \in \mathcal{E}$.

  This addresses all cases.

- *$\mathcal{E}$ is closed under relative complements*: Let $A, B \in \mathcal{E}$. Then there exists i, j such that $A \in \mathcal{E}_i$ or $A \subset \mathbf{R}^d - E_i$, and $B \in \mathcal{E}_j$ or $B \subset \mathbf{R}^d - E_j$.

  Suppose $A \in \mathcal{E}_i$ and $B \in \mathcal{E}_j$. If $i \geq j$, then either $A \cap B = \emptyset$, or $A \subset B$. In either case, $A - B = \emptyset \in \mathcal{E}$. If $i \leq j$, then either $A \cap B = \emptyset$, or $B \subset A$. In the latter case,

$$A - B = \bigcup \{B_i : B_i \in \mathcal{E}_j, B_i \neq B, B_i \subset A\} \cup \{A - E_j\}.$$

  Note that this union is disjoint, and each set in the union is an element of $\mathcal{E}$.

  Suppose $A \in \mathcal{E}_i$, and $B \subset \mathbf{R}^d - E_j$. If $i \geq j$, then $A \subset E_i \subset E_j$, so $A - B = A$. If $i \leq j$, then

$$A - B = (A \cap E_j) \cup (A \cap (\mathbf{R}^d - B)).$$

  The former set can be written as $\bigcup_{A' \in \mathcal{E}_j} : A' \subset A$, and the latter set is a subset of $\mathbf{R}^d - E_j$. This gives $A - B$ as a disjoint union of elements of $\mathcal{E}$.

  If $A \subset \mathbf{R}^d - E_i$, then $A - B \subset \mathbf{R}^d - E_i$, so $A - B \in \mathcal{E}$.

The function μ extends to a function on $\mathcal{E}$ by setting $\mu(E) = 0$ if $E \subset \mathbf{R}^d - E_k$ for some k. This does not conflict with our previous definition, since if $A \in \mathcal{E}_i$, then $A \cap E_j \neq \emptyset$ for all $j \geq 0$. We claim that μ is then a *pre-measure* on the semi-ring:

- It is certainly true that $\mu(\emptyset) = 0$.
- Let $A \in \mathcal{E}$, and suppose $A = \bigcup_i A_i$, where $\{A_i\}$ is an at most countable subcollection of disjoint sets from $\mathcal{E}$. We fix k such that $A \in \mathcal{E}_k$, or $A \subset \mathbf{R}^d - E_k$, and for each i, fix k_i such that $A_i \in \mathcal{E}_{k_i}$, or $A \subset \mathbf{R}^d - E_{k_i}$.

  If $A \in \mathbf{R}^d - E_k$ for some k, then $A_i \in \mathbf{R}^d - E_k$ for each i, and thus $\mu(A_i) = 0$ for all i. Since $\mu(A) = 0$, this means $\mu(A) = \sum \mu(A_i)$.

  Suppose $A \in \mathcal{E}_k$. If there are only *finitely many* sets A_j with $A_j \in \mathcal{E}_k$, then the claim is obvious. Thus it suffices to show this is the only case. Let $\{A_j\}$ be the family of sets with $A_j \in \mathcal{E}_{k_j}$. Then we may find a disjoint family of open sets $\{U_j\}$ with $\overline{A_j} \subset U_j$ for each j. Define

  $$A_\infty = \bigcup \overline{A_j}$$

  Then $\{U_j\}$ is a cover of A_∞. The set A_∞ is compact, since it is bounded and closed by the separability criterion. Since the sets U_j are disjoint, and $U_j \cap A_\infty \neq \emptyset$ for any j, this implies the set of $\{U_j\}$ is finite, and thus the set $\{A_j\}$ is finite.

The Caratheodory extension theorem then guarantees that μ extends to a measure on the σ algebra generated by $\mathcal{E}$. Since

$$\lim_{k \to \infty} \left(\max_{A \in \mathcal{E}_k} \operatorname{diam}(A) \right) = 0,$$

this σ algebra contains all Borel sets. □

For the next lemma, let $\mathcal{Q}^d$ denote the collection of all dyadic cubes in $\mathbf{R}^d$, let $\mathcal{Q}^d_k$ denote the dyadic cubes of length $1/2^k$, and let $Q^* \in \mathcal{Q}^d_k$ denote the parent cube of any $Q \in \mathcal{Q}^d_{k+1}$.

Lemma 2 (Mass Distribution Principle)**.** *Let $f : \mathcal{Q}^d \to [0, \infty)$ be a function such that for any $Q_0 \in \mathcal{Q}^d$,*

$$\sum_{Q^* = Q_0} f(Q) = f(Q_0), \tag{1}$$

Then there exists a regular Borel measure μ supported on

$$\bigcap_{k=1}^{\infty} \left[\bigcup \{Q \in \mathcal{Q}^d_k : f(Q) > 0\} \right]$$

such that for each $Q \in \mathcal{Q}^d$,

$$\mu(Q) \geq f(Q), \tag{2}$$

and for any set E *and* $k \geq 0$,

$$\mu(E) \leq \sum f(Q), \tag{3}$$

where Q *ranges over all cubes in* $\mathcal{Q}_k^d(E(l_k))$.

Proof. For each i, define a regular Borel measure μ_i such that for each $f \in C_c(\mathbf{R}^d)$,

$$\int f d\mu_i = \sum_{Q \in \mathcal{Q}_i^d} \frac{f(Q)}{|Q|} \int_Q f\ dx$$

Then, for each $j \leq i$, $\mu_i(Q) = f(Q)$.

We claim that $\{\mu_i\}$ is a Cauchy sequence in the space of all regular Borel measures on $\mathbf{R}^d$, viewing the space as a locally convex space under the weak topology. Fix $\varphi \in C_c(\mathbf{R}^d)$, and choose some N such that φ is supported on $[-N, N]^d$. Set $Q_0 = [-N, N]^d$. Since φ is compactly supported, φ is *uniformly continuous*, so for each $\varepsilon > 0$, if i is suitably large, there is a sequence of values $\{a_Q : Q \in \mathcal{Q}_i^d(Q_0)\}$ such that if $x \in Q$, $|\varphi(x) - a_Q| \leq \varepsilon$. But this means that

$$\sum (a_Q - \varepsilon)\mathbf{I}_Q \leq \varphi \leq \sum (a_Q + \varepsilon)\mathbf{I}_Q,$$

where Q ranges over cubes in $\mathcal{Q}_i^d(Q_0)$. Thus for any $j \geq i$,

$$\int \varphi\ d\mu_j \leq \sum (a_Q + \varepsilon)\mu_j(Q) = \sum (a_Q + \varepsilon) f(Q)$$

and

$$\int \varphi\ d\mu_j \geq \sum (a_Q - \varepsilon)\mu_j(Q) = \sum (a_Q - \varepsilon) f(Q).$$

In particular, if $j, j' \geq i$,

$$\left| \int \varphi d\mu_j - \int \varphi d\mu_{j'} \right| \leq 2\varepsilon \sum_{Q \in \mathcal{Q}_i(Q_0)} f(Q) = 2\varepsilon \sum_{Q \in \mathcal{Q}_0(Q_0)} f(Q).$$

Since ε and φ were arbitrary, this shows $\{\mu_i\}$ is Cauchy.

Since the space of all regular Borel measures on $\mathbf{R}^d$ is *complete*, the last paragraph implies there exists a regular Borel measure μ such that $\mu_i \to \mu$ weakly. For any $Q \in \mathcal{Q}^d$, since Q is a closed set,

$$\mu(Q) \geq \limsup_{i \to \infty} \mu_i(Q) = f(Q).$$

Conversely, if E is an arbitrary set, then $E \subset E(l_i)^\circ$ for any i, so

$$\mu(E) \leq \liminf_{i \to \infty} \mu_i(E(l_i)^\circ) \leq \liminf_{i \to \infty} \mu_i(E(l_i)) = \sum f(Q),$$

where Q ranges over the cubes in $\mathcal{Q}_i^d(E(l_k))$. $\square$

Remark. *Common approaches to the mass distribution principle consider*

$$\mu(E) = \inf \left\{ \sum_{i=1}^{\infty} f(Q_i) : Q_i \in \mathcal{Q}^d, E \subset \bigcup Q_i \right\}.$$

If $d(E, F) > 0$*, then* (1) *implies that* $\mu(E + F) = \mu(E) + \mu(F)$*, so* μ *is a metric exterior measure, and so all Borel sets are measurable with respect to* μ*. However, given the conditions of this theorem, it is* not *necessarily true that* μ *is even a non-zero measure, let alone that it reflects the values of the functions* f*.*