

# **Cartesian Products Avoiding Patterns**

by

Jacob Denson

BSc. Computing Science, University of Alberta, 2017

A THESIS SUBMITTED IN PARTIAL FULFILLMENT  
OF THE REQUIREMENTS FOR THE DEGREE OF

**Master of Science**

in

THE FACULTY OF SCIENCE

(Mathematics)

The University of British Columbia

(Vancouver)

April 2019

© Jacob Denson, 2019



The following individuals certify that they have read, and recommend to the Faculty of Graduate and Postdoctoral Studies for acceptance, the thesis entitled:

**Cartesian Products Avoiding Patterns**

submitted by **Jacob Denson** in partial fulfillment of the requirements for the degree of **Master of Science** in **Mathematics**.

**Examining Committee:**

Malabika Pramanik, Mathematics

*Supervisor*

Joshua Zahl, Mathematics

*Supervisor*



# Abstract

The pattern avoidance problem seeks to construct a set  $X \subset \mathbf{R}^d$  with large dimension that avoids a prescribed pattern such as three term arithmetic progressions, or more general patterns such as avoiding points  $x_1, \dots, x_n$  such that  $f(x_1, \dots, x_n) = 0$  for a given function  $f$ . Previous work on the subject has considered patterns described by polynomials, or functions  $f$  satisfying certain regularity conditions. We provide an exposition of this work, as well as considering the case of ‘rough patterns’. There are several problems that fit into the framework of rough pattern avoidance. As a first application, if  $Y \subset [0, 1]$  is a set with Minkowski dimension  $\alpha$ , we construct a set  $X \subset [0, 1]$  with Hausdorff dimension  $1 - \alpha$  so that  $X + X$  is disjoint from  $Y$ . As a second application, given a set  $Y$  with dimension close to one, we can construct a subset  $X \subset Y$  of dimension  $1/2$  that avoids isosceles triangles.



# Lay Summary

Imagine looking at a patch of carpet with a microscope. Zooming in, we see the carpet is really a collection of twines tied together. On closer inspection, those twines break off into smaller twines. The carpet is rough at all scales. If you want to model the carpet mathematically such that all small features are accounted for, normal shapes like polyhedra wouldn't suffice. Instead, you need a *fractal*: shapes with structure at small scales. Such models are often useful in applied mathematics and small-scale physics.

It is often easy to construct polyhedra with geometric properties at a single scale, but non-trivial to describe how to construct fractals with properties at many scales, such as not containing the vertices of an isosceles triangles, or not containing any three collinear points. In this thesis, we provide an exposition of previous work in this topic, as well as new random fractal avoidance techniques.





# Preface

At UBC, a preface may be required. Be sure to check the GPS guidelines as they may have specific content to be included.



# Contents

<b>Abstract</b>	<b>iii</b>
<b>Lay Summary</b>	<b>v</b>
<b>Preface</b>	<b>vii</b>
<b>Acknowledgments</b>	<b>xi</b>
<b>1 Introduction</b>	<b>1</b>
<b>2 Background</b>	<b>3</b>
2.1 Configuration Avoidance . . . . .	3
2.2 Minkowski Dimension . . . . .	6
2.3 Hausdorff Dimension . . . . .	6
2.4 Dyadic Scales . . . . .	9
2.5 Frostman Measures . . . . .	11
2.6 Dyadic Fractional Dimension . . . . .	16
2.7 Beyond Hyperdyadics . . . . .	21
<b>3 Related Work</b>	<b>23</b>
3.1 Keleti: A Translate Avoiding Set . . . . .	23
3.2 Generalizing Keleti’s Argument . . . . .	25
3.3 Fraser/Pramanik: Smooth Configurations . . . . .	26
<b>4 Avoiding Rough Sets</b>	<b>33</b>
4.1 Frequently Used Notation and Terminology . . . . .	35
4.2 Avoidance at Discrete Scales . . . . .	37
4.3 Fractal Discretization . . . . .	41
4.4 Dimension Bounds . . . . .	44

4.5	Applications . . . . .	46
4.5.1	Sum-sets avoiding specified sets . . . . .	46
4.5.2	Subsets of Lipschitz curves avoiding isosceles triangles . .	47
<b>5</b>	<b>Conclusions</b>	<b>51</b>
	<b>Bibliography</b>	<b>53</b>

# Acknowledgments

Thank those people who helped you.

Don't forget your parents or loved ones.

You may wish to acknowledge your funding sources.



# Chapter 1

## Introduction

In this thesis, we study a simple family of questions:

*How large can Euclidean sets be not containing geometric patterns?.*

For instance, what is the maximal size of a set  $X \subset \mathbf{R}^d$  that contains no three collinear points? Or what is the maximal size of a set not containing the vertices of an isosceles triangle? Aside from pure geometric interest, these problems provide useful settings to test methods of ergodic theory, additive combinatorics and harmonic analysis.

Sets which avoid the patterns we consider are highly irregular, in many ways behaving like a fractal set. Our understanding of them requires techniques from geometric measure theory. In particular, we use the *fractal dimension* to measure the size of a set when determining how large a pattern avoiding set can be.

At present, many fundamental questions about geometric structure and its relation to fractal dimension remain unsolved. One might expect sets with sufficiently large fractal dimension contain patterns. For example, Theorem 6.8 of [12] shows that any set  $X \subset \mathbf{R}^d$  with Hausdorff dimension exceeding one must contain three collinear points. On the other hand, Theorem 2.3 of [10] constructs a set  $X \subset \mathbf{R}^d$  with full Hausdorff dimension such that no four points in  $X$  form the vertices of a parallelogram. Thus the problem depends on the particular aspects of the configurations involved. For most geometric configurations, it remains unknown at what threshold patterns are guaranteed, or whether such a threshold exists at all. So finding sets with large Hausdorff dimension avoiding patterns is an important topic to determine for which classes of patterns our intuition remains true.

Our goal in this thesis is to derive new methods for constructing sets avoiding patterns. In particular, we expand on a number of general *pattern dissection meth-*

*ods* which have proven useful in the area, originally developed by Keleti but also studied notably by Mathé, Pramanik, and Fraser. The main contribution we give to pattern dissection methods is to use random dissections to avoid patterns which lack structural information. Thus we can expand the utility of interval dissection methods from regular families of patterns to a set of *fractal avoidance problems*, that previous methods were completely unavailable to address. Such problems include finding large  $X$  such that the angles formed by any three distinct points of  $X$  avoid a specified set  $Y$  of angles, where  $Y$  has a fixed fractal dimension.



# Chapter 2

## Background

### 2.1 Configuration Avoidance

We consider an ambient set  $\mathbf{A}$ . It's  $n$ -point configuration space is

$$\mathcal{C}^n(\mathbf{A}) = \{(a_1, \dots, a_n) \in \mathbf{A}^n : a_i \neq a_j \text{ if } i \neq j\}.$$

An  $n$  point configuration, or  $n$  point pattern, is a subset of  $\mathcal{C}^n(\mathbf{A})$ . More generally, we define the general configuration space of  $\mathbf{A}$  as  $\mathcal{C}(\mathbf{A}) = \bigcup_{n=1}^{\infty} \mathcal{C}^n(\mathbf{A})$ , and a pattern, or configuration, on  $\mathbf{A}$  as a subset of  $\mathcal{C}(\mathbf{A})$ .

Our main focus in this thesis is the *pattern avoidance problem*. For a fixed configuration  $\mathcal{C}$  on  $\mathbf{A}$ , we say a set  $X \subset \mathbf{A}$  *avoids*  $\mathcal{C}$  if  $\mathcal{C}(X)$  is disjoint from  $\mathcal{C}$ . The pattern avoidance problem asks to find sets  $X$  of maximal size avoiding a fixed configuration  $\mathcal{C}$ . Often  $\mathcal{C}$  describes algebraic or geometric structure, and the pattern avoidance problem asks to find the maximal size of a set before it is guaranteed to have such structure.

**Example** (Isocoles Triangle Configuration). *Let*

$$\mathcal{C} = \{(x_1, x_2, x_3) \in \mathcal{C}^3(\mathbf{R}^2) : |x_1 - x_2| = |x_1 - x_3|\}.$$

*Then  $\mathcal{C}$  is a 3-point configuration, and a set  $X \subset \mathbf{R}^2$  avoids  $\mathcal{C}$  if and only if it does not contain all three vertices of an isocoles triangle.*

**Example** (Linear Independence Configuration). *Let  $V$  be a vector space over a*

field  $K$ . We set

$$\mathcal{C} = \bigcup_{n=1}^{\infty} \{(x_1, \dots, x_n) \in \mathcal{C}^n(V) : \text{for any } a_1, \dots, a_n \in K, a_1x_1 + \dots + a_nx_n = 0\}.$$

A subset  $X \subset V$  avoids  $\mathcal{C}$  if and only if  $X$  is a linearly independent subset of  $V$ . Note that if  $V$  is infinite dimensional, then  $\mathcal{C}$  cannot be replaced by a  $n$  point configuration for any  $n$ ; arbitrarily large tuples must be considered. We will be interested in the case where  $K = \mathbf{Q}$ , and  $V = \mathbf{R}$ . Our interest here is to find analytically large linearly independent subsets of  $V$ .

Even though our problem formulation assumes configurations are formed by distinct sets of points, one can still formulate avoidance problems involving repeated points by a simple trick.

**Example** (Sum Set Configuration). Let  $G$  be an abelian group, and fix  $Y \subset G$ . Set

$$\mathcal{C}^1 = \{g \in G : g + g \in Y\} \quad \text{and} \quad \mathcal{C}^2 = \{(g_1, g_2) \in G^2 : g_1 + g_2 \in Y\}.$$

Then set  $\mathcal{C} = \mathcal{C}^1 \cup \mathcal{C}^2$ . A set  $X \subset G$  avoids  $\mathcal{C}$  if and only if  $(X + X) \cap Y = \emptyset$ .

Depending on the structure of the ambient space  $\mathbf{A}$  and the configuration  $\mathcal{C}$ , there are various ways of measuring the size of sets  $X \subset \mathbf{A}$  for the purpose of the pattern avoidance problem:

- If  $\mathbf{A}$  is finite, we wish to find a set  $X$  with large cardinality.
- If  $\{\mathbf{A}_n\}$  is an increasing family of finite sets with  $\mathbf{A} = \bigcup_n \mathbf{A}_n$ , we wish to find a set  $X$  such that  $X \cap \mathbf{A}_n$  has large cardinality asymptotically as  $n \rightarrow \infty$ .
- If  $\mathbf{A} = \mathbf{R}^d$ , but  $\mathcal{C}$  is a discrete configuration, then a satisfactory goal is to find a set  $X$  with large Lebesgue measure avoiding  $\mathcal{C}$ .

In this thesis, inspired by results in these three settings, we establish methods for avoiding non-discrete configurations  $\mathcal{C}$  in  $\mathbf{R}^d$ . Here, Lebesgue measure completely fails to measure the size of pattern avoiding solutions, as the next theorem shows, under the often true assumption that  $\mathcal{C}$  is *translation invariant*, i.e. that if  $(a_1, \dots, a_n) \in \mathcal{C}$  and  $b \in \mathbf{R}^d$ ,  $(a_1 + b, \dots, a_n + b) \in \mathcal{C}$ .

**Theorem 1.** Let  $\mathcal{C}$  be a  $n$ -point configuration on  $\mathbf{R}^d$ . Suppose

(A)  $\mathcal{C}$  is translation invariant.

(B) For any  $\varepsilon > 0$ , there is  $(a_1, \dots, a_n) \in \mathcal{C}$  with  $\text{diam}\{a_1, \dots, a_n\} \leq \varepsilon$ .

Then no set with positive Lebesgue measure avoids  $\mathcal{C}$ .

*Proof.* Let  $X \subset \mathbf{R}^d$  have positive Lebesgue measure. The Lebesgue density theorem shows that there exists a point  $x \in X$  such that

$$\lim_{l(Q) \rightarrow 0} \frac{|X \cap Q|}{|Q|} = 1, \quad (2.1)$$

where  $Q$  ranges over all cubes in  $\mathbf{R}^d$  with  $x \in Q$ , and  $l(Q)$  denotes the sidelength of  $Q$ . Fix  $\varepsilon > 0$ , to be specified later, and choose  $r$  small enough that  $|X \cap Q| \geq (1 - \varepsilon)|Q|$  for any cube  $Q$  with  $x \in Q$  and  $l(Q) \leq r$ . Now let  $Q_0$  denote a cube centered at  $x$  with  $l(Q_0) \leq r$ . Applying Property (B), we find  $C = (a_1, \dots, a_n) \in \mathcal{C}$  such that

$$\text{diam}\{a_1, \dots, a_n\} \leq l(Q_0)/2. \quad (2.2)$$

For each  $p \in Q_0$ , let  $C(p) = (a_1(p), \dots, a_n(p))$ , where  $a_i(p) = p + (a_i - a_1)$ . Property (A) implies  $C(p) \in \mathcal{C}$  for each  $p \in \mathbf{R}^d$ . A union bound shows

$$|\{p \in Q_0 : C(p) \notin \mathcal{C}(X)\}| \leq \sum_{i=1}^d |\{p \in Q_0 : a_i(p) \notin X\}|. \quad (2.3)$$

We have  $a_i(p) \notin X$  precisely when  $p + (a_i - a_1) \notin X$ , so

$$|\{p \in Q_0 : a_i(p) \notin X\}| = |(Q_0 + (a_i - a_1)) \cap X^c|. \quad (2.4)$$

Note  $Q_0 + (a_i - a_1)$  is a cube with the same sidelength as  $Q_0$ . Equation (2.2) implies  $|a_i - a_1| \leq l(Q_0)/2$ , so  $x \in Q_0 + (a_i - a_1)$ . Thus (2.1) shows

$$|Q_0 + (a_i - a_1) \cap X^c| \leq \varepsilon |Q_0|. \quad (2.5)$$

Combining (2.3), (2.4), and (2.5), we find

$$|\{p \in Q_0 : C(p) \notin \mathcal{C}(X)\}| \leq \varepsilon d |Q_0|.$$

Provided  $\varepsilon d < 1$ , this means there is  $p \in Q_0$  with  $C(p) \in \mathcal{C}(X)$ . But this means  $X$  does not avoid  $\mathcal{C}$ .  $\square$

Since no set of positive Lebesgue measure can avoid non-discrete configurations, we cannot use the Lebesgue measure to quantify the size of pattern avoiding

sets for many non-discrete configurations. Fortunately, there is a quantity which can distinguish between the size of sets of measure zero. This is the *fractional dimension* of a set.

There are many variants of fractional dimension. Here we choose to use the Minkowski dimension, the Hausdorff dimension, and the Fourier dimension. They assign the same dimension to any smooth manifold with non-vanishing curvature, but vary over more singular sets. One major difference is that Minkowski dimension measures relative density at a single scale, whereas Hausdorff dimension measures relative density at countably many scales. The Fourier dimension is a refinement of the Hausdorff dimension which gives greater control on the set when viewed ‘in the frequency domain’, and therefore gives additional structural information on sets.

## 2.2 Minkowski Dimension

We begin by discussing the Minkowski dimension, which is the simplest to define. Given  $l > 0$ , and a bounded set  $E \subset \mathbf{R}^d$ , we let  $N(l, E)$  denote the *covering number* of  $E$ , i.e. the minimum number of sidelength  $l$  cubes required to cover  $E$ . We define the *lower* and *upper* Minkowski dimension as

$$\dim_{\underline{\mathbf{M}}}(E) = \liminf_{l \rightarrow 0} \frac{\log(N(l, E))}{\log(1/l)} \quad \text{and} \quad \dim_{\overline{\mathbf{M}}}(E) = \limsup_{l \rightarrow 0} \frac{\log(N(l, E))}{\log(1/l)}.$$

If  $\dim_{\underline{\mathbf{M}}}(E) = \dim_{\overline{\mathbf{M}}}(E)$ , then we refer to this common quantity as the *Minkowski dimension* of  $E$ , denoted  $\dim_{\mathbf{M}}(E)$ . Thus  $\dim_{\underline{\mathbf{M}}}(E) < s$  if there exists a sequence of lengths  $\{l_k\}$  converging to zero with  $N(l_k, E) \leq (1/l_k)^s$ , and  $\dim_{\overline{\mathbf{M}}}(E) < s$  if  $N(l, E) \leq (1/l)^s$  for *all* sufficiently small lengths  $l$ .

## 2.3 Hausdorff Dimension

For  $E \subset \mathbf{R}^d$  and  $\delta > 0$ , we define the *Hausdorff content*

$$H_{\delta}^s(E) = \inf \left\{ \sum_{k=1}^{\infty} l(Q_k)^s : E \subset \bigcup_{k=1}^{\infty} Q_k, l(Q_k) \leq \delta \right\}.$$

The  $s$ -dimensional Hausdorff measure of  $E$  is

$$H^s(E) = \lim_{\delta \rightarrow 0} H_\delta^s(E) = \sup_{\delta > 0} H_\delta^s(E).$$

It is easy to see  $H^s$  is an exterior measure on  $\mathbf{R}^d$ , and  $H^s(E \cup F) = H^s(E) + H^s(F)$  if the Hausdorff distance  $d(E, F)$  between  $E$  and  $F$  is positive. So  $H^s$  is actually a metric exterior measure, and the Caratheodory extension theorem shows all Borel sets are measurable with respect to  $H^s$ . Sometimes, it is convenient to use the exterior measure

$$H_\infty^s(E) = \inf \left\{ \sum_{k=1}^{\infty} l(Q_k)^s : E \subset \bigcup_{k=1}^{\infty} Q_k \right\}.$$

The majority of Borel sets which occur in practice fail to be measurable with respect to  $H_\infty^s$ , but the exterior measure  $H_\infty^s$  has the useful property that  $H_\infty^s(E) = 0$  if and only if  $H^s(E) = 0$ .

**Lemma 2.** Consider  $t < s$ , and  $E \subset \mathbf{R}^d$ .

(i) If  $H^t(E) < \infty$ , then  $H^s(E) = 0$ .

(ii) If  $H^s(E) \neq 0$ , then  $H^t(E) = \infty$ .

*Proof.* Suppose that  $H^t(E) = A < \infty$ . Then for any  $\delta > 0$ , there is a cover of  $E$  by a collection of intervals  $\{Q_k\}$ , such that  $l(Q_k) \leq \delta$  for each  $k$ , and

$$\sum l(Q_k)^t \leq A < \infty.$$

But then

$$H_\delta^s(E) \leq \sum l(Q_k)^s \leq \sum l(Q_k)^{s-t} l(Q_k)^t \leq \delta^{s-t} A.$$

As  $\delta \rightarrow 0$ , we conclude  $H^s(E) = 0$ , proving (i). And (ii) is just the contrapositive of (i), and therefore immediately follows.  $\square$

**Corollary 3.** If  $s > d$ ,  $H^s = 0$ .

*Proof.* The measure  $H^d$  is just the Lebesgue measure on  $\mathbf{R}^d$ , so

$$H^d[-N, N]^d = (2N)^d.$$

If  $s > d$ , Lemma 2 shows  $H^s[-N, N]^d = 0$ . By countable additivity, taking  $N \rightarrow \infty$  shows  $H^s(\mathbf{R}^d) = 0$ . Since  $H^s$  is a positive measure,  $H^s(E) = 0$  for all  $E$ .  $\square$

Given any Borel set  $E$ , Corollary 3, combined with Lemma 2, implies there is a unique value  $s_0 \in [0, d]$  such that  $H^s(E) = 0$  for  $s > s_0$ , and  $H^s(E) = \infty$  for  $0 \leq s < s_0$ . We refer to  $s_0$  as the *Hausdorff dimension* of  $E$ , denoted  $\dim_{\mathbf{H}}(E)$ .

**Theorem 4.** *For any bounded set  $E$ ,  $\dim_{\mathbf{H}}(E) \leq \dim_{\mathbf{M}}(E) \leq \dim_{\overline{\mathbf{M}}}(E)$ .*

*Proof.* Given  $l > 0$ , we have a simple bound  $H_l^s(E) \leq N(l, E) \cdot l^s$ . If  $\dim_{\mathbf{M}}(E) < s$ , then there exists a sequence  $\{l_k\}$  with  $l_k \rightarrow 0$ , and  $N(l_k, E) \leq (1/l_k)^s$ . Thus we conclude that

$$H^s(E) = \lim_{k \rightarrow \infty} H_{l_k}^s(E) \leq \lim_{k \rightarrow \infty} N(l_k, E) \cdot l_k^s \leq 1,$$

Thus  $\dim_{\mathbf{H}}(E) \leq s$ . Taking infima over all  $s$ , we find  $\dim_{\mathbf{H}}(E) \leq \dim_{\mathbf{M}}(E)$ .  $\square$

**Remark.** *If  $\dim_{\mathbf{H}}(E) < d$ , then  $|E| = H^d(E) = 0$ . Thus any set with fractional dimension less than  $d$  must have measure zero. This means we can use the dimension as a way of distinguishing between sets of measure zero, which is precisely what we need to study configuration avoidance problem for non-discrete configurations.*

The fact that Hausdorff dimension is defined over simultaneously multiple scales makes it more stable under analytical operations. In particular,

$$\dim_{\mathbf{H}} \left\{ \bigcup E_k \right\} = \sup \{ \dim_{\mathbf{H}}(E_k) \}.$$

This need not be true for the Minkowski dimension; a single point has Minkowski dimension zero, but the rational numbers, which are a countable union of points, have Minkowski dimension one. An easy way to make Minkowski dimension countably stable is to define the *modified Minkowski dimensions*

$$\dim_{\mathbf{MM}}(E) = \inf \left\{ s : E \subset \bigcup_{i=1}^{\infty} E_i, \dim_{\mathbf{M}}(E_i) \leq s \right\}$$

and

$$\dim_{\overline{\mathbf{M}}\mathbf{M}}(E) = \inf \left\{ s : E \subset \bigcup_{i=1}^{\infty} E_i, \dim_{\overline{\mathbf{M}}}(E_i) \leq s \right\}.$$

This notion of dimension, in a disguised form, appears in Chapter BLAH.

## 2.4 Dyadic Scales

It is now useful to introduce the dyadic notation we utilize throughout this thesis. At the cost of losing topological perspective about  $\mathbf{R}^d$ , applying dyadic techniques often allows us to elegantly discretize certain problems in Euclidean space.

Fix an integer  $N$ . The classic family of dyadic cubes with *branching factor*  $N$  is given by setting, for each integer  $k \geq 0$ ,

$$\mathcal{D}_k^d = \left\{ \prod_{i=1}^d \left[ \frac{n_i}{N^k}, \frac{n_i+1}{N^k} \right] : n \in \mathbf{Z}^d \right\},$$

and then setting  $\mathcal{D} = \bigcup_{k \geq 0} \mathcal{D}_k^d$ . Elements of  $\mathcal{D}$  are known as *dyadic cubes*, and elements of  $\mathcal{D}_k^d$  are known as *dyadic cubes of generation  $k$* . The most important properties of the dyadic cubes is that for each  $k$ ,  $\mathcal{D}_k^d$  is a cover of  $\mathbf{R}^d$  by cubes of sidelength  $1/N^k$ , and for any two cubes  $Q, R \in \mathcal{D}$ , either their interiors are disjoint, or one cube is nested in the other. We also introduce some further terminology and notation:

- For each cube  $Q \in \mathcal{D}_{k+1}^d$ , there is a unique cube  $R \in \mathcal{D}_k^d$  such that  $Q \subset R$ . We refer to  $R$  as the *parent* of  $Q$ , and  $Q$  as a *child* of  $R$ . Each cube in  $\mathcal{D}$  has exactly  $N^d$  children. For  $Q \in \mathcal{D}_{k+1}^d$ , we let  $Q^* \in \mathcal{D}_k^d$  denote it's parent.
- We say a set  $E \subset \mathbf{R}^d$  is  $\mathcal{D}_k$  *discretized* if it is a union of cubes in  $\mathcal{D}_k^d$ . If  $E$  is  $\mathcal{D}_k$  discretized, we define

$$\mathcal{D}_k(E) = \{Q \in \mathcal{D}_k^d : Q \subset E\}.$$

Then  $E = \bigcup \mathcal{D}_k(E)$ .

- Given  $k \geq 0$ , and  $E \subset \mathbf{R}^d$ , we let

$$E(1/N^k) = \bigcup \{Q \in \mathcal{D}_k^d : Q \cap E \neq \emptyset\}.$$

Then  $E(1/N^k)$  is the smallest  $\mathcal{D}_k$  discretized set containing  $E$  in it's interior. Our choice of notation invites thinking of  $E(1/N^k)$  as a discretized version of the classic  $1/N^k$  thickening

$$\{x \in \mathbf{R}^d : d(x, E) < 1/N^k\}.$$

We have no need for this type of thickening in this thesis.

Since any cube is covered by at most  $O_d(1)$  cubes in  $\mathcal{D}$  of comparable sidelength, from the perspective of measure theory, working with dyadic cubes is normally sufficient. For instance,

Our main purpose with working with dyadic cubes is to construct *fractal-type sets*. By this, we mean defining sets  $X$  as the intersection of a nested family of sets  $\{X_k\}$ , where each  $X_k$  is  $\mathcal{D}_k$  discretized, and each successive set  $X_{k+1}$  is obtained from  $X_k$  by repeatedly performing a simple, recursive procedure. Such a construction satisfies Property (i), (ii), and (v) of Falconer's definition of a fractal, justifying the name.

**Example.** A classic fractal-type set is the Cantor set  $C$ . We form  $C$  from the family of dyadic cubes with branching factor  $M = 3$ . We initially set  $C_0 = [0, 1]$ . Then, given the  $\mathcal{D}_k$  discretized set  $C_k$ , we consider each  $I \in \mathcal{D}_k^1(C_k)$ , and let  $\mathcal{D}_{k+1}^1(I) = \{I_1, I_2, I_3\}$ , where  $I_1, I_2, I_3$  are listed in order. We set

$$C_{k+1} = \bigcup \{I_1 \cup I_3 : I \in \mathcal{D}_k^1(C_k)\}.$$

Then  $C = \bigcap_{k \geq 0} C_k$  is the Cantor set.

Unfortunately, a *constant* branching factor is not sufficient to describe the types of recursive procedures we discuss in this thesis. This motivates a more general iterated family of cubes. Instead of a single branching factor  $N$ , we fix a sequence of positive integers  $\{N_k : k \geq 1\}$ , with  $N_k \geq 2$  for all  $k$ , which give the branching factor at each stage of the dyadic cubes we define.

- For each  $k \geq 0$ , we define

$$\mathcal{Q}_k^d = \left\{ \prod_{i=1}^d \left[ \frac{m_i}{N_1 \dots N_k}, \frac{m_i + 1}{N_1 \dots N_k} \right] : m \in \mathbf{Z}^d \right\}.$$

These are the *dyadic cubes of generation  $k$* . We let  $\mathcal{Q}^d = \bigcup_{k \geq 0} \mathcal{Q}_k^d$ . Note that any two cubes  $\mathcal{Q}^d$  are either nested within one another, or disjoint from one another.

- We set  $l_k = (N_1 \dots N_k)^{-1}$ . Then  $l_k$  is the sidelength of the cubes in  $\mathcal{Q}_k^d$ .
- Given  $Q \in \mathcal{Q}_{k+1}^d$ , we let  $Q^* \in \mathcal{Q}_k^d$  denote the *parent cube* of  $Q$ , i.e. the unique dyadic cube of generation  $k$  such that  $Q \subset Q^*$ .



- We say a set  $E \subset \mathbf{R}^d$  is  $\mathcal{Q}_k$  discretized if it is a union of cubes in  $\mathcal{Q}_k^d$ . In this case, we let

$$\mathcal{Q}_k(E) = \{Q \in \mathcal{Q}_k^d : Q \subset E\}$$

denote the family of cubes whose union is  $E$ .

- For  $E \subset \mathbf{R}^d$  and  $k \geq 0$ , we let  $E(l_k) = \{Q \in \mathcal{Q}_k^d : Q \cap E = \emptyset\}$ .

Sometimes, our recursive constructions need a family of ‘intermediary’ cubes that lie between the scales  $\mathcal{Q}_k^d$  and  $\mathcal{Q}_{k+1}^d$ . In this case, we consider a supplementary sequence  $\{M_k : k \geq 1\}$  with  $M_k | N_k$  for each  $k$ .

- For  $k \geq 1$ , we define

$$\mathcal{R}_k^d = \left\{ \prod_{i=1}^d \left[ \frac{m_i}{N_1 \dots N_{k-1} M_k}, \frac{m_i + 1}{N_1 \dots N_{k-1} M_k} \right] : m \in \mathbf{Z}^d \right\}.$$

- We set  $r_k = (N_1 \dots N_{k-1} M_k)^{-1}$ . Then  $r_k$  is the sidelength of a cube in  $\mathcal{R}_k^d$ .
- The notions of being  $\mathcal{R}_k^d$  discretized, the collection of cubes  $\mathcal{R}_k^d(E)$ , and the sets  $E(r_k)$ , are defined as should be expected.

The cubes in  $\mathcal{R}_k^d$  are coarser than those in  $\mathcal{Q}_k^d$ , but finer than those in  $\mathcal{Q}_{k-1}^d$ .

**Remark.** We note that there is some notational conflict between the cubes  $\mathcal{D}^d$  and the cubes  $\mathcal{Q}^d$ , but since we never use both families simultaneously in a single argument, it should be clear which notation we are using.

## 2.5 Frostman Measures

It is often easy to upper bound Hausdorff dimension, but non-trivial to *lower bound* the Hausdorff dimension of a given set. A key technique to finding a lower bound is *Frostman’s lemma*, which says that a set has large Hausdorff dimension if and only if it supports a probability measure which obeys a certain decay law on small sets. We say a Borel probability measure  $\mu$  is a *Frostman measure* of dimension  $s$  if it is non-zero, compactly supported, and there exists  $C > 0$  such that for any cube  $Q$ ,  $\mu(Q) \leq C \cdot l(Q)^s$ . The proof of Frostman’s lemma will utilize a technique often useful, known as the *mass distribution principle*. Such a principle was stated in BLAH.

**Lemma 5.** For each  $k \geq 0$ , let  $\mathcal{E}_k$  be a finite collection of non-empty, bounded Borel subsets of  $\mathbf{R}^d$ , such that

- Separation: For each  $k$ , the collection  $\{\bar{A} : A \in \mathcal{E}_k\}$  is disjoint.
- Nesting: For each  $A \in \mathcal{E}_{k+1}$ , there is a unique  $A^* \in \mathcal{E}_k$  such that  $A \subset A^*$ , and for any  $A^* \in \mathcal{E}_k$ , there is at least one  $A \in \mathcal{E}_{k+1}$  with  $A \subset A^*$ .

Let  $E_k = \bigcup \mathcal{E}_k$ , and let  $E_\infty = \bigcap E_k$ . Recursively define a function  $\mu : \bigcup_{k \geq 0} \mathcal{E}_k \rightarrow [0, \infty)$  by setting  $\mu(A) = 1$  if  $A \in \mathcal{E}_0$ , and for each  $A \in \mathcal{E}_{k+1}$ , setting

$$\mu(A) = \frac{\mu(A^*)}{\#\{A' \in \mathcal{E}_{k+1} : A' \subset A^*\}}.$$

Then  $\mu$  extends to a finite Borel measure on  $\mathbf{R}^d$  supported on  $E_\infty$ .

*Proof.* Let  $\mathcal{E} = \bigcup_{k \geq 0} \mathcal{E}_k \cup 2^{\mathbf{R}^d - E_k}$ . Then  $\mathcal{E}$  is a semi-ring of sets:

- $\mathcal{E}$  is closed under intersections: Let  $A, B \in \mathcal{E}$ . Then without loss of generality, swapping  $A$  and  $B$  if necessary, there exists  $i \leq j$  such that  $A \in \mathcal{E}_i$  or  $A \subset \mathbf{R}^d - E_i$ , and  $B \in \mathcal{E}_j$  or  $B \subset \mathbf{R}^d - E_j$ .

If  $A \in \mathcal{E}_i$ , and  $B \in \mathcal{E}_j$ , then either  $A \cap B = \emptyset$ , or  $B \subset A$ , and  $A \cap B = B$ . In either case,  $A \cap B \in \mathcal{E}$ .

If  $A \subset \mathbf{R}^d - E_i$ , then  $A \cap B \subset A \subset \mathbf{R}^d - E_i$ , so  $A \cap B \in \mathcal{E}$ . Similarly, if  $B \subset \mathbf{R}^d - E_j$ , then  $A \cap B \subset \mathbf{R}^d - E_j$ , so  $A \cap B \in \mathcal{E}$ .

This addresses all cases.

- $\mathcal{E}$  is closed under relative complements: Let  $A, B \in \mathcal{E}$ . Then there exists  $i, j$  such that  $A \in \mathcal{E}_i$  or  $A \subset \mathbf{R}^d - E_i$ , and  $B \in \mathcal{E}_j$  or  $B \subset \mathbf{R}^d - E_j$ .

Suppose  $A \in \mathcal{E}_i$  and  $B \in \mathcal{E}_j$ . If  $i \geq j$ , then either  $A \cap B = \emptyset$ , or  $A \subset B$ . In either case,  $A - B = \emptyset \in \mathcal{E}$ . If  $i \leq j$ , then either  $A \cap B = \emptyset$ , or  $B \subset A$ . In the latter case,

$$A - B = \bigcup \{B_i : B_i \in \mathcal{E}_j, B_i \neq B, B_i \subset A\} \cup \{A - E_j\}.$$

Note that this union is disjoint, and each set in the union is an element of  $\mathcal{E}$ .

Suppose  $A \in \mathcal{E}_i$ , and  $B \subset \mathbf{R}^d - E_j$ . If  $i \geq j$ , then  $A \subset E_i \subset E_j$ , so  $A - B = A$ . If  $i \leq j$ , then

$$A - B = (A \cap E_j) \cup (A \cap (\mathbf{R}^d - B)).$$

The former set can be written as  $\bigcup_{A' \in \mathcal{E}_j} A' \subset A$ , and the latter set is a subset of  $\mathbf{R}^d - E_j$ . This gives  $A - B$  as a disjoint union of elements of  $\mathcal{E}$ .

If  $A \subset \mathbf{R}^d - E_i$ , then  $A - B \subset \mathbf{R}^d - E_i$ , so  $A - B \in \mathcal{E}$ .

The function  $\mu$  extends to a function on  $\mathcal{E}$  by setting  $\mu(E) = 0$  if  $E \subset \mathbf{R}^d - E_k$  for some  $k$ . This does not conflict with our previous definition, since if  $A \in \mathcal{E}_i$ , then  $A \cap E_j \neq \emptyset$  for all  $j \geq 0$ . We claim that  $\mu$  is then a *pre-measure* on the semi-ring:

- It is certainly true that  $\mu(\emptyset) = 0$ .
- Let  $A \in \mathcal{E}$ , and suppose  $A = \bigcup_i A_i$ , where  $\{A_i\}$  is an at most countable sub-collection of disjoint sets from  $\mathcal{E}$ . We fix  $k$  such that  $A \in \mathcal{E}_k$ , or  $A \subset \mathbf{R}^d - E_k$ , and for each  $i$ , fix  $k_i$  such that  $A_i \in \mathcal{E}_{k_i}$ , or  $A_i \subset \mathbf{R}^d - E_{k_i}$ .

If  $A \in \mathbf{R}^d - E_k$  for some  $k$ , then  $A_i \in \mathbf{R}^d - E_k$  for each  $i$ , and thus  $\mu(A_i) = 0$  for all  $i$ . Since  $\mu(A) = 0$ , this means  $\mu(A) = \sum \mu(A_i)$ .

Suppose  $A \in \mathcal{E}_k$ . If there are only *finitely many* sets  $A_j$  with  $A_j \in \mathcal{E}_k$ , then the claim is obvious. Thus it suffices to show this is the only case. Let  $\{A_j\}$  be the family of sets with  $A_j \in \mathcal{E}_{k_j}$ . Then we may find a disjoint family of open sets  $\{U_j\}$  with  $\overline{A_j} \subset U_j$  for each  $j$ . Define

$$A_\infty = \bigcup \overline{A_j}$$

Then  $\{U_j\}$  is a cover of  $A_\infty$ . The set  $A_\infty$  is compact, since it is bounded and closed by the separability criterion. Since the sets  $U_j$  are disjoint, and  $U_j \cap A_\infty \neq \emptyset$  for any  $j$ , this implies the set of  $\{U_j\}$  is finite, and thus the set  $\{A_j\}$  is finite.

The Caratheodory extension theorem then guarantees that  $\mu$  extends to a measure on the  $\sigma$  algebra generated by  $\mathcal{E}$ . Since

$$\lim_{k \rightarrow \infty} \left( \max_{A \in \mathcal{E}_k} \text{diam}(A) \right) = 0,$$

this  $\sigma$  algebra contains all Borel sets. □

**Lemma 6** (Mass Distribution Principle). *Let  $f : \mathcal{Q}^d \rightarrow [0, \infty)$  be a function such that for any  $Q_0 \in \mathcal{Q}^d$ ,*

$$\sum_{Q^* = Q_0} f(Q) = f(Q_0), \quad (2.6)$$

Then there exists a regular Borel measure  $\mu$  supported on

$$\bigcap_{k=1}^{\infty} \left[ \bigcup \{Q \in \mathcal{Q}_k^d : f(Q) > 0\} \right]$$

such that for each  $Q \in \mathcal{Q}^d$ ,

$$\mu(Q) \geq f(Q), \quad (2.7)$$

and for any set  $E$  and  $k \geq 0$ ,

$$\mu(E) \leq \sum f(Q), \quad (2.8)$$

where  $Q$  ranges over all cubes in  $\mathcal{Q}_k^d(E(l_k))$ .

*Proof.* For each  $i$ , define a regular Borel measure  $\mu_i$  such that for each  $f \in C_c(\mathbf{R}^d)$ ,

$$\int f d\mu_i = \sum_{Q \in \mathcal{Q}_i^d} \frac{f(Q)}{|Q|} \int_Q f dx$$

Then, for each  $j \leq i$ ,  $\mu_i(Q) = f(Q)$ .

We claim that  $\{\mu_i\}$  is a Cauchy sequence in the space of all regular Borel measures on  $\mathbf{R}^d$ , viewing the space as a locally convex space under the weak topology. Fix  $\varphi \in C_c(\mathbf{R}^d)$ , and choose some  $N$  such that  $\varphi$  is supported on  $[-N, N]^d$ . Set  $Q_0 = [-N, N]^d$ . Since  $\varphi$  is compactly supported,  $\varphi$  is *uniformly continuous*, so for each  $\varepsilon > 0$ , if  $i$  is suitably large, there is a sequence of values  $\{a_Q : Q \in \mathcal{Q}_i^d(Q_0)\}$  such that if  $x \in Q$ ,  $|\varphi(x) - a_Q| \leq \varepsilon$ . But this means that

$$\sum (a_Q - \varepsilon) \mathbf{I}_Q \leq \varphi \leq \sum (a_Q + \varepsilon) \mathbf{I}_Q,$$

where  $Q$  ranges over cubes in  $\mathcal{Q}_i^d(Q_0)$ . Thus for any  $j \geq i$ ,

$$\int \varphi d\mu_j \leq \sum (a_Q + \varepsilon) \mu_j(Q) = \sum (a_Q + \varepsilon) f(Q)$$

and

$$\int \varphi d\mu_j \geq \sum (a_Q - \varepsilon) \mu_j(Q) = \sum (a_Q - \varepsilon) f(Q).$$

In particular, if  $j, j' \geq i$ ,

$$\left| \int \varphi d\mu_j - \int \varphi d\mu_{j'} \right| \leq 2\varepsilon \sum_{Q \in \mathcal{Q}_i(Q_0)} f(Q) = 2\varepsilon \sum_{Q \in \mathcal{Q}_0(Q_0)} f(Q).$$

Since  $\varepsilon$  and  $\varphi$  were arbitrary, this shows  $\{\mu_i\}$  is Cauchy.

Since the space of all regular Borel measures on  $\mathbf{R}^d$  is *complete*, the last paragraph implies there exists a regular Borel measure  $\mu$  such that  $\mu_i \rightarrow \mu$  weakly. For any  $Q \in \mathcal{Q}^d$ , since  $Q$  is a closed set,

$$\mu(Q) \geq \limsup_{i \rightarrow \infty} \mu_i(Q) = f(Q).$$

Conversely, if  $E$  is an arbitrary set, then  $E \subset E(l_i)^\circ$  for any  $i$ , so

$$\mu(E) \leq \liminf_{i \rightarrow \infty} \mu_i(E(l_i)^\circ) \leq \liminf_{i \rightarrow \infty} \mu_i(E(l_i)) = \sum f(Q),$$

where  $Q$  ranges over the cubes in  $\mathcal{Q}_i^d(E(l_k))$ . □

**Remark.** *Common approaches to the mass distribution principle consider*

$$\mu(E) = \inf \left\{ \sum_{i=1}^{\infty} f(Q_i) : Q_i \in \mathcal{Q}^d, E \subset \bigcup Q_i \right\}.$$

*If  $d(E, F) > 0$ , then (2.6) implies that  $\mu(E + F) = \mu(E) + \mu(F)$ , so  $\mu$  is a metric exterior measure, and so all Borel sets are measurable with respect to  $\mu$ . However, given the conditions of this theorem, it is not necessarily true that  $\mu$  is even a non-zero measure, let alone that it reflects the values of the functions  $f$ .*

**Lemma 7** (Frostman's Lemma). *If  $E$  is a Borel set,  $H^s(E) > 0$  if and only if there exists an  $s$  dimensional Frostman measure supported on  $E$ .*

*Proof.* Suppose that  $\mu$  is  $s$  dimensional and supported on  $E$ . If  $H^s(F) = 0$ , then for each  $\varepsilon > 0$  there is a sequence of cubes  $\{Q_k\}$  with  $\sum_{k=1}^{\infty} l(Q_k)^s \leq \varepsilon$ . But then

$$\mu(F) \leq \sum_{k=1}^{\infty} \mu(Q_k) \lesssim \sum_{k=1}^{\infty} l(Q_k)^s \leq \varepsilon.$$

Taking  $\varepsilon \rightarrow 0$ , we conclude  $\mu(F) = 0$ . Thus  $\mu$  is absolutely continuous with respect to  $H^s$ . Since  $\mu(E) > 0$ , this means that  $H^s(E) > 0$ .

To prove the converse, we will suppose for simplicity that  $E$  is compact. We work dyadically, with the classical family of dyadic cubes  $\mathcal{D}$ . By translating, we may assume that  $H^s(E \cap [0, 1]^d) > 0$ , and so without loss of generality we may assume  $E \subset [0, 1]^d$ . For each  $Q \in \mathcal{D}_k^d$ , define  $f^+(Q) = H_\infty^s(E \cap Q)$ . Then  $f^+(Q) \leq 1/2^{ks}$ , and  $f^+$  is subadditive. We try and define a function  $f$  to which we can apply Lemma 6. We initially define  $f$  by setting  $f([0, 1]^d) = f^+([0, 1]^d)$ . Given  $Q \in \mathcal{D}_k^d$ , we enumerate it's children as  $Q_1, \dots, Q_M \in \mathcal{D}_{k+1}^d$ . We then consider any values  $A_1, \dots, A_M \geq 0$  such that

$$A_1 + \dots + A_M = f(Q), \quad (2.9)$$

and for each  $k$ ,

$$A_k \leq f^+(Q_k). \quad (2.10)$$

This is feasible to do because  $f^+(Q_1) + \dots + f^+(Q_M) \geq f^+(Q)$ . We then define  $f(Q_k) = A_k$  for each  $k$ . Equation (2.9) implies the recursive constraint is satisfied, so  $f$  is a well defined function. Furthermore, (2.9) implies (2.6) of Lemma 6, and so the mass distribution principle implies that there exists a measure  $\mu$  supported on  $E$ , satisfying (2.8) and (2.7). In particular, (2.8) implies  $\mu$  is non-zero. For each  $Q \in \mathcal{D}_k^d$ ,  $\#\mathcal{D}_k^d(Q(l_k)) = 3^d = O_d(1)$ , so we can apply (2.7) to conclude

$$\mu(Q) \lesssim_d \max f(Q') \leq 1/2^{ks} = l(Q)^s.$$

where  $Q'$  ranges over the cubes in  $\mathcal{D}_k^d(Q(l_k))$ . Given any cube  $Q$ , we find  $k$  with  $1/2^{k-1} \leq l(Q) \leq 1/2^k$ . Then  $Q$  is covered by  $O_d(1)$  dyadic cubes in  $\mathcal{D}_k^d$ , and so  $\mu(Q) \lesssim l(Q)^s$ . Thus  $\mu$  is a Frostman measure of dimension  $s$ .  $\square$

Frostman's lemma implies that to study the Hausdorff dimension of the set, it suffices to understand the class of measures which can be supported on that set. The Fourier dimension of a set also studies this perspective, by slightly refining the measure bound required by the Frostman dimension in frequency space.

## 2.6 Dyadic Fractional Dimension

We wish to establish results about fractional dimension 'dyadically', working with the family of cubes  $\mathcal{Q}^d$  specified in terms of a sequence  $\{N_k : k \geq 1\}$ . We begin with Minkowski dimension. For each  $m$ , let  $N_{\mathcal{Q}}(m, E)$  denote the minimal number of cubes in  $\mathcal{Q}_m^d$  required to cover  $E$ . This is often easy to calculate, up to a multiplicative constant, by greedily selecting cubes which intersect  $E$ .

**Lemma 8.** *For any set  $E$ ,*

$$N_{\mathcal{Q}}(m, E) \approx_d \#\{Q \in \mathcal{Q}_m^d : Q \cap E \neq \emptyset\} \approx_d N(l_m, E).$$

*Proof.* Let  $\mathcal{E} = \{Q \in \mathcal{Q}_m^d : Q \cap E \neq \emptyset\}$ . Then  $N(l_m, E) \leq N_{\mathcal{Q}}(m, E) \leq \#(\mathcal{E})$ . Conversely, let  $\{Q_k\}$  be a minimal cover of  $E$  by cubes. Then each cube  $Q_k$  intersects at most  $3^d$  cubes in  $\mathcal{Q}_k^d$ , so  $\#(\mathcal{E}) \leq 3^d \cdot N(l_m, E) \leq 3^d \cdot N_{\mathcal{Q}}(m, E)$ .  $\square$

Thus it is natural to ask whether it is true that for any set  $E$ ,

$$\begin{aligned} \dim_{\mathbf{M}}(E) &= \liminf_{k \rightarrow \infty} \frac{\log[N_{\mathcal{Q}}(k, E)]}{\log[1/l_k]} \\ &\text{and} \\ \dim_{\mathbf{M}}(E) &= \limsup_{k \rightarrow \infty} \frac{\log[N_{\mathcal{Q}}(k, E)]}{\log[1/l_k]}. \end{aligned} \tag{2.11}$$

The answer depends on the choice of  $\{N_k\}$ . In particular, we find that a sufficient condition is that

$$N_{k+1} \lesssim_{\varepsilon} (N_1 \dots N_k)^{\varepsilon} \quad \text{for any } \varepsilon > 0. \tag{2.12}$$

We will see that this condition allows us to work dyadically in many scenarios when it comes to fractal dimension.

**Lemma 9.** *If (2.12) holds, then (2.11) holds.*

*Proof.* Fix a length  $l$ , and find  $k$  with  $l_{k+1} \leq l \leq l_k$ . Applying Lemma 8 shows

$$N(l, E) \leq N(l_{k+1}, E) \lesssim_d N_{\mathcal{Q}}(k+1, E)$$

and

$$N(l, E) \geq N(l_k, E) \gtrsim_d N_{\mathcal{Q}}(k, E).$$

Thus

$$\frac{\log[N(l, E)]}{\log[1/l]} \leq \left[ \frac{\log(1/l_{k+1})}{\log(1/l_k)} \right] \frac{\log[N_{\mathcal{Q}}(k+1, E)]}{\log[1/l_{k+1}]} + O_d(1/k)$$

and

$$\frac{\log[N(l, E)]}{\log[1/l]} \geq \left[ \frac{\log(1/l_k)}{\log(1/l_{k+1})} \right] \frac{\log[N_{\mathcal{Q}}(k, E)]}{\log[1/l_k]} + O_d(1/k).$$

Thus, provided that

$$\frac{\log(1/l_{k+1})}{\log(1/l_k)} \rightarrow 1, \tag{2.13}$$

the conclusion of the theorem is true. But (2.13) is equivalent to the condition that

$$\frac{\log(N_{k+1})}{\log(N_1) + \dots + \log(N_k)} \rightarrow 0,$$

and this is equivalent to (2.12).  $\square$

Any constant branching factor satisfies 2.12 for the Minkowski dimension. In particular, we can work fairly freely with the classical dyadic cubes without any problems occuring. But more importantly for our work, we can let the sequence  $\{N_k\}$  increase rapidly.

**Lemma 10.** *If  $N_k = 2^{\lfloor 2^{k\psi(k)} \rfloor}$ , where  $\psi(k)$  is any decreasing sequence of positive numbers tending to zero, such that  $\psi(k) \gtrsim \log(k)/k$ , then (2.12) holds.*

*Proof.* We note that  $\log(N_k) = 2^{k\psi(k)} + O(1)$ . Thus

$$\begin{aligned} \frac{\log(N_{k+1})}{\log(N_1) + \dots + \log(N_k)} &= \frac{2^{(k+1)\psi(k+1)} + O(1)}{2^{\psi(1)} + 2^{2\psi(2)} + \dots + 2^{k\psi(k)} + O(k)} \\ &\lesssim \frac{2^{(k+1)\psi(k+1)}}{2^{\psi(k)} + 2^{2\psi(k)} + \dots + 2^{k\psi(k)}} \\ &\lesssim \frac{2^{(k+1)\psi(k+1)}}{2^{(k+1)\psi(k)}} (2^{\psi(k)} - 1) \\ &\leq (2^{\psi(k)} - 1) \rightarrow 0. \end{aligned}$$

This is equivalent to the fact that  $N_{k+1} \lesssim_\varepsilon (N_1 \dots N_k)^\varepsilon$  for any  $\varepsilon > 0$ .  $\square$

We refer to any sequence  $\{l_k\}$  constructed by  $\{N_k\}$  satisfying the conditions of Lemma 10 as a *subhyperdyadic* sequence. If a sequence is generated by a sequence  $\{N_k = 2^{\lfloor 2^{ck} \rfloor}\}$ , for some fixed  $c > 0$ , the values  $\{l_k\}$  are referred to as *hyperdyadic*. The next (counter) example shows that hyperdyadic sequences are essentially the ‘boundary’ for sequences that can be used to measure the Minkowski dimension.

**Example.** *We consider a multi-scale dyadic construction, utilizing the two families  $\mathcal{Q}^d$  and  $\mathcal{R}^d$ . Fix  $0 \leq c < 1$ , and define  $N_k = 2^{\lfloor 2^{ck} \rfloor}$ , and  $M_k = 2^{\lfloor c2^{ck} \rfloor}$ . Then  $M_k \mid N_k$  for each  $k$ . We recursively define a nested family of sets  $\{E_k\}$ , with each  $E_k$  a  $\mathcal{Q}_k^d$  discretized set, and set  $E = \bigcap E_k$ . We define  $E_0 = [0, 1]$ . Then, given  $E_k$ , we divide each sidelength  $l_k$  dyadic interval in  $E_k$  into  $M_{k+1}$  intervals, and*



then keep the first sidelength  $l_k/M_{k+1}$  interval from this set, which is formed from  $N_{k+1}/M_{k+1}$  sidelength  $l_{k+1}$  dyadic intervals. Thus  $\#(\mathcal{Q}_0(E_0)) = 1$ , and

$$\#(\mathcal{Q}_{k+1}(E_{k+1})) = (N_{k+1}/M_{k+1})\#(\mathcal{Q}_k(E_k))$$

so

$$\#(\mathcal{Q}_k(E_k)) = \frac{N_1 \dots N_k}{M_1 \dots M_k} \quad (2.14)$$

Noting that  $\log(N_i) = 2^{ci} + O(1)$ , and  $\log(M_i) = c2^{ci} + O(1)$ , we conclude that

$$\frac{\log \#(\mathcal{Q}_k(E_k))}{\log(1/l_k)} = \frac{(1-c)(2^c + \dots + 2^{ck}) + O(k)}{(2^c + \dots + 2^{ck}) + O(k)} = 1 - c + O\left(\frac{k}{2^{ck}}\right) \rightarrow 1 - c.$$

On the other hand, if  $r_{k+1} = l_k/M_{k+1}$ , then for each  $k$ ,

$$\#(\mathcal{R}_{k+1}^d(E_k)) = \#(\mathcal{Q}_k(E_k)) = \frac{N_1 \dots N_k}{M_1 \dots M_k},$$

and so

$$\begin{aligned} \frac{\log \#(\mathcal{R}_{k+1}^d(E_k))}{\log(1/r_{k+1})} &= \frac{(1-c)(2^c + \dots + 2^{ck}) + O(k)}{(2^c + \dots + 2^{ck}) + c2^{c(k+1)} + O(k)} \\ &= \frac{\left(\frac{(1-c)2^c}{2^c-1}\right) 2^{ck} + O(k)}{\left(\frac{2^c}{2^c-1} + c2^c\right) 2^{ck} + O(k)} \\ &= \frac{1-c}{1-c+c2^c} + O\left(\frac{k}{2^{ck}}\right) \rightarrow \frac{1-c}{1-c+c2^c} < 1-c. \end{aligned}$$

In particular,

$$\dim_{\mathbf{M}}(E) \neq \liminf_{k \rightarrow \infty} \frac{\log[N(l_k, E)]}{\log(1/l_k)},$$

so measurements at hyperdyadic scales fail to establish general results about the Minkowski dimension.

We now move on to calculating Hausdorff dimension dyadically. The natural quantity to consider is the measure defined for any  $E$  as  $H_{\mathcal{Q}}^s(E) = \lim_{m \rightarrow \infty} H_{\mathcal{Q},m}^s(E)$ ,

where

$$H_{\mathcal{Q},m}^s(E) = \inf \left\{ \sum_k l(Q_k)^s : E \subset \bigcup_k Q_k, Q_k \in \bigcup_{i \geq m} \mathcal{Q}_i^d \text{ for each } k \right\}.$$

A similar argument to the standard Hausdorff measures shows there is a unique  $s_0$  such that  $H_{\mathcal{Q}}^s(E) = \infty$  for  $s < s_0$ , and  $H_{\mathcal{Q}}^s(E) = 0$  for  $s > s_0$ . It is obvious that  $H_{\mathcal{Q}}^s(E) \geq H^s(E)$  for any set  $E$ , so we certainly have  $s_0 \geq \dim_{\mathbf{H}}(E)$ . The next lemma guarantees that  $s_0 = \dim_{\mathbf{H}}(E)$ , under the same conditions on the sequence  $\{N_k\}$  as found in Lemma 9.

**Lemma 11.** *If (2.12) holds, then for any  $\varepsilon > 0$ ,  $H_{\mathcal{Q}}^s(E) \lesssim_{s,\varepsilon} H^{s-\varepsilon}(E)$ .*

*Proof.* Fix  $\varepsilon > 0$  and  $m$ . Let  $E \subset \bigcup Q_k$ , where  $l(Q_k) \leq l_m$  for each  $k$ . Then for each  $k$ , we can find  $i_k$  such that  $l_{i_k+1} \leq l(Q_k) \leq l_{i_k}$ . Then  $Q_k$  is covered by  $O_d(1)$  elements of  $\mathcal{Q}_{i_k}^d$ , and

$$H_{\mathcal{Q},m}^s(E) \lesssim_d \sum l_{i_k}^s \leq \sum (l_{i_k}/l_{i_k+1})^s l(Q_k)^s \leq \sum (l_{i_k}/l_{i_k+1})^s l_{i_k}^\varepsilon l(Q_k)^{s-\varepsilon}. \quad (2.15)$$

By assumption,

$$l_{i_k+1} = \frac{1}{N_1 \dots N_{i_k} N_{i_k+1}} \gtrsim_{s,\varepsilon} (N_1 \dots N_{i_k})^{1+\varepsilon/s} = l_{i_k}^{1+\varepsilon/s}. \quad (2.16)$$

Putting (2.15) and (2.16) together, we conclude that  $H_{\mathcal{Q},m}^s(E) \lesssim_{d,s,\varepsilon} \sum l(Q_k)^{s-\varepsilon}$ . Since  $\{Q_k\}$  was an arbitrary cover of  $E$ , we conclude  $H_{\mathcal{Q},m}^s(E) \lesssim H^{s-\varepsilon}(E)$ , and since  $m$  was arbitrary, that  $H_{\mathcal{Q}}^s(E) \lesssim H^{s-\varepsilon}(E)$ .  $\square$

Finally, we consider computing whether we can establish that a measure is a Frostman measure dyadically.

**Theorem 12.** *If (2.12) holds, and if  $\mu$  is a Borel measure such that  $\mu(Q) \lesssim l(Q)^s$  for each  $Q \in \mathcal{Q}_k^d$ , then  $\mu$  is a Frostman measure of dimension  $s - \varepsilon$  for each  $\varepsilon > 0$ .*

*Proof.* Given a cube  $Q$ , find  $k$  such that  $l_{k+1} \leq l(Q) \leq l_k$ . Then  $Q$  is covered by  $O_d(1)$  cubes in  $\mathcal{Q}_k^d$ , which shows

$$\mu(Q) \lesssim_d l_k^s = [(l_k/l)^s l^\varepsilon] l^{s-\varepsilon} \leq [l_k^{s+\varepsilon}/l_{k+1}^s] l^{s-\varepsilon} = \left[ \frac{N_{k+1}^s}{(N_1 \dots N_k)^\varepsilon} \right] l^{s-\varepsilon} \lesssim_\varepsilon l^{s-\varepsilon}. \quad \square$$

**Remark.** The dyadic construction showing that Minkowski dimension cannot be measured only at hyperdyadic scales also shows that a bound on a measure  $\mu$  on hyperdyadic cubes does not imply the correct bound at all scales. It is easy to show from (2.14) that the canonical measure  $\mu$  for this example satisfies  $\mu(Q) \lesssim l(Q)^{1-c}$  for all  $Q \in \mathcal{Q}_k(E_k)$ , yet we know that for the set constructed in that example,

$$\dim_{\mathbf{H}}(E) \leq \dim_{\mathbf{M}}(E) < 1 - c.$$

Frostman's lemma implies we cannot possibly have  $\mu(Q) \lesssim_{\varepsilon} l(Q)^{1-c-\varepsilon}$  for all  $\varepsilon > 0$  and all cubes  $Q$ .

Let's recognize the utility of this approach from the perspective of a dyadic construction. Suppose we have a sequence of nested sets  $\{E_k\}$ , where  $E_k$  is a  $\mathcal{Q}_k$  discretized subset of  $[0, 1]^d$ , and for each  $Q_0 \in \mathcal{Q}_k(E_k)$ , there is at least one cube  $Q \in \mathcal{Q}_{k+1}(E_{k+1})$  with  $Q^* = Q_0$ . Then we can set  $E = \bigcap E_k$  as a 'limit' of the discretizations  $E_k$ . We can associate with this construction a finite measure  $\mu$ . It is defined by setting  $f([0, 1]^d) = 1$ , and setting, for each  $Q \in \mathcal{Q}_{k+1}(E_{k+1})$ ,

$$f(Q) = \frac{f(Q^*)}{\#\{Q' \in \mathcal{Q}_{k+1}(E_{k+1}) : (Q')^* = Q^*\}}.$$

The mass distribution principle shows that there exists a Borel measure  $\mu$ , which we refer to as the *canonical measure* associated with this construction. For this measure, it is often easy to show from a combinatorial argument that  $f(Q) \lesssim l(Q)^s$  if  $Q \in \mathcal{Q}^d$ , which leads to bounds on the measure  $\mu$ . This makes Theorem 12 useful.

## 2.7 Beyond Hyperdyadics

If we are to use a faster increasing sequence of branching factors than the last section guarantees, we therefore must exploit some extra property of our construction, which is not always present in general sets. Here, we rely on a *uniform mass distribution* between scales. Given the uniformity assumption, the lengths can decrease as fast as desired. We utilize a multi-scale set of dyadic cubes.

**Lemma 13.** *Let  $\mu$  be a measure supported on a set  $E$ . Suppose that*

- (A) *For any  $Q \in \mathcal{Q}_k^d$ ,  $\mu(Q) \lesssim l_k^s$ .*
- (B) *For each  $Q \in \mathcal{R}_k^d$ ,  $\#[\mathcal{Q}_k^d(E(l_k) \cap Q)] \lesssim 1$ .*

(C) For any  $Q \in \mathcal{R}_{k+1}^d$  with parent cube  $Q^* \in \mathcal{Q}_k^d$ ,  $\mu(Q) \lesssim (r_{k+1}/l_k)^d \mu(Q^*)$ .

Then  $\mu$  is a Frostman measure of dimension  $s$ .

*Proof.* We establish the general bound  $\mu(Q) \lesssim l(Q)^s$  for all cubes  $Q$  by separating into two different cases:

- Suppose there is  $k$  with  $r_{k+1} \leq l(Q) \leq l_k$ . Then  $\#(\mathcal{R}_{k+1}^d(Q(r_{k+1}))) \lesssim (l/r_{k+1})^d$ . Properties (A) and (C) imply each of these cubes has measure at most  $O((r_{k+1}/l_k)^d l_k^s)$ , so we obtain that

$$\mu(Q) \lesssim (l/r_{k+1})^d (r_{k+1}/l_k)^d l_k^s = l^d / l_k^{d-s} \lesssim l^s.$$

- Suppose there exists  $k$  with  $l_k \leq l \leq r_k$ . Then  $\#[\mathcal{R}_k^d(Q(r_k))] \lesssim 1$ . Combining this with Property (C) gives  $\#[\mathcal{Q}_k^d(Q(r_k) \cap E(l_k))] \lesssim 1$ . This and Property (A) then shows

$$\mu(Q) \lesssim l_k^s \leq l^s.$$

This addresses all cases, so  $\mu$  is a Frostman measure of dimension  $s$ . □

TODO: ADDRESS OTHER FRACTIONAL DIMENSION CASES?

# Chapter 3

## Related Work

Here, we discuss the main papers which influenced our results. In particular, the work of Keleti on translate avoiding sets, Fraser and Pramanik's work on sets avoiding smooth configurations, Mathé's result on sets avoiding algebraic varieties, and Schmerkin's result on sets with large Fourier dimension avoiding smooth configurations.

### 3.1 Keleti: A Translate Avoiding Set

Keleti's two page paper constructs a full dimensional set  $X \subset [0, 1]$  such that for each  $t \neq 0$ ,  $X$  intersects  $t + X$  in at most one place. The set  $X$  is then said to *avoid translates*. This paper contains the core idea behind the *interval dissection* method adapted in Fraser and Pramanik's paper. We also adapt this technique in our paper, which makes the result of interest.

It is often convenient to avoid certain configurations when they are expressed in terms of an equation, which is also exploited in Fraser and Pramanik's work, and in Mathé's.

**Lemma 14.** *Let  $X$  be a set. Then  $X$  avoids translates if and only if there do not exist values  $x_1 < x_2 \leq x_3 < x_4$  in  $X$  with  $x_2 - x_1 = x_4 - x_3$ .*

*Proof.* Suppose  $(t + X) \cap X$  contains two points  $a < b$ . Without loss of generality, we may assume that  $t > 0$ . If  $a \leq b - t$ , then the equation

$$a - (a - t) = t = b - (b - t)$$

satisfies the constraints, since  $a - t < a \leq b - t < b$  are all elements of  $X$ . We also

have

$$(b - t) - (a - t) = b - a,$$

which satisfies the constraints if  $a - t < b - t \leq a < b$ . This covers all possible cases. Conversely, if there are  $x_1 < x_2 \leq x_3 < x_4$  in  $X$  with

$$x_2 - x_1 = t = x_4 - x_3,$$

then  $X + t$  contains  $x_2 = x_1 + (x_2 - x_1)$  and  $x_4 = x_3 + (x_4 - x_3)$ .  $\square$

The basic, but fundamental idea of the interval dissection technique is to introduce memory into Cantor set constructions. Keleti constructs a nested family of discrete sets  $\{X_k\}$ , with  $X_k$  a  $\mathcal{Q}_k$  discretized set, and with  $X = \bigcap X_k$ . The sequence  $\{N_k\}$  will be specified later, but each  $N_k$  will be a multiple of 10. We initialize  $X_0 = [0, 1]$ . The novel feature of the argument is to add a queue of intervals to the construction, initially just containing  $[0, 1]$ . To construct the sequence  $\{X_k\}$ , Keleti iteratively performs the following procedure:

---

**Algorithm 1** Construction of the Sets  $\{X_k\}$ :

---

Set  $k = 0$ .

**Repeat**

Take off an interval  $I$  from the front of the queue.

**For all**  $J \in \mathcal{Q}_k(X_k)$ :

Order the intervals in  $\mathcal{Q}_{k+1}(J)$  as  $J_1, \dots, J_N$ .

**If**  $J \subset I$ , add all intervals  $J_i$  to  $X_{k+1}$  with  $i \equiv 0$  modulo 10.

**Else** add all  $J_i$  with  $i \equiv 5$  modulo 10.

Add all intervals in  $\mathcal{Q}_{k+1}^d$  to the end of the queue.

Increase  $k$  by 1.

---

Each iteration of the algorithm produces a new set  $X_k$ , and so leaving the algorithm to repeat infinitely produces a sequence  $\{X_k\}$  whose intersection is  $X$ .

**Lemma 15.** *The set  $X$  is translate avoiding.*

*Proof.* If  $X$  is not translate avoiding, there is  $x_1 < x_2 \leq x_3 < x_4$  with  $x_2 - x_1 = x_4 - x_3$ . Since  $l_k \rightarrow 0$ , there is a suitably large integer  $N$  such that  $x_1$  is contained in an interval  $I \in \mathcal{Q}_N$  not containing  $x_2, x_3$ , or  $x_4$ . At stage  $N$  of the algorithm, the interval  $I$  is added to the end of the queue, and at a much later stage  $M$ , the interval  $I$  is retrieved. Find the startpoints  $x_1^\circ, x_2^\circ, x_3^\circ, x_4^\circ \in l_M \mathbf{Z}$  to the intervals in

$\mathcal{Q}_M$  containing  $x_1, x_2, x_3$ , and  $x_4$ . Then we can find  $n$  and  $m$  such that  $x_4^\circ - x_3^\circ = (10n)l_M$ , and  $x_2^\circ - x_1^\circ = (10m + 5)l_M$ . In particular, this means that  $|(x_4^\circ - x_3^\circ) - (x_2^\circ - x_1^\circ)| \geq 5L_M$ . But

$$\begin{aligned} |(x_4^\circ - x_3^\circ) - (x_2^\circ - x_1^\circ)| &= |[(x_4^\circ - x_3^\circ) - (x_2^\circ - x_1^\circ)] - [(x_4 - x_3) - (x_2 - x_1)]| \\ &\leq |x_1^\circ - x_1| + \dots + |x_4^\circ - x_4| \leq 4L_M \end{aligned}$$

which gives a contradiction.  $\square$

It is easy to see from the algorithm that

$$\#(\mathcal{Q}_k(X_k)) = (l_{k-1}/10l_k) \cdot \#(\mathcal{Q}_{k-1}(X_{k-1})).$$

Closing the recursive definition shows

$$\#(\mathcal{Q}_k(X_k)) = \frac{1}{10^k l_k}.$$

In particular, this means  $|X_k| = 1/10^k$ , so  $X$  has measure zero irrespective of our parameters. Nonetheless, the canonical measure  $\mu$  on  $X$  defined with respect to the decomposition  $\{X_k\}$  satisfies  $\mu(I) = 10^k l_k$  for all  $I \in \mathcal{B}(l_k, X)$ . If  $10^k l_k^\varepsilon \lesssim_\varepsilon 1$  for all  $\varepsilon$ , then we can establish the bounds  $\mu(I) \lesssim_\varepsilon l_k^{1-\varepsilon}$  for all  $\varepsilon$ . In particular, this is true if  $N_k = 10^{\lfloor \log(k+2) \rfloor}$ . And because this sequence does not rapidly decrease too fast, we can apply Lemma 12 to show  $\mu$  is a Frostman measure of dimension  $1 - \varepsilon$  for each  $\varepsilon > 0$ , so  $X$  has full Hausdorff dimension.

## 3.2 Generalizing Keleti's Argument

Before we move onto other methods which developed Keleti's argument work, it is useful to dwell on what general properties this argument has:

- *Simplification to a Discrete Problem:* A major part of Keleti's argument is solving a discrete version of the configuration argument. We could summarize the result of Keleti's discrete argument in a lemma.

**Lemma 16.** *Let  $T_1, T_2$  be disjoint,  $\mathcal{Q}_k$  discretized sets. Then we can find  $S_1 \subset T_1$  and  $S_2 \subset T_2$  such that*

- (i) *For each  $k$ ,  $S_k$  is a  $\mathcal{Q}_{k+1}$  discretized subset of  $T_k$ .*

(ii) If  $x_1 \in S_1$  and  $x_2, x_3, x_4 \in S_2$ , then  $x_2 - x_1 \neq x_4 - x_3$ .

(iii) For each cube  $Q \in \mathcal{Q}_k$  with  $Q \subset T_k$ ,

$$\#(\mathcal{Q}_{k+1}(Q \cap S_k)) \geq (1/10) \cdot \#(\mathcal{Q}_{k+1}(Q)).$$

- *Iterative Application of Discrete Solution:* Keleti then repeatedly applies his argument iteratively. In particular, Property (i) of Lemma 16 allows him to apply his argument iteratively. The reason why Keleti obtains a configuration avoiding set in the limit is because of Property (ii). Most importantly the reason why Keleti obtains a set with full Hausdorff dimension, if the sequence  $\{N_k\}$  decreases rapidly enough, is because of Property (iii).

Of course, it is not possible to extend the discrete solution of the configuration argument to general configurations; this part of Keleti's method strongly depends on the arithmetic structure of the configuration. The iterative application of a discrete solution, however, can be applied in generality. In various senses, this technique is applied in all the results we review in this chapter.

### 3.3 Fraser/Pramanik: Smooth Configurations

Inspired by Keleti's result, Pramanik and Fraser obtained a generalization of the queue method which allows one to find sets avoiding  $n + 1$  point configurations given by the zero sets of smooth functions, i.e.

$$\mathcal{C} = \{(x_0, \dots, x_n) \in \mathbf{R}^{dn} : f(x_0, \dots, x_n) = 0\},$$

under mild regularity conditions on the function  $f : [0, 1]^{dn} \rightarrow [0, 1]^m$ . Here, a simple pigeonholing strategy suffices.

**Theorem 17** (Pramanik and Fraser). *Fix  $m \leq d(n - 1)$ . Consider a countable family of functions  $\{f_k : [0, 1]^{dn} \rightarrow [0, 1]^m\}$  each of which being  $C^2$ , and such that for each  $k$ ,  $Df_k$  has full rank at any  $(x_1, \dots, x_n)$  with  $f(x_1, \dots, x_n) = 0$ , where the  $(x_1, \dots, x_n)$  are distinct. Then there exists a set  $X \subset \mathbf{R}^d$  with Hausdorff dimension  $m/(n - 1)$  such that  $X$  avoids the configuration*

$$\mathcal{C} = \bigcup_k \{(x_1, \dots, x_n) \in \mathcal{C}^n(\mathbf{R}^d) : f(x_1, \dots, x_n) = 0\}.$$



**Remark.** For simplicity, we only prove the result for a single function, rather than a countable family of functions. The only major difference between the two approaches is the choice of scales we must choose later on in the argument.

Pramanik and Fraser also simplify to a discrete version of their problem, which they then iteratively apply. In the discrete setting, rather than making a linear shift in one of the intervals we avoid as in Keleti's approach, one must use the smoothness properties of the function to find large segments of an interval avoiding. Corollary 20 gives the discrete solution that Pramanik and Fraser utilizes.

**Lemma 18.** Fix  $n > 1$ . Let  $T \subset [0, 1]^d$  be  $\mathcal{Q}_k$  discretized, and  $T' \subset [0, 1]^{(n-1)d}$  be  $\mathcal{Q}_k$  discretized. Let  $B \subset T \times T'$  be  $\mathcal{Q}_{k+1}$  discretized. Then there exists a  $\mathcal{Q}_{k+1}$  discretized set  $S \subset T$ , and a  $\mathcal{Q}_{k+1}$  discretized set  $B' \subset T'$ , such that

$$(A) \quad (S \times T') \cap B \subset S \times B'.$$

$$(B) \quad \text{For every } Q \in \mathcal{Q}_k^d, \text{ there exists } \mathcal{R}(Q) \subset \mathcal{R}_{k+1}^d(Q), \text{ such that}$$

$$\#(\mathcal{R}(Q)) \geq (1/2) \cdot \#(\mathcal{R}_{k+1}^d(Q)).$$

$$\text{and for each } R \in \mathcal{R}_{k+1}^d(Q),$$

$$\#(\mathcal{Q}_{k+1}(R)) = \begin{cases} 1 & : R \in \mathcal{R}(Q), \\ 0 & : R \notin \mathcal{R}(Q). \end{cases}$$

$$(C) \quad \#(\mathcal{Q}_{k+1}(B')) \leq 2(N_1 \dots N_k)^d (M_{k+1}/N_{k+1})^d \cdot \#(\mathcal{Q}_{k+1}(B)).$$

*Proof.* Fix  $Q_0 \in \mathcal{Q}_k(T)$ . For each  $R \in \mathcal{R}_{k+1}(Q_0)$ , define a slab  $S[R] = R \times T'$ , and for each  $Q \in \mathcal{Q}_{k+1}(Q_0)$ , define a wafer  $W[Q] = Q \times T'$ . We say a wafer  $W[Q]$  is good if

$$\#(\mathcal{Q}_{k+1}(W[Q] \cap B)) \leq (2/N_{k+1}^d) \cdot \#(\mathcal{Q}_{k+1}(B)), \quad (3.1)$$

Then at most  $N_{k+1}^d/2$  wafers are bad. We call a slab good if it contains a wafer which is good. Since a slab is the union of  $(N_{k+1}/M_{k+1})^d$  wafers, at most  $M_{k+1}^d/2 = (1/2) \cdot \#(\mathcal{R}_{k+1}(Q_0))$  slabs are bad. Thus if we set

$$\mathcal{R}(Q_0) = \{R \in \mathcal{R}_{k+1}(Q_0) : S[R] \text{ is good}\},$$

then

$$\#(\mathcal{R}(Q_0)) \geq (1/2) \cdot \#(\mathcal{R}_{k+1}(Q_0)). \quad (3.2)$$

For each  $R \in \mathcal{R}(Q_0)$ , we pick  $Q_R \in \mathcal{Q}_{k+1}(R)$  such that  $W[Q_R]$  is good, and define

$$S = \bigcup \{Q_R : R \in \mathcal{R}(Q_0)\}.$$

Equation (3.2) implies  $S$  satisfies Property (B).

Let  $B'$  be the union of all cubes  $Q' \in \mathcal{Q}_{k+1}(T')$  such that there is  $Q \in \mathcal{Q}_{k+1}(S)$  with  $Q \times Q' \in \mathcal{Q}_{k+1}(B)$ . By definition, Property (A) is then satisfied. For each  $Q \in \mathcal{Q}_{k+1}(S)$ ,  $W[Q]$  is good, so (3.1) implies

$$\#\{Q' : Q \times Q' \in \mathcal{Q}_{k+1}(B)\} \leq (2/N_{k+1}^d) \cdot \#(\mathcal{Q}_{k+1}(B)).$$

But  $\#(\mathcal{Q}_{k+1}(S)) \leq \#(\mathcal{R}_{k+1}(T)) \leq (1/r_{k+1})^d = (N_1 \dots N_k)^d M_{k+1}^d$ , so

$$\begin{aligned} \#(\mathcal{Q}_{k+1}(B')) &\leq \#(\mathcal{Q}_{k+1}(S))[(2/N_{k+1}^d) \cdot \#(\mathcal{Q}_{k+1}(B))] \\ &\leq 2(N_1 \dots N_k)^d (M_{k+1}/N_{k+1})^d \#(\mathcal{Q}_{k+1}(B)), \end{aligned}$$

which establishes Property (C).  $\square$

We apply the lemma recursively  $n - 1$  times to continually reduce the dimensionality of the avoidance problem we are considering. Eventually, we obtain the case where  $n = 0$ , and we are in need of a final technique.

**Lemma 19.** *Fix  $n > 1$ . Let  $T \subset [0, 1]^d$  be  $\mathcal{Q}_k$  discretized, and let  $B \subset T$  be  $\mathcal{Q}_{k+1}$  discretized. Suppose*

$$\# \mathcal{Q}_{k+1}(B) \leq \left[ C \cdot 2^{n-1} (N_1 \dots N_k)^{d(n-1)} (M_{k+1}/N_{k+1})^{d(n-1)} \right] (1/l_{k+1})^{dn-m}.$$

and

$$N_{k+1} > \left[ C \cdot 2^n (N_1 \dots N_k)^{2dn} \right]^{1/m} M_{k+1}^{d(n-1)/m} \quad (3.3)$$

Then there exists a  $\mathcal{Q}_{k+1}$  discretized set  $S \subset T$  such that

(A)  $S \cap B = \emptyset$ .

(B) For each  $Q_0 \in \mathcal{Q}_k(T)$ , there is  $\mathcal{R}(Q_0) \subset \mathcal{R}_{k+1}(Q_0)$  with

$$\#(\mathcal{R}(Q_0)) \geq (1/2) \#(\mathcal{R}_{k+1}(Q_0)),$$

such that for each  $R \in \mathcal{R}_{k+1}(Q_0)$ ,

$$\#(\mathcal{Q}_{k+1}(R \cap S)) = \begin{cases} 1 & : R \in \mathcal{R}(Q_0), \\ 0 & : R \notin \mathcal{R}(Q_0). \end{cases}$$

*Proof.* For each  $Q_0 \in \mathcal{Q}_k(T)$ , we set

$$\mathcal{R}(Q_0) = \{R \in \mathcal{R}_{k+1}(Q_0) : \#(\mathcal{Q}_{k+1}(R \cap B)) \leq (2/M_{k+1}^d) \cdot \#(\mathcal{Q}_{k+1}(B))\}.$$

Since  $\mathcal{R}_{k+1}(Q_0) = M_{k+1}^d$ ,

$$\#(\mathcal{R}(Q_0)) \geq \#(\mathcal{R}_{k+1}(Q_0)) - (M_{k+1}^d/2) \geq (1/2) \cdot \#(\mathcal{R}_{k+1}(T)).$$

Now (3.3) implies that for each  $R \in \mathcal{R}(Q_0)$ ,

$$\begin{aligned} \#(\mathcal{Q}_{k+1}(R \cap B)) &\leq (2/M_{k+1}^d) \cdot \#(\mathcal{Q}_{k+1}(B)) \\ &\leq (2/M_{k+1}^d) \left( C \cdot 2^{n-1} (N_1 \dots N_k)^{2dn} (M_{k+1}/N_{k+1})^{d(n-1)} \right) \\ &= \left[ 2^n C (N_1 \dots N_k)^{2dn} \right] \left( M_{k+1}^{d(n-2)} / N_{k+1}^{m-d} \right) \\ &< (N_{k+1}/M_{k+1})^d \\ &= \mathcal{Q}_{k+1}(R) \end{aligned}$$

Thus for each  $R \in \mathcal{R}(Q_0)$ , we can find  $Q_R \in \mathcal{Q}_{k+1}(R)$  such that  $Q_R \cap B = \emptyset$ . And so if we set

$$S = \bigcup \{Q_R : R \in \mathcal{R}(Q_0), Q_0 \in \mathcal{Q}_k(T)\},$$

then (A) and (B) are satisfied.  $\square$

**Corollary 20.** *Let  $f : [0, 1]^{dn} \rightarrow [0, 1]^m$  be  $C^2$ , and have full rank at every point  $(x_1, \dots, x_n)$  with all  $x_1, \dots, x_n$  distinct, and  $f(x_1, \dots, x_n) = 0$ . Then there exists a universal constant  $C$  depending only on  $f$  such that, if (3.3) is satisfied, then for any disjoint,  $\mathcal{Q}_k$  discretized sets  $T_1, \dots, T_n \subset [0, 1]^d$ , we can find  $\mathcal{Q}_{k+1}$  discretized sets  $S_1 \subset T_1, \dots, S_n \subset T_n$  such that*

(A) *If  $x_1 \in S_1, \dots, x_n \in S_n$ , then  $f(x_1, \dots, x_n) \neq 0$ .*

(B) *For each  $k$ , and for each  $Q_0 \in \mathcal{Q}_k(T_k)$ , there is  $\mathcal{R}(Q_0) \subset \mathcal{R}_{k+1}(Q_0)$  with*

$$\#(\mathcal{R}(Q_0)) \geq (1/2) \cdot \#(\mathcal{R}_{k+1}(Q_0)),$$

and for each  $R \in \mathcal{R}_{k+1}(Q_0)$ ,

$$\#(\mathcal{Q}_{k+1}(R \cap S)) = \begin{cases} 1 & : R \in \mathcal{R}(Q_0), \\ 0 & : R \notin \mathcal{R}(Q_0). \end{cases}$$

*Proof.* Since  $f$  is  $C^2$  and has full rank on the set

$$V(f) = \{(x_1, \dots, x_n) \in \mathcal{C}^n(\mathbf{R}^d) : f(x_1, \dots, x_n) = 0\},$$

the implicit function theorem implies  $V(f)$  is a smooth manifold of dimension  $nd - m$  in  $\mathbf{R}^{dn}$ , and the coarea formula implies the existence of a constant  $C$  such that for each  $k$ ,

$$\#\{Q \in \mathcal{Q}_k^{dn} : Q \cap V(f) \neq \emptyset\} \leq C/l_k^{dn-m}.$$

To apply Lemma 18 and 19, we set

$$B = \#\{Q \in \mathcal{Q}_k^{dn} : Q \cap V(f) \neq \emptyset\}.$$

Applying Lemma 18 iteratively  $n - 1$  times, then finishing with Lemma 19, constructs the sets  $S_1, \dots, S_n$ .  $\square$

**Remark.** The values of  $N_1, \dots, N_k$ , exponentials  $2^k$ , and constants, are roughly insignificant in comparison to  $M_{k+1}$  and  $N_{k+1}$ , when these numbers are hyperdyadic for a suitably large constant, or grow faster than hyperdyadic. In particular, we can roughly read off the Hausdorff dimension of the set  $m/(n - 1)$  we construct from the inverse power of  $M_{k+1}$  in (3.3), i.e. that  $N_{k+1} \gtrsim M_{k+1}^{d(n-1)/m}$ .

Just like in Keleti's proof, Pramanik and Fraser's technique applies a discrete result, Corollary 20, iteratively at many scales to obtain a high dimensional set avoiding the zeroes of a function. We construct a nested family  $\{X_k : k \geq 0\}$  of  $\mathcal{Q}_k$  discretized sets, converging to a set  $X$ , which we will show is translate avoiding. We initialize  $X_0 = [0, 1]$ . Our queue shall consist of  $n$  tuples of disjoint intervals  $(T_1, \dots, T_n)$ , all of the same length, which initially consists of all possible tuples of intervals in  $\mathcal{Q}_1^d([0, 1]^d)$ . To construct the sequence  $\{X_k\}$ , we perform the following iterative procedure:

---

**Algorithm 2** Construction of the Sets  $\{X_k\}$ 


---

Set  $k = 0$

**Repeat**

Take off an  $n$  tuple  $(T'_1, \dots, T'_n)$  from the front of the queue

Set  $T_i = T'_i \cap X_k$  for each  $i$

Apply Corollary 20 to the sets  $T_1, \dots, T_d$ , obtaining  $\mathcal{Q}_{k+1}$  discretized sets  $S_1, \dots, S_n$  satisfying Properties (A), (B), and (C) of that Lemma.

Set  $X_{k+1} = X_k - \bigcup_{i=1}^n T_i - S_i$ .

Add all  $n$  tuples of disjoint cubes  $(T'_1, \dots, T'_n)$  in  $\mathcal{Q}_{k+1}^d(X_{k+1})$  to the back of the queue.

Increase  $k$  by 1.

---

**Lemma 21.** *The set  $X$  constructed by the procedure avoids the configuration*

$$\mathcal{C} = \{(x_1, \dots, x_n) \in \mathcal{C}^n(\mathbf{R}^d) : f(x_1, \dots, x_n) = 0\}.$$

*Proof.* Suppose  $x_1, \dots, x_n \in X$  are distinct. Then at some stage  $k$ ,  $x_1, \dots, x_n$  lie in disjoint cubes  $T'_1, \dots, T'_n \in \mathcal{Q}_k^d(X_k)$ , for some large  $k$ . At this stage,  $(T'_1, \dots, T'_n)$  is added to the back of the queue, and therefore, at some much later stage  $N$ , the tuple  $(T'_1, \dots, T'_n)$  is taken off the front. Sets  $S_1 \subset T'_1, \dots, S_n \subset T'_n$  are constructed satisfying Property (A) of Corollary 20. Since  $x_1, \dots, x_n \in X$ , we must have  $x_i \in S_i$  for each  $i$ , so  $f(x_1, \dots, x_n) \neq 0$ .  $\square$

What remains is to bound the Hausdorff dimension of  $X$ . For this, one must use a sequence of factors  $\{N_k\}$  which is *just* faster than a hyperdyadic sequence of factors.

**Theorem 22.** *If  $M_k = 2^{k \log k}$ ,  $X$  has Hausdorff dimension exceeding  $m/(n-1)$ .*

*Proof.* First, we construct the canonical measure  $\mu$  on  $X$ . Property (B) of Corollary 20 implies that for each  $Q \in \mathcal{Q}_{k+1}^d$ ,  $\mu(Q) \leq 2/M_{k+1}^d \mu(Q^*)$ , which implies that for  $Q \in \mathcal{Q}_k^d$ ,  $\mu(Q) \leq 2^k/(M_k \dots M_1)^d$ . By Lemma 13, it suffices to show that for each  $\varepsilon > 0$ ,

$$\frac{2^k}{(M_{k+1} \dots M_1)^d} \lesssim_{\varepsilon} \frac{1}{(N_1 \dots N_{k+1})^{m/(n-1)(1-\varepsilon)}} \quad (3.4)$$

Set

$$N_{k+1} = A(N_1 \dots N_k)^{2dn/m} M_{k+1}^{d(n-1)/m},$$

where  $A$  is an arbitrary constant not depending on  $k$  so that (3.3) holds. Then inequality (3.4) is then implied if

$$M_{k+1} \gtrsim_{\varepsilon} \left( (N_1 \dots N_k)^{6dn} [2^k A^d] \right)^{1/\varepsilon}.$$

This equation is satisfied if  $M_k = 2^{2^{k^2}}$ .

□

# Chapter 4

## Avoiding Rough Sets

In the last chapter, we saw that many authors have considered the pattern avoidance problem for configurations  $\mathcal{C}$  which take the form of many general classes of smooth shapes; in Mathé’s work,  $\mathcal{C}$  can take the form of an algebraic variety of low degree, and in Pramanik and Fraser’s work,  $\mathcal{C}$  can take the form of a smooth manifold. In this chapter, we consider the pattern avoidance problem for an even more general class of ‘rough’ patterns, that are the countable union of sets with controlled lower Minkowski dimension.

**Theorem 23.** *Let  $\alpha \geq d$ , and suppose  $\mathcal{C} \subset \mathcal{C}^n(\mathbf{R}^d)$  is the countable union of pre-compact sets, each with lower Minkowski dimension at most  $\alpha$ . Then there exists a set  $X \subset [0, 1]^d$  with Hausdorff dimension at least  $(nd - \alpha)/(n - 1)$  avoiding  $\mathcal{C}$ .*

A major question in modern geometric measure theory is whether sufficiently large sets are forced to contain copies of certain patterns. Intuitively, one expects the answer to be yes, and many results in the literature support this intuition. For example, the Lebesgue density theorem implies that a set of positive Lebesgue measure contains an affine copy of every finite set. And any set  $X \subset \mathbf{R}^2$  with Hausdorff dimension exceeding one must contain three collinear points. On the other hand, there is a distinct genre of results that challenges this intuition. Keleti [9] constructs a set  $X \subset \mathbf{R}$  that avoids all solutions of the equation  $x_2 - x_1 = x_4 - x_3$  with  $x_1 < x_2 \leq x_3 < x_4$ , and which consequently does not contain any nontrivial arithmetic progression. Given any triangle, Falconer [?] constructs a full dimensional planar set that does not contain any similar copy of the vertex set of the triangle. Maga [10] provides a set  $X \subset \mathbf{R}^2$  of full Hausdorff dimension such that no four points in  $X$  form the vertices of a parallelogram. The pattern

avoidance problem (informally stated) asks: for a given pattern, how large can the dimension of a set  $X \subset \mathbf{R}^d$  be before it is forced to contain a copy of this pattern?

One way to formalize the notion of a pattern is as follows. If  $d \geq 1$  and  $n \geq 2$  are integers, we define a pattern to be a set  $Z \subset \mathbf{R}^{dn}$ . We say that a set  $X \subset \mathbf{R}^d$  avoids the pattern  $Z$  if for every  $n$ -tuple of distinct points  $x_1, \dots, x_n \in X$ , we have  $(x_1, \dots, x_n) \notin Z$ . For example, a set  $X \subset \mathbf{R}^2$  does not contain three collinear points if and only if it avoids the pattern

$$Z = \{(x_1, x_2, x_3) \in \mathbf{R}^6 : |(x_1 - x_2) \wedge (x_1 - x_3)| = 0\}.$$

Similarly, a set  $X \subset \mathbf{R}^2$  avoids the pattern

$$Z = \{(x_1, x_2, x_3, x_4) \in \mathbf{R}^8 : x_1 + x_4 = x_2 + x_3\}$$

if and only if no four points in  $X$  form the vertices of a (possibly degenerate) parallelogram.

A number of recent articles have established pattern avoidance results for increasingly general patterns. In [11], Máthé constructs a set  $X \subset \mathbf{R}^d$  that avoids a pattern specified by a countable union of algebraic varieties of controlled degree. In [6], Fraser and the second author consider the pattern avoidance problem for countable unions of  $C^1$  manifolds. In this paper, we consider the pattern avoidance problem for an even more general class of ‘rough’ patterns  $Z \subset \mathbf{R}^{dn}$ , that are the countable union of sets with controlled lower Minkowski dimension.

**Theorem 24.** *Let  $\alpha \geq d$ , and let  $Z \subset \mathbf{R}^{dn}$  be a countable union of compact sets, each with lower Minkowski dimension at most  $\alpha$ . Then there exists a set  $X \subset [0, 1]^d$  with Hausdorff dimension at least  $(nd - \alpha)/(n - 1)$  such that whenever  $x_1, \dots, x_n \in X$  are distinct, we have  $(x_1, \dots, x_n) \notin Z$ .*

### Remarks.

1. When  $\alpha < d$ , the pattern avoidance problem is trivial, since  $X = [0, 1]^d - \pi(Z)$  is full dimensional and solves the pattern avoidance problem, where  $\pi(x_1, \dots, x_n) = x_1$  is a projection map from  $\mathbf{R}^{dn}$  to  $\mathbf{R}^d$ . We will therefore assume that  $\alpha \geq d$  in our proof of the theorem. Note that obtaining a full dimensional set in the case  $\alpha = d$ , however, is still interesting.
2. Theorem 24 is trivial when  $\alpha = dn$ , since we can set  $X = \emptyset$ . We will therefore assume that  $\alpha < dn$  in our proof of the theorem.



3. When  $Z$  is a countable union of smooth manifolds in  $\mathbf{R}^{nd}$  of co-dimension  $m$ , we have  $\alpha = nd - m$ . In this case Theorem 24 yields a set in  $\mathbf{R}^d$  with Hausdorff dimension at least  $(nd - \alpha)/(n - 1) = m/(n - 1)$ . This recovers Theorem 1.1 and 1.2 from [6], making Theorem 24 a generalization of these results.
4. Since Theorem 24 does not require any regularity assumptions on the set  $Z$ , it can be applied in contexts that cannot be addressed using previous methods. Two such applications, new to the best of our knowledge, have been recorded in Section 4.5; see Theorems 32 and 33 there.

The set  $X$  in Theorem 24 is obtained by constructing a sequence of approximations to  $X$ , each of which avoids the pattern  $Z$  at different scales. For a sequence of lengths  $l_k \searrow 0$ , we construct a nested family of sets  $\{X_k\}$ , where  $X_k$  is a union of cubes of sidelength  $l_k$  that avoids  $Z$  at scales close to  $l_k$ . The set  $X = \bigcap X_k$  avoids  $Z$  at all scales. While this proof strategy is not new, our method for constructing the sets  $\{X_k\}$  has several innovations that simplify the analysis of the resulting set  $X = \bigcap X_k$ . In particular, through a probabilistic selection process we are able to avoid the complicated queuing techniques used in [9] and [6], that required storage of data from each step of the iterated construction, to be retrieved at a much later stage of the construction process.

At the same time, our construction continues to share certain features with [6]. For example, between each pair of scales  $l_{k-1}$  and  $l_k$ , we carefully select an intermediate scale  $r_k$ . The set  $X_k \subset X_{k-1}$  avoids  $Z$  at scale  $l_k$ , and it is ‘evenly distributed’ at scale  $r_k$ ; the set  $X_k$  is a union of cubes of length  $l_k$  whose midpoints resemble (a large subset of) generalized arithmetic progression of step size  $r_k$ . The details of a single step of this construction are described in Section 4.2. In Section 4.3, we explain how the length scales  $l_k$  and  $r_k$  for  $X$  are chosen, and prove its avoidance property. In Section 4.4 we analyze the size of  $X$  and show that it satisfies the conclusions of Theorem 24.

## 4.1 Frequently Used Notation and Terminology

- (A) A *dyadic length* is a number  $l$  of the form  $2^{-k}$  for some non-negative integer  $k$ .
- (B) Given a length  $l > 0$ , we let  $\mathcal{B}_l^d$  denote the set of all closed cubes in  $\mathbf{R}^d$  with

sidelength  $l$  and corners on the lattice  $(l \cdot \mathbf{Z})^d$ , i.e.

$$\mathcal{B}_l^d = \{[a_1, a_1 + l] \times \cdots \times [a_d, a_d + l] : a_k \in l \cdot \mathbf{Z}\}.$$

- (C) A set  $E \subset \mathbf{R}^d$  is  $l$  discretized if it is a union of cubes in  $\mathcal{B}_l^d$ . For any set  $E \subset \mathbf{R}^d$ , and any length  $l > 0$ , we let

$$E(l) = \bigcup \{I \in \mathcal{B}_l^d : I \cap E \neq \emptyset\}.$$

Then  $E(l)$  is the smallest  $l$  discretized set with  $E \subset E(l)^\circ$ . Given an  $l$  discretized set  $E$ , we let

$$\mathcal{B}_l^d(E) = \{I \in \mathcal{B}_l^d : I \subset E\}.$$

Then  $E = \bigcup \mathcal{B}_l^d(E)$ .

- (D) The *lower Minkowski dimension* of a bounded set  $Z \subset \mathbf{R}^d$  is defined as

$$\dim_{\mathbf{M}}(Z) = \liminf_{l \rightarrow 0} \frac{\log [\#\mathcal{B}_l^d(Z(l))]}{\log[1/l]}.$$

- (E) If  $\alpha \geq 0$  and  $\delta > 0$ , we define the dyadic Hausdorff content of a set  $E \subset \mathbf{R}^d$  as

$$H_\delta^\alpha(E) = \inf \left\{ \sum_{k=1}^{\infty} l_k^\alpha : E \subset \bigcup_{k=1}^{\infty} I_k \right\},$$

where the infimum is taken over all families of cubes  $\{I_k\}$  such that for all  $k$ ,  $I_k \in \mathcal{B}_{l_k}^d$ , and  $l_k$  is a dyadic length with  $l_k \leq \delta$ . The  $\alpha$ -dimensional dyadic Hausdorff measure  $H^\alpha$  on  $\mathbf{R}^d$  is  $H^\alpha(E) = \lim_{\delta \rightarrow 0} H_\delta^\alpha(E)$ , and the *Hausdorff dimension* of a set  $E$  is  $\dim_{\mathbf{H}}(E) = \inf\{\alpha \geq 0 : H^\alpha(E) = 0\}$ .

- (F) Given  $K \in \mathcal{B}_l^{dn}$ , we can decompose  $K$  as  $K_1 \times \cdots \times K_n$  for unique cubes  $K_1, \dots, K_n \in \mathcal{B}_l^d$ . We say  $K$  is *strongly non-diagonal* if the cubes  $K_1, \dots, K_n$  are distinct. Strongly non-diagonal cubes will play an important role in Section 4.2, when we solve a discrete version of Theorem 24.

- (G) Adopting the terminology of [8], we say a collection of sets  $\{U_k\}$  is a *strong cover* of a set  $E$  if  $E \subset \limsup U_k$ , which means every element of  $E$  is contained in infinitely many of the sets  $U_k$ . This idea will be useful in Section

4.3.

(H) A *Frostman measure* of dimension  $\alpha$  is a non-zero compactly supported probability measure  $\mu$  on  $\mathbf{R}^d$  such that for every dyadic cube  $I$  of sidelength  $l$ ,  $\mu(I) \lesssim l^\alpha$ . Note that a measure  $\mu$  satisfies this inequality for every cube  $I$  if and only if it satisfies the inequality for cubes whose sidelengths are dyadic lengths. *Frostman's lemma* says that

$$\dim_{\mathbf{H}}(E) = \sup \left\{ \alpha : \begin{array}{l} \text{there is a Frostman measure of} \\ \text{dimension } \alpha \text{ supported on } E \end{array} \right\}.$$

## 4.2 Avoidance at Discrete Scales

In this section we describe a method for avoiding  $Z$  at a single scale. We apply this technique in Section 4.3 at many scales to construct a set  $X$  avoiding  $Z$  at all scales. This single scale avoidance technique is the core building block of our construction, and the efficiency with which we can avoid  $Z$  at a single scale has direct consequences on the Hausdorff dimension of the set  $X$  obtained in Theorem 24.

At a single scale, we solve a discretized version of the problem, where all sets are unions of cubes at two dyadic lengths  $l > s$ . In this discrete setting,  $Z$  is replaced by a discretized version of itself, a union of cubes in  $\mathcal{B}_s^{dn}$  denoted by  $G$ . Given a set  $E$ , which is a union of cubes in  $\mathcal{B}_l^d$ , our goal is to construct a set  $F \subset E$  that is a union of cubes in  $\mathcal{B}_s^d$ , such that  $F^n$  is disjoint from strongly non-diagonal cubes (see Definition (F)) in  $\mathcal{B}_s^{dn}(G)$ . Using the setup introduced at the end of the introduction, we will later choose  $l = l_k$ ,  $s = l_{k+1}$ , and  $E = X_k$ . The set  $X_{k+1}$  will be defined as the set  $F$  constructed.

In order to ensure the final set  $X$  obtained in Theorem 24 has large Hausdorff dimension regardless of the rapid decay of scales used in the construction of  $X$ , it is crucial that  $F$  is uniformly distributed at intermediate scales between  $l$  and  $s$ . We achieve this by decomposing  $E$  into sub-cubes in  $\mathcal{B}_r^d$  for some intermediate scale  $r \in [s, l]$ , and distributing  $F$  as evenly among these intermediate sub-cubes as possible. This is possible assuming a mild regularity condition on the number of cubes in  $G$ , i.e. Equation (4.1).

**Lemma 25.** *Fix two distinct dyadic lengths  $l$  and  $s$ , with  $l > s$ . Let  $E \subseteq [0, 1]^d$  be a nonempty,  $l$  discretized set, and let  $G \subset \mathbf{R}^{dn}$  be a nonempty  $s$  discretized set*

such that

$$(l/s)^d \leq \#\mathcal{B}_s^{dn}(G) \leq \frac{1}{2}(l/s)^{dn}. \quad (4.1)$$

Then there exists a dyadic length  $r \in [s, l]$  of size

$$r \sim \left( l^{-d} s^{dn} \#\mathcal{B}_s^{dn}(G) \right)^{\frac{1}{d(n-1)}}, \quad (4.2)$$

and an  $s$  discretized set  $F \subset E$  satisfying the following three properties:

- (A) Avoidance: For any  $n$  distinct cubes  $I_1, \dots, I_n \in \mathcal{B}_s^d(F)$ ,  $I_1 \times \dots \times I_n \notin \mathcal{B}_s^d(G)$ .
- (B) Non-Concentration: For any  $I \in \mathcal{B}_r^d(E)$ ,  $\#(\mathcal{B}_s^d(F \cap I)) \leq 1$ .
- (C) Large Size: For every  $I \in \mathcal{B}_l^d(E)$ ,  $\#(\mathcal{B}_s^d(F \cap I)) \geq \#(\mathcal{B}_r^d(E \cap I))/2$ .

**Remark.** Property (A) says that  $F$  avoids strongly non-diagonal cubes in  $\mathcal{B}_s^{dn}(G)$ . Properties (B) and (C) together imply that for every  $I \in \mathcal{B}_l^d(E)$ , at least half of the cubes in  $\mathcal{B}_r^d(I)$  contribute a single sub-cube of sidelength  $s$  to  $F$ ; the rest contribute none.

*Proof.* Let  $r$  be the smallest dyadic length at least as large as  $R$ , where

$$R = \left( 2l^{-d} s^{dn} \#\mathcal{B}_s^{dn}(G) \right)^{\frac{1}{d(n-1)}}. \quad (4.3)$$

This choice of  $r$  satisfies (4.2). The inequalities in (4.1) ensure that  $r \in [s, l]$ ; more precisely, the left inequality in (4.1) implies  $R$  is bounded from below by  $s$ , and the right inequality implies  $R$  is bounded from above by  $l$ . The minimality of  $r$  ensures  $s \leq r \leq l$ .

For each  $I \in \mathcal{B}_r^d(E)$ , let  $J_I$  be an element of  $\mathcal{B}_s^d(I)$  chosen uniformly at random; these choices are independent as  $I$  ranges over the elements of  $\mathcal{B}_r^d(E)$ . Define

$$U = \bigcup \left\{ J_I : I \in \mathcal{B}_r^d(E) \right\},$$

and

$$\mathcal{K}(U) = \{K \in \mathcal{B}_s^{dn}(G) : K \in U^n, K \text{ strongly non-diagonal}\}.$$

Note that the sets  $U$  and  $\mathcal{K}(U)$  are random sets, in the sense that they depend on the random variables  $\{J_I\}$ . Define

$$F(U) = \bigcup \left( \mathcal{B}_s^d(U) - \{\pi(K) : K \in \mathcal{K}(U)\} \right), \quad (4.4)$$

where  $\pi: \mathbf{R}^{dn} \rightarrow \mathbf{R}^d$  is the projection map  $(x_1, \dots, x_n) \mapsto x_1$ , for  $x_i \in \mathbf{R}^d$ . Given any strongly non-diagonal cube  $K = J_1 \times \dots \times J_n \in \mathcal{B}_s^{dn}(G)$ , either  $K \notin \mathcal{B}_s^{dn}(U^n)$ , or  $K \in \mathcal{B}_s^{dn}(U^n)$ . If the former occurs then  $K \notin \mathcal{B}_s^{dn}(F(U)^n)$  since  $F(U) \subset U$ , so  $\mathcal{B}_s^{dn}(F(U)^n) \subset \mathcal{B}_s^{dn}(U^n)$ . If the latter occurs then  $K \in \mathcal{K}(U)$ , and since  $\pi(K) = J_1$ ,  $J_1 \notin \mathcal{B}_s^d(F(U))$ . In either case,  $K \notin \mathcal{B}_s^{dn}(F(U)^n)$ , so  $F(U)$  satisfies Property (A). By construction,  $U$  contains at most one sub cube  $J \in \mathcal{B}_s^{dn}$  for each  $I \in \mathcal{B}_l^{dn}(E)$ . Since  $F(U) \subset U$ ,  $F(U)$  satisfies Property (B). Thus the set  $F(U)$  satisfies Properties (A) and (B) regardless of which values are assumed by the random variables  $\{J_I\}$ . Next we will show that with non-zero probability, the set  $F(U)$  satisfies Property (C).

For each cube  $J \in \mathcal{B}_s^d(E)$ , there is a unique ‘parent’ cube  $J^* \in \mathcal{B}_r^d(E)$  such that  $J \subset J^*$ . Since  $\#\mathcal{B}_s^d(J^*) = (r/s)^d$ , and  $J_{J^*}$  is chosen uniformly at random from  $\mathcal{B}_s^d(J^*)$ ,

$$\mathbf{P}(J \subset U) = \mathbf{P}(J_{J^*} = J) = (s/r)^d.$$

The cubes  $\{J_I\}$  are chosen independently, so if  $J_1, \dots, J_n$  are distinct cubes in  $\mathcal{B}_s^d(E)$ , then

$$\mathbf{P}(J_1, \dots, J_n \in U) = \begin{cases} (s/r)^{dn} & \text{if } J_1, \dots, J_n \text{ have distinct parents,} \\ 0 & \text{otherwise.} \end{cases} \quad (4.5)$$

Let  $K = J_1 \times \dots \times J_n \in \mathcal{B}_s^{dn}(G)$ . If the cubes  $J_1, \dots, J_n$  are distinct, we deduce from (4.5) that

$$\mathbf{P}(K \subset U^n) = \mathbf{P}(J_1, \dots, J_n \in U) \leq (s/r)^{dn}. \quad (4.6)$$

By (4.6), linearity of expectation, and (4.3),

$$\mathbf{E}(\#\mathcal{K}(U)) = \sum_{K \in \mathcal{B}_s^{dn}(G)} \mathbf{P}(K \subset U^n) \leq \#\mathcal{B}_s^{dn}(G) \cdot (s/r)^{dn} \leq 0.5 \cdot (l/r)^d.$$

In particular, there exists at least one (non-random) set  $U_0$  such that

$$\#\mathcal{K}(U_0) \leq \mathbf{E}(\#\mathcal{K}(U)) \leq 0.5 \cdot (l/r)^d. \quad (4.7)$$

In other words,  $F(U_0) \subset U_0$  is obtained by removing at most  $0.5 \cdot (l/r)^d$  cubes in  $\mathcal{B}_s^d$  from  $U_0$ . For each  $I \in \mathcal{B}_l^d(E)$ , we know that  $\#\mathcal{B}_s^d(I \cap U_0) = (l/r)^d$ . Combining

this with (4.7), we arrive at the estimate

$$\begin{aligned}
\#\mathcal{B}_s^d(I \cap F(U_0)) &= \mathcal{B}_s^d(I \cap U_0) - \#\{\pi(K) : K \in \mathcal{K}(U_0), \pi(K) \in F(U_0)\} \\
&\geq \mathcal{B}_s^d(I \cap U_0) - \#\mathcal{K}(U_0) \\
&\geq (l/r)^d - 0.5 \cdot (l/r)^d \geq 0.5 \cdot (l/r)^d
\end{aligned}$$

In other words,  $F(U_0)$  satisfies Property (C). Setting  $F = F(U_0)$  completes the proof.  $\square$

**Remarks.**

1. While Lemma 25 uses probabilistic arguments, the conclusion of the lemma is not a probabilistic statement. In particular, one can find a suitable  $F$  constructively by checking every possible choice of  $U$  (there are finitely many) to find one particular choice  $U_0$  which satisfies (4.7), and then defining  $F$  by (4.4). Thus the set we obtain in Theorem 24 exists by purely constructive means.
2. At this point, it is possible to motivate the numerology behind the dimension bound  $\dim(X) \geq (dn - \alpha)/(n - 1)$  from Theorem 24, albeit in the context of Minkowski dimension. We will pause to do so here before returning to the proof of Theorem 24. For simplicity, let  $\alpha > d$ , and suppose that  $Z \subset \mathbf{R}^{dn}$  satisfies

$$\#\mathcal{B}_s^{dn}(Z(s)) \sim s^{-\alpha} \quad \text{for every } s \in (0, 1]. \quad (4.8)$$

Let  $l = 1$ ,  $E = [0, 1]^d$ , and let  $s > 0$  be a small parameter. If  $s$  is chosen sufficiently small compared to  $d, n$ , and  $\alpha$ , then (4.1) is satisfied with  $G = \bigcup \mathcal{B}_s^{dn}(Z(s))$ . We can then apply Lemma 25 to find a dyadic scale  $r \sim s^{(dn-\alpha)/d(n-1)}$  and a set  $F$  that avoids the strongly non-diagonal cubes of  $\mathcal{B}_s^{dn}(Z(s))$ . The set  $F$  is a union of approximately  $r^{-d} \sim s^{-(dn-\alpha)/(n-1)}$  cubes of sidelength  $s$ . Thus informally, the set  $F$  resembles a set with Minkowski dimension  $\alpha$  when viewed at scale  $s$ .

The set  $X$  constructed in Theorem 24 will be obtained by applying Lemma 25 iteratively at many scales. At each of these scales,  $X$  will resemble a set of Minkowski dimension  $(dn - \alpha)/(n - 1)$ . A careful analysis of the construction (performed in Section 4.4) shows that  $X$  actually has Hausdorff dimension at least  $(dn - \alpha)/(n - 1)$ .

3. Lemma 25 is the core method in our avoidance technique. The remaining argument is fairly modular. If, for a special case of  $Z$ , one can improve

the result of Lemma 25 so that  $r$  is chosen on the order of  $s^{\beta/d}$ , then the remaining parts of our paper can be applied near verbatim to yield a set  $X$  with Hausdorff dimension  $\beta$ , as in Theorem 24.

### 4.3 Fractal Discretization

In this section we construct the set  $X$  from Theorem 24 by applying Lemma 25 at many scales. Let us start by fixing a strong cover  $Z$  that we will work with in the sequel.

**Lemma 26.** *Let  $Z \subset \mathbf{R}^{dn}$  be a countable union of bounded sets with Minkowski dimension at most  $\alpha$ , and let  $\varepsilon_k \searrow 0$  with  $2\varepsilon_k < dn - \alpha$  for all  $k$ . Then there exists a sequence of dyadic lengths  $\{l_k\}$  and a sequence of sets  $\{Z_k\}$ , such that*

- (A) Strong Cover: *The interiors  $\{Z_k^\circ\}$  of the sets  $\{Z_k\}$  form a strong cover of  $Z$ .*
- (B) Discreteness: *For all  $k \geq 0$ ,  $Z_k$  is an  $l_k$  discretized subset of  $\mathbf{R}^{dn}$ .*
- (C) Sparsity: *For all  $k \geq 0$ ,  $l_k^{-d} \leq \#\mathcal{B}_{l_k}^{dn}(Z_k) \leq l_k^{-\alpha-\varepsilon_k}$ .*
- (D) Rapid Decay: *For all  $k > 1$ ,*

$$l_k^{dn-\alpha-\varepsilon_k} \leq 0.5 \cdot l_{k-1}^{dn}, \quad (4.9)$$

$$l_k^{\varepsilon_k} \leq l_{k-1}^{2d}. \quad (4.10)$$

*Proof.* We can write  $Z \subset \bigcup_{i=1}^{\infty} Y_i$ , with  $\dim_{\mathbf{M}}(Y_i) \leq \alpha$  for each  $i$ . Without loss of generality, we may assume that for any  $l$ ,

$$\#\mathcal{B}_l^{nd}(Y_i(l)) \geq l^{-d}. \quad (4.11)$$

If (4.11) fails to be satisfied for some set  $Y_i$ , we consider the  $d$  dimensional hyperplane

$$H = \{(x_1, \dots, x_1) : x_1 \in [0, 1]^d\}.$$

and replace  $Y_i$  with  $Y_i \cup H$ . Since  $\dim_{\mathbf{M}}(Y_i) \leq \alpha$  for each index  $i$ . Let  $\{i_k\}$  be a sequence of integers that repeats each integer infinitely often.

The lengths  $\{l_k\}$  and sets  $\{Z_k\}$  are defined inductively. As a base case, set  $l_0 = 1$  and  $Z_0 = [0, 1]^{nd}$ . Suppose that the lengths  $l_0, \dots, l_{k-1}$  have been chosen. Since

$\dim_{\mathbf{M}}(Y_{i_k}) \leq \alpha$ , and  $\varepsilon_k > 0$ , Definition (D) implies that there exists arbitrarily small lengths  $l$  that satisfy

$$\#\mathcal{B}_l^{dn}(Y_{i_k}(l)) \leq l^{-\alpha-\varepsilon_k}. \quad (4.12)$$

In particular, we can choose a dyadic length  $l = l_k$  small enough to satisfy (4.9), (4.10), and (4.12). With this choice of  $l_k$ , Property (D) is satisfied. Define  $Z_k = Y_{i_k}(l_k)$ . This choice of  $Z_k$  clearly satisfies Property (B), and Property (C) is implied by (4.11) and (4.12).

It remains to verify that the sets  $\{Z_k^\circ\}$  strongly cover  $Z$ . Fix a point  $z \in Z$ . Then there exists an index  $i$  such that  $z \in Y_i$ , and there is a subsequence  $k_1, k_2, \dots$  such that  $i_{k_j} = i$  for each  $j$ . But then  $z \in Y_i \subset Z_{i_{k_j}}^\circ$ , so  $z$  is contained in each of the sets  $Z_{i_{k_j}}^\circ$ , and thus  $z \in \limsup Z_i^\circ$ .  $\square$

To construct  $X$ , we consider a nested, decreasing family of sets  $\{X_k\}$ , where each  $X_k$  is an  $l_k$  discretized subset of  $\mathbf{R}^d$ . We then set  $X = \bigcap X_k$ . The goal is to choose  $X_k$  such that  $X_k^n$  does not contain any *strongly non diagonal* cubes in  $Z_k$ .

**Lemma 27.** *Let  $\{l_k\}$  be a sequence of positive numbers converging to zero, and let  $Z \subset \mathbf{R}^{dn}$ . Let  $\{Z_k\}$  be a sequence of sets, with each  $Z_k$  an  $l_k$  discretized subset of  $\mathbf{R}^{dn}$ , such that the interiors  $\{Z_k^\circ\}$  strongly cover  $Z$ . For each index  $k$ , let  $X_k$  be an  $l_k$  discretized subset of  $\mathbf{R}^d$ . Suppose that for each  $k$ ,  $\mathcal{B}_{l_k}^{dn}(X_k^n) \cap \mathcal{B}_{l_k}^{dn}(Z_k)$  contains no strongly non diagonal cubes. If  $X = \bigcap X_k$ , then for any distinct  $x_1, \dots, x_n \in X$ , we have  $(x_1, \dots, x_n) \notin Z$ .*

*Proof.* Let  $z \in Z$  be a point with distinct coordinates  $z_1, \dots, z_n$ . Define

$$\Delta = \{(w_1, \dots, w_n) \in \mathbf{R}^{dn} : \text{there exists } i \neq j \text{ such that } w_i = w_j\}.$$

Then  $d(\Delta, z) > 0$ , where  $d$  is the Hausdorff distance between  $\Delta$  and  $z$ . Since  $\{Z_k\}$  strongly covers  $Z$ , there is a subsequence  $\{k_m\}$  such that  $z \in Z_{k_m}^\circ$  for every index  $m$ . Since  $l_k$  converges to 0 and thus  $l_{k_m}$  converges to 0, if  $m$  is sufficiently large then  $\sqrt{dn} \cdot l_{k_m} < d(\Delta, z)$ . Note that  $\sqrt{dn} \cdot l_{k_m}$  is the diameter of a cube in  $\mathcal{B}_{l_{k_m}}^{dn}$ . For such a choice of  $m$ , any cube  $I \in \mathcal{B}_{l_{k_m}}^d$  which contains  $z$  is strongly non-diagonal. Furthermore,  $z \in Z_{k_m}^\circ$ . Since  $X_{k_m}$  and  $Z_{k_m}$  share no cube which contains  $z$ , this implies  $z \notin X_{k_m}$ . In particular, this means  $z \notin X^n$ .  $\square$

All that remains is to apply the discrete lemma to choose the sets  $X_k$ .



**Lemma 28.** *Given a sequence of dyadic length scales  $\{l_k\}$  obeying, (4.9), (4.10), and (4.12) as above, there exists a sequence of sets  $\{X_k\}$  and a sequence of dyadic intermediate scales  $\{r_k\}$  with  $l_k \leq r_k \leq l_{k-1}$  for each  $k \geq 1$ , such that each set  $X_k$  is an  $l_k$  discretized subset of  $[0, 1]^d$  such that  $\mathcal{B}_{l_k}^{dn}(X_k^d) \cap \mathcal{B}_{l_k}^{dn}(Z_k)$  contains no strongly non diagonal cubes. Furthermore, for each index  $k \geq 1$  we have*

$$r_k \lesssim l_k^{(dn-\alpha-\varepsilon_k)/d(n-1)}, \quad (4.13)$$

$$\#\mathcal{B}_{l_k}^d(X_k \cap I) \geq 0.5 \cdot (l_{k-1}/r_k)^d \quad \text{for each } I \in \mathcal{B}_{l_{k-1}}^d(X_{k-1}), \quad (4.14)$$

$$\#\mathcal{B}_{l_k}^d(X_k \cap I) \leq 1 \quad \text{for each } I \in \mathcal{B}_{r_k}^d(X_{k-1}). \quad (4.15)$$

*Proof.* We construct  $X_k$  by induction, using Lemma 25 at each step. Set  $X_0 = [0, 1]^d$ . Next, suppose that the sets  $X_0, \dots, X_{k-1}$  have been defined. Our goal is to apply Lemma 25 to  $E = X_{k-1}$  and  $G = Z_k$  with  $l = l_{k-1}$  and  $s = l_k$ . This will be possible once we verify the hypothesis (4.1), which in this case takes the form

$$(l_{k-1}/l_k)^d \leq \#\mathcal{B}_{l_k}^{dn}(Z_k) \leq 0.5 \cdot (l_{k-1}/l_k)^{dn}. \quad (4.16)$$

The right hand side follows from Property (C) of Lemma 26 and (4.10). On the other hand, Property (C) and the fact that  $l_{k-1} \leq 1$  implies that

$$(l_{k-1}/l_k)^d \leq l_k^{-d} \leq \#\mathcal{B}_{l_k}^{dn}(Z_k),$$

establishing the left inequality in (4.16). Applying Lemma 25 as described above now produces a dyadic length

$$r \sim (l_{k-1}^{-d} l_k^{dn} \#\mathcal{B}_{l_k}^{dn}(Z_k))^{\frac{1}{d(n-1)}} \quad (4.17)$$

and an  $l_k$  discretized set  $F \subset X_{k-1}$ . The set  $F$  satisfies Properties (A), (B), and (C) from the statement of Lemma 25. Define  $r_k = r$  and  $X_k = F$ . The estimate (4.13) on  $r_k$  follows from (4.17) using the known bounds (4.10) and (4.12):

$$r_k \lesssim (l_{k-1}^{-d} l_k^{dn-\alpha-0.5\varepsilon_k})^{\frac{1}{d(n-1)}} = (l_{k-1}^{-d} l_k^{0.5\varepsilon_k} l_k^{dn-\alpha-\varepsilon_k})^{\frac{1}{d(n-1)}} = (l_{k-1}^{-2d} l_k^{\varepsilon_k})^{\frac{1}{2d(n-1)}} l_k^{\frac{dn-\alpha-\varepsilon_k}{d(n-1)}} \lesssim l_k^{\frac{dn-\alpha-\varepsilon_k}{d(n-1)}}.$$

The requirements (4.14) and (4.15) follow from Properties (B) and (C) of Lemma 25 respectively.  $\square$

Now we have defined the sets  $\{X_k\}$ , we set  $X = \bigcap X_k$ . Since  $X_k$  avoids strongly non-diagonal cubes in  $Z_k$ , Lemma 27 implies that if  $x_1, \dots, x_n \in X$  are distinct,

then  $(x_1, \dots, x_n) \notin Z$ . To finish the proof of Theorem 24, we must show that  $\dim_{\mathbf{H}}(X) \geq (dn - \alpha)/(n - 1)$ . This will be done in the next section.

## 4.4 Dimension Bounds

To complete the proof of Theorem 24, we must show that  $\dim_{\mathbf{H}}(X) \geq (dn - \alpha)/(n - 1)$ . In view of Definition (H), we will do this by constructing a Frostman measure of appropriate dimension supported on  $X$ . We start by recursively defining a positive function  $f : \bigcup_{i=1}^{\infty} \mathcal{B}_{l_i}^d \rightarrow [0, 1]$ . Set  $f([0, 1]^d) = 1$ , and  $f(I) = 0$  for every other  $I \in \mathcal{B}_{l_0}^d$ . Suppose now that  $f(I)$  has been defined for all cubes  $I \in \mathcal{B}_{l_{k-1}}^d$ , and let  $I \in \mathcal{B}_{l_k}^d$ . Consider the unique ‘parent cube’  $I^* \in \mathcal{B}_{l_{k-1}}^d$  for which  $I \subset I^*$ . Define

$$f(I) = \begin{cases} f(I^*)/\#\mathcal{B}_{l_k}^d(X_k \cap I^*) & \text{if } I \subset X_k, \\ 0 & \text{otherwise.} \end{cases} \quad (4.18)$$

Observe that  $f(I) > 0$  if and only if  $I \cap X \neq \emptyset$ , and for each index  $k \geq 1$ , if  $I^* \in \mathcal{B}_{l_{k-1}}^d$ ,

$$\sum \{f(I) : I \in \mathcal{B}_{l_k}^d(I^*)\} = \sum \{f(I) : I \in \mathcal{B}_{l_k}^d(X_k \cap I^*)\} = f(I^*). \quad (4.19)$$

In particular, for each index  $k$  we have

$$\sum_{I \in \mathcal{B}_{l_k}^d} f(I) = 1.$$

By a standard argument involving weak convergence (CITE PRAMANIK FRASER HERE?), there exists a Borel probability measure  $\mu$  supported on  $\bigcap X_k = X$ , such that for each  $I \in \bigcup_{i=1}^{\infty} \mathcal{B}_{l_i}^d$ ,  $\mu(I) \geq f(I)$ , and for any Borel set  $E$ , and any  $k \geq 0$ ,

$$\mu(E) \leq \sum \{f(I) : I \in \mathcal{B}_{l_k}^d(E(l_k))\}. \quad (4.20)$$

To complete the proof of Theorem 24 we will show that  $\mu$  is a Frostman measure of dimension  $(dn - \alpha)/(n - 1) - \varepsilon$  for every  $\varepsilon > 0$ .

**Lemma 29.** *For each  $k \geq 1$ , if  $I \in \mathcal{B}_{l_k}^d$ ,*

$$\mu(I) \lesssim l_k^{\frac{dn-\alpha}{n-1}-\eta_k}, \quad \text{where} \quad \eta_k = \frac{n+1}{2(n-1)} \cdot \varepsilon_k \searrow 0 \text{ as } k \rightarrow \infty.$$

*Proof.* Let  $I \in \mathcal{B}_{l_k}^d$  and let  $I^* \in \mathcal{B}_{l_{k-1}}^d$  be the parent cube of  $I$ . If  $f(I) > 0$ , then  $I \subset X_k$ , and combining (4.18), (4.14), and (4.13) with the fact that  $f(I^*) \leq 1$ , we obtain

$$f(I) = \frac{f(I^*)}{\#\mathcal{B}_{l_k}^d(X_k \cap I^*)} \leq 2 \left( \frac{r_k}{l_{k-1}} \right)^d \lesssim \frac{l_k^{\frac{dn-\alpha-\varepsilon_k}{n-1}}}{l_{k-1}^d} \leq l_k^{\frac{dn-\alpha}{n-1}-\eta_k} (l_k^{\varepsilon_k/2}/l_{k-1}^d) \leq l_k^{\frac{dn-\alpha}{n-1}-\eta_k}. \quad (4.21)$$

Given any cube  $I_0 \in \mathcal{B}_{l_k}^d$ , we have  $\#\mathcal{B}_{l_k}^d(I_0(l_k)) \lesssim 1$ , so we can apply (4.20) to conclude that

$$\mu(I_0) \leq \sum \{f(I) : I \in \mathcal{B}_{l_k}^d(I_0(l_k))\} \lesssim l_k^{\frac{dn-\alpha}{n-1}-\eta_k}. \quad \square$$

**Corollary 30.** For each  $k \geq 1$  and each  $I \in \mathcal{B}_{r_k}^d$ ,  $\mu(I) \lesssim (r_k/l_{k-1})^d l_{k-1}^{\frac{dn-\alpha}{n-1}-\eta_{k-1}}$ .

*Proof.* We begin by noting that  $\mathcal{B}_{r_k}^d(I(r_k)) \lesssim 1$ . If we combine this with (4.15), we find that

$$\#\mathcal{B}_{l_k}^d(I(r_k) \cap X_k) \lesssim 1. \quad (4.22)$$

For each  $J \in \mathcal{B}_{l_k}^d(I(r_k) \cap X_k)$ , let  $J^* \in \mathcal{B}_{l_{k-1}}^d$  denote the parent cube of  $J$ . Equation (4.21) of Lemma 29 combined with (4.18) and (4.14) implies

$$f(J) = \frac{f(J^*)}{\#\mathcal{B}_{l_k}^d(X_k \cap J^*)} \lesssim (r_k/l_{k-1})^d l_{k-1}^{\frac{dn-\alpha}{n-1}-\eta_k}.$$

Thus (4.20) and (4.22) show that

$$\mu(I) \lesssim (r_k/l_{k-1})^d l_{k-1}^{\frac{dn-\alpha}{n-1}-\eta_k}. \quad \square$$

Lemma 29 and Corollary 30 allow us to control the behavior of  $\mu$  at all scales.

**Lemma 31.** For every  $\alpha \in [d, dn)$ , and for each  $\varepsilon > 0$ , there is a constant  $C_\varepsilon$  so that for all dyadic lengths  $l \in (0, 1]$  and all  $I \in \mathcal{B}_l^d$ , we have

$$\mu(I) \leq C_\varepsilon l^{\frac{dn-\alpha}{n-1}-\varepsilon}. \quad (4.23)$$

*Proof.* Fix  $\varepsilon > 0$ . Since  $\eta_k \searrow 0$  as  $k \rightarrow \infty$ , there is a constant  $C_\varepsilon$  so that  $l_k^{-\eta_k} \leq C_\varepsilon l_k^{-\varepsilon}$  for each  $k \geq 1$ . For instance, if  $\varepsilon_k$  is decreasing, we could choose  $C_\varepsilon = l_{k_0}^{-\eta_{k_0}}$ ,

where  $k_0$  is the largest integer for which  $\eta_{k_0} \geq \varepsilon$ . Next, let  $k$  be the (unique) index so that  $l_{k+1} \leq l < l_k$ . We will split the proof of (4.23) into two cases, depending on the position of  $l$  within  $[l_{k+1}, l_k]$ .

*Case 1:* If  $r_{k+1} \leq l \leq l_k$ , we can cover  $I$  by  $(l/r_{k+1})^d$  cubes in  $\mathcal{B}_{r_{k+1}}^d$ . A union bound combined with Corollary 30 gives

$$\begin{aligned} \mu(I) &\lesssim (l/r_{k+1})^d (r_{k+1}/l_k)^d l_k^{\frac{dn-\alpha}{n-1}-\eta_k} \\ &= (l/l_k)^d l_k^{\frac{dn-\alpha}{n-1}-\eta_k} \\ &= l^{\frac{dn-\alpha}{n-1}} (l/l_k)^{\frac{\alpha-d}{n-1}} l_k^{-\eta_k} \\ &\leq l^{\frac{dn-\alpha}{n-1}-\eta_k} \\ &\leq C_\varepsilon l^{\frac{dn-\alpha}{n-1}-\varepsilon}. \end{aligned} \tag{4.24}$$

The penultimate inequality is a consequence of our assumption  $\alpha \geq d$ .

*Case 2:* If  $l_{k+1} \leq l \leq r_{k+1}$ , we can cover  $I$  by a single cube in  $\mathcal{B}_{r_{k+1}}^d$ . By (4.15), each cube in  $\mathcal{B}_{r_{k+1}}^d$  contains at most one cube  $I_0 \in \mathcal{B}_{l_{k+1}}^d(X_{k+1})$ , so by Lemma 29,

$$\mu(I) \leq \mu(I_0) \lesssim l_{k+1}^{\frac{dn-\alpha}{n-1}-\eta_{k+1}} \leq C_\varepsilon l_{k+1}^{\frac{dn-\alpha}{n-1}-\varepsilon} \leq C_\varepsilon l^{\frac{dn-\alpha}{n-1}-\varepsilon}. \quad \square$$

Applying Frostman's lemma to Lemma 31 gives  $\dim_{\mathbf{H}}(X) \geq \frac{dn-\alpha}{n-1} - \varepsilon$  for every  $\varepsilon > 0$ , which concludes the proof of Theorem 24.

## 4.5 Applications

As discussed in the introduction, Theorem 24 generalizes Theorems 1.1 and 1.2 from [6]. In this section, we present two applications of Theorem 24 in settings where previous methods do not yield any results.

### 4.5.1 Sum-sets avoiding specified sets

**Theorem 32.** *Let  $Y \subset \mathbf{R}^d$  be a countable union of sets of Minkowski dimension at most  $\beta < d$ . Then there exists a set  $X \subset \mathbf{R}^d$  with Hausdorff dimension at least  $d - \beta$  such that  $X + X$  is disjoint from  $Y$ .*

*Proof.* Define  $Z = Z_1 \cup Z_2$ , where

$$Z_1 = \{(x, y) : x + y \in Y\} \quad \text{and} \quad Z_2 = \{(x, y) : y \in Y/2\}.$$

Since  $Y$  is a countable union of sets of Minkowski dimension at most  $\beta$ ,  $Z$  is a countable union of sets with lower Minkowski dimension at most  $d + \beta$ . Applying Theorem 24 with  $n = 2$  and  $\alpha = d + \beta$  produces a set  $X \subset \mathbf{R}^d$  with Hausdorff dimension  $2d - (d + \beta) = d - \beta$  such that  $(x, y) \notin Z$  for all  $x, y \in X$  with  $x \neq y$ . We claim that  $X + X$  is disjoint from  $Y$ . To see this, first suppose  $x, y \in X$ ,  $x \neq y$ . Since  $X$  avoids  $Z_1$ , we conclude that  $x + y \notin Y$ . Suppose now that  $x = y \in X$ . Since  $X$  avoids  $Z_2$ , we deduce that  $X \cap (Y/2) = \emptyset$ , and thus for any  $x \in X$ ,  $x + x = 2x \notin Y$ . This completes the proof.  $\square$

### 4.5.2 Subsets of Lipschitz curves avoiding isosceles triangles

In [6], Fraser and the second author prove that there exists a set  $S \subset [0, 1]$  with dimension  $\log_3 2$  such that for any simple  $C^2$  curve  $\gamma: [0, 1] \rightarrow \mathbf{R}^n$  with bounded non-vanishing curvature,  $\gamma(S)$  does not contain the vertices of an isosceles triangle. Our method enables us to obtain a result that works for Lipschitz curves with small Lipschitz constants. The dimensional bound that we provide is slightly worse than [6] ( $1/2$  instead of  $\log_3 2$ ), and the set we obtain only works for a single Lipschitz curve, not for many curves simultaneously.

**Theorem 33.** *Let  $f: [0, 1] \rightarrow \mathbf{R}^{n-1}$  be Lipschitz with*

$$\|f\|_{Lip} := \sup\{|f(x) - f(y)|/|x - y| : x, y \in [0, 1], x \neq y\} < 1.$$

*Then there is a set  $X \subset [0, 1]$  of Hausdorff dimension  $1/2$  so that the set  $\{(t, f(t)) : t \in X\}$  does not contain the vertices of an isosceles triangle.*

**Corollary 34.** *Let  $f: [0, 1] \rightarrow \mathbf{R}^{n-1}$  be  $C^1$ . Then there is a set  $X \subset [0, 1]$  of Hausdorff dimension  $1/2$  so that the set  $\{(t, f(t)) : t \in X\}$  does not contain the vertices of an isosceles triangle.*

*Proof of Corollary 34.* The graph of any  $C^1$  function can be locally expressed, after possibly a translation and rotation, as the graph of a Lipschitz function with small Lipschitz constant. In particular, there exists an interval  $I \subset [0, 1]$  of positive length so that the graph of  $f$  restricted to  $I$ , after being suitably translated, rotated, and dilated, is the graph of a Lipschitz function  $g: [0, 1] \rightarrow \mathbf{R}^{n-1}$  with

Lipschitz constant at most  $1/2$ . Since isosceles triangles remain invariant under these transformations, the corollary is a consequence of Theorem 33.  $\square$

*Proof of Theorem 33.* Set

$$Z = \left\{ (x_1, x_2, x_3) \in [0, 1]^3 : \begin{array}{l} \text{The three points } p_j = (x_j, f(x_j)), 1 \leq j \leq 3 \\ \text{form the vertices of an isosceles triangle} \end{array} \right\}. \quad (4.25)$$

In the next lemma, we show  $Z$  has lower Minkowski dimension at most two. By Theorem 24, there is a set  $X \subset [0, 1]$  of Hausdorff dimension  $1/2$  so that for each distinct  $x_1, x_2, x_3 \in X$ , we have  $(x_1, x_2, x_3) \notin Z$ . This is precisely the statement that for each  $x_1, x_2, x_3 \in X$ , the points  $(x_1, f(x_1))$ ,  $(x_2, f(x_2))$ , and  $(x_3, f(x_3))$  do not form the vertices of an isosceles triangle.  $\square$

**Lemma 35.** *Let  $f: [0, 1] \rightarrow \mathbf{R}^{n-1}$  be Lipschitz with  $\|f\|_{Lip} < 1$ . Then the set  $Z$  given by (4.25) has Minkowski dimension at most two.*

*Proof.* First, notice that three points  $p_1, p_2, p_3 \in \mathbf{R}^n$  form an isosceles triangle, with  $p_3$  as the apex, if and only if  $p_3 \in H_{p_1, p_2}$ , where

$$H_{p_1, p_2} = \left\{ x \in \mathbf{R}^n : \left( x - \frac{p_1 + p_2}{2} \right) \cdot (p_2 - p_1) = 0 \right\}. \quad (4.26)$$

To prove  $Z$  has Minkowski dimension at most two, it suffices to show that the set

$$W = \{x \in [0, 1]^3 : p_3 = (x_3, f(x_3)) \in H_{p_1, p_2}\}$$

has upper Minkowski dimension at most 2. This is because  $Z$  is the union of three copies of  $W$ , obtained by permuting coordinates. To bound the upper Minkowski dimension of  $W$ , we prove the estimate

$$\#(\mathcal{B}_\delta^3(W(\delta))) \leq C\delta^{-2} \log(1/\delta) \quad \text{for all } 0 < \delta < 1, \quad (4.27)$$

where  $C$  is a constant independent of  $\delta$ , and  $\delta$  is a dyadic scale.

Fix  $0 < \delta < 1$ . Note that

$$\#(\mathcal{B}_\delta^3(W)) = \sum_{k=0}^{1/\delta} \sum_{\substack{I_1, I_2 \in \mathcal{B}_\delta^1[0, 1] \\ d(I_1, I_2) = k\delta}} \#(\mathcal{B}_\delta^3(W(\delta) \cap I_1 \times I_2 \times [0, 1])). \quad (4.28)$$

Our next task is to bound each of the summands in (4.28). Let  $I_1, I_2 \in \mathcal{B}_\delta^1[0, 1]$ , and let  $k = \delta^{-1}d(I_1, I_2)$ . Let  $x_1$  be the midpoint of  $I_1$ , and  $x_2$  the midpoint of  $I_2$ . Let  $(y_1, y_2, y_3) \in W \cap I_1 \times I_2 \times [0, 1]$ . Then it follows from (4.26) that

$$\left(y_3 - \frac{y_1 + y_2}{2}\right) \cdot (y_2 - y_1) + \left(f(y_3) - \frac{f(y_2) + f(y_1)}{2}\right) \cdot (f(y_2) - f(y_1)) = 0.$$

We know  $|x_1 - y_1|, |x_2 - y_2| \leq \delta/2$ , so

$$\begin{aligned} & \left| \left(y_3 - \frac{y_1 + y_2}{2}\right) (y_2 - y_1) - \left(y_3 - \frac{x_1 + x_2}{2}\right) (x_2 - x_1) \right| \\ & \leq \frac{|y_1 - x_1| + |y_2 - x_2|}{2} |y_2 - y_1| + \left(|y_1 - x_1| + |y_2 - x_2|\right) \left|y_3 - \frac{x_1 + x_2}{2}\right| \\ & \leq (\delta/2) \cdot 1 + \delta \cdot 1 \leq 3\delta/2. \end{aligned} \tag{4.29}$$

Conversely, we know  $|f(x_1) - f(y_1)|, |f(x_2) - f(y_2)| \leq \delta/2$  because  $\|f\|_{\text{Lip}} < 1$ , and a similar calculation yields

$$\begin{aligned} & \left| \left(f(y_3) - \frac{f(y_1) + f(y_2)}{2}\right) \cdot (f(y_2) - f(y_1)) \right. \\ & \quad \left. - \left(f(y_3) - \frac{f(x_1) + f(x_2)}{2}\right) \cdot (f(x_2) - f(x_1)) \right| \leq 3\delta/2. \end{aligned} \tag{4.30}$$

Putting (4.29) and (4.30) together, we conclude that

$$\left| \left(y_3 - \frac{x_1 + x_2}{2}\right) (x_2 - x_1) + \left(f(y_3) - \frac{f(x_2) + f(x_1)}{2}\right) \cdot (f(x_2) - f(x_1)) \right| \leq 3\delta. \tag{4.31}$$

Since  $|(x_2 - x_1, f(x_2) - f(x_1))| \geq |x_2 - x_1| \geq k\delta$ , we can interpret (4.31) as saying the point  $(y_3, f(y_3))$  is contained in a  $3/k$  thickening of the hyperplane  $H_{(x_1, f(x_1)), (x_2, f(x_2))}$ . Given another value  $y' \in W \cap I_1 \cap I_2 \cap [0, 1]$ , it satisfies a variant of the inequality (4.31), and we can subtract the difference between the two inequalities to conclude

$$\left| (y_3 - y'_3) (x_2 - x_1) + (f(y_3) - f(y'_3)) \cdot (f(x_2) - f(x_1)) \right| \leq 6\delta. \tag{4.32}$$

The triangle difference inequality applied with (4.32) implies

$$\begin{aligned} (f(y_3) - f(y'_3)) \cdot (f(x_2) - f(x_1)) &\geq |y_3 - y'_3| |x_2 - x_1| - 6\delta \\ &= (k+1)\delta \cdot |y_3 - y'_3| - 6\delta. \end{aligned} \quad (4.33)$$

Conversely,

$$\begin{aligned} (f(y_3) - f(y'_3)) \cdot (f(x_2) - f(x_1)) &\leq \|f\|_{\text{Lip}}^2 |y_3 - y'_3| |x_2 - x_1| \\ &= \|f\|_{\text{Lip}}^2 \cdot (k+1)\delta \cdot |y_3 - y'_3|. \end{aligned} \quad (4.34)$$

Combining (4.33) and (4.34) and rearranging, we see that

$$|y_3 - y'_3| \leq \frac{6}{(k+1)(1 - \|f\|_{\text{Lip}}^2)} \lesssim \frac{1}{k+1}, \quad (4.35)$$

where the implicit constant depends only on  $\|f\|_{\text{Lip}}$  (and blows up as  $\|f\|_{\text{Lip}}$  approaches 1). We conclude that

$$\#\mathcal{B}_\delta^3(W(\delta) \cap I_1 \times I_2 \times [0, 1]) \lesssim \frac{1}{k+1}, \quad (4.36)$$

which holds uniformly over any value of  $k$ .

We are now ready to bound the sum from (4.28). Note that for each value of  $k$ , there are at most  $2/\delta$  pairs  $(I_1, I_2)$  with  $d(I_1, I_2) = k\delta$ . Indeed, there are  $1/\delta$  choices for  $I_1$  and then at most two choices for  $I_2$ . Equation (4.36) shows

$$\#\mathcal{B}_\delta^3(W(\delta)) = \sum_{k=0}^{1/\delta} \sum_{\substack{I_1, I_2 \in \mathcal{B}_\delta^1[0, 1] \\ d(I_1, I_2) = k\delta}} \#\mathcal{B}_\delta^3(W(\delta) \cap I_1 \times I_2 \times [0, 1]) \lesssim \delta^{-2} \sum_{k=0}^{1/\delta} \frac{1}{k+1} \lesssim \delta^{-2} \log(1/\delta).$$

In the above inequalities, the implicit constants depend on  $\|f\|_{\text{Lip}}$ , but they are independent of  $\delta$ . This establishes (4.27) and completes the proof.  $\square$



## **Chapter 5**

### **Conclusions**



# Bibliography

- [1] K. Athreya. Large deviation rates for branching processes–i. single type case. *Ann. Appl. Probab.*, 4:779–790, 1994.
- [2] J. Balogh, R. Morris, and W. Samotij. Independent sets in hypergraphs. *J. Amer. Math. Soc.*, 28:669–709, 2015.
- [3] J. Biggins and N. Bingham. Large deviations in the supercritical branching process. *Adv. in Appl. Probab.*, 25:757–772.
- [4] K. Dickman. On the frequency of numbers containing prime factors of a certain relative magnitude.
- [5] K. Falconer. *Fractal Geometry: Mathematical Foundations and Applications*. Wiley, 2003.
- [6] R. Fraser and M. Pramanik. Large sets avoiding patterns. *Anal. PDE*, 11: 1083–1111, 2018. → pages 34, 35, 46, 47
- [7] T. Harris. *The Theory of Branching Processes*. Springer-Verlag, 1963.
- [8] N. H. Katz and T. Tao. Some connections between Falconer’s distance set conjecture, and sets of Furstenberg type. *New York J. Math.*, 7:149–187, 2001. → page 36
- [9] T. Keleti. A 1-dimensional subset of the reals that intersects each of its translates in at most a single point. *Real Anal. Exchange*, 24:843–845, 1999. → pages 33, 35
- [10] P. Maga. Full dimensional sets without given patterns. *Real Anal. Exchange*, 36:79–90, 2010. → pages 1, 33

- [11] A. Máthé. Sets of large dimension not containing polynomial configurations. *Adv. Math.*, 316:691–709, 2017. → page 34
- [12] P. Mattila. *Fourier analysis and Hausdorff dimension*. Cambridge University Press, 2015. → page 1
- [13] I. Z. Ruzsa. Difference sets without squares.
- [14] B. Sudakov, E. Szemerédi, and V. Vu. On a question of erdős and moser.
- [15] T. Tao. 245 C, Notes 5: Hausdorff dimension, 2009. URL <https://terrytao.wordpress.com/2009/05/19/245c-notes-5-hausdorff-dimension-optional/>.