# Multipliers of Elliptic Operators on Compact Manifolds

Necessary and Sufficient Conditions For Boundedness

by

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# Thesis Title

Thesis Subtitle

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## Abstract

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# Dedication

To mum and dad

# Declaration

I declare that..

# Acknowledgements

I want to thank...

Supported by grant numbers...

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# Notation

- We use the normalization of the Fourier transform given by the formula

$$\widehat{f}(\xi) = \int f(x) e^{-2\pi i \xi \cdot x}\, dx.$$

- We use the translation operators $\mathrm{Trans}_y f(x) = f(x-y)$, and $L^\infty$-normalized dilations $\mathrm{Dil}_t f(x) = f(x/t)$. Overloading notation, we also consider dyadic dilations of the form $\mathrm{Dil}_j f(x) = f(x/2^j)$. The dyadic dilation operator will only be used along with symbols that stand for integers, like $n$, $m$, $j$, or $k$, whereas the other dilation operator will be used in all other cases, so the operator used should be clear from context.

- On $\mathbb{R}^d$, we use $\partial_j$ to denote the usual partial derivative operators, and $D_j$ to denote the self-adjoint normalization $D_j f = (2\pi i)^{-1} \partial_j$. This notation has the convenience that for a polynomial $P$,

$$P(D_j)\{f\} = \int P(\xi) \widehat{f}(\xi) e^{2\pi i \xi \cdot x}\, d\xi,$$

  and simplifies many formulas associated with Fourier integral operators.

- We will often use the Japanese bracket $\langle x \rangle := (1 + |x|^2)^{1/2}$ for $x \in \mathbb{R}^d$.

- A function $f : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}$ is a *symbol of order $s$* if it satisfies bounds of the form

$$|\partial_x^\alpha \partial_\theta^\beta f(x,\theta)| \lesssim_{\alpha,\beta} \langle \theta \rangle^{s-|\beta|}$$

  for all multi-indices $\alpha$ and $\beta$.

- For a measure space $X$, $L^\infty(X)$ is the Banach space of essentially-bounded functions, defined almost everywhere. For a set $X$, $l^\infty(X)$ is the Banach space of bounded functions on $X$, defined everywhere.

- All operators we consider are Schwartz operators between manifolds equipped with a canonical measure. Abusing notation, we thus identify an operator with the distribution that gives it's Schwartz kernel. Thus, for a Schwartz operator $A$ from a manifold $Y$ to a manifold $X$, we might write

$$Af(x) = \int_Y A(x,y) f(y)\, dy.$$

- For $1 \leq p \leq \infty$ and $s \in \mathbb{R}$, we consider the Sobolev spaces

$$\|f\|_{W^{s,p}(\mathbb{R}^d)} = \|(-\Delta)^{s/2} f\|_{L^p(\mathbb{R}^d)},$$

and for $1 \leq r \leq \infty$, we consider the Besov spaces

$$\|f\|_{B^{s,p}_r(\mathbb{R}^d)} = \|P_k f\|_{l^r_k L^p(\mathbb{R}^d)},$$

where $\{P_k\}$ are Littlewood-Paley projections.

# Introduction

Let $P$ be an elliptic linear operator on a compact manifold $M$, such that we can associate a functional calculus $a \mapsto a(P)$ for functions $a$ on the real line. In this thesis, we study the following question:

*What conditions on a guarantee an operator $a(P)$ to be bounded.*

Aside from testing our ability to understand the operator $P$ and the relations of it's eigenfunctions, the question has applications in the study of various partial differential equations associated with the operator $P$, and is closely related to the geometry of $M$, in particular, the way waves propogate and interact on the manifold.

Moreover, the study of multipliers on a manifold provides a useful setting with which to test and develop methods of harmonic analysis in a 'variable-coefficient setting', where a lack of symmetry and translation invariance forces us to introduce more robust methods than are required in the Euclidean setting, i.e. as compared to studying multipliers of the Laplace operator on $\mathbb{R}^d$ using the Fourier transform.

At present, there are many obstacles related to the global geometry of manifolds, which prevent an understanding of spectral multiplier operators on a general manifold. Over the years, sufficient conditions for boundedness had been established, but no characterizations of boundedness were known on any compact manifold, aside from the translation-invariant case when studying the Laplace operator on the torus. This thesis describes the first such characterizations beyond this setting, for a limited range of Lebesgue spaces, and on manifolds all of whose geodesics are closed and have common length. In particular, we find characterizations of functions $a$ whose dilates induce uniformly bounded multiplier operators for spherical harmonic expansions on $S^d$.

We obtain these results by expanding on techniques for understanding Fourier integral operators on manifolds. In Chapter TODO, we discuss the general theory of Fourier integral operators, as well as a discussion of the best results currently known for multipliers of the Laplacian $\Delta$ on $\mathbb{R}^d$, which can also be described as *radial Fourier multipliers on Euclidean Space*. In Chapter TODO, we give an exposition of methods more closely associated with the problem. Chapters TODO and TODO discuss our main contribution to the problem (TODO More Here). Chapter TODO discusses methods we hope to develop in the future.

# Part I

# Background

# Chapter 1

# Multipliers of an Elliptic Operator

Let us now more precisely describe the problem of this thesis. Let $X$ be a compact manifold equipped with a volume density, and let $P$ be a classical elliptic operator on $X$ of order $s > 0$, formally positive-definite in the sense that

$$\langle Pf, g\rangle = \langle f, Pg\rangle \quad \text{and} \quad \langle Pf, f\rangle \geq 0 \qquad \text{for all } f, g \in C^\infty(M).$$

By general properties of elliptic operators, $1 + P$ is an isomorphism between the Sobolev space $H^s(X)$ and $L^2(X)$, and so the inverse $(1 + P)^{-1}$, viewed as a map from $L^2(X)$ to itself, is compact by a form of the Rellich-Kondrachov embedding theorem. By the spectral theorem for compact operators, there exists a discrete set $\Lambda \subset [0, \infty)$, and a decomposition

$$L^2(X) = \bigoplus_{\lambda \in \Lambda} \mathcal{V}_\lambda,$$

where $\mathcal{V}_\lambda$ is a finite dimensional subspace of $C^\infty(X)$, such that $Pf = \lambda f$ for all $f \in \mathcal{V}_\lambda$. We use this decomposition to define a functional calculus for the operator $P$; given a bounded function $a : \Lambda \to \mathbb{C}$, we can define an operator $a(P)$ on $L^2(X)$ so that $a(P)f = a(\lambda)f$ for all $f \in \mathcal{V}_\lambda$. Our goal is to study the regularity of such operators, in terms of properties of the function $a$.

In the sequel, we will assume the operator $P$ is an elliptic operator *of order one*. This does not restrict the scope of our analysis; given a formally positive elliptic operator $P$ of order $s$, the operator $P^{1/s}$ defined by the functional calculus above is a formally positive elliptic operator of order one[1], and the spectral theory of $P$ is identical with the spectral theory of $P^{1/s}$. Fixing the order of our operator reflects the fact that the theory of Fourier integral operators we will eventually employ is most elegant when the resulting oscillatory integrals have homogeneous phases of order one. To prevent being overly wordy, in all that follows by an 'elliptic operator' we will mean a classical elliptic operator of order one which is formally positive-definite in the sense above.

The primary example of elliptic operators for our purposes are obtained from a Laplace-Beltrami operator on a compact Riemannian manifold $M$; the operator $-\Delta$ is self-adjoint

[1] See Theorem 3.3.1 of [21] for a proof of this fact, based on a technique of [20] initially developed for elliptic differential operators, but which can be straightforwardly adapted to pseudodifferential operators.

and positive-definite with respect to the volume form $dV$ on $M$, because of the 'integration by parts' identity $\langle \Delta f, g\rangle = -\langle \nabla f, \nabla g\rangle$. Since we restrict our attention to elliptic operators of order one, the archetypical object of study is the operator $P = \sqrt{-\Delta}$.

To study the regularity of $a(P)$, we introduce the spaces $M^p(X)$, consisting of all functions $a : \Lambda \to \mathbb{C}$ for which the operator $a(P)$ is bounded on $L^p(X)$. The space $M^p(X)$ is then equipped with a norm

$$\|a\|_{M^p(X)} = \sup \left\{ \frac{\|a(P)f\|_{L^p(X)}}{\|f\|_{L^p(X)}} : f \in C^\infty(X) \right\}.$$

Orthogonality implies $M^2(X)$ is isometrically equal to $l^\infty(\Lambda)$, but the structure of the other spaces $M^p(X)$ is unclear.

The discrete spectrum of the operator $P$ makes it difficult to apply methods of oscillatory integrals to the problem. Fortunately, certain semiclassical heuristics tell us that these problems disappear when we restrict the problem to 'high frequency inputs'. In order to take advantage of this fact, rather than studying what conditions ensure the operators $a(P)$ are bounded, we study conditions that ensure the operators $a_R(P) = a(P/R)$ are uniformly bounded, for functions $a : [0,\infty) \to \mathbb{C}$. To prevent pathological examples from arising under dilation, we restrict ourselves to *regulated functions a*, i.e. functions such that

$$a(\lambda_0) = \lim_{\delta \to 0} \fint_{|\lambda - \lambda_0| \leq \delta} a(\lambda)\, d\lambda \quad \text{for all } \lambda_0 \in [0,\infty).$$

Define the set $M^p_{\text{Dil}}(X)$ of all regulated functions $a$ such that $\|a\|_{M^p_{\text{Dil}}(X)} = \sup_R \|a_R\|_{M^p(X)}$ is finite. With notation introduced, we now state precisely the problem studied in this thesis:

*Can one find simple conditions on a function a,*
*necessary and sufficient to be contained in $M^p_{Dil}(X)$ for $p \neq 2$.*

For $p = 2$, orthogonality gives the isometric equivalence $M^2_{\text{Dil}}(X) = L^\infty[0,\infty)$.

**Lemma 1.1.** *For any regulated a and any compact manifold X, $\|a\|_{M^2_{Dil}(X)} = \|a\|_{L^\infty[0,\infty)}$.*

*Proof.* Given $f \in L^2(X)$, if we write $f = \sum f_\lambda$ with $f_\lambda \in \mathcal{V}_\lambda$, then for any $R > 0$,

$$\|a_R(P)f\|_{L^2(X)} = \Big\| \sum\nolimits_\lambda a_R(\lambda) f_\lambda \Big\|_{L^2(X)} = \left( \sum\nolimits_\lambda |a_R(\lambda)|^2 \|f_\lambda\|^2_{L^2(X)} \right)^{1/2}$$

Conversely,

$$\|f\|_{L^2(X)} = \left( \sum\nolimits_\lambda \|f_\lambda\|^2_{L^2(X)} \right)^{1/2}.$$

Setting $\|f_\lambda\|_{L^2(X)} = c_\lambda$, the quantity $\|a_R\|_{M^2(X)}$ is thus the smallest constant such that

$$\left( \sum\nolimits_\lambda |a_R(\lambda)|^2 c_\lambda^2 \right)^{1/2} \leq \|a_R\|_{M^2(X)} \left( \sum\nolimits_\lambda c_\lambda^2 \right)^{1/2}.$$

It is now clear that $\|a_R\|_{M^2(X)} = \|a_R\|_{l^\infty(\Lambda)}$, and thus (now using the fact that $a$ is regulated),

$$\|a\|_{M^2_{\text{Dil}}(X)} = \sup\nolimits_R \|a_R\|_{l^\infty(\Lambda)} = \|a\|_{L^\infty[0,\infty)} \qquad \square.$$

Before the results of this thesis, no conditions had been proved necessary and sufficient for boundedness for $p \neq 2$ and $X \neq \mathbb{T}^d$, aside from the isometric , which holds by orthogonality. We will obtain results under two main assumptions, which we call Assumption A and Assumption B, the first being a curvature condition on the principal symbol of the $P$, and the second an assumption about the operator's eigenvalues.

**Assumption A**: For each $x_0 \in M$, the cosphere

$$S_{x_0} = \{\xi \in T^*_{x_0} M : p(x_0, \xi) = 1\}$$

is a hypersurface in $T^*_x M$ with non-vanishing Gauss curvature.

**Assumption B**: The spectrum of the operator $P$ is contained in an arithmetic progression.

When $P = \sqrt{-\Delta}$ on a Riemannian manifold $M$, Assumption A always holds, because the cospheres defined above are ellipsoids. For more general operators, Assumption A implies a geometry on the manifold $M$, and Assumption B is closely related to this geometry, in particular, that geodesics with respect to this geometry are all closed and have common length.

Under Assumption A and Assumption B, we find necessary and sufficient conditions for a function to be contained in $M^{p,q}_{\text{Dil}}(M)$ for $1/p - 1/2 > 1/(d+1)$ and for $p \leq q < 2$. Fix $\chi \in C_c^\infty(\mathbb{R}^d)$. For $s \geq 0$ and $1 \leq p \leq \infty$, define the norm

$$\|a\|_{R^{s,p}[0,\infty)} = \sup_{R>0} \left( \int_0^\infty \left| \widehat{a_R}(t) \langle t \rangle^s \right|^p \, dt \right)^{1/p} \quad \text{where} \quad a_R(\lambda) = \chi(\lambda) a(R\lambda).$$

Here $\widehat{a_R}(t) = \int_0^\infty a_R(\lambda) \cos(2\pi\lambda t)$ is the cosine transform of $a_\rho$, i.e. the Fourier transform of the even extension of $a$ to a function on $\mathbb{R}$. The main result of this thesis is that if $s = (d-1)(1/p - 1/2)$, then for $1/p - 1/2 > 1/(d+1)$, $M^p_{\text{Dil}}(X)$ is isomorphically equal to $R^{s,p}[0,\infty)$. This is the first such characterization for $p \neq 2$ and $X \neq \mathbb{T}^d$.

**Theorem 1.2.** *Suppose $X$ is a manifold, and $P$ is an elliptic operator on $X$ satisfying Assumption A and Assumption B. Then for $1/p - 1/2 > 1/(d+1)$, if $s = (d-1)(1/p-1/2)$, then for any regulated $a$, $\|a\|_{M^p_{Dil}(X)} \sim \|a\|_{R^{s,p}[0,\infty)}$.*

One may view control on the $R^{s,p}$ norm as a smoothness condition on the multiplier $a$; indeed the Hausdorff-Young inequality implies a Sobolev space estimate

$$\|a_R\|_{W^{s,p'}[0,\infty)} \lesssim \|a\|_{R^{s,p}[0,\infty)}. \tag{1.1}$$

However, the space $R^{s,p}[0,\infty)$ is not equal to a Sobolev or Besov space, though we do have the Besov space estimate

$$\|a\|_{R^{s,p}[0,\infty)} \lesssim \sup_R \|a_R\|_{B_p^{s+1/p-1/2,2}[0,\infty)}. \tag{1.2}$$

The result is somewhat intuitive. Inequality (1.1) tells us that elements of $R^{s,p}$ have $s$ derivatives in $L^{p'}$, though with 'some extra control' occuring on the other side of the

Fourier transform. Sobolev embedding heuristics tell us that having $s + 1/p - 1/2$ derivatives in $L^2$ is sufficient to have $s$ derivatives in $L^{p'}$. The presence of $L^2$ norms on the right hand side of (1.2) allows us to losslessly convert between estimates on the Fourier transform side, allowing us to recover the information on the Fourier side and obtain (1.2).

For most manifolds $X$ and $P$, elements of $\mathcal{V}_\lambda$ are difficult to describe explicitly; even for the relatively simple case of the sphere $S^d$ many questions about the geometric behaviour of eigenfunctions remain open. Many arguments in harmonic analysis involve an interplay between spatial and frequential control, and without explicit descriptions of eigenfunctions, spatial control becomes difficult. Nonetheless, we will find we can obtain some spatial control by utilizing the wave equation on $M$, which carries geometric information via the behaviour of wave propogation. Under the curvature assumptions we make, wave propogation has sufficient smoothing properties to match certain necessary conditions that multipliers need in order to be bounded. And Assumption B implies that the wave equation associated with the operator $P$ is periodic, which simplifies the large time analysis of the wave equation.

In the next chapter, we will begin our study by describing what is currently known for the boundedness problem on $\mathbb{T}^d$, where eigenfunctions are explicit, and one can study the behaviour of spectral multipliers via the Fourier transform and radial convolution operators on $\mathbb{R}^d$. Our analysis here will help gain intuition and motivate potential hypotheses in the more general setting.

# Chapter 2

# Radial Multipliers on Euclidean Space

Consider the elliptic operator $P = \sqrt{-\Delta}$ on $\mathbb{T}^d$, where $\Delta = \partial_1^2 + \cdots + \partial_d^2$ is the usual Laplacian. In such a setting, we have an explicit basis for the eigenfunctions of $\Delta$: for a given eigenvalue $\lambda > 0$, the space $\mathcal{V}_\lambda$ has an orthonormal basis consisting of the exponentials $e^{2\pi i n \cdot x}$, where $n \in \mathbb{Z}^d$ and $|n| = \lambda$. Since the Fourier series of a function gives the expansion of the function in this basis, it follows that we can expand the spectral multiplier operator $T = a(P)$ using a Fourier series, i.e. writing

$$T f(x) = \sum\nolimits_{n \in \mathbb{Z}^d} a(|n|) \widehat{f}(n) e^{2\pi i n \cdot x}.$$

Thus a multiplier of $\Delta$ on $\mathbb{T}^d$ is nothing more than a Fourier multiplier operator on $\mathbb{T}^d$ whose symbol is radial. Methods of transference[1] show that $\|a\|_{M^p(\mathbb{R}^d)} \sim \|a\|_{M^p_{\text{Dil}}(\mathbb{T}^d)}$, where $\|a\|_{M^p(\mathbb{R}^d)}$ is the operator norm of the Fourier multiplier on $\mathbb{R}^d$ given by

$$T f(x) = \int_{\mathbb{R}^d} a(|\xi|) \widehat{f}(\xi) e^{2\pi i \xi \cdot x}\, d\xi$$

on $L^p(\mathbb{R}^d)$. In this section we will focus on operators of this type, which we call *radial Fourier multiplier operators*.

The study of the regularity of Fourier multiplier operators has proved central to the development of modern harmonic analysis and the theory of linear partial differential operators. This is because essentially any translation invariant operator $T$ on $\mathbb{R}^d$ is a Fourier multiplier operator, i.e. we can find a tempered distribution $m$ on $\mathbb{R}^d$, the *symbol* of $T$, such that for any Schwartz function $f$,

$$T f(x) = \int_{\mathbb{R}^d} m(\xi) \widehat{f}(\xi) e^{2\pi i \xi \cdot x}\, d\xi.$$

The study of translation invariant operators emerges from classical questions in analysis, such as the convergence of Fourier series, and problems in mathematical physics related to the study of the heat, wave, and Schrödinger equations. These physical equations also

[1] See Section 3.6.2 of Grafakos [5], based on methods of de Leeuw [15]

often have *rotational* symmetry, so it is natural to restrict our attention to translation-invariant operators which are also rotation-invariant. These operators are precisely the family of radial multiplier operators. If $m(\xi) = a(|\xi|)$ for a tempered distribution on $[0, \infty)$ we will write $T = a(P)$, where $P = \sqrt{-\Delta}$, since $Tf = a(\lambda)f$ whenever $\Delta f = -\lambda^2 f$.

## 2.1 Convolution Kernels of Fourier Multipliers

It is often useful to study spatial representations of these operators, since one can often exploit certain geometric information about the behaviour of operators. Given any translation invariant operator $T$ on $\mathbb{R}^d$, we can associate a tempered distribution $k$, the *convolution kernel* of $T$, such that

$$Tf(y) = \int_{\mathbb{R}^d} k(x) f(y - x)\, dx \qquad \text{for any Schwartz } f \in \mathcal{S}(\mathbb{R}^d).$$

If $T$ is radial, then so is $k$, and so we can write $k(x) = b(|x|)$ for some distribution $b$ on $[0, \infty)$, and then we have a representation

$$T = \int_0^\infty b(r) S_r\, dr, \quad \text{where} \quad S_r f(x) = \int_{|y|=r} f(x + y)\, dy$$

are the *spherical averaging operators*.

With the notation as above, the function $k$ is the Fourier transform of $m$, and the function $b$ is a Bessel transform of $a$, i.e. $b = \mathcal{B}_d a$, where

$$\mathcal{B}_d a(r) = \int_0^\infty s^{\frac{d-1}{2}} J_{\frac{d-1}{2}}(2\pi s) a(s)\, ds,$$

and where

$$J_\alpha(\lambda) = \frac{(\lambda/2)^\alpha}{\Gamma(\alpha + 1/2)} \int_{-1}^1 e^{i\lambda s} (1 - s^2)^{\alpha - 1/2}\, ds.$$

Using the theory of stationary phase, for each $d$ we can write

$$J_d(\lambda) = e^{2\pi i \lambda} s_1(\lambda) + e^{-2\pi i \lambda} s_2(\lambda).$$

for symbols $s_1$ and $s_2$ of order $-1/2$. The presence of $e^{2\pi i\lambda}$ and $e^{-2\pi i\lambda}$ allows one to relate the Bessel transform of a function to it's Fourier transform, to a certain extent. In particular, we record the following result of Garrigos and Seeger [4].

**Theorem 2.1.** *Suppose $d > 1$ and $1 < p < 2d/(d + 1)$, and suppose $a : [0, \infty) \to \mathbb{C}$ has compact support away from the origin. Then, with implicit constants depending on the support of a,*

$$\left( \int_0^\infty |\mathcal{B}_d a(t)|^p t^{d-1} \right)^{1/p} \sim_{p,d,\phi} \left( \int_0^\infty |\widehat{a}(t)|^p \langle t \rangle^{(d-1)(1-p/2)}\, dt \right)^{1/p},$$

*where $\widehat{a}(t) = \int_0^\infty a(\lambda) \cos(2\pi\lambda t)\, dt$ is the cosine transform of a.*

## 2.2 The Radial Multiplier Conjecture

The general study of the boundedness properties of Fourier multiplier operators in multiple variables was initiated in the 1950s, as connections of the theory to partial differential equations became more fully realized[2]. It was quickly realized that the most fundamental estimates were $L^p \to L^q$ estimates for such operators. It is therefore natural to introduce the space $M^p(\mathbb{R}^d)$, consisting of all symbols $m$ which induce a Fourier multiplier operator $T$ bounded on $L^p(\mathbb{R}^d)$. Duality implies that $M^p(\mathbb{R}^d)$ is isometric to $M^{p'}(\mathbb{R}^d)$, where $p$ and $p'$ are conjugates, so it suffices to study the spaces $M^p(\mathbb{R}^d)$ where $1 \leq p \leq 2$ or when $2 \leq p \leq \infty$. We only know simple characterizations of $M^p(\mathbb{R}^d)$ for very particular $p$:

- The spaces $M^1(\mathbb{R}^d) = M^\infty(\mathbb{R}^d)$ can be characterized, by virtue of the fact that the boundedness of operators with domain $L^1(\mathbb{R}^d)$ or range $L^\infty(\mathbb{R}^d)$ is often simple; we have
$$M^1(\mathbb{R}^d) = \widehat{M}(\mathbb{R}^d),$$
Here $M(\mathbb{R}^d)$ is the space of all finite signed Borel measures, equipped with the total variation norm. The proof follows from Schur's Lemma for integral operators, which often gives tight estimates to bound operators with domain $L^1$ or range $L^\infty$.

- The unitary nature of the Fourier transform also allows for the characterization $M^2(\mathbb{R}^d) = L^\infty(\mathbb{R}^d)$. The proof follows from Parseval's identity.

It is perhaps surprising that these are the *only* known characterizations of the spaces $M^p(\mathbb{R}^d)$. No necessary and simple conditions for boundedness are known for any other values of $p$, and perhaps no simple characterization exists.

Despite the lack of a characterization of the classes $M^p(\mathbb{R}^d)$, it is surprising that we *can* conjecture a characterization of the subspace of $M^p(\mathbb{R}^d)$ consisting of *radial symbols*, for an appropriate range of exponents. The conjectured range of estimates was first suggested by a result of [4], concerning the boundedness of a radial Fourier multiplier $T$ with symbol $a(|\cdot|)$ *restricted to radial functions*, i.e. such that the norm

$$\|a\|_{M^p_{\text{Rad}}(\mathbb{R}^d)} = \sup \left\{ \frac{\|a(P)f\|_{L^q(\mathbb{R}^d)}}{\|f\|_{L^p(\mathbb{R}^d)}} : f \text{ is radial} \right\}$$

is finite. For any $\chi \in C_c^\infty(\mathbb{R})$, if we define $k_R$ to be the Fourier transform of $\chi(|\cdot|)a(R|\cdot|)$, then the identity $k_R = a(RP)\{\widehat{\chi}\}$ and dilation symmetry imply that

$$\|k_R\|_{L^p(\mathbb{R}^d)} \lesssim \|a\|_{M^p_{\text{Rad}}(\mathbb{R}^d)}, \tag{2.1}$$

with implicit constants depending on $\chi$. For $d > 1$, and for $1/p - 1/2 > 1/2d$, Garrigos and Seeger proved [4] the converse bound

$$\|a\|_{M^p_{\text{Rad}}(\mathbb{R}^d)} \lesssim \sup_{R>0} \|k_R\|_{L^p(\mathbb{R}^d)}, \tag{2.2}$$

[2]See [9] for a more detailed overview of what was known at this time.

Theorem 2.1 implies that

$$\sup_{R>0} \|k_R\|_{L^p(\mathbb{R}^d)} \sim \|a\|_{R^{s,p}[0,\infty)}, \tag{2.3}$$

where $s = (d-1)(1/p-1/2)$. Thus Garrigos and Seeger have proved that the isomorphism $M^p_{\text{Rad}}(\mathbb{R}^d) = R^{s,p}[0,\infty)$ holds in the range above.

For any other value of $p$, $M^p_{\text{Rad}}(\mathbb{R}^d)$ is a proper subset of $R^{s,p}[0,\infty)$, as the following counterexamples show:

- For $p = 1$, the inclusion 'fails by a logarithm', as we can see by the characterization of $M^( \mathbb{R}^d)$ in the last section; if $a$ is supported on an interval $I$ we have the converse inequality $\|a\|_{M^1(\mathbb{R}^d)} \lesssim \log|I| \; \|a\|_{R^{s,p}[0,\infty)}$ where dependence on $I$ is in general sharp, and cannot be removed.

- Suppose $p \geq 2d/(d+1)$. Then $R^{s,p}[0,\infty)$ contains all elements of the Besov space $B^{1/2,2}_p[0,\infty)$ supported on $[1/2,2]$, and thus, in particular, must contain unbounded functions for $p > 1$ (because the Sobolev embedding theorem fails when used at the endpoint to embed into $L^\infty$). Since $M^p(\mathbb{R}^d) \subset M^2(\mathbb{R}^d) = L^\infty[0,\infty)$, $M^p(\mathbb{R}^d)$ cannot contain unbounded functions, and thus $M^p(\mathbb{R}^d)$ is a proper subset of $R^{s,p}(\mathbb{R}^d)$.

It is natural to conjecture that the same constraint continues to hold when we remove the constraint that our inputs $f$ are radial, i.e so that for a radial function $m(\xi) = a(|\xi|)$, for $d > 1$, $1/2d < 1/p - 1/2 < 1/2$, and for $s = (d-1)(1/p-1/2)$,

$$\|m\|_{M^p(\mathbb{R}^d)} \sim_{p,d} \|a\|_{R^{p,s}[0,\infty)}. \tag{2.4}$$

We call this the *radial multiplier conjecture* on $\mathbb{R}^d$.

We now know, by results of Heo, Nazarov, and Seeger [7] that the radial multiplier conjecture is true when $d \geq 4$ and when $1/p - 1/2 > 1/(d+1)$. A summary of the proof strategies of this argument is provided in the following section, and a major part of our bounds for spectral multipliers follow by adapting this argument to the non-Euclidean setting. Partial improvements were obtained by Cladek [2] for symbols $m$ compactly supported away from the origin, obtaining results for compactly supported multipliers when $d = 4$ and $1/p - 1/2 > 11/36$, and establishing *restricted weak type* bounds when $d = 3$ and $1 < p < 11/26$. But the radial multiplier conjecture has not been resolved fully in any dimension $d$, we do not have any strong type $L^p$ bounds when $d = 3$, and no bounds whatsoever are known when $d = 2$.

## 2.3 Radial-Multiplier Bounds by Density Decompositions

In this section and the following, we give an overview of the proof of the radial multiplier bounds obtained by Heo, Nazarov, and Seeger in [7]. The main tool we will take away for applications to the proof of Theorem 1.2 is the method of *density decompositions*.

**Theorem 2.2.** *Suppose* $1 < p < 2(d-1)/(d+1)$*, and* $m(\xi) = a(|\xi|)$ *is a radial function. Then* $\|m\|_{M^p(\mathbb{R}^d)} \lesssim \|a\|_{R^{p,s}[0,\infty)}$*, where* $s = (d-1)(1/p-1/2)$.

In this section, we establish a single scale version of this result, as described in the following lemma.

**Lemma 2.3.** *Suppose* $1 < p < 2(d-1)/(d+1)$*, and* $k$ *is a radial function with Fourier transform supported on* $1/2 \leq |\xi| \leq 2$*. Then*

$$\|k * f\|_{L^p(\mathbb{R}^d)} \lesssim \|k\|_{L^p(\mathbb{R}^d)} \|f\|_{L^p(\mathbb{R}^d)} \tag{2.5}$$

We reduce Lemma (2.3) to an inequality for sums of functions oscillating on spheres. Let $\sigma_r$ be the surface measure for the sphere of radius $r$ centered at the origin in $\mathbb{R}^d$. Also fix a nonzero, radial, compactly supported function $\psi \in \mathcal{S}(\mathbb{R}^d)$ whose Fourier transform is non-negative, and vanishes to high order at the origin. Given $x \in \mathbb{R}^d$ and $r \geq 1$, define $\chi_{x,r} = \mathrm{Trans}_x(\sigma_r * \psi)$. Then $\chi_{x,r}$ is a smooth function adapted to a thickness $O(1)$ annulus of radius $r$ centered at $x$, which is *slightly oscillating*. We will verify the following lemma.

**Lemma 2.4.** *For any* $a : \mathbb{R}^d \times [1,\infty) \to \mathbb{C}$*, and if* $1/p - 1/2 > 1/(d+1)$*,*

$$\left\| \int_{\mathbb{R}^d} \int_1^\infty a(x,r) \chi_{x,r} \, dx \, dr \right\|_{L^p(\mathbb{R}^d)} \lesssim \left( \int_{\mathbb{R}^d} \int_1^\infty |a(x,r)|^p r^{d-1} dr dx \right)^{1/p}.$$

*The implicit constant here depends on* $p$*,* $d$*, and* $\psi$*.*

*Proof of Inequality* (2.3) *from Lemma 2.4.* Suppose $k(\cdot) = b(|\cdot|)$ for some function $b : [0,\infty) \to \mathbb{C}$. If we set $a(x,r) = f(x)b(r)$ for any function $f : \mathbb{R}^d \to \mathbb{C}$, then

$$k * \psi * f = \int_{\mathbb{R}^d} \int_1^\infty a(x,r) \chi_{x,r} \, dx \, dr, \tag{2.6}$$

Lemma 2.4 thus says that

$$\|k * \psi * f\|_{L^p(\mathbb{R}^d)} \lesssim \|k\|_{L^p(\mathbb{R}^d)} \|f\|_{L^p(\mathbb{R}^d)}. \tag{2.7}$$

If we choose $\psi$ so that $\widehat{\psi}$ is non-vanishing on the support of $k$, then the function $1/\widehat{\psi}(\cdot)$ is smooth on the support of $m$; if $T$ is a Fourier multiplier operator with a smooth, compactly supported symbol agreeing with $1/\widehat{\psi}(\cdot)$ on the support of $m$, then $T$ is bounded on $L^p(\mathbb{R}^d)$, and $T(k * \psi * f) = k * f$, and so we conclude that

$$\|k * \psi\|_{L^p(\mathbb{R}^d)} = \|T(k * \psi * f)\|_{L^p(\mathbb{R}^d)} \lesssim \|k * \psi * f\|_{L^p(\mathbb{R}^d)} \lesssim \|k\|_{L^p(\mathbb{R}^d)} \|f\|_{L^p(\mathbb{R}^d)}, \tag{2.8}$$

which completes the argument. □

Next, we consider a discretization of Lemma 2.4.

**Theorem 2.5.** *Fix a 1-separated set* $\mathcal{E} \subset \mathbb{R}^d \times [1,\infty)$*. Then for any* $a : \mathcal{E} \to \mathbb{C}$ *and if* $1/p - 1/2 > 1/(d+1)$*,*

$$\left\| \sum_{(x,r)\in\mathcal{E}} a(x,r) \chi_{x,r} \right\|_{L^p(\mathbb{R}^d)} \lesssim \left( \sum_{(x,r)\in\mathcal{E}} |a(x,r)|^p r^{d-1} \right)^{1/p},$$

*where the implicit constant is independent of* $\mathcal{E}$*.*

*Proof of Lemma 2.4 from Lemma 2.5.* For any $a : \mathbb{R}^d \times [1,\infty) \to \mathbb{C}$, if we consider the vector-valued function $\mathbf{a}(x,r) = a(x,r)\chi_{x,r}$, then

$$\int_{\mathbb{R}^d} \int_1^\infty \mathbf{a}(x,r)\, dr\, dx = \int_{[0,1)^d} \int_0^1 \sum\nolimits_{n \in \mathbb{Z}^d} \sum\nolimits_{m>0} \mathrm{Trans}_{n,m} \mathbf{a}(x,r)\, dr\, dx \tag{2.9}$$

The triangle inequality and the increasing property of norms on $[0,1)^d \times [0,1]$ imply that

$$\begin{aligned} &\left\| \int_{\mathbb{R}^d} \int_1^\infty \mathbf{a}(x,r)\, dr\, dx \right\|_{L^p(\mathbb{R}^d)} \\ &\quad \le \int_{[0,1)^d} \int_0^1 \left\| \sum\nolimits_{n \in \mathbb{Z}^d} \sum\nolimits_{m>0} \mathrm{Trans}_{n,m} \mathbf{a}(x,r) \right\|_{L^p(\mathbb{R}^d)} dr\, dx \\ &\quad \lesssim \int_{[0,1)^d} \int_0^1 \left( \sum\nolimits_{n \in \mathbb{Z}^d} \sum\nolimits_{m>0} |a(x-n, r+m)|^p r^{d-1} \right)^{1/p} dr\, dx \qquad (2.10) \\ &\quad \le \left( \int_{[0,1)^d} \int_0^1 \sum\nolimits_{n \in \mathbb{Z}^d} \sum\nolimits_{m>0} |a(x-n, r+m)|^p r^{d-1}\, dr\, dx \right)^{1/p} \\ &\quad = \left( \int_{\mathbb{R}^d} \int_1^\infty |a(x,r)|^p r^{d-1} dr dx \right)^{1/p}, \end{aligned}$$

which completes the proof. □

Lemma 2.5 can be further reduced by considering it as a bound on the operator

$$a \mapsto \sum\nolimits_{(x,r) \in \mathcal{E}} a(x,r)\chi_{x,r}.$$

In particular, applying a real interpolation, since we are proving a result for an open range of $p$, it suffices for us to prove a restricted strong type bound. Given any discretized set $\mathcal{E}$, let $\mathcal{E}_k$ be the set of $(x,r) \in \mathcal{E}$ with $2^k \le r < 2^{k+1}$. Then Lemma 2.5 is implied by the following Lemma.

**Lemma 2.6.** *For $1/p - 1/2 > 1/(d+1)$ and $k \ge 1$,*

$$\left\| \sum\nolimits_{(x,r) \in \mathcal{E}} \chi_{x,r} \right\|_{L^p(\mathbb{R}^d)} \lesssim \left( \sum\nolimits_{k \ge 1} 2^{k(d-1)} \#(\mathcal{E}_k) \right)^{1/p}.$$

*Remark.* Note that if $r \sim 2^k$, then $\|\chi_{x,r}\|_{L^p(\mathbb{R}^d)} \sim 2^{k(d-1)/p}$, and so Lemma 2.6 says

$$\left\| \sum\nolimits_{(x,r) \in \mathcal{E}} \chi_{x,r} \right\|_{L^p(\mathbb{R}^d)} \lesssim_p \left( \sum\nolimits_{(x,r) \in \mathcal{E}} \|\chi_{x,r}\|_{L^p(\mathbb{R}^d)}^p \right)^{1/p}. \tag{2.11}$$

Thus we are proving a $p$th root cancellation bound.

To control these sums, we apply a 'density decomposition', somewhat analogous to a Calderon Zygmund decomposition, which will enable us to obtain $L^2$ bounds. We say a 1-separated set $\mathcal{E}$ in a metric space $X$ is of *density type* $(u,r)$ if $\#(B \cap \mathcal{E}) \le u \cdot \mathrm{diam}(B)$ for each ball $B$ in $X$ with diameter at most $r$.

**Theorem 2.7.** *For any family of 1-separated sets $\mathcal{E}_k \subset \mathbb{R}^d \times [2^k, 2^{k+1})$, there exists a decomposition $\mathcal{E}_k = \bigcup_{m=1}^{\infty} \mathcal{E}_k(2^m)$ with the following properties:*

- *For each $m$, $\mathcal{E}_k(2^m)$ has density type $(2^m, 2^k)$.*
- *If $B$ is a ball in $\mathbb{R}^{d+1}$ of radius $r \leq 2^k$ containing at least $2^m \cdot r$ points of $\mathcal{E}_k$, then*

$$B \cap \mathcal{E}_k \subset \bigcup_{m' \geq m} \mathcal{E}_k(2^{m'}).$$

- *For each $m$, there are disjoint balls $\{B_i\}$ in $\mathbb{R}^{d+1}$ with radii $\{r_i\}$, such that*

$$\sum_i r_i \leq 2^{-m} \# \mathcal{E}_k,$$

  *such that $r_i \leq 2^k$ for all $i$, and such that $\bigcup B_i^*$ covers $\bigcup_{m' \geq m} \mathcal{E}_k(2^{m'})$, where $B_i^*$ denotes the ball with the same center as $B_i$ but 5 times the radius.*

*Proof.* Define a function $M : \mathcal{E}_k \to [0, \infty)$ by setting

$$M(x, r) = \sup \left\{ \frac{\#(\mathcal{E}_k \cap B)}{\mathrm{rad}(B)} : (x, r) \in B \text{ and } \mathrm{rad}(B) \leq 2^k \right\}.$$

We can establish a kind of weak $L^1$ estimate for $M$ using a Vitali type argument. Let

$$\widehat{\mathcal{E}_k}(2^m) = \{(x, r) \in \mathcal{E}_k : M(x, r) \geq 2^m\}.$$

We can therefore cover $\widehat{\mathcal{E}_k}(2^m)$ by a family of balls $\{B\}$ such that $\#(\mathcal{E}_k \cap B) \geq 2^m \mathrm{rad}(B)$. The Vitali covering lemma allows us to find a disjoint subcollection of balls $B_1, \dots, B_N$ such that $B_1^*, \dots, B_N^*$ covers $\widehat{\mathcal{E}_k}(2^m)$. We find that

$$\#(\mathcal{E}_k) \geq \sum_i \#(B_i \cap \mathcal{E}_k) \geq 2^m \sum_i \mathrm{rad}(B_i), \tag{2.12}$$

Setting $\mathcal{E}_k = \widehat{\mathcal{E}_k}(2^m) - \bigcup_{k' > k} \widehat{\mathcal{E}_{k'}}(2^m)$ thus gives the required result. □

*Remark.* We will apply the Lemma with $\mathbb{R}^d \times [0, \infty)$ replaced by $X \times [0, \infty)$, where $X$ is a compact, $d$-dimensional manifold equipped with a Finsler metric. A version of the Vitali covering lemma also holds for this metric space, so that the same proof allows one to perform a density decomposition in this setting.

To prove Lemma 2.6, we perform a decomposition of $\mathcal{E}_k$ for each $k$, into the sets $\mathcal{E}_k(2^m)$, and then define $\mathcal{E}^m = \bigcup_{k \geq 1} \mathcal{E}_k^m$. For appropriate exponents, we will prove $L^p$ bounds on the functions

$$F^m = \sum_{(x,r) \in \mathcal{E}^m} \chi_{x,r}$$

which are exponentially decaying in $m$, i.e. that

$$\|F^m\|_{L^p(\mathbb{R}^d)} \lessapprox 2^{m\left(\frac{1}{d+1} - (1/p - 1/2)\right)} \left( \sum_k 2^{k(d-1)} \#(\mathcal{E}_k) \right)^{1/p},$$

where the implicit constant in the inequality can depend polynomially on $m$. Thus, in the range $1/p - 1/2 > 1/(d+1)$, there is geometric decay in $m$, decaying much quicker than the polynomial implicit constant, and so we may sum in $m$ using the triangle inequality to conclude that

$$\|F\|_{L^p(\mathbb{R}^d)} \lesssim \left(\sum_k 2^{k(d-1)}\#(\mathcal{E}_k)\right)^{1/p},$$

proving Lemma 2.6. To get the bound on $F^m$, we interpolate between an $L^2$ bound for $F^m$, and an $L^0$ bound (i.e. a bound on the measure of the support of $F^m$). First, we calculate the support of $F^m$.

**Lemma 2.8.** *For each $k$,*

$$|supp(F_k^m)| \lesssim 2^{-m}2^{k(d-1)}\#\mathcal{E}_k.$$

*Thus we have*

$$|supp(F^m)| \leq \sum_k |supp(F_k^m)| \lesssim \sum_k 2^{-m}2^{k(d-1)}\#\mathcal{E}_k.$$

*Proof.* We recall that for each $k$ and $m$, we can find disjoint balls $B_1, \dots, B_N$ with radii $r_1, \dots, r_N \leq 2^k$ such that

$$\sum_{i=1}^N r_i \leq 2^{-m}\#\mathcal{E}_k,$$

where $\mathcal{E}_k(2^m)$ is covered by the expanded balls $B_1^* \cup \dots \cup B_N^*$. If we write

$$F_{k,i}^m = \sum_{(x,r)\in\mathcal{E}_k(2^m)\cap B_i^*} \chi_{x,r},$$

then $\text{supp}(F_k^m) \subset \bigcup_i \text{supp}(F_{k,i}^m)$. For each $(x,r) \in B_i^* \cap \mathcal{E}_k(2^m)$, the support of $\chi_{x,r}$, an annulus of thickness $O(1)$ and radius $r$, is contained in an annulus of thickness $O(r_i)$ and radius $O(2^k)$ with the same centre as $B_i$. Thus we conclude that

$$|\text{supp}(F_{k,i}^m)| \lesssim r_i 2^{k(d-1)},$$

and it follows that

$$|\text{supp}(F_k^m)| \leq \sum_i r_i 2^{k(d-1)} \leq 2^{-m}2^{k(d-1)}\#\mathcal{E}_k. \qquad \square$$

Interpolating, it suffices to prove the following $L^2$ estimate on the function $F^m$.

**Lemma 2.9.** *Suppose $\mathcal{E} = \bigcup_k \mathcal{E}_k$ is a 1-separated set, where $\mathcal{E}_k \subset \mathbb{R}^d \times [2^k, 2^{k+1})$ is a set of density type $(2^m, 2^k)$. Then*

$$\left\|\sum_{(x,r)\in\mathcal{E}} \chi_{x,r}\right\|_{L^2(\mathbb{R}^d)} \lesssim \sqrt{m} \cdot 2^{\frac{m}{d-1}} \left(\sum_k 2^{k(d-1)}\#(\mathcal{E}_k)\right)^{1/2}.$$

The $L^2$ bound in Lemma 2.9 gets worse and worse as $m$ grows, whereas the $L^0$ bound in Lemma 2.8 gets better and better, since annuli are concentrating in a small set, which is bad from the perspective of constructive interference, but absolutely fine from the perspective of a support bound. To prove the $L^2$ bound, we require an analysis of the interference patterns of pairs of the functions $\chi_{x,r}$, as provided by the following lemma.

**Lemma 2.10.** *For any $N > 0$, $x_1, x_2 \in \mathbb{R}^d$ and $r_1, r_2 \geq 1$,*

$$|\langle \chi_{x_1,r_1}, \chi_{x_2,r_2} \rangle| \lesssim_N \left( \frac{r_1 r_2}{\langle (x_1, r_1) - (x_2, r_2) \rangle} \right)^{\frac{d-1}{2}} \sum_{\pm,\pm} \langle |x_1 - x_2| \pm r_1 \pm r_2 \rangle^{-N}.$$

*Remark.* Suppose $r_1 \leq r_2$. Then Lemma 2.10 implies that $\chi_{x_1,r_1}$ and $\chi_{x_2,r_2}$ are roughly uncorrelated, except when they are supported on annuli that roughly have the same radii and centers, and in addition, one of the following two properties hold:

- $r_1 + r_2 \approx |x_1 - x_2|$, which holds when the two annuli are 'approximately' externally tangent to one another.
- $r_2 - r_1 \approx |x_1 - x_2|$, which holds when the two annuli are 'approximately' internally tangent to one another.

Heo, Nazarov, and Seeger do not exploit the tangency information, though utilizing the tangencies seems important to improve the results they obtain. Cladek exploits this tangency information further, to obtain improved results.

*Proof.* We write

$$\begin{aligned}
\langle \chi_{x_1 r_1}, \chi_{x_2 r_2} \rangle &= \langle \widehat{\chi}_{x_1 r_1}, \widehat{\chi}_{x_2 r_2} \rangle \\
&= \int_{\mathbb{R}^d} \widehat{\sigma_{r_1} * \psi}(\xi) \cdot \overline{\widehat{\sigma_{r_2} * \psi}(\xi)} e^{2\pi i (x_2 - x_1) \cdot \xi} \, d\xi \\
&= (r_1 r_2)^{d-1} \int_{\mathbb{R}^d} \widehat{\sigma}(r_1 \xi) \overline{\widehat{\sigma}(r_2 \xi)} |\widehat{\psi}(\xi)|^2 e^{2\pi i (x_2 - x_1) \cdot \xi} \, d\xi.
\end{aligned}$$

Define functions $A$ and $B$ such that $B(|\xi|) = \widehat{\sigma}(\xi)$, and $A(|\xi|) = |\widehat{\psi}(\xi)|^2$. Then

$$\langle \chi_{x_1,r_1}, \chi_{x_2,r_2} \rangle = C_d (r_1 r_2)^{d-1} \int_0^\infty s^{d-1} A(s) B(r_1 s) B(r_2 s) B(|x_2 - x_1| s) \, ds.$$

Using well known asymptotics for $\widehat{\sigma}$, we have, for any $N > 0$,

$$B(s) = s^{-(d-1)/2} \sum_{n=0}^{N-1} (c_{n,+} e^{2\pi i s} + c_{n,-} e^{-2\pi i s}) s^{-n} + O_N(s^{-N}).$$

Now, assuming $A(s)$ vanishes to order $\gtrsim N$ at the origin, we conclude that

$$\begin{aligned}
\langle \chi_{x_1 r_1}, \chi_{x_2 r_2} \rangle = C_d &\left( \frac{r_1 r_2}{|x_1 - x_2|} \right)^{(d-1)/2} \sum_{n,\tau} c_{n,\tau} r_1^{-n_1} r_2^{-n_2} |x_2 - x_1|^{-n_3} \\
&\left\{ \int_0^\infty A(s) s^{-(d-1)/2} s^{-n_1 - n_2 - n_3} e^{2\pi i (\tau_1 r_1 + \tau_2 r_2 + \tau_3 |x_2 - x_1|) s} \, ds \right\} \\
&\lesssim_N \left( \frac{r_1 r_2}{|x_1 - x_2|} \right)^{\frac{d-1}{2}} \left( 1 + \frac{1}{|x_1 - x_2|^N} \right) \sum_\tau (1 + |\tau_1 r_1 + \tau_2 r_2 + \tau_3 |x_2 - x_1||)^{-5N} \\
&\lesssim_N \left( \frac{r_1 r_2}{|x_1 - x_2|} \right)^{\frac{d-1}{2}} \left( 1 + \frac{1}{|x_1 - x_2|^N} \right) \sum_\tau (1 + |\tau_1 r_1 + \tau_2 r_2 + |x_2 - x_1||)^{-5N}.
\end{aligned}$$

This gives the result provided that $1 + |x_1 - x_2| \geq |r_1 - r_2|/10$ and $|x_1 - x_2| \geq 1$. If $1+|x_1-x_2| \leq |r_1-r_2|/10$, then the supports of $\chi_{x_1,r_1}$ and $\chi_{x_2,r_2}$ are disjoint, so the inequality is trivial. On the other hand, if $|x_1 - x_2| \leq 1$, then the bound is trivial by the last sentence unless $|r_1 - r_2| \leq 10$, and in this case the inequality reduces to the simple inequality

$$\langle \chi_{x_1,r_1}, \chi_{x_2,r_2} \rangle \lesssim_N (r_1 r_2)^{(d-1)/2}.$$

But this follows immediately from the Cauchy-Schwartz inequality. □

The exponent $(d-1)/2$ in Lemma 2.10 is too weak to apply almost orthogonality directly to obtain $L^2$ bounds on $\sum_{(x,r)\in\mathcal{E}_k} \chi_{xr}$ on it's own, but together with the density decomposition assumption we will be able to obtain Lemma 2.9.

*Proof of Lemma 2.9.* Without loss of generality, we may assume that the set of $k$ such that $\mathcal{E}_k \neq \emptyset$ is 10-separated. Write

$$F = \sum_{(x,r)\in\mathcal{E}} \chi_{x,r}$$

and $F_k = \sum_{(x,r)\in\mathcal{E}_k} \chi_{x,r}$. First, we deal with $F_{\lesssim m} = \sum_{k \leq 10m} F_k$ trivially, i.e. writing

$$\|F\|_{L^2(\mathbb{R}^d)} \lesssim m^{1/2} \left( \sum_{k\leq 10m} \|F_k\|^2_{L^2(\mathbb{R}^d)} + \| \sum_{k>10m} F_k \|_{L^2(\mathbb{R}^d)} \right)^{1/2}.$$

We then decompose

$$\| \sum_{k>10m} F_k \|^2_{L^2(\mathbb{R}^d)} \leq \sum_{k>10m} \|F_k\|^2_{L^2(\mathbb{R}^d)} + 2 \sum_{k'>k>10m} |\langle F_k, F_{k'} \rangle|.$$

Let us analyze $\langle F_k, F_{k'} \rangle$. The term will become a sum of the form $\langle \chi_{x,r}, \chi_{y,s} \rangle$, where $r \sim 2^k$ and $s \sim 2^{k'}$. Because of our assumption of being 10-separated, we have $r \leq s/2^{10}$. If $\langle \chi_{x,r}, \chi_{y,s} \rangle \neq 0$, then since the support of $\chi_{y,s}$ is an annulus of radius $s$ centered at $y$, with thickness $O(1)$, and $\chi_{x,r}$ has support on an annulus of radius $r$ centered at $x$, with thickness $O(1)$, the fact that $r$ is comparatively smaller than $s$ implies that $(x, r)$ must be contained in the annulus of radius $s$ centered at $y$, with thickness $O(2^k)$. Such an annulus is covered by $O(2^{(k'-k)(d-1)})$ balls of radius $2^k$. Each ball can only contain $2^{k+m}$ points $(x, r)$, and so there can be at most

$$O(2^{k'(d-1)} 2^{-k(d-1)} 2^{k+m}) = O(2^{k'(d-1)-k(d-2)+m}).$$

pairs $(x, r) \in \mathcal{E}_k$ for which $\langle \chi_{x,r}, \chi_{y,s} \rangle \neq 0$. For such pairs we have

$$|\langle \chi_{x,r}, \chi_{y,s} \rangle| \lesssim \left( \frac{2^k 2^{k'}}{2^{k'}} \right)^{\frac{d-1}{2}} = 2^{\frac{k(d-1)}{2}}.$$

Thus we conclude that

$$|\langle F_k, \chi_{y,s} \rangle| \lesssim 2^{-k(\frac{d-3}{2})+k'(d-1)+m}.$$

Summing over $10m < k < k'$, we conclude that since $d \geq 4$,

$$\sum_{10m<k<k'} |\langle F_k, \chi_{y,s} \rangle| \lesssim 2^{k'(d-1)+m} \sum_{10m<k<k'} 2^{-k\frac{d-3}{2}} \lesssim 2^{k'(d-1)+m} 2^{-5m} \lesssim 2^{k'(d-1)}.$$

But this means that

$$\sum_{10m<k<k'} |\langle F_k, F_{k'}\rangle| \lesssim 2^{k'(d-1)} \cdot \#(\mathcal{E}_{k'}).$$

This means that

$$\|\sum_{k>10m} F_k\|^2_{L^2(\mathbb{R}^d)} \lesssim \sum_{k>10m} \|F_k\|^2_{L^2(\mathbb{R}^d)} + \sum_{k'} 2^{k'(d-1)}\#(\mathcal{E}_{k'}),$$

and it now suffices to deal with estimates the $\|F_k\|_{L^2(\mathbb{R}^d)}$, i.e. the interactions of functions supported on radii of comparable magnitude. To deal with these, we further decompose the radii, writing $[2^k, 2^{k+1})$ as the disjoint union of intervals $I_{k,\mu} = [2^k+(\mu-1)2^{am}, 2^k+\mu 2^{am}]$, for some $a$ to be chosen later. These interval induces a decomposition $\mathcal{E}_k = \bigcup_\mu \mathcal{E}_{k,\mu}$. Again, incurring a constant loss at most, we may assume that the $\mu$ such that $\mathcal{E}_{k,\mu} \neq \emptyset$ are 10 separated. We write $F_k = \sum F_{k,\mu}$, and we have

$$\|F_k\|^2_{L^2(\mathbb{R}^d)} = \sum_\mu \|F_{k,\mu}\|^2_{L^2(\mathbb{R}^d)} + \sum_{\mu<\mu'} |\langle F_{k,\mu}, F_{k,\mu'}\rangle|.$$

We now consider $\chi_{x,r}$ and $\chi_{y,s}$ with $r \in I_{k,\mu}$ and $s \in I_{k,\mu'}$. Then we must have $|x-y| \lesssim 2^k$ and $2^{am} \leq |r-s| \lesssim 2^k$, and so we have

$$\begin{aligned} |\sum_{\mu<\mu'} \langle F_{k,\mu}, \chi_{y,s}\rangle| &\lesssim 2^{k(d-1)} \sum_{\substack{(x,r)\in\mathcal{E}_k \\ 2^{am}\leq|(x,r)-(y,s)|\lesssim 2^k}} |(x,r)-(y,s)|^{-\frac{d-1}{2}} \\ &\lesssim 2^{k(d-1)} \sum_{am\leq l\leq k} 2^{-l(d-1)/2}\#\{(x,r)\in\mathcal{E}_k : |(x,r)-(y,s)| \sim 2^l\}. \end{aligned}$$

Using the density assumption,

$$\#\{(x,r)\in\mathcal{E}_k : |(x,r)-(y,s)| \sim 2^l\} \lesssim 2^{l+m}$$

and so we obtain that, again using the assumption that $d \geq 4$,

$$|\sum_{\mu<\mu'} \langle F_{k,\mu}, \chi_{y,s}\rangle| \lesssim 2^{k(d-1)} 2^{m(1-a(d-3)/2)}.$$

Now summing over all $(y,s)$, we obtain that

$$|\sum_{\mu<\mu'} \langle F_{k,\mu}, F_{k,\mu'}\rangle| \lesssim 2^{k(d-1)} 2^{m(1-a(d-3)/2)}\#(\mathcal{E}_{k,\mu'}).$$

and now summing over $\mu'$ gives that

$$\|F_k\|^2_{L^2(\mathbb{R}^d)} \lesssim \sum_\mu \|F_{k,\mu}\|^2_{L^2(\mathbb{R}^d)} + 2^{k(d-1)} 2^{m(1-a(d-3)/2)}\#\mathcal{E}_k,$$

which is a good enough bound if we pick $a$ to be large enough. Now we are left to analyze $\|F_{k,\mu}\|_{L^2(\mathbb{R}^d)}$, i.e. analyzing interactions between annuli which have radii differing from one another by at most $O(2^{am})$. Since the family of all possible radii are discrete, the set $\mathcal{R}_{k,\mu}$ of all possible radii has cardinality $O(2^{am})$. We do not really have any orthogonality to play with here, so we just apply Cauchy-Schwartz, writing $F_{k,\mu} = \sum_{r\in\mathcal{R}_{k,\mu}} F_{k,\mu,r}$, to write

$$\|F_{k,\mu}\|^2_{L^2(\mathbb{R}^d)} \lesssim 2^{am} \sum_r \|F_{k,\mu,r}\|^2_{L^2(\mathbb{R}^d)}.$$

Recall that $\chi_{x,r} = \mathrm{Trans}_x(\sigma_r * \psi)$, where $\psi$ is a compactly supported function whose Fourier transform is non-negative and vanishes to high order at the origin. In particular, we now make the additional assumption that $\psi = \psi_\circ * \psi_\circ$ for some other compactly function $\psi_\circ$ whose Fourier transform is non-negative and vanishes to high order at the origin. Then we find that $F_{k,\mu,r}$ is equal to the convolution of the function

$$A_r = \sum_{(x,r)\in\mathcal{E}} \mathrm{Trans}_x \psi_\circ$$

with the function $\sigma_r * \psi_\circ$. Using the standard asymptotics for the Fourier transform of $\sigma_r$, i.e. that for $|\xi| \geq 1$,

$$|\widehat{\sigma_r}(\xi)| \lesssim r^{d-1}(1 + r|\xi|)^{-\frac{d-1}{2}},$$

and since $|\widehat{\psi_\circ}(\xi)| \lesssim_N |\xi|^N$, we get that if $r \geq 1$, then for $|\xi| \leq 1/r$,

$$|\widehat{\sigma_r}(\xi)\widehat{\psi_\circ}(\xi)| \lesssim_N r^{d-1-N}$$

and for $|\xi| \geq 1/r$,

$$|\widehat{\sigma_r}(\xi)\widehat{\psi_\circ}(\xi)| \lesssim_N r^{\frac{d-1}{2}}|\xi|^{-N}.$$

Thus in particular,the $L^\infty$ norm of the Fourier transform of $\sigma_r * \psi_\circ$ is $O(r^{(d-1)/2})$. Now the functions $\psi_\circ$ are compactly supported, so since the set of $x$ such that $(x, r) \in \mathcal{E}$ is one-separated, we find that

$$\|A_r\|_{L^2(\mathbb{R}^d)} \lesssim \#\{x : (x, r) \in \mathcal{E}\}^{1/2}.$$

But this means that

$$\|F_{k,\mu,r}\|_{L^2(\mathbb{R}^d)} = \|A_r * (\sigma_r * \psi_\circ)\|_{L^2(\mathbb{R}^d)} \lesssim r^{\frac{d-1}{2}}\#\{x : (x, r) \in \mathcal{E}\}^{1/2}.$$

Thus we have that

$$\|F_{k,\mu}\|^2_{L^2(\mathbb{R}^d)} = 2^{am} \cdot \#\mathcal{E}_{k,\mu} \cdot 2^{k(d-1)}.$$

Summing over $\mu$ gives that

$$\|F_k\|^2_{L^2(\mathbb{R}^d)} = 2^{k(d-1)}\#\mathcal{E}_k(2^{am} + 2^{m(1-a(d-3)/2)}).$$

Picking $a = 2/(d - 1)$ optimizes this bound, giving

$$\|F_k\|_{L^2(\mathbb{R}^d)} \lesssim 2^{m/(d-1)}2^{k(d-1)/2}(\#\mathcal{E}_k)^{1/2}.$$

Plugging this into the estimates we got for $F$ gives the required bound. □

This completes a proof of the single scale estimates of the paper. The paper then uses an atomic decomposition method to combine these scales and thus complete the proof of Theorem 2.2. Rather than discuss these methods, we instead discuss an iteration of this method, developed by the same authors in a follow up paper [6].

## 2.4 Combining Scales with Atomic Decompositions

We now sketch a proof as to how we can complete the proof of Theorem 2.2, being deliberately vague, as our goal is to build intuition about the techniques, in order to carry out analogous methods more rigorously in the compact manifold setting later. If $T = m(P)$, then the last section proves that if we write $T = \sum T_j$, where $T_j$ is the radial Fourier multiplier operator with convolution kernel $k_j(\cdot/2^j)$, then the results of the last section (appropriately rescaled), imply that

$$\|T_j\|_{L^p(\mathbb{R}^d)\to L^q(\mathbb{R}^d)} \lesssim \|k_j\|_{L^q(\mathbb{R}^d)}. \tag{2.13}$$

The goal of this section is to prove that we can bound the sum of the operators $T_j$ by the suprema of the quantities on the right hand side in (2.13) which, morally speaking, is a bound of the form

$$\left\|\sum_j T_j\right\| \lesssim \sup_j \|T_j\|.$$

We must thus show that the operators $\{T_j\}$ do not constructively interfere with one another to a significant extent.

Atomic decompositions are a powerful way to control interactions between operators. The method involves decomposing elements of function spaces into more elementary components, which we call *atoms*, that have controlled size, spatial support, and oscillatory properties. Such atoms are chosen via careful, non-linear selection processes, often related to stopping times, to ensure certain cancellation conditions hold. We use a variant of this method, involving a use of a Whitney decomposition rather than a stopping time argument, is used to obtain an atomic decomposition where atoms do not cluster together at each frequency scale, in the sense that the doubles of the cubes upon which the atoms are supported have the bounded overlap property.

Since we are controlling different operators supported on different dyadic frequency ranges on functions in $L^p(\mathbb{R}^d)$ for $1 < p < \infty$, it is natural to consider the Littlewood-Paley inequality $\|Sf\|_{L^p(\mathbb{R}^d)} \lesssim \|f\|_{L^p(\mathbb{R}^d)}$, where

$$Sf(x) = \left(\sum_j |P_j f(x)|^2\right)^{1/2},$$

and the operators $P_j$ are Littlewood-Paley projections, Fourier multiplier operators with symbol $\psi(\cdot/2^j)$, where $\psi \in C_c^\infty(\mathbb{R}^d)$ is equal to one on the support of the function $\chi$ used to decompose $T$ into the operators $T_j$ above, so that $T_j f = T_j f_j$, where $f_j = P_j f$. Roughly speaking, our atomic decomposition will be of the following form: for each dyadic number $H$, we consider a family of dyadic cubes $\mathcal{W}_H$, whose union is the set $\{x : |Sf(x)| \sim H\}$, and whose doubles have the bounded overlap property. We will then obtain a decomposition $f = \sum a_{j,H,W}$, where $a_{j,H,W}$ has Fourier support on $|\xi| \sim 2^j$, is supported on the cube $W$, and for each fixed $H$, $(\sum_j |\sum_W |a_{j,H,W}| |^2)^{1/2} \sim H$.

The advantage of this decomposition is that it controls how many *local interactions* the atoms $\{a_{j,H,W}\}$ have. To see why this might be useful, suppose we were considering a Fourier multiplier operator $T$ with the property that $Ta_{j,H,W}$ is essentially supported on the cube $W^*$ obtained by doubling the sidelengths of the cube $W$, but maintaining the same centre. The bounded overlap property of the dyadic cubes $\mathcal{W}_H$ would then imply an almost orthogonality bound for each $H$, that

$$\begin{aligned}\left\|\sum_{j,W} T_j a_{j,H,W}\right\|_{L^2(\mathbb{R}^d)} &\lesssim \left(\sum_{j,W} \|T_j a_{j,H,W}\|_{L^2(\mathbb{R}^d)}^2\right)^{1/2} \\ &\lesssim \left(\sum_{j,W} \|a_{j,H,W}\|_{L^2(\mathbb{R}^d)}^2\right)^{1/2} \\ &= \left\|\left(\sum_{j,W} |a_{j,H,W}|^2\right)^{1/2}\right\|_{L^2(\mathbb{R}^d)} \lesssim H|\Omega_H|^{1/2}\end{aligned}$$

Hölder's inequality then justifies that

$$\left\|\sum_{j,W} T_j a_{j,H,W}\right\|_{L^1(\mathbb{R}^d)} \lesssim |\Omega_H|^{1/2} \left\|\sum_{j,W} T_j a_{j,H,W}\right\|_{L^2(\mathbb{R}^d)} \lesssim H|\Omega_H|.$$

Real interpolation allows us to sum in $H$, and conclude that for $1 < p < 2$,

$$\|Tf\|_{L^p(\mathbb{R}^d)} = \left\|\sum_{H,j,W} T_j a_{j,H,W}\right\|_{L^p(\mathbb{R}^d)} \lesssim \left(\sum \left[H|\Omega_H|^{1/p}\right]^p\right)^{1/p} \lesssim \|Sf\|_{L^p(\mathbb{R}^d)} \lesssim \|f\|_{L^p(\mathbb{R}^d)}.$$

and thus we have proven the boundedness of such operators.

Such a simple proof will not suffice for our analysis, since the class of multipliers we are considering is not pseudolocal at all; the kernels $k_j$ are only assumed to be uniformly bounded in $L^p(\mathbb{R}^d)$, and need not satisfy any decay bound as $|x| \to \infty$ (though the atomic decomposition methods above can be used to prove bounds for other more pseudolocal multipliers, e.g. obtaining an alternate proof of the endpoint results of [18]). We will, however, be able to exploit the above calculations to control *close range interactions* of general multipliers, i.e. for any multiplier $T$, writing

$$Tf = \sum T_j a_{j,H,W} = \sum T_{j,W,\mathrm{Short}} a_{j,H,W} + T_{j,W,\mathrm{Long}} a_{j,H,W},$$

where

$$T_{j,W,\mathrm{Short}}(x,y) = \mathbb{I}_{W^*}(y)\; T_{j,W}(x,y) \quad \text{and} \quad T_{j,W,\mathrm{Short}}(x,y) = (1 - \mathbb{I}_{W^*}(y))\; T_{j,W}(x,y).$$

The above analysis can be adapted to show that $\|\sum T_{j,W,\mathrm{Short}} a_{j,H,W}\|_{L^q(\mathbb{R}^d)} \lesssim \|f\|_{L^p(\mathbb{R}^d)}$, and it remains to find a way to control the long range interactions between atoms.

There are several related methods to control these interactions, the most elegant formulation provided in the paper [6]. The idea is to take a singular scale estimate, and upgrade this to an estimate with a geometrically decaying constant term when our inputs have amplitudes that are locally constant on a much larger scale than is required by the uncertainty principle and the frequency support of the inputs, namely, that

$$\left\|\sum_H \sum_{W \in \mathcal{W}_{H,l-j}} T_{j,W,\mathrm{Long}} * a_{H,W}\right\|_{L^q(\mathbb{R}^d)} \lesssim 2^{-l\varepsilon} \left(\sum_H \sum_{W \in \mathcal{W}_{H,l-j}} \|a_{H,W}\|_{L^\infty(\mathbb{R}^d)}^p |W|\right)^{1/p},$$

where $\mathcal{W}_{H,a}$ are the set of all cubes of sidelength $2^a$. Summing in frequencies using $L^p$ orthogonality, we find that

$$\left\|\sum_j \sum_H \sum_{W\in\mathcal{W}_{H,l-j}} T_{j,W,\mathrm{Long}} a_{j,H,W}\right\|_{L^p(\mathbb{R}^d)} \lesssim 2^{-l\varepsilon}\left(\sum_j \sum_H \sum_{W\in\mathcal{W}_{H,l-j}} \|a_{j,H,W}\|^p_{L^\infty(\mathbb{R}^d)}|W|\right)^{1/p},$$

For a fixed $l$, and each $W \in \mathcal{W}_H$, there exists a unique $j$ such that $W \in \mathcal{W}_{H,l-j}$, which implies that the supports of the functions $\{a_{j,H,W}\}$ in the sum above are almost disjoint, and thus the simple bound $\|a_{j,H,W}\|_{L^\infty(\mathbb{R}^d)} \lesssim H$ gives that

$$\sum_j \sum_{W\in\mathcal{W}_{H,l-j}} \|a_{j,H,W}\|^p_{L^\infty(\mathbb{R}^d)}|W| \leq H^p|\Omega_H|,$$

and thus that

$$\left\|\sum_j \sum_H \sum_{W\in\mathcal{W}_{H,l-j}} T_{j,W,\mathrm{Long}} a_{j,H,W}\right\|_{L^p(\mathbb{R}^d)} \lesssim 2^{-l\varepsilon}\left(\sum_H H^p|\Omega_H|\right)^{1/p} \lesssim 2^{-l\varepsilon}\|f\|_{L^p(\mathbb{R}^d)}.$$

Summing in $l$ trivially using the triangle inequality and the geometric decay in $l$ gives that

$$\left\|\sum_j \sum_H \sum_{W\in\mathcal{W}_H} T_{j,W,\mathrm{Long}} a_{j,H,W}\right\|_{L^p(\mathbb{R}^d)} \lesssim \|f\|_{L^p(\mathbb{R}^d)},$$

which controls the long-range interactions in full.

## 2.5 Boundedness of Quasi-Radial Multipliers and Local Smoothing Inequalities

Kim [12] has extended the bounds of Heo-Nazarov-Seeger to quasiradial multipliers, i.e. multipliers of the form $(a \circ r)(\xi)$, where $r : \mathbb{R}^d \to [0,\infty)$ is a smooth, homogeneous function of order one, such that the cosphere $S = \{\xi : r(\xi) = 1\}$ is a hypersurface with non-vanishing Gauss curvature. Such multipliers retain the translation and dilation symmetries of the family of radial Fourier multiplier operators, but are no longer *rotation-invariant*.

Using the same identities as for radial Fourier multiplier operators,, one can verify that

$$\|a \circ r\|_{M^p(\mathbb{R}^d)} \gtrsim \sup_R \|k_R\|_{L^q(\mathbb{R}^d)},$$

where $k_j$ is the Fourier transform of $\chi(\cdot)(a\circ r)(R\cdot)$, with $\chi$ as in that section. In an analysis of Bochner-Riesz type quasi-radial multipliers, Lee and Seeger [14] showed that

$$\sup_R \|k_R\|_{L^p(\mathbb{R}^d)} \sim \|a\|_{R^{p,s}[0,\infty)},$$

Kim [12] has verified that in the range $1/p - 1/2 > 1/(d-1)$,

$$\|a \circ r\|_{M^{p,q}(\mathbb{R}^d)} \lesssim \|a\|_{R^{q,s}_d[0,\infty)},$$

thus generalizing the results of [7] to the setting of quasi-radial multipliers.

The proof is an adaption of the proof of [6]. However, it is more difficult to work directly with convolution kernels $k$ in this problem, since, unlike for radial multipliers, whose kernels are also radial, the kernels $k$ corresponding to quasi-radial Fourier multiplier operators need not be quasi-radial. It is here that we introduce a technique that has proved essential to an analysis of multiplier operators in settings lacking the full symmetry of Euclidean space: a reduction of the study of multipliers to an analysis of wave equations. Using the Fourier inversion formula, we write

$$\begin{aligned}(a \circ r)f(x) &= \int a(r(\xi))e^{2\pi i\xi\cdot(x-y)}f(y)\,dy\,dx\\ &= \iint \widehat{a}(t)e^{2\pi i[tr(\xi)+\xi\cdot(x-y)]}f(y)\,dy\,dx\,dt\\ &= \int \widehat{a}(t)(w_t * f)(x)\,dt\end{aligned}$$

where $\widehat{w_t}(\xi) = e^{2\pi itr(\xi)}$. As $t$ varies, $u = w_t * f$ solves the wave equation $\partial_t u = 2\pi iPu$, where $P$ is the Fourier multiplier operator whose symbol is the function $r$. Define $\chi_{x,t} = \text{Trans}_x\left(\int_{-\infty}^{\infty}\psi(s)w_{t-s}\,ds\right)$, where the Fourier transform of $\psi$ is non-negative and vanishing to high order at the origin. Then, using oscillatory integral techniques akin to Lemma 2.10, one can obtain inner product estimates on the quantities $\langle\chi_{x_1,t_1},\chi_{x_2,t_2}\rangle$ that show such terms are negligible unless $|x_1 - x_2| + |t_1 - t_2| \lesssim 1$. We can then adapt the proof of [6], performing a density decomposition using the Euclidean metric on $\mathbb{R}^{d+1}$, and thus obtain single scale bounds for the quasi-radial multipliers. An atomic decomposition analogous to that discussed in Section 2.4 then yields Kim's result.

We note that the endpoint analysis of radial multipliers we have been discussing is very closely related to the regularity of solutions to wave equations. In particular, if $\widehat{w_t} = e^{2\pi it|\xi|}$, then for any function $f$, $u = w_t * f$ solves the half-wave equation $\partial_t u = 2\pi i\sqrt{-\Delta}u$. The Littlewood-Paley pieces $(P_k u)(\cdot,t)$ are Fourier multipliers with symbol $\chi(\cdot/2^k)e^{2\pi it|\xi|}$, which are radial, and thus can be written in terms of spherical averages. The analysis of Lemma 2.4 can then be used to show that for $1/2 - 1/q > 1/(d-1)$,

$$\|P_k u\|_{L^q(\mathbb{R}^d\times[1,2])} \lesssim 2^{ks}\|f\|_{L^q(\mathbb{R}^d)} \quad \text{where } s = (d-1)(1/2-1/q) - 1/q.$$

With some more work involving atomic decompositions, Heo, Nazarov, and Seeger are able to combine the frequency scales, and thus prove the local smoothing estimate

$$\|u\|_{L^q(\mathbb{R}^d\times[1,2])} \lesssim \|f\|_{W^{s,q}(\mathbb{R}^d)}.$$

These are the current sharpest *endpoint* local smoothing bounds (they do not lose any $\varepsilon$ in the smoothness parameter).

To end our discussion of radial multipliers, notice that the Fourier multiplier operator $P$ is above is an elliptic operator on $\mathbb{R}^d$, satisfying the Assumption A we introduced in Chapter 1. For such an operator, the partial differential equation $\partial_t u = 2\pi iPu$ is hyperbolic, and it's solutions have wavelike properties. We will rely on a similar decomposition method used for the quasi-radial multiplier with symbol $a \circ r$ on manifolds, together with a study of the geometry of the manifold upon which we study these multipliers, to extend the methods we have discussed in the previous three sections to compact manifolds. We will now return to discuss this setting in more detail.

# Chapter 3

# Wave Equations on Compact Manifolds

## 3.1 Geometries Induced by Elliptic Operators

As alluded to in the last section, we now plan to study spectral multipliers of an elliptic operator $P$ on a compact manifold $X$ via the use of solutions of the wave equation $\partial_t u = 2\pi i P u$ on $M$, which allow us to exploit geometric information about the manifold $M$ and thus extend the results of Heo, Nazarov, and Seeger to the setting of compact manifolds when Assumption A and Assumption B of Chapter 1 hold.

But what is the geometry of the manifold we should be using? If the elliptic operator $P = \sqrt{-\Delta}$ is induced by a Riemannian metric on $X$, the geometric structure is clear. Given a more general operator $P$, we will use the principal symbol of $P$ to give $X$ a *Finsler geometry*. In this section we describe the bare eessentials of Finsler geometry needed for the arguments which occur later on in the thesis.

Let $X$ be a $d$-dimensional manifold. We denote an element of the tangent bundle $TX$ by $(x, v)$, where $x \in X$ and $v \in T_x X$. A *Finsler metric* on $X$ is a homogeneous function $F : TX \to [0, \infty)$, which is smooth on $TX - 0$, and strictly convex, in the sense that for $(x, v) \in TX - 0$, the Hessian matrix with entries

$$g_{ij}(x, v) = \frac{1}{2} \frac{\partial^2 F^2}{\partial v^i \partial v^j}(x, v) \tag{3.1}$$

is positive-definite. Then for each $x$ and $v$, the coefficients (3.1) give rise to an inner product $\langle \cdot, \cdot \rangle_{(x,v)}$ on $T_x X$ which approximates the Finsler metric near $v$ up to second order.

Given a metric $F$ on $X$, we define a dual metric $F_* : T^*X \to [0, \infty)$ by equipping $T_x^* X$ with the norm dual to the norm on $T_x X$, i.e. setting $F_*(x, \xi) = \sup\{\xi(v) : F(x, v) = 1\}$. Then $F_*$ is also strictly convex, in the sense that for an element $(x, \xi) \in T^*X - 0$, the Hessian matrix with entries

$$g^{ij}(x, \xi) = \frac{1}{2} \frac{\partial^2 F_*^2}{\partial \xi_i \partial \xi_j}(x, \xi) \tag{3.2}$$

is positive definite, and so the Finsler metric gives rise to an inner product on $T_x^* X$ for each $(x, \xi) \in T^*X$ which approximates $F_*$ up to second order.

If $X$ is a Finsler manifold, we define the *Legendre transform* $\mathcal{L} : TX - 0 \to T^*X - 0$, by defining it's behaviour as a homogeneous diffeomorphism $\mathcal{L}_x$ from $T_xX - \{0\} \to T_x^*X - \{0\}$ on each fiber. We define the map so that for $v, w \in T_xX$, $\langle \mathcal{L}_x(v), w \rangle = \langle v, w \rangle_{(x,v)}$. In coordinates, $\mathcal{L}_x(v)_i = \sum_j g_{ij}(x, v)v^j$. If $\xi = \mathcal{L}_x(v)$, then the matrix with entries $g_{ij}(x, v)$ is the inverse of the matrix with entries $g^{ij}(x, \xi)$, so that the inverse of $\mathcal{L}_x$ is given in coordinates by $\mathcal{L}_x^{-1}(\xi)^i = \sum_j g^{ij}(x, \xi)\xi_j$. The Legendre transform is the Finsler variant of the musical isomorphism in Riemannian geometry, though the Legendre transform is in general only *homogeneous*, rather than linear.

Now let $P$ be an elliptic operator on a manifold $X$ satisfying Assumption A. The principal symbol of $P$ is a well-defined coordinate independent function $p : T^*M \to [0, \infty)$, and the following lemma applies.

**Lemma 3.1.** *Suppose $p : T^*M \to [0, \infty)$ is homogeneous, and for each $x \in M$, the cosphere $S_x^* = \{\xi \in T_x^*M : p(x, \xi) = 1\}$ has non-vanishing Gaussian curvature. Then the function*

$$F(x, v) = \{\xi(v) : p(x, \xi) = 1\}$$

*is a Finsler metric on $M$, and the dual metric $F_*$ on $T^*M$ is equal to $p$.*

*Proof.* For each $x \in U_0$, the cosphere $S_x^* = \{\xi \in T_x^*M : p(x, \xi) = 1\}$ has non-vanishing Gaussian curvature. We claim that all principal curvatures of $S_x^*$ must actually be *positive*. This follows from a simple modification of an argument found in Chapter 2 of [8]. Indeed, if we fix an arbitrary point $v_0 \in T_x^*M$, and consider the smallest closed ball $B \subset T_x^*M$ centered at $v_0$ and containing $S_x^*$, then the sphere $\partial B$ must share the same tangent plane as $S_x^*$ at some point. All principal curvatures of $\partial B$ are positive, and at this point all principal curvatures of $S_x^*$ must be greater than the principal curvatures of $\partial B$, since $S_x^*$ curves away faster than $\partial B$ in all directions. By continuity, we conclude that the principal curvatures are everywhere positive. Thus for each $x \in M$ and $\xi \in T_x^*M - \{0\}$, the coefficients $g^{ij}(x, \xi) = (1/2)(\partial^2 p^2 / \partial\xi_i \partial\xi_j)$ form a positive-definite matrix. But inverting the procedure by which we constructed $F^*$ shows that the dual norm $F(x, v) = \sup_{\xi \in S_x^*} \xi(v)$ is also strictly convex, and thus gives a Finsler metric on $M$ with $F_* = p$. ◻

A Finsler metric gives a length to each tangent vector on the manifold, and can thus be used to define the lengths of curves $c : I \to U_0$ by the formula $L(c) = \int_I F(c, \dot{c})\, dt$. An analysis of length minimizing curves naturally leads to a theory of geodesics on a Finsler manifold. The theory of geodesics on Finsler manifolds is similar to the Riemannian case, except for the interesting quirk that a geodesic from a point $p$ to a point $q$ need not necesarily be a geodesic when considered as a curve from $q$ to $p$, and so we must consider *forward* and *backward* geodesics (the optimal path to descend a hill need not be the optimal path to ascend a hill, if your speed depends on the slope of the hill). We define the *forward distance* $d_+ : X \times X \to [0, \infty)$ by taking the infima of paths between points. This function is a *quasi-metric*, as it satisfies the triangle inequality, but is not necessarily symmetric. We define the *backward distance* $d_- : X \times X \to [0, \infty)$ by setting $d_-(p, q) = d_+(q, p)$. A *constant speed speed forward geodesic* on a Finsler manifold is a curve $c : I \to U_0$ which, in coordinates, satisfies the geodesic equation

$$\ddot{c}^a = -\sum\nolimits_{j,k} \gamma^a_{jk}(c, \dot{c})\dot{c}^j \dot{c}^k \quad \text{for } 1 \leq i \leq d, \tag{3.3}$$

where $\gamma^a_{jk} = \sum_i (g^{ai}/2)(\partial_{x_k} g_{ij} + \partial_{x_j} g_{ik} - \partial_{x_i} g_{jk})$ are the *formal Christoffel symbols*. We call the reversal of such a geodesic a *backward geodesic*. A curve $c$ is a geodesic if and only if it's Legendre transform $(x, \xi) = \mathcal{L}(c, \dot{c})$ satisfies

$$\dot{x} = (\partial_\xi F_*)(x, \xi) \quad \text{and} \quad \dot{\xi} = -(\partial_x F_*)(x, \xi), \tag{3.4}$$

equations that will reoccur in our study of wave propogation, and will tell us that high frequency wave packet solutions to the equation $\partial_t u = 2\pi i P$ travel along geodesics of the Finsler metric. Sufficiently short geodesics on a Finsler manifold are length minimizing. In particular, if $X$ is a compact Finsler manifold, then there exists $\varepsilon_X > 0$ such that if $c$ is a geodesic between two points $x_0$ and $x_1$ on $X$, and $L(c) < \varepsilon_X$, then $d_+(x_0, x_1) = L(c)$.

The last piece of technology we must consider before our analysis of critical points are *variation formulas* for the length of curves on Finsler manifolds. Consider a function $A : (-\varepsilon, \varepsilon) \times [0, 1] \to U_0$, such that for each $s$, the function $t \mapsto A(s, t)$ is a constant speed geodesic, and such that all such geodesics begin at the same point, i.e. so that $A(s, 0)$ is independent of $s$. Define the vector field $V(s) = (\partial_s A)(s, 1)$ and $T(s) = (\partial_t A)(s, 1)$. Then if $L(s)$ gives the arclength of the geodesic $t \mapsto A(s, t)$, the first variation formula for energy tells us

$$L'(s) = L(s)^{-1} \sum\nolimits_{i,j} g_{ij}(A(s, 1), T(s)) V^i(s) T^j(s). \tag{3.5}$$

Details can be found in Chapter 5 of [1]. We will consider *second variations*, i.e. estimates on $L''$, but we will not need the second variation formula to obtain the estimates we need, relying only on differentiating the equation (3.5) rather than explicitly working with Jacobi fields as is common in the analysis of second variations.

To conclude, we see that any elliptic operator $P$ on a manifold $M$ gives rise to a Finsler metric on $M$ which reflects the behaviour of the principal symbol of the operator. We will see this metric arises in the study of the wave equation $\partial_t u = 2\pi i P u$ on $M$. In particular, high-frequency wave packet solutions to the wave equation travel along *geodesics* in $M$, and so an analysis of the Finsler geometry will be necessary to understand the interactions of different wave packets, which will give us an analogue of Lemma 2.10 on compact manifolds for *small time interactions* between the wave packets when Assumption A holds. We will control the large time behaviour for the wave equation via a reduction to the local smoothing inequality, which is sufficient to obtain an analogue of the results [7] on manifolds, when Assumption B is true.

## 3.2 An Analogue of the Radial Multiplier Conjecture on Compact Manifolds

We now return to the study of spectral multipliers on compact manifolds. In a heuristic sense, one can think of the Fourier multipliers studied in the last Chapter as a limiting case of multipliers in the 'high frequency limit', i.e. so that for any elliptic operator on a manifold $X$ with principal symbol $p(x_0, \xi)$ as $R \to \infty$, locally around $x_0 \in X$ the behaviour of the operators $P/R$ becomes more and more like the Fourier multiplier operator on $\mathbb{R}^d$ with norm $p(x_0, \cdot)$ as $R \to \infty$. In particular, the following transference principle of Mitjagin [17] holds.

**Theorem 3.2.** *If $P$ is an elliptic operator on a compact manifold $X$ with principal symbol $p$, then for any regulated function $a$ and each $x_0 \in X$, $\|a(p(x_0, \cdot))\|_{M^p(\mathbb{R}^d)} \lesssim \|a\|_{M^p_{Dil}(X)}$.*

Thus if $P$ satisfies Assumption A, it follows that

$$M^p_{\mathrm{Dil}}(X) \subset M^p(\mathbb{R}^d) \subset R^{p,s}(\mathbb{R}^d),$$

with $s = (d-1)(1/p - 1/2)$. We might conjecture that one can reverse this inclusion in the same range as for the radial multiplier conjecture, i.e. proving that $M^p_{\mathrm{Dil}}(X) = R^{p,s}(\mathbb{R}^d)$ for $1/p - 1/2 > 1/2d$. However, this is not possible on a general manifold, and in fact, fails in the full range whenever the geometry of $X$ induced by $P$ does not have constant sectional curvature. It is not clear what the range of the conjecture should be for an arbitrary manifold, though it is likely true that the conjecture holds on manifolds with constant sectional curvature. TODO EDIT THIS.

## 3.3 Fourier Integral Operator Techniques

For any operator $P$ to which we can establish a robust functional calculus, and for any regulated function $a$, the identity

$$a(P) = \int_{-\infty}^{\infty} \widehat{a}(t) e^{2\pi i t P}$$

holds, obtained by plugging $P$ into the usual Fourier inversion formula. As $t$ varies, the operators $e^{2\pi i t P}$ solve the wave equation $\partial_t u = 2\pi i P u$. Thus we can study multipliers of $P$ in terms of the wave equation $\partial_t u = 2\pi i P u$. For a general operator $P$, it is difficult to understand the behaviour of the operator $e^{2\pi i t P}$ given it's abstract definition via the functional calculus. However if $P$ is elliptic, then $e^{2\pi i t P}$ is a *Fourier integral operator*, which roughly means that the kernel of the operator is well approximated by oscillatory integral representation *for high frequency inputs*. In this section, we briefly describe the relevant components we will need from the theory of Fourier integral operators for work in the latter part of the thesis.

To motivate the definition of Fourier integral operators, consider an operator $A$ from $\mathbb{R}^m$ to $\mathbb{R}^n$, whose kernel is a distribution given by an oscillatory integral of the form

$$A(x, y) = \int_{\mathbb{R}^p} s(x, y, \theta) e^{i\Phi(x,y,\theta)}\, d\theta,$$

where $s$ is a symbol of some order $\mu$, and $\Phi$ is smooth, and homogeneous of order one in the $\theta$ variable. If we localize this kernel in a small neighborhood of $x_0 \in \mathbb{R}^n$ and $y_0 \in \mathbb{R}^m$, defining $A_{x_0,y_0}(x,y) = \phi(x-x_0)A(x,y)\psi(y-y_0)$, then the principle of non-stationary phase tells us that for $|\xi_0| = |\eta_0| = 1$, we have rapid decay of the form

$$|\widehat{A}_{x_0,y_0}(R\xi_0, R\eta_0)| \lesssim_N R^{-N} \quad \text{for all } N > 0$$

unless $(x_0, \xi_0, y_0, \eta_0)$ lies in the set

$$\Lambda_\Phi = \Big\{(x, \nabla_x \Phi(x,y,\theta), y, \nabla_y \Phi(x,y,\theta)) : \nabla_\theta \Phi(x,y,\theta) = 0\Big\}.$$

We also consider the *canonical relation*

$$C_\Phi = \Big\{(x, \nabla_x \Phi(x,y,\theta), y, -\nabla_y \Phi(x,y,\theta)) : \nabla_\theta \Phi(x,y,\theta) = 0\Big\},$$

defined so that for wave packets $f_{y_0}(y) = \psi(y-y_0)e^{2\pi i R\eta_0 \cdot y}$ and $g_{x_0}(x) = \phi(x-x_0)e^{2\pi i R\xi_0 \cdot x}$,

$$|\langle g_{x_0}, A f_{y_0}\rangle| \lesssim_N R^{-N} \quad \text{for all } N > 0$$

unless $(x_0, \xi_0, y_0, \eta_0) \in C_\Phi$. The canonical relation of an oscillatory integral operator thus tells us the location where high-frequency wave packets are mapped in phase space. The main lesson of the theory of Fourier integral operators is that, to a large extent, the canonical relation *determines* the behaviour of the operator $A$.

Under the assumption that the vectors $\partial_{\theta_1}\{\nabla_{x,y,\xi}\Phi\}, \dots \partial_{\theta_N}\{\nabla_{x,y,\xi}\Phi\}$ are everywhere linearly independent, $C_\Phi$ is a smooth, $n+m$ dimensional manifold[1], and then for $(x_0, \xi_0, y_0, \eta_0) \in C_\Phi$ the principle of stationary phase implies that $|\langle g_{x_0}, A f_{y_0}\rangle| \lesssim R^{\mu + p/2 - (n+m)/2}$.

A Fourier integral operator is precisely an operator whose kernel is microlocally expressible in oscillatory integrals of the form above. Let $X$ be an $n$ dimensional manifold, and $Y$ an $m$ dimensional manifold. If $C$ is a $n+m$ 'Lagrangian' submanifold of $T^*X \times T^*Y$, we say an operator $A$ is a *Fourier integral operator* with canonical relation $C$ and order $\nu$ if, locally, the operator $A$ can, be expressed as a finite sum of oscillatory integral operators of the form above, with an amplitude given by a symbol $\mu$, a phase $\Phi$ in $N$ phase variables, with $C_\Phi \subset C$ and with $\nu = \mu - N/2 + (n+m)/4$.

The main examples of

The advantage of this perspective is that the canonical relation is truely an 'invariant' of the operator $A$, rather than the phase which is used to define the operator. The *equivalence of phase theorem* says that for any phase $\Phi$ such that locally, $C$ and $C_\Phi$ agree around a pair of points $(x_0, \xi_0)$ and $(y_0, \eta_0)$

## 3.4 BLEH

We will be able to control the behaviour of the operators $e^{2\pi i t P}$ with respect to the geometry of $X$ by using the theory of Fourier integral operators, which allows us to approximate the

[1]It is actually a *Lagrangian submanifold* of $T^*\mathbb{R}^n \times T^*\mathbb{R}^m$. Thus the canonical relation of any Fourier integral operator must be a Lagrangian submanifold, though we will only use this fact implicitly

high-frequency behaviour of the kernel of the operator $e^{2\pi i t P}$ to a high degree of precision by an oscillatory integral.

Prove local regularity theorems for a class of Fourier integral operators in $I^\mu(Z, Y; C)$, where $Y$ is a manifold of dimension $n \geq 2$, and $Z$ is a manifold of dimension $n+1$, which naturally arise from the study of wave equations. A consequence of this result will be a local smoothing result for solutions to the wave equation, i.e. that if $2 < p < \infty$, then there is $\delta$ depending on $p$ and $n$, such that if $T : Y \to Y \times \mathbb{R}$ is the solution operator to the wave equation, and $Y$ is a compact manifold whose geodesics are periodic, then $T$ is continuous from from $L^p_c(Y)$ to $L^p_{\alpha,\text{loc}}(Y \times \mathbb{R})$ for $\alpha \leq -(n-1)|1/2 - 1/p| + \delta$. Such a result is called local smoothing, since if we define $Tf(t,x) = T_t f(x)$, then the operator $T_t$ is, for each $t$, a Fourier integral operator of order zero, with canonical relation

$$C_t = \{(x, y; \xi, \xi) : x = y + t\widehat{\xi}\},$$

where $\widehat{\xi} = \xi/|\xi|$ is the normalization of $\xi$. Standard results about the regularity of hyperbolic partial differential equations show that each of the operators $T_t$ is continuous from $L^p_c(Y)$ To $L^p_{\alpha,\text{loc}}(Y \times \mathbb{R})$ for $\alpha \leq -(n-1)|1/2 - 1/p|$, and that this bound is sharp. Thus $T$ is *smoothing* in the $t$ variable, so that for any $f \in L^p$, the functions $T_t f$ 'on average' gain a regularity of $\delta$ over the worst case regularity at each time. The local smoothing conjecture states that this result is true for any $\delta < 1/p$.

The class of Fourier integral operators studied are those satisfying the following condition: as is standard, the canonical relation $C$ is a conic Lagrangian manifold of dimension $2n+1$. The fact that $C$ is Lagrangian implies $C$ is locally parameterized by $(\nabla_\zeta H(\zeta, \eta), \nabla_\eta H(\zeta, \eta), \zeta, \eta)$, where $H$ is a smooth, real homogeneous function of order one. If we assume $C \to T^*Y$ is a submersion, then $D_\xi[\nabla_\eta H(\zeta, \eta)]$ has full rank, which implies $D_\eta[\nabla \xi H(\zeta, \eta)] = (D_\xi[\nabla_\eta H(\zeta, \eta)])^t$ has full rank, and thus the projection $C \to T^*Z$ is an immersion. We make the further assumption that the projection $C \to Z$ is a submersion, from which it follows that for each $z$ in the image of this projection, the projection of points in $C$ onto $T^*_z Z$ is a conic hypersurface $\Gamma_z$ of dimension $n$. The final assumption we make is that all principal curvatures of $\Gamma_z$ are non-vanishing.

## 3.5 Periodic Geodesics and Assumption B

Using the Fourier integral operator methods above, we can prove that TODO

## 3.6 Relation to Previous Work

TODO: Lee Seeger exploited density decomposition to obtain local smoothing estimates for FIOs with cinematic curvature.

TODO: Kim exploited Sogge's Eigenfunction bounds to obtain optimal $L^2$ based multiplier bounds.

Large time control of the wave equation proved difficult.

# Part II

# New Results

# Chapter 4

# The Boundedness of Spectral Multipliers with Compact Support

In this chapter, we begin proving the main results for the analogue of the radial multiplier conjecture for elliptic operators on compact manifolds, which we introduced in Section TODO.

**Theorem 4.1.** *Let $X$ be a compact manifold of dimension $d$, and let $P$ be an elliptic operator satisfying Assumption A and Assumption B of section TODO. Then for any regulated function $a$ supported on* $[1/2, 2]$*,*

$$\|a\|_{M^p_{Dil}(X)} \sim \|a\|_{R^{s,p}_0[0,\infty)},$$

*where $s = (d-1)(1/q - 1/2)$.*

We have seen in Section TODO how to lower bounding the left hand side of TODO by the right hand side. In Lemma 4.2 of Section 4.1 that same section, we show that

$$\sup_{R \leq 1} \|a_R\|_{M^p(X)} \lesssim \|a\|_{l^\infty[0,\infty)} \lesssim \|a\|_{R^{s,p}_0[0,\infty)}. \tag{4.1}$$

On the other hand, the upper bound

$$\sup_{R \geq 1} \|a_R\|_{M^p(X)} \lesssim \|a\|_{R^{s,p}_0[0,\infty)}. \tag{4.2}$$

requires a more in depth analysis than (4.1). We write

$$a_R(P) = \int_{-\infty}^{\infty} R\widehat{a}(Rt) e^{2\pi i t P}\, dt, \tag{4.3}$$

which reduces $a_R(P)$ to studying averages associated with solutions to the half-wave equation $\partial_t = 2\pi i P$ on $M$. We prove Proposition 4.4 using several new estimates for understanding these averages, including:

(A) Quasi-orthogonality estimates for averages of solutions to the half-wave equation on $M$, discussed in Section 4.2, which arise from a connection between the theory of pseudodifferential operators whose principal symbols satisfy the curvature condition of Theorem 4.1, and Finsler metrics on the manifold $M$.

(B) Variants of the density-decomposition arguments which we introduced in Section TODO, which apply the quasi-orthogonality estimates obtained in Section 4.2 with a geometric argument which controls the ‘small time behavior’ of solutions to the half-wave equation.

(C) A new strategy to reduce the ‘large time behavior’ of the half-wave equation to an endpoint local smoothing inequality for the half-wave equation on $M$, described in Section 4.4.

Equations (4.1) and (4.2) immediately imply Theorem 4.1.

## 4.1 Preliminary Setup

We now begin with the details of the proof of Theorem 4.1, following the path laid out at the end of the last section. We fix some geometric constant $\varepsilon_X \in (0,1)$, matching the constant given in the statement of Theorem 4.4. Our goal is then to prove inequalities (4.1) and (4.2). Proving (4.1) is simple because the operators $a_R(P)$ are smoothing operators, uniformly for $0 < R < 1$, and $X$ is a compact manifold, because the spectrum of $X$ is discrete, and low eigenvalue eigenfunctions are uniformly smooth.

**Lemma 4.2.** *Let $X$ be a compact $d$-dimensional manifold, let $P$ be a classical elliptic self-adjoint pseudodifferential operator of order one. If $1 < p < 2d/(d+1)$, and if $a$ is a regulated function, then*

$$\sup_{R\leq 1} \|a_R\|_{M^p_{Dil}(X)} \lesssim \|a\|_{l^\infty(\mathbb{R})} \lesssim \|a\|_{R^{q,s}[0,\infty)}. \tag{4.4}$$

*Proof.* Let $T_R = m_R(P)$. The set $\Lambda \cap [0,2]$ is finite. For each $\lambda \in \Lambda$, choose a finite orthonormal basis $\mathcal{E}_\lambda$. Then we can write

$$T_R = \sum_{\lambda\in\Lambda} \sum_{e\in\mathcal{E}_\lambda} \langle f, e\rangle e. \tag{4.5}$$

Since $\mathcal{V}_\lambda \subset C^\infty(M)$, Hölder’s inequality implies

$$\|\langle f, e\rangle e\|_{L^p(M)} \leq \|f\|_{L^p(M)} \|e\|_{L^{p'}(M)} \|e\|_{L^p(M)} \lesssim_\lambda \|f\|_{L^p(M)}. \tag{4.6}$$

But this means that

$$\|T_R f\|_{L^p(M)} \leq \sum_{\lambda\in\Lambda_P\cap[0,2]} \sum_{e\in\mathcal{E}_\lambda} |m(\lambda/R)| \|\langle f, e\rangle e\|_{L^p(M)} \lesssim \|m\|_{l^\infty[0,\infty)}. \tag{4.7}$$

The proof is completed by noting that for $1 < p < 2d/(d+1)$, Sobolev embedding theorems imply that

$$\|a\|_{L^\infty[0,\infty)} \lesssim \|a\|_{W^{s,p'}[0,\infty)} \lesssim \|a\|_{R_0^{s,p}[0,\infty)}.$$ □

We now begin to reduce the analysis of $a_R(P)$ for $R \geq 1$ to the study of the wave equation. Fix a bump function $q \in C_c^\infty(\mathbb{R})$ with $\mathrm{supp}(q) \subset [1/2, 4]$ and $q(\lambda) = 1$ for $\lambda \in [1, 2]$, and define $Q_R = q(P/R)$. We write

$$T_R = Q_R \circ T_R \circ Q_R = \int_{\mathbb{R}} R\widehat{m}(Rt)(Q_R \circ e^{2\pi itP} \circ Q_R)\, dt, \tag{4.8}$$

and view the operators $(Q_R \circ e^{2\pi itP} \circ Q_R)$ as ‘frequency localized’ wave propagators.

Replacing the operator $P$ by $aP + b$ for appropriate $a, b \in \mathbb{R}$, we may assume without loss of generality that all eigenvalues of $P$ are integers. It follows that $e^{2\pi i(t+n)P} = e^{2\pi itP}$ for any $t \in \mathbb{R}$ and $n \in \mathbb{Z}$. Let $I_0$ denote the interval $[-1/2, 1/2]$. We may then write

$$T_R = \int_{I_0} b_R(t)(Q_R \circ e^{2\pi itP} \circ Q_R)\, dt, \tag{4.9}$$

where $b_R : I_0 \to \mathbb{C}$ is the periodic function

$$b_R(t) = \sum\nolimits_{n\in\mathbb{Z}} R\widehat{m}(R(t + n)). \tag{4.10}$$

We split our analysis of $T_R$ into two regimes: regime I and regime II. In regime I, we analyze the behaviour of the wave equation over times $0 \leq |t| \leq \varepsilon_M$ by decomposing this time interval into length $1/R$ pieces, and analyzing the interactions of the wave equations between the different intervals. In regime II, we analyze the behaviour of the wave equation over times $\varepsilon_M \leq |t| \leq 1$. Here we need not perform such a decomposition, since the boundedness of $C_p(m)$ gives better control on the function $b_R$ over these times.

**Lemma 4.3.** *Fix $\varepsilon > 0$. Let $\mathcal{T}_R = \mathbb{Z}/R \cap [-\varepsilon, \varepsilon]$ and define $I_t = [t - 1/R, t + 1/R]$. For a function $m : [0,\infty) \to \mathbb{C}$, define a periodic function $b : I_0 \to \mathbb{C}$ by setting*

$$b(t) = \sum\nolimits_{n\in\mathbb{Z}} R\widehat{m}(R(t + n)). \tag{4.11}$$

*Then we can write $b = \left(\sum_{t_0\in\mathcal{T}_R} b^I_{t_0}\right) + b^{II}$, where*

$$\mathit{supp}(b^I_{t_0}) \subset I_{t_0} \quad \mathit{and} \quad \mathit{supp}(b^{II}_R) \subset I_0 \setminus [-\varepsilon, \varepsilon]. \tag{4.12}$$

*Moreover, we have*

$$\left(\sum\nolimits_{t_0\in\mathcal{T}_R} \left[\|b^I_{t_0}\|_{L^p(I_0)} \langle Rt_0\rangle^{\alpha(p)}\right]^p\right)^{1/p} \lesssim R^{1/p'} C_p(m) \tag{4.13}$$

*and*

$$\|b^{II}\|_{L^p(I_0)} \lesssim R^{1/p'-\alpha(p)} C_p(m). \tag{4.14}$$

The proof is a simple calculation which we relegate to the appendix.

The following proposition, a kind of $L^p$ square root cancellation bound, implies (4.2) once we take Lemma 4.3 into account. Since we already proved (4.1), proving this proposition completes the proof of Theorem 4.1.

**Proposition 4.4.** *Let $P$ be a classical elliptic self-adjoint pseudodifferential operator of order one on a compact manifold $M$ of dimension $d$ satisfying the assumptions of Theorem 4.1. Fix $R > 0$ and suppose $1 < p < 2(d-1)/(d+1)$. Then there exists $\varepsilon_M > 0$ with the following property. Consider any function $b : I_0 \to \mathbb{C}$, and suppose we can write $b = \sum_{t_0 \in \mathcal{T}_R} b^I_{t_0} + b^{II}$, where $supp(b^I_{t_0}) \subset I_{t_0}$ and $supp(b^{II}_R) \subset I_0 \setminus [-\varepsilon_M, \varepsilon_M]$. Define operators $T^I = \sum_{t_0 \in \mathcal{T}_R} T^I_{t_0}$ and $T^{II}$, where*

$$T^I_{t_0} = \int b^I_{t_0}(t)(Q_R \circ e^{2\pi i t P} \circ Q_R)\, dt \;\; and \;\; T^{II} = \int b^{II}(t)(Q_R \circ e^{2\pi i t P} \circ Q_R)\, dt.$$

*Then*

$$\|T^I\|_{L^p \to L^p} \lesssim R^{-1/p'} \left( \sum_{t_0 \in \mathcal{T}_R} \left[ \|b^I_{t_0}\|_{L^p(I_0)} \langle R t_0 \rangle^{\alpha(p)} \right]^p \right)^{1/p} \tag{4.15}$$

*and*

$$\|T^{II}\|_{L^p \to L^p} \lesssim R^{\alpha(p) - 1/p'} \|b^{II}\|_{L^p(I_0)}. \tag{4.16}$$

Proposition 4.4 splits the main bound of the paper into two regimes: regime I and regime II. Noting that we require weaker bounds in (4.16) than in (4.15), the operator $T^{II}$ will not require as refined an analysis as for the operator $T^I$, and we obtain bounds on $T^{II}$ by a reduction to an endpoint local smoothing inequality in Section 4.4. On the other hand, to obtain more refined estimates for the $L^p$ norms of quantities of the form $f = T^I u$, we consider a decompositions of the form $u = \sum_{x_0} u_{x_0}$, where $u_{x_0} : M \to \mathbb{C}$ is supported on a ball $B(x, 1/R)$ of radius $1/R$ centered at $x_0$. We then have $f = \sum_{(x_0,t_0)} f_{x_0,t_0}$, where $f_{x_0,t_0} = T^I_{t_0} u_{x_0}$. To control $f$ we must establish an $L^p$ square root cancellation bound for the functions $\{f_{x_0,t_0}\}$. In the next section, we study the $L^2$ quasi-orthogonality of these functions, which we use as a starting point to obtain the required square root cancellation.

## 4.2 Quasi-Orthogonality Estimates For Wave-Packets

The discussion at the end of Section 4.1 motivates us to consider estimates for functions obtained by taking averages of the wave equation over a small time interval, with initial conditions localized to a particular part of space. In this section, we study the $L^2$ orthogonality of such quantities. We do not exploit periodicity of the Hamiltonian flow in this section since we are only dealing with estimates for the half wave-equation for *small times*. Our results here thus hold for any manifold $M$, and any operator $P$ whose principal symbol has cospheres with non-vanishing Gaussian curvature.

In order to prove the required quasi-orthogonality estimates of the functions $\{f_{x_0,t_0}\}$, we find a new connection between *Finsler geometry* on the manifold $M$ and the behaviour of the operator $P$. We will not assume any knowledge of Finsler geometry in the sequel, describing results from the literature needed when required and trying to relate the tools we use to their Riemannian analogues. Moreover, in the case where $p(x,\xi) = \sqrt{\sum g^{jk}(x)\xi_j\xi_k}$ for some Riemannian metric $g$ on $M$, the Finsler geometry we study will simply be the Riemannian geometry on $M$ given by the metric $g$. We begin this section by briefly outlining the relevant concepts of Finsler geometry required to state Theorem 4.5, relegating more precise details to Subsection 4.2.1 where Finsler geometry arises more explicitly in our proofs.

For our purposes, Finsler geometry is very akin to Riemannian geometry, though instead of a smoothly varying *inner product* being given on the tangent spaces of $M$, in Finsler geometry we are given a smoothly varying *vector space norm* $F : TM \to [0,\infty)$ on the tangent spaces of $M$. Many results in Riemannian geometry carry over to the Finsler setting. In particular, we can define the length of a curve $c : I \to M$ via it's derivative $c' : I \to TM$ by the formula

$$L(c) = \int_I F(c'), \tag{4.17}$$

and thus get a theory of metric, geodesics, and curvature as in the Riemannian setting. The main quirk of the theory for us is that the metric induced from this length function is not symmetric. Indeed, let

$$d_+(p_0, p_1) = \inf \{L(c) : c(0) = p_0 \text{ and } c(1) = p_1\}, \tag{4.18}$$

and set $d_-(p_0, p_1) = d_+(p_1, p_0)$, i.e. so that

$$d_+(p_0, p_1) = \inf \{L(c) : c(0) = p_1 \text{ and } c(1) = p_0\}. \tag{4.19}$$

In the Riemannian setting, $d_+$ and $d_-$ are both equal to the usual Riemannian metric, but on a general Finsler manifold one has $d_+ \neq d_-$, and these functions are distinct quasi-metrics on $M$ (though a compactness argument shows $d_M^+(x_0, x_1) \sim d_M^-(x_0, x_1)$). Geodesics from a point $p_0$ to a point $p_1$ need not be geodesics from $p_1$ to $p_0$ when reversed. A metric can be obtained by setting $d_M = (d_M^+ + d_M^-)/2$, and we will use this as the canonical metric on $M$ in what follows.

Finsler geometry arises here because, if $p : T^*M \to [0,\infty)$ is a homogeneous function satisfying the curvature assumptions of Theorem 4.1, then a Finsler metric arises by taking the 'dual norm' of $p$, i.e. setting

$$F(x, v) = \sup \{\xi(v) : \xi \in T_x^*M \text{ and } p(x, \xi) = 1\}. \tag{4.20}$$

Thus $p$ induces quasimetrics $d_+$ and $d_-$ on $M$ via the metric $F$ as defined above. We now use these quasimetrics to state Proposition 4.5.

**Proposition 4.5.** *Let $M$ be a compact manifold of dimension $d$, and let $P$ be a classical elliptic self-adjoint pseudodifferential operator of order one whose principal symbol satisfies the curvature assumptions of Theorem 4.1. Then there exists $\varepsilon_M > 0$ such that for all $R \geq 0$, the following estimates hold:*

- *(Pointwise Estimates) Fix $|t_0| \leq \varepsilon_M$ and $x_0 \in M$. Consider any two measurable functions $c : \mathbb{R} \to \mathbb{C}$ and $u : M \to \mathbb{C}$, with $\|c\|_{L^1(\mathbb{R})} \leq 1$ and $\|u\|_{L^1(M)} \leq 1$, and with supp$(c) \subset I_{t_0}$ and supp$(u) \subset B(x_0, 1/R)$. Define $S : M \to \mathbb{C}$ by setting*

$$S = \int c(t)(Q_R \circ e^{2\pi i t P} \circ Q_R)\{u\}\, dt. \tag{4.21}$$

*Then for any $K \geq 0$, and any $x \in M$,*

$$|S(x)| \lesssim_K \frac{R^d}{\langle R d_M(x_0, x)\rangle^{\frac{d-1}{2}}} \max_{\pm} \left\langle R \left| t_0 \pm d_M^{\pm}(x, x_0) \right| \right\rangle^{-K}. \tag{4.22}$$

- *(Quasi-Orthogonality Estimates) Fix $|t_0 - t_1| \leq \varepsilon_M$, and $x_0, x_1 \in M$. Consider any two pairs of functions $c_0, c_1 : \mathbb{R} \to \mathbb{C}$ and $u_0, u_1 : M \to \mathbb{C}$ such that, for each $\nu \in \{0, 1\}$, $\|c_\nu\|_{L^1(\mathbb{R})} \leq 1$, $\|u_\nu\|_{L^1(M)} \leq 1$, supp$(c_\nu) \subset I_{t_\nu}$, and supp$(u_\nu) \subset B(x_\nu, 1/R)$. Define $S_\nu : M \to \mathbb{C}$ by setting*

$$S_\nu = \int c_\nu(t)(Q_R \circ e^{2\pi i t P} \circ Q_R)\{u_\nu\}\, dt. \tag{4.23}$$

*Then for any $K \geq 0$,*

$$|\langle S_0, S_1\rangle| \lesssim_K \frac{R^d}{\langle R d_M(x_0, x_1)\rangle^{\frac{d-1}{2}}} \max_{\pm} \left\langle R \left| (t_0 - t_1) \pm d^{\pm}(x_0, x_1) \right| \right\rangle^{-K}. \tag{4.24}$$

*Remark.* Estimate (4.24) is a variable coefficient analogue of Lemma 3.3 of [7]. The pointwise estimate tells us that the function $S$ is concentrated on a geodesic annulus of radius $|t_0|$ centered at $x_0$ and thickness $O(1/R)$. The quasi-orthogonality estimate tells us that the two functions $S_0$ and $S_1$ are only significantly correlated with one another if the two annuli on which the majority of the support of $S_0$ and $S_1$ lie are internally or externally tangent to one another, depending on whether $t_0$ and $t_1$ have the same or opposite sign respectively.

*Proof of Proposition 4.5.* To simplify notation, in the following proof we will suppress the use of $R$ as an index, for instance, writing $Q$ for $Q_R$. For both the pointwise and quasi-orthogonality estimates, we want to consider the operators in coordinates, so we can use the *Lax-Hörmander Parametrix* to understand the wave propagators in terms of various oscillatory integrals, though we use a slight variant of the usual parametrix which will help us in the stationary phase arguments which occur when manipulating the resulting oscillatory integrals.

Start by covering $M$ by a finite family of suitably small open sets $\{V_\alpha\}$, such that for each $\alpha$, there is a coordinate chart $U_\alpha$ compactly containing $V_\alpha$ and with $N(V_\alpha, 1.1\varepsilon_M) \subset U_\alpha$. Let $\{\eta_\alpha\}$ be a partition of unity subordinate to $\{V_\alpha\}$. It will be convenient to define $V_\alpha^* = N(V_\alpha, 0.01\varepsilon_M)$ for each $\alpha$. The next lemma allows us to approximate the operator $Q$, and the propagators $e^{2\pi i t P}$ with operators which have more explicit representations in the coordinate system $\{U_\alpha\}$, with an error negligible to the results of Proposition 4.5.

**Lemma 4.6.** *Suppose $\varepsilon_M$ is suitably small, depending on $M$ and $P$. For each $\alpha$, and $|t| \leq \varepsilon_M$, there exists Schwartz operators $Q_\alpha$ and $W_\alpha(t)$, each with kernels supported on $U_\alpha \times V_\alpha^*$, such that the following holds:*

- *For any* $u : M \to \mathbb{C}$ *with* $\|u\|_{L^1(M)} \leq 1$ *and* $supp(u) \subset V_\alpha^*$,

$$supp(Q_\alpha u) \subset N(supp(u), 0.01\varepsilon_M), \tag{4.25}$$

$$supp(W_\alpha(t)u) \subset N(supp(u), 1.01\varepsilon_M), \tag{4.26}$$

$$\|(Q - Q_\alpha)u\|_{L^\infty(M)} \lesssim_N R^{-N} \quad \text{for all } N \geq 0, \tag{4.27}$$

*and*

$$\left\|(Q_\alpha \circ (e^{2\pi i t P} - W_\alpha(t)) \circ Q_\alpha)\{u\}\right\|_{L^\infty(M)} \lesssim_N R^{-N} \quad \text{for all } N \geq 0. \tag{4.28}$$

- *In the coordinate system of* $U_\alpha$*, the operator* $Q_\alpha$ *is a pseudo-differential operator of order zero given by a symbol* $\sigma_\alpha(x, \xi)$*, where*

$$supp(\sigma_\alpha) \subset \{\xi \in \mathbb{R}^d : R/8 \leq |\xi| \leq 8R\}, \tag{4.29}$$

*and* $\sigma_\alpha$ *satisfies derivative estimates of the form*

$$|\partial_x^\lambda \partial_\xi^\kappa \sigma_\alpha(x, \xi)| \lesssim_{\lambda,\kappa} R^{-|\kappa|}. \tag{4.30}$$

- *In the coordinate system* $U_\alpha$*, the operator* $W_\alpha(t)$ *has a kernel* $W_\alpha(t, x, y)$ *with an oscillatory integral representation*

$$W_\alpha(t, x, y) = \int s(t, x, y, \xi) e^{2\pi i [\phi(x,y,\xi) + t p(y,\xi)]}\, d\xi, \tag{4.31}$$

*where the function* $s$ *is compactly supported in* $U_\alpha$*, such that*

$$supp_{t,x,y}(s) \subset \{(t, x, y) : |x - y| \leq C|t|\} \tag{4.32}$$

*for some constant* $C > 0$*, suc that*

$$supp_\xi(s) \subset \{\xi \in \mathbb{R}^d : R/8 \leq |\xi| \leq 8R\}, \tag{4.33}$$

*and such that the function* $s$ *satisfies derivative estimates of the form*

$$|\partial_{t,x,y}^\lambda \partial_\xi^\kappa s| \lesssim_{\lambda,\kappa} R^{-|\kappa|}. \tag{4.34}$$

*The phase function* $\phi$ *is smooth and homogeneous of degree one in the* $\xi$ *variable, and solves the* eikonal equation

$$p\,(x, \nabla_x \phi(x, y, \xi)) = p(y, \xi), \tag{4.35}$$

*subject to the constraint that* $\phi(x, y, \xi) = 0$ *if* $\xi \cdot (x - y) = 0$.

We relegate the proof of Lemma 4.6 to the appendix, the proof being a technical and mostly conventional calculation involving a manipulation of oscillatory integrals using integration by parts.

Let us now proceed with the proof of the pointwise bounds in Proposition 4.5 using Lemma 4.6. Given $u : M \to \mathbb{C}$, write $u = \sum_\alpha u_\alpha$, where $u_\alpha = \eta_\alpha u$. Lemma 4.6 implies that if we define

$$S_\alpha = \int c(t)(Q_{j,\alpha} \circ W_{j,\alpha}(t) \circ Q_{j,\alpha})\{u_\alpha\}\, dt, \tag{4.36}$$

then for all $N \geq 0$,

$$\left\| S - \textstyle\sum_\alpha S_\alpha \right\|_{L^\infty(M)} \lesssim_N R^{-N}. \tag{4.37}$$

This error is negligible to the pointwise bounds we want to obtain in Proposition 4.5 if we choose $N \geq K - \frac{d+1}{2}$, since the compactness of $M$ implies that $d_M(x, x_0) \lesssim 1$ for all $x \in M$, and so

$$R^{-N} \lesssim R^{\left(\frac{d+1}{2} - K\right)} \lesssim \frac{R^d}{\langle R d_M(x, x_0) \rangle^{\frac{d-1}{2}}} \left\langle R \left| |t_0| \pm d_M^{\pm}(x, x_0) \right| \right\rangle^{-K}. \tag{4.38}$$

We bound each of the functions $\{S_\alpha\}$ separately, combining the estimates using the triangle inequality. We continue by expanding out the implicit integrals in the definition of $S_\alpha$. In the coordinate system $U_\alpha$, we can write

$$\begin{aligned} S_\alpha(x) = \int c(t)\sigma(x, \eta) &e^{2\pi i \eta \cdot (x-y)} \\ &s(t, y, z, \xi) e^{2\pi i [\phi(y,z,\xi) + t p(z,\xi)]} \\ &\quad \sigma(z, \theta) e^{2\pi i \theta \cdot (z-w)} (\eta_\alpha u)(w) \\ &\quad\quad dt\, dy\, dz\, dw\, d\theta\, d\xi\, d\eta. \end{aligned} \tag{4.39}$$

The integral in (4.39) looks highly complicated, but can be simplified considerably by noticing that most variables are quite highly localized. In particular, oscillation in the $\eta$ variable implies that the amplitude is negligible unless $|x - y| \lesssim 1/R$, oscillation in the $\theta$ variable implies that the amplitude is negligible unless $|z - w| \lesssim 1/R$, and the support of $u$ implies that $|w - x_0| \lesssim 1/R$. Define

$$k_1(t, x, z, \xi) = \int \sigma(x, \eta) s(t, y, z, \xi) e^{2\pi i [\eta \cdot (x-y) + \phi(y,z,\xi) - \phi(x,z,\xi)]}\, dy\, d\eta, \tag{4.40}$$

and

$$\begin{aligned} k_2(t, \xi) = \int k_1(t, x, z, \xi) &\sigma(z, \theta)(\eta_\alpha u)(w) \\ &e^{2\pi i [\theta \cdot (z-w) + \phi(x,z,\xi) - \phi(x,x_0,\xi) + t p(z,\xi) - t p(x_0,\xi)]}\, d\theta\, dw, \end{aligned} \tag{4.41}$$

and then set

$$a(x, \xi) = \int c(t) k_2(t, R\xi) e^{2\pi i [(t - t_0) p(x_0, R\xi)]}\, dt\, dz, \tag{4.42}$$

so that $\mathrm{supp}_\xi(a) \subset \{\xi : 1/8 \leq |\xi| \leq 8\}$, and

$$S_\alpha(x) = R^d \int a(x,\xi) e^{2\pi i R[\phi(x,x_0,\xi)+t_0 p(x_0,\xi)]}\, d\xi. \tag{4.43}$$

Integrating by parts in $\eta$ and $\theta$ in (4.40) and (4.41) gives that for all multi-indices $\alpha$,

$$|\partial_\xi^\alpha k_1(t,x,z,\xi)| \lesssim_\alpha R^{-|\alpha|} \quad \text{and} \quad |\partial_\xi^\alpha k_2(z,\xi)| \lesssim_\alpha R^{-|\alpha|}. \tag{4.44}$$

Using the bounds in (4.44) with the fact that $\mathrm{supp}(c)$ is contained in a $O(1/R)$ neighborhood of $t_0$ in (4.42) then implies $|\partial_\xi^\alpha a(x,\xi)| \lesssim_\alpha 1$ for all $\alpha$.

We now account for angular oscillation of the integral by working in a kind of 'polar coordinate' system. First we find $\lambda : V_\alpha^* \times S^{d-1} \to (0,\infty)$ such that for all $|\xi| = 1$,

$$p(x_0, \lambda(x_0,\xi)\xi) = 1. \tag{4.45}$$

If $\tilde{a}(x,\rho,\eta) = a(x,\rho\lambda(x_0)\xi)\det[\lambda(x_0,\xi)I + \xi(\nabla_\xi\lambda)(x_0,\xi)^T]$, then

$$S_\alpha(x) = R^d \int_0^\infty \rho^{d-1} \int_{|\xi|=1} \tilde{a}(x,\rho,\xi) e^{2\pi i R\rho[t_0+\phi(x,x_0,\lambda(x_0,\xi)\xi)]}\, d\xi\, d\rho. \tag{4.46}$$

Define $\Phi : S^{d-1} \to \mathbb{R}$ by setting $\Phi(\xi) = \phi(x,x_0,\lambda(x_0,\xi)\xi)$. We claim that, in the $\xi$ variable, $\Phi$ has exactly two critical points $|\xi^+|^{-1}\xi^+$ and $|\xi^-|^{-1}\xi^-$, where $\xi^+ \in S^*_{x_0}$ is the covector corresponding to the forward geodesic from $x_0$ to $x$, and $\xi^- \in S^*_{x_0}$ is the covector corresponding to the backward geodesic from $x_0$ to $x$. Moreover,

$$\Phi(|\xi^+|^{-1}\xi^+) = d_M^+(x_0,x) \quad \text{and} \quad \Phi(|\xi^-|^{-1}\xi^-) = -d_M^-(x_0,x), \tag{4.47}$$

and the Hessian at each of these points is non-degenerate, with each eigenvalue of the Hessian having magnitude exceeding a constant multiple of $d_M^\pm(x_0,x)$. We prove that these properties hold for $\Phi$ in Proposition 4.7 of the following section, via a series of geometric arguments. It then follows from the principle of stationary phase that

$$S_\alpha(x) = \sum_\pm \frac{R^d}{\left\langle R d_M^\pm(x_0,x)\right\rangle^{\frac{d-1}{2}}} \int_0^\infty \rho^{\frac{d-1}{2}} a_\pm(x,\rho) e^{2\pi i R\rho[t_0 \pm d_M^\pm(x_0,x)]}\, d\rho, \tag{4.48}$$

where $a_\pm$ is supported on $|\rho| \sim 1$, and for all $\alpha$, $|\partial_\rho^\alpha a_\pm| \lesssim_\alpha 1$. Integrating by parts in the $\rho$ variable if $t_0 \pm d_M^\pm(x_0,x)$ is large, we conclude that

$$|S_\alpha(x)| \lesssim \frac{R^d}{\langle R d_M(x_0,x)\rangle^{\frac{d-1}{2}}} \sum_\pm \langle R|t_0 \pm d_M(x_0,x)|\rangle^{-K}. \tag{4.49}$$

Combining (4.37) and (4.49) completes the proof of the pointwise bounds.

The quasi-orthogonality arguments are obtained by a largely analogous method, and so we only sketch the proof. One major difference is that we can use the self-adjointness of the operators $Q$, and the unitary group structure of $\{e^{2\pi itP}\}$, to write

$$\begin{aligned}\langle S_0, S_1\rangle &= \int c_0(t)c_1(s)\langle (Q\circ e^{2\pi itP}\circ Q)\{u_0\}, (Q\circ e^{2\pi isP}\circ Q)\{u_1\}\rangle \\ &= \int c_0(t)c_1(s)\langle (Q^2\circ e^{2\pi i(t-s)P}\circ Q^2)\{u_0\}, u_1\rangle \\ &= \int c(t)\langle (Q^2\circ e^{2\pi itP}\circ Q^2)\{u_0\}, u_1\rangle,\end{aligned} \tag{4.50}$$

where $c(t) = \int c_0(u)c_1(u-t)\,du$, by Young's inequality, satisfies

$$\|c\|_{L^1(\mathbb{R})} \lesssim \|c_0\|_{L^1(\mathbb{R})}\|c_1\|_{L^1(\mathbb{R})} \leq 1 \tag{4.51}$$

and $\operatorname{supp}(c) \subset [(t_0-t_1)-4/R, (t_0-t_1)+4/R]$. After this, one proceeds exactly as in the proof of the pointwise estimate. We write the inner product as

$$\sum\nolimits_\alpha \int c(t)\langle (Q^2\circ e^{2\pi itP}\circ Q^2)\{\eta_\alpha u_0\}, u_1\rangle. \tag{4.52}$$

Then we use Lemma 4.6 to replace $Q^2\circ e^{2\pi itP}\circ Q^2$ with $Q_\alpha^2\circ W_\alpha(t)\circ Q_\alpha^2$ using Lemma 4.6, modulo a negligible error. The integral

$$\sum\nolimits_\alpha \int c(t)\langle (Q_\alpha^2\circ W_\alpha(t)\circ Q_\alpha^2)\{\eta_\alpha u_0\}, u_1\rangle \tag{4.53}$$

is then only non-zero if both the supports of $u_0$ and $u_1$ are compactly contained in $U_\alpha$. Thus we can switch to the coordinate system of $U_\alpha$, in which we can express the inner product by oscillatory integrals of the exact same kind as those occurring in the pointwise estimate. Integrating away the highly localized variables as in the pointwise case, and then applying stationary phase in polar coordinates proves the required estimates. □

### 4.2.1 Analysis of Critical Points

In this section, we now classify the critical points of the kinds of functions that arose in Proposition 4.5.

**Proposition 4.7.** *Fix a bounded open set $U_0 \subset \mathbb{R}^d$. Consider a Finsler metric $F : U_0 \times \mathbb{R}^d \to [0,\infty)$ on $U_0$, and it's dual metric $F_* : U_0\times\mathbb{R}^d \to [0,\infty)$, which extends to a Finsler metric on an open set containing the closure of $U_0$. Fix a suitably small constant $r > 0$. Let $U$ be an open subset of $U_0$ with diameter at most $r$ which is geodesically convex (any two points are joined by a minimizing geodesic). Let $\phi : U\times U\times\mathbb{R}^d \to \mathbb{R}$ solve the eikonal equation*

$$F_*(x, \nabla_x\phi(x,y,\xi)) = F_*(y,\xi), \tag{4.54}$$

*such that $\phi(x,y,\xi) = 0$ for $x \in H(y,\xi)$, where $H(y,\xi) = \{x \in U : \xi \cdot (x-y) = 0\}$. For each $x_0, x_1 \in U$, let $S^*_{x_0} = \{\xi \in \mathbb{R}^d : F_*(x_0,\xi) = 1\}$ be the cosphere at $x_0$, and define $\Psi : S^*_{x_0} \to \mathbb{R}$ by setting*

$$\Psi(\xi) = \phi(x_1, x_0, \xi). \tag{4.55}$$

*Then the function $\Psi$ has exactly two critical points, at $\xi^+$ and $\xi^-$, where the Legendre transform of $\xi^+$ is the tangent vector of the forward geodesic from $x_0$ to $x_1$, and the Legendre transform of $\xi^-$ is the tangent vector of the backward geodesic from $x_1$ to $x_0$. Moreover,*

$$\Psi(\xi^+) = d_M^+(x_0,x_1) \quad \text{and} \quad \Psi(\xi^-) = -d_M^-(x_0,x_1), \tag{4.56}$$

*and the Hessians $H_+$ and $H_-$ of $\Psi$ at these critical points, viewed as quadratic maps from $T_{\xi_\pm} S^*_{x_0} \to \mathbb{R}$ satisfy*

$$H_+(\zeta) \geq C d_M^+(x_0,x_1)|\zeta| \quad \text{and} \quad H_-(\zeta) \leq -C d_M^-(x_0,x_1)|\zeta|, \tag{4.57}$$

*where the implicit constant is uniform in $x_0$ and $x_1$.*

If $\Psi$ is as above, then the function $\Phi : S^{d-1} \to \mathbb{R}$ obtained by setting $\Phi(\xi) = \phi(x, x_0, F_*(x,\xi)^{-1}\xi)$ is precisely the kind of function that arose as a phase in Proposition 4.5, where $F_*$ was the principal symbol $p$ of the pseudodifferential operator we were considering. Since critical points and the Hessians of maps at critical points are stable under diffeomorphisms, and the map $\xi \mapsto F_*(x,\xi)^{-1}\xi$ is a diffeomorphism from $S^*_{x_0}$ to $S^{d-1}$, classifying the critical points of the map $\Psi$ implies the required properties of the map $\Phi$ used in Proposition 4.5.

In order to prove 4.7, we rely on a geometric interpretation of $\Psi$ following from Hamilton-Jacobi theory.

**Lemma 4.8.** *Consider the setup to Proposition 4.7. For any $\xi \in S^*_{x_0}$,*

$$|\Psi(\xi)| = \begin{cases} \text{the length of the shortest curve from } H(x_0,\xi) \text{ to } x_1 & : \text{if } \Psi(\xi) > 0, \\ \text{the length of the shortest curve from } x_1 \text{ to } H(x_0,\xi) & : \text{if } \Psi(\xi) < 0. \end{cases}$$

*Proof.* We rely on a construction of $\phi$ from Proposition 3.7 of [**Treves2**], which we briefly describe. Fix $x_0$ and $\xi$. Then there is a unique covector field $\omega : H(x_0,\xi) \to \mathbb{R}^d$ which is everywhere perpendicular to $H(x_0,\xi)$, with $\omega(x_0) = \xi$ and with $F_*(x,\omega(x)) = F_*(x_0,\xi)$ for all $x \in H(x_0,\xi)$. There exists a unique point $x(\xi) \in H(x_0,\xi)$ and a unique $t(\xi) \in \mathbb{R}$, such that the unit speed geodesic $\gamma$ on $M$ with $\gamma(0) = x(\xi)$ and $\gamma'(0) = \mathcal{L}^{-1}(x(\xi), \omega(x(\xi)))$ satisfies $\gamma(t(\xi)) = x_1$. We then have $\Psi(\xi) = t(\xi)$. If $t(\xi)$ is negative, then $\gamma|_{[t(\xi),0]}$ is a geodesic from $x_1$ to $x_0$, and if $t(\xi)$ is positive, $\gamma|_{[0,t(\xi)]}$ is a geodesic from $x_0$ to $x_1$. Because $\gamma$ is a geodesic, the geometric interpretation then follows if $U$ is a suitably small neighborhood such that geodesics are length minimizing. □

It follows immediately from Lemma 4.8 that

$$\Psi(\xi_+) = d_M^+(x_0,x_1) \quad \text{and} \quad \Psi(\xi_-) = -d_M^-(x_0,x_1). \tag{4.58}$$

A simple geometric argument also shows $\Psi(\xi^+)$ is the maximum value of $\Psi$ on $S^*_{x_0}$, and $\Psi(\xi^-)$ is the minimum value on $S^*_{x_0}$, so that these two points are both critical. Indeed, the point $x_0$ lies in $H(x_0,\xi)$ for all $\xi \in \mathbb{R}^d - \{0\}$. Thus the shortest curve from $x_0$ to $x_1$ is always longer than the shortest curve from $H(x_0,\xi)$ to $x_1$. Similarily, the shortest curve from $x_1$ to $x_0$ is always longer than the shortest curve from $x_1$ to $H(x_0,\xi)$. Thus

$$-d_M^-(x_0,x_1) \leq \Psi(\xi) \leq d_M^+(x_0,x_1), \tag{4.59}$$

and so $\Psi(\xi^-) \leq \Psi(\xi) \leq \Psi(\xi^+)$ for all $\xi \in$. All that remains is to prove that $\xi^+$ and $\xi^-$ are the *only* critical points of $\Psi$, and that these critical points are appropriately non-degenerate.

In order to simplify proofs, we employ a structural symmetry to reduce the number of cases we need to analyze. Namely, if one defines the reverse Finsler metric $F_\rho(x,v) = F(x,-v)$, then $F^*_\rho(x,\xi) = F_*(x,-\xi)$, and so the associated function $\Psi^\rho$ which is the analogue of $\Psi$ for $F^\rho$ satisfies $\Psi^\rho(\xi) = -\Psi(-\xi)$. The critical points of $\Psi$ and $\Psi^\rho$ are thus directly related to one another, which allows us without loss of generality to study only points with $\Psi \leq 0$ (and thus only study geodesics beginning at $x_1$).

*Proof that $\xi^+$ and $\xi^-$ are the only critical points.* Fix $\xi^* \in T_{x_0}M - \{\xi^\pm\}$. Using the notation defined above, let $x_* = x(\xi^*)$ and $t_* = t(\xi^*)$. Using the symmetry above, we may assume without loss of generality that $\Psi(\xi^*) \leq 0$. Since $\xi^-$ is not perpendicular to $H(x_0,\xi^*)$ at $x_0$, we have $x_* \neq x_0$. If $\Psi(\xi^*) < 0$, let $\gamma$ be the unique unit speed forward geodesic with $\gamma(0) = x_1$ and $\gamma(t_*) = x_*$. If $\Psi(\xi^*) = 0$, let $\gamma$ be the unique unit speed forward geodesic with $\gamma'(0)$ equal to the Legendre transform of $\xi^*$. Pick $\eta$ such that $\eta \cdot (x_* - x_0) \neq 0$, and then, for $t$ suitably close to $t_*$, define a smooth map

$$\xi(t) = \frac{\xi^* + a(t)\eta}{F_*(x_0,\xi^* + a(t)\eta)}, \tag{4.60}$$

into $S^*_{x_0}$, where

$$a(t) = -\frac{\xi^* \cdot (\gamma(t) - x_0)}{\eta \cdot (\gamma(t) - x_0)} \tag{4.61}$$

is defined so that $\gamma(t) \in H(x_0,\xi(t))$. Then $\Psi(\xi(t)) = -t$, and differentiation at $t = t_*$ gives $D\Psi(\xi^*)(\xi'(t_*)) = -1$. In particular, $D\Psi(\xi^*) \neq 0$, so $\xi^*$ is not a critical point. $\square$

We now analyze the non-degeneracy of the critical points $\xi^+$ and $\xi^-$.

*Proof that $\xi^+$ and $\xi^-$ are non-degenerate.* Using symmetry, it suffices without loss of generality to analyze the critical point $\xi^-$ rather than $\xi^+$. Let $H_- : T_{\xi^-}S^*_{x_0} \to \mathbb{R}$ be the Hessian of $\Psi$ at $\xi^-$. Let $v_0 = \mathcal{L}^{-1}_{x_0}\xi^-$, and using the notation of the last argument, let $l = t(\xi^-)$. Consider a curve $\xi(a)$ valued in $S^*_{x_0}$ with $\xi(0) = \xi^-$ and $\zeta = \xi'(0)$ for some $\zeta \in T^*_{\xi^-}S_{x_0}$. Then the second derivative of $\Psi(\xi(a))$ at $a = 0$ is $H_-(\zeta)$, and so our proof would be complete if we could show that the function $L(a) = -\Psi(\xi(a))$, which is the length of the shortest geodesic from $x_1$ to $H(x_0,\xi(a))$, satisfies $L''(0) \geq Cl|\zeta|$, for a constant $C > 0$ uniform in $x_0$, $x_1$, and $\zeta$.

Consider the partial function $\mathrm{Exp}_{x_1} : \mathbb{R}^d \to U_0$ obtained from the exponential map at $x_1$. If $r$ is chosen suitably small, there exists a neighborhood $V$ of the origin in $\mathbb{R}^d$ such that $E = \mathrm{Exp}_{x_1}|_V$ is a diffeomorphism between $V$ and $U$. Unlike in Riemannian manifolds, in general $E$ is only $C^1$ at the origin, though smooth on $V - \{0\}$. However, for $|v| = 1$ and $t > 0$ we can write $E(tv) = (\pi \circ \varphi)(x_1, \mathcal{L}_{x_1} v, t)$, where the partial function $\varphi : (T^*U_0 - 0) \times \mathbb{R} \to (T^*U_0 - 0)$ is the flow induced by the the Hamiltonian vector field $(\partial_\xi F_*, -\partial_x F_*)$ on $T^*U_0 - 0$, and $\pi$ is the projection map from $T^*U_0 - 0$ to $U_0$. Where defined, $\varphi$ is a smooth function since the Hamiltonian vector field is smooth, and so it follows by homogeneity and the precompactness of $U$ that the partial derivatives $(\partial^\alpha E_x)(v)$ are uniformly bounded for $v \neq 0$ and $x \in U$. It thus follows from the inverse function theorem that there exists a constant $A > 0$ such that for all $x_1$ and $x$ in $U$ with $x \neq x_1$,

$$|\partial_j G_{x_1}(x)| \leq A \quad \text{and} \quad |(\partial_j \partial_k G_{x_1})(x)| \leq A. \tag{4.62}$$

We can also pick $A$ to be large enough that for all $x \in U$, and all $v, w \in \mathbb{R}^d - \{0\}$,

$$A^{-1}|w|^2 \leq \sum\nolimits_{ij} g_{ij}(x, v) w^i w^j \leq A|w|^2. \tag{4.63}$$

and such that for all $x \in U$ and $v \in \mathbb{R}^d - \{0\}$,

$$|\partial_{x_k} g_{ij}(x, v)| \leq A. \tag{4.64}$$

Since $\zeta \in T_{\xi^-} S^*_{x_0}$, and $S^*_{x_0} = \{\xi : F_*(x_0, \xi)^2 = 1\}$, by Euler's homogeneous function theorem we have

$$\sum\nolimits_{ij} g^{ij}(x_0, \xi^-) \xi'(0) \xi^-_j = \frac{1}{2} \sum\nolimits_{i,j} \frac{\partial^2 F_*^2}{\partial \xi_i \partial \xi_j}(x_0, \xi^-) \zeta_i \xi^-_j = \frac{1}{2} \sum \frac{\partial F_*^2}{\partial \xi_j}(x_0, \xi^-) \zeta_j = 0. \tag{4.65}$$

Differentiating $F_*(x_0, \xi(a))^2 = 1$ twice with respect to $a$ at $a = 0$ yields that

$$\sum\nolimits_{i,j} g_{ij}(x_0, \xi^-) \xi''_i(0) \xi^-_j = - \sum\nolimits_{i,j} g_{ij}(x_0, \xi^-) \zeta^i \zeta^j \tag{4.66}$$

Define vectors $n(a)$ by setting $n^i(a) = \sum g^{ij}(x_0, \xi^-) \xi_j(a)$. Then $n(a)$ is the normal vector to $H(x_0, \xi(a))$ with respect to the inner product with coefficients $g_{ij}(x_0, v_0)$. Let $u(a)$ be the orthogonal projection of $v_0$ onto the hyperplane $\{v : \xi(a) \cdot v = 0\}$ with respect to the inner product $g_{ij}(x_0, v_0)$. If we define

$$c(a) = \sum g_{ij}(x_0, v_0) v_0^i n^j(a) = \sum g^{ij}(x_0, \xi^-) \xi^-_i \xi_j(a), \tag{4.67}$$

then $u(a) = v_0 - c(a)n(a)$. Note that (4.65) and (4.66) imply $c(0) = 1$, $c'(0) = 0$, and $c''(0) \leq -|\zeta|^2/A$. Also $n(0) = v_0$ and $|n'(0)| \leq A|\zeta|$.

Let $x(s) = x_0 + su(a)$ and let $R(s)$ be the length of the geodesic from $x_1$ to $x(s)$. To control $R(s)$, define $y(s) = G(x(s))$, and consider the variation $A(s, t) = E(ty(s))$, defined so that $t \mapsto A(s, t)$ is the geodesic from $x_1$ to $x(s)$. The Gauss Lemma for Finsler manifolds (see Lemma 6.1.1 of [1]) implies that

$$A^{-1} R(s) \leq |y(s)| \leq AR(s). \tag{4.68}$$

Define $T(s) = (\partial_t A)(s,1)$ and $V(s) = (\partial_s A)(s,1)$. Then (3.5) implies

$$R(s)R'(s) = \sum\nolimits_{i,j} g_{ij}(x(s),T(s))V^i(s)T^j(s). \tag{4.69}$$

Again, the Gauss Lemma implies

$$A^{-1}R(s) \leq |T(s)| \leq AR(s). \tag{4.70}$$

We can write

$$T^i(s) = \sum\nolimits_{i,j} (\partial_j E^i)(y(s))y^j(s) \tag{4.71}$$

and

$$V^i(s) = \sum\nolimits_{i,j,k} (\partial_j E^i)(y(s))(\partial_k G^j)(x(s))u^k(a) = u^i(a). \tag{4.72}$$

In particular, $T(0) = v_0$, so $R'(0) = \sum g_{ij}(x_0,v_0)u^i(a)v_0^j = 1 - c(a)^2$. Cauchy-Schwartz applied to (4.69) also tells us that

$$|R'(s)| \lesssim_A |u(a)|. \tag{4.73}$$

Note that $u(0) = 0$, so if $a$ is small enough, then we have $|u(a)| \leq l/2$. Taylor's theorem applied to (4.73), noting $R(0) = l$ then gives that for $|s| \leq 1$,

$$l/2 \leq R(s) \leq 2l. \tag{4.74}$$

Differentiating (4.69) tells us that

$$\begin{aligned} R'(s)^2 + R''(s)R(s) = \sum\nolimits_{i,j,k} &\left[ (\partial_{x_k} g_{ij})(x(s),T(s))u^i(a)T^j(s)u^k(a) \right] \\ &+ \left[ (\partial_{v_k} g_{ij})(x(s),T(s))u^i(a)T^j(s)(\partial_s T^k)(s) \right] \\ &+ \left[ g_{ij}(x(s),T(s))u^i(a)(\partial_s T^j)(s) \right]. \end{aligned} \tag{4.75}$$

Write the right hand side as I + II + III. Using (4.64), (4.70), (4.74), and the triangle inequality gives

$$|\mathrm{I}| \lesssim R(s)|u(a)|^2 \lesssim l|u(a)|^2. \tag{4.76}$$

Since $\partial_{v_k} g_{ij} = (1/2)(\partial^3 F^2/\partial v_i \partial v_j \partial v_k)$, applying Euler's homogeneous function theorem when summing over $j$ implies that

$$\mathrm{II} = 0. \tag{4.77}$$

Applying Cauchy-Schwarz and (4.62), we find

$$|\mathrm{III}| \lesssim |u(a)||T'(s)| \lesssim |u(a)|^2[|y(s)| + 1] \lesssim (l+1)|u(a)|^2. \tag{4.78}$$

But now combining (4.76), (4.77), (4.78), and rearranging (4.75) shows that

$$|R''(s)| \lesssim l^{-1}|u(a)|^2 + (1 + l^{-1})|u(a)|^2 \lesssim l^{-1}|u(a)|^2. \tag{4.79}$$

Taylor's theorem implies there exists $B > 0$ depending only on $A$ and $d$ such that

$$|R(s) - (R(0) + sR'(0))| \leq Bl^{-1}|u(a)|^2 s^2. \tag{4.80}$$

Since $R(0) = l$, $R'(0) = 1 - c(a)^2$, $c(0) = 1$, $u(0) = 0$, $c'(0) = 0$, $|u'(0)| \leq A|\zeta|$, and $c''(0) \leq -|\zeta|^2/A$, we conclude from (4.80) that as $a \to 0$, if $s > 0$ then

$$\begin{aligned} R(-s) &\leq l - sR'(0) + Bl^{-1}|u(a)|^2 s^2 \\ &\leq l - s(A^{-1}|\zeta|^2 a^2) + A^2 B l^{-1}|\zeta|^2 s^2 a^2 + O(a^3(s + l^{-1}s^2)). \end{aligned} \tag{4.81}$$

For all $s$, $L(a) \leq R(s)$. Optimizing by picking $s = l/2A^3B$ gives

$$L(a) \leq R(-s) \leq l - l|\zeta|^2 a^2/4A^4 B + O(a^3). \tag{4.82}$$

Taking $a \to 0$ and using that $L(0) = l$ gives that $L''(0) \leq -l|\zeta|^2/4A^4B$, so setting $C = 1/4A^4B$, we find we have proved what was required. $\square$

## 4.3 Analysis of Regime I via Density Methods

### 4.3.1 $L^2$ Estimates For Bounded Density Inputs

We now begin obtaining bounds for the operator $T^I$ specified in Proposition 4.4 by using the quasi-orthogonality estimates of Proposition 4.5. Define a metric $d_M = d_M^+ + d_M^-$ on $M$. Given an input $u : M \to \mathbb{C}$, we consider a maximal $1/R$ separated subset $\mathcal{X}_R$ of $M$, and then consider a decomposition $u = \sum_{x_0 \in \mathcal{X}_R} u_{x_0}$ with respect to some partition of unity, where $\text{supp}(u_{x_0}) \subset B(x_0, 1/R)$. The balls $\{B(x_0, 1/R) : x_0 \in \mathcal{X}_R\}$ have finite overlap, and so

$$\|u\|_{L^p(M)} \sim \left( \sum_{x_0 \in \mathcal{X}_R} \|u_{x_0}\|_{L^p(M)}^p \right)^{1/p}. \tag{4.83}$$

If we set $f_{x_0,t_0} = T_{t_0}^I \{u_{x_0}\}$, then

$$\left\| T^I u \right\|_{L^p(M)} = \left\| \sum_{(x_0,t_0) \in \mathcal{X}_R \times \mathcal{T}_R} f_{x_0,t_0} \right\|_{L^p(M)}. \tag{4.84}$$

In this subsection, we use the quasi-orthogonality estimates of the last section to obtain $L^2$ estimates on partial sums of a family of functions of the form $\{S_{x_0,t_0}\}$, which are essentially $L^1$ normalized versions of the functions $\{f_{x_0,t_0}\}$, under a density assumption on the set of indices we are summing over. Namely, we say a set $\mathcal{E} \subset \mathcal{X}_R \times \mathcal{T}_R$ has *density type* $(A_0, A_1)$ if for any set $B \subset \mathcal{X}_R \times \mathcal{T}_R$ with $1/R \leq \text{diam}(B) \leq A_1/R$,

$$\#(\mathcal{E} \cap B) \leq RA_0 \ \text{diam}(B).[1] \tag{4.85}$$

To obtain $L^p$ bounds from these $L^2$ bounds, in the next section we will perform a *density decomposition* to break up $\mathcal{X}_R \times \mathcal{T}_R$ into families of indices with controlled density, and then apply Proposition 4.9 on each subfamily to control (4.84) via an interpolation.

[1] This definition of density is chosen because it is 'scale-invariant' as we change the parameter $R$. Indeed, if $M = \mathbb{R}^d$, $\mathcal{X}_R = (\mathbb{Z}/R)^d$, and $\mathcal{T}_R = \mathbb{Z}/R$, then a set $\mathcal{E} \subset (\mathbb{Z}/R)^d \times (\mathbb{Z}/R)$ has density type $(A_0, A_1)$ if and only if $R\,\mathcal{E} \subset \mathbb{Z}^d \times \mathbb{Z}$ has density type $(A_0, A_1)$.

**Proposition 4.9.** *Fix $A \geq 1$. Consider a set $\mathcal{E} \subset \mathcal{X}_R \times \mathcal{T}_R$. Suppose that for each $(x_0, t_0) \in \mathcal{E}$, we pick two measurable functions $b_{t_0} : I_0 \to \mathbb{R}$ and $u_{x_0} : M \to \mathbb{R}$, supported on $I_{t_0}$ and $B(x_0, 1/R)$ respectively, such that $\|b_{t_0}\|_{L^1(I_0)} \leq 1$ and $\|u_{x_0}\|_{L^1(M)} \leq 1$. Define*

$$S_{x_0,t_0} = \int b_{t_0}(t)(Q_R \circ e^{2\pi i t P} \circ Q_R) u_{x_0} \; dt. \tag{4.86}$$

*Write $\mathcal{E} = \bigcup_{k=0}^{\infty} \mathcal{E}_k$, where*

$$\mathcal{E}_0 = \{(x,t) \in \mathcal{E} : 0 \leq |t| \leq 1/R\} \tag{4.87}$$

*and for $k > 0$, define*

$$\mathcal{E}_k = \{(x,t) \in \mathcal{E} : 2^{k-1}/R < |t| \leq 2^k/R\}. \tag{4.88}$$

*Suppose that for each $k$, the set $\mathcal{E}_k$ has density type $(A, 2^k)$. Then*

$$\Big\| \sum_k \sum_{(x_0,t_0) \in \mathcal{E}_k} 2^{k\frac{d-1}{2}} S_{x_0,t_0} \Big\|_{L^2(M)}^2 \lesssim R^d \log(A) A^{\frac{2}{d-1}} \sum_k 2^{k(d-1)} \# \mathcal{E}_k. \tag{4.89}$$

*Remark.* If $\|b_{t_0}\|_{L^1(I_0)} \sim 1$ and $\|u_{x_0}\|_{L^1(M)} \sim 1$, then locally constancy from the uncertainty principle and energy conservation of the wave equation tell us that morally,

$$\|S_{x_0,t_0}\|_{L^2(M)}^2 \sim R^d t_0^{d-1}. \tag{4.90}$$

If $\|b_{t_0}\|_{L^1(I_0)} \sim 1$ and $\|u_{x_0}\|_{L^1(M)} \sim 1$ for all $(x_0, t_0) \in \mathcal{E}$, this means that Proposition 4.9 is morally equivalent to

$$\Big\| \sum_{(x_0,t_0) \in \mathcal{E}} S_{x_0,t_0} \Big\|_{L^2(M)} \lesssim \sqrt{\log(A)} A^{\frac{1}{d-1}} \left( \sum_{(x_0,t_0) \in \mathcal{E}} \|S_{x_0,t_0}\|_{L^2(M)}^2 \right)^{1/2}. \tag{4.91}$$

Thus Proposition 4.9 is a kind of square root cancellation bound, albeit with an implicit constant which grows as the set $\mathcal{E}$ increases in density, a necessity given that the functions $\{S_{x_0,t_0}\}$ are not almost-orthogonal to one another.

*Proof.* Write $F = \sum_k F_k$, where

$$F_k = 2^{k\frac{d-1}{2}} \sum_{(x_0,t_0) \in \mathcal{E}_k} S_{x_0,t_0}. \tag{4.92}$$

Our goal is to bound $\|F\|_{L^2(M)}$. Applying Cauchy-Schwarz, we have

$$\|F\|_{L^2(M)}^2 \leq \log(A) \left( \sum_{k \leq \log(A)} \|F_k\|_{L^2(M)}^2 + \Big\| \sum_{k \geq \log(A)} F_k \Big\|_{L^2(M)}^2 \right). \tag{4.93}$$

Without loss of generality, increasing the implicit constant in the final result by applying the triangle inequality, we can assume that $\{k : \mathcal{E}_k \neq \emptyset\}$ is 10-separated, and that all values of $t$ with $(x,t) \in \mathcal{E}$ are positive. Thus if $F_k$ and $F_{k'}$ are both nonzero functions, then $k = k'$ or $|k - k'| \geq 10$.

Let us estimate $\langle F_k, F_{k'}\rangle$ for $k \geq k' + 10$. We write

$$\langle F_k, F_{k'}\rangle = \sum\nolimits_{(x_0,t_0)\in\mathcal{E}_k} \sum\nolimits_{(x_1,t_1)\in\mathcal{E}_{k'}} 2^{k\frac{d-1}{2}} 2^{k'\frac{d-1}{2}} \langle S_{x_0,t_0}, S_{x_1,t_1}\rangle. \tag{4.94}$$

For each $(x_0, t_0) \in \mathcal{E}_k$, and each $k' \leq k - 10$, consider the set

$$\mathcal{G}_0(x_0, t_0, k') = \{(x_1, t_1) \in \mathcal{E}_{k'} : |(t_0 - t_1) - d_M^-(x_0, x_1)| \leq 2^{k'+5}/R\}, \tag{4.95}$$

Also consider the sets of indices

$$\mathcal{G}_l^+(x_0, t_0, k') = \{(x_1, t_1) \in \mathcal{E}_{k'} : 2^l/R < |(t_0 - t_1) + d_M^+(x_0, x_1)| \leq 2^{l+1}/R\}. \tag{4.96}$$

and

$$\mathcal{G}_l^-(x_0, t_0, k') = \{(x_1, t_1) \in \mathcal{E}_{k'} : 2^l/R < |(t_0 - t_1) - d_M^-(x_0, x_1)| \leq 2^{l+1}/R\}. \tag{4.97}$$

If we set

$$\mathcal{G}_0(x_0, t_0, k') = (\mathcal{G}_l^+(x_0, t_0, k') \cup \mathcal{G}_l^-(x_0, t_0, k')) \tag{4.98}$$

and

$$\begin{aligned} \mathcal{G}_l(x_0, t_0, k') = \Big(&\mathcal{G}_l^+(x_0, t_0, k') \cup \mathcal{G}_l^-(x_0, t_0, k')\Big) \\ &- \bigcup\nolimits_{r<l} \Big(\mathcal{G}_r^+(x_0, t_0, k') \cup \mathcal{G}_r^-(x_0, t_0, k')\Big). \end{aligned} \tag{4.99}$$

Then $\mathcal{E}_{k'}$ is covered by $\mathcal{G}_0(x_0, t_0, k')$ and $\mathcal{G}_l(x_0, t_0, k')$ for $k' + 5 \leq l \leq 10 \log R$. Define

$$B_0(x_0, t_0, k') = \sum\nolimits_{(x_1,t_1)\in\mathcal{G}_0(x_0,t_0,k')} 2^{k\frac{d-1}{2}} 2^{k'\frac{d-1}{2}} |\langle S_{x_0,t_0}, S_{x_1,t_1}\rangle|, \tag{4.100}$$

and

$$B_l(x_0, t_0, k') = \sum\nolimits_{(x_1,t_1)\in\mathcal{G}_l(x_0,t_0,k')} 2^{k\frac{d-1}{2}} 2^{k'\frac{d-1}{2}} |\langle S_{x_0,t_0}, S_{x_1,t_1}\rangle|. \tag{4.101}$$

We thus have

$$\langle F_k, F_{k'}\rangle \leq \sum\nolimits_{(x_0,t_0)\in\mathcal{E}_k} B_0(x_0, t_0, k') + \sum\nolimits_{(x_0,t_0)\in\mathcal{E}_k} \sum\nolimits_{k'+5\leq l\leq 10\log R} B_l(x_0, t_0, k'). \tag{4.102}$$

Using the density properties of $\mathcal{E}$, we can control the size of the index sets $\mathcal{G}_\bullet(x_0, t_0, k')$, and thus control the quantities $B_\bullet(x_0, t_0, k')$. The rapid decay of Proposition 4.5 means that only $\mathcal{G}_0(x_0, t_0, k')$ needs to be estimated rather efficiently:

- Start by bounding the quantities $B_0(x_0, t_0, k')$. If $(x_1, t_1) \in \mathcal{G}_0(x_0, t_0, k')$, then

  $$|d_M^-(x_0, x_1) - (t_0 - t_1)| \leq 2^{k'+5}/R, \tag{4.103}$$

  Thus if we consider $\mathrm{Ann}(x_0, t_0, k') = \{x_1 : |d_M^-(x_0, x_1) - t_0| \leq 2^{k'+8}/R\}$, which is a geodesic annulus of radius $\sim 2^k/R$ and thickness $O(2^{k'}/R)$, then

  $$\mathcal{G}_0(x_0, t_0, k') \subset \mathrm{Ann}(x_0, t_0, k') \times [2^{k'}/R, 2^{k'+1}/R]. \tag{4.104}$$

The latter set is covered by $O(2^{(k-k')(d-1)})$ balls of radius $2^{k'}/R$, and so the density properties of $\mathcal{E}_{k'}$ implies that

$$\#(\mathcal{G}_0(x_0, t_0, k')) \lesssim A 2^{(k-k')(d-1)} 2^{k'}. \tag{4.105}$$

Since $k \geq k' + 10$, for $(x_1, t_1) \in \mathcal{G}_0(x_0, t_0, k')$ we have $d_M(x_0, x_1) \gtrsim 2^k/R$ and so

$$\langle S_{x_0,t_0}, S_{x_1,t_1} \rangle \lesssim R^d 2^{-k\left(\frac{d-1}{2}\right)}. \tag{4.106}$$

But putting together (4.105) and (4.106) gives that

$$\begin{aligned} B_0(x_0, t_0, k') &\leq (2^{k\frac{d-1}{2}} 2^{k'\frac{d-1}{2}})(A 2^{(k-k')(d-1)} 2^{k'})(R^d 2^{-k\left(\frac{d-1}{2}\right)}) \\ &= A R^d 2^{k(d-1)} 2^{-k'\frac{d-3}{2}}. \end{aligned}$$

Thus for each $k$, since $d \geq 4$,

$$\sum\nolimits_{(x_0,t_0)\in\mathcal{E}_k} \sum\nolimits_{k'\in[\log(A),k-10]} B_0(x_0, t_0, k') \lesssim R^d 2^{k(d-1)} \#\mathcal{E}_k. \tag{4.107}$$

- Next we bound $B_l(x_0, t_0, k')$ for $k' + 5 \leq l \leq k - 5$. The set $\mathcal{G}_l^+(x_0, t_0, k')$ is empty in this case. Thus

$$\mathcal{G}_l(x_0, t_0, k') \subset (\mathrm{Ann} \cup \mathrm{Ann}') \times [t_0 - 2^{k'}/R, t_0 + 2^{k'}/R], \tag{4.108}$$

where

$$\mathrm{Ann} = \{x \in M : |d_M^-(x_0, x) - (t_0 - 2^{k'})/R| \leq 100 \cdot 2^l/R\} \tag{4.109}$$

and

$$\mathrm{Ann}' = \{x \in M : |d_M^-(x_0, x) - (t_0 + 2^{k'})/R| \leq 100 \cdot 2^l/R\}, \tag{4.110}$$

These are geodesic annuli of thickness $O(2^l/R)$ and radius $\sim 2^k$. Thus $\mathcal{G}_l(x_0, t_0, k')$ is covered by $O(2^{(l-k')} 2^{(k-k')(d-1)})$ balls of radius $2^{k'}/R$, and the density of $\mathcal{E}_{k'}$ implies that

$$\#(\mathcal{G}_l(x_0, t_0, k')) \lesssim RA\ 2^{(l-k')} 2^{(k-k')(d-1)} 2^{k'}/R = A 2^l 2^{(k-k')(d-1)}. \tag{4.111}$$

For $(x_1, t_1) \in \mathcal{G}_l(x_0, t_0, k')$, $d_M(x_0, x_1) \sim 2^k/R$, and thus Proposition 4.5 implies

$$|\langle S_{x_0,t_0}, S_{x_1,t_1} \rangle| \lesssim R^d 2^{-k\frac{d-1}{2}} 2^{-lK}. \tag{4.112}$$

Thus for any $K \geq 0$,

$$\begin{aligned} B_l(x_0, t_0, k') &\lesssim_K \left(A 2^l 2^{(k-k')(d-1)}\right) R^d 2^{k\frac{d-1}{2}} 2^{k'\frac{d-1}{2}} \left(2^{-k\frac{d-1}{2}} 2^{-lK}\right) \\ &\lesssim A R^d 2^l 2^{k(d-1)} 2^{-k'\frac{d-1}{2}} 2^{-lK}. \end{aligned} \tag{4.113}$$

Picking $K > 1$, we conclude that

$$\sum_{(x_0,t_0)\in\mathcal{E}_k} \sum_{k'\in[\log(A),k-10]} \sum_{l\in[k'+10,k-5]} B_l(x_0, t_0, k') \lesssim R^d 2^{k(d-1)} \#\mathcal{E}_k. \tag{4.114}$$

- Finally, let's bound $B_l(x_0, t_0, k')$ for $k - 5 \leq l \leq 10 \log R$. If either $(x_1, t_1) \in \mathcal{G}_l^-(x_0, t_0, k')$ or $(x_1, t_1) \in \mathcal{G}_l^+(x_0, t_0, k')$, then $d_M(x_0, x_1) \lesssim 2^l/R$. So $\mathcal{G}_l(x_0, t_0, k')$ is covered by $O(2^{(l-k')d})$ balls of radius $2^{k'}/R$, and thus

$$\#(\mathcal{G}_l(x_0, t_0, k')) \lesssim RA\, 2^{(l-k')d}(2^{k'}/R) = A2^{(l-k')d}2^{k'}. \tag{4.115}$$

For $(x_1, t_1) \in \mathcal{G}_l(x_0, t_0, k')$, we have no good control over $d_M(x_0, t_1)$ aside from the trivial estimate $d_M(x_0, x_1) \lesssim 1$. Thus Proposition 4.5 yields a bound of the form

$$|\langle S_{x_0,t_0}, S_{x_1,t_1}\rangle| \lesssim R^d 2^{-lK}. \tag{4.116}$$

Thus we conclude that

$$\begin{aligned} B_l(x_0, t_0, k') &\lesssim_N R^d 2^{k\frac{d-1}{2}} 2^{k'\frac{d-1}{2}} \left(A2^{(l-k')d}2^{k'}\right)\left(2^{-lN}\right) \\ &= AR^d 2^{k\frac{d-1}{2}} 2^{-k'\frac{d-1}{2}} 2^{-lN} \end{aligned} \tag{4.117}$$

Picking $K > d$, we conclude that

$$\sum\nolimits_{(x_0,t_0)\in\mathcal{E}_k} \sum\nolimits_{k'\in[\log(A),k-10]} \sum\nolimits_{l\in[k+10,\log R]} B_l(x_0, t_0, k') \lesssim R^d. \tag{4.118}$$

The three bounds (4.107), (4.114) and (4.118) imply that

$$\sum\nolimits_k \sum\nolimits_{k'\in[\log(A),k]} |\langle F_k, F_{k'}\rangle| \lesssim R^d \sum\nolimits_k 2^{k(d-1)}\#\mathcal{E}_k. \tag{4.119}$$

In particular, combining (4.119) with (4.93), we have

$$\|F\|_{L^2(M)}^2 \lesssim \log(A)\left(\sum\nolimits_k \|F_k\|_{L^2(M)}^2 + R^d \sum\nolimits_k 2^{k(d-1)}\#\mathcal{E}_k\right). \tag{4.120}$$

Next, consider some parameter $a$ to be determined later, and decompose the interval $[2^k/R, 2^{k+1}/R]$ into the disjoint union of length $A^a/R$ intervals of the form

$$I_{k,\mu} = [2^k/R + (\mu - 1)A^a/R, 2^k/R + \mu A^a/R] \quad \text{for } 1 \leq \mu \leq 2^k/A^a. \tag{4.121}$$

We thus consider a further decomposition $\mathcal{E}_k = \bigcup \mathcal{E}_{k,\mu}$, where $F_k = \sum F_{k,\mu}$. As before, increasing the implicit constant in the Proposition, we may assume without loss of generality that the set $\{\mu : \mathcal{E}_{k,\mu} \neq \emptyset\}$ is 10-separated. We now estimate

$$\sum\nolimits_{\mu\geq\mu'+10} |\langle F_{k,\mu}, F_{k,\mu'}\rangle|. \tag{4.122}$$

For $(x_0, t_0) \in \mathcal{E}_{k,\mu}$ and $l \geq 1$, define

$$\mathcal{H}_l(x_0, t_0, \mu') = \left\{(x_1, t_1) \in \mathcal{E}_{k,\mu'} : \frac{2^l A^a}{2R} \leq \max(d_M(x_0, x_1), t_0 - t_1) \leq \frac{2^l A^a}{R}\right\}. \tag{4.123}$$

Then $\bigcup_{l\geq1} \mathcal{H}_l(x_0, t_0, \mu')$ covers $\bigcup_{\mu\geq\mu'+10} \mathcal{E}_{k,\mu'}$. Set

$$B_l'(x_0, t_0, \mu') = \sum\nolimits_{(x_1,t_1)\in\mathcal{H}_l(x_0,t_0,\mu')} 2^{k(d-1)}|\langle S_{x_0,t_0}, S_{x_1,t_1}\rangle|. \tag{4.124}$$

Then

$$\langle F_{k,\mu}, F_{k,\mu'}\rangle \leq \sum\nolimits_{(x_0,t_0)\in\mathcal{E}_{k,\mu}} \sum\nolimits_l B_l'(x_0,t_0,\mu'). \tag{4.125}$$

We now bound the constants $B_l'$. Pick a constant $r$ such that $d_M \leq 2^r d_M^+$ and $d_M \leq 2^r d_M^-$. As in the estimates of the quantities $B_l$, the quantities where $l$ is large have negligible magnitude:

- For $l \leq k - a\log_2 A + 10r$, we have $2^l A^a/R \lesssim 2^k/R$. The set $\mathcal{H}_l(x_0,t_0,\mu')$ is covered by $O(1)$ balls of radius $2^l A^a/R$, and density properties imply

$$\#\mathcal{H}_l(x_0,t_0,\mu') \lesssim (RA)(2^l A^a/R) = A^{a+1}2^l \tag{4.126}$$

  For $(x_1,t_1) \in \mathcal{H}_l(x_0,t_0,\mu')$, we claim that

$$2^{k(d-1)}|\langle S_{x_0,t_0}, S_{x_1,t_1}\rangle| \lesssim R^d 2^{k(d-1)}(2^l A^a)^{-\frac{d-1}{2}}. \tag{4.127}$$

  Indeed, for such tuples we have

$$d_M(x_0,x_1) \gtrsim 2^l A^a/R \quad \text{or} \quad \min\nolimits_\pm |d_M^\pm(x_0,x_1) - (t_0-t_1)| \gtrsim 2^l A^a/R, \tag{4.128}$$

  and the estimate follows from Proposition 4.5 in either case. Since $d \geq 4$, we conclude that

$$\begin{aligned}\sum\nolimits_{l\in[1,k-a\log_2 A+10]} & B_l'(x_0,t_0,,\mu') \\ &\lesssim \sum\nolimits_{l\in[1,k-a\log_2 A+10]} R^d(2^{k(d-1)})(2^l A^a)^{-\frac{d-1}{2}}(A^{a+1}2^l) \\ &\lesssim \sum\nolimits_{l\in[1,k-a\log_2 A+10]} R^d 2^{k(d-1)} 2^{-l\frac{d-3}{2}} A^{1-a\left(\frac{d-3}{2}\right)} \\ &\lesssim R^d 2^{k(d-1)} A^{1-a\left(\frac{d-3}{2}\right)}.\end{aligned} \tag{4.129}$$

- For $l > k - a\log_2 A + 10r$, a tuple $(x_1,t_1) \in \mathcal{E}_k$ lies in $\mathcal{H}_l(x_0,t_0,\mu')$ if and only if $2^l A^a/2R \leq d_M(x_0,x_1) \leq 2^l A^a/R$, since we always have

$$t_0 - t_1 \leq 2^{k+1}/R < 2^{l+r}A^a/8R. \tag{4.130}$$

  and so $d_M(x_0,x_1) \geq 2^l A^a/2R$. And so

$$|(t_0-t_1) - d_M^-(x_0,x_1)| \geq 2^{-r}2^l A^a/2R - 2^{k+1}/R \geq 2^{-r}2^l A^a/4R.$$

  Also $|(t_0-t_1) + d_M^+(x_0,x_1)| \geq |d_M^+(x_0,x_1) \geq 2^{-r}2^l A^a/4R$. Thus we conclude from Proposition 4.5 that

$$2^{k(d-1)}|\langle S_{x_0,t_0}, S_{x_1,t_1}\rangle| \lesssim_K R^d 2^{k(d-1)}(2^l A^a)^{-K}. \tag{4.131}$$

  Now $\mathcal{H}_l(x_0,t_0,\mu')$ is covered by $O((2^{l-k}A^a)^d)$ balls of radius $2^k/R$, and the density properties of $\mathcal{E}_k$ thus imply that

$$\#(\mathcal{H}_l(x_0,t_0,\mu')) \lesssim (RA)(2^{l-k}A^a)^d(2^k/R) \lesssim A^{1+ad}2^{ld}2^{-k(d-1)}. \tag{4.132}$$

Thus, picking $K > \max(d, 1 + ad)$, we conclude that

$$\begin{aligned}&\sum\nolimits_{l \geq k - a\log_2 A + 10} B'_l(x_0, t_0, \mu')\\ &\quad \lesssim R^d \sum\nolimits_{l \geq k - a\log_2 A + 10} (2^{k(d-1)})(2^l A^a)^{-M} A^{1+ad} 2^{ld} 2^{-k(d-1)} \lesssim R^d.\end{aligned} \tag{4.133}$$

Combining (4.129) and (4.133), and then summing over the tuples $(x_0, t_0) \in \mathcal{E}_{k,\mu}$, we conclude that

$$\sum\nolimits_{\mu \geq \mu' + 10} |\langle F_{k,\mu}, F_{k,\mu'} \rangle| \lesssim R^d \left(1 + 2^{k(d-1)} A^{1 - a\left(\frac{d-3}{2}\right)}\right) \#\mathcal{E}_{k,\mu}. \tag{4.134}$$

Now summing in $\mu$, (4.134) implies that

$$\|F_k\|^2_{L^2(M)} \lesssim \sum\nolimits_{\mu} \|F_{k,\mu}\|^2_{L^2(M)} + R^d \left(1 + 2^{k(d-1)} A^{1 - a\left(\frac{d-3}{2}\right)}\right) \#\mathcal{E}_k. \tag{4.135}$$

The functions in the sum defining $F_{k,\mu}$ are highly coupled, and it is difficult to use anything except Cauchy-Schwarz to break them apart. Since $\#(\mathcal{T}_R \cap I_{k,\mu}) \sim A^a$, if we set $F_{k,\mu} = \sum_{t \in \mathcal{T}_R \cap I_{k,\mu}} F_{k,\mu,t}$, where

$$F_{k,\mu,t} = \sum\nolimits_{(x_0,t) \in \mathcal{E}_{k,\mu}} 2^{k\frac{d-1}{2}} S_{x_0,t}. \tag{4.136}$$

Then Cauchy-Schwarz implies that

$$\|F_{k,\mu}\|^2_{L^2(M)} \lesssim A^a \sum\nolimits_{t \in \mathcal{T}_R \cap I_{k,\mu}} \|F_{k,\mu,t}\|^2_{L^2(M)}. \tag{4.137}$$

Since the elements of $\mathcal{X}_R$ are $1/R$ separated, the functions in the sum defining $F_{k,\mu,t}$ are quite orthogonal to one another; Proposition 4.5 implies that for $x_0 \neq x_1$,

$$|\langle S_{x_0,t}, S_{x_1,t} \rangle| \lesssim R^d (R d_M(x_0, x_1))^{-K} \quad \text{for all } K \geq 0. \tag{4.138}$$

Thus

$$\|F_{k,\mu,t}\|^2_{L^2(M)} \lesssim R^d 2^{k(d-1)} \#(\mathcal{E}_k \cap (M \times \{t\})). \tag{4.139}$$

But this means that

$$A^a \sum\nolimits_{t \in \mathcal{T}_R \cap I_{k,\mu}} \|F_{k,\mu,t}\|^2_{L^2(M)} \lesssim R^d 2^{k(d-1)} A^a \#\mathcal{E}_{k,\mu}. \tag{4.140}$$

Thus (4.135), (4.137), and (4.140) imply that

$$\begin{aligned}\|F_k\|^2_{L^2(M)} &\lesssim \sum\nolimits_{\mu} \|F_{k,\mu}\|^2_{L^2(M)} + R^d \left(1 + 2^{k(d-1)} A^{1 - a\left(\frac{d-3}{2}\right)}\right) \#\mathcal{E}_k\\ &\lesssim R^d \left(2^{k(d-1)} A^a + (1 + 2^{k(d-1)} A^{1 - a\left(\frac{d-3}{2}\right)}\right) \#\mathcal{E}_k.\end{aligned} \tag{4.141}$$

Optimizing by picking $a = 2/(d-1)$ gives that

$$\|F_k\|^2_{L^2(M)} \lesssim R^d 2^{k(d-1)} A^{\frac{2}{d-1}} \#\mathcal{E}_k. \tag{4.142}$$

The proof is completed by combining (4.120) with (4.142). □

### 4.3.2 $L^p$ Estimates Via Density Decompositions

Combining the $L^2$ analysis of Section 4.3 with a density decomposition argument, we can now prove the following Lemma, which completes the analysis of the operator $T^I$ in Proposition 4.4.

**Lemma 4.10.** *Using the notation of Proposition 4.4, let $T^I = \sum_{t_0 \in \mathcal{T}_R} T^I_{t_0}$, where*

$$T^I_{t_0} = b^I_{t_0}(t)(Q_R \circ e^{2\pi i t P} \circ Q_R)\, dt. \tag{4.143}$$

*Then for $1 \le p < 2(d-1)/(d+1)$,*

$$\|T^I u\|_{L^p(M)} \lesssim R^{-1/p'} \left( \sum_{t_0 \in \mathcal{T}_R} \left[ \|b^I_{t_0}\|_{L^p(I_0)} \langle R t_0 \rangle^{\alpha(p)} \right]^p \right)^{1/p} \|u\|_{L^p(M)}. \tag{4.144}$$

We prove Lemma 4.10 via a *density decomposition* argument, adapted from the methods of [7]. Given a function $u : M \to \mathbb{C}$, we use a partition of unity to write

$$u = \sum_{x_0 \in \mathcal{X}_R} u_{x_0}, \tag{4.145}$$

where $u_{x_0}$ is supported on $B(x_0, 1/R)$, and

$$\left( \sum_{x_0 \in \mathcal{X}_R} \|u_{x_0}\|^p_{L^1(M)} \right)^{1/p} \lesssim R^{-d/p'} \left( \sum_{x_0 \in \mathcal{X}_R} \|u_{x_0}\|^p_{L^p(M)} \right)^{1/p} \lesssim R^{-d/p'} \|u\|_{L^p(M)}. \tag{4.146}$$

Define

$$\mathcal{X}_a = \{ x_0 \in \mathcal{X}_R : 2^{a-1} < \|u_{x_0}\|_{L^1(M)} \le 2^a \} \tag{4.147}$$

and let

$$\mathcal{T}_b = \{ t_0 \in \mathcal{T}_R : 2^{b-1} < \|b^I_{t_0}\|_{L^1(M)} \le 2^b \}. \tag{4.148}$$

Define functions $f_{x_0,t_0} = T^I_{t_0} u_{x_0}$. Lemma 4.10 follows from the following result.

**Lemma 4.11.** *Fix $u \in L^p(M)$, and consider $\mathcal{X}_a$, $\mathcal{T}_b$, and $\{f_{x_0,t_0}\}$ as above. For any function $c : \mathcal{X}_R \times \mathcal{T}_R \to \mathbb{C}$, and $1 < p < 2(d-1)/(d+1)$,*

$$\begin{aligned} &\left\| \sum_{a,b} \sum_{(x_0,t_0) \in \mathcal{X}_a \times \mathcal{T}_b} 2^{-(a+b)} \langle R t_0 \rangle^{\frac{d-1}{2}} c(x_0,t_0) f_{x_0,t_0} \right\|_{L^p(M)} \\ &\qquad \lesssim R^{d/p'} \left( \sum_{a,b} \sum_{(x_0,t_0) \in \mathcal{X}_a \times \mathcal{T}_b} |c(x_0,t_0)|^p \langle R t_0 \rangle^{d-1} \right)^{1/p}. \end{aligned} \tag{4.149}$$

To see how Lemma 4.11 implies Lemma 4.10, set $c(x_0,t_0) = 2^{a+b} \langle R t_0 \rangle^{-\frac{d-1}{2}}$ for $x_0 \in \mathcal{X}_a$ and $t_0 \in \mathcal{T}_b$. Then Lemma 4.11 implies that

$$\begin{aligned} &\|T^I u\|_{L^p(M)} \\ &\quad = \left\| \sum f_{x_0,t_0} \right\|_{L^p(M)} \\ &\quad \lesssim R^{d/p'} \left( \sum_{(x_0,t_0)} \left[ \|b^I_{t_0}\|_{L^1(\mathbb{R})} \|u_{x_0}\|_{L^1(M)} \langle R t_0 \rangle^{\alpha(p)} \right]^p \right)^{1/p} \\ &\quad \lesssim R^{-1/p'} \left( \sum_{t_0} \left[ \|b^I_{t_0}\|_{L^p(I_0)} \langle R t_0 \rangle^{\alpha(p)} \right]^p \right)^{1/p} \left( \sum_{x_0} \left[ \|u_{x_0}\|_{L^1(M)} R^{d/p'} \right]^p \right)^{1/p} \\ &\quad \lesssim R^{-1/p'} \left( \sum_{t_0} \left[ \|b^I_{t_0}\|_{L^p(I_0)} \langle R t_0 \rangle^{\alpha(p)} \right]^p \right)^{1/p} \|u\|_{L^p(M)}. \end{aligned} \tag{4.150}$$

Thus we have proved Lemma 4.10. We take the remainder of this section to prove Lemma 4.11 using a density decomposition argument.

*Proof of Lemma 4.11.* For $p = 1$, this inequality follows simply by applying the triangle inequality, and applying the pointwise estimates of Proposition 4.5. By methods of interpolation, to prove the result for $p > 1$, we thus only need only prove a restricted strong type version of this inequality. In other words, we can restrict $c$ to be the indicator function of a set $\mathcal{E} \subset \mathcal{X}_R \times \mathcal{T}_R$. Write $\mathcal{E} = \bigcup_{k \geq 0} \mathcal{E}_{k,a,b}$, where

$$\mathcal{E}_{0,a,b} = \{(x,t) \in \mathcal{E} \cap (\mathcal{X}_a \times \mathcal{T}_b) : |t| \leq 1/R\} \tag{4.151}$$

and for $k > 0$, let

$$\mathcal{E}_{k,a,b} = \{(x,t) \in \mathcal{E} \cap (\mathcal{X}_a \times \mathcal{T}_b) : 2^{k-1}/R < |t| \leq 2^k/R\}. \tag{4.152}$$

Write

$$F_k = \sum_{a,b} \sum_{(x_0,t_0) \in \mathcal{E}_{k,a,b}} 2^{k\left(\frac{d-1}{2}\right)} 2^{-(a+b)} f_{x_0,t_0}. \tag{4.153}$$

Our proof will be completed if we can show that

$$\Big\| \sum_k F_k \Big\|_{L^p(M)} \lesssim R^{d(1-1/p)} \Big( \sum_k 2^{k(d-1)} \# \mathcal{E}_k \Big)^{1/p}. \tag{4.154}$$

To prove (4.154), we perform a density decomposition on the sets $\{\mathcal{E}_k\}$. For $u \geq 0$, let $\widehat{\mathcal{E}_k}(u)$ be the set of all points $(x_0,t_0) \in \mathcal{E}_k$ that are contained in a ball $B$ with $\mathrm{rad}(B) \leq 2^k/100R$, such that $\#(\mathcal{E}_k \cap B) \geq R2^u \mathrm{rad}(B)$. Then define

$$\mathcal{E}_k(u) = \widehat{\mathcal{E}_k}(u) - \bigcup_{u' > u} \widehat{\mathcal{E}_k}(u'). \tag{4.155}$$

Because the set $\mathcal{E}_k$ is $1/R$ discretized, we have

$$\mathcal{E}_k = \bigcup_{u \geq 0} \mathcal{E}_k(u). \tag{4.156}$$

Moreover $\mathcal{E}_k(u)$ has density type $(R2^u, 2^k/100R)$, and thus by a covering argument, also has density type $(C_d R 2^u, 2^k/R)$ for $C_d = 1000^d$. Furthermore, there are disjoint balls $B_{k,u,1}, \dots, B_{k,u,N_{k,u}}$ of radius at most $2^k/100R$ such that

$$\sum_n \mathrm{rad}(B_{k,u,n}) \leq 2^{-u}/R \# \mathcal{E}_k. \tag{4.157}$$

and such that $\mathcal{E}_k(u)$ is covered by the balls $\{B^*_{k,u,n}\}$, where, for a ball $B$, $B^*$ denotes the ball with the same center as $B$, but 5 times the radius. Now write

$$F_{k,u} = \sum_{a,b} \sum_{(x_0,t_0) \in \mathcal{E}_{k,a,b}(u)} 2^{k\left(\frac{d-1}{2}\right)} 2^{-(a+b)} f_{x_0,t_0}. \tag{4.158}$$

Using the density assumption on $\mathcal{E}_k(u)$, we can apply Lemma 4.9 of the last section, which implies that, with $S_{x_0,t_0} = 2^{-(a+b)} f_{x_0,t_0}$ for $(x_0,t_0) \in \mathcal{E}_{k,a,b}(u)$,

$$\Big\| \sum_k F_{k,u} \Big\|_{L^2(M)} \lesssim R^{d/2} \left( u^{1/2} 2^{u\left(\frac{1}{d-1}\right)} \right) \Big( \sum_k 2^{k(d-1)} \# \mathcal{E}_k \Big)^{1/2}. \tag{4.159}$$

Let $(y_{k,u,n}, t_{k,u,n})$ denote the center of $B_{k,u,n}$. Then

$$\sum\nolimits_{(x_0,t_0)\in[B_{k,u,n}\cap\mathcal{E}_k(u)]} 2^{-(a+b)} f_{x_0,t_0} \tag{4.160}$$

has mass concentrated on the geodesic annulus $\mathrm{Ann}_{k,u,n} \subset M$ with center $y_{k,u,n}$, with radius $t_{k,u,n} \sim 2^k/R$, and with thickness 5 $\mathrm{rad}(B_{k,u,n})$. Thus

$$\sum\nolimits_n |\mathrm{Ann}_{k,u,n}| \lesssim \sum\nolimits_n (2^k/R)^{d-1}\mathrm{rad}(B_{k,u,n}) \leq (2^k/R)^{d-1}R^{-1}2^{-u}\#\mathcal{E}_k. \tag{4.161}$$

If we set $\Lambda_u = \bigcup_k \bigcup_n \mathrm{Ann}_{k,u,n}$, then

$$|\Lambda_u| \lesssim R^{-d}2^{-u} \sum\nolimits_k 2^{k(d-1)}\#\mathcal{E}_k \tag{4.162}$$

Since $1/p - 1/2 > 1/(d-1)$, so that $\alpha(p) > 1$, Hölder's inequality implies that

$$\begin{aligned} \Big\| \sum\nolimits_k F_{k,u} \Big\|_{L^p(\Lambda_u)} &\lesssim |\Lambda_u|^{1/p-1/2} \Big\| \sum\nolimits_k F_{k,u} \Big\|_{L^2(\Lambda_{k,u})} \\ &\lesssim \Big( R^{-d}2^{-u} \sum\nolimits_k 2^{k(d-1)}\#\mathcal{E}_k \Big)^{1/p-1/2} \\ &\qquad \Big( R^d \Big( u 2^{u\left(\frac{2}{d-1}\right)} \Big) \sum\nolimits_k 2^{k(d-1)}\#\mathcal{E}_k \Big)^{1/2} \\ &= R^{d(1-1/p)} \Big( u^{1/2} 2^{-u\left(\frac{\alpha(p)-1}{d-1}\right)} \Big) \Big( \sum\nolimits_k 2^{k(d-1)}\#\mathcal{E}_k \Big)^{1/p} \\ &\lesssim R^{d(1-1/p)} 2^{-u\varepsilon} \Big( \sum\nolimits_k 2^{k(d-1)}\#\mathcal{E}_k \Big)^{1/p} \end{aligned} \tag{4.163}$$

for some suitable small $\varepsilon > 0$. For each $(x_0, t_0) \in \mathcal{E}_{k,a,b}(u) \cap B_{k,u,n}$, we calculate using the pointwise bounds for the functions $\{f_{x_0,t_0}\}$ that

$$\begin{aligned} \|2^{-(a+b)} f_{x_0,t_0}\|_{L^1(\Lambda(u)^c)} &= R^d \int_{\mathrm{Ann}_R^c} \langle R d_M(x,x_0)\rangle^{-\left(\frac{d-1}{2}\right)} \langle R|t_0 - d_M(x,x_0)|\rangle^{-M}\, dx \\ &\lesssim R^{\left(\frac{d+1}{2}-M\right)} \int_{5\,\mathrm{rad}(B_{k,u,n})}^{O(1)} (t_{k,u,n}+s)^{\left(\frac{d-1}{2}\right)} s^{-M}\, ds \\ &\lesssim R^{\left(\frac{d+1}{2}-M\right)} \mathrm{rad}(B_{k,u,n})^{1-M} t_{k,u,n}^{\left(\frac{d-1}{2}\right)} \\ &\lesssim 2^{k\left(\frac{d-1}{2}\right)} (R\mathrm{rad}(B_{k,u,n}))^{1-M}. \end{aligned} \tag{4.164}$$

Thus

$$\|2^{k\left(\frac{d-1}{2}\right)} 2^{-(a+b)} f_{x_0,t_0}\|_{L^1(\mathrm{Ann}_n^c)} \lesssim 2^{k(d-1)} (R\mathrm{rad}(B_{k,u,n}))^{1-M} \tag{4.165}$$

Because the set of points in $\mathcal{E}_k$ is $1/R$ separated, there are at most $O((R\mathrm{rad}(B_{k,u,n}))^{d+1})$ points in $\mathcal{E}_k(u) \cap B_{k,u,n}$, and so the triangle inequality implies that

$$\begin{aligned} &\Big\| \sum\nolimits_{a,b} \sum\nolimits_{(x_0,t_0)\in\mathcal{E}_{k,a,b}(u)\cap B_{k,u,n}} 2^{k\left(\frac{d-1}{2}\right)} 2^{-(a+b)} f_{x_0,t_0} \Big\|_{L^1(\Lambda(u)^c)} \\ &\qquad \lesssim 2^{k(d-1)} (R\mathrm{rad}(B_{k,u,n}))^{d+2-M}. \end{aligned} \tag{4.166}$$

Since $\#\mathcal{E}_k \cap B_{k,u,n} \geq R2^u \operatorname{rad}(B_{k,u,n})$, and $\mathcal{E}_k$ is $1/R$ discretized, we must have

$$\operatorname{rad}(B_{k,u,n}) \geq (2^u/2^d)^{\frac{1}{d-1}} \; 1/R. \tag{4.167}$$

Thus

$$\begin{aligned}
\Big\| \sum\nolimits_k F_{k,u} \Big\|_{L^1(\Lambda(u)^c)} &\lesssim_M \sum\nolimits_k \sum\nolimits_n 2^{k(d-1)} (R\operatorname{rad}(B_{k,u,n}))^{d+2-M} \\
&\lesssim \sum\nolimits_k 2^{k(d-1)} \Big(R \min\nolimits_n \operatorname{rad}(B_{k,u,n})\Big)^{d+1-M} \\
&\qquad \Big(\sum\nolimits_n R\operatorname{rad}(B_{k,u,n})\Big) \\
&\lesssim \sum\nolimits_k 2^{k(d-1)} 2^{u\left(\frac{d+1-M}{d-1}\right)} \left(2^{-u} \#\mathcal{E}_k\right) \\
&\lesssim 2^{u\left(\frac{2-M}{d-1}\right)} \sum\nolimits_k 2^{k(d-1)} \#\mathcal{E}_k
\end{aligned} \tag{4.168}$$

Picking $M > 2+(1-1/p)(1/p-1/2)^{-1}$, and interpolating with the bounds on $\| \sum_k F_{k,u} \|_{L^2(M)}$ yields that

$$\begin{aligned}
\Big\| \sum\nolimits_k F_{k,u} \Big\|_{L^p(\Lambda(u)^c)} &\lesssim \Big(2^{u\left(\frac{2-M}{d-1}\right)}\Big)^{2/p-1} \Big(R^{d/2} \Big(u^{1/2} 2^{u\left(\frac{1}{d-1}\right)}\Big)\Big)^{2(1-1/p)} \\
&\qquad \Big(\sum 2^{k(d-1)} \#\mathcal{E}_k\Big)^{1/p} \\
&\lesssim 2^{-u\varepsilon} R^{d(1-1/p)} \sum\nolimits_k \Big(\sum 2^{k(d-1)} \#\mathcal{E}_k\Big)^{1/p}.
\end{aligned} \tag{4.169}$$

So now we know

$$\Big\| \sum\nolimits_k F_{k,u} \Big\|_{L^p(M)} \lesssim R^{d(1-1/p)} 2^{-u\varepsilon} \Big(\sum\nolimits_k 2^{k(d-1)} \#\mathcal{E}_k\Big)^{1/p}. \tag{4.170}$$

The exponential decay in $u$ allows us to sum in $u$ to obtain that

$$\Big\| \sum\nolimits_u \sum\nolimits_k F_{k,u} \Big\|_{L^p(M)} \lesssim R^{d(1-1/p)} \Big(\sum\nolimits_k 2^{k(d-1)} \#\mathcal{E}_k\Big)^{1/p}. \tag{4.171}$$

This is precisely the bound we were required to prove. □

## 4.4 Analysis of Regime II via Local Smoothing

In this section, we bound the operators $\{T^{II}\}$, by a reduction to an endpoint local smoothing inequality, namely, the inequality that

$$\|e^{2\pi i t P} f\|_{L^{p'}(M) L_t^{p'}(I_0)} \lesssim \|f\|_{L^{p'}_{\alpha(p)-1/p'}}. \tag{4.172}$$

This inequality is proved in Corollary 1.2 of [13] for $1 < p < 2(d-1)/(d+1)$ for classical elliptic pseudodifferential operators $P$ satisfying the cosphere assumption of Theorem 4.1. The range of $p$ here is also precisely the range of $p$ in Theorem 4.1. Alternatively, Lemma 4.11 can be used to prove (4.172) independently of [13] in the same range by a generalization of the method of Section 10 of [7].

**Lemma 4.12.** *Using the notation of Proposition 4.4, let*

$$T^{II} = \int b^{II}(t)(Q_R \circ e^{2\pi itP} \circ Q_R)\, dt. \tag{4.173}$$

*For* $1 < p < 2(d-1)/(d+1)$*, we then have*

$$\|T^{II}u\|_{L^p(M)} \lesssim R^{\alpha(p)-1/p'} \|b^{II}\|_{L^p(I_0)} \|u\|_{L^p(M)}. \tag{4.174}$$

*Proof.* For each $R$, the *class* of operators of the form $\{T^{II}\}$ formed from a given function $b^{II}$ is closed under taking adjoints. Indeed, if $T^{II}$ is obtained from $b^{II}$, then $(T^{II})^*$ is obtained from the multiplier $\overline{b^{II}}$. Because of this self-adjointness, if we can prove that

$$\|T^{II}u\|_{L^{p'}(M)} \lesssim R^{\alpha(p)-1/p'} \|b^{II}\|_{L^p(I_0)} \|u\|_{L^{p'}(M)}, \tag{4.175}$$

then we obtain the required result by duality. We apply this duality because it is easier to exploit local smoothing inequalities in $L^{p'}(M)$ since now $p' > 2$.

We begin by noting that the operators $\{Q_R\}$, being a bounded family of order zero pseudo-differential operators, are uniformly bounded on $L^{p'}(M)$. Thus

$$\begin{aligned} \|T^{II}u\|_{L^{p'}(M)} &= \Big\| Q_R \circ \Big( \int_{I_0} b^{II}(t) e^{2\pi itP}(Q_R u) \Big) \Big\|_{L^{p'}(M)} \\ &\lesssim \Big\| \Big( \int_{I_0} b^{II}(t) e^{2\pi itP}(Q_R u) \Big) \Big\|_{L^{p'}(M)}. \end{aligned} \tag{4.176}$$

Applying Hölder and Minkowski's inequalities, we find that

$$\|T^{II}u\|_{L^{p'}(M)} \leq \|b^{II}\|_{L^p(\mathbb{R})} \Big\| \Big( \int_{I_0} |e^{2\pi itP}(Q_R u)|^{p'} \Big)^{1/p'} \Big\|_{L^{p'}(M)}. \tag{4.177}$$

Applying the endpoint local smoothing inequality (4.172), we conclude that

$$\begin{aligned} \|T^{II}u\|_{L^{p'}(M)} &\lesssim \|b^{II}\|_{L^p(\mathbb{R})} \|e^{2\pi iP}(Q_R u)\|_{L^{p'}_t L^{p'}_x} \\ &\lesssim \|b^{II}\|_{L^p(\mathbb{R})} \|Q_R u\|_{L^q_{\alpha(p)-1/p'}(M)}, \end{aligned} \tag{4.178}$$

Bernstein's inequality for compact manifolds (see [21], Section 3.3) gives

$$\|Q_R u\|_{L^q_{\alpha(p)-1/p'}(M)} \lesssim R^{\alpha(p)-1/p'} \|u\|_{L^p(M)}. \tag{4.179}$$

Thus we conclude that

$$\|T^{II}u\|_{L^{p'}(M)} \lesssim R^{\alpha(p)-1/p'} \|b^{II}\|_{L^p(I_0)} \|u\|_{L^{p'}(M)}, \tag{4.180}$$

which completes the proof. □

Combining Lemma 4.10 and Lemma 4.12 completes the proof of Proposition 4.4, and thus of inequality (4.2). Since (4.1) was already proven as a consequence of Lemma 4.2, this completes the proof of Theorem 4.1, and thus the main results of the paper.

## 4.5 Appendix

In this appendix, we provide proofs of Lemmas 4.3 and 4.6.

*Proof of Lemma 4.3.* The intervals $\{I_{t_0} : t_0 \in \mathcal{T}_R\}$ cover $[-\varepsilon, \varepsilon]$, and so we may consider an associated partition $\mathbb{I}_{[-\varepsilon,\varepsilon]} = \sum_{t_0} \chi_{t_0}$ where $\mathrm{supp}(\chi_{t_0}) \subset I_{t_0}$ and $|\chi_{t_0}| \leq 1$. Define $b^I_{t_0} = \chi_{t_0} b_j$ and $b^{II} = (1 - \mathbb{I}_{[-\varepsilon,\varepsilon]})b$. Then $b = \sum_{t_0} b^I_{t_0} + b^{II}$, and the support assumptions are satisfied. It remains to prove the required norm bounds for these choices. For each $n \in \mathbb{Z}$, define a function $b_n : I_0 \to \mathbb{C}$ by setting $b_n(t) = R\widehat{m}(R(t+n))$. Then $b = \sum_n b_n$. Moreover,

$$\begin{aligned}
&\left(\sum_{n\neq 0} \left[\langle Rn\rangle^{\alpha(p)} \|b_n\|_{L^p(I_0)}\right]^p\right)^{1/p} \\
&\qquad \sim \left(\int_{|t|\geq 1/2} \left[\langle Rt\rangle^{\alpha(p)} |R\widehat{m}(Rt)|\right]^p\right)^{1/p} \\
&\qquad = R^{1/p'} \left(\int_{|t|\geq R/2} \left[|t|^{\alpha(p)} \widehat{m}(t)\right]^p\right)^{1/p} \leq R^{1/p'} C_p(m).
\end{aligned} \tag{4.181}$$

Write $b^I_{t_0} = \sum_n b^I_{t_0,n}$ and $b^{II} = \sum_n b^{II}_n$, where $b^I_{t_0,n} = \chi_{t_0} b_n$ and $b^{II}_n = \mathbb{I}_{I_0 \setminus [-\varepsilon,\varepsilon]} b_n$. Then

$$\begin{aligned}
\|b^{II}_0\|_{L^p(I_0)} &= \left(\int_{\varepsilon\leq|t|\leq 1/2} |R\widehat{m}(Rt)|^p\right)^{1/p} \\
&= R^{1/p'} \left(\int_{R\varepsilon\leq|t|\leq R/2} |\widehat{m}(t)|^p\right)^{1/p} \lesssim R^{1/p'-\alpha(p)} C_p(m).
\end{aligned} \tag{4.182}$$

Using (4.181), (4.182), and Hölder's inequality, we conclude that

$$\begin{aligned}
\|b^{II}\|_{L^p(I_0)} &\leq \sum\nolimits_n \|b^{II}_n\|_{L^p(I_0)} \\
&\leq \|b^{II}_0\|_{L^p(I_0)} + \sum\nolimits_{n\neq 0} \left[|Rn|^{\alpha(p)} \|b^{II}_n\|_{L^p(I_0)}\right] \frac{1}{|Rn|^{\alpha(p)}} \\
&\leq \|b^{II}_0\|_{L^p(I_0)} + R^{-\alpha(p)} \left(\sum\nolimits_{n\neq 0} \left[|Rn|^{\alpha(p)} \|b_n\|_{L^p(I_0)}\right]^p\right)^{1/p} \\
&\lesssim R^{1/p'-\alpha(p)} C_p(m).
\end{aligned} \tag{4.183}$$

A similar calculation shows that

$$\begin{aligned}
\|b^I_{t_0}\|_{L^p(I_0)} &\leq \sum\nolimits_n \|b^I_{t_0,n}\|_{L^p(I_0)} \\
&= \|b^I_{t_0,0}\|_{L^p(I_0)} + \sum\nolimits_{n\neq 0} \|b^I_{t_0,n}\|_{L^p(I_0)} \\
&\lesssim \|b^I_{t_0,0}\|_{L^p(I_0)} + R^{-\alpha(p)} \Big(\sum\nolimits_{n\neq 0} |Rn|^{\alpha(p)} \|b^I_{t_0,n}\|^p_{L^p(I_0)}\Big)^{1/p} \\
&\lesssim \|b^I_{t_0,0}\|_{L^p(I_0)} + \Big(\sum\nolimits_{n\neq 0} |Rn|^{\alpha(p)} \|b^I_{t_0,n}\|^p_{L^p(I_0)}\Big)^{1/p}.
\end{aligned} \tag{4.184}$$

Using (4.184), we calculate that

$$\begin{aligned}
&\left( \sum_{t_0 \in \mathcal{T}_R} \left[ \|b^I_{t_0}\|_{L^p(I_0)} \langle R t_0 \rangle^{\alpha(p)} \right]^p \right)^{1/p} \\
&\quad \lesssim \left( \sum_{t_0 \in \mathcal{T}_R} \left[ \|b^I_{t_0,0}\|_{L^p(I_0)} \langle R t_0 \rangle^{\alpha(p)} \right]^p + \sum_{n \neq 0} \left[ |Rn|^{\alpha(p)} \|b^I_{t_0,n}\|_{L^p(I_0)} \right]^p \right)^{1/p} \\
&\quad \lesssim \left( \int_{\mathbb{R}} \left[ \langle Rt \rangle^{\alpha(p)} R \widehat{m}(Rt) \right]^p \, dt \right)^{1/p} \\
&\quad \lesssim R^{1/p'} C_p(m).
\end{aligned} \tag{4.185}$$

Since each function $b^I_{t_0}$ is supported on a length $1/R$ interval, we have

$$\|b^I_{t_0}\|_{L^1(I_0)} \lesssim R^{-1/p'} \|b^I_{t_0}\|_{L^p(I_0)}, \tag{4.186}$$

and substituting this inequality into (4.185) completes the proof. □

*Proof of Lemma 4.6.* For each $\alpha$, given our choice of $\varepsilon_M$, the Lax-Hörmander Parametrix construction (see Theorem 4.1.2 of [21]) guarantees that we can find operators $\tilde{W}_\alpha(t)$ and $\tilde{R}_\alpha(t)$ for $|t| \leq \varepsilon_M$, such that for $u \in L^1(M)$ with $\text{supp}(u) \subset V^*_\alpha$,

$$e^{2\pi i t P} u = \tilde{W}_\alpha(t) u + \tilde{R}_\alpha(t) u, \tag{4.187}$$

where $\tilde{R}_\alpha(t)$ has a smooth kernel, and the kernel of $\tilde{W}_\alpha(t)$ is given in coordinates by

$$\tilde{W}_\alpha(t)(x,y) = \int s_0(t,x,y,\xi) e^{2\pi i [\phi(x,y,\xi) + t p(y,\xi)]} \, d\xi, \tag{4.188}$$

for an order zero symbol $s_0$ with

$$\text{supp}_{x,y,\xi}(s_0) \subset \{(x,y,\xi) \in U_\alpha \times V^*_\alpha \times \mathbb{R}^d : d_M(x,y) \leq 1.01\varepsilon_M \text{ and } |\xi| \leq 1\}, \tag{4.189}$$

and an order one symbol $\phi$, homogeneous in $\xi$ of order one, solving the Eikonal equation and vanishing for $x \in \Sigma_\alpha(y,\xi)$ as required by the lemma. The only difference here compared to Theorem 4.1.2 of [21] is that in that construction the function $\phi$ there is chosen to vanish for $x \in \tilde{\Sigma}_\alpha(y,\xi)$, where $\tilde{\Sigma}_\alpha(y,\xi) = \{x : \xi \cdot (x-y) = 0\}$. The only property of this choice that is used in the proof is that the perpendicular vector to $\tilde{\Sigma}_\alpha(y,\xi)$ at $y$ is $\xi$, and this is also true of the hypersurfaces $\Sigma_\alpha(y,\xi)$ that we have specified, so that there is no problem making this modification.

In the remainder of the proof it will be convenient to fix a orthonormal basis $\{e_k\}$ of eigenfunctions for $P$, such that $\Delta e_k = \lambda_k e_k$ for a non-decreasing sequence $\{\lambda_k\}$. We fix $u \in L^1(M)$ with $\text{supp}(u) \subset V^*_\alpha$ and $\|u\|_{L^1(M)} \leq 1$.

We begin by mollifying the functions $Q$. We proceed here with a similar approach to Theorem 4.3.1 of [21]. We fix $\rho \in C_c^\infty(\mathbb{R})$ equal to one in a neighborhood of the origin and with $\rho(t) = 0$ for $|t| \geq \varepsilon_M/2$. We write

$$\begin{aligned} Q &= \int R\widehat{q}(Rt)e^{2\pi itP}\,dt \\ &= \int R\widehat{q}(Rt)\Big\{\rho(t)\tilde{W}(t) + \rho(t)\tilde{R}(t) + (1-\rho(t))e^{2\pi itP}\Big\}\,dt \\ &= Q_I + Q_{II} + Q_{III}. \end{aligned} \tag{4.190}$$

The rapid decay of $\widehat{q}$ implies that the function $\psi(t) = R\widehat{q}(Rt)(1-\rho(t))$ satisfies $\|\partial_t^N\psi\|_{L^1(\mathbb{R})} \lesssim_M R^{-M}$, and so

$$|\widehat{\psi}(\lambda)| \lesssim_{N,M} R^{-M}\lambda^{-N}. \tag{4.191}$$

But since $Q_{III} = \widehat{\psi}(-P)$, we can write the kernel of $Q_{III}$ as

$$Q_{III}(x,y) = \sum\nolimits_\lambda \widehat{\psi}(-\lambda_k)e_k(x)\overline{e_k(y)}, \tag{4.192}$$

Sobolev embedding and (4.191) imply that $|Q_{III}(x,y)| \lesssim_N R^{-N}$ and thus

$$\|Q_{III}u\|_{L^\infty(M)} \lesssim_N R^{-N}. \tag{4.193}$$

Integration by parts, using the fact that $q$ vanishes near the origin, yields that

$$\left|\int R\widehat{q}(Rt)\rho(t)\tilde{R}(t,x,y)\right| \lesssim_N R^{-N}, \tag{4.194}$$

and thus

$$\|Q_{II}u\|_{L^\infty(M)} \lesssim_N R^{-N}. \tag{4.195}$$

Now we expand

$$Q_I = \iint R\widehat{q}(Rt)\rho(t)s_0(t,x,y,\xi)e^{2\pi i[\phi(x,y,\xi)+tp(y,\xi)]}\,d\xi\,dt. \tag{4.196}$$

We perform a Fourier series expansion, writing

$$c_n(x,y,\xi) = \int \rho(t)s_0(t,x,y,\xi)e^{-2\pi int}\,dt. \tag{4.197}$$

Then the symbol estimates for $s_0$, and the compact support of $\rho$ imply that

$$|\partial_{x,y}^\alpha\partial_\xi^\beta c_n(x,y,\xi)| \lesssim_{\alpha,\beta,N} |n|^{-N}\langle\xi\rangle^{-\beta}. \tag{4.198}$$

Using Fourier inversion we can write

$$\begin{aligned} Q_I(x,y) &= \iint \sum\nolimits_n R\widehat{q}(Rt)c_n(x,y,\xi)e^{2\pi i[\phi(x,y,\xi)+t[n+p(y,\xi)]]}\,d\xi\,dt \\ &= \int \sum\nolimits_n q\Big((n+p(y,\xi))/R\Big)c_n(x,y,\xi)e^{2\pi i\phi(x,y,\xi)}\,d\xi \\ &= \int \tilde{\sigma}_\alpha(x,y,\xi)e^{2\pi i\phi(x,y,\xi)}\,d\xi, \end{aligned} \tag{4.199}$$

where

$$\tilde{\sigma}_\alpha(x,y,\xi) = \sum_{n\in\mathbb{Z}} q\left(\frac{n+p(y,\xi)}{R}\right) c_n(x,y,\xi). \tag{4.200}$$

The $n$th term of this sum is supported on $R/4 - n \leq p(y,\xi) \leq 4R - n$, so in particular, if $n > 4R$ then the term vanishes. For $n \leq 4R$, we have estimates of the form

$$\left|\partial_{x,y}^\alpha \partial_\xi^\beta \left\{ q\left(\frac{n+p(y,\xi)}{R}\right) c_n(x,y,\xi)\right\}\right| \lesssim_{\alpha,\beta,N} |n|^{-N}. \tag{4.201}$$

and for $-4R \leq n \leq R/8$,

$$\left|\partial_{x,y}^\alpha \partial_\xi^\beta \left\{ q\left(\frac{n+p(y,\xi)}{R}\right) c_n(x,y,\xi)\right\}\right| \lesssim_{\alpha,\beta,N} |n|^{-N} R^{-\beta}. \tag{4.202}$$

But this means that if we define

$$\sigma_\alpha(x,y,\xi) = \sum\nolimits_{-4R\leq n\leq R/8} q\left(\frac{n+p(y,\xi)}{R}\right) c_n(x,y,\xi). \tag{4.203}$$

and define

$$Q_\alpha(x,y) = \int \sigma_\alpha(x,y,\xi) e^{2\pi i\phi(x,y,\xi)}\, d\xi \tag{4.204}$$

then

$$\left|\partial_{x,y}^\alpha \partial_\xi^\beta \{\tilde{\sigma}_\alpha - \sigma\}(x,y,\xi)\right| \lesssim_{\alpha,\beta,N,M} R^{-N}\langle\xi\rangle^{-M}, \tag{4.205}$$

and so

$$\|(Q_I - Q_\alpha)u\|_{L^\infty(M)} \lesssim_N R^{-N}. \tag{4.206}$$

Combining (4.193), (4.195), and (4.206), we conclude that

$$\|(Q - Q_\alpha)u\|_{L^\infty(M)} \lesssim_N R^{-N}. \tag{4.207}$$

Since $\sigma_\alpha$ is supported on $|\xi| \sim R$, we have verified the required properties of $Q_\alpha$.

Using the bounds on $Q - Q_\alpha$ obtained above, we see that

$$\left\|[(Q\circ e^{2\pi itP}\circ Q) - (Q_\alpha \circ e^{2\pi itP}\circ Q_\alpha)]\{u\}\right\|_{L^\infty(M)} \lesssim_N R^{-N}. \tag{4.208}$$

Since $\tilde{R}_\alpha(t)$ has a smooth kernel, we also see that

$$\begin{aligned}&\left\|(Q_\alpha\circ(e^{2\pi itP} - \tilde{W}_\alpha(t))\circ Q_\alpha)\{u\}\right\|_{L^\infty(M)}\\ &\quad = \left\|(Q_\alpha \circ \tilde{R}_\alpha(t)\circ Q_\alpha)\{u\}\right\|_{L^\infty(M)} \lesssim_N R^{-N}.\end{aligned} \tag{4.209}$$

We now write

$$\begin{aligned}&(Q_\alpha \circ \tilde{W}_\alpha(T))(x,y)\\ &\quad = \int \sigma_\alpha(x,\xi) s_0(t,z,y,\eta) e^{2\pi i[\phi(x,z,\xi)+\phi(z,y,\eta)+tp(y,\eta)]}\, d\xi\, d\eta\, dz.\end{aligned} \tag{4.210}$$

The phase of this equation has gradient in the $z$ variable with magnitude $\gtrsim R$ for $|\eta| \ll R$, and $\gtrsim R|\xi|$ if $|\eta| \gg R$. Thus, if we define $s(t,x,y,\xi) = s_0(t,x,y,\xi)\chi(\xi/R)$ where $\mathrm{supp}(\chi) \subset [1/8, 8]$, and then define

$$W_\alpha(t)(x,y) = \int s(t,x,y,\xi) e^{2\pi i[\phi(x,y,\xi)+tp(y,\xi)]}\, d\xi, \tag{4.211}$$

then we may integrate by parts in the $z$ variable to conclude that

$$\left|(Q_R \circ (\tilde{W}_\alpha(t) - W_\alpha(t)))(x,y)\right| \lesssim_N R^{-N}, \tag{4.212}$$

and thus

$$\|(Q_R \circ (\tilde{W}_\alpha(t) - W_\alpha(t)) \circ Q_R)u\|_{L^\infty(M)} \lesssim R^{-N}. \tag{4.213}$$

This proves the required estimates for the operators $W_\alpha$. □

# Chapter 5

# Combining Scales Via Atomic Decompositions

# Chapter 6

# Future Work

## 6.1 Exploiting Tangency Bounds in $\mathbb{R}^3$ and $\mathbb{R}^4$

The results of Heo, Nazarov, and Seeger only apply when $d \geq 4$. Cladek found a method to get an initial radial multiplier conjecture result in $\mathbb{R}^3$, and an improvmeent of the bounds obtained by Heo, Nazarov, and Seeger when $d = 3$. The idea is to exploit the fact that one need only prove a version of 2.5 for a set $\mathcal{E} = \mathcal{E}_X \times \mathcal{E}_R$, where $\mathcal{E}_X$ is a one-separated family of points, and $\mathcal{E}_R$ are a family of radii. One can then exploit this Cartesian product structure when analyzing functions of the form

$$F = \sum_{(x,r)\in\mathcal{E}} \chi_{x,r},$$

in particular, improving upon the result of [7].

### 6.1.1 Result in 3 Dimensions

As in [7], Cladek first performs a density decomposition, i.e. writing

$$F = \sum F_k^m$$

where

$$F_k^m = \sum_{(x,r)\in\mathcal{E}_k(2^m)} \chi_{x,r}.$$

Cladek then interpolates between an $L^0$ bound and an $L^2$ bound on the resulting functions. The $L^0$ bound is exactly the same bound used in [7].

**Theorem 6.1.** *For the function $F$, we have*

$$|supp(F_k^m)| \lesssim 2^{-m} 4^k \#\mathcal{E}_k$$

*and thus*

$$|supp(F^m)| \lesssim \sum_k 2^{-m} 4^k \#\mathcal{E}_k.$$

The $L^2$ bound is improved upon, which is what allows us to obtain a new result in three dimensions.

**Lemma 6.2.** *Suppose $\mathcal{E} = \bigcup_k \mathcal{E}_k$ is a one-separated set, where $\mathcal{E}_k \subset \mathbb{R}^d \times [2^k, 2^{k+1})$ is a set of density type $(2^m, 2^k)$. Then*

$$\left\| \sum_{(x,r)\in\mathcal{E}} \chi_{x,r} \right\|_{L^2(\mathbb{R}^d)} \lesssim_\varepsilon 2^{[(11/13)+\varepsilon]m} \sum_k 4^k \# \mathcal{E}_k.$$

Interpolation thus yields that for a set of density type $2^m$ as in this Lemma,

$$\| \sum_{(x,r)\in\mathcal{E}} \chi_{x,r} \|_{L^p(\mathbb{R}^d)} \lesssim_\varepsilon 2^{-m(1/p-12/13-\varepsilon)} (\sum_k 4^k \# \mathcal{E}_k)^{1/p}.$$

If $1 < p < 13/12$, this sum is favorable in $m$, and may be summed without harm to prove the radial multiplier conjecture for unit scale radial multipliers in this range.

*Proof of Lemma 6.2.* Write

$$F_k = \sum_{(x,r)\in\mathcal{E}_k} \chi_{x,r}.$$

As before, we can throw away terms for $k \leq 10m$, i.e. obtaining that

$$\| \sum F_k \|_{L^2(\mathbb{R}^d)} \lesssim m^{1/2} \left( \sum_k \|F_k\|_{L^2(\mathbb{R}^d)}^2 + \sum_{10m<k<k'} |\langle F_k, F_{k'} \rangle| \right)^{1/2}.$$

Our proof thus splits into two cases: where the radii are incomparable, and where the radii are comparable.

TODO: □

### 6.1.2 Results in 4 Dimensions

TODO

## 6.2 Manifolds With Periodic Geodesic Flow

## 6.3 Abstract Reductions In Manifolds With Constant Sectional Curvature

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