Multipliers of Elliptic Operators on Compact Manifolds

Necessary and Sufficient Conditions For Boundedness

by

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Abstract

For any bounded, regulated function $a : [0, \infty) \to \mathbb{C}$, consider the family of operators $\{T_R\}$ on the sphere S^d such that $T_R f = a(k/R) f$ for any spherical harmonic f of degree k. This thesis discusses the work of the author on characterizing the functions a for which the operators $\{T_R\}$ are uniformly bounded on $L^p(S^d)$ in the range $1/(d-1) < |1/p - 1/2| < 1/2$. More generally, we obtain analogous results in the more general setting of multiplier operators for eigenfunction expansions of an elliptic pseudodifferential operator P on a compact manifold X, under curvature assumptions on the principal symbol of P, and assuming the eigenvalues of P are contained in an arithmetic progression, assuming that the function a defining the eigenfunction expansion is compactly supported.

One consequence of our result are new transference principles controlling the L^p boundedness of the multiplier operators associated with a function a, in terms of the L^p operator norm of the radial Fourier multiplier operator on $\mathbb{R}^d$ with symbol $a(|\cdot|) : \mathbb{R}^d \to \mathbb{C}$, the first nontrivial result of it's kind. The latter results have been published in [4], and the former results have yet to be described elsewhere. In order to prove these results, we obtain new quasi-orthogonality estimates for averages of solutions to the half-wave equation $\partial_t - iP = 0$, via a connection between pseudodifferential operators satisfying an appropriate curvature condition and Finsler geometry, as well as utilizing atomic decompositions on manifolds.

In Part I of the thesis, we review prior work on endpoint bounds relevant to the new results stated, in particular, endpoint bounds for radial and quasi-radial Fourier multiplier operators on Euclidean space, as well as Finsler geometry and Fourier integral operators on compact manifolds. In Part II, we prove the results stated above.

Contents

Notation

- We use the normalization of the Fourier transform given by the formula

$$\widehat{f}(\xi) = \int f(x) e^{-2\pi i \xi \cdot x}\, dx.$$

- We use the translation operators $\mathrm{Trans}_y f(x) = f(x-y)$, and L^∞-normalized dilations $\mathrm{Dil}_t f(x) = f(x/t)$. Overloading notation, we also consider dyadic dilations of the form $\mathrm{Dil}_j f(x) = f(x/2^j)$. The dyadic dilation operator will only be used along with symbols that stand for integers, like n, m, j, or k, whereas the other dilation operator will be used in all other cases, so the operator used should be clear from context.

- On $\mathbb{R}^d$, we use ∂_j to denote the usual partial derivative operators, and D_j to denote the self-adjoint normalization $D_j f = (2\pi i)^{-1}\partial_j$. This notation has the convenience that for a polynomial P,

$$P(D_j)\{f\} = \int P(\xi)\widehat{f}(\xi) e^{2\pi i \xi \cdot x}\, d\xi,$$

  and simplifies many formulas associated with Fourier integral operators.

- We will often use the Japanese bracket $\langle x \rangle := (1 + |x|^2)^{1/2}$ for $x \in \mathbb{R}^d$.

- A function $f : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}$ is a *symbol of order s* if it satisfies bounds of the form

$$|\partial_x^\alpha \partial_\theta^\beta f(x,\theta)| \lesssim_{\alpha,\beta} \langle \theta \rangle^{s-|\beta|}$$

  for all multi-indices α and β.

- For a measure space X, $L^\infty(X)$ is the Banach space of essentially-bounded functions, defined almost everywhere. For a set X, $l^\infty(X)$ is the Banach space of bounded functions on X, defined everywhere.

- All operators we consider are Schwartz operators between manifolds equipped with a canonical measure. Abusing notation, we thus identify an operator with the distribution that gives it's Schwartz kernel. Thus, for a Schwartz operator A from a manifold Y to a manifold X, we might write

$$Af(x) = \int_Y A(x,y) f(y)\, dy.$$

- For $1 \leq p \leq \infty$ and $s \in \mathbb{R}$, we consider the Sobolev spaces

$$\|f\|_{W^{s,p}(\mathbb{R}^d)} = \|(-\Delta)^{s/2} f\|_{L^p(\mathbb{R}^d)},$$

and for $1 \leq r \leq \infty$, we consider the Besov spaces

$$\|f\|_{B^{s,p}_r(\mathbb{R}^d)} = \|P_k f\|_{l^r_k L^p(\mathbb{R}^d)},$$

where $\{P_k\}$ are Littlewood-Paley projections.

- For $1/p + 1/p' = 1$, we can identify $f \in L^{p'}(X)$ as an element of $(L^p(X))^*$ either via the pairing $\langle f, g\rangle = \int f(x)g(x)$, or via the pairing $\langle f, g\rangle = \int \overline{f(x)}g(x)\, dx$. Given an operator A from $L^p(Y)$ to $L^q(X)$, we can take the dual to obtain an operator from $L^q(X)^*$ to $L^p(Y)^*$, and the two pairings above give two possibly different linear maps from $L^{q'}(X)$ to $L^{p'}(X)$. We denote the first by A^t, and the second by A^*. As examples, for a Fourier multiplier on $\mathbb{R}^d$ given by the integral expression

$$Af(x) = \int m(\xi)\widehat{f}(\xi)e^{2\pi i\xi\cdot x}\, d\xi,$$

we have

$$A^t f(x) = \int m(-\xi)\ \widehat{f}(x)e^{2\pi i\xi\cdot x}\, d\xi \quad \text{and} \quad A^* f(x) = \int \overline{m(\xi)}\ \widehat{f}(x)e^{2\pi i\xi\cdot x}\, d\xi.$$

Introduction

Let P be an elliptic linear operator on a compact manifold M, such that we can associate a functional calculus $a \mapsto a(P)$ for functions a on the real line. In this thesis, we study the following question:

What conditions on a guarantee an operator $a(P)$ to be bounded.

Aside from testing our ability to understand the operator P and the relations of it's eigenfunctions, the question has applications in the study of various partial differential equations associated with the operator P, and is closely related to the geometry of M, in particular, the way waves propogate and interact on the manifold.

Moreover, the study of multipliers on a manifold provides a useful setting with which to test and develop methods of harmonic analysis in a 'variable-coefficient setting', where a lack of symmetry and translation invariance forces us to introduce more robust methods than are required in the Euclidean setting, i.e. as compared to studying multipliers of the Laplace operator on $\mathbb{R}^d$ using the Fourier transform.

At present, there are many obstacles related to the global geometry of manifolds, which prevent an understanding of spectral multiplier operators on a general manifold. Over the years, sufficient conditions for boundedness had been established, but no characterizations of boundedness were known on any compact manifold, aside from the translation-invariant case when studying the Laplace operator on the torus. This thesis describes the first such characterizations beyond this setting, for a limited range of Lebesgue spaces, and on manifolds all of whose geodesics are closed and have common length. In particular, we find characterizations of functions a whose dilates induce uniformly bounded multiplier operators for spherical harmonic expansions on S^d. We obtain these results by expanding on techniques for understanding Fourier integral operators on manifolds.

In Chapter 1, we describe the setup to problem we are to discuss in more detail. In Chapter 2, we discuss results known for radial and quasi-radial multipliers which are analogoue to the endpoint bounds we will establish for spectral multipliers on manifolds. In Chapter 3, we return to the study of spectral multipliers on manifolds, introducing the background on Finsler geometry and Fourier integral operators we will need in the proofs to come. In Chapter 4, we discuss our new results for spectral multipliers with a compactly supported symbol, and in Chapter 5, we discuss new results about combining different frequency scales to bound more general spectral multiplier operators.

Part I

Background

Chapter 1

Multipliers of an Elliptic Operator

Let us now more precisely describe the problem of this thesis. Let X be a compact manifold equipped with a volume density, and let P be a classical elliptic operator on X of order $s > 0$, formally positive-definite in the sense that $\langle Pf, g\rangle = \langle f, Pg\rangle$ and $\langle Pf, f\rangle \geq 0$ hold for all $f, g \in C^\infty(X)$. By general properties of elliptic operators, $1 + P$ is an isomorphism between the Sobolev space $H^s(X)$ and $L^2(X)$, and so the inverse $(1 + P)^{-1}$, viewed as a map from $L^2(X)$ to itself, is compact by a form of the Rellich-Kondrachov embedding theorem. By the spectral theorem for compact operators, there exists a discrete set $\Lambda \subset [0, \infty)$, and a decomposition

$$L^2(X) = \bigoplus_{\lambda\in\Lambda} \mathcal{V}_\lambda, \tag{1.0.1}$$

where $\mathcal{V}_\lambda$ is a finite dimensional subspace of $C^\infty(X)$, such that $Pf = \lambda f$ for all $f \in \mathcal{V}_\lambda$. We use this decomposition to define a functional calculus for the operator P; given a bounded function $a : \Lambda \to \mathbb{C}$, we can define an operator $a(P)$ on $L^2(X)$ so that $a(P)f = a(\lambda)f$ for all $f \in \mathcal{V}_\lambda$. Our goal is to study the regularity of such operators, in terms of properties of the function a.

In the sequel, we will assume the operator P is an elliptic operator *of order one*. This does not restrict the scope of our analysis; given a formally positive elliptic operator P of order s, the operator $P^{1/s}$ defined by the functional calculus above is a formally positive elliptic operator of order one*, and the spectral theory of P is identical with the spectral theory of $P^{1/s}$. Fixing the order of our operator reflects the fact that the theory of Fourier integral operators we will eventually employ is most elegant when the resulting oscillatory integrals have homogeneous phases of order one. To prevent being overly wordy, in all that follows by an 'elliptic operator' we will mean a classical elliptic operator of order one which is formally positive-definite in the sense above.

The primary example of elliptic operators for our purposes are obtained from a Laplace-Beltrami operator on a compact Riemannian manifold M; the operator $-\Delta$ is self-adjoint and positive-definite with respect to the volume form dV on M, because of the 'integration

*See Theorem 3.3.1 of [20] for a proof of this fact, based on a technique of [19] initially developed for elliptic differential operators, but which can be straightforwardly adapted to pseudodifferential operators.

by parts' identity $\langle \Delta f, g\rangle = -\langle \nabla f, \nabla g\rangle$. Since we restrict our attention to elliptic operators of order one, the archetypical object of study is the operator $P = \sqrt{-\Delta}$.

To study the regularity of $a(P)$, we introduce the spaces $M^p(X)$, consisting of all functions $a : \Lambda \to \mathbb{C}$ for which the operator $a(P)$ is bounded on $L^p(X)$. The space $M^p(X)$ is then equipped with a norm

$$\|a\|_{M^p(X)} = \sup\left\{ \frac{\|a(P)f\|_{L^p(X)}}{\|f\|_{L^p(X)}} : f \in C^\infty(X) \right\}. \tag{1.0.2}$$

$M^2(X)$ is isometrically equal to $l^\infty(\Lambda)$, as the calculations below in Lemma 1.1 show, but the structure of the other spaces $M^p(X)$ is unclear.

The discrete spectrum of the operator P makes it difficult to apply methods of oscillatory integrals to the problem. Fortunately, certain semiclassical heuristics tell us that these problems disappear when we restrict the problem to 'high frequency inputs'. In order to take advantage of this fact, rather than studying what conditions ensure the operators $a(P)$ are bounded, we study conditions that ensure the operators $a_R(P)$ are uniformly bounded, for functions $a : [0,\infty) \to \mathbb{C}$ and $a_R(\lambda) = a(\lambda/R)$. To prevent pathological examples from arising, we restrict ourselves to *regulated functions* a, i.e. functions such that

$$a(\lambda_0) = \lim_{\delta \to 0} ⨍_{|\lambda-\lambda_0|\leq\delta} a(\lambda)\, d\lambda \quad \text{for all } \lambda_0 \in [0,\infty). \tag{1.0.3}$$

Define the set $M^p_{\mathrm{Dil}}(X)$ of all regulated functions a such that $\|a\|_{M^p_{\mathrm{Dil}}(X)} = \sup_R \|a_R\|_{M^p(X)}$ is finite. With notation introduced, we now state precisely the problem studied in this thesis:

Can one find simple conditions on a function a,
necessary and sufficient to be contained in $M^p_{Dil}(X)$ for $p \neq 2$.

For $p = 2$, orthogonality gives the isometric equivalence $M^2_{\mathrm{Dil}}(X) = L^\infty[0,\infty)$.

Lemma 1.1. *For any regulated a and any compact manifold X, $\|a\|_{M^2_{Dil}(X)} = \|a\|_{L^\infty[0,\infty)}$.*

Proof. Given $f \in L^2(X)$, if we write $f = \sum f_\lambda$ with $f_\lambda \in \mathcal{V}_\lambda$, then for any $R > 0$,

$$\|a_R(P)f\|_{L^2(X)} = \Big\| \sum\nolimits_\lambda a_R(\lambda) f_\lambda \Big\|_{L^2(X)} = \left(\sum\nolimits_\lambda |a_R(\lambda)|^2 \|f_\lambda\|^2_{L^2(X)} \right)^{1/2} \tag{1.0.4}$$

Conversely,

$$\|f\|_{L^2(X)} = \left(\sum\nolimits_\lambda \|f_\lambda\|^2_{L^2(X)} \right)^{1/2}. \tag{1.0.5}$$

Setting $\|f_\lambda\|_{L^2(X)} = c_\lambda$, the quantity $\|a_R\|_{M^2(X)}$ is thus the smallest constant such that

$$\left(\sum\nolimits_\lambda |a_R(\lambda)|^2 c_\lambda^2 \right)^{1/2} \leq \|a_R\|_{M^2(X)} \left(\sum\nolimits_\lambda c_\lambda^2 \right)^{1/2}. \tag{1.0.6}$$

holds for all sequences $\{c_\lambda\}$. It is then clear that $\|a_R\|_{M^2(X)} = \|a_R\|_{l^\infty(\Lambda)}$, and thus (now using the fact that a is regulated),

$$\|a\|_{M^2_{\mathrm{Dil}}(X)} = \sup_R \|a_R\|_{l^\infty(\Lambda)} = \|a\|_{L^\infty[0,\infty)}, \tag{1.0.7}$$

which completes the proof. □

Before the results of this thesis, no simple conditions had been proved necessary and sufficient for inclusion in $M^p_{\mathrm{Dil}}(X)$ for $p \neq 2$ and $X \neq \mathbb{T}^d$. The main result of this thesis is that under two assumptions on X and the operator P, which we call Assumption A and Assumption B, we can find and prove such a condition is necessary and sufficient. The first assumption is a curvature condition on the principal symbol of the P, and the second is an assumption about the operator's eigenvalues.

Assumption A: For each $x_0 \in M$, the cosphere

$$S_{x_0} = \{\xi \in T^*_{x_0}M : p(x_0, \xi) = 1\}$$

is a hypersurface in $T^*_x M$ with non-vanishing Gauss curvature.

Assumption B: The spectrum of the operator P is contained in an arithmetic progression.

When $P = \sqrt{-\Delta}$ on a Riemannian manifold M, Assumption A always holds, because the cospheres defined above are ellipsoids. To find an explicit example of an operator satisfying Assumption B as well, we look to $X = S^d$. On S^d, recall that the spherical harmonics are restrictions of harmonic, homogeneous polynomials on $\mathbb{R}^{d+1}$. Define the operator P_{SH} by setting $P_{\mathrm{SH}} f = kf$ for all degree k spherical harmonics of degree k. We can write $P_{\mathrm{SH}} = \alpha(\sqrt{-\Delta})$, whjere Δ is the Laplace-Beltrami for the usual Riemannian geometry on S^d, and $\alpha(\lambda) = \sqrt{\lambda^2 + d^2/4} - d/2$. Since α is a symbol of order 1 with order one term λ, P_{SH} is an elliptic operator of order one, and P_{SH} and $\sqrt{-\Delta}$ share the same principal symbol*. Thus P satisfies Assumption A, and the operator also satisfies Assumption B by definition.

More generally, we will see in Section 3.2 that if an operator P satisfies Assumption A then it gives X a geometry, and Assumption B is closely related to this geometry, in particular, that geodesics with respect to this geometry are all closed and have commensurable lengths. If P satisfies Assumption A, and the geometry on X is such that all geodesics of X are closed, and the lengths of these closed geodesics all divide a common period, then there exists† an elliptic operator $\tilde{P}$ on X satisfying both assumptions, and such that $P - \tilde{P}$ is a pseudodifferential operator of order -1, so we can view $\tilde{P}$ as a pertubation of P. We can study the regularity of a subfamily of multipliers of P by representing them as multipliers of $\tilde{P}$.

Under Assumption A and Assumption B, we find necessary and sufficient conditions for a function to be contained in $M^{p,q}_{\mathrm{Dil}}(M)$ for $1/p - 1/2 > 1/(d+1)$ and for $p \leq q < 2$. Fix $\chi \in C_c^\infty(\mathbb{R}^d)$. For $s \geq 0$ and $1 \leq p \leq \infty$, define the norm

$$\|a\|_{R^{s,p}[0,\infty)} = \sup_{R>0} \left(\int_0^\infty \left| \widehat{a_R}(t) \langle t \rangle^s \right|^p \, dt \right)^{1/p} \quad \text{where} \quad a_R(\lambda) = \chi(\lambda) a(R\lambda). \tag{1.0.8}$$

Here $\widehat{a_R}(t) = \int_0^\infty a_R(\lambda) \cos(2\pi\lambda t)$ is the cosine transform of a_R, i.e. the Fourier transform of the even extension of a to a function on $\mathbb{R}$. The main result of this thesis is that for

*See Theorem 4.3.1 of [20] for details.

†See Lemma 29.2.1 of [11], discussed in slightly more detail (but not proved) in Section 3.4.

$1/p - 1/2 > 1/(d-1)$, the elements of $M^p_{\text{Dil}}(X)$ compactly supported away from 0 are equal to the elements of $R^{s,p}[0,\infty)$ compactly supported away from 0 for appropriate s. This is the first such characterization for $p \neq 2$ and $X \neq \mathbb{T}^d$.

Theorem 1.2. *Suppose X is a manifold, and P is an elliptic operator on X satisfying Assumption A and Assumption B. Then for $1/p-1/2 > 1/(d-1)$, if $s = (d-1)(1/p-1/2)$, then for any regulated a with supp$(a) \subset [1/2, 2]$,*

$$\|a\|_{M^p_{Dil}(X)} \sim \|a\|_{R^{s,p}[0,\infty)}.$$

We also provide a robust method for combining frequency scales to obtain a full characterization of $M^p_{\text{Dil}}(X)$, under the assumption of certain additional estimates. We are only able to obtain such estimates when we have a greater understanding of the conjugate points of the manifold X. But we are able to use the method to fully characterize $M^p_{\text{Dil}}(S^d)$ in the range covered by Theorem 1.2.

Theorem 1.3. *Consider S^d, equipped with the operator P_{SH}. Then if $1/p-1/2 > 1/(d-1)$ and $s = (d-1)(1/p-1/2)$, $M^p_{Dil}(S^d) = R^{s,p}[0,\infty)$.*

One may view control on the $R^{s,p}$ norm as a smoothness condition on the multiplier a; indeed the Hausdorff-Young inequality implies a Sobolev space estimate

$$\|a_R\|_{W^{s,p'}[0,\infty)} \lesssim \|a\|_{R^{s,p}[0,\infty)}. \tag{1.0.9}$$

However, the space $R^{s,p}[0,\infty)$ is not equal to a Sobolev or Besov space, though we do have the Besov space estimate

$$\|a\|_{R^{s,p}[0,\infty)} \lesssim \sup_R \|a_R\|_{B_p^{s+1/p-1/2,2}[0,\infty)}. \tag{1.0.10}$$

The result is somewhat intuitive. Inequality (1.0.9) tells us that elements of $R^{s,p}$ have s derivatives in $L^{p'}$, though with 'some extra control' occuring on the other side of the Fourier transform. Sobolev embedding heuristics tell us that having $s + 1/p - 1/2$ derivatives in L^2 is sufficient to have s derivatives in $L^{p'}$. The presence of L^2 norms on the right hand side of (1.0.10) allows us to losslessly convert between estimates on the Fourier transform side, allowing us to recover the information on the Fourier side and obtain (1.0.10).

For most manifolds X and P, elements of $\mathcal{V}_\lambda$ are difficult to describe explicitly; even for the relatively simple case of the sphere S^d many questions about the geometric behaviour of eigenfunctions remain open. Many arguments in harmonic analysis involve an interplay between spatial and frequential control, and without explicit descriptions of eigenfunctions, spatial control becomes difficult. Nonetheless, we will find we can obtain some spatial control by utilizing the wave equation on M, which carries geometric information via the behaviour of wave propogation. Under the curvature assumptions we make, wave propogation has sufficient smoothing properties to match certain necessary conditions that multipliers need in order to be bounded. And Assumption B implies that the wave equation associated with the operator P is periodic, which simplifies the large time analysis of the wave equation.

In the next chapter, we will begin our study by describing what is currently known for the boundedness problem on $\mathbb{T}^d$, where eigenfunctions are explicit, and one can study the behaviour of spectral multipliers via the Fourier transform and radial convolution operators on $\mathbb{R}^d$. Our analysis here will help gain intuition and motivate potential hypotheses in the more general setting.

Chapter 2

Radial Multipliers on Euclidean Space

Consider the elliptic operator $P = \sqrt{-\Delta}$ on $\mathbb{T}^d$, where $\Delta = \partial_1^2 + \cdots + \partial_d^2$ is the usual Laplacian. In such a setting, we have an explicit basis for the eigenfunctions of Δ: for a given eigenvalue $\lambda > 0$, the space $\mathcal{V}_\lambda$ has an orthonormal basis consisting of the exponentials $e^{2\pi i n \cdot x}$, where $n \in \mathbb{Z}^d$ and $|n| = \lambda$. Since the Fourier series of a function gives the expansion of the function in this basis, it follows that we can expand the spectral multiplier operator $T = a(P)$ using a Fourier series, i.e. writing

$$Tf(x) = \sum_{n \in \mathbb{Z}^d} a(|n|) \widehat{f}(n) e^{2\pi i n \cdot x}. \tag{2.0.1}$$

Thus a multiplier of Δ on $\mathbb{T}^d$ is nothing more than a Fourier multiplier operator on $\mathbb{T}^d$ whose symbol is radial. Methods of transference* show that $\|a\|_{M^p(\mathbb{R}^d)} \sim \|a\|_{M^p_{\mathrm{Dil}}(\mathbb{T}^d)}$, where $\|a\|_{M^p(\mathbb{R}^d)}$ is the operator norm of the Fourier multiplier on $\mathbb{R}^d$ given by

$$Tf(x) = \int_{\mathbb{R}^d} a(|\xi|) \widehat{f}(\xi) e^{2\pi i \xi \cdot x}\, d\xi \tag{2.0.2}$$

on $L^p(\mathbb{R}^d)$. In this section we will focus on operators of this type, which we call *radial Fourier multiplier operators*.

The study of the regularity of Fourier multiplier operators has proved central to the development of modern harmonic analysis and the theory of linear partial differential operators. This is because essentially any translation invariant operator T on $\mathbb{R}^d$ is a Fourier multiplier operator, i.e. we can find a tempered distribution m on $\mathbb{R}^d$, the *symbol* of T, such that for any Schwartz function f,

$$Tf(x) = \int_{\mathbb{R}^d} m(\xi) \widehat{f}(\xi) e^{2\pi i \xi \cdot x}\, d\xi. \tag{2.0.3}$$

The study of translation invariant operators emerges from classical questions in analysis, such as the convergence of Fourier series, and problems in mathematical physics related to the study of the heat, wave, and Schrödinger equations. These physical equations also

*See Section 3.6.2 of Grafakos [6], based on methods of de Leeuw [15] often have *rotational* symmetry, so it is natural to restrict our attention to translation-invariant operators which are also rotation-invariant. These operators are precisely the family of radial multiplier operators. If $m(\xi) = a(|\xi|)$ for a tempered distribution on $[0,\infty)$ we will write $T = a(P)$, where $P = \sqrt{-\Delta}$, since $Tf = a(\lambda)f$ whenever $\Delta f = -\lambda^2 f$.

2.1 Convolution Kernels of Fourier Multipliers

It is often useful to study spatial representations of these operators, since one can often exploit certain geometric information about the behaviour of operators. Given any translation invariant operator T on $\mathbb{R}^d$, we can associate a tempered distribution k, the *convolution kernel* of T, such that

$$Tf(y) = \int_{\mathbb{R}^d} k(x) f(y-x)\, dx \qquad \text{for all } f \in \mathcal{S}(\mathbb{R}^d). \tag{2.1.1}$$

If T is radial, then so is k, and so we can write $k(x) = b(|x|)$ for some distribution b on $[0,\infty)$, and then we have a representation

$$T = \int_0^\infty b(r) S_r\, dr, \quad \text{where} \quad S_r f(x) = \int_{|y|=r} f(x+y)\, dy \tag{2.1.2}$$

are the *spherical averaging operators*.

With the notation as above, the function k is the Fourier transform of m, and the function b is a Bessel transform of a, i.e. $b = \mathcal{B}_d a$, where

$$\mathcal{B}_d a(r) = \int_0^\infty s^{\frac{d-1}{2}} J_{\frac{d-1}{2}}(2\pi s) a(s)\, ds, \tag{2.1.3}$$

and where

$$J_\alpha(\lambda) = \frac{(\lambda/2)^\alpha}{\Gamma(\alpha+1/2)} \int_{-1}^1 e^{i\lambda s}(1-s^2)^{\alpha-1/2}\, ds. \tag{2.1.4}$$

Using the theory of stationary phase, for each d we can write

$$J_d(\lambda) = e^{2\pi i\lambda} s_1(\lambda) + e^{-2\pi i\lambda} s_2(\lambda). \tag{2.1.5}$$

for symbols s_1 and s_2 of order $-1/2$. The presence of $e^{2\pi i\lambda}$ and $e^{-2\pi i\lambda}$ allows one to relate the Bessel transform of a function to it's Fourier transform, to a certain extent. In particular, we record the following result of Garrigos and Seeger [5].

Theorem 2.1. *Suppose $d > 1$ and $1/2d < 1/p - 1/2 < 1/2$, and suppose $a : [0,\infty) \to \mathbb{C}$ has compact support away from the origin. Then, with implicit constants depending on the support of a,*

$$\left(\int_0^\infty |\mathcal{B}_d a(t)|^p t^{d-1} \right)^{1/p} \sim_{p,d,\phi} \left(\int_0^\infty |\widehat{a}(t)|^p \langle t \rangle^{(d-1)(1-p/2)}\, dt \right)^{1/p},$$

where $\widehat{a}(t) = \int_0^\infty a(\lambda)\cos(2\pi\lambda t)\, dt$ is the cosine transform of a.

2.2 The Radial Multiplier Conjecture

The general study of the boundedness properties of Fourier multiplier operators in multiple variables was initiated in the 1950s, as connections of the theory to partial differential equations became more fully realized*. It was quickly realized that the most fundamental estimates were $L^p \to L^q$ estimates for such operators. It is therefore natural to introduce the space $M^p(\mathbb{R}^d)$, consisting of all symbols m which induce a Fourier multiplier operator T bounded on $L^p(\mathbb{R}^d)$. Duality implies that $M^p(\mathbb{R}^d)$ is isometric to $M^{p'}(\mathbb{R}^d)$, where p and p' are conjugates, so it suffices to study the spaces $M^p(\mathbb{R}^d)$ where $1 \leq p \leq 2$ or when $2 \leq p \leq \infty$. We only know simple characterizations of $M^p(\mathbb{R}^d)$ for very particular p:

- The spaces $M^1(\mathbb{R}^d) = M^\infty(\mathbb{R}^d)$ can be characterized, by virtue of the fact that the boundedness of operators with domain $L^1(\mathbb{R}^d)$ or range $L^\infty(\mathbb{R}^d)$ is often simple; we have $M^1(\mathbb{R}^d) = \widehat{M}(\mathbb{R}^d)$, where $M(\mathbb{R}^d)$ is the space of all finite signed Borel measures, equipped with the total variation norm. The proof follows from Schur's Lemma for integral operators, which often gives tight estimates to bound operators with domain L^1 or range L^∞.

- The unitary nature of the Fourier transform also allows for the characterization $M^2(\mathbb{R}^d) = L^\infty(\mathbb{R}^d)$. The proof follows from Parseval's identity.

It is perhaps surprising that these are the *only* known characterizations of the spaces $M^p(\mathbb{R}^d)$. No necessary and simple conditions for boundedness are known for any other values of p, and perhaps no simple characterization exists.

Despite the lack of a characterization of the classes $M^p(\mathbb{R}^d)$, it is surprising that we *can* conjecture a characterization of the subspace of $M^p(\mathbb{R}^d)$ consisting of *radial symbols*, for an appropriate range of exponents. The conjectured range of estimates was first suggested by a result of [5], concerning the boundedness of a radial Fourier multiplier T with symbol $a(|\cdot|)$ *restricted to radial functions*, i.e. such that the norm

$$\|a\|_{M^p_{\text{Rad}}(\mathbb{R}^d)} = \sup\left\{ \frac{\|a(P)f\|_{L^q(\mathbb{R}^d)}}{\|f\|_{L^p(\mathbb{R}^d)}} : f \text{ is radial} \right\} \tag{2.2.1}$$

is finite. For any $\chi \in C_c^\infty(\mathbb{R})$, if we define k_R to be the Fourier transform of $\chi(|\cdot|)a(R|\cdot|)$, then the identity $k_R = a(RP)\{\widehat{\chi}\}$ and dilation symmetry imply that

$$\|k_R\|_{L^p(\mathbb{R}^d)} \lesssim \|a\|_{M^p_{\text{Rad}}(\mathbb{R}^d)}, \tag{2.2.2}$$

with implicit constants depending on χ. For $d > 1$, and for $1/p - 1/2 > 1/2d$, Garrigos and Seeger proved [5] the converse bound

$$\|a\|_{M^p_{\text{Rad}}(\mathbb{R}^d)} \lesssim \sup_{R>0} \|k_R\|_{L^p(\mathbb{R}^d)}, \tag{2.2.3}$$

Theorem 2.1 implies that

$$\sup_{R>0} \|k_R\|_{L^p(\mathbb{R}^d)} \sim \|a\|_{R^{s,p}[0,\infty)}, \tag{2.2.4}$$

*See [10] for a more detailed overview of what was known at this time.

where $s = (d-1)(1/p-1/2)$. Thus Garrigos and Seeger have proved that the isomorphism $M^p_{\mathrm{Rad}}(\mathbb{R}^d) = R^{s,p}[0,\infty)$ holds in the range above.

For any other value of p, $M^p_{\mathrm{Rad}}(\mathbb{R}^d)$ is a proper subset of $R^{s,p}[0,\infty)$, as the following counterexamples show:

- For $p = 1$, the inclusion 'fails by a logarithm', as we can see by the characterization of $M^(\mathbb{R}^d)$ in the last section; if a is supported on an interval I we have the converse inequality $\|a\|_{M^1(\mathbb{R}^d)} \lesssim \log |I| \; \|a\|_{R^{s,p}[0,\infty)}$ where dependence on I is in general sharp, and cannot be removed.

- Suppose $p \geq 2d/(d+1)$. Then $R^{s,p}[0,\infty)$ contains all elements of the Besov space $B^{1/2,2}_p[0,\infty)$ supported on $[1/2, 2]$, and thus, in particular, must contain unbounded functions for $p > 1$ (because the Sobolev embedding theorem fails when used at the endpoint to embed into L^∞). Since $M^p(\mathbb{R}^d) \subset M^2(\mathbb{R}^d) = L^\infty[0,\infty)$, $M^p(\mathbb{R}^d)$ cannot contain unbounded functions, and thus $M^p(\mathbb{R}^d)$ is a proper subset of $R^{s,p}(\mathbb{R}^d)$.

It is natural to conjecture the same bounds hold when we remove the constraint that the inputs are radial, i.e so that for a radial function $m(\xi) = a(|\xi|)$ and $1/2d < 1/p - 1/2 < 1/2$,

$$\|m\|_{M^p(\mathbb{R}^d)} \sim_{p,d} \|a\|_{R^{p,s}[0,\infty)} \quad \text{where } s = (d-1)(1/p-1/2). \tag{2.2.5}$$

We call this the *radial multiplier conjecture* on $\mathbb{R}^d$.

We now know, by results of Heo, Nazarov, and Seeger [8] that the radial multiplier conjecture is true when $d \geq 4$ and when $1/(d-1) < 1/p - 1/2 < 1/2$. A summary of the proof strategies of this argument is provided in the following two sections, and a major part of our bounds for spectral multipliers follow by adapting this argument to the non-Euclidean setting. Partial improvements were obtained by Cladek [3] for symbols m compactly supported away from the origin, obtaining results for compactly supported multipliers when $d = 4$ and $1/p - 1/2 > 11/36$, and establishing *restricted weak type* bounds when $d = 3$ and $1 < p < 11/26$. But the radial multiplier conjecture has not been resolved fully in any dimension d, we do not have any strong type L^p bounds when $d = 3$, and no bounds whatsoever are known when $d = 2$.

2.3 Radial-Multiplier Bounds by Density Decompositions

In this section and the following, we give an overview of the proof of the radial multiplier bounds obtained by Heo, Nazarov, and Seeger in [8].

Theorem 2.2. *Suppose* $1 < p < 2(d-1)/(d+1)$*, and* $m(\xi) = a(|\xi|)$ *is radial. Then*

$$\|m\|_{M^p(\mathbb{R}^d)} \lesssim \|a\|_{R^{p,s}[0,\infty)},$$

where $s = (d-1)(1/p-1/2)$.

The main tool we will take away for application to the proof of Theorem 1.2 is the method of *density decompositions*, and the method of *atomic decompositions* used to combine frequency scales in the problem. We give an overview of the density decomposition argument in this section to establish a single scale version of Theorem 2.2, as described in the following lemma, and discuss the particular atomic decomposition method in the following section.

Lemma 2.3. *Suppose* $1 < p < 2(d-1)/(d+1)$*, and* k *is a radial function with Fourier transform supported on* $1/2 \leq |\xi| \leq 2$*. Then*

$$\|k * f\|_{L^p(\mathbb{R}^d)} \lesssim \|k\|_{L^p(\mathbb{R}^d)} \|f\|_{L^p(\mathbb{R}^d)}$$

We reduce Lemma 2.3 to an inequality for sums of functions oscillating on spheres. Let σ_r be the surface measure for the sphere of radius r centered at the origin in $\mathbb{R}^d$. Also fix a nonzero, radial, compactly supported function $\psi \in \mathcal{S}(\mathbb{R}^d)$ whose Fourier transform is non-negative, and vanishes to high order at the origin. Given $x \in \mathbb{R}^d$ and $r \geq 1$, define $\chi_{x,r} = \text{Trans}_x(\sigma_r * \psi)$. Then $\chi_{x,r}$ is a smooth function adapted to a thickness $O(1)$ annulus of radius r centered at x, which is *slightly oscillating*. We will verify the following lemma.

Lemma 2.4. *For any* $a : \mathbb{R}^d \times [1,\infty) \to \mathbb{C}$*, and if* $1 < p < 2(d-1)/(d+1)$*,*

$$\left\| \int_{\mathbb{R}^d} \int_1^\infty a(x,r) \chi_{x,r}\, dx\, dr \right\|_{L^p(\mathbb{R}^d)} \lesssim \left(\int_{\mathbb{R}^d} \int_1^\infty |a(x,r)|^p r^{d-1} dr dx \right)^{1/p}.$$

The implicit constant here depends on p*,* d*, and* ψ*.*

Proof of Inequality (2.3) *from Lemma 2.4.* Suppose $k(\cdot) = b(|\cdot|)$ for some function $b : [0,\infty) \to \mathbb{C}$. If we set $a(x,r) = f(x)b(r)$ for any function $f : \mathbb{R}^d \to \mathbb{C}$, then

$$k * \psi * f = \int_{\mathbb{R}^d} \int_1^\infty a(x,r) \chi_{x,r}\, dx\, dr, \tag{2.3.1}$$

Lemma 2.4, applied to (2.3.1), says that

$$\|k * \psi * f\|_{L^p(\mathbb{R}^d)} \lesssim \|k\|_{L^p(\mathbb{R}^d)} \|f\|_{L^p(\mathbb{R}^d)}. \tag{2.3.2}$$

If we choose ψ so that $\widehat{\psi}$ is non-vanishing on the support of k, then the function $1/\widehat{\psi}(\cdot)$ is smooth on the support of m; if T is a Fourier multiplier operator with a smooth, compactly supported symbol agreeing with $1/\widehat{\psi}(\cdot)$ on the support of m, then the convolution kernel of T is Schwartz, and so T is bounded on $L^p(\mathbb{R}^d)$. Since $T(k * \psi * f) = k * f$, we conclude

$$\|k * f\|_{L^p(\mathbb{R}^d)} = \|T(k * \psi * f)\|_{L^p(\mathbb{R}^d)} \lesssim \|k * \psi * f\|_{L^p(\mathbb{R}^d)} \lesssim \|k\|_{L^p(\mathbb{R}^d)} \|f\|_{L^p(\mathbb{R}^d)}, \tag{2.3.3}$$

which completes the argument. □

Next, we consider a discretization of Lemma 2.4.

Lemma 2.5. *Fix a 1-separated set $\mathcal{E} \subset \mathbb{R}^d \times [1,\infty)$. Then for any $a : \mathcal{E} \to \mathbb{C}$, and for $1 < p < 2(d-1)/(d+1)$,*

$$\Big\| \sum_{(x,r)\in\mathcal{E}} a(x,r)\chi_{x,r} \Big\|_{L^p(\mathbb{R}^d)} \lesssim \Big(\sum_{(x,r)\in\mathcal{E}} |a(x,r)|^p r^{d-1} \Big)^{1/p},$$

where the implicit constant is independent of $\mathcal{E}$.

Proof of Lemma 2.4 from Lemma 2.5. For any $a : \mathbb{R}^d \times [1,\infty) \to \mathbb{C}$, if we consider the vector-valued function $\mathbf{a}(x,r) = a(x,r)\chi_{x,r}$, then

$$\int_{\mathbb{R}^d} \int_1^\infty \mathbf{a}(x,r)\, dr\, dx = \int_{[0,1)^d} \int_0^1 \sum_{n\in\mathbb{Z}^d} \sum_{m>0} \mathrm{Trans}_{n,m}\mathbf{a}(x,r)\, dr\, dx \tag{2.3.4}$$

The triangle inequality and the increasing property of norms on $[0,1)^d \times [0,1]$ imply that

$$\begin{aligned} &\Big\| \int_{\mathbb{R}^d} \int_1^\infty \mathbf{a}(x,r)\, dr\, dx \Big\|_{L^p(\mathbb{R}^d)} \\ &\leq \int_{[0,1)^d} \int_0^1 \Big\| \sum_{n\in\mathbb{Z}^d} \sum_{m>0} \mathrm{Trans}_{n,m}\mathbf{a}(x,r) \Big\|_{L^p(\mathbb{R}^d)}\, dr\, dx \\ &\lesssim \int_{[0,1)^d} \int_0^1 \Big(\sum_{n\in\mathbb{Z}^d} \sum_{m>0} |a(x-n, r+m)|^p r^{d-1} \Big)^{1/p}\, dr\, dx \\ &\leq \Big(\int_{[0,1)^d} \int_0^1 \sum_{n\in\mathbb{Z}^d} \sum_{m>0} |a(x-n, r+m)|^p r^{d-1}\, dr\, dx \Big)^{1/p} \\ &= \Big(\int_{\mathbb{R}^d} \int_1^\infty |a(x,r)|^p r^{d-1} dr dx \Big)^{1/p}, \end{aligned} \tag{2.3.5}$$

which completes the proof. ◻

Lemma 2.5 can be further reduced by considering it as a bound on the operator

$$a \mapsto \sum_{(x,r)\in\mathcal{E}} a(x,r)\chi_{x,r}. \tag{2.3.6}$$

In particular, since Lemma 2.5 is an estimate for an open interval of L^p spaces, by using real interpolation methods it suffices to prove a restricted strong type bound L^p bound for $1 < p < 2(d-1)/(d+1)$. Given any discretized set $\mathcal{E}$, let $\mathcal{E}_k$ be the set of $(x,r) \in \mathcal{E}$ with $2^k \leq r < 2^{k+1}$. Then Lemma 2.5 is implied by the following Lemma.

Lemma 2.6. *For $1 < p < 2(d-1)/(d+1)$ and $k \geq 1$,*

$$\Big\| \sum_{(x,r)\in\mathcal{E}} \chi_{x,r} \Big\|_{L^p(\mathbb{R}^d)} \lesssim \Big(\sum_{k\geq 1} 2^{k(d-1)} \#(\mathcal{E}_k) \Big)^{1/p}.$$

Remark. Note that if $r \sim 2^k$, then $\|\chi_{x,r}\|_{L^p(\mathbb{R}^d)} \sim 2^{k(d-1)/p}$, and so Lemma 2.6 says

$$\Big\| \sum_{(x,r)\in\mathcal{E}} \chi_{x,r} \Big\|_{L^p(\mathbb{R}^d)} \lesssim_p \Big(\sum_{(x,r)\in\mathcal{E}} \|\chi_{x,r}\|^p_{L^p(\mathbb{R}^d)} \Big)^{1/p}. \tag{2.3.7}$$

Thus we are proving a pth root cancellation bound for the functions $\{\chi_{x,r}\}$.

To control these sums, we apply a 'density decomposition', which splits a discrete set into different parts which are either spread out, or clustered on small sets. The density decomposition will enable us to obtain L^2 bounds. We say a 1-separated set $\mathcal{E}$ in a metric space X is of *density type* (u, r) if $\#(B \cap \mathcal{E}) \leq u \cdot \operatorname{diam}(B)$ for each ball B in X with diameter at most r.

Lemma 2.7. *For any family of 1-separated sets $\mathcal{E}_k \subset \mathbb{R}^d \times [2^k, 2^{k+1})$, there exists a decomposition $\mathcal{E}_k = \bigcup_{m=1}^{\infty} \mathcal{E}_k(2^m)$ with the following properties:*

- *For each m, $\mathcal{E}_k(2^m)$ has density type $(2^m, 2^k)$.*
- *If B is a ball in $\mathbb{R}^{d+1}$ of radius $r \leq 2^k$ containing at least $2^m \cdot r$ points of $\mathcal{E}_k$, then*
$$B \cap \mathcal{E}_k \subset \bigcup_{m' \geq m} \mathcal{E}_k(2^{m'}).$$
- *For each m, there are disjoint balls $\{B_i\}$ in $\mathbb{R}^{d+1}$ with radii $\{r_i\}$, such that*
$$\sum_i r_i \leq 2^{-m} \# \mathcal{E}_k,$$
such that $r_i \leq 2^k$ for all i, and such that $\bigcup B_i^*$ covers $\bigcup_{m' \geq m} \mathcal{E}_k(2^{m'})$, where B_i^* denotes the ball with the same center as B_i but 5 times the radius.

Proof. Define a function $M : \mathcal{E}_k \to [0, \infty)$ by setting

$$M(x, r) = \sup \left\{ \frac{\#(\mathcal{E}_k \cap B)}{\operatorname{rad}(B)} : (x, r) \in B \text{ and } \operatorname{rad}(B) \leq 2^k \right\}. \tag{2.3.8}$$

We can establish a kind of weak L^1 estimate for M using a Vitali type argument. Let

$$\widehat{\mathcal{E}_k}(2^m) = \{(x, r) \in \mathcal{E}_k : M(x, r) \geq 2^m\}. \tag{2.3.9}$$

We can therefore cover $\widehat{\mathcal{E}_k}(2^m)$ by a family of balls $\{B\}$ such that $\#(\mathcal{E}_k \cap B) \geq 2^m \operatorname{rad}(B)$. The Vitali covering lemma allows us to find a disjoint subcollection of balls $B_1, \dots, B_N$ such that $B_1^*, \dots, B_N^*$ covers $\widehat{\mathcal{E}_k}(2^m)$. We find that

$$\#(\mathcal{E}_k) \geq \sum_i \#(B_i \cap \mathcal{E}_k) \geq 2^m \sum_i \operatorname{rad}(B_i), \tag{2.3.10}$$

Setting $\mathcal{E}_k = \widehat{\mathcal{E}_k}(2^m) - \bigcup_{k' > k} \widehat{\mathcal{E}_{k'}}(2^m)$ thus gives the required result. □

Remark. We will apply the Lemma with $\mathbb{R}^d \times [0, \infty)$ replaced by $X \times [0, \infty)$, where X is a compact, d-dimensional manifold equipped with a Finsler metric. A version of the Vitali covering lemma also holds for this metric space, so that the same proof allows one to perform a density decomposition in this setting.

To prove Lemma 2.6, we perform a decomposition of $\mathcal{E}_k$ for each k, into the sets $\mathcal{E}_k(2^m)$, and then define $\mathcal{E}^m = \bigcup_{k\geq 1} \mathcal{E}_k^m$. For appropriate exponents, we prove L^p bounds on the functions $F^m = \sum_{(x,r)\in\mathcal{E}^m} \chi_{x,r}$ which are exponentially decaying in m, i.e. that

$$\|F^m\|_{L^p(\mathbb{R}^d)} \lessapprox 2^{m\left(\frac{1}{d-1}-(1/p-1/2)\right)} \left(\sum_k 2^{k(d-1)}\#(\mathcal{E}_k)\right)^{1/p}, \tag{2.3.11}$$

where the implicit constant in the inequality can depend polynomially on m. Thus, in the range $1/p - 1/2 > 1/(d-1)$, i.e. for $1 < p < 2(d-1)/(d+1)$ there is geometric decay in m, decaying much quicker than the polynomial implicit constant, and so we may sum in m using the triangle inequality to conclude that

$$\|F\|_{L^p(\mathbb{R}^d)} \lesssim \left(\sum_k 2^{k(d-1)}\#(\mathcal{E}_k)\right)^{1/p}, \tag{2.3.12}$$

proving Lemma 2.6. To get the bound on F^m, we interpolate between an L^2 bound for F^m, and an L^0 bound (i.e. a bound on the measure of the support of F^m). First, we calculate the support of F^m.

Lemma 2.8. *For each k,*

$$|supp(F_k^m)| \lesssim 2^{-m}2^{k(d-1)}\#\mathcal{E}_k.$$

Thus we have

$$|supp(F^m)| \leq \sum_k |supp(F_k^m)| \lesssim \sum_k 2^{-m}2^{k(d-1)}\#\mathcal{E}_k.$$

Proof. We recall that for each k and m, we can find disjoint balls $B_1, \dots, B_N$ with radii $r_1, \dots, r_N \leq 2^k$ such that

$$\sum_{i=1}^N r_i \leq 2^{-m}\#\mathcal{E}_k, \tag{2.3.13}$$

where $\mathcal{E}_k(2^m)$ is covered by the expanded balls $B_1^* \cup \dots \cup B_N^*$. If we write

$$F_{k,i}^m = \sum_{(x,r)\in\mathcal{E}_k(2^m)\cap B_i^*} \chi_{x,r}, \tag{2.3.14}$$

then $\mathrm{supp}(F_k^m) \subset \bigcup_i \mathrm{supp}(F_{k,i}^m)$. For each $(x,r) \in B_i^* \cap \mathcal{E}_k(2^m)$, the support of $\chi_{x,r}$, an annulus of thickness $O(1)$ and radius r, is contained in an annulus of thickness $O(r_i)$ and radius $O(2^k)$ with the same centre as B_i. Thus we conclude that

$$|\mathrm{supp}(F_{k,i}^m)| \lesssim r_i 2^{k(d-1)}, \tag{2.3.15}$$

and it follows that

$$|\mathrm{supp}(F_k^m)| \leq \sum_i r_i 2^{k(d-1)} \leq 2^{-m}2^{k(d-1)}\#\mathcal{E}_k, \tag{2.3.16}$$

which completes the proof. □

Interpolating, it suffices to prove the following L^2 estimate on the function F^m.

Lemma 2.9. *Suppose $\mathcal{E} = \bigcup_k \mathcal{E}_k$ is a 1-separated set, where $\mathcal{E}_k \subset \mathbb{R}^d \times [2^k, 2^{k+1})$ is a set of density type $(2^m, 2^k)$. Then*

$$\left\|\sum_{(x,r)\in\mathcal{E}} \chi_{x,r}\right\|_{L^2(\mathbb{R}^d)} \lesssim \sqrt{m} \cdot 2^{\frac{m}{d-1}} \left(\sum_k 2^{k(d-1)}\#(\mathcal{E}_k)\right)^{1/2}.$$

The L^2 bound in Lemma 2.9 gets worse and worse as m grows, whereas the L^0 bound in Lemma 2.8 gets better and better, since annuli are concentrating in a small set, which is bad from the perspective of constructive interference, but absolutely fine from the perspective of a support bound. To prove the L^2 bound, we require an analysis of the interference patterns of pairs of the functions $\chi_{x,r}$, as provided by the following lemma.

Lemma 2.10. *For any $N > 0$, $x_1, x_2 \in \mathbb{R}^d$ and $r_1, r_2 \geq 1$,*

$$|\langle \chi_{x_1,r_1}, \chi_{x_2,r_2} \rangle| \lesssim_N \left(\frac{r_1 r_2}{\langle (x_1, r_1) - (x_2, r_2) \rangle} \right)^{\frac{d-1}{2}} \sum\nolimits_{\pm,\pm} \langle |x_1 - x_2| \pm r_1 \pm r_2 \rangle^{-N}.$$

Remark. Suppose $r_1 \leq r_2$. Then Lemma 2.10 implies that χ_{x_1,r_1} and χ_{x_2,r_2} are roughly uncorrelated, except when they are supported on annuli that roughly have the same radii and centers, and in addition, one of the following two properties hold:

- $r_1 + r_2 \approx |x_1 - x_2|$, which holds when the two annuli are 'approximately' externally tangent to one another.
- $r_2 - r_1 \approx |x_1 - x_2|$, which holds when the two annuli are 'approximately' internally tangent to one another.

Heo, Nazarov, and Seeger do not exploit the tangency information, though utilizing the tangencies seems important to improve the results they obtain. Cladek exploits this tangency information further, to obtain improved results.

Proof. We write

$$\begin{aligned} \langle \chi_{x_1 r_1}, \chi_{x_2 r_2} \rangle &= \langle \widehat{\chi}_{x_1 r_1}, \widehat{\chi}_{x_2 r_2} \rangle \\ &= \int_{\mathbb{R}^d} \widehat{\sigma_{r_1} * \psi}(\xi) \cdot \overline{\widehat{\sigma_{r_2} * \psi}(\xi)} e^{2\pi i (x_2 - x_1)\cdot \xi} \, d\xi \\ &= (r_1 r_2)^{d-1} \int_{\mathbb{R}^d} \widehat{\sigma}(r_1 \xi) \overline{\widehat{\sigma}(r_2 \xi)} |\widehat{\psi}(\xi)|^2 e^{2\pi i (x_2 - x_1)\cdot \xi} \, d\xi. \end{aligned} \tag{2.3.17}$$

Define functions A and B such that $B(|\xi|) = \widehat{\sigma}(\xi)$, and $A(|\xi|) = |\widehat{\psi}(\xi)|^2$. Then

$$\langle \chi_{x_1,r_1}, \chi_{x_2,r_2} \rangle = C_d (r_1 r_2)^{d-1} \int_0^\infty s^{d-1} A(s) B(r_1 s) B(r_2 s) B(|x_2 - x_1| s) \, ds. \tag{2.3.18}$$

Using well known asymptotics for $\widehat{\sigma}$, we have, for any $N > 0$,

$$B(s) = s^{-(d-1)/2} \sum\nolimits_{n=0}^{N-1} (c_{n,+} e^{2\pi i s} + c_{n,-} e^{-2\pi i s}) s^{-n} + O_N(s^{-N}). \tag{2.3.19}$$

Write

$$a_{n,\tau} = \int_0^\infty A(s) s^{-\frac{d-1}{2} - n_1 - n_2 - n_3} e^{2\pi i (\tau_1 r_1 + \tau_2 r_2 + \tau_3 |x_2 - x_1|) s} \, ds. \tag{2.3.20}$$

Assuming $A(s)$ vanishes to suitably high order at the origin, depending on N, we find that

$$\begin{aligned}\langle \chi_{x_1 r_1}, \chi_{x_2 r_2} \rangle &= C_d \left(\frac{r_1 r_2}{|x_1 - x_2|} \right)^{(d-1)/2} \sum\nolimits_{n,\tau} a_{n,\tau} c_{n,\tau} r_1^{-n_1} r_2^{-n_2} |x_2 - x_1|^{-n_3} \\ &\lesssim \left(\frac{r_1 r_2}{|x_1 - x_2|} \right)^{\frac{d-1}{2}} \left(1 + \frac{1}{|x_1 - x_2|^N} \right) \sum\nolimits_{\tau} \langle \tau_1 r_1 + \tau_2 r_2 + \tau_3 |x_2 - x_1| \rangle^{-N}.\end{aligned} \tag{2.3.21}$$

This gives the result provided that $1 + |x_1 - x_2| \geq |r_1 - r_2|/10$ and $|x_1 - x_2| \geq 1$. If $1 + |x_1 - x_2| \leq |r_1 - r_2|/10$, then the supports of χ_{x_1, r_1} and χ_{x_2, r_2} are disjoint, so the inequality is trivial. On the other hand, if $|x_1 - x_2| \leq 1$, then the bound is trivial by the last sentence unless $|r_1 - r_2| \leq 10$, and in this case the inequality reduces to the simple inequality

$$\langle \chi_{x_1, r_1}, \chi_{x_2, r_2} \rangle \lesssim_N (r_1 r_2)^{(d-1)/2}. \tag{2.3.22}$$

But this follows immediately from the Cauchy-Schwartz inequality. □

The exponent $\frac{d-1}{2}$ in Lemma 2.10 is too weak to apply almost orthogonality directly to obtain L^2 bounds on $\sum_{(x,r) \in \mathcal{E}_k} \chi_{xr}$ on it's own, but together with the density decomposition assumption we will be able to obtain Lemma 2.9.

Proof of Lemma 2.9. Without loss of generality, we may assume that the set of k such that $\mathcal{E}_k \neq \emptyset$ is 10-separated. Write

$$F = \sum\nolimits_{(x,r) \in \mathcal{E}} \chi_{x,r} \tag{2.3.23}$$

and $F_k = \sum_{(x,r) \in \mathcal{E}_k} \chi_{x,r}$. First, we deal with $F_{\lesssim m} = \sum_{k \leq 10m} F_k$ trivially, i.e. writing

$$\|F\|_{L^2(\mathbb{R}^d)} \lesssim m^{1/2} \left(\sum\nolimits_{k \leq 10m} \|F_k\|_{L^2(\mathbb{R}^d)}^2 + \| \sum\nolimits_{k > 10m} F_k \|_{L^2(\mathbb{R}^d)} \right)^{1/2}. \tag{2.3.24}$$

We then decompose

$$\left\| \sum\nolimits_{k > 10m} F_k \right\|_{L^2(\mathbb{R}^d)}^2 \leq \sum\nolimits_{k > 10m} \|F_k\|_{L^2(\mathbb{R}^d)}^2 + 2 \sum\nolimits_{k' > k > 10m} |\langle F_k, F_{k'} \rangle|. \tag{2.3.25}$$

Let us analyze $\langle F_k, F_{k'} \rangle$. The term will become a sum of the form $\langle \chi_{x,r}, \chi_{y,s} \rangle$, where $r \sim 2^k$ and $s \sim 2^{k'}$. Because of our assumption of being 10-separated, we have $r \leq s/2^{10}$. If $\langle \chi_{x,r}, \chi_{y,s} \rangle \neq 0$, then since the support of $\chi_{y,s}$ is an annulus of radius s centered at y, with thickness $O(1)$, and $\chi_{x,r}$ has support on an annulus of radius r centered at x, with thickness $O(1)$, the fact that r is comparatively smaller than s implies that (x, r) must be contained in the annulus of radius s centered at y, with thickness $O(2^k)$. Such an annulus is covered by $O(2^{(k'-k)(d-1)})$ balls of radius 2^k. Each ball can only contain 2^{k+m} points (x, r), and so there can be at most

$$O(2^{k'(d-1)} 2^{-k(d-1)} 2^{k+m}) = O(2^{k'(d-1) - k(d-2) + m}). \tag{2.3.26}$$

pairs $(x,r)\in\mathcal{E}_k$ for which $\langle\chi_{x,r},\chi_{y,s}\rangle\neq 0$. For such pairs we have

$$|\langle\chi_{x,r},\chi_{y,s}\rangle|\lesssim\left(\frac{2^k2^{k'}}{2^{k'}}\right)^{\frac{d-1}{2}}=2^{\frac{k(d-1)}{2}}. \tag{2.3.27}$$

Thus we conclude that

$$|\langle F_k,\chi_{y,s}\rangle|\lesssim 2^{-k(\frac{d-3}{2})+k'(d-1)+m}. \tag{2.3.28}$$

Summing over $10m<k<k'$, we conclude that since $d\geq 4$,

$$\sum\nolimits_{10m<k<k'}|\langle F_k,\chi_{y,s}\rangle|\lesssim 2^{k'(d-1)+m}\sum\nolimits_{10m<k<k'}2^{-k\frac{d-3}{2}}\lesssim 2^{k'(d-1)+m}2^{-5m}\lesssim 2^{k'(d-1)}. \tag{2.3.29}$$

But this means that

$$\sum\nolimits_{10m<k<k'}|\langle F_k,F_{k'}\rangle|\lesssim 2^{k'(d-1)}\cdot\#(\mathcal{E}_{k'}). \tag{2.3.30}$$

This means that

$$\left\|\sum\nolimits_{k>10m}F_k\right\|^2_{L^2(\mathbb{R}^d)}\lesssim\sum\nolimits_{k>10m}\|F_k\|^2_{L^2(\mathbb{R}^d)}+\sum\nolimits_{k'}2^{k'(d-1)}\#(\mathcal{E}_{k'}), \tag{2.3.31}$$

and it now suffices to deal with estimates the $\|F_k\|_{L^2(\mathbb{R}^d)}$, i.e. the interactions of functions supported on radii of comparable magnitude. To deal with these, we further decompose the radii, writing $[2^k,2^{k+1})$ as the disjoint union of intervals $I_{k,\mu}=[2^k+(\mu-1)2^{am},2^k+\mu 2^{am}]$, for some a to be chosen later. These interval induces a decomposition $\mathcal{E}_k=\bigcup_\mu\mathcal{E}_{k,\mu}$. Again, incurring a constant loss at most, we may assume that the μ such that $\mathcal{E}_{k,\mu}\neq\emptyset$ are 10 separated. We write $F_k=\sum F_{k,\mu}$, and we have

$$\|F_k\|^2_{L^2(\mathbb{R}^d)}=\sum\nolimits_\mu\|F_{k,\mu}\|^2_{L^2(\mathbb{R}^d)}+\sum\nolimits_{\mu<\mu'}|\langle F_{k,\mu},F_{k,\mu'}\rangle|. \tag{2.3.32}$$

We now consider $\chi_{x,r}$ and $\chi_{y,s}$ with $r\in I_{k,\mu}$ and $s\in I_{k,\mu'}$. Then we must have $|x-y|\lesssim 2^k$ and $2^{am}\leq|r-s|\lesssim 2^k$, and so we have

$$\begin{aligned}\left|\sum\nolimits_{\mu<\mu'}\langle F_{k,\mu},\chi_{y,s}\rangle\right|&\lesssim 2^{k(d-1)}\sum\nolimits_{\substack{(x,r)\in\mathcal{E}_k\\ 2^{am}\leq|(x,r)-(y,s)|\lesssim 2^k}}|(x,r)-(y,s)|^{-\frac{d-1}{2}}\\&\lesssim 2^{k(d-1)}\sum\nolimits_{am\leq l\leq k}2^{-l(d-1)/2}\#\{(x,r)\in\mathcal{E}_k:|(x,r)-(y,s)|\sim 2^l\}.\end{aligned} \tag{2.3.33}$$

Using the density assumption,

$$\#\{(x,r)\in\mathcal{E}_k:|(x,r)-(y,s)|\sim 2^l\}\lesssim 2^{l+m} \tag{2.3.34}$$

and so we obtain that, again using the assumption that $d\geq 4$,

$$|\sum\nolimits_{\mu<\mu'}\langle F_{k,\mu},\chi_{y,s}\rangle|\lesssim 2^{k(d-1)}2^{m(1-a(d-3)/2)}. \tag{2.3.35}$$

Now summing over all (y,s), we obtain that

$$\left|\sum\nolimits_{\mu<\mu'}\langle F_{k,\mu},F_{k,\mu'}\rangle\right|\lesssim 2^{k(d-1)}2^{m(1-a(d-3)/2)}\#(\mathcal{E}_{k,\mu'}). \tag{2.3.36}$$

and now summing over μ' gives that

$$\|F_k\|_{L^2(\mathbb{R}^d)}^2 \lesssim \sum_\mu \|F_{k,\mu}\|_{L^2(\mathbb{R}^d)}^2 + 2^{k(d-1)} 2^{m(1-a(d-3)/2)} \# \mathcal{E}_k, \tag{2.3.37}$$

which is a good enough bound if we pick a to be large enough. Now we are left to analyze $\|F_{k,\mu}\|_{L^2(\mathbb{R}^d)}$, i.e. analyzing interactions between annuli which have radii differing from one another by at most $O(2^{am})$. Since the family of all possible radii are discrete, the set $\mathcal{R}_{k,\mu}$ of all possible radii has cardinality $O(2^{am})$. We do not really have any orthogonality to play with here, so we just apply Cauchy-Schwartz, writing $F_{k,\mu} = \sum_{r \in \mathcal{R}_{k,\mu}} F_{k,\mu,r}$, to write

$$\|F_{k,\mu}\|_{L^2(\mathbb{R}^d)}^2 \lesssim 2^{am} \sum_r \|F_{k,\mu,r}\|_{L^2(\mathbb{R}^d)}^2. \tag{2.3.38}$$

Recall that $\chi_{x,r} = \mathrm{Trans}_x(\sigma_r * \psi)$, where ψ is a compactly supported function whose Fourier transform is non-negative and vanishes to high order at the origin. In particular, we now make the additional assumption that $\psi = \psi_\circ * \psi_\circ$ for some other compactly function $\psi_\circ$ whose Fourier transform is non-negative and vanishes to high order at the origin. Then we find that $F_{k,\mu,r}$ is equal to the convolution of the function

$$A_r = \sum_{(x,r)\in\mathcal{E}} \mathrm{Trans}_x \psi_\circ \tag{2.3.39}$$

with the function $\sigma_r * \psi_\circ$. Using the standard asymptotics for the Fourier transform of σ_r, i.e. that for $|\xi| \geq 1$,

$$|\widehat{\sigma_r}(\xi)| \lesssim r^{d-1}(1 + r|\xi|)^{-\frac{d-1}{2}}, \tag{2.3.40}$$

and since $|\widehat{\psi_\circ}(\xi)| \lesssim_N |\xi|^N$, we get that if $r \geq 1$, then for $|\xi| \leq 1/r$,

$$|\widehat{\sigma_r}(\xi)\widehat{\psi_\circ}(\xi)| \lesssim_N r^{d-1-N} \tag{2.3.41}$$

and for $|\xi| \geq 1/r$,

$$|\widehat{\sigma_r}(\xi)\widehat{\psi_\circ}(\xi)| \lesssim_N r^{\frac{d-1}{2}} |\xi|^{-N}. \tag{2.3.42}$$

Thus in particular,the L^∞ norm of the Fourier transform of $\sigma_r * \psi_\circ$ is $O(r^{(d-1)/2})$. Now the functions $\psi_\circ$ are compactly supported, so since the set of x such that $(x, r) \in \mathcal{E}$ is one-separated, we find that

$$\|A_r\|_{L^2(\mathbb{R}^d)} \lesssim \#\{x : (x, r) \in \mathcal{E}\}^{1/2}. \tag{2.3.43}$$

But this means that

$$\|F_{k,\mu,r}\|_{L^2(\mathbb{R}^d)} = \|A_r * (\sigma_r * \psi_\circ)\|_{L^2(\mathbb{R}^d)} \lesssim r^{\frac{d-1}{2}} \#\{x : (x, r) \in \mathcal{E}\}^{1/2}. \tag{2.3.44}$$

Thus we have that

$$\|F_{k,\mu}\|_{L^2(\mathbb{R}^d)}^2 = 2^{am} \cdot \#\mathcal{E}_{k,\mu} \cdot 2^{k(d-1)}. \tag{2.3.45}$$

Summing over μ gives that

$$\|F_k\|_{L^2(\mathbb{R}^d)}^2 = 2^{k(d-1)} \#\mathcal{E}_k (2^{am} + 2^{m(1-a(d-3)/2)}). \tag{2.3.46}$$

Picking $a = 2/(d-1)$ optimizes this bound, giving

$$\|F_k\|_{L^2(\mathbb{R}^d)} \lesssim 2^{m/(d-1)} 2^{k(d-1)/2} (\#\mathcal{E}_k)^{1/2}. \tag{2.3.47}$$

Combining (2.3.31) and (2.3.47) completes the proof. □

This completes a proof of the single scale estimates of the paper. The paper then uses an atomic decomposition method to combine these scales and thus complete the proof of Theorem 2.2. Rather than discuss these methods, we instead discuss an iteration of this method, developed by the same authors in a follow up paper [7].

2.4 Combining Scales with Atomic Decompositions

We now sketch a proof as to how we can complete the proof of Theorem 2.2. Our arguments are deliberately vague, as our goal is to build intuition about the techniques, in order to carry out analogous methods more rigorously in the compact manifold setting later. If $T = m(P)$, then Lemma 2.3, appropriately rescale, implies that if we write $T = \sum T_j$, where T_j is the radial Fourier multiplier operator with convolution kernel $k_j(\cdot/2^j)$, then

$$\|T_j\|_{L^p(\mathbb{R}^d)\to L^q(\mathbb{R}^d)} \lesssim \|k_j\|_{L^q(\mathbb{R}^d)}. \tag{2.4.1}$$

The goal of this section is to prove that we can bound the sum of the operators T_j by $\sup_j \|k_j\|_{L^q(\mathbb{R}^d)}$, which morally speaking, is a bound of the form

$$\left\|\sum\nolimits_j T_j\right\| \lesssim \sup\nolimits_j \|T_j\|. \tag{2.4.2}$$

We must thus show that the operators $\{T_j\}$ do not constructively interfere with one another to a significant extent.

Atomic decompositions are a powerful way to control interactions between operators. The method involves decomposing functions into more elementary components, which we call *atoms*, that have controlled size, spatial support, and oscillatory properties. Such atoms are chosen via careful, non-linear selection processes, often related to stopping times, to ensure certain cancellation conditions hold. We use a variant of this method, involving a use of a Whitney decomposition rather than a stopping time argument, to obtain an atomic decomposition where atoms do not cluster, in an appropriate sense.

Since we are controlling different operators supported on different dyadic frequency ranges on functions in $L^p(\mathbb{R}^d)$ for $1 < p < \infty$, it is natural to consider the Littlewood-Paley inequality $\|Sf\|_{L^p(\mathbb{R}^d)} \lesssim \|f\|_{L^p(\mathbb{R}^d)}$, where

$$Sf(x) = \left(\sum\nolimits_j |P_j f(x)|^2\right)^{1/2}, \tag{2.4.3}$$

and the operators P_j are Littlewood-Paley projections, Fourier multiplier operators with symbol $\psi(\cdot/2^j)$, where $\psi \in C_c^\infty(\mathbb{R}^d)$ is equal to one on the support of the function χ used to decompose T into the operators T_j above, so that $T_j f = T_j f_j$, where $f_j = P_j f$. Roughly speaking, our atomic decomposition will be of the following form: for each dyadic number H, we consider a family of dyadic cubes $\mathcal{W}_H$, whose union is the set $\{x : |Sf(x)| \sim H\}$, and whose doubles have the bounded overlap property. We will then obtain a decomposition $f_j = \sum a_{j,H,W}$, where $a_{j,H,W}$ has Fourier support on $|\xi| \sim 2^j$, is supported on the cube W, and for each fixed H,

$$\left(\sum\nolimits_j \left|\sum\nolimits_W |a_{j,H,W}(x)|\right|^2\right)^{1/2} \sim H \quad \text{if } |Sf(x)| \sim H. \tag{2.4.4}$$

The advantage of this decomposition is that it controls how many *local interactions* the atoms $\{a_{j,H,W}\}$ have. To see why this might be useful, suppose we were considering a Fourier multiplier operator T with the property that $Ta_{j,H,W}$ is essentially supported on the cube W^* obtained by doubling the sidelengths of the cube W, but maintaining the same centre. The bounded overlap property of the dyadic cubes $\mathcal{W}_H$ would then imply an almost orthogonality bound for each H, that

$$\begin{aligned}\left\|\sum\nolimits_{j,W} T_j a_{j,H,W}\right\|_{L^2(\mathbb{R}^d)} &\lesssim \left(\sum\nolimits_{j,W} \|T_j a_{j,H,W}\|_{L^2(\mathbb{R}^d)}^2\right)^{1/2} \\ &\lesssim \left(\sum\nolimits_{j,W} \|a_{j,H,W}\|_{L^2(\mathbb{R}^d)}^2\right)^{1/2} \\ &= \left\|\left(\sum\nolimits_{j,W} |a_{j,H,W}|^2\right)^{1/2}\right\|_{L^2(\mathbb{R}^d)} \lesssim H|\Omega_H|^{1/2}\end{aligned} \tag{2.4.5}$$

Hölder's inequality then justifies that

$$\left\|\sum\nolimits_{j,W} T_j a_{j,H,W}\right\|_{L^1(\mathbb{R}^d)} \lesssim |\Omega_H|^{1/2} \left\|\sum\nolimits_{j,W} T_j a_{j,H,W}\right\|_{L^2(\mathbb{R}^d)} \lesssim H|\Omega_H|. \tag{2.4.6}$$

Real interpolation allows us to obtain an l^p bound in H, so that for $1 < p < 2$,

$$\|Tf\|_{L^p(\mathbb{R}^d)} \lesssim \left(\sum \left[H|\Omega_H|^{1/p}\right]^p\right)^{1/p} \lesssim \|Sf\|_{L^p(\mathbb{R}^d)} \lesssim \|f\|_{L^p(\mathbb{R}^d)}. \tag{2.4.7}$$

and thus we have proven the boundedness of such operators.

Such a simple proof will not suffice for our analysis, since the class of multipliers we are considering is not pseudolocal at all; the kernels k_j are only assumed to be uniformly bounded in $L^p(\mathbb{R}^d)$, and need not satisfy any decay bound as $|x| \to \infty$. Nonetheless, the argument above can be used to prove bounds for other more pseudolocal multipliers, for instance, obtaining an alternate proof of the endpoint results of [18]. We will, however, be able to exploit the above calculations to control *close range interactions* of general multipliers, i.e. for any multiplier T, writing

$$Tf = \sum T_j\{a_{j,H,W}\} = \sum T_{j,W,\mathrm{Short}}\{a_{j,H,W}\} + T_{j,W,\mathrm{Long}}\{a_{j,H,W}\}, \tag{2.4.8}$$

where

$$T_{j,W,\mathrm{Short}}(x,y) = \mathbb{I}_{W^*}(x)\, T_{j,W}(x,y) \quad \text{and} \quad T_{j,W,\mathrm{Long}}(x,y) = \mathbb{I}_{(W^*)^c}(x)\, T_{j,W}(x,y). \tag{2.4.9}$$

The calculations of the previous paragraph can be adapted to show that

$$\left\|\sum\nolimits_{j,W,H} T_{j,W,\mathrm{Short}}\{a_{j,H,W}\}\right\|_{L^q(\mathbb{R}^d)} \lesssim \|f\|_{L^p(\mathbb{R}^d)}, \tag{2.4.10}$$

and it remains to find a way to control the long range interactions between atoms.

There are several related methods to control these interactions, the most elegant formulation provided in the paper [7]. The idea is to take a singular scale estimate, and upgrade this to an estimate with a geometrically decaying constant term when our inputs have amplitudes that are locally constant on a much larger scale than is required by the uncertainty principle and the frequency support of the inputs, namely, that

$$\left\|\sum_H \sum_{W\in\mathcal{W}_{H,l-j}} T_{j,W,\mathrm{Long}}\{a_{j,H,W}\}\right\|_{L^q(\mathbb{R}^d)} \lesssim 2^{-l\varepsilon}\left(\sum_H \sum_{W\in\mathcal{W}_{H,l-j}} \|a_{H,W}\|^p_{L^\infty(\mathbb{R}^d)}|W|\right)^{1/p}, \quad (2.4.11)$$

where $\mathcal{W}_{H,a}$ are the set of all cubes of sidelength 2^a. Summing in j using L^p orthogonality, we find that

$$\left\|\sum_{j,H} \sum_{W\in\mathcal{W}_{H,l-j}} T_{j,W,\mathrm{Long}} a_{j,H,W}\right\|_{L^p(\mathbb{R}^d)} \lesssim 2^{-l\varepsilon}\left(\sum_{j,H} \sum_{W\in\mathcal{W}_{H,l-j}} \|a_{j,H,W}\|^p_{L^\infty(\mathbb{R}^d)}|W|\right)^{1/p}. \quad (2.4.12)$$

For a fixed l, and each $W \in \mathcal{W}_H$, there exists a unique j such that $W \in \mathcal{W}_{H,l-j}$, which implies that the supports of the functions $\{a_{j,H,W}\}$ in the sum above are almost disjoint, and thus the simple bound $\|a_{j,H,W}\|_{L^\infty(\mathbb{R}^d)} \lesssim H$ gives that

$$\sum_j \sum_{W\in\mathcal{W}_{H,l-j}} \|a_{j,H,W}\|^p_{L^\infty(\mathbb{R}^d)}|W| \leq H^p|\Omega_H|, \quad (2.4.13)$$

and thus that

$$\left\|\sum_{j,H} \sum_{W\in\mathcal{W}_{H,l-j}} T_{j,W,\mathrm{Long}} a_{j,H,W}\right\|_{L^p(\mathbb{R}^d)} \lesssim 2^{-l\varepsilon}\left(\sum_H H^p|\Omega_H|\right)^{1/p} \lesssim 2^{-l\varepsilon}\|f\|_{L^p(\mathbb{R}^d)}. \quad (2.4.14)$$

Summing in l trivially using the triangle inequality and the geometric decay in l gives that

$$\left\|\sum_j \sum_H \sum_{W\in\mathcal{W}_H} T_{j,W,\mathrm{Long}} a_{j,H,W}\right\|_{L^p(\mathbb{R}^d)} \lesssim \|f\|_{L^p(\mathbb{R}^d)}, \quad (2.4.15)$$

which controls the long-range interactions in full, completing the proof of Theorem 2.2.

2.5 Quasi-Radial Multipliers and Local Smoothing

Kim [12] has extended the bounds of Heo-Nazarov-Seeger to quasi-radial multipliers, i.e. multipliers of the form $(a \circ r)(\xi)$, where $r : \mathbb{R}^d \to [0,\infty)$ is a smooth, homogeneous function of order one, such that the cosphere $S = \{\xi : r(\xi) = 1\}$ is a hypersurface with non-vanishing Gauss curvature. Such multipliers retain the translation and dilation symmetries of the family of radial Fourier multiplier operators, but are no longer *rotation-invariant*.

Using the same identities as for radial Fourier multiplier operators, one can verify that

$$\|a \circ r\|_{M^p(\mathbb{R}^d)} \gtrsim \sup_R \|k_R\|_{L^q(\mathbb{R}^d)}, \quad (2.5.1)$$

where k_R is the Fourier transform of $\chi(\cdot)(a \circ r)(R\cdot)$, for some fixed $\chi \in C_c^\infty(\mathbb{R})$. In a paper analyzing quasi-radial multipliers of Bochner-Riesz type, Lee and Seeger [14] showed

$$\sup_R \|k_R\|_{L^p(\mathbb{R}^d)} \sim \|a\|_{R^{p,s}[0,\infty)}, \quad (2.5.2)$$

Thus

$$\|a \circ r\|_{M^p(\mathbb{R}^d)} \gtrsim \|a\|_{R^{p,s}[0,\infty)}. \tag{2.5.3}$$

Kim [12] has showed the converse, in the same range $1/p - 1/2 > 1/(d-1)$ as considered by Heo, Nazarov and Seeger in the last section.

Theorem 2.11. *For $1/p - 1/2 > 1/(d-1)$, and $r : \mathbb{R}^d \to [0,\infty)$ is smooth and homogeneous, then for any regulated function a,*

$$\|a \circ r\|_{M^{p,q}(\mathbb{R}^d)} \lesssim \|a\|_{R^{q,s}_d[0,\infty)}.$$

The proof is an adaption of the proof of [7]. However, it is more difficult to work directly with convolution kernels k in this problem, since, unlike for radial multipliers, whose kernels are also radial, the kernels k corresponding to quasi-radial Fourier multiplier operators need not be quasi-radial. It is here that we introduce a technique that has proved essential to an analysis of multiplier operators in settings lacking the full symmetry of Euclidean space: a reduction of the study of multipliers to an analysis of wave equations. Using the Fourier inversion formula, we write

$$\begin{aligned} (a \circ r) f(x) &= \int a(r(\xi)) e^{2\pi i \xi \cdot (x-y)} f(y)\, dy\, dx \\ &= \iint \widehat{a}(t) e^{2\pi i [t r(\xi) + \xi \cdot (x-y)]} f(y)\, dy\, dx\, dt \\ &= \int \widehat{a}(t) (w_t * f)(x)\, dt \end{aligned} \tag{2.5.4}$$

where $\widehat{w_t}(\xi) = e^{2\pi i t r(\xi)}$. As t varies, $u = w_t * f$ solves the wave equation $\partial_t u = 2\pi i P u$, where P is the Fourier multiplier operator whose symbol is the function r. Define $\chi_{x,t} = \text{Trans}_x \left(\int_{-\infty}^{\infty} \psi(s) w_{t-s}\, ds \right)$, where the Fourier transform of ψ is non-negative and vanishing to high order at the origin. Then, using oscillatory integral techniques akin to Lemma 2.10, one can obtain inner product estimates on the quantities $\langle \chi_{x_1,t_1}, \chi_{x_2,t_2} \rangle$ that show such terms are negligible unless $|x_1 - x_2| + |t_1 - t_2| \lesssim 1$. We can then adapt the proof of [7], performing a density decomposition using the Euclidean metric on $\mathbb{R}^{d+1}$, and thus obtain single scale bounds for the quasi-radial multipliers. An atomic decomposition analogous to that discussed in Section 2.4 then yields Kim's result.

We note that the endpoint analysis of radial multipliers we have been discussing is very closely related to the regularity of solutions to wave equations. In particular, if $\widehat{w_t} = e^{2\pi i t |\xi|}$, then for any function f, $u = w_t * f$ solves the half-wave equation $\partial_t u = 2\pi i \sqrt{-\Delta} u$. The Littlewood-Paley pieces $(P_k u)(\cdot, t)$ are Fourier multipliers with symbol $\chi(\cdot/2^k) e^{2\pi i t|\xi|}$, which are radial, and thus can be written in terms of spherical averages. The analysis of Lemma 2.4 can then be used to show that for $1/2 - 1/q > 1/(d-1)$,

$$\|P_k u\|_{L^q(\mathbb{R}^d \times [1,2])} \lesssim 2^{ks} \|f\|_{L^q(\mathbb{R}^d)} \quad \text{where } s = (d-1)(1/2 - 1/q) - 1/q. \tag{2.5.5}$$

With some more work involving atomic decompositions, Heo, Nazarov, and Seeger are able to combine the frequency scales, and thus prove the local smoothing estimate

$$\|u\|_{L^q(\mathbb{R}^d \times [1,2])} \lesssim \|f\|_{W^{s,q}(\mathbb{R}^d)}. \tag{2.5.6}$$

These are the current sharpest *endpoint* local smoothing bounds (they do not lose any ε in the smoothness parameter).

To end our discussion of radial multipliers, notice that the Fourier multiplier operator P is above is an elliptic operator on $\mathbb{R}^d$, satisfying the Assumption A we introduced in Chapter 1. For such an operator, the partial differential equation $\partial_t u = 2\pi i P u$ is hyperbolic, and it's solutions have wavelike properties. We will rely on a similar decomposition method used for the quasi-radial multiplier with symbol $a \circ r$ on manifolds, together with a study of the geometry of the manifold upon which we study these multipliers, to extend the methods we have discussed in the previous three sections to compact manifolds. We will now return to discuss this setting in more detail.

Chapter 3

Wave Equations on Compact Manifolds

3.1 An Analogue of the Radial Multiplier Conjecture on Compact Manifolds

We now return to the study of spectral multipliers on compact manifolds. In a heuristic sense, one can think of the Fourier multipliers studied in the last Chapter as a limiting case of multipliers in the 'high frequency limit', i.e. so that for any elliptic operator on a manifold X with principal symbol $p(x_0, \xi)$ as $R \to \infty$, locally around $x_0 \in X$ the behaviour of the operators P/R becomes more and more like the Fourier multiplier operator on $\mathbb{R}^d$ with norm $p(x_0, \cdot)$ as $R \to \infty$. In particular, the following transference result of Mitjagin [17] holds.

Theorem 3.1. *If P is an elliptic operator on a d-dimensional compact manifold X. For each $x_0 \in X$, if we choose a basis, identifying $T^*_{x_0}X$ with $\mathbb{R}^d$, and let $p_{x_0} : \mathbb{R}^d \to [0, \infty)$ be the principal symbol of p restricted to $T^*_{x_0}X$, then for any regulated function a,*

$$\|a \circ p_{x_0}\|_{M^p(\mathbb{R}^d)} \lesssim \|a\|_{M^p_{Dil}(X)}.$$

Thus if P satisfies Assumption A, it follows from Theorem 3.1 and (2.5.3) that

$$\|a\|_{M^p_{\mathrm{Dil}}(X)} \gtrsim \|a\|_{R^{p,s}(\mathbb{R}^d)}, \tag{3.1.1}$$

with $s = (d-1)(1/p - 1/2)$. We might conjecture that one can reverse this inclusion in the same range as for the radial multiplier conjecture, i.e. proving that $\|a\|_{M^p_{\mathrm{Dil}}(X)} \lesssim \|a\|_{R^{p,s}(\mathbb{R}^d)}$ for $1/p - 1/2 > 1/2d$. However, this is not possible on a general manifold, and in fact, fails in the full range whenever the geometry of X induced by P, which we introduce in the next section, does not have constant sectional curvature. It is not clear what the range of the conjecture should be for an arbitrary manifold, though it is likely true that the conjecture holds on manifolds with constant sectional curvature. TODO EDIT THIS.

3.2 Geometries Induced by Elliptic Operators

We plan to study spectral multipliers of an elliptic operator P on a compact manifold X via the use of solutions of the wave equation $\partial_t u = 2\pi i P u$ on M, which allow us to exploit geometric information about the manifold M and thus extend the results of Heo, Nazarov, and Seeger to the setting of compact manifolds when Assumption A and Assumption B of Chapter 1 hold. But what is the geometry of the manifold we should be using? If the elliptic operator $P = \sqrt{-\Delta}$ is induced by a Riemannian metric on X, the geometric structure is clear, since X already has a Riemannian geometry. Given a more general operator P, we will use the principal symbol of P to give X a *Finsler geometry*. In this section we describe the bare essentials of Finsler geometry needed for the arguments which occur later on in the thesis. We mainly refer to [2] for a reference to further details.

Let X be a d-dimensional manifold. We denote an element of the tangent bundle TX by (x, v), where $x \in X$ and $v \in T_x X$, and an element of the cotangent bundle T^*X by (x, ξ), with $\xi \in T_x^* X$. A *Finsler metric* on X is a homogeneous function $F : TX \to [0, \infty)$, which is smooth on $TX - 0$, and such that for each $x \in X$, the function $F_x : T_x^* X \to [0, \infty)$ is a *strictly convex* norm, in the sense that for $(x, v) \in TX - 0$, the Hessian of F_x^2 in the v variable is positive-definite, i.e. the $(2, 0)$ tensor $g(x, v)$ given in coordinates by

$$g_{ij}(x, v) = \frac{1}{2} \frac{\partial^2 F^2}{\partial v^i \partial v^j}(x, v). \tag{3.2.1}$$

These coordinates also give rise to an inner product on $T_x X$ which approximates the Finsler metric near v up to second order. We record the *fundamental inequality for Finsler metrics*.

Lemma 3.2. *For any $v, w \in T_x X$, $\sum g_{ij}(x, v) v^i w^j \leq F(x, v) F(x, w)$.*

Proof. By Euler's homogeneous function theorem, we can write

$$\sum g_{ij}(x, v) v^i w^j = (1/2) \sum \partial_{v_i} F^2(x, v) w^i = F(x, v) \sum \partial_{v_i} F(x, v) w^i. \tag{3.2.2}$$

To complete the proof, we must thus show that $\sum \partial_{v_i} F(x, v) w^i \leq F(x, w)$. But this follows by the triangle inequality $F(x, v + tw) \leq F(x, v) + tF(x, w)$. □

A Finsler metric gives each of the tangent spaces of a manifold the structure of a norm space. Such a structure naturally gives the dual space a dual norm $F_* : T^*X \to [0, \infty)$. The strict convexity of F gives rise to a *Legendre transform* on X.

Lemma 3.3. *For a Finsler manifold X, define a homogeneous map from TX to T^*X, defined in coordinates by setting*

$$\mathcal{L}(x, v)_i = \sum g_{ij}(x, v) v^j.$$

Then $\mathcal{L}$ is a diffeomorphism from $TX - 0$ to $T^*X - 0$, and for each $x \in X$ and each unit vector $v \in T_x X$, $\mathcal{L}(x, v)$ is the unique unit vector ξ such that $\langle \xi, v \rangle = 1$.

Proof. Begin by defining a map $\mathcal{L}^{-1}$, which is homogeneous, and for each unit vector $\xi \in T_x^*X$, is the unique unit vector $v \in T_xX$ such that $\langle \xi, v \rangle = 1$. The existence follows from the definition of the dual norm, and the uniqueness follows from the strict convexity of the Finsler norm. By the method of Lagrangian multipliers, if ξ is a unit vector, and $(x, v) = \mathcal{L}^{-1}(x, \xi)$, it must be true in coordinates that $\xi_i = \sum g_{ij}(x, v)v^j$ for each i, which shows that the map $\mathcal{L}^{-1}$ is injective. Moreover, given any unit vector $v \in T_xX$, the vector $\xi \in T_x^*X$ given in coordinates by $\xi_i = \sum g_{ij}(x, v)v^j$ is a unit vector, and satisfies $\langle \xi, v \rangle = 1$, by Euler's homogeneous function theorem and the fundamental inequality. Thus we conclude that $\mathcal{L}^{-1}(x, \xi) = (x, v)$, which shows that the map $\mathcal{L}^{-1}$ is surjective, and the inverse is given by the map $\mathcal{L}$ defined above. By Euler's homogeneous function theorem, we calculate that $\partial_{v_i}\mathcal{L}_j = g_{ij}$, and since the values g_{ij} define a positive-definite (and thus invertible) matrix, it follows that $\mathcal{L}$ is a diffeomorphism. □

The Legendre transform is the Finsler variant of the musical isomorphism in Riemannian geometry, though the musical isomorphism of Riemannian geometry is linear rather than just homogeneous.

Corollary 3.4. *The norm F_* is strictly convex, and if $\mathcal{L}(x, v) = (x, \xi)$, and we define*

$$g^{ij}(x, \xi) = \frac{1}{2}\frac{\partial^2 F_*^2}{\partial \xi_i \partial \xi_j}(x, \xi).$$

then

$$\sum_j g^{ij}(x, \xi) g_{jk}(x, v) = \delta^i_k.$$

Proof. The equation implies strict convexity, because it implies that the matrix with entries $g^{ij}(x, \xi)$ is the inverse of the matrix with entries $g_{ij}(x, v)$, and the inverse of a positive definite matrix is positive definite. If v is a unit vector, then a form of the fundamental inequality (applied to F_* rather than F) implies that the vector $w \in T_xX$ defined in coordinates by $w^i = g^{ij}(x, \xi)\xi_j$ is a unit vector, and Euler's homogeneous function implies that $\langle \xi, w \rangle = 1$. Thus $w = v$, and so we conclude by homogeneity that in general $\mathcal{L}^{-1}(x, \xi)^i = \sum g^{ij}(x, \xi)\xi_j$. We calculate that $\partial_i \mathcal{L}^{-1}(x, \xi)^j = g^{ij}(x, \xi)$, and then differentiating the identity $\mathcal{L}^{-1} \circ \mathcal{L} = I$ implies the required claim. □

Now let P be an elliptic operator on a manifold X satisfying Assumption A. The principal symbol of P is a function $p : T^*M \to [0, \infty)$, and the following lemma applies.

Lemma 3.5. *Suppose $p : T^*M \to [0, \infty)$ is homogeneous, and for each $x \in M$, the cosphere $S_x^* = \{\xi \in T_x^*M : p(x, \xi) = 1\}$ has non-vanishing Gaussian curvature. Then*

$$F(x, v) = \{\xi(v) : p(x, \xi) = 1\}$$

is a Finsler metric on M, and the dual metric F_* on T^*M is equal to p.

Proof. For each $x \in U_0$, the cosphere $S_x^* = \{\xi \in T_x^*M : p(x,\xi) = 1\}$ has non-vanishing Gaussian curvature. We claim that all principal curvatures of S_x^* must actually be *positive*. This follows from a simple modification of an argument found in Chapter 2 of [9]. Indeed, if we fix an arbitrary point $v_0 \in T_x^*M$, and consider the smallest closed ball $B \subset T_x^*M$ centered at v_0 and containing S_x^*, then the sphere ∂B must share the same tangent plane as S_x^* at some point. All principal curvatures of ∂B are positive, and at this point all principal curvatures of S_x^* must be greater than the principal curvatures of ∂B, since S_x^* curves away faster than ∂B in all directions. By continuity, we conclude that the principal curvatures are everywhere positive. Thus for each $x \in M$ and $\xi \in T_x^*M - \{0\}$, the coefficients

$$g^{ij}(x,\xi) = (1/2)(\partial^2 p^2/\partial\xi_i\partial\xi_j) \tag{3.2.3}$$

form a positive-definite matrix. But inverting the procedure of the previous two lemmas shows that the dual norm $F(x,v) = \sup_{\xi\in S_x^*}\xi(v)$ is also strictly convex, and thus gives a Finsler metric on M with $F_* = p$. □

A Finsler metric gives a length to each tangent vector on the manifold, and can thus be used to define the lengths of curves $c : I \to U_0$ by the formula

$$L(c) = \int_I F(c,\dot{c}), \tag{3.2.4}$$

which is invariant under reparameterization. An analysis of length minimizing curves naturally leads to a theory of geodesics on a Finsler manifold, i.e. to a theory of critical points in the space of paths between two points. The theory of geodesics on Finsler manifolds is similar to the Riemannian case, except for the interesting quirk that a geodesic from a point p to a point q need not necesarily be a geodesic when considered as a curve from q to p, and so we must consider *forward* and *backward* geodesics*. We define the *forward distance* $d_+ : X \times X \to [0,\infty)$ by taking the infima of paths between points. This function is a *quasi-metric*, as it satisfies the triangle inequality, but is not necessarily symmetric. We define the *backward distance* $d_- : X \times X \to [0,\infty)$ by setting $d_-(p,q) = d_+(q,p)$. On a general Finsler manifold one has $d_+ \neq d_-$, and these functions are distinct quasi-metrics on M (though a compactness argument shows both are proportional to one another on compact manifolds). Geodesics from a point p_0 to a point p_1 need not be geodesics from p_1 to p_0 when reversed. A metric can be obtained by setting $d_X = d_X^+ + d_X^-$, and we will use this definition as the canonical metric on a Finsler manifold X.

A standard approach to an analysis of geodesics in Riemannian geometry is to rely on the *fundamental theorem of Riemannian geometry*, which posits the existence of a torsion-free linear connection on the tangent bundle of a Riemannian manifold X compatible with

*If walking up a hill is more strenuous than walking down a hill, then the optimal path to climb from the bottom of a mountain to it's summit need not be the optimal path to descend from the peak to the bottom. Indeed, Matsumoto formulated such a problem by constructing a Finsler metric upon the surface of the mountain such that geodesics correspond to optimal paths of ascent and descent [16]. Similar Finsler manifolds can be constructed to model optimal paths taken sailing under the influence of varying wind conditions, the theory being known as Zermelo's navigation problem.

the metric, and using the connection to derive a first variation formula. A similar approach works for Finsler geometry, aside from the fact that (a) the connection is not defined on the tangent bundle and (b) the analogue of the fundamental theorem in general *fails*. Because of this, there is no canonical connection that is used in the Finsler geometry literature, and depending on the application and personal preference one of several can be used, the most popular being the Cartan, Berwald, and Chern connections, the first being metric-compatible, but which can have torsion, and the latter two being torsion free, but which are almost metric-compatible. We use the Chern connection in what follows.

To define the Chern connection, recall that for a vector bundle B over a manifold X, a connection on B is an association, with each $s \in \Gamma(B)$ and $v \in T_x X$, of a vector $\nabla_v(s) \in B_x$, which is linear in v and s, and such that for each $X \in \Gamma(X)$ and a section $s \in \Gamma(B)$, $\nabla_X(s) \in \Gamma(B)$ is smooth. We will be studying connections on the pullback bundle $\pi^*(TX)$, viewed as a vector bundle over TX, where $\pi : TX \to X$ is the projection map. The bundle $\pi^*(TX)$ can be viewed as a subbundle of $T(TX)$. This bundle has a distinguished section $l \in \Gamma(\pi^*(TX))$, given in coordinates by $l(x,v)_i = v_i$. Define the Cartan tensor A, a symmetric 3-tensor given in coordinates by

$$A_{ijk} = \frac{F}{2} \frac{\partial^3 F^2}{\partial v^i \partial v^j \partial v^k}. \tag{3.2.5}$$

Note that if X is a Riemannian manifold, and $F(x,v) = \langle v, v\rangle$ is the Finsler metric induced by the Riemannian metric, then the Cartan tensor A vanishes identically, but on a general Finsler manifold this need not be the case.

Proposition 3.6. *If X is a Finsler manifold, and consider the projection map $\pi : TX \to X$. Then there exists a unique linear connection ∇ on the pullback bundle $\pi^*(TM)$, viewed as a bundle over TM, which is torsion-free, in the sense that for any $X, Y \in \Gamma(\pi^*(TM))$, identifying X and Y with vector fields on TM,*

$$\nabla_X Y - \nabla_Y X = [X, Y],$$

and almost metric compatible, in the sense that for $X, Y, Z \in \Gamma(\pi^*(TM))$,

$$X(g(Y,Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z) + A(\nabla_X l, Y, Z).$$

Proof. We do not prove this proposition, but merely define the connection. Define the formal Christoffel symbols

$$\gamma^l_{jk} = \sum\nolimits_i \frac{g^{li}}{2}\left(\frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{ki}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^i}\right), \tag{3.2.6}$$

the geodesic spray coefficients and nonlinear connection coefficients

$$G^i = \sum\nolimits_{j,k} \gamma^i_{jk} v^j v^k \quad \text{and} \quad N^i_j = \frac{1}{2}\frac{\partial G^i}{\partial v^j}, \tag{3.2.7}$$

the horizontal vectors

$$\frac{\delta}{\delta x^a} = \frac{\partial}{\partial x^a} - \sum\nolimits_b N_a^b \frac{\partial}{\partial v^b}, \tag{3.2.8}$$

and the Chern connection coefficients

$$\Gamma^i_{jk} = \sum\nolimits_s \frac{g^{is}}{2}\left(\frac{\delta g_{sj}}{\delta x^k} + \frac{\delta g_{ks}}{\delta x^j} - \frac{\delta g_{jk}}{\delta x^s}\right). \tag{3.2.9}$$

Finally, the connection is defined by setting

$$\left(\nabla_{\partial_{x^j}} X\right)^i = \frac{\partial X^i}{\partial x^j} + \Gamma^i_{jk} X^k \quad \text{and} \quad \left(\nabla_{\partial_{v^j}} X\right)^i = \frac{\partial X^I}{\partial v^j}. \tag{3.2.10}$$

See Theorem 2.4.1 of [2] for details of the proof that this connection is torsion free and almost metric compatible, as well as the uniqueness of the connection. □

Now we derive a first variation formula for the energy of a curve.

Proposition 3.7. *Consider a variation $c : (-\varepsilon, \varepsilon) \times I \to X$ on a Finsler manifold, not necessarily fixed at the endpoints. Define $T = \partial_t c$ be the tangent field, and $U = \partial_s c$ the variation field. Define*

$$E(s) = \frac{1}{2}\int_I F(c, \dot{c})^2 = \frac{1}{2}\int_I g_T(T, T).$$

Then

$$E'(s) = g_T(U, T)|_{\partial I} - \int_I g_T(U, \nabla_T T).$$

Proof. Almost metric compatibility gives that

$$\partial_s\{g_T(T, T)\} = U\{g_T(T, T)\} = 2g_T(\nabla_U T, T), \tag{3.2.11}$$

and since $[U, T] = 0$, being torsion free and almost metric compatibility implies that

$$g_T(\nabla_U T, T) = g_T(\nabla_T U, T) = \partial_t\{g_T(U, T)\} - g_T(U, \nabla_T T). \tag{3.2.12}$$

The Cartan tensor does not appear here, since $A_T(X, Y, Z) = 0$ if $T \in \{X, Y, Z\}$, by the Euler homogeneous function theorem. Differentiating under the integral sign thus gives that

$$E'(s) = \frac{1}{2}\int_I \partial_s\{g_T(T, T)\} = g_T(U, T)|_{\partial I} - \int_I g_T(U, \nabla_T T), \tag{3.2.13}$$

which finishes the proof. □

A differential formula for geodesics immediately follows.

Corollary 3.8. *A constant speed curve $c : I \to X$ is a geodesic (i.e. a critical point of any length variation fixing endpoints) if and only if it satisfies the differential equation*

$$\ddot{c}^a = -\sum_{j,k} \gamma^a_{jk}(c,\dot{c})\dot{c}^j\dot{c}^k \quad \textit{for all } 1 \leq a \leq d.$$

Alternatively, if we set $(x,\xi) = \mathcal{L}(c,\dot{c})$, then c is a geodesic if and only if it satisfies Hamilton's equations for motion

$$\dot{x} = -(\partial_\xi F_*)(x,\xi) \quad \textit{and} \quad \dot{\xi} = (\partial_x F_*)(x,\xi).$$

Proof. The proof of the first equation follows by expanding the equation $\nabla_T T = 0$, which is necessary by the first variation formula for a curve to be a geodesic. To obtain the second, we apply the theory of Euler-Lagrange equations and Hamiltonian mechanics, as described in Chapters 14 and 15 of [1]. The Euler-Lagrange equation for the energy functional

$$E(c) = \frac{1}{2}\int_I F(c,\dot{c})^2 \tag{3.2.14}$$

is equal to

$$\frac{1}{2}\frac{\partial F^2}{\partial x^i}(c,\dot{c}) = \frac{1}{2}\frac{\partial^2 F^2}{\partial v^i \partial x^j}(c,\dot{c})\dot{c}^j + g_{ij}(c,\dot{c})\ddot{c}^j \tag{3.2.15}$$

and applying the Legendre transform to get Hamilton's equations of motions, we obtain the second pair of equations. □

Remark. The Hamiltonian equations derived in this corollary will reoccur in our study of wave propogation, and will tell us that high frequency wave packet solutions to the equation $\partial_t u = 2\pi i P$ travel along geodesics of the Finsler metric.

Just as in the Riemannian case, one can also obtain a second variation formula involving a theory of Jacobi fields along geodesics, but this takes us too far afield. Since we will not use the second variation formula in what follows, we simply assume one of it's consequences, that geodesics are length minimizing up until the development of cut points on a manifold, which is at least as big as the *injectivity radius* of the manifold, i.e. the minimum time at which the geodesic flow defined by the Hamiltonian equations above fails to be injective from a point on the manifold.

3.3 Fourier Integral Operator Techniques

For any operator P to which we can establish a robust functional calculus, and for any regulated function a, the identity

$$a(P) = \int_{-\infty}^{\infty} \widehat{a}(t)e^{2\pi i t P} \tag{3.3.1}$$

holds, obtained by plugging P into the usual Fourier inversion formula. As t varies, the operators $e^{2\pi i t P}$ solve the wave equation $\partial_t u = 2\pi i P u$. Thus we can study multipliers of

P in terms of the wave equation $\partial_t u = 2\pi i P u$. For a general operator P, it is difficult to understand the behaviour of the operator $e^{2\pi i t P}$ given it's abstract definition via the functional calculus. However if P is elliptic, then $e^{2\pi i t P}$ is a *Fourier integral operator*, which roughly means that the kernel of the operator is well approximated by oscillatory integral representation *for high frequency inputs*. In this section, we briefly describe the relevant components we will need from the theory of Fourier integral operators for work in the latter part of the thesis.

We will require the barest notions of microlocal analysis, i.e. the theory of decomposition of functions in space and frequency simultaneously (a decomposition in *phase space*). Temporarily define a *conical multiplier at a point* $(x_0, \xi_0) \in T^* \mathbb{R}^d$ to be an operator of the form

$$(\Psi_{x_0,\xi_0} f)(x) = \int \phi(\xi) e^{i\xi\cdot(x-y)} \psi(y) f(y)\, dy\, d\xi, \tag{3.3.2}$$

where ϕ, ψ are smooth functions, with ϕ homogeneous of order zero, equal to one in a neighborhood of ξ_0, and ψ with compact support equal to one in a neighborhood of x_0. Such an operator is precisely a composition of a Fourier multiplier that localized it's input in a particular frequency direction, with an operator that takes a smooth cutoff of a function. We will say such a multiplier is *microlocally supported* on $\text{supp}(\psi) \times \text{supp}(\phi)$.

To motivate the definition of Fourier integral operators, consider an operator A from $\mathbb{R}^m$ to $\mathbb{R}^n$, whose kernel is a distribution given by an oscillatory integral of the form

$$A(x, y) = \int_{\mathbb{R}^p} s(x, y, \theta) e^{i\Phi(x,y,\theta)}\, d\theta, \tag{3.3.3}$$

where s is a symbol of some fixed order, and Φ is smooth, and homogeneous of order one in the θ variable. Define the *canonical relation* of A, the set

$$C_\Phi = \Big\{ (x, \nabla_x \Phi(x, y, \theta), y, -\nabla_y \Phi(x, y, \theta)) : \nabla_\theta \Phi(x, y, \theta) = 0 \Big\}, \tag{3.3.4}$$

Under the assumption that the vectors $\partial_{\theta_1}\{\nabla_{x,y,\xi}\Phi\}, \dots \partial_{\theta_N}\{\nabla_{x,y,\xi}\Phi\}$ are everywhere linearly independent when $\nabla_\theta \Phi = 0$, C_Φ is a smooth, $n + m$ dimensional manifold*. Such a phase Φ is called *nondegenerate*, and we will only deal with such phases in the sequel. If $(x_0, \xi_0, y_0, \eta_0) \notin C_\Phi$, then for conical multipliers Ψ_{x_0,ξ_0} and Ψ_{y_0,η_0} of suitably small support, integration by parts justifies that the Fourier transform of $\Psi_{x_0,\xi_0} \circ A \circ \Psi_{y_0,\eta_0}$ is rapidly decaying, and thus the kernel of the operator $\Psi_{x_0,\xi_0} \circ A \circ \Psi_{y_0,\eta_0}$ is smooth. The canonical relation of an oscillatory integral operator thus tells us pairs of points in phase space the location where high-frequency wave packets are mapped in phase space. The main lesson of the theory of Fourier integral operators is that, to a large extent, the canonical relation *determines* the behaviour of the operator A, rather than other features of the phase Φ.

A Fourier integral operator is precisely an operator whose kernel is microlocally expressible in oscillatory integrals of the form above. A *Fourier integral operator* of order μ from $\mathbb{R}^m$ to $\mathbb{R}^n$ associated with a $n + m$ dimensional 'Lagrangian' submanifold C of

*It is actually a *Lagrangian submanifold* of $T^* \mathbb{R}^n \times T^* \mathbb{R}^m$. Thus the canonical relation of any Fourier integral operator must be a Lagrangian submanifold, though we will only use this fact implicitly

$T^*\mathbb{R}^n \times T^*\mathbb{R}^m$ is an operator A such that for each $(x_0, \xi_0) \in T^*\mathbb{R}^n$ and $(y_0, \eta_0) \in T^*\mathbb{R}^m$, there exists conical multipliers Ψ_{x_0,ξ_0} and Ψ_{y_0,η_0} at (x_0, ξ_0) and at (y_0, η_0) such that the operator $\Psi_{x_0,\xi_0} \circ A \circ \Psi_{y_0,\eta_0}$ is expressed as an oscillatory integral operator of the kind studied in the previous paragraph, with an amplitude given by a symbol of order $\nu + p/2 - (n+m)/4$.

A Fourier integral operator from an m-dimensional manifold Y to an n-dimensional manifold X associated with an $n+m$ dimensional Lagrangian submanifold C of $T^*X \times T^*Y$ is precisely an operator which, when localized in any pair of coordinate systems for X and Y, is a Fourier integral operator of the form above. The manifold C is truly a geometric invariant of the operator A, because it is in direct correspondence to the *wavefront set* of the operator A, which gives the location and direction of singularities of the kernel of the operator. The *equivalence of phase theorem* for Fourier integral operators tells us that the particular phase used to define Fourier integral operators is largely irrelevant to the analysis of the operator, as long as the canonical relation is shared by the phase.

Theorem 3.9. *Let A be a Fourier integral operator of order ν between two manifolds Y^m and X^n, associated with the canonical relation C. Localize A around $x_0 \in X$ and $y_0 \in Y$, so we may consider the operator as a Fourier integral operator from $\mathbb{R}^m$ to $\mathbb{R}^n$. Fix $(x_0, \xi_0, y_0, \eta_0) \in C$, and consider two conical multipliers Ψ_{x_0,ξ_0} and Ψ_{y_0,η_0} microlocally supported on two sets $\Gamma_{x_0,\xi_0} \subset T^*\mathbb{R}^n$ and $\Gamma_{y_0,\eta_0} \subset T^*\mathbb{R}^m$. Consider $p \geq 0$, and a non-degenerate phase $\Phi : \Gamma_{x_0,\xi_0} \times \Gamma_{y_0,\eta_0} \times \mathbb{R}^p$ with $C_\Phi = C \cap (\Gamma_{x_0,\xi_0} \times \Gamma_{y_0,\eta_0})$. Then there exists a symbol s of order $\nu + p/2 - (n+m)/4$, such that*

$$\Psi_{x_0,\xi_0} \circ A \circ \Psi_{y_0,\eta_0} = \int s(\cdot, \cdot, \theta) e^{i\Phi(x,y,\theta)}\, d\theta + C^\infty(\mathbb{R}^n \times \mathbb{R}^m),$$

i.e. $\Psi_{x_0,\xi_0} \circ A \circ \Psi_{y_0,\eta_0}$ differs from the oscillatory integral operator on the right hand side by a smoothing operator.

Proof. See Theorem TODO of [11]. □

The simplest class of Fourier integral operators are the *pseudodifferential operators*. They are precisely the Fourier integral operators from a manifold X to itself, whose canonical relation is the diagonal $\Delta_X = \{(x, \xi, x, \xi) : (x, \xi) \in T^*X\}$. In any coordinate system, and for any diffeomorphism $\phi : \mathbb{R}^d \to \mathbb{R}^d$, the phase function $\Phi(x, y, \xi) = \xi \cdot (\phi(x) - \phi(y))$ parameterizes Δ_X, and so since $p = d$, $n = d$, and $m = d$, any pseudodifferential operator of order ν on $\mathbb{R}^d$ can be written, modulo a smooth moperator, as an oscillatory integral operator of the form

$$\int s(x, y, \xi) e^{i\xi \cdot (\phi(x) - \phi(y))}\, d\xi, \tag{3.3.5}$$

where s is a symbol of order ν.

For our purposes, the most important Fourier integral operators are associated with solutions to wave equations on manifolds.

Theorem 3.10. *Let P be an elliptic operator on a manifold X satisfying Assumption A, with principal symbol $p : T^*X \to \mathbb{R}$. Let $\{\alpha_t\}$ be the (co) geodesic flow on T^*X given by* *the Finsler metric on X induced by p. Define the canonical relation $C \subset T^*(X \times \mathbb{R}) \times T^*X$ by settting*

$$C = \{(x, t, \xi, \tau, x', \xi') : (x, \xi) = \alpha_t(x', \xi') \quad \text{and} \quad \tau = p(x', \xi')\}.$$

Then the solution operator $Af(x,t) = e^{2\pi itP} f(x)$ from X to $X \times \mathbb{R}$, which takes an initial condition f on X to a solution u to the wave equation $\partial_t u = 2\pi iPu$ is a Fourier integral operator of order $1/4$ associated with the canonical relation C. For each fixed t, the operator $e^{2\pi itP}$ is a Fourier integral operator of order 0 associated with the canonical relation

$$C_t = \{(x, \xi; x', \xi') : (x, \xi) = \alpha_t(x', \xi')\}.$$

Proof. It suffices to construct a parametrix for the half-wave equation which is given by oscillatory integrals that fit into the canonical relations above. One can find such parametrices in many sources, such as TODO of [11] or TODO of [20]. □

Given this theorem, the equivalence of phase theorem implies multiple useful oscillatory integral representations for the wave propogators $e^{2\pi itP}$.

- Consider a smooth hypersurface Σ of $T^*X \times X$, such that for each $(x, \xi) \in T^*X$, the slice $\Sigma_{x,\xi} = \{x' : (x, \xi, x') \in \Sigma\}$ is a smooth hypersurface in X passing through (x, ξ) conormal to ξ. The theory of Hamilton-Jacobi equations implies the local existence of a homogeneous function ϕ satisfying the *Eikonal equation*

$$|\nabla\phi(x, x', \xi)| = p(x', \xi) \tag{3.3.6}$$

  and such that for $x' \in \Sigma_{x,\xi}$, $\phi(x, x', \xi) = 0$ and $\nabla\phi(x, x', \xi)$ is conormal to $\Sigma_{x,\xi}$. If we define $\Phi(x, t, x', \xi) = \phi(x, x', \xi) + tp(x, \xi)$, then C_Φ agrees with the canonical relation C for the solution operator A, and so there exists a symbol s of order zero such that, in coordinates, for x' in a neighborhood of x, and suitably small t,

$$Af(x, t, x') = e^{2\pi itP} f(x) = \int s(x, t, x', \xi) e^{i[\phi(x,x',\xi) + tp(x,\xi)]} \, d\xi. \tag{3.3.7}$$

- Suppose that $t > 0$ is smaller than the smallest cut time on the manifold X with respect to it's Finsler metric. Then the phase $\Phi(x, x', t, \tau) = \tau(d_+(x, x') - t)$ has canonical relation agreeing with C_t, and so there exists a symbol s of order $-\frac{d-1}{2}$ such that

$$Af(x, t, x') = e^{2\pi itP} f(x) = \int s(x, t, x', \tau) e^{i\tau(d_+(x,x') - t)} \, d\tau. \tag{3.3.8}$$

  The problem with this oscillatory integral representation is that it begins to break down as $t \to 0$, and as t approaches times at which cut points develop on X.

- Because of the semigroup property $C_t \circ C_s = C_{t+s}$, we can obtain oscillatory integral representations of $e^{2\pi itP}$ by composing the oscillatory integral representations here. We do not carry out the details, because as $t \to \infty$ the number of compositions required grows linearly, which means the oscillatory integrals are harder and harder to control in a uniform way as $t \to \infty$, and we will not use such compositions in our arguments.

We will primarily use the first and second representations in what follows.

3.4 Periodic Geodesics and Assumption B

Using the Fourier integral operator methods we have described, we can now discuss the relation of Assumption B to the geometry of the manifold X induced by the operator P. Suppose that all the eigenvalues of P occur in an arithmetic progression of the form $\{a + nb\}$. Then simply by the functional calculus, we know that $e^{2\pi i P/b} = e^{2\pi i a/b} I$. In particular, it follows that the canonical relation of the operator $e^{2\pi i P/b}$ must be equal to the canonical relation of the identity, which is the diagonal $\Delta_X \subset T^*X \times T^*X$. But Theorem 3.10 says that the canonical relation is equal to $\{(x, \xi; x', \xi') : (x, \xi) = \alpha_{1/b}(x', \xi')\}$, and so it follows that $\alpha_{1/b}$ is the identity map; thus all geodesics are closed, and their length is equal to $1/bn$ for some integer n.

The converse is *almost* true. Suppose that all the geodesics of the Finsler geometry on X are closed, and their length is equal to $1/bn$ for some integer n. Then it follows that the canonical relation of the operator $e^{2\pi i P/b}$ is equal to Δ_X, and thus this operator is a pseudodifferential operator of order zero. With slightly more work, one can show that the principal symbol of the operator $e^{2\pi i P/b}$ is equal to $e^{2\pi i a}$, where a is the *Maslov index* of the geodesic flow on X (see e.g. BLEH). We can then use the fact that $e^{2\pi i P/b} - e^{2\pi i a} I$ is a pseudodifferential operator of order -1 to control the eigenvalues of the operator P; namely, if λ is an eigenvalue of P, and e_λ is an L^2 normalized eigenfunction, then

$$\lambda |e^{2\pi i \lambda/b} - e^{2\pi i a}| = \|(e^{2\pi i P/b} - e^{2\pi i a}) e_\lambda\|_{H^1(X)} \lesssim \|e_\lambda\|_{L^2(X)} = 1, \tag{3.4.1}$$

and so $|e^{2\pi i \lambda/b} - e^{2\pi i a}| \leq 1/\lambda$, which implies λ clusters near the arithmetic progression $\{a + nb\}$. In what follows, we need the fact that the wave equation is *completely periodic* in order to control the wave equation for large times, but I hope to address the more general case in future work.

3.5 Relation to Previous Work

TODO.

Part II

New Results

Chapter 4

The Boundedness of Spectral Multipliers with Compact Support

In this chapter, we prove Theorem 1.2, which we restate below:

Theorem 1.2. *Suppose X is a manifold, and P is an elliptic operator on X satisfying Assumption A and Assumption B. Then for $1/p - 1/2 > 1/(d-1)$, if $s = (d-1)(1/p - 1/2)$, then for any regulated a with $supp(a) \subset [1/2, 2]$,*

$$\|a\|_{M^p_{Dil}(X)} \sim \|a\|_{R^{s,p}[0,\infty)}.$$

In Section 3.1 we saw that for operators P satisfying Assumption A, $\|a\|_{M^p_{\text{Dil}}(X)} \gtrsim \|a\|_{R^{s,p}[0,\infty)}$, and so it suffices to prove that $\|a\|_{M^p_{\text{Dil}}(X)} \lesssim \|a\|_{R^{s,p}[0,\infty)}$. In Section 4.1, we will see from elementary functional analysis that

$$\sup_{R \leq 1} \|a_R\|_{M^p(X)} \lesssim \|a\|_{l^\infty[0,\infty)} \lesssim \|a\|_{R^{s,p}[0,\infty)}. \tag{4.0.1}$$

On the other hand, the upper bound

$$\sup_{R \geq 1} \|a_R\|_{M^p(X)} \lesssim \|a\|_{R^{s,p}[0,\infty)}. \tag{4.0.2}$$

requires a more in depth analysis than (4.0.1). It is here we apply the wave equation transform, writing

$$a_R(P) = \int_{-\infty}^{\infty} R\widehat{a}(Rt) e^{2\pi i t P}\, dt, \tag{4.0.3}$$

which reduces $a_R(P)$ to studying averages associated with solutions to the half-wave equation $\partial_t = 2\pi i P$ on M. We obtain (4.0.2) by studying averages to the wave equation using several methods, including:

(A) Quasi-orthogonality estimates for averages of solutions to the half-wave equation on M, discussed in Section 4.2, which arise from a study of the geometry of the Finsler metric on X.

(B) Variants of the density-decomposition arguments, which we introduced in Section 2.3, to control the 'small time behavior' of solutions to the half-wave equation. We carry out these arguments in section 4.3

(C) A new strategy to reduce the 'large time behavior' of the half-wave equation to an endpoint local smoothing inequality for the half-wave equation on M, described in Section 4.4.

Equations (4.0.1) and (4.0.2) thus imply Theorem 1.2.

4.1 Preliminary Setup

Fix some geometric constant $\varepsilon_X \in (0,1)$, matching the constant given in the statement of Proposition 4.3. Our goal is to prove inequalities (4.0.1) and (4.0.2). Proving (4.0.1) is simple because the operators $a_R(P)$ are smoothing operators, uniformly for $0 < R < 1$, because the spectrum of X is discrete, low eigenvalue eigenfunctions are uniformly smooth, and X is compact.

Lemma 4.1. *Let X be a compact d-dimensional manifold, and suppose P is an elliptic operator on X. If $1/2d < 1/p - 1/2 \leq 1/2$, and if a is a regulated function, then*

$$\sup_{R\leq 1} \|a_R\|_{M^p_{Dil}(X)} \lesssim \|a\|_{l^\infty(\Lambda)} \lesssim \|a\|_{R^{p,s}[0,\infty)}.$$

Proof. Let $T_R = m_R(P)$. Recall that Λ is the set of eigenvalues of P. The set $\Lambda \cap [0,2]$ is finite. For each $\lambda \in \Lambda$, choose a finite orthonormal basis $\mathcal{E}_\lambda$. Then we can write

$$T_R = \sum_{\lambda\in\Lambda} \sum_{e\in\mathcal{E}_\lambda} \langle f, e\rangle e. \tag{4.1.1}$$

Since $\mathcal{V}_\lambda \subset C^\infty(M)$, Hölder's inequality implies

$$\|\langle f, e\rangle e\|_{L^p(M)} \leq \|f\|_{L^p(M)} \|e\|_{L^{p'}(M)} \|e\|_{L^p(M)} \lesssim_\lambda \|f\|_{L^p(M)}. \tag{4.1.2}$$

But this means that

$$\|T_R f\|_{L^p(M)} \leq \sum_{\lambda\in\Lambda_P\cap[0,2]} \sum_{e\in\mathcal{E}_\lambda} |m(\lambda/R)| \|\langle f, e\rangle e\|_{L^p(M)} \lesssim \|m\|_{l^\infty(\Lambda)}. \tag{4.1.3}$$

For $1/p - 1/2 > 1/2d$, the Sobolev embedding theorem and (1.0.9) imply that

$$\|a\|_{l^\infty(\Lambda)} \lesssim \|a\|_{W^{s,p'}[0,\infty)} \lesssim \|a\|_{R^{s,p}[0,\infty)}, \tag{4.1.4}$$

which completes the proof. ◻

We now begin to reduce the analysis of $a_R(P)$ for $R \geq 1$ to the study of the wave equation. Fix a bump function $q \in C_c^\infty(\mathbb{R})$ with $\text{supp}(q) \subset [1/4, 4]$ and $q(\lambda) = 1$ for $\lambda \in [1/2, 2]$, and define $Q_R = q(P/R)$. We write

$$T_R = Q_R \circ T_R \circ Q_R = \int_{\mathbb{R}} R\widehat{m}(Rt)(Q_R \circ e^{2\pi i t P} \circ Q_R)\, dt, \tag{4.1.5}$$

and view the operators $(Q_R \circ e^{2\pi itP} \circ Q_R)$ as 'frequency localized' wave propagators.

Using Assumption B, replacing the operator P by $aP + b$ for appropriate $a, b \in \mathbb{R}$ if necessary, we may assume without loss of generality that all eigenvalues of P are integers. It follows that $e^{2\pi i(t+n)P} = e^{2\pi itP}$ for any $t \in \mathbb{R}$ and $n \in \mathbb{Z}$. Let I_0 denote the interval $[-1/2, 1/2]$. We may then write

$$T_R = \int_{I_0} b_R(t)(Q_R \circ e^{2\pi itP} \circ Q_R)\, dt, \tag{4.1.6}$$

where $b_R : I_0 \to \mathbb{C}$ is the periodic function

$$b_R(t) = \sum\nolimits_{n\in\mathbb{Z}} R\widehat{m}(R(t+n)). \tag{4.1.7}$$

We split our analysis of T_R into two regimes: regime I and regime II. In regime I, we analyze the behaviour of the wave equation over times $0 \leq |t| \leq \varepsilon_X$ by decomposing this time interval into length $1/R$ pieces, and analyzing the interactions of the wave equations between the different intervals. In regime II, we analyze the behaviour of the wave equation over times $\varepsilon_X \leq |t| \leq 1$. Here we need not perform such a decomposition, since the $R^{s,p}$ norm gives better control on the function b_R over these times.

Lemma 4.2. *Fix $\varepsilon > 0$. Let $\mathcal{T}_R = \mathbb{Z}/R \cap [-\varepsilon, \varepsilon]$ and define $I_t = [t - 1/R, t + 1/R]$. For any function $a : [0, \infty) \to \mathbb{C}$, define a periodic function $b : I_0 \to \mathbb{C}$ by setting*

$$b(t) = \sum\nolimits_{n\in\mathbb{Z}} R\widehat{m}(R(t+n)).$$

Then we can write $b = \left(\sum_{t_0\in\mathcal{T}_R} b^I_{t_0}\right) + b^{II}$, where

$$supp(b^I_{t_0}) \subset I_{t_0} \quad and \quad supp(b^{II}_R) \subset I_0 \setminus [-\varepsilon, \varepsilon].$$

Moreover, we have

$$\left(\sum\nolimits_{t_0\in\mathcal{T}_R} \left[\|b^I_{t_0}\|_{L^p(I_0)} \langle Rt_0\rangle^{\alpha(p)}\right]^p\right)^{1/p} \lesssim R^{1/p'} \|a\|_{R^{s,p}[0,\infty)}$$

and

$$\|b^{II}\|_{L^p(I_0)} \lesssim R^{1/p' - (d-1)(1/p-1/2)} \|a\|_{R^{s,p}[0,\infty)}.$$

Proof of Lemma 4.2. The intervals $\{I_{t_0} : t_0 \in \mathcal{T}_R\}$ cover $[-\varepsilon, \varepsilon]$, and so we may consider an associated partition $\mathbb{I}_{[-\varepsilon,\varepsilon]} = \sum_{t_0} \chi_{t_0}$ where $\text{supp}(\chi_{t_0}) \subset I_{t_0}$ and $|\chi_{t_0}| \leq 1$. Define $b^I_{t_0} = \chi_{t_0} b_j$ and $b^{II} = (1 - \mathbb{I}_{[-\varepsilon,\varepsilon]})b$. Then $b = \sum_{t_0} b^I_{t_0} + b^{II}$, and the support assumptions are satisfied. It remains to prove the required norm bounds for these choices. For each $n \in \mathbb{Z}$, define a function $b_n : I_0 \to \mathbb{C}$ by setting $b_n(t) = R\widehat{m}(R(t+n))$. Then $b = \sum_n b_n$. Moreover,

$$\begin{aligned}
&\left(\sum_{n\neq 0} \left[\langle Rn\rangle^{\alpha(p)} \|b_n\|_{L^p(I_0)}\right]^p\right)^{1/p} \\
&\quad \sim \left(\int_{|t|\geq 1/2} \left[\langle Rt\rangle^{\alpha(p)} |R\widehat{m}(Rt)|\right]^p\right)^{1/p} \\
&\quad = R^{1/p'} \left(\int_{|t|\geq R/2} \left[|t|^{\alpha(p)} \widehat{m}(t)\right]^p\right)^{1/p} \leq R^{1/p'} C_p(m).
\end{aligned} \tag{4.1.8}$$

Write $b^I_{t_0} = \sum_n b^I_{t_0,n}$ and $b^{II} = \sum_n b^{II}_n$, where $b^I_{t_0,n} = \chi_{t_0} b_n$ and $b^{II}_n = \mathbb{I}_{I_0 \setminus [-\varepsilon,\varepsilon]} b_n$. Then

$$\begin{aligned} \|b^{II}_0\|_{L^p(I_0)} &= \left(\int_{\varepsilon \le |t| \le 1/2} |R\widehat{m}(Rt)|^p \right)^{1/p} \\ &= R^{1/p'} \left(\int_{R\varepsilon \le |t| \le R/2} |\widehat{m}(t)|^p \right)^{1/p} \lesssim R^{1/p' - \alpha(p)} C_p(m). \end{aligned} \tag{4.1.9}$$

Using (4.1.8), (4.1.9), and Hölder's inequality, we conclude that

$$\begin{aligned} \|b^{II}\|_{L^p(I_0)} &\le \sum\nolimits_n \|b^{II}_n\|_{L^p(I_0)} \\ &\le \|b^{II}_0\|_{L^p(I_0)} + \sum\nolimits_{n \ne 0} \left[|Rn|^{\alpha(p)} \|b^{II}_n\|_{L^p(I_0)} \right] \frac{1}{|Rn|^{\alpha(p)}} \\ &\le \|b^{II}_0\|_{L^p(I_0)} + R^{-\alpha(p)} \left(\sum\nolimits_{n \ne 0} \left[|Rn|^{\alpha(p)} \|b_n\|_{L^p(I_0)} \right]^p \right)^{1/p} \\ &\lesssim R^{1/p' - \alpha(p)} C_p(m). \end{aligned} \tag{4.1.10}$$

A similar calculation shows that

$$\begin{aligned} \|b^I_{t_0}\|_{L^p(I_0)} &\le \sum\nolimits_n \|b^I_{t_0,n}\|_{L^p(I_0)} \\ &= \|b^I_{t_0,0}\|_{L^p(I_0)} + \sum\nolimits_{n \ne 0} \|b^I_{t_0,n}\|_{L^p(I_0)} \\ &\lesssim \|b^I_{t_0,0}\|_{L^p(I_0)} + R^{-\alpha(p)} \Big(\sum\nolimits_{n \ne 0} |Rn|^{\alpha(p)} \|b^I_{t_0,n}\|^p_{L^p(I_0)} \Big)^{1/p} \\ &\lesssim \|b^I_{t_0,0}\|_{L^p(I_0)} + \Big(\sum\nolimits_{n \ne 0} |Rn|^{\alpha(p)} \|b^I_{t_0,n}\|^p_{L^p(I_0)} \Big)^{1/p}. \end{aligned} \tag{4.1.11}$$

Using (4.1.11), we calculate that

$$\begin{aligned} &\left(\sum_{t_0 \in \mathcal{T}_R} \left[\|b^I_{t_0}\|_{L^p(I_0)} \langle Rt_0 \rangle^{\alpha(p)} \right]^p \right)^{1/p} \\ &\quad \lesssim \left(\sum_{t_0 \in \mathcal{T}_R} \left[\|b^I_{t_0,0}\|_{L^p(I_0)} \langle Rt_0 \rangle^{\alpha(p)} \right]^p + \sum_{n \ne 0} \left[|Rn|^{\alpha(p)} \|b^I_{t_0,n}\|_{L^p(I_0)} \right]^p \right)^{1/p} \\ &\quad \lesssim \left(\int_{\mathbb{R}} \left[\langle Rt \rangle^{\alpha(p)} R\widehat{m}(Rt) \right]^p dt \right)^{1/p} \\ &\quad \lesssim R^{1/p'} C_p(m). \end{aligned} \tag{4.1.12}$$

Since each function $b^I_{t_0}$ is supported on a length $1/R$ interval, we have

$$\|b^I_{t_0}\|_{L^1(I_0)} \lesssim R^{-1/p'} \|b^I_{t_0}\|_{L^p(I_0)}, \tag{4.1.13}$$

and substituting this inequality into (4.1.12) completes the proof. □

The following proposition, a kind of L^p square root cancellation bound, implies (4.0.2) once we take Lemma 4.2 into account. Since we already proved (4.0.1), proving this proposition completes the proof of Theorem 1.2, and will take the remainder of the chapter.

Proposition 4.3. *Let P be a classical elliptic self-adjoint pseudodifferential operator of order one on a compact manifold M of dimension d satisfying the assumptions of Theorem 1.2. Fix $R > 0$ and suppose $1/(d-1) < 1/p - 1/2 < 1/2$. Then there exists $\varepsilon_X > 0$ with the following property. Consider any function $b : I_0 \to \mathbb{C}$, and suppose we can write $b = \sum_{t_0 \in \mathcal{T}_R} b^I_{t_0} + b^{II}$, where $supp(b^I_{t_0}) \subset I_{t_0}$ and $supp(b^{II}_R) \subset I_0 \setminus [-\varepsilon_X, \varepsilon_X]$. Define operators $T^I = \sum_{t_0 \in \mathcal{T}_R} T^I_{t_0}$ and T^{II}, where*

$$T^I_{t_0} = \int b^I_{t_0}(t)(Q_R \circ e^{2\pi i t P} \circ Q_R)\, dt \;\text{ and }\; T^{II} = \int b^{II}(t)(Q_R \circ e^{2\pi i t P} \circ Q_R)\, dt.$$

Then

$$\|T^I\|_{L^p \to L^p} \lesssim R^{-1/p'} \left(\sum_{t_0 \in \mathcal{T}_R} \left[\|b^I_{t_0}\|_{L^p(I_0)} \langle R t_0 \rangle^{\alpha(p)} \right]^p \right)^{1/p} \tag{4.1.14}$$

and

$$\|T^{II}\|_{L^p \to L^p} \lesssim R^{\alpha(p) - 1/p'} \|b^{II}\|_{L^p(I_0)}. \tag{4.1.15}$$

Remark. One should view this proposition as a discretization of Theorem 1.2, and thus Proposition 4.3 should be compared to Lemma 2.5 in the argument of Heo, Nazarov, and Seeger.

Proposition 4.3 splits the main bound of the paper into two regimes: regime I and regime II. Noting that we require weaker bounds in (4.1.15) than in (4.1.14), the operator T^{II} will not require as refined an analysis as for the operator T^I, and we obtain bounds on T^{II} by a reduction to an endpoint local smoothing inequality in Section 4.4. On the other hand, to obtain more refined estimates for the L^p norms of quantities of the form $f = T^I u$, we consider a decompositions of the form $u = \sum_{x_0} u_{x_0}$, where $u_{x_0} : M \to \mathbb{C}$ is supported on a ball $B(x, 1/R)$ of radius $1/R$ centered at x_0. We then have $f = \sum_{(x_0,t_0)} f_{x_0,t_0}$, where $f_{x_0,t_0} = T^I_{t_0} u_{x_0}$. To control f we must establish an L^p square root cancellation bound for the functions $\{f_{x_0,t_0}\}$. In the next section, we study the L^2 quasi-orthogonality of these functions, which we use as a starting point to obtain the required square root cancellation.

4.2 Quasi-Orthogonality Estimates For Wave-Packets

The discussion at the end of Section 4.1 motivates us to consider estimates for functions obtained by taking averages of the wave equation over a small time interval, with initial conditions localized to a particular part of space. In this section, we study the L^2 orthogonality of such quantities. One should compare the results of this section to the results of Lemma 2.10. We do not exploit periodicity of the Hamiltonian flow in this section since we are only dealing with estimates for the half wave-equation for *small times*. Our results here thus hold for any manifold M, and any operator P whose principal symbol has cospheres with non-vanishing Gaussian curvature. In order to prove the required quasi-orthogonality estimates of the functions $\{f_{x_0,t_0}\}$, we find a new connection between the Finsler geometry we introduced in section 3.2 and the behaviour of the operator P. In particular, recall the quasimetrics d_+ and d_- introduced in that section.

Proposition 4.4. *Let M be a compact manifold of dimension d, and let P be a classical elliptic self-adjoint pseudodifferential operator of order one whose principal symbol satisfies the curvature assumptions of Theorem 1.2. Then there exists $\varepsilon_X > 0$ such that for all $R \geq 0$, the following estimates hold:*

- *(Pointwise Estimates) Fix $|t_0| \leq \varepsilon_X$ and $x_0 \in M$. Consider any two measurable functions $c : \mathbb{R} \to \mathbb{C}$ and $u : M \to \mathbb{C}$, with $\|c\|_{L^1(\mathbb{R})} \leq 1$ and $\|u\|_{L^1(M)} \leq 1$, and with $supp(c) \subset I_{t_0}$ and $supp(u) \subset B(x_0, 1/R)$. Define $S : M \to \mathbb{C}$ by setting*

$$S = \int c(t)(Q_R \circ e^{2\pi i t P} \circ Q_R)\{u\}\, dt.$$

  Then for any $K \geq 0$, and any $x \in M$,

$$|S(x)| \lesssim_K \frac{R^d}{\langle R d_X(x_0, x)\rangle^{\frac{d-1}{2}}} \max_\pm \left\langle R\left|t_0 \pm d_X^\pm(x, x_0)\right|\right\rangle^{-K}.$$

- *(Quasi-Orthogonality Estimates) Fix $|t_0 - t_1| \leq \varepsilon_X$, and $x_0, x_1 \in M$. Consider any two pairs of functions $c_0, c_1 : \mathbb{R} \to \mathbb{C}$ and $u_0, u_1 : M \to \mathbb{C}$ such that, for each $\nu \in \{0,1\}$, $\|c_\nu\|_{L^1(\mathbb{R})} \leq 1$, $\|u_\nu\|_{L^1(M)} \leq 1$, $supp(c_\nu) \subset I_{t_\nu}$, and $supp(u_\nu) \subset B(x_\nu, 1/R)$. Define $S_\nu : M \to \mathbb{C}$ by setting*

$$S_\nu = \int c_\nu(t)(Q_R \circ e^{2\pi i t P} \circ Q_R)\{u_\nu\}\, dt.$$

  Then for any $K \geq 0$,

$$|\langle S_0, S_1\rangle| \lesssim_K \frac{R^d}{\langle R d_X(x_0, x_1)\rangle^{\frac{d-1}{2}}} \max_\pm \left\langle R\left|(t_0 - t_1) \pm d^\pm(x_0, x_1)\right|\right\rangle^{-K}.$$

Remark. Just as in Lemma 2.10, the pointwise estimate of Lemma 4.4 tells us that the function S is concentrated on a geodesic annulus of radius $|t_0|$ centered at x_0 and thickness $O(1/R)$. The quasi-orthogonality estimate tells us that the two functions S_0 and S_1 are only significantly correlated with one another if the two annuli on which the majority of the support of S_0 and S_1 lie are internally or externally tangent to one another, depending on whether t_0 and t_1 have the same or opposite sign respectively.

Proof of Proposition 4.4. To simplify notation, in the following proof we will suppress the use of R as an index, for instance, writing Q for Q_R. For both the pointwise and quasi-orthogonality estimates, we want to consider the operators in coordinates, so we can use Theorem 3.9 to understand the wave propagators in terms of various oscillatory integrals.

Start by covering M by a finite family of suitably small open sets $\{V_\alpha\}$, such that for each α, there is a coordinate chart U_α compactly containing V_α and with $N(V_\alpha, 1.1\varepsilon_X) \subset U_\alpha$. Let $\{\eta_\alpha\}$ be a partition of unity subordinate to $\{V_\alpha\}$. It will be convenient to define $V_\alpha^* = N(V_\alpha, 0.01\varepsilon_X)$ for each α. The next lemma allows us to approximate the operator Q, and the propagators $e^{2\pi i t P}$ with operators which have more explicit representations in the coordinate system $\{U_\alpha\}$, with an error negligible to the results of Proposition 4.4.

Lemma 4.5. *Suppose ε_X is suitably small, depending on M and P. For each α, and $|t| \leq \varepsilon_X$, there exists Schwartz operators Q_α and $W_\alpha(t)$, each with kernels supported on $U_\alpha \times V_\alpha^*$, such that the following holds:*

- *For any $u : M \to \mathbb{C}$ with $\|u\|_{L^1(M)} \leq 1$ and supp$(u) \subset V_\alpha^*$,*
$$\mathit{supp}(Q_\alpha u) \subset N(\mathit{supp}(u), 0.01\varepsilon_X),$$
$$\mathit{supp}(W_\alpha(t)u) \subset N(\mathit{supp}(u), 1.01\varepsilon_X),$$
$$\|(Q - Q_\alpha)u\|_{L^\infty(M)} \lesssim_N R^{-N} \quad \text{for all } N \geq 0,$$
and
$$\left\|(Q_\alpha \circ (e^{2\pi i t P} - W_\alpha(t)) \circ Q_\alpha)\{u\}\right\|_{L^\infty(M)} \lesssim_N R^{-N} \quad \text{for all } N \geq 0.$$
- *In the coordinate system of U_α, the operator Q_α is a pseudo-differential operator of order zero given by a symbol $\sigma_\alpha(x,\xi)$, where*
$$\mathit{supp}(\sigma_\alpha) \subset \{\xi \in \mathbb{R}^d : R/8 \leq |\xi| \leq 8R\},$$
and σ_α satisfies derivative estimates of the form
$$|\partial_x^\lambda \partial_\xi^\kappa \sigma_\alpha(x,\xi)| \lesssim_{\lambda,\kappa} R^{-|\kappa|}.$$
- *In the coordinate system U_α, the operator $W_\alpha(t)$ has a kernel $W_\alpha(t,x,y)$ with an oscillatory integral representation*
$$W_\alpha(t,x,y) = \int s(t,x,y,\xi) e^{2\pi i [\phi(x,y,\xi) + t p(y,\xi)]}\, d\xi,$$
where the function s is compactly supported in U_α, such that
$$\mathit{supp}_{t,x,y}(s) \subset \{(t,x,y) : |x-y| \leq C|t|\}$$
for some constant $C > 0$, such that
$$\mathit{supp}_\xi(s) \subset \{\xi \in \mathbb{R}^d : R/8 \leq |\xi| \leq 8R\},$$
and such that the function s satisfies derivative estimates of the form
$$|\partial_{t,x,y}^\lambda \partial_\xi^\kappa s| \lesssim_{\lambda,\kappa} R^{-|\kappa|}.$$
The phase function ϕ is smooth and homogeneous of degree one in the ξ variable, and solves the eikonal equation
$$p\,(x, \nabla_x \phi(x,y,\xi)) = p(y,\xi),$$
subject to the constraint that $\phi(x,y,\xi) = 0$ if $\xi \cdot (x-y) = 0$.

Proof. For each α, given our choice of ε_X, Theorems 3.9 and 3.10 guarantee that we can find operators $\tilde{W}_\alpha(t)$ and $\tilde{R}_\alpha(t)$ for $|t| \leq \varepsilon_X$, such that for $u \in L^1(M)$ with $\operatorname{supp}(u) \subset V_\alpha^*$,

$$e^{2\pi i t P} u = \tilde{W}_\alpha(t) u + \tilde{R}_\alpha(t) u, \tag{4.2.1}$$

where $\tilde{R}_\alpha(t)$ has a smooth kernel, and the kernel of $\tilde{W}_\alpha(t)$ is given in coordinates by

$$\tilde{W}_\alpha(t)(x,y) = \int s_0(t,x,y,\xi) e^{2\pi i [\phi(x,y,\xi) + t p(y,\xi)]}\, d\xi, \tag{4.2.2}$$

for an order zero symbol s_0 with

$$\operatorname{supp}_{x,y,\xi}(s_0) \subset \{(x,y,\xi) \in U_\alpha \times V_\alpha^* \times \mathbb{R}^d : d_X(x,y) \leq 1.01\varepsilon_X \text{ and } |\xi| \leq 1\}, \tag{4.2.3}$$

and an order one symbol ϕ, homogeneous in ξ of order one, solving the Eikonal equation (3.3.6) and vanishing for $x \in \Sigma_\alpha(y,\xi)$ as required by the lemma. In the remainder of the proof it will be convenient to fix a orthonormal basis $\{e_k\}$ of eigenfunctions for P, such that $\Delta e_k = \lambda_k e_k$ for a non-decreasing sequence $\{\lambda_k\}$. We fix $u \in L^1(M)$ with $\operatorname{supp}(u) \subset V_\alpha^*$ and $\|u\|_{L^1(M)} \leq 1$.

We begin by mollifying the functions Q. We proceed here with a similar approach to Theorem 4.3.1 of [20]. We fix $\rho \in C_c^\infty(\mathbb{R})$ equal to one in a neighborhood of the origin and with $\rho(t) = 0$ for $|t| \geq \varepsilon_X/2$. We write

$$\begin{aligned} Q &= \int R\widehat{q}(Rt) e^{2\pi i t P}\, dt \\ &= \int R\widehat{q}(Rt) \Big\{ \rho(t)\tilde{W}(t) + \rho(t)\tilde{R}(t) + (1-\rho(t)) e^{2\pi i t P} \Big\}\, dt \\ &= Q_I + Q_{II} + Q_{III}. \end{aligned} \tag{4.2.4}$$

The rapid decay of $\widehat{q}$ implies that the function $\psi(t) = R\widehat{q}(Rt)(1-\rho(t))$ satisfies $\|\partial_t^N \psi\|_{L^1(\mathbb{R})} \lesssim_X R^{-M}$, and so

$$|\widehat{\psi}(\lambda)| \lesssim_{N,M} R^{-M} \lambda^{-N}. \tag{4.2.5}$$

But since $Q_{III} = \widehat{\psi}(-P)$, we can write the kernel of Q_{III} as

$$Q_{III}(x,y) = \sum\nolimits_\lambda \widehat{\psi}(-\lambda_k) e_k(x) \overline{e_k(y)}, \tag{4.2.6}$$

Sobolev embedding and (4.2.5) imply that $|Q_{III}(x,y)| \lesssim_N R^{-N}$ and thus

$$\|Q_{III} u\|_{L^\infty(M)} \lesssim_N R^{-N}. \tag{4.2.7}$$

Integration by parts, using the fact that q vanishes near the origin, yields that

$$\left| \int R\widehat{q}(Rt)\rho(t)\tilde{R}(t,x,y) \right| \lesssim_N R^{-N}, \tag{4.2.8}$$

and thus

$$\|Q_{II} u\|_{L^\infty(M)} \lesssim_N R^{-N}. \tag{4.2.9}$$

Now we expand

$$Q_I = \iint R\widehat{q}(Rt)\rho(t)s_0(t,x,y,\xi)e^{2\pi i[\phi(x,y,\xi)+tp(y,\xi)]}\,d\xi\,dt. \tag{4.2.10}$$

We perform a Fourier series expansion, writing

$$c_n(x,y,\xi) = \int \rho(t)s_0(t,x,y,\xi)e^{-2\pi int}\,dt. \tag{4.2.11}$$

Then the symbol estimates for s_0, and the compact support of ρ imply that

$$|\partial^\alpha_{x,y}\partial^\beta_\xi c_n(x,y,\xi)| \lesssim_{\alpha,\beta,N} |n|^{-N}\langle\xi\rangle^{-\beta}. \tag{4.2.12}$$

Using Fourier inversion we can write

$$\begin{aligned} Q_I(x,y) &= \iint \sum\nolimits_n R\widehat{q}(Rt)c_n(x,y,\xi)e^{2\pi i[\phi(x,y,\xi)+t[n+p(y,\xi)]]}\,d\xi\,dt \\ &= \int \sum\nolimits_n q\big((n+p(y,\xi))/R\big)c_n(x,y,\xi)e^{2\pi i\phi(x,y,\xi)}\,d\xi \\ &= \int \tilde{\sigma}_\alpha(x,y,\xi)e^{2\pi i\phi(x,y,\xi)}\,d\xi, \end{aligned} \tag{4.2.13}$$

where

$$\tilde{\sigma}_\alpha(x,y,\xi) = \sum_{n\in\mathbb{Z}} q\left(\frac{n+p(y,\xi)}{R}\right)c_n(x,y,\xi). \tag{4.2.14}$$

The nth term of this sum is supported on $R/4 - n \leq p(y,\xi) \leq 4R - n$, so in particular, if $n > 4R$ then the term vanishes. For $n \leq 4R$, we have estimates of the form

$$\left|\partial^\alpha_{x,y}\partial^\beta_\xi\left\{q\left(\frac{n+p(y,\xi)}{R}\right)c_n(x,y,\xi)\right\}\right| \lesssim_{\alpha,\beta,N} |n|^{-N}. \tag{4.2.15}$$

and for $-4R \leq n \leq R/8$,

$$\left|\partial^\alpha_{x,y}\partial^\beta_\xi\left\{q\left(\frac{n+p(y,\xi)}{R}\right)c_n(x,y,\xi)\right\}\right| \lesssim_{\alpha,\beta,N} |n|^{-N}R^{-\beta}. \tag{4.2.16}$$

But this means that if we define

$$\sigma_\alpha(x,y,\xi) = \sum\nolimits_{-4R\leq n\leq R/8} q\left(\frac{n+p(y,\xi)}{R}\right)c_n(x,y,\xi). \tag{4.2.17}$$

and define

$$Q_\alpha(x,y) = \int \sigma_\alpha(x,y,\xi)e^{2\pi i\phi(x,y,\xi)}\,d\xi \tag{4.2.18}$$

then

$$\left|\partial^\alpha_{x,y}\partial^\beta_\xi\{\tilde{\sigma}_\alpha - \sigma\}(x,y,\xi)\right| \lesssim_{\alpha,\beta,N,M} R^{-N}\langle\xi\rangle^{-M}, \tag{4.2.19}$$

and so

$$\|(Q_I - Q_\alpha)u\|_{L^\infty(M)} \lesssim_N R^{-N}. \tag{4.2.20}$$

Combining (4.2.7), (4.2.9), and (4.2.20), we conclude that

$$\|(Q - Q_\alpha)u\|_{L^\infty(M)} \lesssim_N R^{-N}. \tag{4.2.21}$$

Since σ_α is supported on $|\xi| \sim R$, we have verified the required properties of Q_α.

Using the bounds on $Q - Q_\alpha$ obtained above, we see that

$$\left\| [(Q \circ e^{2\pi itP} \circ Q) - (Q_\alpha \circ e^{2\pi itP} \circ Q_\alpha)]\{u\} \right\|_{L^\infty(M)} \lesssim_N R^{-N}. \tag{4.2.22}$$

Since $\tilde{R}_\alpha(t)$ has a smooth kernel, we also see that

$$\begin{aligned} &\left\| (Q_\alpha \circ (e^{2\pi itP} - \tilde{W}_\alpha(t)) \circ Q_\alpha)\{u\} \right\|_{L^\infty(M)} \\ &\quad = \left\| (Q_\alpha \circ \tilde{R}_\alpha(t) \circ Q_\alpha)\{u\} \right\|_{L^\infty(M)} \lesssim_N R^{-N}. \end{aligned} \tag{4.2.23}$$

We now write

$$\begin{aligned} &(Q_\alpha \circ \tilde{W}_\alpha(T))(x,y) \\ &\quad = \int \sigma_\alpha(x,\xi) s_0(t,z,y,\eta) e^{2\pi i[\phi(x,z,\xi) + \phi(z,y,\eta) + tp(y,\eta)]}\, d\xi\, d\eta\, dz. \end{aligned} \tag{4.2.24}$$

The phase of this equation has gradient in the z variable with magnitude $\gtrsim R$ for $|\eta| \ll R$, and $\gtrsim R|\xi|$ if $|\eta| \gg R$. Thus, if we define $s(t,x,y,\xi) = s_0(t,x,y,\xi)\chi(\xi/R)$ where $\text{supp}(\chi) \subset [1/8, 8]$, and then define

$$W_\alpha(t)(x,y) = \int s(t,x,y,\xi) e^{2\pi i[\phi(x,y,\xi) + tp(y,\xi)]}\, d\xi, \tag{4.2.25}$$

then we may integrate by parts in the z variable to conclude that

$$\left| (Q_R \circ (\tilde{W}_\alpha(t) - W_\alpha(t)))(x,y) \right| \lesssim_N R^{-N}, \tag{4.2.26}$$

and thus

$$\|(Q_R \circ (\tilde{W}_\alpha(t) - W_\alpha(t)) \circ Q_R)u\|_{L^\infty(M)} \lesssim R^{-N}. \tag{4.2.27}$$

This proves the required estimates for the operators W_α. □

Let us now proceed with the proof of the pointwise bounds in Proposition 4.4 using Lemma 4.5. Given $u : M \to \mathbb{C}$, write $u = \sum_\alpha u_\alpha$, where $u_\alpha = \eta_\alpha u$. Lemma 4.5 implies that if we define

$$S_\alpha = \int c(t)(Q_{j,\alpha} \circ W_{j,\alpha}(t) \circ Q_{j,\alpha})\{u_\alpha\}\, dt, \tag{4.2.28}$$

then for all $N \geq 0$,

$$\left\| S - \textstyle\sum_\alpha S_\alpha \right\|_{L^\infty(M)} \lesssim_N R^{-N}. \tag{4.2.29}$$

This error is negligible to the pointwise bounds we want to obtain in Proposition 4.4 if we choose $N \geq K - \frac{d+1}{2}$, since the compactness of M implies that $d_X(x, x_0) \lesssim 1$ for all $x \in M$, and so

$$R^{-N} \lesssim R^{\left(\frac{d+1}{2} - K\right)} \lesssim \frac{R^d}{\langle R d_X(x, x_0) \rangle^{\frac{d-1}{2}}} \left\langle R \left| |t_0| \pm d_X^{\pm}(x, x_0) \right| \right\rangle^{-K}. \tag{4.2.30}$$

We bound each of the functions $\{S_\alpha\}$ separately, combining the estimates using the triangle inequality. We continue by expanding out the implicit integrals in the definition of S_α. In the coordinate system U_α, we can write

$$\begin{aligned} S_\alpha(x) = \int & c(t)\sigma(x,\eta) e^{2\pi i \eta \cdot (x-y)} \\ & s(t,y,z,\xi) e^{2\pi i [\phi(y,z,\xi) + tp(z,\xi)]} \\ & \sigma(z,\theta) e^{2\pi i \theta \cdot (z-w)} (\eta_\alpha u)(w) \\ & dt\, dy\, dz\, dw\, d\theta\, d\xi\, d\eta. \end{aligned} \tag{4.2.31}$$

The integral in (4.2.31) looks highly complicated, but can be simplified considerably by noticing that most variables are quite highly localized. In particular, oscillation in the η variable implies that the amplitude is negligible unless $|x - y| \lesssim 1/R$, oscillation in the θ variable implies that the amplitude is negligible unless $|z - w| \lesssim 1/R$, and the support of u implies that $|w - x_0| \lesssim 1/R$. Define

$$k_1(t,x,z,\xi) = \int \sigma(x,\eta) s(t,y,z,\xi) e^{2\pi i [\eta \cdot (x-y) + \phi(y,z,\xi) - \phi(x,z,\xi)]}\, dy\, d\eta, \tag{4.2.32}$$

and

$$\begin{aligned} k_2(t,\xi) = \int & k_1(t,x,z,\xi)\sigma(z,\theta)(\eta_\alpha u)(w) \\ & e^{2\pi i [\theta \cdot (z-w) + \phi(x,z,\xi) - \phi(x,x_0,\xi) + tp(z,\xi) - tp(x_0,\xi)]}\, d\theta\, dw, \end{aligned} \tag{4.2.33}$$

and then set

$$a(x,\xi) = \int c(t) k_2(t, R\xi) e^{2\pi i [(t-t_0) p(x_0, R\xi)]}\, dt\, dz, \tag{4.2.34}$$

so that $\operatorname{supp}_\xi(a) \subset \{\xi : 1/8 \leq |\xi| \leq 8\}$, and

$$S_\alpha(x) = R^d \int a(x,\xi) e^{2\pi i R [\phi(x,x_0,\xi) + t_0 p(x_0,\xi)]}\, d\xi. \tag{4.2.35}$$

Integrating by parts in η and θ in (4.2.32) and (4.2.33) gives that for all multi-indices α,

$$|\partial_\xi^\alpha k_1(t,x,z,\xi)| \lesssim_\alpha R^{-|\alpha|} \quad \text{and} \quad |\partial_\xi^\alpha k_2(z,\xi)| \lesssim_\alpha R^{-|\alpha|}. \tag{4.2.36}$$

Using the bounds in (4.2.36) with the fact that $\operatorname{supp}(c)$ is contained in a $O(1/R)$ neighborhood of t_0 in (4.2.34) then implies $|\partial_\xi^\alpha a(x,\xi)| \lesssim_\alpha 1$ for all α.

We now account for angular oscillation of the integral by working in a kind of 'polar coordinate' system. First we find $\lambda : V_\alpha^* \times S^{d-1} \to (0,\infty)$ such that for all $|\xi| = 1$,

$$p(x_0, \lambda(x_0,\xi)\xi) = 1. \tag{4.2.37}$$

If $\tilde{a}(x,\rho,\eta) = a(x,\rho\lambda(x_0)\xi)\det[\lambda(x_0,\xi)I + \xi(\nabla_\xi\lambda)(x_0,\xi)^T]$, then

$$S_\alpha(x) = R^d \int_0^\infty \rho^{d-1} \int_{|\xi|=1} \tilde{a}(x,\rho,\xi) e^{2\pi i R\rho[t_0 + \phi(x,x_0,\lambda(x_0,\xi)\xi)]}\, d\xi\, d\rho. \tag{4.2.38}$$

Define $\Phi : S^{d-1} \to \mathbb{R}$ by setting $\Phi(\xi) = \phi(x,x_0,\lambda(x_0,\xi)\xi)$. We claim that, in the ξ variable, Φ has exactly two critical points $|\xi^+|^{-1}\xi^+$ and $|\xi^-|^{-1}\xi^-$, where $\xi^+ \in S^*_{x_0}$ is the covector corresponding to the forward geodesic from x_0 to x, and $\xi^- \in S^*_{x_0}$ is the covector corresponding to the backward geodesic from x_0 to x. Moreover,

$$\Phi(|\xi^+|^{-1}\xi^+) = d_X^+(x_0,x) \quad \text{and} \quad \Phi(|\xi^-|^{-1}\xi^-) = -d_X^-(x_0,x), \tag{4.2.39}$$

and the Hessian at each of these points is non-degenerate, with each eigenvalue of the Hessian having magnitude exceeding a constant multiple of $d_X^\pm(x_0,x)$. We prove that these properties hold for Φ in Proposition 4.6 of the following section, via a series of geometric arguments. It then follows from the principle of stationary phase that

$$S_\alpha(x) = \sum_\pm \frac{R^d}{\left\langle R d_X^\pm(x_0,x)\right\rangle^{\frac{d-1}{2}}} \int_0^\infty \rho^{\frac{d-1}{2}} a_\pm(x,\rho) e^{2\pi i R\rho[t_0 \pm d_X^\pm(x_0,x)]}\, d\rho, \tag{4.2.40}$$

where $a_\pm$ is supported on $|\rho| \sim 1$, and for all α, $|\partial_\rho^\alpha a_\pm| \lesssim_\alpha 1$. Integrating by parts in the ρ variable if $t_0 \pm d_X^\pm(x_0,x)$ is large, we conclude that

$$|S_\alpha(x)| \lesssim \frac{R^d}{\langle R d_X(x_0,x)\rangle^{\frac{d-1}{2}}} \sum_\pm \langle R|t_0 \pm d_X(x_0,x)|\rangle^{-K}. \tag{4.2.41}$$

Combining (4.2.29) and (4.2.41) completes the proof of the pointwise bounds.

The quasi-orthogonality arguments are obtained by a largely analogous method, and so we only sketch the proof. One major difference is that we can use the self-adjointness of the operators Q, and the unitary group structure of $\{e^{2\pi i tP}\}$, to write

$$\begin{aligned} \langle S_0, S_1\rangle &= \int c_0(t)c_1(s)\langle (Q \circ e^{2\pi i tP} \circ Q)\{u_0\}, (Q \circ e^{2\pi i sP} \circ Q)\{u_1\}\rangle \\ &= \int c_0(t)c_1(s)\langle (Q^2 \circ e^{2\pi i(t-s)P} \circ Q^2)\{u_0\}, u_1\rangle \\ &= \int c(t)\langle (Q^2 \circ e^{2\pi i tP} \circ Q^2)\{u_0\}, u_1\rangle, \end{aligned} \tag{4.2.42}$$

where $c(t) = \int c_0(u)c_1(u-t)\, du$, by Young's inequality, satisfies

$$\|c\|_{L^1(\mathbb{R})} \lesssim \|c_0\|_{L^1(\mathbb{R})}\|c_1\|_{L^1(\mathbb{R})} \leq 1 \tag{4.2.43}$$

and $\mathrm{supp}(c) \subset [(t_0 - t_1) - 4/R, (t_0 - t_1) + 4/R]$. After this, one proceeds exactly as in the proof of the pointwise estimate. We write the inner product as

$$\sum_\alpha \int c(t) \langle (Q^2 \circ e^{2\pi i t P} \circ Q^2)\{\eta_\alpha u_0\}, u_1 \rangle. \tag{4.2.44}$$

Then we use Lemma 4.5 to replace $Q^2 \circ e^{2\pi i t P} \circ Q^2$ with $Q_\alpha^2 \circ W_\alpha(t) \circ Q_\alpha^2$ using Lemma 4.5, modulo a negligible error. The integral

$$\sum_\alpha \int c(t) \langle (Q_\alpha^2 \circ W_\alpha(t) \circ Q_\alpha^2)\{\eta_\alpha u_0\}, u_1 \rangle \tag{4.2.45}$$

is then only non-zero if both the supports of u_0 and u_1 are compactly contained in U_α. Thus we can switch to the coordinate system of U_α, in which we can express the inner product by oscillatory integrals of the exact same kind as those occurring in the pointwise estimate. Integrating away the highly localized variables as in the pointwise case, and then applying stationary phase in polar coordinates proves the required estimates. □

In remains to classify the critical points that arose in Proposition 4.4.

Proposition 4.6. *Fix a bounded open set $U_0 \subset \mathbb{R}^d$. Consider a Finsler metric $F : U_0 \times \mathbb{R}^d \to [0, \infty)$ on U_0, and it's dual metric $F_* : U_0 \times \mathbb{R}^d \to [0, \infty)$, which extends to a Finsler metric on an open set containing the closure of U_0. Fix a suitably small constant $r > 0$. Let U be an open subset of U_0 with diameter at most r which is geodesically convex (any two points are joined by a minimizing geodesic). Let $\phi : U \times U \times \mathbb{R}^d \to \mathbb{R}$ solve the eikonal equation*

$$F_*(x, \nabla_x \phi(x, y, \xi)) = F_*(y, \xi),$$

such that $\phi(x, y, \xi) = 0$ for $x \in H(y, \xi)$, where $H(y, \xi) = \{x \in U : \xi \cdot (x - y) = 0\}$. For each $x_0, x_1 \in U$, let $S^*_{x_0} = \{\xi \in \mathbb{R}^d : F_*(x_0, \xi) = 1\}$ be the cosphere at x_0, and define $\Psi : S^*_{x_0} \to \mathbb{R}$ by setting

$$\Psi(\xi) = \phi(x_1, x_0, \xi).$$

Then the function Ψ has exactly two critical points, at ξ^+ and ξ^-, where the Legendre transform of ξ^+ is the tangent vector of the forward geodesic from x_0 to x_1, and the Legendre transform of ξ^- is the tangent vector of the backward geodesic from x_1 to x_0. Moreover,

$$\Psi(\xi^+) = d_X^+(x_0, x_1) \quad \textit{and} \quad \Psi(\xi^-) = -d_X^-(x_0, x_1),$$

and the Hessians H_+ and H_- of Ψ at these critical points, viewed as quadratic maps from $T_{\xi_\pm} S^*_{x_0} \to \mathbb{R}$ satisfy

$$H_+(\zeta) \geq C d_X^+(x_0, x_1)|\zeta| \quad \textit{and} \quad H_-(\zeta) \leq -C d_X^-(x_0, x_1)|\zeta|,$$

where the implicit constant is uniform in x_0 and x_1.

If Ψ is as above, then the function $\Phi : S^{d-1} \to \mathbb{R}$ obtained by setting $\Phi(\xi) = \phi(x, x_0, F_*(x,\xi)^{-1}\xi)$ is precisely the kind of function that arose as a phase in Proposition 4.4, where F_* was the principal symbol p of the pseudodifferential operator we were considering. Since critical points and the Hessians of maps at critical points are stable under diffeomorphisms, and the map $\xi \mapsto F_*(x,\xi)^{-1}\xi$ is a diffeomorphism from $S^*_{x_0}$ to S^{d-1}, classifying the critical points of the map Ψ implies the required properties of the map Φ used in Proposition 4.4.

In order to prove 4.6, we rely on a geometric interpretation of Ψ following from Hamilton-Jacobi theory.

Lemma 4.7. *Consider the setup to Proposition 4.6. For any $\xi \in S^*_{x_0}$,*

$$|\Psi(\xi)| = \begin{cases} \text{the length of the shortest curve from } H(x_0,\xi) \text{ to } x_1 & : \text{if } \Psi(\xi) > 0, \\ \text{the length of the shortest curve from } x_1 \text{ to } H(x_0,\xi) & : \text{if } \Psi(\xi) < 0. \end{cases}$$

Proof. We rely on a construction of ϕ from Proposition 3.7 of [21], which we briefly describe. Fix x_0 and ξ. Then there is a unique covector field $\omega : H(x_0,\xi) \to \mathbb{R}^d$ which is everywhere perpendicular to $H(x_0,\xi)$, with $\omega(x_0) = \xi$ and with $F_*(x, \omega(x)) = F_*(x_0,\xi)$ for all $x \in H(x_0,\xi)$. There exists a unique point $x(\xi) \in H(x_0,\xi)$ and a unique $t(\xi) \in \mathbb{R}$, such that the unit speed geodesic γ on M with $\gamma(0) = x(\xi)$ and $\gamma'(0) = \mathcal{L}^{-1}(x(\xi), \omega(x(\xi)))$ satisfies $\gamma(t(\xi)) = x_1$. We then have $\Psi(\xi) = t(\xi)$. If $t(\xi)$ is negative, then $\gamma|_{[t(\xi),0]}$ is a geodesic from x_1 to x_0, and if $t(\xi)$ is positive, $\gamma|_{[0,t(\xi)]}$ is a geodesic from x_0 to x_1. Because γ is a geodesic, the geometric interpretation then follows if U is a suitably small neighborhood such that geodesics are length minimizing. $\square$

It follows immediately from Lemma 4.7 that

$$\Psi(\xi_+) = d_X^+(x_0, x_1) \quad \text{and} \quad \Psi(\xi_-) = -d_X^-(x_0, x_1). \tag{4.2.46}$$

A simple geometric argument also shows $\Psi(\xi^+)$ is the maximum value of Ψ on $S^*_{x_0}$, and $\Psi(\xi^-)$ is the minimum value on $S^*_{x_0}$, so that these two points are both critical. Indeed, the point x_0 lies in $H(x_0,\xi)$ for all $\xi \in \mathbb{R}^d - \{0\}$. Thus the shortest curve from x_0 to x_1 is always longer than the shortest curve from $H(x_0,\xi)$ to x_1. Similarily, the shortest curve from x_1 to x_0 is always longer than the shortest curve from x_1 to $H(x_0,\xi)$. Thus

$$-d_X^-(x_0, x_1) \leq \Psi(\xi) \leq d_X^+(x_0, x_1), \tag{4.2.47}$$

and so $\Psi(\xi^-) \leq \Psi(\xi) \leq \Psi(\xi^+)$ for all $\xi \in$. All that remains is to prove that ξ^+ and ξ^- are the *only* critical points of Ψ, and that these critical points are appropriately non-degenerate.

In order to simplify proofs, we employ a structural symmetry to reduce the number of cases we need to analyze. Namely, if one defines the reverse Finsler metric $F_\rho(x, v) = F(x, -v)$, then $F^*_\rho(x,\xi) = F_*(x, -\xi)$, and so the associated function Ψ^ρ which is the analogue of Ψ for F^ρ satisfies $\Psi^\rho(\xi) = -\Psi(-\xi)$. The critical points of Ψ and Ψ^ρ are thus directly related to one another, which allows us without loss of generality to study only points with $\Psi \leq 0$ (and thus only study geodesics beginning at x_1).

Proof that ξ^+ and ξ^- are the only critical points. Fix $\xi^* \in T_{x_0}M - \{\xi^\pm\}$. Using the notation defined above, let $x_* = x(\xi^*)$ and $t_* = t(\xi^*)$. Using the symmetry above, we may assume without loss of generality that $\Psi(\xi^*) \leq 0$. Since ξ^- is not perpendicular to $H(x_0, \xi^*)$ at x_0, we have $x_* \neq x_0$. If $\Psi(\xi^*) < 0$, let γ be the unique unit speed forward geodesic with $\gamma(0) = x_1$ and $\gamma(t_*) = x_*$. If $\Psi(\xi^*) = 0$, let γ be the unique unit speed forward geodesic with $\gamma'(0)$ equal to the Legendre transform of ξ^*. Pick η such that $\eta \cdot (x_* - x_0) \neq 0$, and then, for t suitably close to t_*, define a smooth map

$$\xi(t) = \frac{\xi^* + a(t)\eta}{F_*(x_0, \xi^* + a(t)\eta)}, \tag{4.2.48}$$

into $S^*_{x_0}$, where

$$a(t) = -\frac{\xi^* \cdot (\gamma(t) - x_0)}{\eta \cdot (\gamma(t) - x_0)} \tag{4.2.49}$$

is defined so that $\gamma(t) \in H(x_0, \xi(t))$. Then $\Psi(\xi(t)) = -t$, and differentiation at $t = t_*$ gives $D\Psi(\xi^*)(\xi'(t_*)) = -1$. In particular, $D\Psi(\xi^*) \neq 0$, so ξ^* is not a critical point. □

We now analyze the non-degeneracy of the critical points ξ^+ and ξ^-.

Proof that ξ^+ and ξ^- are non-degenerate. Using symmetry, it suffices without loss of generality to analyze the critical point ξ^- rather than ξ^+. Let $H_- : T_{\xi^-}S^*_{x_0} \to \mathbb{R}$ be the Hessian of Ψ at ξ^-. Let $v_0 = \mathcal{L}_{x_0}^{-1}\xi^-$, and using the notation of the last argument, let $l = t(\xi^-)$. Consider a curve $\xi(a)$ valued in $S^*_{x_0}$ with $\xi(0) = \xi^-$ and $\zeta = \xi'(0)$ for some $\zeta \in T^*_{\xi^-}S_{x_0}$. Then the second derivative of $\Psi(\xi(a))$ at $a = 0$ is $H_-(\zeta)$, and so our proof would be complete if we could show that the function $L(a) = -\Psi(\xi(a))$, which is the length of the shortest geodesic from x_1 to $H(x_0, \xi(a))$, satisfies $L''(0) \geq Cl|\zeta|$, for a constant $C > 0$ uniform in x_0, x_1, and ζ.

Consider the partial function $\mathrm{Exp}_{x_1} : \mathbb{R}^d \to U_0$ obtained from the exponential map at x_1. If r is chosen suitably small, there exists a neighborhood V of the origin in $\mathbb{R}^d$ such that $E = \mathrm{Exp}_{x_1}|_V$ is a diffeomorphism between V and U. Unlike in Riemannian manifolds, in general E is only C^1 at the origin, though smooth on $V - \{0\}$. However, for $|v| = 1$ and $t > 0$ we can write $E(tv) = (\pi \circ \varphi)(x_1, \mathcal{L}_{x_1}v, t)$, where the partial function $\varphi : (T^*U_0 - 0) \times \mathbb{R} \to (T^*U_0 - 0)$ is the flow induced by the the Hamiltonian vector field $(\partial_\xi F_*, -\partial_x F_*)$ on $T^*U_0 - 0$, and π is the projection map from $T^*U_0 - 0$ to U_0. Where defined, φ is a smooth function since the Hamiltonian vector field is smooth, and so it follows by homogeneity and the precompactness of U that the partial derivatives $(\partial^\alpha E_x)(v)$ are uniformly bounded for $v \neq 0$ and $x \in U$. It thus follows from the inverse function theorem that there exists a constant $A > 0$ such that for all x_1 and x in U with $x \neq x_1$,

$$|\partial_j G_{x_1}(x)| \leq A \quad \text{and} \quad |(\partial_j \partial_k G_{x_1})(x)| \leq A. \tag{4.2.50}$$

We can also pick A to be large enough that for all $x \in U$, and all $v, w \in \mathbb{R}^d - \{0\}$,

$$A^{-1}|w|^2 \leq \sum_{ij} g_{ij}(x, v) w^i w^j \leq A|w|^2. \tag{4.2.51}$$

and such that for all $x \in U$ and $v \in \mathbb{R}^d - \{0\}$,

$$|\partial_{x_k} g_{ij}(x, v)| \leq A. \tag{4.2.52}$$

Since $\zeta \in T_{\xi^-} S^*_{x_0}$, and $S^*_{x_0} = \{\xi : F_*(x_0, \xi)^2 = 1\}$, by Euler's homogeneous function theorem we have

$$\sum_{ij} g^{ij}(x_0, \xi^-)\xi'(0)\xi^-_j = \frac{1}{2}\sum_{i,j} \frac{\partial^2 F_*^2}{\partial \xi_i \partial \xi_j}(x_0, \xi^-)\zeta_i \xi^-_j = \frac{1}{2}\sum \frac{\partial F_*^2}{\partial \xi_j}(x_0, \xi^-)\zeta_j = 0. \tag{4.2.53}$$

Differentiating $F_*(x_0, \xi(a))^2 = 1$ twice with respect to a at $a = 0$ yields that

$$\sum_{i,j} g_{ij}(x_0, \xi^-)\xi''_i(0)\xi^-_j = -\sum_{i,j} g_{ij}(x_0, \xi^-)\zeta^i \zeta^j \tag{4.2.54}$$

Define vectors $n(a)$ by setting $n^i(a) = \sum g^{ij}(x_0, \xi^-)\xi_j(a)$. Then $n(a)$ is the normal vector to $H(x_0, \xi(a))$ with respect to the inner product with coefficients $g_{ij}(x_0, v_0)$. Let $u(a)$ be the orthogonal projection of v_0 onto the hyperplane $\{v : \xi(a) \cdot v = 0\}$ with respect to the inner product $g_{ij}(x_0, v_0)$. If we define

$$c(a) = \sum g_{ij}(x_0, v_0)v_0^i n^j(a) = \sum g^{ij}(x_0, \xi^-)\xi^-_i \xi_j(a), \tag{4.2.55}$$

then $u(a) = v_0 - c(a)n(a)$. Note that (4.2.53) and (4.2.54) imply $c(0) = 1$, $c'(0) = 0$, and $c''(0) \leq -|\zeta|^2/A$. Also $n(0) = v_0$ and $|n'(0)| \leq A|\zeta|$.

Let $x(s) = x_0 + su(a)$ and let $R(s)$ be the length of the geodesic from x_1 to $x(s)$. To control $R(s)$, define $y(s) = G(x(s))$, and consider the variation $A(s, t) = E(ty(s))$, defined so that $t \mapsto A(s, t)$ is the geodesic from x_1 to $x(s)$. The Gauss Lemma for Finsler manifolds (see Lemma 6.1.1 of [2]) implies that

$$A^{-1}R(s) \leq |y(s)| \leq AR(s). \tag{4.2.56}$$

Define $T(s) = (\partial_t A)(s, 1)$ and $V(s) = (\partial_s A)(s, 1)$. Then (**??**) implies

$$R(s)R'(s) = \sum_{i,j} g_{ij}(x(s), T(s))V^i(s)T^j(s). \tag{4.2.57}$$

Again, the Gauss Lemma implies

$$A^{-1}R(s) \leq |T(s)| \leq AR(s). \tag{4.2.58}$$

We can write

$$T^i(s) = \sum_{i,j} (\partial_j E^i)(y(s))y^j(s) \tag{4.2.59}$$

and

$$V^i(s) = \sum_{i,j,k} (\partial_j E^i)(y(s))(\partial_k G^j)(x(s))u^k(a) = u^i(a). \tag{4.2.60}$$

In particular, $T(0) = v_0$, so $R'(0) = \sum g_{ij}(x_0, v_0)u^i(a)v_0^j = 1 - c(a)^2$. Cauchy-Schwartz applied to (4.2.57) also tells us that

$$|R'(s)| \lesssim_A |u(a)|. \tag{4.2.61}$$

Note that $u(0) = 0$, so if a is small enough, then we have $|u(a)| \leq l/2$. Taylor's theorem applied to (4.2.61), noting $R(0) = l$ then gives that for $|s| \leq 1$,

$$l/2 \leq R(s) \leq 2l. \tag{4.2.62}$$

Differentiating (4.2.57) tells us that

$$\begin{aligned} R'(s)^2 + R''(s)R(s) = \sum\nolimits_{i,j,k} &\Big[(\partial_{x_k} g_{ij})(x(s), T(s))u^i(a)T^j(s)u^k(a)\Big] \\ &+ \Big[(\partial_{v_k} g_{ij})(x(s), T(s))u^i(a)T^j(s)(\partial_s T^k)(s)\Big] \\ &+ \Big[g_{ij}(x(s), T(s))u^i(a)(\partial_s T^j)(s)\Big]. \end{aligned} \tag{4.2.63}$$

Write the right hand side as $\mathrm{I} + \mathrm{II} + \mathrm{III}$. Using (4.2.52), (4.2.58), (4.2.62), and the triangle inequality gives

$$|\mathrm{I}| \lesssim R(s)|u(a)|^2 \lesssim l|u(a)|^2. \tag{4.2.64}$$

Since $\partial_{v_k} g_{ij} = (1/2)(\partial^3 F^2/\partial v_i \partial v_j \partial v_k)$, applying Euler's homogeneous function theorem when summing over j implies that

$$\mathrm{II} = 0. \tag{4.2.65}$$

Applying Cauchy-Schwarz and (4.2.50), we find

$$|\mathrm{III}| \lesssim |u(a)||T'(s)| \lesssim |u(a)|^2[|y(s)| + 1] \lesssim (l + 1)|u(a)|^2. \tag{4.2.66}$$

But now combining (4.2.64), (4.2.65), (4.2.66), and rearranging (4.2.63) shows that

$$|R''(s)| \lesssim l^{-1}|u(a)|^2 + (1 + l^{-1})|u(a)|^2 \lesssim l^{-1}|u(a)|^2. \tag{4.2.67}$$

Taylor's theorem implies there exists $B > 0$ depending only on A and d such that

$$|R(s) - (R(0) + sR'(0))| \leq Bl^{-1}|u(a)|^2 s^2. \tag{4.2.68}$$

Since $R(0)$ =l, $R'(0) = 1 - c(a)^2$, $c(0) = 1$, $u(0) = 0$, $c'(0) = 0$, $|u'(0)| \leq A|\zeta|$, and $c''(0) \leq -|\zeta|^2/A$, we conclude from (4.2.68) that as $a \to 0$, if $s > 0$ then

$$\begin{aligned} R(-s) &\leq l - sR'(0) + Bl^{-1}|u(a)|^2 s^2 \\ &\leq l - s(A^{-1}|\zeta|^2 a^2) + A^2 Bl^{-1}|\zeta|^2 s^2 a^2 + O(a^3(s + l^{-1}s^2)). \end{aligned} \tag{4.2.69}$$

For all s, $L(a) \leq R(s)$. Optimizing by picking $s = l/2A^3B$ gives

$$L(a) \leq R(-s) \leq l - l|\zeta|^2 a^2/4A^4 B + O(a^3). \tag{4.2.70}$$

Taking $a \to 0$ and using that $L(0) = l$ gives that $L''(0) \leq -l|\zeta|^2/4A^4B$, so setting $C = 1/4A^4B$, we find we have proved what was required. □

4.3 Analysis of Regime I via Density Methods

We now begin obtaining bounds for the operator T^I specified in Proposition 4.3 by using the quasi-orthogonality estimates of Proposition 4.4. Define a metric $d_X = d_X^+ + d_X^-$ on M. Given an input $u : M \to \mathbb{C}$, we consider a maximal $1/R$ separated subset $\mathcal{X}_R$ of M, and then consider a decomposition $u = \sum_{x_0 \in \mathcal{X}_R} u_{x_0}$ with respect to some partition of unity, where $\mathrm{supp}(u_{x_0}) \subset B(x_0, 1/R)$. The balls $\{B(x_0, 1/R) : x_0 \in \mathcal{X}_R\}$ have finite overlap, and so

$$\|u\|_{L^p(M)} \sim \left(\sum\nolimits_{x_0 \in \mathcal{X}_R} \|u_{x_0}\|_{L^p(M)}^p \right)^{1/p}. \tag{4.3.1}$$

If we set $f_{x_0,t_0} = T^I_{t_0}\{u_{x_0}\}$, then

$$\left\| T^I u \right\|_{L^p(M)} = \left\| \sum\nolimits_{(x_0,t_0) \in \mathcal{X}_R \times \mathcal{T}_R} f_{x_0,t_0} \right\|_{L^p(M)}. \tag{4.3.2}$$

In this subsection, we use the quasi-orthogonality estimates of the last section to obtain L^2 estimates on partial sums of a family of functions of the form $\{S_{x_0,t_0}\}$, which are essentially L^1 normalized versions of the functions $\{f_{x_0,t_0}\}$, under a density assumption on the set of indices we are summing over. Namely, we say a set $\mathcal{E} \subset \mathcal{X}_R \times \mathcal{T}_R$ has *density type* (A_0, A_1) if for any set $B \subset \mathcal{X}_R \times \mathcal{T}_R$ with $1/R \leq \mathrm{diam}(B) \leq A_1/R$,

$$\#(\mathcal{E} \cap B) \leq R A_0 \; \mathrm{diam}(B).^* \tag{4.3.3}$$

To obtain L^p bounds from these L^2 bounds, in the next section we will perform a *density decomposition* to break up $\mathcal{X}_R \times \mathcal{T}_R$ into families of indices with controlled density, and then apply Proposition 4.8 on each subfamily to control (4.3.2) via an interpolation.

Proposition 4.8. *Fix $A \geq 1$. Consider a set $\mathcal{E} \subset \mathcal{X}_R \times \mathcal{T}_R$. Suppose that for each $(x_0, t_0) \in \mathcal{E}$, we pick two measurable functions $b_{t_0} : I_0 \to \mathbb{R}$ and $u_{x_0} : M \to \mathbb{R}$, supported on I_{t_0} and $B(x_0, 1/R)$ respectively, such that $\|b_{t_0}\|_{L^1(I_0)} \leq 1$ and $\|u_{x_0}\|_{L^1(M)} \leq 1$. Define*

$$S_{x_0,t_0} = \int b_{t_0}(t)(Q_R \circ e^{2\pi i t P} \circ Q_R) u_{x_0} \, dt.$$

Write $\mathcal{E} = \bigcup_{k=0}^\infty \mathcal{E}_k$, where

$$\mathcal{E}_0 = \{(x,t) \in \mathcal{E} : 0 \leq |t| \leq 1/R\}$$

and for $k > 0$, define

$$\mathcal{E}_k = \{(x,t) \in \mathcal{E} : 2^{k-1}/R < |t| \leq 2^k/R\}.$$

Suppose that for each k, the set $\mathcal{E}_k$ has density type $(A, 2^k)$. Then

$$\left\| \sum\nolimits_k \sum\nolimits_{(x_0,t_0) \in \mathcal{E}_k} 2^{k\frac{d-1}{2}} S_{x_0,t_0} \right\|_{L^2(M)}^2 \lesssim R^d \log(A) A^{\frac{2}{d-1}} \sum\nolimits_k 2^{k(d-1)} \#\mathcal{E}_k.$$

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*This definition of density is chosen because it is 'scale-invariant' as we change the parameter R, matching the definition given in Section 2.3 when $R = 1$. Indeed, if $M = \mathbb{R}^d$, $\mathcal{X}_R = (\mathbb{Z}/R)^d$, and $\mathcal{T}_R = \mathbb{Z}/R$, then a set $\mathcal{E} \subset (\mathbb{Z}/R)^d \times (\mathbb{Z}/R)$ has density type (A_0, A_1) if and only if $R\ \mathcal{E} \subset \mathbb{Z}^d \times \mathbb{Z}$ has density type (A_0, A_1).

Remark. If $\|b_{t_0}\|_{L^1(I_0)} \sim 1$ and $\|u_{x_0}\|_{L^1(M)} \sim 1$, then locally constancy from the uncertainty principle and energy conservation of the wave equation tell us that morally,

$$\|S_{x_0,t_0}\|_{L^2(M)}^2 \sim R^d t_0^{d-1}. \tag{4.3.4}$$

If $\|b_{t_0}\|_{L^1(I_0)} \sim 1$ and $\|u_{x_0}\|_{L^1(M)} \sim 1$ for all $(x_0,t_0) \in \mathcal{E}$, this means that Proposition 4.8 is morally equivalent to

$$\Big\| \sum\nolimits_{(x_0,t_0)\in\mathcal{E}} S_{x_0,t_0} \Big\|_{L^2(M)} \lesssim \sqrt{\log(A)} A^{\frac{1}{d-1}} \left(\sum\nolimits_{(x_0,t_0)\in\mathcal{E}} \|S_{x_0,t_0}\|_{L^2(M)}^2 \right)^{1/2}. \tag{4.3.5}$$

Thus Proposition 4.8 is a kind of square root cancellation bound, albeit with an implicit constant which grows as the set $\mathcal{E}$ increases in density, a necessity given that the functions $\{S_{x_0,t_0}\}$ are not almost-orthogonal to one another.

Proof. Write $F = \sum_k F_k$, where

$$F_k = 2^{k\frac{d-1}{2}} \sum\nolimits_{(x_0,t_0)\in\mathcal{E}_k} S_{x_0,t_0}. \tag{4.3.6}$$

Our goal is to bound $\|F\|_{L^2(M)}$. Applying Cauchy-Schwarz, we have

$$\|F\|_{L^2(M)}^2 \leq \log(A) \left(\sum\nolimits_{k\leq\log(A)} \|F_k\|_{L^2(M)}^2 + \Big\| \sum\nolimits_{k\geq\log(A)} F_k \Big\|_{L^2(M)}^2 \right). \tag{4.3.7}$$

Without loss of generality, increasing the implicit constant in the final result by applying the triangle inequality, we can assume that $\{k : \mathcal{E}_k \neq \emptyset\}$ is 10-separated, and that all values of t with $(x,t) \in \mathcal{E}$ are positive. Thus if F_k and $F_{k'}$ are both nonzero functions, then $k = k'$ or $|k - k'| \geq 10$.

Let us estimate $\langle F_k, F_{k'} \rangle$ for $k \geq k' + 10$. We write

$$\langle F_k, F_{k'} \rangle = \sum\nolimits_{(x_0,t_0)\in\mathcal{E}_k} \sum\nolimits_{(x_1,t_1)\in\mathcal{E}_{k'}} 2^{k\frac{d-1}{2}} 2^{k'\frac{d-1}{2}} \langle S_{x_0,t_0}, S_{x_1,t_1} \rangle. \tag{4.3.8}$$

For each $(x_0,t_0) \in \mathcal{E}_k$, and each $k' \leq k - 10$, consider the set

$$\mathcal{G}_0(x_0,t_0,k') = \{(x_1,t_1) \in \mathcal{E}_{k'} : |(t_0 - t_1) - d_X^-(x_0,x_1)| \leq 2^{k'+5}/R\}, \tag{4.3.9}$$

Also consider the sets of indices

$$\mathcal{G}_l^+(x_0,t_0,k') = \{(x_1,t_1) \in \mathcal{E}_{k'} : 2^l/R < |(t_0 - t_1) + d_X^+(x_0,x_1)| \leq 2^{l+1}/R\}. \tag{4.3.10}$$

and

$$\mathcal{G}_l^-(x_0,t_0,k') = \{(x_1,t_1) \in \mathcal{E}_{k'} : 2^l/R < |(t_0 - t_1) - d_X^-(x_0,x_1)| \leq 2^{l+1}/R\}. \tag{4.3.11}$$

If we set

$$\mathcal{G}_0(x_0,t_0,k') = (\mathcal{G}_l^+(x_0,t_0,k') \cup \mathcal{G}_l^-(x_0,t_0,k')) \tag{4.3.12}$$

and

$$\mathcal{G}_l(x_0,t_0,k') = \left(\mathcal{G}_l^+(x_0,t_0,k') \cup \mathcal{G}_l^-(x_0,t_0,k')\right) \\ - \bigcup_{r<l}\left(\mathcal{G}_r^+(x_0,t_0,k') \cup \mathcal{G}_r^-(x_0,t_0,k')\right). \tag{4.3.13}$$

Then $\mathcal{E}_{k'}$ is covered by $\mathcal{G}_0(x_0,t_0,k')$ and $\mathcal{G}_l(x_0,t_0,k')$ for $k'+5 \le l \le 10\log R$. Define

$$B_0(x_0,t_0,k') = \sum\nolimits_{(x_1,t_1)\in\mathcal{G}_0(x_0,t_0,k')} 2^{k\frac{d-1}{2}} 2^{k'\frac{d-1}{2}} |\langle S_{x_0,t_0}, S_{x_1,t_1}\rangle|, \tag{4.3.14}$$

and

$$B_l(x_0,t_0,k') = \sum\nolimits_{(x_1,t_1)\in\mathcal{G}_l(x_0,t_0,k')} 2^{k\frac{d-1}{2}} 2^{k'\frac{d-1}{2}} |\langle S_{x_0,t_0}, S_{x_1,t_1}\rangle|. \tag{4.3.15}$$

We thus have

$$\langle F_k, F_{k'}\rangle \le \sum\nolimits_{(x_0,t_0)\in\mathcal{E}_k} B_0(x_0,t_0,k') + \sum\nolimits_{(x_0,t_0)\in\mathcal{E}_k} \sum\nolimits_{k'+5\le l\le 10\log R} B_l(x_0,t_0,k'). \tag{4.3.16}$$

Using the density properties of $\mathcal{E}$, we can control the size of the index sets $\mathcal{G}_\bullet(x_0,t_0,k')$, and thus control the quantities $B_\bullet(x_0,t_0,k')$. The rapid decay of Proposition 4.4 means that only $\mathcal{G}_0(x_0,t_0,k')$ needs to be estimated rather efficiently:

- Start by bounding the quantities $B_0(x_0,t_0,k')$. If $(x_1,t_1)\in\mathcal{G}_0(x_0,t_0,k')$, then

  $$|d_X^-(x_0,x_1) - (t_0-t_1)| \le 2^{k'+5}/R, \tag{4.3.17}$$

  Thus if we consider $\mathrm{Ann}(x_0,t_0,k') = \{x_1 : |d_X^-(x_0,x_1) - t_0| \le 2^{k'+8}/R\}$, which is a geodesic annulus of radius $\sim 2^k/R$ and thickness $O(2^{k'}/R)$, then

  $$\mathcal{G}_0(x_0,t_0,k') \subset \mathrm{Ann}(x_0,t_0,k') \times [2^{k'}/R, 2^{k'+1}/R]. \tag{4.3.18}$$

  The latter set is covered by $O(2^{(k-k')(d-1)})$ balls of radius $2^{k'}/R$, and so the density properties of $\mathcal{E}_{k'}$ implies that

  $$\#(\mathcal{G}_0(x_0,t_0,k')) \lesssim A2^{(k-k')(d-1)}2^{k'}. \tag{4.3.19}$$

  Since $k \ge k'+10$, for $(x_1,t_1)\in\mathcal{G}_0(x_0,t_0,k')$ we have $d_X(x_0,x_1) \gtrsim 2^k/R$ and so

  $$\langle S_{x_0,t_0}, S_{x_1,t_1}\rangle \lesssim R^d 2^{-k\left(\frac{d-1}{2}\right)}. \tag{4.3.20}$$

  But putting together (4.3.19) and (4.3.20) gives that

  $$\begin{aligned} B_0(x_0,t_0,k') &\le (2^{k\frac{d-1}{2}}2^{k'\frac{d-1}{2}})(A2^{(k-k')(d-1)}2^{k'})(R^d2^{-k\left(\frac{d-1}{2}\right)}) \\ &= AR^d 2^{k(d-1)}2^{-k'\frac{d-3}{2}}. \end{aligned}$$

  Thus for each k, since $d \ge 4$,

  $$\sum\nolimits_{(x_0,t_0)\in\mathcal{E}_k} \sum\nolimits_{k'\in[\log(A),k-10]} B_0(x_0,t_0,k') \lesssim R^d 2^{k(d-1)} \#\mathcal{E}_k. \tag{4.3.21}$$

- Next we bound $B_l(x_0, t_0, k')$ for $k' + 5 \leq l \leq k - 5$. The set $\mathcal{G}_l^+(x_0, t_0, k')$ is empty in this case. Thus

$$\mathcal{G}_l(x_0, t_0, k') \subset (\text{Ann} \cup \text{Ann}') \times [t_0 - 2^{k'}/R, t_0 + 2^{k'}/R], \tag{4.3.22}$$

where

$$\text{Ann} = \{x \in M : |d_X^-(x_0, x) - (t_0 - 2^{k'})/R| \leq 100 \cdot 2^l/R\} \tag{4.3.23}$$

and

$$\text{Ann}' = \{x \in M : |d_X^-(x_0, x) - (t_0 + 2^{k'})/R| \leq 100 \cdot 2^l/R\}, \tag{4.3.24}$$

These are geodesic annuli of thickness $O(2^l/R)$ and radius $\sim 2^k$. Thus $\mathcal{G}_l(x_0, t_0, k')$ is covered by $O(2^{(l-k')}2^{(k-k')(d-1)})$ balls of radius $2^{k'}/R$, and the density of $\mathcal{E}_{k'}$ implies that

$$\#(\mathcal{G}_l(x_0, t_0, k')) \lesssim RA\ 2^{(l-k')}2^{(k-k')(d-1)}2^{k'}/R = A2^l 2^{(k-k')(d-1)}. \tag{4.3.25}$$

For $(x_1, t_1) \in \mathcal{G}_l(x_0, t_0, k')$, $d_X(x_0, x_1) \sim 2^k/R$, and thus Proposition 4.4 implies

$$|\langle S_{x_0,t_0}, S_{x_1,t_1}\rangle| \lesssim R^d 2^{-k\frac{d-1}{2}} 2^{-lK}. \tag{4.3.26}$$

Thus for any $K \geq 0$,

$$\begin{aligned} B_l(x_0, t_0, k') &\lesssim_K \left(A2^l 2^{(k-k')(d-1)}\right) R^d 2^{k\frac{d-1}{2}} 2^{k'\frac{d-1}{2}} \left(2^{-k\frac{d-1}{2}} 2^{-lK}\right) \\ &\lesssim AR^d 2^l 2^{k(d-1)} 2^{-k'\frac{d-1}{2}} 2^{-lK}. \end{aligned} \tag{4.3.27}$$

Picking $K > 1$, we conclude that

$$\sum_{(x_0,t_0)\in\mathcal{E}_k} \sum_{k'\in[\log(A),k-10]} \sum_{l\in[k'+10,k-5]} B_l(x_0, t_0, k') \lesssim R^d 2^{k(d-1)} \#\mathcal{E}_k. \tag{4.3.28}$$

- Finally, let's bound $B_l(x_0, t_0, k')$ for $k - 5 \leq l \leq 10 \log R$. If either $(x_1, t_1) \in \mathcal{G}_l^-(x_0, t_0, k')$ or $(x_1, t_1) \in \mathcal{G}_l^+(x_0, t_0, k')$, then $d_X(x_0, x_1) \lesssim 2^l/R$. So $\mathcal{G}_l(x_0, t_0, k')$ is covered by $O(2^{(l-k')d})$ balls of radius $2^{k'}/R$, and thus

$$\#(\mathcal{G}_l(x_0, t_0, k')) \lesssim RA\ 2^{(l-k')d}(2^{k'}/R) = A2^{(l-k')d}2^{k'}. \tag{4.3.29}$$

For $(x_1, t_1) \in \mathcal{G}_l(x_0, t_0, k')$, we have no good control over $d_X(x_0, t_1)$ aside from the trivial estimate $d_X(x_0, x_1) \lesssim 1$. Thus Proposition 4.4 yields a bound of the form

$$|\langle S_{x_0,t_0}, S_{x_1,t_1}\rangle| \lesssim R^d 2^{-lK}. \tag{4.3.30}$$

Thus we conclude that

$$\begin{aligned} B_l(x_0, t_0, k') &\lesssim_N R^d 2^{k\frac{d-1}{2}} 2^{k'\frac{d-1}{2}} \left(A2^{(l-k')d}2^{k'}\right)\left(2^{-lN}\right) \\ &= AR^d 2^{k\frac{d-1}{2}} 2^{-k'\frac{d-1}{2}} 2^{-lN} \end{aligned} \tag{4.3.31}$$

Picking $K > d$, we conclude that

$$\sum\nolimits_{(x_0,t_0)\in\mathcal{E}_k} \sum\nolimits_{k'\in[\log(A),k-10]} \sum\nolimits_{l\in[k+10,\log R]} B_l(x_0, t_0, k') \lesssim R^d. \tag{4.3.32}$$

The three bounds (4.3.21), (4.3.28) and (4.3.32) imply that

$$\sum_k \sum_{k' \in [\log(A), k]} |\langle F_k, F_{k'} \rangle| \lesssim R^d \sum_k 2^{k(d-1)} \# \mathcal{E}_k. \tag{4.3.33}$$

In particular, combining (4.3.33) with (4.3.7), we have

$$\|F\|^2_{L^2(M)} \lesssim \log(A) \left(\sum_k \|F_k\|^2_{L^2(M)} + R^d \sum_k 2^{k(d-1)} \# \mathcal{E}_k \right). \tag{4.3.34}$$

Next, consider some parameter a to be determined later, and decompose the interval $[2^k/R, 2^{k+1}/R]$ into the disjoint union of length A^a/R intervals of the form

$$I_{k,\mu} = [2^k/R + (\mu - 1)A^a/R, 2^k/R + \mu A^a/R] \quad \text{for } 1 \leq \mu \leq 2^k/A^a. \tag{4.3.35}$$

We thus consider a further decomposition $\mathcal{E}_k = \bigcup \mathcal{E}_{k,\mu}$, where $F_k = \sum F_{k,\mu}$. As before, increasing the implicit constant in the Proposition, we may assume without loss of generality that the set $\{\mu : \mathcal{E}_{k,\mu} \neq \emptyset\}$ is 10-separated. We now estimate

$$\sum_{\mu \geq \mu' + 10} |\langle F_{k,\mu}, F_{k,\mu'} \rangle|. \tag{4.3.36}$$

For $(x_0, t_0) \in \mathcal{E}_{k,\mu}$ and $l \geq 1$, define

$$\mathcal{H}_l(x_0, t_0, \mu') = \left\{ (x_1, t_1) \in \mathcal{E}_{k,\mu'} : \frac{2^l A^a}{2R} \leq \max(d_X(x_0, x_1), t_0 - t_1) \leq \frac{2^l A^a}{R} \right\}. \tag{4.3.37}$$

Then $\bigcup_{l \geq 1} \mathcal{H}_l(x_0, t_0, \mu')$ covers $\bigcup_{\mu \geq \mu' + 10} \mathcal{E}_{k,\mu'}$. Set

$$B'_l(x_0, t_0, \mu') = \sum_{(x_1, t_1) \in \mathcal{H}_l(x_0, t_0, \mu')} 2^{k(d-1)} |\langle S_{x_0,t_0}, S_{x_1,t_1} \rangle|. \tag{4.3.38}$$

Then

$$\langle F_{k,\mu}, F_{k,\mu'} \rangle \leq \sum_{(x_0,t_0) \in \mathcal{E}_{k,\mu}} \sum_l B'_l(x_0, t_0, \mu'). \tag{4.3.39}$$

We now bound the constants B'_l. Pick a constant r such that $d_X \leq 2^r d_X^+$ and $d_X \leq 2^r d_X^-$. As in the estimates of the quantities B_l, the quantities where l is large have negligible magnitude:

- For $l \leq k - a \log_2 A + 10r$, we have $2^l A^a/R \lesssim 2^k/R$. The set $\mathcal{H}_l(x_0, t_0, \mu')$ is covered by $O(1)$ balls of radius $2^l A^a/R$, and density properties imply

  $$\# \mathcal{H}_l(x_0, t_0, \mu') \lesssim (RA)(2^l A^a/R) = A^{a+1} 2^l \tag{4.3.40}$$

  For $(x_1, t_1) \in \mathcal{H}_l(x_0, t_0, \mu')$, we claim that

  $$2^{k(d-1)} |\langle S_{x_0,t_0}, S_{x_1,t_1} \rangle| \lesssim R^d 2^{k(d-1)} (2^l A^a)^{-\frac{d-1}{2}}. \tag{4.3.41}$$

  Indeed, for such tuples we have

  $$d_X(x_0, x_1) \gtrsim 2^l A^a/R \quad \text{or} \quad \min_\pm |d_X^\pm(x_0, x_1) - (t_0 - t_1)| \gtrsim 2^l A^a/R, \tag{4.3.42}$$

and the estimate follows from Proposition 4.4 in either case. Since $d \geq 4$, we conclude that

$$\begin{aligned} &\sum\nolimits_{l \in [1, k - a \log_2 A + 10]} B_l'(x_0, t_0, , \mu') \\ &\qquad \lesssim \sum\nolimits_{l \in [1, k - a \log_2 A + 10]} R^d (2^{k(d-1)})(2^l A^a)^{-\frac{d-1}{2}} (A^{a+1} 2^l) \\ &\qquad \lesssim \sum\nolimits_{l \in [1, k - a \log_2 A + 10]} R^d 2^{k(d-1)} 2^{-l\frac{d-3}{2}} A^{1 - a\left(\frac{d-3}{2}\right)} \\ &\qquad \lesssim R^d 2^{k(d-1)} A^{1 - a\left(\frac{d-3}{2}\right)}. \end{aligned} \tag{4.3.43}$$

- For $l > k - a \log_2 A + 10r$, a tuple $(x_1, t_1) \in \mathcal{E}_k$ lies in $\mathcal{H}_l(x_0, t_0, \mu')$ if and only if $2^l A^a / 2R \leq d_X(x_0, x_1) \leq 2^l A^a / R$, since we always have

$$t_0 - t_1 \leq 2^{k+1}/R < 2^{l+r} A^a / 8R. \tag{4.3.44}$$

  and so $d_X(x_0, x_1) \geq 2^l A^a / 2R$. And so

$$|(t_0 - t_1) - d_X^-(x_0, x_1)| \geq 2^{-r} 2^l A^a / 2R - 2^{k+1}/R \geq 2^{-r} 2^l A^a / 4R.$$

  Also $|(t_0 - t_1) + d_X^+(x_0, x_1)| \geq |d_X^+(x_0, x_1) \geq 2^{-r} 2^l A^a / 4R$. Thus we conclude from Proposition 4.4 that

$$2^{k(d-1)} |\langle S_{x_0, t_0}, S_{x_1, t_1} \rangle| \lesssim_K R^d 2^{k(d-1)} (2^l A^a)^{-K}. \tag{4.3.45}$$

  Now $\mathcal{H}_l(x_0, t_0, \mu')$ is covered by $O((2^{l-k} A^a)^d)$ balls of radius $2^k/R$, and the density properties of $\mathcal{E}_k$ thus imply that

$$\#(\mathcal{H}_l(x_0, t_0, \mu')) \lesssim (RA)(2^{l-k} A^a)^d (2^k/R) \lesssim A^{1+ad} 2^{ld} 2^{-k(d-1)}. \tag{4.3.46}$$

  Thus, picking $K > \max(d, 1 + ad)$, we conclude that

$$\begin{aligned} &\sum\nolimits_{l \geq k - a \log_2 A + 10} B_l'(x_0, t_0, \mu') \\ &\quad \lesssim R^d \sum\nolimits_{l \geq k - a \log_2 A + 10} (2^{k(d-1)})(2^l A^a)^{-M} A^{1+ad} 2^{ld} 2^{-k(d-1)} \lesssim R^d. \end{aligned} \tag{4.3.47}$$

Combining (4.3.43) and (4.3.47), and then summing over the tuples $(x_0, t_0) \in \mathcal{E}_{k,\mu}$, we conclude that

$$\sum\nolimits_{\mu \geq \mu' + 10} |\langle F_{k,\mu}, F_{k,\mu'} \rangle| \lesssim R^d \left(1 + 2^{k(d-1)} A^{1 - a\left(\frac{d-3}{2}\right)}\right) \# \mathcal{E}_{k,\mu}. \tag{4.3.48}$$

Now summing in μ, (4.3.48) implies that

$$\|F_k\|_{L^2(M)}^2 \lesssim \sum\nolimits_\mu \|F_{k,\mu}\|_{L^2(M)}^2 + R^d \left(1 + 2^{k(d-1)} A^{1 - a\left(\frac{d-3}{2}\right)}\right) \# \mathcal{E}_k. \tag{4.3.49}$$

The functions in the sum defining $F_{k,\mu}$ are highly coupled, and it is difficult to use anything except Cauchy-Schwarz to break them apart. Since $\#(\mathcal{T}_R \cap I_{k,\mu}) \sim A^a$, if we set $F_{k,\mu} = \sum_{t \in \mathcal{T}_R \cap I_{k,\mu}} F_{k,\mu,t}$, where

$$F_{k,\mu,t} = \sum\nolimits_{(x_0, t) \in \mathcal{E}_{k,\mu}} 2^{k\frac{d-1}{2}} S_{x_0, t}. \tag{4.3.50}$$

Then Cauchy-Schwarz implies that

$$\|F_{k,\mu}\|_{L^2(M)}^2 \lesssim A^a \sum\nolimits_{t\in\mathcal{T}_R\cap I_{k,\mu}} \|F_{k,\mu,t}\|_{L^2(M)}^2. \tag{4.3.51}$$

Since the elements of $\mathcal{X}_R$ are $1/R$ separated, the functions in the sum defining $F_{k,\mu,t}$ are quite orthogonal to one another; Proposition 4.4 implies that for $x_0 \neq x_1$,

$$|\langle S_{x_0,t}, S_{x_1,t}\rangle| \lesssim R^d (R d_X(x_0,x_1))^{-K} \quad \text{for all } K \geq 0. \tag{4.3.52}$$

Thus

$$\|F_{k,\mu,t}\|_{L^2(M)}^2 \lesssim R^d 2^{k(d-1)} \#(\mathcal{E}_k \cap (M \times \{t\})). \tag{4.3.53}$$

But this means that

$$A^a \sum\nolimits_{t\in\mathcal{T}_R\cap I_{k,\mu}} \|F_{k,\mu,t}\|_{L^2(M)}^2 \lesssim R^d 2^{k(d-1)} A^a \#\mathcal{E}_{k,\mu}. \tag{4.3.54}$$

Thus (4.3.49), (4.3.51), and (4.3.54) imply that

$$\begin{aligned}\|F_k\|_{L^2(M)}^2 &\lesssim \sum\nolimits_\mu \|F_{k,\mu}\|_{L^2(M)}^2 + R^d \left(1 + 2^{k(d-1)} A^{1-a\left(\frac{d-3}{2}\right)}\right) \#\mathcal{E}_k \\ &\lesssim R^d \left(2^{k(d-1)} A^a + (1 + 2^{k(d-1)} A^{1-a\left(\frac{d-3}{2}\right)}\right) \#\mathcal{E}_k.\end{aligned} \tag{4.3.55}$$

Optimizing by picking $a = 2/(d-1)$ gives that

$$\|F_k\|_{L^2(M)}^2 \lesssim R^d 2^{k(d-1)} A^{\frac{2}{d-1}} \#\mathcal{E}_k. \tag{4.3.56}$$

The proof is completed by combining (4.3.34) with (4.3.56). □

Combining the L^2 analysis of Section 4.3 with a density decomposition argument, we can now prove the following Lemma, which completes the analysis of the operator T^I in Proposition 4.3.

Lemma 4.9. *Using the notation of Proposition 4.3, let $T^I = \sum_{t_0\in\mathcal{T}_R} T^I_{t_0}$, where*

$$T^I_{t_0} = b^I_{t_0}(t)(Q_R \circ e^{2\pi i t P} \circ Q_R)\, dt.$$

Then for $1 \leq p < 2(d-1)/(d+1)$,

$$\|T^I u\|_{L^p(M)} \lesssim R^{-1/p'} \left(\sum\nolimits_{t_0\in\mathcal{T}_R} \left[\|b^I_{t_0}\|_{L^p(I_0)} \langle R t_0\rangle^{\alpha(p)}\right]^p\right)^{1/p} \|u\|_{L^p(M)}.$$

We prove Lemma 4.9 via a *density decomposition* argument, adapted from the methods of [8]. Given a function $u : M \to \mathbb{C}$, we use a partition of unity to write

$$u = \sum\nolimits_{x_0\in\mathcal{X}_R} u_{x_0}, \tag{4.3.57}$$

where u_{x_0} is supported on $B(x_0, 1/R)$, and

$$\left(\sum\nolimits_{x_0\in\mathcal{X}_R} \|u_{x_0}\|_{L^1(M)}^p\right)^{1/p} \lesssim R^{-d/p'} \left(\sum\nolimits_{x_0\in\mathcal{X}_R} \|u_{x_0}\|_{L^p(M)}^p\right)^{1/p} \lesssim R^{-d/p'} \|u\|_{L^p(M)}. \tag{4.3.58}$$

Define

$$\mathcal{X}_a = \{x_0 \in \mathcal{X}_R : 2^{a-1} < \|u_{x_0}\|_{L^1(M)} \leq 2^a\} \tag{4.3.59}$$

and let

$$\mathcal{T}_b = \{t_0 \in \mathcal{T}_R : 2^{b-1} < \|b^I_{t_0}\|_{L^1(M)} \leq 2^b\}. \tag{4.3.60}$$

Define functions $f_{x_0,t_0} = T^I_{t_0} u_{x_0}$. Lemma 4.9 follows from the following result.

Lemma 4.10. *Fix $u \in L^p(M)$, and consider $\mathcal{X}_a$, $\mathcal{T}_b$, and $\{f_{x_0,t_0}\}$ as above. For any function $c : \mathcal{X}_R \times \mathcal{T}_R \to \mathbb{C}$, and $1 < p < 2(d-1)/(d+1)$,*

$$\begin{aligned} &\left\| \sum\nolimits_{a,b} \sum\nolimits_{(x_0,t_0)\in \mathcal{X}_a \times \mathcal{T}_b} 2^{-(a+b)} \langle Rt_0 \rangle^{\frac{d-1}{2}} c(x_0,t_0) f_{x_0,t_0} \right\|_{L^p(M)} \\ &\qquad \lesssim R^{d/p'} \left(\sum\nolimits_{a,b} \sum\nolimits_{(x_0,t_0)\in \mathcal{X}_a \times \mathcal{T}_b} |c(x_0,t_0)|^p \langle Rt_0 \rangle^{d-1} \right)^{1/p}. \end{aligned}$$

To see how Lemma 4.10 implies Lemma 4.9, set $c(x_0,t_0) = 2^{a+b} \langle Rt_0 \rangle^{-\frac{d-1}{2}}$ for $x_0 \in \mathcal{X}_a$ and $t_0 \in \mathcal{T}_b$. Then Lemma 4.10 implies that

$$\begin{aligned} &\|T^I u\|_{L^p(M)} \\ &\quad = \left\| \sum f_{x_0,t_0} \right\|_{L^p(M)} \\ &\quad \lesssim R^{d/p'} \left(\sum\nolimits_{(x_0,t_0)} \left[\|b^I_{t_0}\|_{L^1(\mathbb{R})} \|u_{x_0}\|_{L^1(M)} \langle Rt_0 \rangle^{\alpha(p)} \right]^p \right)^{1/p} \\ &\quad \lesssim R^{-1/p'} \left(\sum\nolimits_{t_0} \left[\|b^I_{t_0}\|_{L^p(I_0)} \langle Rt_0 \rangle^{\alpha(p)} \right]^p \right)^{1/p} \left(\sum\nolimits_{x_0} \left[\|u_{x_0}\|_{L^1(M)} R^{d/p'} \right]^p \right)^{1/p} \\ &\quad \lesssim R^{-1/p'} \left(\sum\nolimits_{t_0} \left[\|b^I_{t_0}\|_{L^p(I_0)} \langle Rt_0 \rangle^{\alpha(p)} \right]^p \right)^{1/p} \|u\|_{L^p(M)}. \end{aligned} \tag{4.3.61}$$

Thus we have proved Lemma 4.9. We take the remainder of this section to prove Lemma 4.10 using a density decomposition argument.

Proof of Lemma 4.10. For $p = 1$, this inequality follows simply by applying the triangle inequality, and applying the pointwise estimates of Proposition 4.4. By methods of interpolation, to prove the result for $p > 1$, we thus only need only prove a restricted strong type version of this inequality. In other words, we can restrict c to be the indicator function of a set $\mathcal{E} \subset \mathcal{X}_R \times \mathcal{T}_R$. Write $\mathcal{E} = \bigcup_{k \geq 0} \mathcal{E}_{k,a,b}$, where

$$\mathcal{E}_{0,a,b} = \{(x,t) \in \mathcal{E} \cap (\mathcal{X}_a \times \mathcal{T}_b) : |t| \leq 1/R\} \tag{4.3.62}$$

and for $k > 0$, let

$$\mathcal{E}_{k,a,b} = \{(x,t) \in \mathcal{E} \cap (\mathcal{X}_a \times \mathcal{T}_b) : 2^{k-1}/R < |t| \leq 2^k/R\}. \tag{4.3.63}$$

Write

$$F_k = \sum\nolimits_{a,b} \sum\nolimits_{(x_0,t_0)\in \mathcal{E}_{k,a,b}} 2^{k\left(\frac{d-1}{2}\right)} 2^{-(a+b)} f_{x_0,t_0}. \tag{4.3.64}$$

Our proof will be completed if we can show that

$$\Big\| \sum_k F_k \Big\|_{L^p(M)} \lesssim R^{d(1-1/p)} \Big(\sum_k 2^{k(d-1)} \# \mathcal{E}_k \Big)^{1/p}. \tag{4.3.65}$$

To prove (4.3.65), we perform a density decomposition on the sets $\{\mathcal{E}_k\}$. For $u \geq 0$, let $\widehat{\mathcal{E}_k}(u)$ be the set of all points $(x_0, t_0) \in \mathcal{E}_k$ that are contained in a ball B with $\text{rad}(B) \leq 2^k/100R$, such that $\#(\mathcal{E}_k \cap B) \geq R2^u \text{rad}(B)$. Then define

$$\mathcal{E}_k(u) = \widehat{\mathcal{E}_k}(u) - \bigcup_{u'>u} \widehat{\mathcal{E}_k}(u'). \tag{4.3.66}$$

Because the set $\mathcal{E}_k$ is $1/R$ discretized, we have

$$\mathcal{E}_k = \bigcup_{u \geq 0} \mathcal{E}_k(u). \tag{4.3.67}$$

Moreover $\mathcal{E}_k(u)$ has density type $(R2^u, 2^k/100R)$, and thus by a covering argument, also has density type $(C_d R 2^u, 2^k/R)$ for $C_d = 1000^d$. Furthermore, there are disjoint balls $B_{k,u,1}, \dots, B_{k,u,N_{k,u}}$ of radius at most $2^k/100R$ such that

$$\sum_n \text{rad}(B_{k,u,n}) \leq 2^{-u}/R \# \mathcal{E}_k. \tag{4.3.68}$$

and such that $\mathcal{E}_k(u)$ is covered by the balls $\{B^*_{k,u,n}\}$, where, for a ball B, B^* denotes the ball with the same center as B, but 5 times the radius. Now write

$$F_{k,u} = \sum_{a,b} \sum_{(x_0,t_0) \in \mathcal{E}_{k,a,b}(u)} 2^{k\left(\frac{d-1}{2}\right)} 2^{-(a+b)} f_{x_0,t_0}. \tag{4.3.69}$$

Using the density assumption on $\mathcal{E}_k(u)$, we can apply Lemma 4.8 of the last section, which implies that, with $S_{x_0,t_0} = 2^{-(a+b)} f_{x_0,t_0}$ for $(x_0, t_0) \in \mathcal{E}_{k,a,b}(u)$,

$$\Big\| \sum_k F_{k,u} \Big\|_{L^2(M)} \lesssim R^{d/2} \left(u^{1/2} 2^{u\left(\frac{1}{d-1}\right)} \right) \Big(\sum_k 2^{k(d-1)} \# \mathcal{E}_k \Big)^{1/2}. \tag{4.3.70}$$

Let $(y_{k,u,n}, t_{k,u,n})$ denote the center of $B_{k,u,n}$. Then

$$\sum_{(x_0,t_0) \in [B_{k,u,n} \cap \mathcal{E}_k(u)]} 2^{-(a+b)} f_{x_0,t_0} \tag{4.3.71}$$

has mass concentrated on the geodesic annulus $\text{Ann}_{k,u,n} \subset M$ with center $y_{k,u,n}$, with radius $t_{k,u,n} \sim 2^k/R$, and with thickness 5 $\text{rad}(B_{k,u,n})$. Thus

$$\sum_n |\text{Ann}_{k,u,n}| \lesssim \sum_n (2^k/R)^{d-1} \text{rad}(B_{k,u,n}) \leq (2^k/R)^{d-1} R^{-1} 2^{-u} \# \mathcal{E}_k. \tag{4.3.72}$$

If we set $\Lambda_u = \bigcup_k \bigcup_n \text{Ann}_{k,u,n}$, then

$$|\Lambda_u| \lesssim R^{-d} 2^{-u} \sum_k 2^{k(d-1)} \# \mathcal{E}_k \tag{4.3.73}$$

Since $1/p - 1/2 > 1/(d-1)$, so that $\alpha(p) > 1$, Hölder's inequality implies that

$$\begin{aligned}
\Big\| \sum_k F_{k,u} \Big\|_{L^p(\Lambda_u)} &\lesssim |\Lambda_u|^{1/p-1/2} \Big\| \sum_k F_{k,u} \Big\|_{L^2(\Lambda_{k,u})} \\
&\lesssim \Big(R^{-d} 2^{-u} \sum_k 2^{k(d-1)} \# \mathcal{E}_k \Big)^{1/p-1/2} \\
&\qquad \Big(R^d \Big(u 2^{u\left(\frac{2}{d-1}\right)} \Big) \sum_k 2^{k(d-1)} \# \mathcal{E}_k \Big)^{1/2} \\
&= R^{d(1-1/p)} \Big(u^{1/2} 2^{-u\left(\frac{\alpha(p)-1}{d-1}\right)} \Big) \Big(\sum_k 2^{k(d-1)} \# \mathcal{E}_k \Big)^{1/p} \\
&\lesssim R^{d(1-1/p)} 2^{-u\varepsilon} \Big(\sum_k 2^{k(d-1)} \# \mathcal{E}_k \Big)^{1/p}
\end{aligned} \tag{4.3.74}$$

for some suitable small $\varepsilon > 0$. For each $(x_0, t_0) \in \mathcal{E}_{k,a,b}(u) \cap B_{k,u,n}$, we calculate using the pointwise bounds for the functions $\{f_{x_0,t_0}\}$ that

$$\begin{aligned}
\| 2^{-(a+b)} f_{x_0,t_0} \|_{L^1(\Lambda(u)^c)} &= R^d \int_{\mathrm{Ann}_R^c} \langle R d_X(x, x_0) \rangle^{-\left(\frac{d-1}{2}\right)} \langle R |t_0 - d_X(x, x_0)| \rangle^{-M} \, dx \\
&\lesssim R^{\left(\frac{d+1}{2} - M\right)} \int_{5 \,\mathrm{rad}(B_{k,u,n})}^{O(1)} (t_{k,u,n} + s)^{\left(\frac{d-1}{2}\right)} s^{-M} \, ds \\
&\lesssim R^{\left(\frac{d+1}{2} - M\right)} \mathrm{rad}(B_{k,u,n})^{1-M} t_{k,u,n}^{\left(\frac{d-1}{2}\right)} \\
&\lesssim 2^{k\left(\frac{d-1}{2}\right)} (R \mathrm{rad}(B_{k,u,n}))^{1-M}.
\end{aligned} \tag{4.3.75}$$

Thus

$$\| 2^{k\left(\frac{d-1}{2}\right)} 2^{-(a+b)} f_{x_0,t_0} \|_{L^1(\mathrm{Ann}_n^c)} \lesssim 2^{k(d-1)} (R \mathrm{rad}(B_{k,u,n}))^{1-M} \tag{4.3.76}$$

Because the set of points in $\mathcal{E}_k$ is $1/R$ separated, there are at most $O((R\mathrm{rad}(B_{k,u,n}))^{d+1})$ points in $\mathcal{E}_k(u) \cap B_{k,u,n}$, and so the triangle inequality implies that

$$\begin{aligned}
\Big\| \sum_{a,b} \sum_{(x_0,t_0) \in \mathcal{E}_{k,a,b}(u) \cap B_{k,u,n}} 2^{k\left(\frac{d-1}{2}\right)} 2^{-(a+b)} f_{x_0,t_0} \Big\|_{L^1(\Lambda(u)^c)} \\
\lesssim 2^{k(d-1)} (R \mathrm{rad}(B_{k,u,n}))^{d+2-M}.
\end{aligned} \tag{4.3.77}$$

Since $\# \mathcal{E}_k \cap B_{k,u,n} \geq R 2^u \, \mathrm{rad}(B_{k,u,n})$, and $\mathcal{E}_k$ is $1/R$ discretized, we must have

$$\mathrm{rad}(B_{k,u,n}) \geq (2^u / 2^d)^{\frac{1}{d-1}} \; 1/R. \tag{4.3.78}$$

Thus

$$\begin{aligned}
\Big\| \sum_k F_{k,u} \Big\|_{L^1(\Lambda(u)^c)} &\lesssim_X \sum_k \sum_n 2^{k(d-1)} (R \mathrm{rad}(B_{k,u,n}))^{d+2-M} \\
&\lesssim \sum_k 2^{k(d-1)} \Big(R \min_n \mathrm{rad}(B_{k,u,n}) \Big)^{d+1-M} \\
&\qquad \Big(\sum_n R \mathrm{rad}(B_{k,u,n}) \Big) \\
&\lesssim \sum_k 2^{k(d-1)} 2^{u\left(\frac{d+1-M}{d-1}\right)} (2^{-u} \# \mathcal{E}_k) \\
&\lesssim 2^{u\left(\frac{2-M}{d-1}\right)} \sum_k 2^{k(d-1)} \# \mathcal{E}_k
\end{aligned} \tag{4.3.79}$$

Picking $M > 2+(1-1/p)(1/p-1/2)^{-1}$, and interpolating with the bounds on $\| \sum_k F_{k,u}\|_{L^2(M)}$ yields that

$$\begin{aligned}\Big\| \sum\nolimits_k F_{k,u}\Big\|_{L^p(\Lambda(u)^c)} &\lesssim \left(2^{u\left(\frac{2-M}{d-1}\right)}\right)^{2/p-1} \left(R^{d/2}\left(u^{1/2}2^{u\left(\frac{1}{d-1}\right)}\right)\right)^{2(1-1/p)} \\ &\qquad \left(\sum 2^{k(d-1)}\#\mathcal{E}_k\right)^{1/p} \\ &\lesssim 2^{-u\varepsilon}R^{d(1-1/p)}\sum\nolimits_k\left(\sum 2^{k(d-1)}\#\mathcal{E}_k\right)^{1/p}.\end{aligned} \tag{4.3.80}$$

So now we know

$$\Big\| \sum\nolimits_k F_{k,u}\Big\|_{L^p(M)} \lesssim R^{d(1-1/p)}2^{-u\varepsilon}\left(\sum\nolimits_k 2^{k(d-1)}\#\mathcal{E}_k\right)^{1/p}. \tag{4.3.81}$$

The exponential decay in u allows us to sum in u to obtain that

$$\Big\| \sum\nolimits_u\sum\nolimits_k F_{k,u}\Big\|_{L^p(M)} \lesssim R^{d(1-1/p)}\left(\sum\nolimits_k 2^{k(d-1)}\#\mathcal{E}_k\right)^{1/p}. \tag{4.3.82}$$

This is precisely the bound we were required to prove. □

4.4 Analysis of Regime II via Local Smoothing

In this section, we bound the operators $\{T^{II}\}$, by a reduction to an endpoint local smoothing inequality, namely, the inequality that

$$\|e^{2\pi itP}f\|_{L^{p'}(M)L^{p'}_t(I_0)} \lesssim \|f\|_{L^{p'}_{\alpha(p)-1/p'}}. \tag{4.4.1}$$

This inequality is proved in Corollary 1.2 of [13] for $1 < p < 2(d-1)/(d+1)$ for classical elliptic pseudodifferential operators P satisfying the cosphere assumption of Theorem 1.2. The range of p here is also precisely the range of p in Theorem 1.2. Alternatively, Lemma 4.10 can be used to prove (4.4.1) independently of [13] in the same range by a generalization of the method of Section 10 of [8].

Lemma 4.11. *Using the notation of Proposition 4.3, let*

$$T^{II} = \int b^{II}(t)(Q_R \circ e^{2\pi itP} \circ Q_R)\, dt.$$

For $1 < p < 2(d-1)/(d+1)$*, we then have*

$$\|T^{II}u\|_{L^p(M)} \lesssim R^{\alpha(p)-1/p'}\|b^{II}\|_{L^p(I_0)}\|u\|_{L^p(M)}.$$

Proof. For each R, the *class* of operators of the form $\{T^{II}\}$ formed from a given function b^{II} is closed under taking adjoints. Indeed, if T^{II} is obtained from b^{II}, then $(T^{II})^*$ is obtained from the multiplier $\overline{b^{II}}$. Because of this self-adjointness, if we can prove that

$$\|T^{II}u\|_{L^{p'}(M)} \lesssim R^{\alpha(p)-1/p'}\|b^{II}\|_{L^p(I_0)}\|u\|_{L^{p'}(M)}, \tag{4.4.2}$$

then we obtain the required result by duality. We apply this duality because it is easier to exploit local smoothing inequalities in $L^{p'}(M)$ since now $p' > 2$.

We begin by noting that the operators $\{Q_R\}$, being a bounded family of order zero pseudo-differential operators, are uniformly bounded on $L^{p'}(M)$. Thus

$$\begin{aligned} \|T^{II}u\|_{L^{p'}(M)} &= \Big\| Q_R \circ \Big(\int_{I_0} b^{II}(t) e^{2\pi i t P}(Q_R u) \Big) \Big\|_{L^{p'}(M)} \\ &\lesssim \Big\| \Big(\int_{I_0} b^{II}(t) e^{2\pi i t P}(Q_R u) \Big) \Big\|_{L^{p'}(M)}. \end{aligned} \tag{4.4.3}$$

Applying Hölder and Minkowski's inequalities, we find that

$$\|T^{II}u\|_{L^{p'}(M)} \leq \|b^{II}\|_{L^p(\mathbb{R})} \Big\| \Big(\int_{I_0} |e^{2\pi i t P}(Q_R u)|^{p'} \Big)^{1/p'} \Big\|_{L^{p'}(M)}. \tag{4.4.4}$$

Applying the endpoint local smoothing inequality (4.4.1), we conclude that

$$\begin{aligned} \|T^{II}u\|_{L^{p'}(M)} &\lesssim \|b^{II}\|_{L^p(\mathbb{R})} \|e^{2\pi i P}(Q_R u)\|_{L^{p'}_t L^{p'}_x} \\ &\lesssim \|b^{II}\|_{L^p(\mathbb{R})} \|Q_R u\|_{L^q_{\alpha(p)-1/p'}(M)}, \end{aligned} \tag{4.4.5}$$

Bernstein's inequality for compact manifolds (see [20], Section 3.3) gives

$$\|Q_R u\|_{L^q_{\alpha(p)-1/p'}(M)} \lesssim R^{\alpha(p)-1/p'} \|u\|_{L^p(M)}. \tag{4.4.6}$$

Thus we conclude that

$$\|T^{II}u\|_{L^{p'}(M)} \lesssim R^{\alpha(p)-1/p'} \|b^{II}\|_{L^p(I_0)} \|u\|_{L^{p'}(M)}, \tag{4.4.7}$$

which completes the proof. □

Combining Lemma 4.9 and Lemma 4.11 completes the proof of Proposition 4.3, and thus of inequality (4.0.2). Since (4.0.1) was already proven as a consequence of Lemma 4.1, this completes the proof of Theorem 1.2.

Chapter 5

Combining Scales Via Atomic Decompositions

In this chapter, we discuss a prospective method of combining frequency scales for spectral multipliers. We take an operator P satisfying Assumptions A and B, and suppose P satisfies a third assumption, which is still sufficient to prove Theorem 1.3.

Assumption C: The operator

$$\cos(2\pi tP) = \frac{e^{2\pi itP} + e^{-2\pi itP}}{2}$$

which is a propogator for the wave equation $\partial_t^2 u = -4\pi^2 P^2$, has *finite propogation speed*, i.e. the kernel of $\cos(2\pi tP)$ is supported on $\{(x,y) : d(x,y) \leq Ct\}$ for some $C > 0$.

Under these assumptions, we obtain an analogue of the results of [7], proving that we can combine frequency scales under an assumption which is *slightly stronger* than controlling each part of the operator separately. We adapt the techniques from the study of Fourier multipliers introduced in Section 2.4.

Consider a pair of maximal 2^{-k} separated subsets $\mathcal{X}_k$ and $\mathcal{T}_k$ of X and of $[0,1]$. Fix a family of functions $\mathfrak{b} = \{b_{t_0}\}$ and $\mathfrak{u} = \{u_{x_0}\}$ with $\|b_{t_0}\|_{L^1(\mathbb{R})} \leq 1$ and $\|u_{x_0}\|_{L^1(X)} \leq 1$ for each $x_0 \in \mathcal{X}_k$ and $t_0 \in \mathcal{T}_k$. Also fix a bump function $q \in C_c^\infty(\mathbb{R})$ with $\operatorname{supp}(q) \subset [1/4, 4]$ and $q(\lambda) = 1$ for $\lambda \in [1/2, 2]$, and define $Q_k = q(P/2^k)$. For each $(x_0, t_0) \in \mathcal{X}_k \times \mathcal{T}_k$, define the function

$$f_{x_0,t_0} = \int b_{t_0}(t)(Q_k \circ \cos(2\pi itP) \circ Q_k)\{u_{x_0}\}.$$

Finally, define an operator A_k from functions on $\mathcal{X}_k \times \mathcal{T}_k$ to functions on X by

$$A_k\{c\} = \sum\nolimits_{(x_0,t_0)\in\mathcal{X}_k\times\mathcal{T}_k} \langle 2^k t_0\rangle^{\frac{d-1}{2}} c(x_0,t_0) f_{x_0,t_0}.$$

We assume uniform bounds on such operators.

Assumption Wave-Bound(p, d): There exists a constant $C > 0$ such that

$$\|A_k\{c\}\|_{L^p(X)} \leq C\, 2^{k\left(\frac{d+1}{p}\right)} \left(\sum\nolimits_{(x_0,t_0) \in \mathcal{X}_k \times \mathcal{T}_k} |c(x_0, t_0)|^p \langle 2^k t_0 \rangle^{d-1} \right)^{1/p}.$$

uniformly in k, $\mathfrak{b}$, and $\mathfrak{u}$.

As we have seen in the previous chapter, such a bound naturally arises in the study of averages of the wave equation. In particular, Wave-Bound(p, d) implies that for any function a, if $a_k = \chi(t) a(2^k t)$ for some $\chi \in C_c^\infty(\mathbb{R})$ with $1 = \sum \chi(\lambda/2^k)$ for $\lambda \neq 0$, then

$$\|a_k(P/2^k)\|_{L^p(X) \to L^p(X)} \lesssim \|a\|_{R^{s,p}[0,\infty)} \quad \text{for } s = (d-1)(1/p - 1/2), \tag{5.0.1}$$

uniformly in k. The main result of this paper is that one can also 'sum these bounds' to bound $a(P) = \sum a_k(P/2^k)$.

Theorem 5.1. *Let X be a manifold, and let P be an elliptic operator on X satisfying Assumptions A, B, and C. Suppose $1 \leq p \leq q$, with $1/q - 1/2 > 1/2d$ and Wave-Bound(q, d) is true for P. Then for any regulated function a, if $s = (d-1)(1/p - 1/2)$,*

$$\|a\|_{M^p_{Dil}(X)} \lesssim \|a\|_{R^{s,p}[0,\infty)}.$$

Before we prove this theorem, let us see how it implies Theorem 1.3, restated below.

Theorem 1.3. *Consider S^d, equipped with the operator P_{SH}. Then if $1/p - 1/2 > 1/(d-1)$ and $s = (d-1)(1/p - 1/2)$, $M^p_{Dil}(S^d) = R^{s,p}[0,\infty)$.*

Proof. Lemma 4.3 implies □

Since the right hand side of the inequality in the theorem is invariant under dilations of a, it suffices to prove a bound of the form $\|a(P)\|_{L^p(X) \to L^p(X)} \lesssim \|a\|_{R^{s,q}[0,\infty)}$. To prove Theorem 5.1, we use an analogue of the technique of atomic decompositions introduced in Section 2.4. The following lemma describes the properties of this atomic decomposition useful to us, and is proved in an appendix.

Lemma 5.2. *Consider coordinate charts $\{U_\alpha\}$ covering X. Then, for any measurable function $u : X \to \mathbb{C}$, if we consider the dyadic decomposition $u = \sum u_k$, where $u_k = Q_k u$ is a quasimode with eigenvalue 2^k, then we have a further decomposition*

$$u_k = \sum\nolimits_\alpha \sum\nolimits_H \sum\nolimits_{W \in \mathcal{W}_{\alpha,H}} a_{\alpha,k,H,W},$$

where H ranges over powers of 2. For each α and H, $\mathcal{W}_{\alpha,H}$ is a family of almost disjoint dyadic cubes in the coordinate system U_α whose union is a set $\Omega_{\alpha,H}$, such that the following properties hold:

- *The 10-fold dilates $\{W^* : W \in \mathcal{W}_{\alpha,H}\}$ have the bounded overlap property.*
- *If $l(W) = 2^l$, then $a_{\alpha,k,H,W} = 0$ for $k < -l$.*

- *For each W, supp$(a_{\alpha,k,H,W}) \subset W$, but as H varies, the functions $\{a_{\alpha,k,H,W}\}$ have disjoint support.*
- *For each H,*

$$\left(\sum_k \sum_\alpha \sum_{W\in\mathcal{W}_{\alpha,H}} \|a_{\alpha,k,H,W}\|_{L^2(X)}^2\right)^{1/2} \lesssim H|\Omega_{\alpha,H}|,$$

- *For any choice of indices $k(\alpha, W)$ for each α and W, we have*

$$\left(\sum_\alpha \sum_{W\in\mathcal{W}_{\alpha,H}} |W|\|a_{\alpha,k(\alpha,W),H,W}\|_{L^\infty(X)}^p\right)^{1/p} \lesssim H|\Omega_{\alpha,H}|.$$

- *For each α,*

$$\left(\sum_H H^p|\Omega_{\alpha,H}|\right)^{1/p} \lesssim \|u\|_{L^p(X)}.$$

To exploit this atomic decomposition, we write

$$a(P)u = \sum_k m_k(P/2^k)\{u_k\} = \sum_\alpha \sum_k \sum_H \sum_{W\in\mathcal{W}_{\alpha,H}} m_k(P/2^k)\{a_{\alpha,k,H,W}\}.$$

We regroup this sum as

$$\sum_\alpha \sum_k \sum_H \sum_{l\geq 0} \sum_{\substack{W\in\mathcal{W}_{\alpha,H}\\ l(W)=l-k}} a_k(P/2^k)\{a_{\alpha,k,H,W}\}.$$

For each k and l, we write $a_k(P/2^k) = T_{k,l,\text{Short}} + T_{k,l,\text{Long}}$, where

$$T_{k,l,\text{Short}} = \int_0^\infty \chi(2^{k-l}t)2^k\widehat{a}_k(2^kt)\cos(2\pi itP)$$

and

$$T_{k,l,\text{Long}} = \int_0^\infty (1-\chi(2^{k-l}t))2^k\widehat{a}_k(2^kt)\cos(2\pi itP).$$

For $f = Tu$, we thus write $f = f_{\text{Short}} + f_{\text{Long}}$, where

$$f_{\text{Short}} = \sum_\alpha \sum_k \sum_H \sum_{l\geq 0} \sum_{\substack{W\in\mathcal{W}_{\alpha,H}\\ l(W)=l-k}} T_{k,l,\text{Short}}\{a_{\alpha,k,H,W}\} = \sum f_{\alpha,k,H,W,\text{Short}}.$$

and

$$f_{\text{Long}} = \sum_\alpha \sum_k \sum_H \sum_{l\geq 0} \sum_{\substack{W\in\mathcal{W}_{\alpha,H}\\ l(W)=l-k}} T_{k,l,\text{Long}}\{a_{\alpha,k,H,W}\} = \sum f_{\alpha,k,H,W,\text{Long}},$$

and analyze each part using separate techniques.

5.1 Short Range Bounds

To obtain short range bounds, we exploit the propogation speed of the operators $\cos(2\pi itP)$, and the bounded overlap of the sets $W_{\alpha,H}$ for a fixed H. To obtain an L^q bound for f_{Short}, we interpolate between an L^2 bound and an L^1 bound, at a fixed quantity H. So write $f_{\text{Short}} = \sum_\alpha \sum_k \sum_H f_{\alpha,k,H}$. In L^2, different frequencies are orthogonal, so that

$$\|f_{H,\text{Short}}\|_{L^2(X)} \lesssim \left(\sum_k \left\|\sum_{W\in\mathcal{W}_{\alpha,H}} f_{\alpha,k,H,W,\text{Short}}\right\|_{L^2(X)}^2\right)^{1/2}.$$

We note that by the finite propogation speed of $\cos(2\pi tP)$, $\langle f_{\alpha,k,H,W,\text{Short}}, f_{\alpha,k,H,W',\text{Short}}\rangle = 0$ if $W^* \cap (W')^* = \emptyset$. By the bounded overlap property of the sets $\{W^*\}$, these functions are almost orthogonal, and so we conclude that

$$\|f_{H,\text{Short}}\|_{L^2(X)} \lesssim \left(\sum_k \sum_{W\in\mathcal{W}_{\alpha,H}} \left\|f_{\alpha,k,H,W,\text{Short}}\right\|_{L^2(X)}^2\right)^{1/2}.$$

We can write $T_{k,l,\text{Short}} = a_{k,l}(P/2^k)$, where

$$a_{k,l}(t) = [2^l\widehat{\chi}(2^l\cdot) * a_k].$$

Now

$$\|a_{k,l}(P/2^k)\|_{L^2(X)\to L^2(X)} = \|a_{k,l}\|_{L^\infty(\mathbb{R})} \lesssim \|a_k\|_{L^\infty(\mathbb{R})} \lesssim \|a\|_{R^{s,p}[0,\infty)},$$

and so

$$\|f_{\alpha,k,H,W}\|_{L^2(X)} = \|T_{k,l,\text{Short}}\{a_{\alpha,k,H,W}\}\|_{L^2(X)} \lesssim \|a\|_{R^{s,p}[0,\infty)}\|a_{\alpha,k,H,W}\|_{L^2(X)}.$$

But we thus conclude that

$$\|f_{H,\text{Short}}\|_{L^2(X)} \lesssim \|a\|_{R^{s,p}[0,\infty)} 2^H |\Omega_H|^{1/2}.$$

For any $r \in [1,2]$, we can then use Hölder's inequality, since (again by finite propogation speed) that $f_{H,\text{Short}}$ is supported on a set $\{x : (M\chi_{\Omega_H})(x) \geq 1/10^d\}$, which by the weak L^1 boundedness of M has measure $O(|\Omega_H|)$, to concclude that

$$\|f_{H,\text{Short}}\|_{L^1(X)} \lesssim \|a\|_{R^{s,p}[0,\infty)}|\Omega_H|^{1/2}2^H|\Omega_H|^{1/2} = \|a\|_{R^{s,p}[0,\infty)}2^H|\Omega_H|.$$

Performing a real interpolation between $r = 1$ and $r = 2$, which allows us to sum over the dyadic height scales H, we conclude that

$$\|f_{\text{Short}}\|_{L^p(X)} \lesssim \left(\sum H^p|\Omega_H|\right)^{1/p} \lesssim \|u\|_{L^p(X)}.$$

This completes the analysis of the short range interactions.

5.2 Long Range Bounds

The long range interactions require slightly more work. Define an operator

$$S_{l,k}C = \sum_{z_0} \sum_{t_0 \geq 2^{l-k}} \langle 2^k t_0 \rangle^{\frac{d-1}{2}} C(z_0, t_0) \sum_{x_0 \in Q(z_0)} F_{x_0,t_0},$$

where $U_{z_0} = \sum_{x_0 \in Q(z_0)} u_{x_0}$ and $F_{x_0,t_0} = \int a_{t_0} \cos(2\pi t P)\{U_{z_0}\}$. The bound Wave-Bound$(q,d)$ implies an exponential decay in l on the L^p operator norm of $S_{l,k}$.

Lemma 5.3. *Suppose* $1 \leq p \leq q$*, and that Wave-Bound*(q,d) *is true. Then*

$$\begin{aligned} &\|S_{l,k}\{C\}\|_{L^p(X)} \\ &\quad \lesssim 2^{k\frac{d+1}{p'}} 2^{-l\varepsilon} \left(\sum_{z_0} \sum_{t_0 \geq 2^{l-k}} |C(z_0,t_0)|^p [2^{(l-k)d} \|U_{z_0,t_0}\|^p_{L^\infty(X)}] \langle 2^k t_0 \rangle^{d-1} \right)^{1/p}, \end{aligned}$$

where

$$\varepsilon = -\left(\frac{d-1}{2}\right)\left(\frac{1/p - 1/q}{1 - 1/q}\right),$$

which, in particular, is positive for $p < q$.

Proof. By applying the triangle inequality, we may assume that the function C is 10-separated in the z_0 variable. Thus each x_0 is contained in a unique sidelength cube $Q(z(x_0))$ with $z(x_0)$ in $\text{supp}_z(C)$, or is not contained in any such cube. If we define $c(x_0,t_0) = C(z(x_0),t_0)\|u_{x_0}\|_{L^q(X)}$, then $S_{l,k}\{C\} = A_k\{c\}$, we can apply Wave-Bound(q,d) to conclude that

$$\|S_{l,k}C\|_{L^q(X)} \lesssim 2^{k\frac{d+1}{q^*}} \left(\sum_{x_0} \sum_{t_0 \geq 2^{l-k}} |C(z(x_0),t_0)|^q \|u_{x_0}\|^q_{L^q(X)} \langle 2^k t_0 \rangle^{d-1} \right)^{1/q}$$

Since the supports of the functions $\{u_{x_0}\}$ are almost disjoint, and since U_{z_0} is supported on a set of measure $2^{(l-k)d}$,

$$\sum_{z(x_0)=z_0} \|u_{x_0}\|^q_{L^q(X)} \lesssim \|U_{z_0}\|^q_{L^q(X)} \lesssim 2^{(l-k)d} \|U_{z_0}\|^q_{L^\infty(X)}.$$

Substituting this bound into BLAH yields that

$$\|S_{l,k}C\|_{L^q(X)} \lesssim 2^{k\frac{d+1}{q'}} \left(\sum_{z_0} \sum_{t_0 \geq 2^{l-k}} |C(z_0,t_0)|^q \left[2^{(l-k)d} \|U_{z_0}\|^q_{L^\infty(X)}\right] \langle 2^k t_0 \rangle^{d-1} \right)^{1/q}.$$

We will interpolate this bound with an L^1 bound with exponential decay, which will yield the result. By finite propogation speed, F_{t_0,z_0} is supported on a set of measure $O(2^{l-k} t_0^{d-1})$, an annulus of width $O(2^{l-k})$ based upon on a sphere of radius t_0. Thus

$$\begin{aligned} \|F_{t_0,z_0}\|_{L^1(X)} &\lesssim \left(2^{l-k} t_0^{d-1}\right)^{1/2} \|F_{t_0,z_0}\|_{L^2(X)} \\ &\lesssim 2^{\frac{l-k}{2}} t_0^{\frac{d-1}{2}} \|U_{z_0}\|_{L^2(X)} \\ &\lesssim 2^{(l-k)\left(\frac{d+1}{2}\right)} t_0^{\frac{d-1}{2}} \|U_{z_0}\|_{L^\infty(X)}. \end{aligned}$$

Here we used the fact that $F_{t_0,z_0} = \int b_{t_0}(t)(Q_k \circ e^{2\pi itP} \circ Q_k)\{U_{z_0}\}$, that $\|b_{t_0}\|_{L^1(\mathbb{R})} \le 1$, and that by energy conservation $\|Q_k \circ e^{2\pi itP} \circ Q_k\|_{L^2(X)\to L^2(X)} \lesssim 1$. so that we can apply the triangle inequality to conclude that $\|T^I_{t_0}\|_{L^2(X)\to L^2(X)} \lesssim \|b_{t_0}\|_{L^1(X)} \le 1$. Substituting this bound into BLAH we conclude that

$$\begin{aligned}
&\|S_{l,k}C\|_{L^1(X)}\\
&\quad\lesssim 2^{(l-k)\left(\frac{d+1}{2}\right)} \sum_{z_0} \sum_{t_0 \ge 2^{l-k}} t_0^{\frac{d-1}{2}} |C(z_0,t_0)| \|U_{z_0}\|_{L^\infty(X)} \langle 2^k t_0\rangle^{\frac{d-1}{2}}\\
&\quad\lesssim 2^{-l\left(\frac{d-1}{2}\right)} \sum_{z_0} \sum_{t_0 \ge 2^{l-k}} |C(z_0,t_0)| \left[2^{(l-k)d} \|U_{z_0}\|_{L^\infty(X)}\right] \langle 2^k t_0\rangle^{d-1}.
\end{aligned}$$

The argument is concluded by interpolation. □

We now use this lemma to control the function f_{Long}. Because of the exponential decay in l given by the lemma above, we may sum in l trivially using the triangle inequality for $1 \le q < p$. Interpolating L^2 orthogonality and the triangle inequality, we find that if $f_{\text{Long}} = \sum_k f_{\text{Long},k}$, then

$$\left\|\sum_k f_{\text{Long},k}\right\|_{L^p(X)} = \left(\sum \|f_{\text{Long},k}\|^p_{L^p(X)}\right)^{1/p}.$$

Now write $f_{\text{Long},k} = \sum f_{\text{Long},\alpha,l,k}$ with

$$f_{\text{Long},\alpha,l,k} = \sum_H \sum_{\substack{W \in \mathcal{W}_{\alpha,H}\\ l(W)=l-k}} T_{k,l,\text{Long}}\{a_{\alpha,k,H,W}\}.$$

Applying Lemma BLAH with l and k as above, $C(z_0,t_0) = \langle 2^k t_0\rangle^{-\frac{d-1}{2}} \|b_{t_0}\|_{L^q(X)}$, and $U_{z_0} = \sum_H a_{\alpha,k,H,W}$, where $W = [z_0, z_0 + 2^{l-k}]$ we conclude that

$$\begin{aligned}
\|f_{\text{Long},\alpha,l,k}\|_{L^p(X)} &\lesssim 2^{-l\varepsilon} 2^{-k/p'} \left(\sum_{z_0} \sum_{t_0 \ge 2^{l-k}} \langle 2^k t_0\rangle^{(d-1)(1-p/2)} \|b_{t_0}\|^p_{L^p(X)} [2^{(l-k)d} \|U_{z_0}\|^p_{L^\infty(X)}]\right)^{1/p}\\
&\lesssim \|a\|_{R^{s,p}[0,\infty)} 2^{-l\varepsilon} \left(\sum_{z_0} 2^{(l-k)d} \|U_{z_0}\|^p_{L^\infty(X)}\right)^{1/p}.
\end{aligned}$$

Summing in k using the L^p orthogonality above, using BLAH to obtain L^p orthogonality since each Whitney cube contributes a single frequency for a fixed l, and summing over α trivially, we conclude that

$$\|f_{\text{Long},l}\|_{L^p(X)} \lesssim C_p 2^{-l\varepsilon} \|u\|_{L^p(X)}.$$

Summing in l trivially gives $\|f_{\text{Long}}\|_{L^p(X)} \lesssim C_p \|u\|_{L^p(X)}$, completing the proof of the long range estimates, and thus the proof.

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