Multipliers of Elliptic Operators on Compact Manifolds

Necessary and Sufficient Conditions For Boundedness

by

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Abstract

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Dedication

To mum and dad

Declaration

I declare that..

Acknowledgements

I want to thank...

Supported by grant numbers...

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Notation

- We use the normalization of the Fourier transform
$$\widehat{f}(\xi) = \int f(x) e^{-2\pi i \xi \cdot x}\, dx$$
which is standard in classical analysis.
- Let Trans_y be the translation operator by y, i.e. so that $\mathrm{Trans}_y f(x) = f(x-y)$.
- We let Dil_t be the L^∞-normalized dilation operator on functions, i.e.
$$\mathrm{Dil}_t f(x) := f(x/t).$$
Overloading notation, we also consider dyadic dilations
$$\mathrm{Dil}_j f(x) := f(x/2^j).$$
The dyadic dilation operator will only be used along with symbols that stand for integers, like n, m, j, or k, whereas the other dilation operator will be used in all other cases, so which dilation we use should be clear from the context.
- On $\mathbb{R}^d$, we use ∂_j to denote the usual partial derivative operators, and D_j to denote the self-adjoint normalization $D_j f := (2\pi i)^{-1}\partial_j$. This notation has the convenience that for a polynomial P,
$$P(D_j)\{f\} = \int P(\xi)\widehat{f}(\xi) e^{2\pi i \xi \cdot x}\, d\xi,$$
and simplifies many formulas associated with Fourier integral operators.
- We will often use the Japanese bracket $\langle x \rangle := (1+|x|^2)^{1/2}$ for $x \in \mathbb{R}^d$.
- A function $f : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}$ is a *symbol of order s* if it satisfies bounds of the form
$$|\partial_x^\alpha \partial_\theta^\beta f(x,\theta)| \lesssim_{\alpha,\beta} \langle \theta \rangle^{s-|\beta|}$$
for all multi-indices α and β.
- For a measure space X, $L^\infty(X)$ is the Banach space of essentially-bounded functions, defined almost everywhere. For a set X, $l^\infty(X)$ is the Banach space of bounded functions on X.

Introduction

Let P be an elliptic linear operator on a compact manifold M, such that we can associate a functional calculus $a \mapsto a(P)$ for functions a on the real line. In this thesis, we study the following question:

What conditions on a are necessary and sufficient for the operator $a(P)$ to be bounded.

Aside from testing our ability to understand the operator P and the relations of it's eigenfunctions, the question has applications in the study of various partial differential equations associated with the operator P, and is closely related to the geometry of M, in particular, the way waves propogate and interact on the manifold.

Moreover, the study of multipliers on a manifold provides a useful setting with which to test and develop methods of harmonic analysis in a 'variable-coefficient setting', where a lack of symmetry and translation invariance forces us to introduce more robust methods than are required in the Euclidean setting, i.e. as compared to studying multipliers of the translation-invariant Laplace operator Δ on $\mathbb{R}^d$ using the Fourier transform.

At present, there are many obstacles related to the global geometry of manifolds, which prevent an understanding of spectral multiplier operators on a general manifold. Before the results discussed in this thesis, sufficient conditions for boundedness had been established, but no necessary and sufficient conditions for boundedness of multipliers were known on any compact manifold, aside from the translation-invariant case where $M = \mathbb{T}^d$. Some results are obtained for general manifolds in this thesis, but our current inability to understand the large-time behaviour of the wave equation to a suitable degree on a general manifold means we must restrict our attention to manifolds whose geodesics are all closed, and have the same length. For a limited range of exponents, we obtain efficient estimates for multipliers on such manifolds. In particular, on the sphere S^d, we find necessary and sufficient conditions for multiplier operators of spherical harmonic expansions to bounded on $L^p(S^d)$.

We obtain these results by expanding on techniques in the theory of Fourier integral operators. In Chapter TODO, we discuss the general theory of Fourier integral operators, as well as a discussion of the best results currently known for multipliers of the Laplacian Δ on $\mathbb{R}^d$, which can also be described as *radial Fourier multipliers on Euclidean Space*. In Chapter TODO, we give an exposition of methods more closely associated with the problem. Chapters TODO and TODO discuss our main contribution to the problem (TODO More Here). Chapter TODO discusses methods we hope to develop in the future.

Part I

Background

Chapter 1

Multipliers of an Elliptic Operator

Let us now more precisely describe the problem of this thesis. Let X be a compact manifold equipped with a volume density $d\omega$, and let P be an elliptic operator on X of order $a > 0$, formally positive-definite in the sense that

$$\langle Pf, g \rangle = \langle f, Pg \rangle \quad \text{and} \quad \langle Pf, f \rangle \geq 0 \qquad \text{for all } f, g \in C^\infty(M).$$

By general properties of elliptic operators, $1 + P$ is an isomorphism between the Sobolev space $H^a(X)$ and $L^2(X)$, and so the inverse $(1+P)^{-1}$, viewed as a map from $L^2(X)$ to itself, is compact by a form of the Rellich-Kondrachov embedding theorem. By the spectral theorem for compact operators, there exists a discrete set $\Lambda \subset [0, \infty)$, and an orthogonal decomposition

$$L^2(X) = \bigoplus_{\lambda \in \Lambda} \mathcal{V}_\lambda,$$

where $\mathcal{V}_\lambda$ is a finite dimensional subspace of $C^\infty(X)$, such that $Pf = \lambda f$ for all $f \in \mathcal{V}_\lambda$. The Weyl Law for such operators tells us that $\#(\Lambda \cap [0, R]) = \text{Vol}(M) R^d + O(R^{d-1})$, where

$$\text{Vol}(M) = \int_{T^*M} \mathbb{I}[p(x,\xi) \leq 1] \; d\xi \; dx,$$

the function $p : T^*M \to [0, \infty)$ being the principal symbol of P. See TODO for details.

We use this decomposition to define a functional calculus for the operator P; given a bounded function $a : \Lambda \to \mathbb{C}$, we can define an operator $m(P)$ on $L^2(X)$ so that $a(P)f = a(\lambda)f$ for all $f \in \mathcal{V}_\lambda$; such an operator is bounded on $L^2(X)$ by orthogonality. To study further regularity of $m(P)$, we introduce the spaces $M^{p,q}(X)$, consisting of all functions $a : [0, \infty) \to \mathbb{C}$ for which the operator $a(P)$ is bounded from $L^p(X)$ to $L^q(X)$. The space $M^{p,q}(X)$ is then a seminorm space with respect to the seminorm

$$\|a\|_{M^{p,q}(X)} := \sup \left\{ \frac{\|a(P)f\|_{L^q(X)}}{\|f\|_{L^p(X)}} : f \in C^\infty(X) \right\}.$$

For notational convenience, we write $M^p(X) = M^{p,p}(X)$. As mentioned above, orthogonality immediate implies that $\|a\|_{M^2(X)} = \|a\|_{l^\infty(\Lambda)}$.

In the sequel, we will restrict our study to formally positive elliptic operators *of order one*; this does not restrict the scope of our analysis; given a formally positive elliptic operator P of order s, the operator $P^{1/s}$ defined by the functional calculus is a formally positive elliptic operator of order one[1], and the spectral theory of P is identical with the spectral theory of $P^{1/s}$. Fixing the order of our operator reflects the fact that the theory of Fourier integral operators we will eventually employ is most elegant when the resulting oscillatory integrals have homogeneous phases of order one.

The primary example of such operators are obtained from a Laplace-Beltrami operator on a compact Riemannian manifold M; the operator $-\Delta$ is self-adjoint and positive-definite with respect to the volume form dV on M, because of the 'integration by parts' identity $\langle \Delta f, g \rangle = -\langle \nabla f, \nabla g \rangle$. Since we restrict to elliptic operators of order one, the canonical object of study is the operator $P = \sqrt{-\Delta}$.

The discrete spectrum of operators on a compact manifold makes it difficult to apply methods of oscillatory integrals to the problem. Fortunately, certain semiclassical heuristics tell us that these problems disappear for 'high frequency inputs'. In order to take advantage of this fact, rather than studying what conditions ensure the operators $m(P)$ are bounded, we study conditions that ensure the operators $a_R(P) = a(P/R)$ are uniformly bounded as $R \to \infty$, since these operators depend more and more on 'high frequency behaviour' in the limit. To prevent pathological examples from arising under dilation, we restrict ourselves to *regulated functions* a, i.e. functions $a : [0, \infty) \to \mathbb{C}$ such that

$$a(\lambda_0) = \lim_{\delta \to 0} \fint_{|\lambda - \lambda_0| \leq \delta} a(\lambda)\, d\lambda \quad \text{for all } \lambda_0 \in [0, \infty).$$

Define the closed subspace $M^{p,q}_{\mathrm{Dil}}(X) \subset M^{p,q}(X)$ of regulated functions $a : [0, \infty) \to \mathbb{C}$ such that the norm $\|a\|_{M^{p,q}_{\mathrm{Dil}}(X)} = \sup_R \|a_R\|_{M^{p,q}(X)}$ is finite.

With notation introduced, we can now more precisely state the main problem that will be studied in this thesis:

What conditions are necessary and sufficient for
a function to be contained in $M^{p,q}_{Dil}(X)$ for $(p, q) \neq (2, 2)$.

Before the results of this thesis, no such results were known for $X \neq \mathbb{T}^d$, aside from the simple bound $\|a\|_{M^2_{\mathrm{Dil}}(X)} = \|a\|_{L^\infty[0,\infty)}$ which holds by orthogonality. We will obtain results under two main assumptions. The first assumption is a curvature condition: for each $x_0 \in M$, the cosphere

$$S_{x_0} = \{\xi \in T^*_{x_0} M : p(x_0, \xi) = 1\}$$

must be a hypersurface in $T^*_x M$ with non-vanishing Gauss curvature. The second assumption we make is that the eigenvalues of the operator P are contained in an arithmetic progression.

For most manifolds M and P, elements of $\mathcal{V}_\lambda$ are difficult to describe explicitly; even for the relatively simple case of the sphere S^d many problems about such functions (e.g.

[1]See Theorem 3.3.1 of [17] for details, based on a technique of [16] initially developed for elliptic differential operators.

such as problems involving nodal domains) remain open. Many arguments in harmonic analysis involve an interplay between spatial and frequential control, and without explicit descriptions of eigenfunctions, spatial control becomes difficult. Nonetheless, we will find we can obtain some spatial control by utilizing the wave equation on M, which carries geometric information via the behaviour of wave propogation. Under the curvature assumptions we make, wave propogation has sufficient smoothing properties to match certain necessary conditions that multipliers need in order to be bounded. And the fact that the eigenvalues of the operator P are contained in an arithmetic progression implies that the wave equation is periodic, which simplifies the large time analysis of the wave equation.

We begin this thesis by describing what is currently known for the boundedness problem on $\mathbb{T}^d$, where eigenfunctions are explicit, and one can study the behaviour of spectral multipliers via the Fourier transform and radial convolution operators on $\mathbb{R}^d$, in order to gain intuition and motivate potential hypotheses in the general setting.

Chapter 2

Radial Multipliers on Euclidean Space

Consider the elliptic operator $P = \sqrt{-\Delta}$ on $\mathbb{T}^d$, where $\Delta = \partial_1^2 + \cdots + \partial_d^2$ is the usual Laplacian. In such a setting, we have an explicit basis for the eigenfunctions of Δ: for a given eigenvalue $\lambda > 0$, the space $\mathcal{V}_\lambda$ has an orthonormal basis consisting of the exponentials $e^{2\pi i n \cdot x}$, where $n \in \mathbb{Z}^d$ and $|n| = \lambda$. Since the Fourier series of a function gives the expansion of the function in this basis, it follows that we can expand the spectral multiplier operator $T = a(P)$ using a Fourier series, i.e. writing

$$Tf(x) = \sum_{n \in \mathbb{Z}^d} a(|n|)\widehat{f}(n)e^{2\pi i n \cdot x}.$$

Thus a multiplier of Δ on $\mathbb{T}^d$ is nothing more than a Fourier multiplier operator on $\mathbb{T}^d$ whose symbol is radial, i.e. depending only on the magnitude of the frequency. Methods of transference[1] show that the operators $m_R(P)$ are uniformly bounded from $L^p(\mathbb{R}^d)$ to $L^q(\mathbb{R}^d)$ if and only if the Fourier multiplier operator on $\mathbb{R}^d$ given by

$$Tf(x) = \int_{\mathbb{R}^d} a(|\xi|)\widehat{f}(\xi)e^{2\pi i \xi \cdot x}\, d\xi$$

is bounded from $L^p(\mathbb{R}^d)$ to $L^q(\mathbb{R}^d)$, and so in this section we will focus on operators of this type, which we call *radial Fourier multiplier operators*.

The study of the regularity of Fourier multiplier operators has proved central to the development of modern harmonic analysis and the theory of linear partial differential operators. This is because essentially any translation invariant operator T is a Fourier multiplier operator, i.e. we can find a tempered distribution m on $\mathbb{R}^d$, the *symbol* of T, such that for any Schwartz function f,

$$Tf(x) = \int_{\mathbb{R}^d} m(\xi)\widehat{f}(\xi)e^{2\pi i \xi \cdot x}\, d\xi.$$

The study of translation invariant operators emerges from classical questions in analysis, such as the convergence of Fourier series, and problems in mathematical physics related

[1] See Section 3.6.2 of Grafakos [4], based on methods of de Leeuw [12]

to the study of the heat, wave, and Schrödinger equations. These physical equations also often have *rotational* symmetry, so it is natural to restrict our attention to translation-invariant operators which are also rotation-invariant. These operators are precisely the family of radial multiplier operators.

2.1 Convolution Kernels of Fourier Multipliers

It is often useful to study spatial representations of these operators, since one can often exploit certain geometry to obtain useful results. Given any translation invariant operator $T = a(P)$ on $\mathbb{R}^d$, we can associate a tempered distribution k, the *convolution kernel* of T, such that

$$Tf(y) = \int_{\mathbb{R}^d} k(x)f(y-x)\,dx \qquad \text{for any Schwartz } f \in \mathcal{S}(\mathbb{R}^d).$$

If T is radial, then so is k, and so we can write $k(x) = b(|x|)$ for some distribution b on $[0,\infty)$, and then we have a representation

$$T = \int_0^\infty b(r)S_r\,dr, \quad \text{where} \quad S_r f(x) := \int_{|y|=r} f(x+y)\,dy$$

are the *spherical averaging operators*. Thus problems about radial Fourier multiplier operators are also connected to spherical averaging problems.

With the notation as above, the function k is the Fourier transform of m, i.e. $k = \mathcal{F}_d m$, and the function b is a Bessel transform of a, i.e. $b = \mathcal{B}_d a$, where

$$\mathcal{B}_d a(r) := (2\pi)r^{1-d/2} \int_0^\infty a(\lambda)J_{d/2-1}(2\pi\lambda r)\lambda^{d/2}\,d\lambda.$$

Here $J_{d/2-1}$ is the Bessel function of order $d/2-1$, given by the formula

$$J_\alpha(\lambda) := \frac{(\lambda/2)^\alpha}{\Gamma(\alpha+1/2)} \int_{-1}^1 e^{i\lambda s}(1-s^2)^{\alpha-1/2}\,ds.$$

Using the theory of stationary phase, for each d we can write

$$J_d(\lambda) = e^{2\pi i\lambda}s_1(\lambda) + e^{-2\pi i\lambda}s_2(\lambda).$$

for symbols s_1 and s_2 of order $-1/2$. The presence of $e^{2\pi i\lambda}$ and $e^{-2\pi i\lambda}$ allows one to relate the Bessel transform of h to it's Fourier transform, to a certain extent. In particular, we record the following result of Garrigos and Seeger [3].

Theorem 2.1. *Suppose $d > 1$ and $1 < p < 2d/(d+1)$, and fix $\phi \in C_c^\infty[0,\infty)$. Then for any function $a : [0,\infty) \to \mathbb{C}$,*

$$\left(\int_0^\infty |\mathcal{B}_d\{\phi a\}(t)|^p t^{d-1}\right)^{1/p} \sim_{p,d,\phi} \left(\int_0^\infty |C_d\{\phi a\}(t)|^p \langle t\rangle^{(d-1)(1-p/2)}\,dt\right)^{1/p},$$

where $C_d f(t) = \int_0^\infty f(\lambda)\cos(2\pi\lambda t)\,dt$ is the cosine transform of f.

2.2 Multipliers on Euclidean Space

The general study of the boundedness properties of Fourier multiplier operators in multiple variables was initiated in the 1950s, as connections of the theory to partial differential equations became more fully realized[2]. It was quickly realized that the most fundamental estimates were $L^p \to L^q$ estimates for such operators. It is therefore natural to introduce the space $M^{p,q}(\mathbb{R}^d)$, consisting of all symbols m which induce a Fourier multiplier operator T bounded from $L^p(\mathbb{R}^d)$ to $L^q(\mathbb{R}^d)$, with the operator norm of T giving a norm on the space. The transference principles mentioned in the previous section imply that for any regulated function m,

$$\|a\|_{M^{p,q}_{\text{Dil}}(\mathbb{T}^d)} = \|a(|\cdot|)\|_{M^{p,q}(\mathbb{R}^d)}.$$

Duality implies that $M^{p,q}(\mathbb{R}^d)$ is isometric to $M^{q^*,p^*}(\mathbb{R}^d)$, where p^* and q^* are the conjugates to p and q. And the fact these operators are translation invariant, together with Littlewood's Principle, implies that $M^{p,q}(\mathbb{R}^d) = \{0\}$, unless $q \geq p$. Combining these reductions, we see that it suffices to study the spaces $M^{p,q}(\mathbb{R}^d)$ where $1 \leq p \leq 2$ and where $q \geq p$, or alternatively, the spaces $M^{p,q}(\mathbb{R}^d)$, where $2 \leq p \leq \infty$ and $q \geq p$. We only know simple characterizations of $M^{p,q}(\mathbb{R}^d)$ for very particular p and q:

- The spaces $M^{1,q}(\mathbb{R}^d) = M^{q^*,\infty}(\mathbb{R}^d)$ are characterized by virtue of the fact that the study of the boundedness of operators with domain $L^1(\mathbb{R}^d)$ or range $L^\infty(\mathbb{R}^d)$ is often simple; we have

  $$M^{1,q}(\mathbb{R}^d) = \widehat{L^q}(\mathbb{R}^d) \quad \text{for } q > 1, \quad \text{and} \quad M^{1,1}(\mathbb{R}^d) = \widehat{M}(\mathbb{R}^d),$$

  Here $M(\mathbb{R}^d)$ is the space of all finite signed Borel measures, equipped with the total variation norm. The proof follows from Schur's Lemma, which often gives tight estimates when bounding for operators in terms of the L^1 norm of their input.

- The unitary nature of the Fourier transform implies $M^2(\mathbb{R}^d) = L^\infty(\mathbb{R}^d)$. The proof follows from Parseval's identity.

It is perhaps surprising that these are the *only* known characterizations of the spaces $M^{p,q}(\mathbb{R}^d)$. No necessary and simple conditions for boundedness are known for any other values of p and q, and perhaps no simple characterization exists.

2.3 The Radial Multiplier Conjecture

Despite the continuing lack of a complete characterization of the classes $M^{p,q}(\mathbb{R}^d)$, it is surprising that we *can* conjecture a characterization of the subspace of $M^{p,q}(\mathbb{R}^d)$ consisting of *radial symbols*, for an appropriate range of exponents. The conjectured range of estimates was first suggested by the result of [3], which concerned radial multipliers m

[2] See [7] for a discussion of what was discovered at this time, and for a more detailed exposition of the content of this section.

whose associated operator T is bounded from the L^p norm to the L^q norm *restricted to radial functions*, i.e. such that the norm

$$\|m\|_{M^{p,q}_{\text{Rad}}(\mathbb{R}^d)} = \sup \left\{ \frac{\|m(P)f\|_{L^q(\mathbb{R}^d)}}{\|f\|_{L^p(\mathbb{R}^d)}} : f \text{ is radial} \right\}$$

is finite. For $d > 1$, in the range $1/p - 1/2 > 1/2d$, for $p \leq q < 2$, and for a radial multiplier m, Garrigos and Seeger proved that

$$\|m\|_{M^{p,q}_{\text{Rad}}(\mathbb{R}^d)} \sim \sup_{j \in \mathbb{Z}} 2^{jd(1/p - 1/q)} \|k_j\|_{L^q(\mathbb{R}^d)}, \tag{2.1}$$

where $m_j(t) = \chi(t) m(t/2^j)$ for a smooth compactly supported χ with $1 = \sum \chi(t/2^j)$, and k_j is the convolution kernel of $m_j(P)$.

The upper bound of (2.1) follows from rescaling the convolution identity $k_j * \psi = k_j$, where ψ is a radial Schwartz function whose Fourier transform is equal to one on the Fourier support of χ. The hard part of the result of [3] is that such a condition is sufficient. The range of p and q in this result is sharp. The result cannot hold for $p \geq 2d/(d+1)$, because, adapting the analysis of Fefferman [2] to a family of multiplier operators whose symbols are smooth and adapted to a δ neighborhood of an annulus as $\delta \to 0$, such a result would imply all Kakeya sets have positive measure. TODO: WHAT HAPPENS WHEN $q > 2$.

It is natural to conjecture that the same constraint continues to hold when we remove the constraint that our inputs f are radial, i.e so that for a radial function m, for $d > 1$, $1/2d < 1/p - 1/2 < 1/2$, and for $p \leq q < 2$,

$$\|m\|_{M^{p,q}(\mathbb{R}^d)} \sim_{p,q,d} \sup_{j \in \mathbb{Z}} 2^{jd(1/p - 1/q)} \|k_j\|_{L^q(\mathbb{R}^d)}.$$

However, for $1/2d < 1/p - 1/2 < 1/(d+1)$ an additional counterexample is obtained by considering the multipliers supported on δ annuli above, and testing them against smooth functions whose Fourier transforms are adapted to δ caps on the annuli. These counterexamples show that the condition $q < (d-1)/(d+1) \cdot p'$ is also necessary. In the sequel, we call this characterization of the radial functions in $M^{p,q}(\mathbb{R}^d)$ the *radial multiplier conjecture* on $\mathbb{R}^d$.

We now know, by results of Heo, Nazarov, and Seeger [6] that the radial multiplier conjecture is true when $d \geq 4$ and when $1/p - 1/2 > 1/(d+1)$. A summary of the proof strategies of this argument is provided in Section **??**, and a major part of our bounds for spectral multipliers follow by adapting this argument to the non-Euclidean setting. Partial improvements were obtained by Cladek [1] for symbols m compactly supported away from the origin, obtaining results for $d = 4$ and $1 < p < 36/29$, and establishing *restricted weak type* bounds when $d = 3$ and $1 < p < 13/12$. Cladek's argument is described in Section **??**. But the radial multiplier conjecture has not yet been completely resolved in any dimension d, we do not have any strong type L^p bounds when $d = 3$, and no bounds whatsoever are known when $d = 2$.

2.4 Radial-Multiplier Bounds By Density Decompositions

In this section, we give an overview of the proof of the radial multiplier bounds obtained by Heo, Nazarov, and Seeger in [6].

Theorem 2.2. *Suppose $1 < p < 2(d-1)/(d+1)$, and $m : \mathbb{R}^d \to \mathbb{C}$ is a radial function. Then for $p \leq q \leq 2$,*

$$\|m\|_{M^{p,q}(\mathbb{R}^d)} \lesssim \sup_{j \in \mathbb{Z}} 2^{jd(1/p-1/q)} \|k_j\|_{L^q(\mathbb{R}^d)}, \tag{2.2}$$

where $m_j(t) = \chi(t)m(t/2^j)$ for a smooth compactly supported χ with $1 = \sum \chi(t/2^j)$, and k_j is the convolution kernel of $m_j(P)$.

We begin by establishing a single scale version of this result, i.e. that if the Fourier transform of k is supported on the annulus $|\xi| \sim 1$, then

$$\|k * f\|_{L^p(\mathbb{R}^d)} \lesssim \|k\|_{L^p(\mathbb{R}^d)} \|f\|_{L^p(\mathbb{R}^d)}. \tag{2.3}$$

We reduce this to an inequality for sums of functions oscillating on spheres. Let σ_r be the surface measure for the sphere of radius r centered at the origin in $\mathbb{R}^d$. Also fix a nonzero, radial, compactly supported function $\psi \in \mathcal{S}(\mathbb{R}^d)$ whose Fourier transform is non-negative, and vanishes to high order at the origin. Given $x \in \mathbb{R}^d$ and $r \geq 1$, define $\chi_{x,r} = \text{Trans}_x(\sigma_r * \psi)$. Then $\chi_{x,r}$ is a smooth function adapted to a thickness $O(1)$ annulus of radius r centered at x, which is *slightly oscillating*. We will verify the following lemma.

Lemma 2.3. *For any $a : \mathbb{R}^d \times [1, \infty) \to \mathbb{C}$, and $1 \leq p < p_d$,*

$$\left\| \int_{\mathbb{R}^d} \int_1^\infty a(x,r) \chi_{x,r} \, dx \, dr \right\|_{L^p(\mathbb{R}^d)} \lesssim \left(\int_{\mathbb{R}^d} \int_1^\infty |a(x,r)|^p r^{d-1} dr dx \right)^{1/p}.$$

The implicit constant here depends on p, d, and ψ.

Lemma 2.3 implies (2.3). Suppose $k(\cdot) = b(|\cdot|)$ for some function $b : [0, \infty) \to \mathbb{C}$. If we set $a(x,r) = f(x)b(r)$ for any function $f : \mathbb{R}^d \to \mathbb{C}$, then

$$k * \psi * f = \int_{\mathbb{R}^d} \int_1^\infty a(x,r) \chi_{x,r} \, dx \, dr,$$

Lemma 2.3 thus says that

$$\|k * \psi * f\|_{L^p(\mathbb{R}^d)} \lesssim \|k\|_{L^p(\mathbb{R}^d)} \|f\|_{L^p(\mathbb{R}^d)}.$$

If we choose ψ so that $\widehat{\psi}$ is non-vanishing on the support of k, then the function $1/\widehat{\psi}(\cdot)$ is smooth on the support of m; if T is a Fourier multiplier operator with a smooth, compactly supported symbol agreeing with $1/\widehat{\psi}(\cdot)$ on the support of m, then T is bounded on $L^p(\mathbb{R}^d)$, and $T(k * \psi * f) = k * f$, and so we conclude that

$$\|k * \psi\|_{L^p(\mathbb{R}^d)} = \|T(k * \psi * f)\|_{L^p(\mathbb{R}^d)} \lesssim \|k * \psi * f\|_{L^p(\mathbb{R}^d)} \lesssim \|k\|_{L^p(\mathbb{R}^d)} \|f\|_{L^p(\mathbb{R}^d)}.$$

To prove Lemma 2.3, it suffices to prove the following discretized estimate where we replace integrals with sums.

Theorem 2.4. *Fix a 1-separated set $\mathcal{E} \subset \mathbb{R}^d \times [1,\infty)$. Then for any $a : \mathcal{E} \to \mathbb{C}$ and $1 \le p < 2(d-1)/(d+1)$,*

$$\Big\| \sum_{(x,r)\in\mathcal{E}} a(x,r)\chi_{x,r} \Big\|_{L^p(\mathbb{R}^d)} \lesssim \Big(\sum_{(x,r)\in\mathcal{E}} |a(x,r)|^p r^{d-1} \Big)^{1/p},$$

where the implicit constant is independent of $\mathcal{E}$.

Proof of Lemma 2.3 from Lemma 2.4. For any $a : \mathbb{R}^d \times [1,\infty) \to \mathbb{C}$, if we consider the vector-valued function $\mathbf{a}(x,r) = a(x,r)\chi_{x,r}$, then

$$\int_{\mathbb{R}^d} \int_1^\infty \mathbf{a}(x,r)\, dr\, dx = \int_{[0,1)^d} \int_0^1 \sum_{n\in\mathbb{Z}^d} \sum_{m>0} \mathrm{Trans}_{n,m}\mathbf{a}(x,r)\, dr\, dx$$

The triangle inequality and the increasing property of norms on $[0,1)^d \times [0,1]$ imply that

$$\begin{aligned}
&\Big\| \int_{\mathbb{R}^d} \int_1^\infty \mathbf{a}(x,r)\, dr\, dx \Big\|_{L^p(\mathbb{R}^d)} \\
&\quad\le \int_{[0,1)^d} \int_0^1 \Big\| \sum_{n\in\mathbb{Z}^d} \sum_{m>0} \mathrm{Trans}_{n,m}\mathbf{a}(x,r) \Big\|_{L^p(\mathbb{R}^d)}\, dr\, dx \\
&\quad\lesssim \int_{[0,1)^d} \int_0^1 \Big(\sum_{n\in\mathbb{Z}^d} \sum_{m>0} |a(x-n, r+m)|^p r^{d-1} \Big)^{1/p}\, dr\, dx \\
&\quad\le \Big(\int_{[0,1)^d} \int_0^1 \sum_{n\in\mathbb{Z}^d} \sum_{m>0} |a(x-n, r+m)|^p r^{d-1}\, dr\, dx \Big)^{1/p} \\
&\quad= \Big(\int_{\mathbb{R}^d} \int_1^\infty |a(x,r)|^p r^{d-1} dr dx \Big)^{1/p}. \qquad \square
\end{aligned}$$

Lemma 2.4 is further reduced by considering it as a bound on the operator

$$a \mapsto \sum_{(x,r)\in\mathcal{E}} a(x,r)\chi_{x,r}.$$

In particular, applying real interpolation, since we are proving a result for an open range of p, it suffices for us to prove a restricted strong type bound. Given any discretized set $\mathcal{E}$, let $\mathcal{E}_k$ be the set of $(x,r) \in \mathcal{E}$ with $2^k \le r < 2^{k+1}$. Then Lemma 2.4 is implied by the following Lemma.

Lemma 2.5. *For any $1 \le p < 2(d-1)/(d+1)$ and $k \ge 1$,*

$$\Big\| \sum_{(x,r)\in\mathcal{E}} \chi_{x,r} \Big\|_{L^p(\mathbb{R}^d)} \lesssim \Big(\sum_{k\ge 1} 2^{k(d-1)} \#(\mathcal{E}_k) \Big)^{1/p}.$$

Remark. Note that if $r \sim 2^k$, then $\|\chi_{x,r}\|_{L^p(\mathbb{R}^d)} \sim 2^{k(d-1)/p}$, and so Lemma 2.5 says

$$\Big\| \sum_{(x,r)\in\mathcal{E}} \chi_{x,r} \Big\|_{L^p(\mathbb{R}^d)} \lesssim_p \Big(\sum_{(x,r)\in\mathcal{E}} \|\chi_{x,r}\|_{L^p(\mathbb{R}^d)}^p \Big)^{1/p}.$$

Thus we are proving a pth root cancellation bound for the norm of sums of functions supported on different annuli with slight oscillation.

To control these sums, we apply a ‘density decomposition’, somewhat analogous to a Calderon Zygmund decomposition, which will enable us to obtain L^2 bounds. We say a 1-separated set $\mathcal{E}$ in a metric space X is of *density type* (u, r) if

$$\#(B \cap \mathcal{E}) \leq u \cdot \mathrm{diam}(B)$$

for each ball B in X with diameter at most r.

Theorem 2.6. *For any 1-separated set $\mathcal{E}_k \subset \mathbb{R}^d \times [2^k, 2^{k+1})$, there exists a decomposition $\mathcal{E}_k = \bigcup_{m=1}^{\infty} \mathcal{E}_k(2^m)$ with the following properties:*

- *For each m, $\mathcal{E}_k(2^m)$ has density type $(2^m, 2^k)$.*
- *If B is a ball of radius $r \leq 2^k$ containing at least $2^m \cdot r$ points of $\mathcal{E}_k$, then*

$$B \cap \mathcal{E}_k \subset \bigcup\nolimits_{m' \geq m} \mathcal{E}_k(2^{m'}).$$

- *For each m, there are disjoint balls $\{B_i\}$, with radii $\{r_i\}$, each at most 2^k, such that*

$$\sum\nolimits_i r_i \leq 2^{-m} \# \mathcal{E}_k,$$

and such that $\bigcup B_i^*$ covers $\bigcup_{m' \geq m} \mathcal{E}_k(2^{m'})$, where B_i^* denotes the ball with the same center as B_i but 5 times the radius.

Proof. Define a function $M : \mathcal{E}_k \to [0, \infty)$ by setting

$$M(x, r) = \sup \left\{ \frac{\#(\mathcal{E}_k \cap B)}{\mathrm{rad}(B)} : (x, r) \in B \text{ and } \mathrm{rad}(B) \leq 2^k \right\}.$$

We can establish a kind of weak L^1 estimate for M using a Vitali type argument. Let

$$\widehat{\mathcal{E}_k}(2^m) = \{(x, r) \in \mathcal{E}_k : M(x, r) \geq 2^m\}.$$

We can therefore cover $\widehat{\mathcal{E}_k}(2^m)$ by a family of balls $\{B\}$ such that $\#(\mathcal{E}_k \cap B) \geq 2^m \mathrm{rad}(B)$. The Vitali covering lemma allows us to find a disjoint subcollection of balls $B_1, \dots, B_N$ such that $B_1^*, \dots, B_N^*$ covers $\widehat{\mathcal{E}_k}(2^m)$. We find that

$$\#(\mathcal{E}_k) \geq \sum_i \#(B_i \cap \mathcal{E}_k) \geq 2^m \sum_i \mathrm{rad}(B_i),$$

Setting $\mathcal{E}_k = \widehat{\mathcal{E}_k}(2^m) - \bigcup_{k' > k} \widehat{\mathcal{E}_{k'}}(2^m)$ thus gives the required result. □

To prove Lemma 2.5, we perform a decomposition of $\mathcal{E}_k$ for each k, into the sets $\mathcal{E}_k(2^m)$, and then define $\mathcal{E}^m = \bigcup_{k \geq 1} \mathcal{E}_k^m$. For appropriate exponents, we will prove L^p bounds on the functions

$$F^m = \sum_{(x,r) \in \mathcal{E}^m} \chi_{x,r}$$

which are exponentially decaying in m, i.e. that

$$\|F^m\|_{L^p(\mathbb{R}^d)} \lesssim m \cdot 2^{-m(1/p-1/p_d)} \left(\sum_k 2^{k(d-1)} \#(\mathcal{E}_k) \right)^{1/p}.$$

Thus summing in m using the triangle inequality gives a bound on $F = \sum_m F^m$, in the range $1 < p < p_d$, i.e. that

$$\|F\|_{L^p(\mathbb{R}^d)} \lesssim \left(\sum_k 2^{k(d-1)} \#(\mathcal{E}_k) \right)^{1/p},$$

proving Lemma 2.5. To get the bound on F^m, we interpolate between an L^2 bound for F^m, and an L^0 bound (i.e. a bound on the measure of the support of F^m). First, we calculate the support of F^m.

Lemma 2.7. *For each k,*

$$|supp(F_k^m)| \lesssim 2^{-m} 2^{k(d-1)} \#\mathcal{E}_k.$$

Thus we have

$$|supp(F^m)| \leq \sum_k |supp(F_k^m)| \lesssim \sum_k 2^{-m} 2^{k(d-1)} \#\mathcal{E}_k.$$

Proof. We recall that for each k and m, we can find disjoint balls $B_1, \dots, B_N$ with radii $r_1, \dots, r_N \leq 2^k$ such that

$$\sum_{i=1}^N r_i \leq 2^{-m} \#\mathcal{E}_k,$$

where $\mathcal{E}_k(2^m)$ is covered by the expanded balls $B_1^* \cup \dots \cup B_N^*$. If we write

$$F_{k,i}^m = \sum_{(x,r) \in \mathcal{E}_k(2^m) \cap B_i^*} \chi_{x,r},$$

then $\text{supp}(F_k^m) \subset \bigcup_i \text{supp}(F_{k,i}^m)$. For each $(x,r) \in B_i^* \cap \mathcal{E}_k(2^m)$, the support of $\chi_{x,r}$, an annulus of thickness $O(1)$ and radius r, is contained in an annulus of thickness $O(r_i)$ and radius $O(2^k)$ with the same centre as B_i. Thus we conclude that

$$|\text{supp}(F_{k,i}^m)| \lesssim r_i 2^{k(d-1)},$$

and it follows that

$$|\text{supp}(F_k^m)| \leq \sum_i r_i 2^{k(d-1)} \leq 2^{-m} 2^{k(d-1)} \#\mathcal{E}_k. \qquad \square$$

Interpolating, it suffices to prove the following L^2 estimate on the function F^m.

Lemma 2.8. *Suppose $\mathcal{E} = \bigcup_k \mathcal{E}_k$ is a 1-separated set, where $\mathcal{E}_k \subset \mathbb{R}^d \times [2^k, 2^{k+1})$ is a set of density type $(2^m, 2^k)$. Then*

$$\left\| \sum_{(x,r) \in \mathcal{E}} \chi_{x,r} \right\|_{L^2(\mathbb{R}^d)} \lesssim \sqrt{m} \cdot 2^{\frac{m}{d-1}} \left(\sum_k 2^{k(d-1)} \#(\mathcal{E}_k) \right)^{1/2}.$$

The L^2 bound in Lemma 2.8 gets worse and worse as m grows, whereas the L^0 bound in Lemma 2.7 gets better and better, since annuli are concentrating in a small set, which is bad from the perspective of constructive interference, but absolutely fine from the perspective of a support bound. Interpolation gives a bound exponentially decaying in m for $1 < p < p_d$.

To prove the L^2 bound, we require an analysis of the interference patterns of pairs of the functions $\chi_{x,r}$, as provided by the following lemma.

Lemma 2.9. *For any $N > 0$, $x_1, x_2 \in \mathbb{R}^d$ and $r_1, r_2 \geq 1$,*

$$|\langle \chi_{x_1,r_1}, \chi_{x_2,r_2} \rangle| \lesssim_N \left(\frac{r_1 r_2}{\langle (x_1, r_1) - (x_2, r_2) \rangle} \right)^{\frac{d-1}{2}} \sum\nolimits_{\pm,\pm} \langle |x_1 - x_2| \pm r_1 \pm r_2 \rangle^{-N}.$$

Remark. Suppose $r_1 \leq r_2$. Then Lemma 2.9 implies that χ_{x_1,r_1} and χ_{x_2,r_2} are roughly uncorrelated, except when they are supported on annuli that roughly have the same radii and centers, and in addition, one of the following two properties hold:

- $r_1 + r_2 \approx |x_1 - x_2|$, which holds when the two annuli are 'approximately' externally tangent to one another.
- $r_2 - r_1 \approx |x_1 - x_2|$, which holds when the two annuli are 'approximately' internally tangent to one another.

Heo, Nazarov, and Seeger do not exploit the tangency information, though utilizing the tangencies seems important to improve the results they obtain. Cladek exploits this tangency information further, to obtain improved results.

Proof. We write

$$\begin{aligned}
\langle \chi_{x_1 r_1}, \chi_{x_2 r_2} \rangle &= \langle \widehat{\chi}_{x_1 r_1}, \widehat{\chi}_{x_2 r_2} \rangle \\
&= \int_{\mathbb{R}^d} \widehat{\sigma_{r_1} * \psi}(\xi) \cdot \overline{\widehat{\sigma_{r_2} * \psi}(\xi)} e^{2\pi i (x_2 - x_1) \cdot \xi} \, d\xi \\
&= (r_1 r_2)^{d-1} \int_{\mathbb{R}^d} \widehat{\sigma}(r_1 \xi) \overline{\widehat{\sigma}(r_2 \xi)} |\widehat{\psi}(\xi)|^2 e^{2\pi i (x_2 - x_1) \cdot \xi} \, d\xi.
\end{aligned}$$

Define functions A and B such that $B(|\xi|) = \widehat{\sigma}(\xi)$, and $A(|\xi|) = |\widehat{\psi}(\xi)|^2$. Then

$$\langle \chi_{x_1,r_1}, \chi_{x_2,r_2} \rangle = C_d (r_1 r_2)^{d-1} \int_0^\infty s^{d-1} A(s) B(r_1 s) B(r_2 s) B(|x_2 - x_1| s) \, ds.$$

Using well known asymptotics for the Fourier transform for the spherical measure, we have

$$B(s) = s^{-(d-1)/2} \sum_{n=0}^{N-1} (c_{n,+} e^{2\pi i s} + c_{n,-} e^{-2\pi i s}) s^{-n} + O_N(s^{-N}).$$

But now, assuming $A(s)$ vanishes to order $\gtrsim N$ at the origin, we conclude that

$$\begin{aligned}\langle \chi_{x_1 r_1}, \chi_{x_2 r_2} \rangle &= C_d \left(\frac{r_1 r_2}{|x_1 - x_2|} \right)^{(d-1)/2} \sum_{n,\tau} c_{n,\tau} r_1^{-n_1} r_2^{-n_2} |x_2 - x_1|^{-n_3} \\ &\qquad \left\{ \int_0^\infty A(s) s^{-(d-1)/2} s^{-n_1 - n_2 - n_3} e^{2\pi i (\tau_1 r_1 + \tau_2 r_2 + \tau_3 |x_2 - x_1|) s}\, ds \right\} \\ &\lesssim_N \left(\frac{r_1 r_2}{|x_1 - x_2|} \right)^{\frac{d-1}{2}} \left(1 + \frac{1}{|x_1 - x_2|^N} \right) \sum_\tau (1 + |\tau_1 r_1 + \tau_2 r_2 + \tau_3 |x_2 - x_1||)^{-5N} \\ &\lesssim_N \left(\frac{r_1 r_2}{|x_1 - x_2|} \right)^{\frac{d-1}{2}} \left(1 + \frac{1}{|x_1 - x_2|^N} \right) \sum_\tau (1 + |\tau_1 r_1 + \tau_2 r_2 + |x_2 - x_1||)^{-5N}.\end{aligned}$$

This gives the result provided that $1 + |x_1 - x_2| \geq |r_1 - r_2|/10$ and $|x_1 - x_2| \geq 1$. If $1 + |x_1 - x_2| \leq |r_1 - r_2|/10$, then the supports of χ_{x_1,r_1} and χ_{x_2,r_2} are disjoint, so the inequality is trivial. On the other hand, if $|x_1 - x_2| \leq 1$, then the bound is trivial by the last sentence unless $|r_1 - r_2| \leq 10$, and in this case the inequality reduces to the simple inequality

$$\langle \chi_{x_1,r_1}, \chi_{x_2,r_2} \rangle \lesssim_N (r_1 r_2)^{(d-1)/2}.$$

But this follows immediately from the Cauchy-Schwartz inequality. □

The exponent $(d-1)/2$ in Lemma 2.9 is too weak to apply almost orthogonality directly to obtain L^2 bounds on $\sum_{(x,r)\in\mathcal{E}_k} \chi_{xr}$ on it's own, but together with the density decomposition assumption we will be able to obtain Lemma 2.8.

Proof of Lemma 2.8. Without loss of generality, we may assume that the set of k such that $\mathcal{E}_k \neq \emptyset$ is 10-separated. Write

$$F = \sum_{(x,r)\in\mathcal{E}} \chi_{x,r}$$

and $F_k = \sum_{(x,r)\in\mathcal{E}_k} \chi_{x,r}$. First, we deal with $F_{\lesssim m} = \sum_{k \leq 10m} F_k$ trivially, i.e. writing

$$\|F\|_{L^2(\mathbb{R}^d)} \lesssim m^{1/2} \left(\sum_{k \leq 10m} \|F_k\|_{L^2(\mathbb{R}^d)}^2 + \| \sum_{k > 10m} F_k \|_{L^2(\mathbb{R}^d)} \right)^{1/2}.$$

We then decompose

$$\| \sum_{k > 10m} F_k \|_{L^2(\mathbb{R}^d)}^2 \leq \sum_{k > 10m} \|F_k\|_{L^2(\mathbb{R}^d)}^2 + 2 \sum_{k' > k > 10m} |\langle F_k, F_{k'} \rangle|.$$

Let us analyze $\langle F_k, F_{k'} \rangle$. The term will become a sum of the form $\langle \chi_{x,r}, \chi_{y,s} \rangle$, where $r \sim 2^k$ and $s \sim 2^{k'}$. Because of our assumption of being 10-separated, we have $r \leq s/2^{10}$. If $\langle \chi_{x,r}, \chi_{y,s} \rangle \neq 0$, then since the support of $\chi_{y,s}$ is an annulus of radius s centered at y, with thickness $O(1)$, and $\chi_{x,r}$ has support on an annulus of radius r centered at x, with thickness $O(1)$, the fact that r is comparatively smaller than s implies that (x, r) must be contained in the annulus of radius s centered at y, with thickness $O(2^k)$. Such an annulus is covered by $O(2^{(k'-k)(d-1)})$ balls of radius 2^k. Each ball can only contain 2^{k+m} points (x,r), and so there can be at most

$$O(2^{k'(d-1)}2^{-k(d-1)}2^{k+m}) = O(2^{k'(d-1)-k(d-2)+m}).$$

pairs $(x,r) \in \mathcal{E}_k$ for which $\langle \chi_{x,r}, \chi_{y,s}\rangle \neq 0$. For such pairs we have

$$|\langle \chi_{x,r}, \chi_{y,s}\rangle| \lesssim \left(\frac{2^k 2^{k'}}{2^{k'}}\right)^{\frac{d-1}{2}} = 2^{\frac{k(d-1)}{2}}.$$

Thus we conclude that

$$|\langle F_k, \chi_{y,s}\rangle| \lesssim 2^{-k(\frac{d-3}{2})+k'(d-1)+m}.$$

Summing over $10m < k < k'$, we conclude that since $d \geq 4$,

$$\sum_{10m<k<k'} |\langle F_k, \chi_{y,s}\rangle| \lesssim 2^{k'(d-1)+m} \sum_{10m<k<k'} 2^{-k\frac{d-3}{2}} \lesssim 2^{k'(d-1)+m}2^{-5m} \lesssim 2^{k'(d-1)}.$$

But this means that

$$\sum_{10m<k<k'} |\langle F_k, F_{k'}\rangle| \lesssim 2^{k'(d-1)} \cdot \#(\mathcal{E}_{k'}).$$

This means that

$$\|\sum_{k>10m} F_k\|^2_{L^2(\mathbb{R}^d)} \lesssim \sum_{k>10m} \|F_k\|^2_{L^2(\mathbb{R}^d)} + \sum_{k'} 2^{k'(d-1)}\#(\mathcal{E}_{k'}),$$

and it now suffices to deal with estimates the $\|F_k\|_{L^2(\mathbb{R}^d)}$, i.e. the interactions of functions supported on radii of comparable magnitude. To deal with these, we further decompose the radii, writing $[2^k, 2^{k+1})$ as the disjoint union of intervals $I_{k,\mu} = [2^k+(\mu-1)2^{am}, 2^k+\mu 2^{am}]$, for some a to be chosen later. These interval induces a decomposition $\mathcal{E}_k = \bigcup_\mu \mathcal{E}_{k,\mu}$. Again, incurring a constant loss at most, we may assume that the μ such that $\mathcal{E}_{k,\mu} \neq \emptyset$ are 10 separated. We write $F_k = \sum F_{k,\mu}$, and we have

$$\|F_k\|^2_{L^2(\mathbb{R}^d)} = \sum_\mu \|F_{k,\mu}\|^2_{L^2(\mathbb{R}^d)} + \sum_{\mu<\mu'} |\langle F_{k,\mu}, F_{k,\mu'}\rangle|.$$

We now consider $\chi_{x,r}$ and $\chi_{y,s}$ with $r \in I_{k,\mu}$ and $s \in I_{k,\mu'}$. Then we must have $|x-y| \lesssim 2^k$ and $2^{am} \leq |r-s| \lesssim 2^k$, and so we have

$$\begin{aligned} |\sum_{\mu<\mu'} \langle F_{k,\mu}, \chi_{y,s}\rangle| &\lesssim 2^{k(d-1)} \sum_{\substack{(x,r)\in\mathcal{E}_k \\ 2^{am}\leq|(x,r)-(y,s)|\lesssim 2^k}} |(x,r)-(y,s)|^{-\frac{d-1}{2}} \\ &\lesssim 2^{k(d-1)} \sum_{am\leq l\leq k} 2^{-l(d-1)/2}\#\{(x,r)\in\mathcal{E}_k : |(x,r)-(y,s)| \sim 2^l\}. \end{aligned}$$

Using the density assumption,

$$\#\{(x,r) \in \mathcal{E}_k : |(x,r) - (y,s)| \sim 2^l\} \lesssim 2^{l+m}$$

and so we obtain that, again using the assumption that $d \geq 4$,

$$|\sum_{\mu<\mu'} \langle F_{k,\mu}, \chi_{y,s}\rangle| \lesssim 2^{k(d-1)} 2^{m(1-a(d-3)/2)}.$$

Now summing over all (y,s), we obtain that

$$|\sum_{\mu<\mu'} \langle F_{k,\mu}, F_{k,\mu'}\rangle| \lesssim 2^{k(d-1)} 2^{m(1-a(d-3)/2)} \#(\mathcal{E}_{k,\mu'}).$$

and now summing over μ' gives that

$$\|F_k\|^2_{L^2(\mathbb{R}^d)} \lesssim \sum_\mu \|F_{k,\mu}\|^2_{L^2(\mathbb{R}^d)} + 2^{k(d-1)} 2^{m(1-a(d-3)/2)} \#\mathcal{E}_k,$$

which is a good enough bound if we pick a to be large enough. Now we are left to analyze $\|F_{k,\mu}\|_{L^2(\mathbb{R}^d)}$, i.e. analyzing interactions between annuli which have radii differing from one another by at most $O(2^{am})$. Since the family of all possible radii are discrete, the set $\mathcal{R}_{k,\mu}$ of all possible radii has cardinality $O(2^{am})$. We do not really have any orthogonality to play with here, so we just apply Cauchy-Schwartz, writing $F_{k,\mu} = \sum_{r\in\mathcal{R}_{k,\mu}} F_{k,\mu,r}$, to write

$$\|F_{k,\mu}\|^2_{L^2(\mathbb{R}^d)} \lesssim 2^{am} \sum_r \|F_{k,\mu,r}\|^2_{L^2(\mathbb{R}^d)}.$$

Recall that $\chi_{x,r} = \text{Trans}_x(\sigma_r * \psi)$, where ψ is a compactly supported function whose Fourier transform is non-negative and vanishes to high order at the origin. In particular, we now make the additional assumption that $\psi = \psi_\circ * \psi_\circ$ for some other compactly function $\psi_\circ$ whose Fourier transform is non-negative and vanishes to high order at the origin. Then we find that $F_{k,\mu,r}$ is equal to the convolution of the function

$$A_r = \sum_{(x,r)\in\mathcal{E}} \text{Trans}_x \psi_\circ$$

with the function $\sigma_r * \psi_\circ$. Using the standard asymptotics for the Fourier transform of σ_r, i.e. that for $|\xi| \geq 1$,

$$|\widehat{\sigma_r}(\xi)| \lesssim r^{d-1}(1 + r|\xi|)^{-\frac{d-1}{2}},$$

and since $|\widehat{\psi_\circ}(\xi)| \lesssim_N |\xi|^N$, we get that if $r \geq 1$, then for $|\xi| \leq 1/r$,

$$|\widehat{\sigma_r}(\xi)\widehat{\psi_\circ}(\xi)| \lesssim_N r^{d-1-N}$$

and for $|\xi| \geq 1/r$,

$$|\widehat{\sigma_r}(\xi)\widehat{\psi_\circ}(\xi)| \lesssim_N r^{\frac{d-1}{2}} |\xi|^{-N}.$$

Thus in particular,the L^∞ norm of the Fourier transform of $\sigma_r * \psi_\circ$ is $O(r^{(d-1)/2})$. Now the functions $\psi_\circ$ are compactly supported, so since the set of x such that $(x, r) \in \mathcal{E}$ is one-separated, we find that

$$\|A_r\|_{L^2(\mathbb{R}^d)} \lesssim \#\{x : (x, r) \in \mathcal{E}\}^{1/2}.$$

But this means that

$$\|F_{k,\mu,r}\|_{L^2(\mathbb{R}^d)} = \|A_r * (\sigma_r * \psi_\circ)\|_{L^2(\mathbb{R}^d)} \lesssim r^{\frac{d-1}{2}} \#\{x : (x, r) \in \mathcal{E}\}^{1/2}.$$

Thus we have that

$$\|F_{k,\mu}\|^2_{L^2(\mathbb{R}^d)} = 2^{am} \cdot \#\mathcal{E}_{k,\mu} \cdot 2^{k(d-1)}.$$

Summing over μ gives that

$$\|F_k\|^2_{L^2(\mathbb{R}^d)} = 2^{k(d-1)} \#\mathcal{E}_k (2^{am} + 2^{m(1-a(d-3)/2)}).$$

Picking $a = 2/(d-1)$ optimizes this bound, giving

$$\|F_k\|_{L^2(\mathbb{R}^d)} \lesssim 2^{m/(d-1)} 2^{k(d-1)/2} (\#\mathcal{E}_k)^{1/2}.$$

Plugging this into the estimates we got for F gives the required bound. $\square$

This completes a proof of the single scale estimates of the paper. The paper then uses an atomic decomposition method to combine these scales and thus complete the proof of Theorem 2.2. Rather than discuss these methods, we instead discuss an iteration of this method developed by the same authors in a follow up paper [5].

2.5 Combining Scales With Atomic Decompositions

Consider a radial Fourier multiplier operator T as in Theorem 2.2. The last section proves that if we write $T = \sum T_j$, where T_j is the radial Fourier multiplier operator with convolution kernel $k_j(\cdot/2^j)$, then applying Bernstein's inequality and the results of the last section (appropriately rescaled), we conclude that

$$\|T_j\|_{L^p(\mathbb{R}^d) \to L^q(\mathbb{R}^d)} \lesssim 2^{jd(1/p-1/q)} \|T_j\|_{L^q(\mathbb{R}^d) \to L^q(\mathbb{R}^d)} \lesssim 2^{jd(1/p-1/q)} \|k_j\|_{L^q(\mathbb{R}^d)}. \tag{2.4}$$

The goal of this section is to prove that we can bound the sum of the operators T_j by the suprema of the quantities on the right hand side in (2.4) which, morally speaking, is a bound of the form

$$\left\| \sum\nolimits_j T_j \right\| \lesssim \sup\nolimits_j \|T_j\|,$$

and thus implies that the operators $\{T_j\}$ do not constructively interfere with one another to a significant extent.

The classical example of combining scales of this form occurs in variants of Hörmander-Mikhlin type multipliers. Here one can use the Littlewood-Paley inequality

$$\left\| \left(\sum_j |f_j|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^d)} \lesssim \left\| \sum_j f_j \right\|_{L^p(\mathbb{R}^d)}$$

which implies that for the average x, the square root cancellation bound

$$\left| \sum_j f_j(x) \right| \sim \left(\sum_j |f_j(x)|^2 \right)^{1/2}$$

holds pointwise. Multipliers of Hörmander-Mikhlin type are *pseudolocal*, i.e. the

Thus $\sup_j \|T_j\|_{L^p(\mathbb{R}^d) \to L^q(\mathbb{R}^d)} \lesssim \sup_j 2^{jd(1/p-1/q)} \|k_j\|_{L^q(\mathbb{R}^d)}$, and our goal is to find a way to prove that the operators $\{T_j\}$ do not interact significantly enough, so that

The last section proves that if T is a Fourier multiplier operator, and we consider a dyadic decomposition $T = \sum T_j$, where T_j is a Fourier multiplier operator

2.6 Exploiting Tangency Bounds in $\mathbb{R}^3$ and $\mathbb{R}^4$

The results of Heo, Nazarov, and Seeger only apply when $d \geq 4$. Cladek found a method to get an initial radial multiplier conjecture result in $\mathbb{R}^3$, and an improvmeent of the bounds obtained by Heo, Nazarov, and Seeger when $d = 3$. The idea is to exploit the fact that one need only prove a version of 2.4 for a set $\mathcal{E} = \mathcal{E}_X \times \mathcal{E}_R$, where $\mathcal{E}_X$ is a one-separated family of points, and $\mathcal{E}_R$ are a family of radii. One can then exploit this Cartesian product structure when analyzing functions of the form

$$F = \sum_{(x,r) \in \mathcal{E}} \chi_{x,r},$$

in particular, improving upon the result of [6].

2.6.1 Result in 3 Dimensions

As in [6], Cladek first performs a density decomposition, i.e. writing

$$F = \sum F_k^m$$

where

$$F_k^m = \sum_{(x,r) \in \mathcal{E}_k(2^m)} \chi_{x,r}.$$

Cladek then interpolates between an L^0 bound and an L^2 bound on the resulting functions. The L^0 bound is exactly the same bound used in [6].

Theorem 2.10. *For the function F, we have*

$$|supp(F_k^m)| \lesssim 2^{-m}4^k\#\mathcal{E}_k$$

and thus

$$|supp(F^m)| \lesssim \sum_k 2^{-m}4^k\#\mathcal{E}_k.$$

The L^2 bound is improved upon, which is what allows us to obtain a new result in three dimensions.

Lemma 2.11. *Suppose $\mathcal{E} = \bigcup_k \mathcal{E}_k$ is a one-separated set, where $\mathcal{E}_k \subset \mathbb{R}^d \times [2^k, 2^{k+1})$ is a set of density type $(2^m, 2^k)$. Then*

$$\left\| \sum_{(x,r)\in\mathcal{E}} \chi_{x,r} \right\|_{L^2(\mathbb{R}^d)} \lesssim_\varepsilon 2^{[(11/13)+\varepsilon]m} \sum_k 4^k\#\mathcal{E}_k.$$

Interpolation thus yields that for a set of density type 2^m as in this Lemma,

$$\| \sum_{(x,r)\in\mathcal{E}} \chi_{x,r}\|_{L^p(\mathbb{R}^d)} \lesssim_\varepsilon 2^{-m(1/p-12/13-\varepsilon)}(\sum_k 4^k\#\mathcal{E}_k)^{1/p}.$$

If $1 < p < 13/12$, this sum is favorable in m, and may be summed without harm to prove the radial multiplier conjecture for unit scale radial multipliers in this range.

Proof of Lemma 2.11. Write

$$F_k = \sum_{(x,r)\in\mathcal{E}_k} \chi_{x,r}.$$

As before, we can throw away terms for $k \leq 10m$, i.e. obtaining that

$$\| \sum F_k\|_{L^2(\mathbb{R}^d)} \lesssim m^{1/2} \left(\sum_k \|F_k\|^2_{L^2(\mathbb{R}^d)} + \sum_{10m<k<k'} |\langle F_k, F_{k'}\rangle| \right)^{1/2}.$$

Our proof thus splits into two cases: where the radii are incomparable, and where the radii are comparable.

TODO: □

2.6.2 Results in 4 Dimensions

TODO

Chapter 3

The Radial Multiplier Conjecture on Compact Manifolds

We now return to the study of spectral multipliers on compact manifolds. In a certain sense, one can think of the Euclidean situation studied in the last Chapter as a limiting case of multipliers in the 'high frequency limit'. In particular, a transference principle of Mitjagin [14] (See [8] for a similar result, written in English and available online) shows that if X is a compact Riemannian manifold, and $h : (0,\infty) \to \mathbb{C}$ is regulated, then

$$\|h(|\cdot|)\|_{M^{p,q}(\mathbb{R}^d)} \lesssim_{X,p,q} \|h\|_{M^{p,q}_{\text{Dil}}(X)}.$$

Thus, in order for h to be an element of $M^{p,q}_{\text{Dil}}(X)$, it is necessary for the radial multiplier $h(|\cdot|)$ to be bounded from $L^p(\mathbb{R}^d)$ to $L^q(\mathbb{R}^d)$, and thus it is necessary for the quantities

Whether an analogous result remains true for more general Riemannian manifolds remains unclear, since the family of eigenfunctions to the Laplacian can take on various different forms on these manifolds, that can look quite different to the Euclidean case (TODO: Does the existence of low dimension Kakeya sets on certain manifolds show that the radial multiplier conjecture cannot be true in general).

Directly translating the assumptions of the radial multiplier conjecture to this setting yields the following statement: If $h : [0,\infty) \to \mathbb{R}$ is a function supported at the unit scale, and we define

$$C_q(h) = \left(\int |\widehat{h}(s)|^q \langle s \rangle^{(d-1)(1-q/2)}\, ds \right)^{1/q},$$

then for what values of p and q is is true that the inequality

$$\|h\|_{M^{p,q}_{\text{Dil}}(X)} \lesssim C_q(h)$$

holds. Mitjagin's result implies that we require $1 < p < 2d/(d+1)$ and $p \leq q < 2$, and we conjecture that, perhaps under appropriate assumptions on X, we can achieve similar ranges of exponents as have been obtained for the Euclidean radial multiplier conjecture.

On general compact manifolds, there are difficulties arising from a generalization of the radial multiplier conjecture, connected to the fact that analogues of the Kakeya

/ Nikodym conjecture are false in this general setting [13]. But these problems do not arise for constant curvature manifolds, like the sphere. The sphere also has over special properties which make it especially amenable to analysis, such as the fact that solutions to the wave equation on spheres are periodic. Best of all, there are already results which achieve the analogue of [3] on the sphere. Thus it seems reasonable that current research techniques can obtain interesting results for radial multipliers on the sphere, at least in the ranges established in [6] or even those results in [1].

3.1 Connections to Local Smoothing Bounds

3.2 Fourier Integral Operator Techniques

3.3 Mockenhaupt Seeger Sogge: Exploiting Periodicity

The main goal of the paper *Local Smoothing of Fourier Integral Operators and Carleson-Sjölin Estimates* is to prove local regularity theorems for a class of Fourier integral operators in $I^\mu(Z, Y; C)$, where Y is a manifold of dimension $n \geq 2$, and Z is a manifold of dimension $n + 1$, which naturally arise from the study of wave equations. A consequence of this result will be a local smoothing result for solutions to the wave equation, i.e. that if $2 < p < \infty$, then there is δ depending on p and n, such that if $T : Y \to Y \times \mathbb{R}$ is the solution operator to the wave equation, and Y is a compact manifold whose geodesics are periodic, then T is continuous from from $L^p_c(Y)$ to $L^p_{\alpha,\mathrm{loc}}(Y \times \mathbb{R})$ for $\alpha \leq -(n-1)|1/2 - 1/p| + \delta$. Such a result is called local smoothing, since if we define $Tf(t, x) = T_t f(x)$, then the operator T_t is, for each t, a Fourier integral operator of order zero, with canonical relation

$$C_t = \{(x, y; \xi, \xi) : x = y + t\widehat{\xi}\},$$

where $\widehat{\xi} = \xi/|\xi|$ is the normalization of ξ. Standard results about the regularity of hyperbolic partial differential equations show that each of the operators T_t is continuous from $L^p_c(Y)$ To $L^p_{\alpha,\mathrm{loc}}(Y \times \mathbb{R})$ for $\alpha \leq -(n-1)|1/2 - 1/p|$, and that this bound is sharp. Thus T is *smoothing* in the t variable, so that for any $f \in L^p$, the functions $T_t f$ 'on average' gain a regularity of δ over the worst case regularity at each time. The local smoothing conjecture states that this result is true for any $\delta < 1/p$.

The class of Fourier integral operators studied are those satisfying the following condition: as is standard, the canonical relation C is a conic Lagrangian manifold of dimension $2n + 1$. The fact that C is Lagrangian implies C is locally parameterized by $(\nabla_\zeta H(\zeta, \eta), \nabla_\eta H(\zeta, \eta), \zeta, \eta)$, where H is a smooth, real homogeneous function of order one. If we assume $C \to T^*Y$ is a submersion, then $D_\xi[\nabla_\eta H(\zeta, \eta)]$ has full rank, which implies $D_\eta[\nabla \xi H(\zeta, \eta)] = (D_\xi[\nabla_\eta H(\zeta, \eta)])^t$ has full rank, and thus the projection $C \to T^*Z$ is an immersion. We make the further assumption that the projection $C \to Z$ is a submersion, from which it follows that for each z in the image of this projection, the projection of points in C onto $T^*_z Z$ is a conic hypersurface Γ_z of dimension n. The final assumption we make is that all principal curvatures of Γ_z are non-vanishing.

Remark. The projection properties of C imply that, in $T^*(Z \times Y)$, there exists a smooth phase ϕ defined on an open subset of $Z \times T^*Y$, homogeneous in T^*Y, such that locally we can write C as $(z, \nabla_z\phi(z,\eta), \nabla_\eta\phi(z,\eta), \eta)$ for $\eta \neq 0$. Then, working locally on conic sets,

$$\Gamma_z = \{(\nabla_z\phi(z,\eta))\},$$

and the curvature condition becomes that the Hessian $H_{\eta\eta}\langle\nabla_z\phi, \nu\rangle$ has constant rank $n-1$, where ν is the normal vector to Γ_z. This is a natural homogeneous analogue of the Carleson-Sjölin condition for non-homogeneous oscillatory integral operators, i.e. the Carleson-Sjölin condition is allowed to assume $H_{\eta\eta}\phi$ has rank n, which cannot be possible in our case, since ϕ is homogeneous here. An approach using the analytic interpolation method of Stein or the Strichartz / Fractional Integral approach generalizes the Carleson-Sjölin theorem to show that for any smooth, non-homogeneous phase function $\Phi : \mathbb{R}^{n+1} \times \mathbb{R}^n \to \mathbb{R}$, and any compactly supported smooth amplitude a on $\mathbb{R}^{n+1} \times \mathbb{R}^n$. Consider the operators

$$T_\lambda f(z) = \int a(z,y) e^{2\pi i\lambda\Phi(z,y)} f(y)\, dy.$$

If the associated canonical relation C, if C projects submersively onto $T^*\mathbb{R}^n$, so that for each $z \in \mathbb{R}^{n+1}$ in the image of the projection map C, the set $S_z \subset \mathbb{R}^{n+1}$ obtained from the inverse image of the projection of $C \to Z$ at z is a n dimensional hypersurface with k non-vanishing curvatures. Then for $1 \leq p \leq 2$,

$$\|T_\lambda f\|_{L^q(\mathbb{R}^{n+1})} \lesssim \lambda^{-(n+1)/q} \|f\|_{L^p(\mathbb{R}^n)}.$$

where $q = p^*(1 + 2/k)$.

Remark. We can also see these assumptions as analogues in the framework of cinematic curvature, splitting the z coordinates into 'time-like' and 'space-like' parts. Working locally, because $C \to T^*Y$ is a submersion, we can consider coordinates $z = (x,t)$ so that, with the phase ϕ introduced above, $D_x(\nabla_\eta\phi)$ has full rank n, and that $\partial_t\phi(x,t,\eta) \neq 0$. Then for each $z = (x,t)$, we can locally write $\partial_t\phi(x,t,\eta) = q(x,t,\nabla_x\phi(x,t,\eta))$, homogeneous in η, and then

$$C = \{(x,t,y;\xi,\tau,\eta) : (x,\xi) = \chi_t(y,\eta), \tau = q(x,t,\xi)\},$$

where χ_t is a canonical transformation. Our curvature conditions becomes that $H_{\xi\xi}q$ has full rank $n-1$. This is the cinematic curvature condition introduced by Sogge.

Under these assumptions, the paper proves that any Fourier integral operator T in $I^{\mu-1/4}(Z,Y;C)$ maps $L^2_c(Y)$ to $L^q_{\text{loc}}(Z)$ if

$$2\left(\frac{n+1}{n-1}\right) \leq q < \infty \quad \text{and} \quad \mu \leq -n(1/2 - 1/q) + 1/q.$$

and maps $L^p_c(Y)$ to $L^p_{\text{loc}}(Z)$ if

$$p > 2 \quad \text{and} \quad \mu \leq -(n-1)(1/2 - 1/p) + \delta(p,n).$$

If we introduce time and space variables locally as in the remark above, any operator in $I^{\mu-1/4}(Z, Y; C)$ can be written locally as a finite sum of operators of the form

$$Tf(x) = \int_{-\infty}^{\infty} T_t f(x),$$

where

$$T_t f(x) = \int a(t, x, \eta) e^{2\pi i \phi(x,t,y,\eta)} f(y)\, dy\, d\eta.$$

is a Fourier integral operator whose canonical relation is a locally a canonical graph, then the general theory implies that each of the maps T_t maps $L^2_c(Y)$ to $L^q_{\text{loc}}(X)$ if

$$2 \leq q \leq \infty \quad \text{and} \quad \mu \leq -n(1/2 - 1/q)$$

so that here we get local smoothing of order $1/q$, and also maps $L^p_c(Y)$ to $L^p_{\text{loc}}(X)$ if

$$1 < p < \infty \quad \text{and} \quad \mu \leq -(n-1)|1/p - 1/2|$$

so we get $\delta(p, n)$ smoothing. A consequence of the smoothing, via Sobolev embedding, is a maximal theorem result for the operator T_t, i.e. that for any finite interval I, the operator

$$Mf = \sup_{t \in I} |T_t f|$$

maps $L^p_c(Y)$ to $L^p_{\text{loc}}(X)$ if $\mu < -(n-1)(1/2 - 1/p) - (1/p - \delta(p, n))$. If the local smoothing conjecture held, we would conclude that, except at the endpoint T^* has the same $L^p_c(Y)$ to $L^p_{\text{loc}}(X)$ mapping properties as each of the operators T_t. We also get square function estimates, such that for any finite interval I, if we consider

$$Sf(x) = \left(\int_I |T_t f(x)|^2\, dt \right)^{1/2},$$

then for

$$2\frac{n+1}{n-1} \leq q < \infty \quad \text{and} \quad \mu \leq -n(1/2 - 1/q) + 1/2,$$

the operator S is bounded from $L^2_c(Y)$ to $L^q_{\text{loc}}(X)$.

Our main reason to focus on this paper is the results of the latter half of the paper applying these techniques to radial multipliers on compact manifolds with periodic geodesics. Thus we consider a compact Riemannian manifold M, such that the geodesic flow is periodic with minimal period $2\pi \cdot \Pi$. We consider $m \in L^\infty(\mathbb{R})$, such that $\sup_{s>0} \|\beta \cdot \text{Dil}_s m\|_{L^2_\alpha(\mathbb{R})} = A_\alpha$ is finite for some $\alpha > 1/2$ and some $\beta \in C^\infty_c(\mathbb{R})$. We define a 'radial multiplier' operator

$$Tf = \sum_\lambda m(\lambda) E_\lambda f$$

where E_λ is the projection of f onto the space of eigenfunctions for the operator $\sqrt{-\Delta}$ on M with eigenvalue λ. We can also write this operator as $m(\sqrt{-\Delta})$. Then the wave propogation operator $e^{2\pi i t \sqrt{-\Delta}}$ is periodic of period Π. The Weyl formula tells us that the number of eigenvalues of $\sqrt{-\Delta}$ which are smaller than λ is equal to $V(M) \cdot \lambda^n + O(\lambda^{n-1})$.

Theorem 3.1. *Let $m \in L^2_\alpha(\mathbb{R})$ be supported on $(1,2)$, and assume $\alpha > 1/2$, then for $2 \le p \le 4$, $f \in L^p(M)$, and for any integer k,*

$$\left\| \sup_{2^k \le \tau \le 2^{k+1}} |Dil_\tau m(\sqrt{-\Delta}) f| \right\|_{L^p(M)} \lesssim_\alpha \|m\|_{L^2_\alpha(M)} \|f\|_{L^p(M)}.$$

Proof. To understand the radial multipliers we apply the Fourier transform, writing

$$T_\tau f = (\mathrm{Dil}_\tau m)(\sqrt{-\Delta}) f = m(\sqrt{-\Delta}/\tau) f = \int_{-\infty}^{\infty} \tau \widehat{m}(t\tau) e^{2\pi i t \sqrt{-\Delta}} f \, dt.$$

If we define $\beta \in C_c^\infty((1/2,8))$ such that $\beta(s) = 1$ for $1 \le s \le 4$, and set $L_k f = \mathrm{Dil}_{2^k} \beta(\sqrt{-\Delta}) f$, then for $2^k \le \tau \le 2^{k+1}$

$$T_\tau f = (\mathrm{Dil}_\tau m)(\sqrt{-\Delta}) f = (\mathrm{Dil}_\tau m \cdot \mathrm{Dil}_{2^k} \beta)(\sqrt{-\Delta}) = T_\tau L_k f.$$

so Cauchy-Schwartz implies that

$$\begin{aligned} |T_\tau f(x)| &= \left| \int_{-\infty}^{\infty} \tau \widehat{m}(\tau) e^{2\pi i t \sqrt{-\Delta}} L_k f(x) \, dt \right| \\ &\le \|m\|_{L^2_\alpha(M)} \left(\int_{-\infty}^{\infty} \frac{\tau}{(1 + |t\tau|^2)^\alpha} |e^{2\pi i t \sqrt{-\Delta}} L_k f(x)|^2 \right)^{1/2} \\ &\le \|m\|_{L^2_\alpha(M)} \left(\int_{-\infty}^{\infty} \frac{2^k}{(1 + |2^k t|^2)^\alpha} |e^{2\pi i t \sqrt{-\Delta}} L_k f(x)|^2 \right)^{1/2} \end{aligned}$$

Because of periodicity, if we set $w_k(t) = 2^k/(1 + |2^k t|^2)^\alpha$, it suffices to prove that for $\alpha > 1/2$,

$$\left\| \left(\int_0^\Pi w_k(t) |e^{2\pi i t \sqrt{-\Delta}} L_k f(x)|^2 \, dt \right)^{1/2} \right\|_{L^p(M)} \lesssim_{\alpha,p} \|f\|_{L^p(M)}.$$

This is a weighted combination of the wave propogators, roughly speaking, assigning weight 2^k for $t \lesssim 1/2^k$, and assigning weight $1/t$ to values $t \gtrsim 1/2^k$.

For a fixed $0 < \delta$, we can split this using a partition of unity into a region where $t \gtrsim \delta$ and a region where $t \lesssim \delta$, where δ is independent of k. For each t, the wave propogation $e^{2\pi i t \sqrt{-\Delta}}$ is a Fourier integral operator of order zero (we have an explicit formula for small t, and the composition calculus for Fourier integral operators can then be used to give a representation of the propogation operators for all times t, such that the symbols of these operators are locally uniformly bounded in S^0). Thus the square function estimate above can be applied in the region where $t \gtrsim \delta$, because the weighted square integral above has weight $O_\delta(1)$ uniformly in k.

Next, we move onto the region $t \lesssim 1/2^k$. The symbol of the operator $e^{2\pi i t \sqrt{-\Delta}}$

Finally we move onto the region $1/2^k \lesssim t \lesssim \delta$. On this region we have $w_k(t) \sim 1/t$, which hints we should try using dyadic estimates. In particular, suppose that for $\gamma \le \delta$, we have a family of dyadic estimates of the form

$$\left\| \left(\int_\gamma^{2\gamma} |e^{2\pi i t \sqrt{-\Delta}} L_k f|^2 \, dt \right)^{1/2} \right\|_{L^p(M)} \lesssim \gamma^{1/2} (1 + \gamma 2^k)^\varepsilon \cdot \|f\|_{L^p(M)}.$$

Summing over the $O(k)$ dyadic numbers between $1/2^k$ and δ gives

$$\left\| \left(\int_{1/2^k \lesssim t \lesssim \delta} |e^{2\pi i t \sqrt{-\Delta}} L_k f|^2 \; \frac{dt}{t} \right)^{1/2} \right\|_{L^p(M)} \lesssim 2^{\varepsilon k} \|f\|_{L^p(M)}$$

If we were able to obtain this inequality for some $\varepsilon > 0$, then we could bound that for all $0 < \gamma < \Pi/2$

If we localize near $t \lesssim 1/2^k$ by multiplying by $\phi(2^k t)$ for some compactly supported smooth ϕ supported on $|t| \lesssim 1$, then for t on the support of $\phi(2^k t)$ we have a weight proportional to 2^k, and rescaling shows that it suffices to bound the quantities

$$\left\| \left(\int \phi(t) |e^{2\pi i (t/2^k) \sqrt{-\Delta}} L_k f(x)|^2 \; dt \right)^{1/2} \right\|$$

the family of functions

$$\left\| \left(\int |\phi(t) e^{2\pi i (t/2^k) \sqrt{-\Delta}} L_k f(x)|^2 \; Dt \right)^{1/2} \right\|_{L^p_x} \lesssim \sup \|e^{2\pi i (t/2^k) \sqrt{-\Delta}} L_k f\|_{L^p_x}$$

$a_k(t) = 2^{-k/2} \widehat{\phi}(t/2^k) \beta(\tau/2^k)$

it suffices to uniformly bound quantities of the form

$$\left\| \left(\int 2^k \phi(2^k t) |e^{2\pi i \sqrt{-\Delta}} L_k f(x)|^2 \; dt \right)^{1/2} \right\|_{L^p(M)} \lesssim_{\alpha,p} \|f\|_{L^p(M)}$$

We now apply a dyadic decomposition to deal with the smaller values of t. Let us assume for simplicity of notation that $\delta < 1$, and then consider a partition of unity $1 = \sum_{j=1}^\infty \phi(2^j t)$ for $0 \leq t \leq 1$, and such that ϕ is localized near $1/4 \leq t \leq 2$, then our goal is to bound the quantities

$$\left\| \left(\int_{-\infty}^\infty \phi(2^j t) \frac{2^k}{(1 + |2^k t|^2)^\alpha} |A_t L_k f(x)|^2 \; dt \right)^{1/2} \right\|_{L^p(M)},$$

which are each proportional to

$$s$$

$\square$

3.4 Lee Seeger: Decomposition Arguments for FIOs with Cinematic Curvature

Let's now discuss a paper [11] entitled *Lebesgue Space Estimates For a Class of Fourier Integral Operators Associated With Wave Propogation*. In this paper, Lee and Seeger prove a variable coefficient version of the result of Heo, Nazarov, and Seeger, i.e. generalizing that result as it applies to sharp local smoothing on $\mathbb{R}^d$ to the local smoothing of Fourier integral operators satisfying the cinematic curvature condition.

We consider two manifolds Y and Z, of dimension d and $d+1$ respectively, and consider a Fourier integral operator

$$T : \mathcal{D}(Y) \to \mathcal{D}^*(Z)$$

of order $\mu - 1/4$, whose canonical relation $C \subset T^*Y \times T^*Z$ satisfies the following properties:

- The projection map $\pi_R : C \to T^*Y$ is a submersion. It follows that around any point $p = (z_0, y_0; \zeta_0, \eta_0)$ we can choose coordinate systems y and (x,t) on Y and Z respectively, centered at z_0 and y_0, such that $\zeta_0 = dx_1$, $\eta_0 = dy_1$, and the tangent plane $T_pC \subset T_p(T^*Y \times T^*Z)$ is the common zero set of the equations

  $$dx = dy \quad \text{and} \quad d\xi = d\eta \quad \text{and} \quad d\tau = 0,$$

  i.e. the tangent space has a basis given by

  $$\{\partial_t\} \cup \{\partial_{x_j} + \partial_{y_j} : 1 \leq j \leq d\} \cup \{\partial_{\xi_j} + \partial_{\eta_j} : 1 \leq j \leq d\}.$$

  This means in particular that the projection map $\pi_L : C \to Z$ is a submersion, and we can locally define a function $\phi(z,\eta)$, homogeneous in η, such that

  $$C = \Big\{ (z, \nabla_z \phi(z,\eta)) \times (\nabla_\eta \phi(z,\eta), \eta) \in T^*Z \times T^*Y \Big\}.$$

  By assumption on the tangent space of C,

  $$(D_\eta \nabla_x)\phi(0,e_1) = I \quad (D_\eta \partial_t)\phi(0,e_1) = 0.$$

  $$\nabla_\eta \phi(0,e_1) = 0 \quad \nabla_x \phi(0,e_1) = e_1 \quad \partial_t \phi(0,e_1) = 0.$$

  The equivalence of phase theorem implies, after appropriately microlocalizing the inputs and outputs of the operator T, we can write

  $$Tf(x,t) = \int a(x,t,y,\eta) e^{2\pi i [\phi(x,t,\eta) - y\cdot\eta]} f(y)\, d\eta\, dy,$$

  where a is a symbol of order μ, compactly supported in the (x,t,y) variables.

- Because π_L is a submersion, for each $z_0 \in Z$, the set

  $$\Sigma_{z_0} = \pi_L^{-1}(z_0)$$

  is a d dimensional submanifold of C. Because C is conic, Σ_{z_0} is also a conic subset of Z. One can see from our choice of coordinates that the projection map of Σ_{z_0} onto

T^*Z is an immersion, whose image is an immersed hypersurface Γ_{z_0} of $T^*_{z_0}$. Indeed, the tangent plane to Σ_{z_0} at the point above is given in coordinates by

$$dx = dy = dt = d\tau = 0 \quad \text{and} \quad d\xi = d\eta.$$

And this is projected injectively to the plane defined by $d\tau = 0$ in $T^*_{z_0}Z$.

We will make one more assumption about C, the *cinematic curvature condition*, that for each $z_0 \in Z$, the hypersurface Σ_{z_0} is a cone with l nonvanishing principal curvatures, for some $1 \leq l \leq d-1$. Since

$$\Sigma_{z_0} = \left\{(z_0; \nabla_z \phi(z_0, \eta_0)\right\},$$

the curvature assumptions amount to the fact that the Hessian matrix

$$H_\eta\{\partial_t \phi\}$$

has rank at least l on a neighborhood of the microsupport of a.

Given these assumptions, the following result is obtained in [11].

Theorem 3.2. *If*

$$l \geq 3 \quad \text{and} \quad 1/l < 1/2 - 1/q < 1/2 \quad \text{and} \quad \mu \leq d(1/q - 1/2) + 1/2$$

then T maps $L^q_c(Y)$ into $L^q_{loc}(Z)$.

If we take $l = d - 1$, we get the full assumption of 'cinematic curvature' and we can use this to get results about local smoothing of the wave equation on compact Riemannian manifolds, which, as a special case, recovers the local smoothing result of Heo, Nazarov, and Seeger obtained in their paper on radial Fourier multipliers.

Theorem 3.3. *If M^d is a compact Riemannian manifold,*

$$d \geq 4 \quad \text{and} \quad \frac{1}{d-1} < 1/2 - 1/q < 1/2,$$

and we set $\alpha = (d-1)/2 - d/q$, then for any interval I,

$$\|e^{it\sqrt{-\Delta}} f\|_{L^q_t(I) L^q_x(M)} \lesssim_I \|f\|_{L^q_\alpha(M)}.$$

Proof. The solution operator for the half-wave equation is a Fourier integral operator of order $-1/4$, where $\mu = 0$, associated with the canonical relation

$$C = \{(x, t, y; \eta, \omega, \eta) : x = \exp_y(t\hat{\xi}) \text{ and } \omega = |\xi|_g\},$$

where $\hat{\xi} = \xi/|\xi|$. It is immediate that the projection maps $C \to T^*Y$ and $C \to Z$ are submersions. For each $z_0 = (x_0, t_0)$,

$$\Gamma_{z_0} = \{(\xi, \omega) : \omega = |\xi|_g\}$$

is a spherical cone, and thus has $d-1$ nonvanishing principal curvatures. Consider the operator

$$T = L \circ (I - \Delta)^{-\alpha/2}.$$

Then T is a Fourier integral operator of order $-1/4+\alpha$, associated with the same canonical relation C. Applying the main theorem of [11], we conclude that for the α specified, T maps $L^q(M)$ into $L^q_{t,\mathrm{loc}}(\mathbb{R})L^q_x(M)$. But this implies that if $g = (I-\Delta)^{\alpha/2} f$, then

$$\|e^{it\sqrt{-\Delta}} f\|_{L^q_{t,\mathrm{loc}} L^q_x(M)} = \|Tg\|_{L^q_{t,\mathrm{loc}} L^q_x(M)} \lesssim \|g\|_{L^q(M)} = \|f\|_{L^q_\alpha(M)}. \qquad \square$$

Now to the proof. We begin by trying to establish a frequency localized theorem, i.e. that

$$T_R f(x,t) = \int a(x,t,y,\eta)\chi(\eta/R) e^{2\pi i[\phi(x,t,\eta) - y\cdot\eta]}\, f(y)\, d\eta\, dy,$$

then

$$\|T_R f\|_{L^q(\mathbb{R}^d)} \lesssim R^{\mu - d(1/q-1/2)+1/2} \|f\|_{L^q(\mathbb{R}^d)}.$$

For $q = \infty$, the bound

$$\|T_R f\|_{L^\infty(\mathbb{R}^d)} \lesssim R^{\mu + \frac{d-1}{2}} \|f\|_{L^\infty(\mathbb{R}^d)}$$

is a result of Seeger, Sogge, and Stein [15]. By interpolation, it suffices to prove a weak type bound for the operator T_R for $1/l < 1/2 - 1/q < 1/2$. We will actually prove a bound for the adjoint T_R^*, which we can write as

$$T_R^* f(x) = \int a_R(x,y,\xi) e^{2\pi i R[x\cdot\xi - \phi(y,t,\xi)]} f(y,t)\, d\xi\, dy\, dt,$$

where $|\partial_x^\alpha \partial_y^\beta \partial_\xi^\lambda a_R(x,y,\xi)| \lesssim_{\alpha,\beta,\lambda} R^{d+\mu}$ and has support on $|\xi| \sim 1$. Using the fact that a_R is smooth and has compact support in the x and y variables, we can apply a Fourier series argument, which means, without loss of generality, it suffices to study operators of the form

$$T_R^* f(x) = \int a_R(\xi) e^{2\pi i R[x\cdot\xi - \phi(y,t,\xi)]} f(y,t)\, d\xi\, dy\, dt.$$

Then

$$\widehat{T_R^* f}(\xi) = R^{-d} a_R(\xi/R) \int e^{-2\pi i \phi(y,t,\xi)} f(y,t)\, dy\, dt.$$

We will prove a restricted weak type inequality for this operator.

By interpolating this bound, to obtain the result above, it is sufficient to obtain a restricted weak-type bound at the endpoint value of q.

3.4.1 Frequency Localization and Discretization

Let us describe the idea of the proof. Let $K(z,y)$ denote the kernel of T, i.e.

$$K(z,y) = \int a(z,y,\eta) e^{2\pi i[\phi(z,\eta) - y\cdot\eta]}\, d\eta.$$

Without loss of generality, we may assume that a is supported on $|\eta| \geq 1$, since integrals over small frequencies give a smoothing operator. We localize in frequency, writing, working modulo smoothing operators by ignoring small frequencies,

$$K(x,t,y) = \sum_{j \geq 100} 2^{j\mu} K_{2^j}(z,y),$$

where, for $R \geq 1$,

$$\begin{aligned} K_R(z,y) &= R^{-\mu} \int a(z,y,\eta)\chi(\eta/R) e^{2\pi i[\phi(z,\eta) - y\cdot\eta]}\, d\eta \\ &= \int a_R(z,y,\eta/R) e^{2\pi i[\phi(z,\eta)-y\cdot\eta]}\, d\eta \end{aligned}$$

where $a_R(z,y,\eta) = R^{-\mu} a(z,y,R\eta)\chi(\eta)$ is chosen to have uniform compact support, and such that

$$|D^\alpha_{z,y,\eta} a_R(z,y,\eta)| \lesssim_\alpha 1$$

holds uniformly in R. Let T_R be the operator with kernel K_R.

3.4.2 Discretizing the Problem

It is more natural to establish estimates for the adjoint operator T_R^*. We will establish restricted estimates, i.e. bounding the behaviour of $T_R^*\{\chi_E\}$ for a measurable set $E \subset \mathbb{R}^{d+1}$. We have

$$T_R^*\{\chi_E\}(y) = \int \chi_E(x,t) \overline{a_R(z,y,\eta/R)} e^{2\pi i [y\cdot\eta - \phi(z,\eta)]}\, d\eta\, dx\, dt.$$

The majority of the behaviour of T_R^* depends on the behaviour of χ_E at frequencies with magnitude $\Theta(R)$, so we perform a discretization at a scale $1/R$. Taking a Fourier series in the y variable, assuming $\mathrm{supp}_y(a_R)$ is contained in $[-1/2,1/2]^d$, we write

$$a_R(z,y,\eta) = \sum_{\nu \in \mathbb{Z}^d} a_{R,\nu}(z,\eta/R)\chi(y) e^{2\pi i \nu\cdot y},$$

where χ is some smooth, compactly supported function equal to one on $\mathrm{supp}_y(K)$. We may then write

$$T_R^*\{\chi_E\}(y) = \sum_{\nu\in\mathbb{Z}^d} T_{R,\nu}^*\{\chi_E\}(y) e^{2\pi i\nu\cdot y},$$

where $T_{R,\nu}$ is the operator with kernel

$$K_{R,\nu}(z,y) = \int a_{R,\nu}(z,\eta/R)\chi(y) e^{2\pi i[\phi(z,\eta) - y\cdot\eta]}\, d\eta.$$

The smoothness and support properties of the symbols $\{a_R\}$ imply that

$$|D^\alpha_{x,t,\eta} a_{R,\nu}(z,\eta)| \lesssim_{\alpha,N} |\nu|^{-N} \quad \text{for all } N > 0,$$

and so we can likely ignore the interactions between the operators $T^*_{R,\nu}$ as we vary ν, i.e. by using the triangle inequality.

Let us now study the behaviour of the function $T^*_{R,\nu}\{\chi_E\}$ on the set $\operatorname{supp}_y(K)$. To do this, we discretize our operator at a scale $1/R$. Let $\mathcal{Z}_R$ be the set of points on the lattice $R^{-1}\,\mathbb{Z}^{d+1}$ which lies in some neighborhood of $\operatorname{supp}_z(K)$. For $\zeta \in \mathcal{Z}_R$, we let Q_ζ be the sidelength R^{-1} cube centred at ζ. Then $\{Q_\zeta\}$ is an almost disjoint family of cubes covering $\operatorname{supp}_z(K)$. Define

$$a_{R,\nu,\zeta}(\eta) = \int_{Q_\zeta \cap E} a_{R,\nu}(z, \eta/R)\chi(y)e^{2\pi i[\phi(\zeta,\eta)-\phi(z,\eta)]}\, dz.$$

Then if we define

$$S_{R,\nu,\zeta}(y) = \int a_{R,\nu,\zeta}(\eta)e^{2\pi i[y\cdot\eta-\phi(\zeta,\eta)]}\, d\eta,$$

then for $y \in \operatorname{supp}_y(K)$,

$$T^*_{R,\nu}\{\chi_E\}(y) = \sum_\zeta S_{R,\nu,\zeta}(y).$$

Now for $z \in Q_\zeta$ and $|\eta| \sim R$,

$$|\partial^\alpha_\eta\{\phi(\zeta,\eta) - \phi(z,\eta)\}| \lesssim_\alpha R^{-\alpha} \quad \text{for all } \alpha,$$

which allows us to conclude that

$$|D^\alpha_\eta a_{R,\nu,\zeta}(\eta)| \lesssim_{\alpha,N} |Q_\zeta \cap E| \cdot R^{-\alpha}\langle\nu\rangle^{-N} \quad \text{for all } \alpha \text{ and } \nu > 0.$$

We are thus reduced to the analysis of the quantities

$$\sum_{\zeta\in\mathcal{Z}_R} S_{R,\nu,\zeta}.$$

If, for $m \geq 0$, we write

$$Z_{R,m} = \{\zeta : |Q_\zeta \cap E| \sim 2^{-m}(1/R)^{d+1}\},$$

then in the next section, we will obtain estimates of the form

$$\left\| \sum_{\zeta\in Z_{R,m}} S_{R,\nu,\zeta} \right\|_{L^{p,\infty}} \lesssim_N \langle\nu\rangle^{-N}[2^{-m}(1/R)^{d+1}] \cdot R^{\frac{d+1}{2}-\frac{1}{p}}\#(Z_{R,m})^{1/p}.$$

If we set $E_{R,m} = \bigcup_{\zeta\in Z_{R,m}} Q_\zeta \cap E$, then $E = \bigcup_m E_{R,m}$. For each m,

$$\#(Z_{R,m}) \sim 2^m R^{d+1}|E_{R,m}|,$$

and so the estimate above implies that

$$\left\| \sum_{\zeta \in Z_{R,m}} S_{R,\nu,\zeta} \right\|_{L^{p,\infty}} \lesssim_N \langle \nu \rangle^{-N} 2^{-m/q} R^{d(1/p-1/2)-1/2} |E_{R,m}|^{1/p}.$$

Summing in $m \geq 0$, and using Hölder's inequality to prove that for any non-negative numbers $\{a_m\}$, we have

$$\sum_m 2^{-m/q} a_m^{1/p} \lesssim_q \left(\sum a_m\right)^{1/p},$$

we conclude that

$$\begin{aligned} \|T^*_{R,\nu}\{\chi_E\}\|_{L^{p,\infty}} &= \left\| \sum_{\zeta \in \mathcal{Z}_R} S_{R,\nu,\zeta} \right\|_{L^{p,\infty}} \\ &\lesssim \sum_m \left\| \sum_{\zeta \in Z_{R,m}} S_{R,\nu,\zeta} \right\|_{L^{p,\infty}} \\ &\lesssim_N \langle \nu \rangle^{-N} R^{d(1/p-1/2)-1/2} |E|^{1/p}. \end{aligned}$$

Summing over ν, we conclude that

$$\|T^*_R\{\chi_E\}\|_{L^{p,\infty}} \lesssim R^{d(1/p-1/2)-1/2} |E|^{1/p}.$$

Thus we have proved a restricted weak-type bound for the operators T^*_R.

3.4.3 Interactions of Discretized Operators

Our proof now rests on the following problem. We fix a large constant $M > 0$, to be specified later. Our goal is to prove the following Lemma.

Lemma 3.4. *For each $Z \subset \mathcal{Z}_R$, and each $C > 0$, suppose there exists a symbol $a_\zeta(\eta)$ supported on $|\eta| \sim R$, and satisfying derivative bounds of the form*

$$|(\partial_\xi^\alpha a_\zeta)(\eta)| \leq C R^{-\alpha} \quad \textit{for } |\alpha| \leq M.$$

If we define

$$S_\zeta(y) = \int a_\zeta(\eta) e^{2\pi i [y \cdot \eta - \phi(\zeta,\eta)]}\, d\eta,$$

then

$$\left\| \sum_{\zeta \in Z} S_\zeta \right\|_{L^{p,\infty}} \lesssim C R^{\frac{d+1}{2} - \frac{1}{p}} \#(Z)^{1/p},$$

where the implicit constant is independent of C, R, and Z.

Lemma 3.4 is clearly sufficient to prove the estimate mentioned in the last section. So let's now prove it. By linearity, we may assume without loss of generality that $C = 1$.

We will obtain the Lemma by interpolating a combination of more elementary estimates, namely, a pair of L^1 and L^∞ bounds for the sum, which do not take advantage of the curvature in the problem, and an L^2 bound which does take into account this curvature.

To understand the individual behaviour of the functions $\{S_\zeta\}$, we perform a 'double dyadic decomposition', i.e. taking a $R^{-1/2}$ discretized subset Θ_R of unit vectors in $\mathbb{R}^d$, consider some smooth partition of unity adapted to the $O(1/R^{1/2})$ neighborhoods of these unit vectors, and thus consider the associated decomposition $a_\zeta = \sum_\theta a_{\zeta,\theta}$. Applyig non-stationary phase, we get that, for the resulting decomposition $S_\zeta = \sum_\theta S_{\zeta,\theta}$, the function $S_{\zeta,\theta}$ has the majority of it's support on a cap about the point $\nabla_\xi \varphi(\zeta, \theta)$, with thickness R^{-1} in the θ-direction, and thickness $R^{-1/2}$ in the directions orthogonal to θ. Moreover, on this cap, $S_{\zeta,\theta}$ has magnitude $O(R^{(d+1)/2})$. Completely ignoring the interactions between these caps, which do not matter anyhow given we are taking the L^1 norm of the functions, the triangle inequality implies that

$$\|S_\zeta\|_{L^1} \leq \sum_\theta \|S_{\zeta,\theta}\|_{L^1} \lesssim R^{\frac{d-1}{2}} \cdot R^{\frac{d+1}{2}} \cdot R^{-1} R^{-\frac{d-1}{2}} = R^{\frac{d-1}{2}}.$$

We interpolate this bound with some L^2 orthogonality bounds for the family $\{S_\zeta\}$ to prove the required result.

We will use Fourier analysis to obtain these L^2 bounds, which is more simple given that the symbols we now have in our Fourier integral operators are independent of y. Namely, we have

$$\widehat{S}_\zeta(\eta) = a_\zeta(\eta) e^{-2\pi i \phi(\zeta,\eta)}.$$

Fix $\zeta = (x, t)$ and $\zeta' = (x', t')$. By the multiplication formula,

$$\begin{aligned}
\langle S_\zeta, S_{\zeta'} \rangle &= \langle \widehat{S}_\zeta, \widehat{S}_{\zeta'} \rangle \\
&= \int a_\zeta(\eta) a_{\zeta'}(\eta) e^{2\pi i [\phi(\zeta',\eta) - \phi(\zeta,\eta)]}\, d\eta \\
&= R^d \int a_\zeta(R\eta) a_{\zeta'}(R\eta) e^{2\pi i R[\phi(\zeta',\eta) - \phi(\zeta,\eta)]}.
\end{aligned}$$

Let us write $\phi_{\zeta,\zeta'}(\eta)$ for the phase in this oscillatory integral. Then

$$\nabla_\eta \phi_{\zeta,\zeta'}(\eta) = \nabla_\eta \phi(\zeta', \eta) - \nabla_\eta \phi(\zeta, \eta).$$

Provided we have localized our analysis to a suitably small neighborhood of $(0, e_1)$, our assumptions that

$$(D_\eta \nabla_x \phi)(0, e_1) = I \quad \text{and} \quad (D_\eta \partial_t \phi)(0, e_1) = 0$$

imply that

$$|\nabla_\eta \phi_{\zeta,\zeta'}(\eta)| \geq 0.5|x - x'| - 0.1|t - t'|.$$

Thus we conclude using this formula that if $|x - x'| \geq 0.5|t - t'|$, then we have a non-stationary phase, and we can integrate by parts to conclude that for all $N \leq M$,

$$\langle S_\zeta, S_{\zeta'} \rangle \lesssim_N R^{-N}.$$

On the other hand, suppose that $|x - x'| \leq 0.5|t - t'|$. Set $\zeta(s) = \zeta + s(\zeta' - \zeta)$, and write

$$\frac{\phi(\zeta', \eta) - \phi(\zeta, \eta)}{t' - t} = \int_0^1 \left[\partial_t \phi(\zeta_s, \eta) + \left(\frac{x' - x}{t' - t} \right) \cdot \nabla_x \phi(\zeta_s, \eta) \right] ds.$$

It follows that the left hand side is a pertubation of $\partial_t \phi(0, \eta)$. The phase

$$(t, t') \mapsto (t' - t) \cdot \partial_t \phi(0, \eta)$$

has a stationary point at $(0, e_1)$ by assumption that $(D_\eta \partial_t)\phi(0, e_1) = 0$. But the Hessian $H_\eta(\partial_t \phi)$ has rank l at $(0, e_1)$, so the principle of stationary phase 'with parameters' (to account for the pertubation in phase, see Hörmander's FIO I paper for details) implies that

$$\langle S_\zeta, S_{\zeta'} \rangle \lesssim R^d \langle R|t - t'| \rangle^{-l/2} \quad \text{if } |x - x'| \leq 0.5|t - t'|.$$

Putting this bound together with the previous bound, we conclude that for any two ζ and ζ', we have

$$\langle S_\zeta, S_{\zeta'} \rangle \lesssim \frac{R^d}{\langle R|\zeta - \zeta'| \rangle^{l/2}}.$$

Together with a form of the Calderón-Zygmund decomposition introduced by Heo, Nasarov and Seeger, this bound will be sufficient to prove we get the bound required. TODO.

3.5 Kim: L^2-Based Multiplier Bounds Using Sogge's Eigenfunction Bounds

This chapter discusses Jongchon Kim's 2017 paper *Endpoint Bounds for Quasiradial Fourier Multipliers* [10], and his 2018 paper *Endpoint Bounds for a Class of Spectral Multipliers on Compact Manifolds* [9]. These two papers introduce some useful techniques which can be used to generalize some of the results of Heo-Nasarov-Seeger to variable-coefficient settings.

Lets begin with the results of [10]. Given a homogeneous function

$$a : \mathbb{R}^d \to (0, \infty)$$

of degree one, smooth away from the origin, the paper discusses the problem of bounding 'quasiradial' Fourier multipliers with symbols of the form

$$m(\xi) = h(a(\xi)).$$

The homogeneity and non-negativity condition implies that the 'cosphere'

$$\Sigma = \{\xi : a(\xi) = 1\}$$

is a smooth hypersurface in $\mathbb{R}^d$. Under the additional assumption that this surface has everywhere non-vanishing Gaussian curvature, Kim proves that for $d \geq 4$, and $1 < p < 2(d-1)/(d+1)$, if h is a unit scale multiplier, then

$$\|h \circ a\|_{M^p(\mathbb{R}^d)} \lesssim C_p(h).$$

This is analogous to the result of Heo, Nasarov, and Seeger, but applied to multipliers that are now only *quasi-radial* rather than radial.

Similar results are obtained in [9], but in the setting of compact manifolds. Given a smooth, compact manifold M, and a first-order classical, elliptic, formally positive, self-adjoint pseudodifferential operator P, such that the cospheres

$$\Sigma_x = \{\xi \in T_x M : p(x, \xi) = 1\}$$

have non-vanishing Gaussian curvature, Kim proves that, for a unit-scale function h and for $1 < p < 2(d+1)/(d+3)$,

$$\|h\|_{M^{p,q}_{\text{Dil}}(M,P)} \lesssim \|h\|_{B^2_{d(1/p-1/2),q}(\mathbb{R})},$$

This improves results of Seeger and Sogge (1989), who proved the result with $B^{2,q}_{\alpha_p}$ replaced with L^2_α for $\alpha > \alpha_p$, and a result of Seeger (1991), who replaced $B^{2,q}_{\alpha_p}$ with a subspace $R^{2,q}_{\alpha_p}$ consisting of functions which have decompositions similar to the Bochner-Riesz multipliers. TODO: Understand the relation between these two results.

3.5.1 Quasi-Radial Multipliers

Let's begin by describing the new ideas introduced in [10]. Let h and a be as above. If η is supported on $\{1/4 \leq |\xi| \leq 4\}$, and equal to one on $\{1/2 \leq |\xi| \leq 2\}$, then

$$\begin{aligned} m(D)f(x) &= \int h(a(\xi)) e^{2\pi i \xi \cdot (x-y)} f(y)\, dy\, d\xi \\ &= \int h(a(\xi)) \eta(a(\xi)) e^{2\pi i \xi \cdot (x-y)} f(y)\, dy\, d\xi \\ &= \int \widehat{h}(t) \eta(a(\xi)) e^{2\pi i [\xi \cdot (x-y) + t a(\xi)]} f(y)\, dy\, d\xi\, dt \\ &= \int \widehat{h}(t) (K_t * f)(x)\, dx\, dt, \end{aligned}$$

where K_t is the kernel with

$$\widehat{K_t}(\xi) = \eta(a(\xi)) e^{2\pi i t a(\xi)}.$$

Let's start with some L^1 estimates for the kernels $\{K_t\}$, which do not even use the curvature properties of Σ.

Theorem 3.5.

$$\|K_t\|_{L^1(\mathbb{R}^d)} \lesssim \langle t \rangle^{\frac{d-1}{2}}.$$

Proof. Let ψ be the inverse Fourier transform of the function $\eta(a(\cdot))$. Then the Fourier transform of $\text{Dil}_{1/t}\psi$ is equal to $t^{-d}\text{Dil}_t(\eta \circ a)$. We calculate that

$$\begin{aligned} K_t(tx) &= \int (\eta \circ a)(\xi) e^{2\pi i [ta(\xi) + \xi \cdot tx]}\, d\xi \\ &= \int t^{-d} \text{Dil}_t(\eta \circ a)(\xi) e^{2\pi i [a(\xi) + \xi \cdot x]}\, d\xi \\ &= e^{2\pi i a(D)} (\text{Dil}_{1/t}\psi)(x). \end{aligned}$$

Thus

$$\|K_t\|_{L^1(\mathbb{R}^d)} = t^{-d} \|e^{2\pi i a(D)} (\text{Dil}_{1/t}\psi)\|_{L^1(\mathbb{R}^d)}.$$

The operator $e^{2\pi i a(D)}$ is a Fourier integral operator on $\mathbb{R}^d$, whose canonical relation is the conormal bundle to the surface Σ, which is, in particular, a canonical graph. Thus Theorem 2.2 of [15] implies that

$$\|e^{2\pi i a(D)} (\text{Dil}_{1/t} f)\|_{L^1(\mathbb{R}^d)} \lesssim \|(1 + \Delta)^{\frac{d-1}{2}} \{\text{Dil}_{1/t} f\}\|_{H^1(\mathbb{R}^d)} \lesssim t^{\frac{d-1}{2}}.$$

TODO: Finish off. □

By the curvature hypothesis, for each $z \in \mathbb{R}^d - \{0\}$, there are exactly two points ξ_+ and ξ_- on Σ such that z is normal to Σ at these points[1]. Moreover, we can globally parameterize these normal points, such that $z \mapsto \xi_+(z)$ and $z \mapsto \xi_-(z)$ are smooth functions for $z \in \mathbb{R}^d - \{0\}$. We let

$$\psi_+(z) = \xi_+ \cdot z \quad \text{and} \quad \psi_-(z) = \xi_- \cdot z.$$

Now consider the coordinate system $(0, \infty) \times \Sigma \to \mathbb{R}^d$ given by $(\rho, \omega) \mapsto \rho\omega$. In coordinates, we have

$$d\xi = \rho^{d-1} (\omega \cdot n(\omega)) d\rho d\sigma(\omega),$$

We can thus write

$$\begin{aligned} K_t(z) &= \int \eta(a(\xi)) e^{2\pi i [ta(\xi) + \xi \cdot z]}\, d\xi \\ &= \int_0^\infty \int_\Sigma \rho^{d-1} \eta(\rho) \left(\langle \omega, n(\omega) \rangle e^{2\pi i \rho [t + \omega \cdot z]} \right)\, d\sigma(\omega)\, d\rho \\ &= \int_0^\infty \rho^{d-1} \eta(\rho) \Big(b_+(\rho z) e^{2\pi i \rho (t + \psi_+(z))} + b_-(\rho z) e^{-2\pi i \rho (t + \psi_-(z))} \Big)\, d\rho, \end{aligned}$$

[1] To prove at least two such points exist, simply take the maxima and minima of the function $f(\xi) = z \cdot \xi$. The curvature condition on Σ implies that any extremal point ξ^* on Σ is either a global maxima or a global minima, since Σ must, by the curvature condition, always curve away from the hyperplane normal to z at each point.

where b_+ and b_- are symbols of order $-(d-1)/2$, and where b_+ and b_- have principal symbols which are scalar multiples of

$$z \mapsto |z|^{-\frac{d-1}{2}} \langle \xi_+, n(\xi_+) \rangle \quad \text{and} \quad z \mapsto |z|^{-\frac{d-1}{2}} \langle \xi_-, n(\xi_-) \rangle$$

respectively. Exploiting the oscillation of $e^{2\pi i \rho(t+\psi_+(z))}$ and $e^{-2\pi i \rho(t+\psi_-(z))}$, integrating by parts, we conclude that for all $N \geq 0$,

$$|K_t(z)| \lesssim_N (1+|t|+|x|)^{-\frac{d-1}{2}} \sum_{\pm} \langle t + \psi_\pm(x) \rangle^{-N}. \tag{3.1}$$

We note that $\psi_+(x) \geq 0 \geq \psi_-(x)$, that both functions are homogeneous of degree one, and that in the basic situation where Σ is the unit sphere, i.e. when $a(\xi) = |\xi|$,

$$\psi_+(x) = x \quad \text{and} \quad \psi_-(x) = -x.$$

For $t > 0$, we thus see that (3.1) reads

$$|K_t(z)| \lesssim_N (1+|t|+|x|)^{-\frac{d-1}{2}} \langle t + \psi_-(x) \rangle^{-N}.$$

Similarily, we can apply Parseval's identity, together with the fact that

$$\widehat{K_t}(\zeta) = \eta(a(\zeta)) e^{2\pi i t a(\zeta)},$$

to conclude that

$$\begin{aligned} \int K_{t_0}(x-x_0) & \overline{K_{t_1}(x-x_1)}\, dx \\ &= \int |\eta(a(\zeta))|^2 e^{2\pi i [(x_0-x_1)\cdot\zeta + (t_0-t_1)a(\zeta)]}, \end{aligned}$$

and similar stationary phase estimates to above show that for $N \geq 0$,

$$\begin{aligned} \Big| \int K_{t_0}(x-x_0) & \overline{K_{t_1}(x-x_1)}\, dx \Big| \\ &\lesssim_N (1+|x_0-x_1|+|t_0-t_1|)^{-\frac{d-1}{2}} \sum_{\pm} \langle (t_0-t_1) + \phi_\pm(x_0-x_1) \rangle^{-N}. \end{aligned}$$

Remark. Using (3.1), one sees the kernels K_t decay rapidly away from

$$\{x : |t + \psi_-(x)| \leq 1\},$$

a set well approximated by an annulus of thickness $O(1)$ and radius $\sim t$, in particular having measure $O(t^{d-1})$. On this set, K_t has magnitude $O(t^{-\frac{d-1}{2}})$. This gives an alternate method to obtain the L^1 bound

$$\|K_t\|_{L^1(\mathbb{R}^d)} \lesssim t^{d-1} t^{-\frac{d-1}{2}} = t^{\frac{d-1}{2}},$$

that was obtained without the use of the curvature hypothesis.

Together with a discretization argument, and the density decomposition arguments introduced in Heo-Nazarov-Seeger, the result follows. One slight difference is that, since we are using the half-wave equation formalism, our discretized estimates have slightly different weights, corresponding to the definition of $C_p(h)$. Indeed, it is proved that

$$\left\| \sum_{n\geq 2} K_{n+u} * f(n,\cdot) \right\|_{L^p(\mathbb{R}^d)} \lesssim \left\| (1+n+u)^{s_p} f(n+u,y) \right\|_{l^p_n(\mathbb{N}) L^p_y(\mathbb{R}^d)},$$

uniformly for $0 \leq u \leq 1$. Applying Minkowski's inequality, and the fact that $L^1[0,1] \leq L^p[0,1]$,

$$\begin{aligned} \left\| \int_2^\infty (K_t * f)(t,\cdot)\, dt \right\|_{L^p(\mathbb{R}^d)} &\lesssim \left\| \sum_{n\geq 2} K_{n+u} * f(n+u,\cdot) \right\|_{L^1_u[0,1] L^p(\mathbb{R}^d)} \\ &\leq \left\| \sum_{n\geq 2} K_{n+u} * f(n+u,\cdot) \right\|_{L^p_u[0,1] L^p(\mathbb{R}^d)} \\ &\lesssim \left\| (1+n+u)^{s_p} f(n+u,y) \right\|_{L^p_u[0,1] l^p_n(\mathbb{N}) L^p_y(\mathbb{R}^d)} \\ &= \left(\iint_{t\geq 1} (1+t)^{(d-1)(1-p/2)} |f(t,y)|^p \, dt\, dy \right)^{1/p}. \end{aligned}$$

The right hand side of this inequality is comparable to $C_p(h)$ times $\|g\|_{L^p(\mathbb{R}^d)}$ if $f(t,y) = \widehat{m}(t) g(y)$, in which case the computed quantity is essentially the L^p norm of $m(a(D))\{g\}$. This inequality is equivalent to the boundedness of the convolution kernel in terms of $C_p(h)$.

Next, discretization is done in the y-variable. To prove the inequality above, it suffices to show that for functions $b_{n,z}$ concentrated on neighborhoods of $z \in \mathbb{Z}^d$,

$$\left\| \sum_{n\geq 2} \sum_{z\in\mathbb{Z}^d} c(n,z)(K_n * b_{n,z}) \right\|_{L^p(\mathbb{R}^d)} \lesssim \left\| c(n,z)(1+n)^{s_p} \right\|_{l^p_n l^p_z}.$$

For interpolation purposes, it is better to reweight this inequality as

$$\left\| \sum_{n\geq 2} \sum_{z\in\mathbb{Z}^d} (1+n)^{\frac{d-1}{2}} c(n,z)(K_n * b_{n,z}) \right\|_{L^p(\mathbb{R}^d)} \lesssim \left(\sum_n \sum_n |c(n,z)|^p (1+n)^{d-1} \right)^{1/p}.$$

3.5.2 Spectral Multipliers

The result of [9] uses similar techniques, combined with the Lax parametrix to substitute for the convolution kernel decomposition possible in [10] because of the translation invariance of the operators studied.

In the proof, by a partition of unity argument, it suffices to prove the result assuming that our inputs functions f lie in $L^p(\Omega_0)$, where Ω_0 is a compact subset of M contained in a single coordinate chart Ω of M. It will help to fix a compact set Ω_1 whose interior contains Ω_0.

To prove the result, we write

$$m(P/R)f = \int [R\widehat{m}(R\xi)]e^{2\pi itP} f.$$

Write $m = \sum_{j\geq 0} m_j$, where for $j > 0$, m_j is a Littlewood-Paley cutoff on an annulus at a scale 2^j, and m_0 is a Littlewood-Paley cutoff on the unit ball.

For $2^j \lesssim 1$, we can reduce the study of the operators $m_j(P/R)$ to symbol estimates. For $2^j \gtrsim R$, one can use the local constancy property of the multipliers, together with $L^p(M)$ to $L^2(M)$ by using the fact that M is compact, and thus has finite volume, and then apply orthogonality, to obtain the boundedness of $m_j(P/R)$. We mainly concentrate on the new techniques for controlling the interactions between the multipliers $\{m_j : 1 \lesssim j \lesssim \log R\}$.

We're going to need some even, smooth cutoff functions $\eta \prec \eta' \prec \eta''$:

- η is supported on $\{|\lambda| \in [1/4, 4]\}$, and equal to one on $\{|\lambda| \in [1/2, 2]\}$.
- η' is supported on $\{|\lambda| \in [1/6, 6]\}$, and equal to one on $\{|\lambda| \in [1/4, 4]\}$.
- η'' is supported on $\{|\lambda| \in [1/16, 16]\}$, and equal to one on $\{|\lambda| \in [1/8, 8]\}$.

We let η_j, η'_j, and η''_j denote the dilations of these cutoffs by 2^j.

It will help us to localize the function f to the eigenband $\lambda \sim R$. For $f \in L^p(\Omega_0)$, we can write

$$\eta''(P/R)f = S_R f + A_R f,$$

where $\|A_R f\|_{L^p(M)} \lesssim_N R^{-N}\|f\|_{L^p(\Omega_0)}$ for all $f \in L^p(\Omega_0)$, and where $S_R f$ is supported on Ω_1 for $f \in L^p(\Omega_0)$.

We could conclude our argument if we could show that

$$\left\| \sum_{1\lesssim j\lesssim \log R} m_j(P/R)f \right\|_{L^{p,q}(M)} \lesssim \|m\|_{B^{2,q}_{\alpha_p}(\mathbb{R})}\|f\|_{L^p(\Omega_0)},$$

for all functions $f \in L^p(\Omega_0)$. We claim this result follows from the following proposition.

Lemma 3.6. *Fix a family of functions $\{b_j : 1 \lesssim j \lesssim \log R\}$ such that:*

- $\|b_j\|_{L^2(\mathbb{R})} \lesssim 1$.
- $\widehat{b_j}$ *was supported on* $\{|t| \sim 2^j\}$.
- *For any n and M, and $|\lambda| \notin [1/8, 8]$,*

$$|\partial^n b_j(\lambda)| \lesssim_{n,M} 2^{-jM}\langle 2^j\lambda\rangle^{-M}.$$

Then for functions $\{f_j\}$ in $L^p(\Omega_0)$,

$$\left\| \sum_j 2^{jd/2} b_j(P/R)\{S_R f_j\} \right\|_{L^p(\Omega)} \lesssim \left(\sum_j 2^{jd} \|f_j\|_{L^p(M)}^p \right)^{1/p}.$$

It follows simply from this that

$$\left\| \sum_j 2^{jd/2} b_j(P/R) f_j \right\|_{L^p(M)} \lesssim \left(\sum_j 2^{jd} \|f_j\|_{L^p(\Omega_0)}^p \right)^{1/p}.$$

An interpolation lemma (Lemma 2.4 of Lee, Rogers, and Seeger) yields a square function estimate of the form.

$$\left\| \sum_j 2^{-jd(1/p-1/2)} b_j(P/R) f_j \right\|_{L^{p,q}(M)} \lesssim \left\| \left(|f_j|^q \right)^{1/q} \right\|_{L^p(\Omega_0)}$$

How does this result imply the required inequality? We can write

$$m_j = \eta_j(D) m_j = \eta_j(D) \left\{ m_j \eta' \right\} + \eta_j(D) \left\{ m_j (1 - \eta') \right\}.$$

The multipliers $\eta_j(D)\{m_j(1-\eta')\}$ are rapidly decaying (TODO: I Can't see intuitively why this should be the case), so it's easy to bound the corresponding multiplier operators. We therefore reduce our required estimate to

$$\left\| \sum_j [\eta_j(D)\{m_j \eta'\}](P/R) f \right\|_{L^{p,q}(M)} \lesssim \|m\|_{B^{2,q}_{\alpha_p}(\mathbb{R})} \|f\|_{L^p(\Omega)},$$

for f supported on Ω. If $b_j = \|m_j\|_{L^2(\mathbb{R})}^{-1} \eta_j(D)\{m_j \eta'\}$, then $\{b_j\}$ satisfies the assumptions of the lemma above, and the interpolated result, with $f_j = 2^{j\alpha_p} \|m_j\|_{L^2(\mathbb{R})} f$.

The Lemma will be proved by a reduction to a restricted weak-type inequality, namely, that for any family of finite subsets $\mathcal{E}_j$ of $(\mathbb{Z}/R)^d$,

$$\left\| \sum_j 2^{jd/2} \sum_{n \in \mathcal{E}_j} b_j(P/R) \chi_{j,n} \right\|_{L^{p,\infty}(\Omega)} \lesssim R^{-d/p} \left(\sum_j 2^{jd} \# \mathcal{E}_j \right)^{1/p}.$$

We will mainly focus now on the L^2 estimates that imply this restricted weak-type inequality.

For a fixed $\alpha > 0$, we cover $\mathbb{R}^d$ by essentially disjoint cubes with sidelength $2^j/R$, and denote the collection of cubes in $\mathcal{Q}_j$. Let $\mathcal{Q}_j(\lambda)$ be the collection of all $Q \in \mathcal{Q}_j$ such that $Q \cap \# \mathcal{E}_j > \lambda^p$. This allows us to write

$$\mathcal{E}_j = \mathcal{E}_j^{\text{High}}(\lambda) \cup \mathcal{E}_j^{\text{Low}}(\lambda),$$

where $\mathcal{E}_j^{\text{High}}(\lambda)$ is the collection of points in $\mathcal{E}_j$ that lie in a cube Q lying in $\mathcal{Q}_j(\lambda)$, i.e. the set of 'high density points'. Fix $C > 0$ suitably large, and for each cube Q, let Q^* denote the cube with the same center as Q, but C times the sidelength. Then

$$\sum_j \sum_{Q \in \mathcal{Q}_j(\lambda)} |Q| \lesssim \sum_j \sum_{Q \in \mathcal{Q}_j} (2^{jd}/R)\alpha^{-p}\#(\mathcal{E}_j \cap Q) \leq \alpha^{-p} \sum_j 2^{jd} t^{-d} \#(\mathcal{E}_j).$$

This inequality shows that it is fair to throw out the union of the cubes in $\mathcal{Q}_j(\lambda)$ in the analysis of the measure of the set

$$\left\{ x \in \Omega : \left| \sum_j 2^{jd/2} \sum_{Q \in \mathcal{Q}_j(\lambda)} b_j(P/R)\chi_{j,n} \right| > \alpha \right\}.$$

The required bound would follow if we could show that

$$\lambda^{-1} \sum_j 2^{jd/2} \left\| \sum_{Q \in \mathcal{Q}_j(\lambda)} [b_j(P/R)\chi_{j,Q}]\mathbf{I}_{(Q^*)^c} \right\|_{L^1(\Omega)} \lesssim \lambda^{-p} \sum_j 2^{jd} R^{-d} \#\mathcal{E}_j.$$

But (see end of Section 5.1 for more details), this follows from the fact that the Lax parametrix is supported on a small neighborhood of the diagonal.

We are now left with the analysis of $\mathcal{E}_j^{\text{Low}}(\lambda)$. To simplify notation, we will assume that $\mathcal{E}_j = \mathcal{E}_j^{\text{Low}}(\lambda)$ for all j. We will try and control

$$\left\| \sum_j 2^{jd/2} \sum_{n \in \mathcal{E}_j} b_j(P/R)\chi_{j,n} \right\|_{L^2(\Omega)}^2 \lesssim \alpha^{2-p} \log \alpha \sum_j 2^{jd} R^{-d} \#\mathcal{E}_j.$$

Define $G_j = \sum_{n \in \mathcal{E}_j} b_j(P/R) f_{j,n}$. Then, applying the triangle inequality for $j \lesssim \log \alpha$, we conclude that

$$\| \sum_j 2^{jd/2} G_j \|_{L^2(\Omega)}^2 \lesssim \log \alpha \left(\sum_j 2^{jd} \|G_j\|_{L^2(\Omega)}^2 + \sum_j \sum_{\log \alpha \lesssim k \lesssim j} 2^{\frac{k+j}{2}} |\langle G_j, G_k \rangle| \right).$$

We will bound each of these quantities separately.

First, let's bounded $\|G_j\|_{L^2(\Omega)}$.

Let us suppose the Lax parametrix applies on times $|t| \leq \varepsilon$. Times larger than this are dealt with fairly simply

We can apply the Lax parametrix for times $|t| \leq \varepsilon$. The large times are dealt with easily, using compactness to reduce $L^p(M)$ estimates to $L^2(M)$ estimates, and then applying orthogonality. Similarily, small times can also be dealt with by reducing to the study of pseudodifferential operators.

Part II

New Results

Chapter 4

The Boundedness of Spectral Multipliers with Compact Support

4.0.1 Summary of Proof

To conclude the introductory sections of the paper, we summarize the various methods which occur in the proof of Theorem **??**. In Lemma 4.1 of Section 4.1, we prove that

$$\sup_R \|m_R(P)\|_{L^p(M)\to L^p(M)} \gtrsim C_p(m). \tag{4.1}$$

We obtain (4.1) via a relatively simple argument, as might be expected from the discussion in Section **??**. The main difficulty is obtaining the opposite inequality

$$\sup_R \|m_R(P)\|_{L^p(M)\to L^p(M)} \lesssim C_p(m). \tag{4.2}$$

Unlike in the study of Euclidean multipliers, the behavior of the multipliers $m_R(P)$ on a *compact* manifold for $R \lesssim 1$ is relatively benign. In Lemma 4.2 of Section 4.1 that same section, another simple argument shows

$$\sup_{R\leq 1} \|m_R(P)\|_{L^p(M)\to L^p(M)} \lesssim C_p(m). \tag{4.3}$$

In that section we also setup notation for the rest of the paper. As a consequence of Proposition 4.4, stated at the end of Section 4.1,we can conclude that

$$\sup_{R\geq 1} \|m_R(P)\|_{L^p(M)\to L^p(M)} \lesssim C_p(m), \tag{4.4}$$

The upper bound (4.4) requires a more in depth analysis than (4.3). Using the fact that m is regulated, we apply the Fourier inversion formula to write

$$m_R(P) = \int_{-\infty}^{\infty} R\widehat{m}(Rt)e^{2\pi itP}\, dt, \tag{4.5}$$

where $e^{2\pi itP}$ are wave propogators, which as t varies, give solutions to the half-wave equation $\partial_t = 2\pi iP$ on M. Studying $m_R(P)$ thus reduces to studying certain 'weighted averages' of the propagators $\{e^{2\pi itP}\}$. We prove Proposition 4.4 using several new estimates for understanding these averages, including:

(A) Quasi-orthogonality estimates for averages of solutions to the half-wave equation on M, discussed in Section 4.2, which arise from a connection between the theory of pseudodifferential operators whose principal symbols satisfy the curvature condition of Theorem **??**, and Finsler metrics on the manifold M.

(B) Variants of the density-decomposition arguments first used in [6], described in Section **??**, which apply the quasi-orthogonality estimates obtained in Section 4.2 with a geometric argument which controls the 'small time behavior' of solutions to the half-wave equation.

(C) A new strategy to reduce the 'large time behavior' of the half-wave equation to an endpoint local smoothing inequality for the half-wave equation on M, described in Section 4.3.

Equations (4.3) and (4.4) immediately imply (4.2), which together with (4.1) completes the proof of Theorem **??**.

4.1 Preliminary Setup

We now begin with the details of the proof of Theorem **??**, following the path laid out at the end of the introduction. To begin with, we prove the lower bound (4.1).

Lemma 4.1. *Suppose M is a compact manifold of dimension d, and P is a classical elliptic self-adjoint pseudodifferential operator of order one on M satisfying the assumptions of Theorem* **??**. *Then for $1 < p < 2(d-1)/(d+1)$, and for any regulated function $m : [0,\infty) \to \mathbb{C}$,*

$$\sup_R \|m_R(P)\|_{L^p(M)\to L^p(M)} \gtrsim C_p(m). \tag{4.6}$$

Proof. Define the quasiradial Fourier multiplier $m(p(x_0,D))$ on $\mathbb{R}^d$ by

$$m(p(x_0,D))f(x) = \int_{\mathbb{R}^d} m(p(x_0,\xi))\widehat{f}(\xi)e^{2\pi i\xi\cdot x}\,d\xi. \tag{4.7}$$

A result of Mitjagin [14] states that for any regulated m, and any $x_0 \in M$,

$$\sup_R \left\|m_R(P)\right\|_{L^p(M)\to L^p(M)} \gtrsim \left\|m(p(x_0,D))\right\|_{L^p(\mathbb{R}^d)\to L^p(\mathbb{R}^d)}. \tag{4.8}$$

Theorem 1.1 of [10] implies that

$$\left\|m(p(x_0,D))\right\|_{L^p(\mathbb{R}^d)\to L^p(\mathbb{R}^d)} \sim C_p(m) \quad \text{and} \quad \left\|m(|D|)\right\|_{L^p(\mathbb{R}^d)\to L^p(\mathbb{R}^d)} \sim C_p(m). \tag{4.9}$$

Putting together (4.8) and (4.9) then proves the claim. □

In the remainder of the article, we focus on proving the upper bound (4.2). For this purpose, we may assume without loss of generality that $\operatorname{supp}(m) \subset [1,2]$. Let us briefly justify why. Suppose we have proved

$$\sup_R \|m_R(P)\|_{L^p\to L^p} \sim C_p(m) \quad \text{if } \operatorname{supp}(m) \subset [1,2]. \tag{4.10}$$

Let m be an arbitrary function whose support is a compact subset of $(0,\infty)$. Fix $\chi \in C_c^\infty(0,\infty)$ with $\mathrm{supp}(\chi) \subset [1,2]$ and with $\sum \chi(\cdot/2^j) = 1$. If we set $m_j(\lambda) = \chi(\lambda) m(2^j \lambda)$, then $\mathrm{supp}(m_j) \subset [1,2]$ for all j, and $m(\lambda) = \sum m_j(\lambda/2^j)$. Then

$$\widehat{m}_j(t) = \widehat{\chi} * 2^{-j} \widehat{m}(\cdot/2^j), \tag{4.11}$$

and the rapid decay of $\widehat{\chi}$ implies that $C_p(m_j) \lesssim_j C_p(m)$. Thus (4.10) implies

$$\begin{aligned} &\sup_R \|m_j(P/2^j R)\|_{L^p(M)\to L^p(M)} \\ &\qquad = \sup_R \|m_j(P/R)\|_{L^p(M)\to L^p(M)} \lesssim_j C_p(m_j) \lesssim C_p(m). \end{aligned} \tag{4.12}$$

Now the triangle inequality, summing over finitely many j, implies

$$\sup_R \|m(P/R)\|_{L^p(M)\to L^p(M)} \lesssim \sum_j \sup_R \|m_j(P/2^j R)\|_{L^p(M)\to L^p(M)} \lesssim C_p(m), \tag{4.13}$$

which proves Theorem **??** in general.

Given a multiplier m supported on $[1/2,2]$, we define $T_R = m(P/R)$. We fix some geometric constant $\varepsilon_M \in (0,1)$, matching the constant given in the statement of Theorem 4.4. Our goal is then to prove inequalities (4.3) and (4.4). Proving (4.3) is simple because the operators T_R are smoothing operators, uniformly for $0 < R < 1$, and M is a compact manifold.

Lemma 4.2. *Let M be a compact d-dimensional manifold, let P be a classical elliptic self-adjoint pseudodifferential operator of order one. If $1 < p < 2d/(d+1)$, and if $m : [0,\infty) \to \mathbb{C}$ is a regulated function with supp$(m) \subset [1/2,2]$, then*

$$\sup_{R\leq 1} \|T_R\|_{L^p(M)\to L^p(M)} \lesssim C_p(m). \tag{4.14}$$

Proof. Let $T_R = m_R(P)$. The set $\Lambda_P \cap [0,2]$ is finite. For each $\lambda \in \Lambda_P$, choose a finite orthonormal basis $\mathcal{E}_\lambda$. Then we can write

$$T_R = \sum_{\lambda\in\Lambda_P} \sum_{e\in\mathcal{E}_\lambda} \langle f, e\rangle e. \tag{4.15}$$

Since $\mathcal{V}_\lambda \subset C^\infty(M)$, Hölder's inequality implies

$$\|\langle f, e\rangle e\|_{L^p(M)} \leq \|f\|_{L^p(M)} \|e\|_{L^{p'}(M)} \|e\|_{L^p(M)} \lesssim_\lambda \|f\|_{L^p(M)}. \tag{4.16}$$

But this means that

$$\|T_R f\|_{L^p(M)} \leq \sum_{\lambda\in\Lambda_P\cap[0,2]} \sum_{e\in\mathcal{E}_\lambda} |m(\lambda/R)| \|\langle f, e\rangle e\|_{L^p(M)} \lesssim \|m\|_{L^\infty[0,\infty)}. \tag{4.17}$$

The proof is completed by noting that for $1 < p < 2d/(d+1)$, the Sobolev embedding theorem guarantees that $\|m\|_{L^\infty[0,\infty)} \lesssim C_p(m)$. $\square$

Lemma 4.2 is relatively simple to prove because the operators $\{T_R : 0 \leq R \leq 1\}$ are supported on a common, finite dimensional subspace of $L^2(M)$. We thus did not have to perform any analysis of the interactions between different eigenfunctions, because the triangle inequality is efficient enough to obtain a finite bound. For $R > 1$, things are not so simple, and so proving (4.4) requires a more refined analysis of multipliers than (4.3). A standard method, originally due to Hörmander [**Hormander2**], is to apply the Fourier inversion formula to write, for a regulated function $h : \mathbb{R} \to \mathbb{C}$,

$$h(P) = \int_{-\infty}^{\infty} \widehat{h}(t) e^{2\pi i t P}\, dt, \tag{4.18}$$

where $e^{2\pi i t P}$ is the spectral multiplier operator on M which, as t varies, gives solutions to the half-wave equation $\partial_t = 2\pi i P$ on M. Thus we conclude that

$$T_R = \int_{-\infty}^{\infty} R\widehat{m}(Rt) e^{2\pi i t P}\, dt. \tag{4.19}$$

Writing T_R in this form, we are lead to obtain estimates for averages of the wave equation tested against a relatively non-smooth function $R\widehat{m}(Rt)$.

Fix a bump function $q \in C_c^\infty(\mathbb{R})$ with $\operatorname{supp}(q) \subset [1/2, 4]$ and $q(\lambda) = 1$ for $\lambda \in [1, 2]$, and define $Q_R = q(P/R)$. We write

$$T_R = Q_R \circ T_R \circ Q_R = \int_{\mathbb{R}} R\widehat{m}(Rt)(Q_R \circ e^{2\pi i t P} \circ Q_R)\, dt, \tag{4.20}$$

and view the operators $(Q_R \circ e^{2\pi i t P} \circ Q_R)$ as 'frequency localized' wave propagators.

Replacing the operator P by $aP + b$ for appropriate $a, b \in \mathbb{R}$, we may assume without loss of generality that all eigenvalues of P are integers. It follows that $e^{2\pi i (t+n) P} = e^{2\pi i t P}$ for any $t \in \mathbb{R}$ and $n \in \mathbb{Z}$. Let I_0 denote the interval $[-1/2, 1/2]$. We may then write

$$T_R = \int_{I_0} b_R(t)(Q_R \circ e^{2\pi i t P} \circ Q_R)\, dt, \tag{4.21}$$

where $b_R : I_0 \to \mathbb{C}$ is the periodic function

$$b_R(t) = \sum\nolimits_{n \in \mathbb{Z}} R\widehat{m}(R(t+n)). \tag{4.22}$$

We split our analysis of T_R into two regimes: regime I and regime II. In regime I, we analyze the behaviour of the wave equation over times $0 \leq |t| \leq \varepsilon_M$ by decomposing this time interval into length $1/R$ pieces, and analyzing the interactions of the wave equations between the different intervals. In regime II, we analyze the behaviour of the wave equation over times $\varepsilon_M \leq |t| \leq 1$. Here we need not perform such a decomposition, since the boundedness of $C_p(m)$ gives better control on the function b_R over these times.

Lemma 4.3. *Fix $\varepsilon > 0$. Let $\mathcal{T}_R = \mathbb{Z}/R \cap [-\varepsilon, \varepsilon]$ and define $I_t = [t - 1/R, t + 1/R]$. For a function $m : [0, \infty) \to \mathbb{C}$, define a periodic function $b : I_0 \to \mathbb{C}$ by setting*

$$b(t) = \sum\nolimits_{n \in \mathbb{Z}} R\widehat{m}(R(t+n)). \tag{4.23}$$

Then we can write $b = \left(\sum_{t_0 \in \mathcal{T}_R} b^I_{t_0}\right) + b^{II}$*, where*

$$supp(b^I_{t_0}) \subset I_{t_0} \quad and \quad supp(b^{II}_R) \subset I_0 \setminus [-\varepsilon, \varepsilon]. \tag{4.24}$$

Moreover, we have

$$\left(\sum_{t_0 \in \mathcal{T}_R} \left[\|b^I_{t_0}\|_{L^p(I_0)} \langle R t_0 \rangle^{\alpha(p)}\right]^p\right)^{1/p} \lesssim R^{1/p'} C_p(m) \tag{4.25}$$

and

$$\|b^{II}\|_{L^p(I_0)} \lesssim R^{1/p' - \alpha(p)} C_p(m). \tag{4.26}$$

The proof is a simple calculation which we relegate to the appendix.

The following proposition, a kind of L^p square root cancellation bound, implies (4.4) once we take Lemma 4.3 into account. Since we already proved (4.3), proving this proposition completes the proof of Theorem **??**.

Proposition 4.4. *Let P be a classical elliptic self-adjoint pseudodifferential operator of order one on a compact manifold M of dimension d satisfying the assumptions of Theorem* **??**. *Fix $R > 0$ and suppose $1 < p < 2(d-1)/(d+1)$. Then there exists $\varepsilon_M > 0$ with the following property. Consider any function $b : I_0 \to \mathbb{C}$, and suppose we can write $b = \sum_{t_0 \in \mathcal{T}_R} b^I_{t_0} + b^{II}$, where $supp(b^I_{t_0}) \subset I_{t_0}$ and $supp(b^{II}_R) \subset I_0 \setminus [-\varepsilon_M, \varepsilon_M]$. Define operators $T^I = \sum_{t_0 \in \mathcal{T}_R} T^I_{t_0}$ and T^{II}, where*

$$T^I_{t_0} = \int b^I_{t_0}(t)(Q_R \circ e^{2\pi i t P} \circ Q_R)\, dt \ \text{ and } \ T^{II} = \int b^{II}(t)(Q_R \circ e^{2\pi i t P} \circ Q_R)\, dt.$$

Then

$$\|T^I\|_{L^p \to L^p} \lesssim R^{-1/p'} \left(\sum_{t_0 \in \mathcal{T}_R} \left[\|b^I_{t_0}\|_{L^p(I_0)} \langle R t_0 \rangle^{\alpha(p)}\right]^p\right)^{1/p} \tag{4.27}$$

and

$$\|T^{II}\|_{L^p \to L^p} \lesssim R^{\alpha(p) - 1/p'} \|b^{II}\|_{L^p(I_0)}. \tag{4.28}$$

Proposition 4.4 splits the main bound of the paper into two regimes: regime I and regime II. Noting that we require weaker bounds in (4.28) than in (4.27), the operator T^{II} will not require as refined an analysis as for the operator T^I, and we obtain bounds on T^{II} by a reduction to an endpoint local smoothing inequality in Section 4.3. On the other hand, to obtain more refined estimates for the L^p norms of quantities of the form $f = T^I u$, we consider a decompositions of the form $u = \sum_{x_0} u_{x_0}$, where $u_{x_0} : M \to \mathbb{C}$ is supported on a ball $B(x, 1/R)$ of radius $1/R$ centered at x_0. We then have $f = \sum_{(x_0, t_0)} f_{x_0, t_0}$, where $f_{x_0, t_0} = T^I_{t_0} u_{x_0}$. To control f we must establish an L^p square root cancellation bound for the functions $\{f_{x_0, t_0}\}$. In the next section, we study the L^2 quasi-orthogonality of these functions, which we use as a starting point to obtain the required square root cancellation.

4.2 Quasi-Orthogonality Estimates For Solutions to the Half-Wave Equation Obtained From High-Frequency Wave Packets

The discussion at the end of Section 4.1 motivates us to consider estimates for functions obtained by taking averages of the wave equation over a small time interval, with initial conditions localized to a particular part of space. In this section, we study the L^2 orthogonality of such quantities. We do not exploit periodicity of the Hamiltonian flow in this section since we are only dealing with estimates for the half wave-equation for *small times*. Our results here thus hold for any manifold M, and any operator P whose principal symbol has cospheres with non-vanishing Gaussian curvature.

In order to prove the required quasi-orthogonality estimates of the functions $\{f_{x_0,t_0}\}$, we find a new connection between *Finsler geometry* on the manifold M and the behaviour of the operator P. We will not assume any knowledge of Finsler geometry in the sequel, describing results from the literature needed when required and trying to relate the tools we use to their Riemannian analogues. Moreover, in the case where $p(x,\xi) = \sqrt{\sum g^{jk}(x)\xi_j\xi_k}$ for some Riemannian metric g on M, the Finsler geometry we study will simply be the Riemannian geometry on M given by the metric g. We begin this section by briefly outlining the relevant concepts of Finsler geometry required to state Theorem 4.5, relegating more precise details to Subsection 4.2.2 where Finsler geometry arises more explicitly in our proofs.

For our purposes, Finsler geometry is very akin to Riemannian geometry, though instead of a smoothly varying *inner product* being given on the tangent spaces of M, in Finsler geometry we are given a smoothly varying *vector space norm* $F : TM \to [0,\infty)$ on the tangent spaces of M. Many results in Riemannian geometry carry over to the Finsler setting. In particular, we can define the length of a curve $c : I \to M$ via it's derivative $c' : I \to TM$ by the formula

$$L(c) = \int_I F(c'), \tag{4.29}$$

and thus get a theory of metric, geodesics, and curvature as in the Riemannian setting. The main quirk of the theory for us is that the metric induced from this length function is not symmetric. Indeed, let

$$d_+(p_0, p_1) = \inf\{L(c) : c(0) = p_0 \text{ and } c(1) = p_1\}, \tag{4.30}$$

and set $d_-(p_0, p_1) = d_+(p_1, p_0)$, i.e. so that

$$d_+(p_0, p_1) = \inf\{L(c) : c(0) = p_1 \text{ and } c(1) = p_0\}. \tag{4.31}$$

In the Riemannian setting, d_+ and d_- are both equal to the usual Riemannian metric, but on a general Finsler manifold one has $d_+ \neq d_-$, and these functions are distinct quasi-metrics on M (though a compactness argument shows $d_M^+(x_0, x_1) \sim d_M^-(x_0, x_1)$). Geodesics from a point p_0 to a point p_1 need not be geodesics from p_1 to p_0 when reversed. A metric can be obtained by setting $d_M = (d_M^+ + d_M^-)/2$, and we will use this as the canonical metric on M in what follows.

Finsler geometry arises here because, if $p : T^*M \to [0,\infty)$ is a homogeneous function satisfying the curvature assumptions of Theorem **??**, then a Finsler metric arises by taking the 'dual norm' of p, i.e. setting

$$F(x,v) = \sup\{\xi(v) : \xi \in T_x^*M \text{ and } p(x,\xi) = 1\}. \tag{4.32}$$

Thus p induces quasimetrics d_+ and d_- on M via the metric F as defined above. We now use these quasimetrics to state Proposition 4.5.

Proposition 4.5. *Let M be a compact manifold of dimension d, and let P be a classical elliptic self-adjoint pseudodifferential operator of order one whose principal symbol satisfies the curvature assumptions of Theorem* **??**. *Then there exists $\varepsilon_M > 0$ such that for all $R \geq 0$, the following estimates hold:*

- *(Pointwise Estimates) Fix $|t_0| \leq \varepsilon_M$ and $x_0 \in M$. Consider any two measurable functions $c : \mathbb{R} \to \mathbb{C}$ and $u : M \to \mathbb{C}$, with $\|c\|_{L^1(\mathbb{R})} \leq 1$ and $\|u\|_{L^1(M)} \leq 1$, and with $\mathrm{supp}(c) \subset I_{t_0}$ and $\mathrm{supp}(u) \subset B(x_0, 1/R)$. Define $S : M \to \mathbb{C}$ by setting*

$$S = \int c(t)(Q_R \circ e^{2\pi i t P} \circ Q_R)\{u\}\, dt. \tag{4.33}$$

  Then for any $K \geq 0$, and any $x \in M$,

$$|S(x)| \lesssim_K \frac{R^d}{\langle R d_M(x_0,x)\rangle^{\frac{d-1}{2}}} \max_\pm \left\langle R\left|t_0 \pm d_M^\pm(x,x_0)\right|\right\rangle^{-K}. \tag{4.34}$$

- *(Quasi-Orthogonality Estimates) Fix $|t_0 - t_1| \leq \varepsilon_M$, and $x_0, x_1 \in M$. Consider any two pairs of functions $c_0, c_1 : \mathbb{R} \to \mathbb{C}$ and $u_0, u_1 : M \to \mathbb{C}$ such that, for each $\nu \in \{0,1\}$, $\|c_\nu\|_{L^1(\mathbb{R})} \leq 1$, $\|u_\nu\|_{L^1(M)} \leq 1$, $\mathrm{supp}(c_\nu) \subset I_{t_\nu}$, and $\mathrm{supp}(u_\nu) \subset B(x_\nu, 1/R)$. Define $S_\nu : M \to \mathbb{C}$ by setting*

$$S_\nu = \int c_\nu(t)(Q_R \circ e^{2\pi i t P} \circ Q_R)\{u_\nu\}\, dt. \tag{4.35}$$

  Then for any $K \geq 0$,

$$|\langle S_0, S_1\rangle| \lesssim_K \frac{R^d}{\langle R d_M(x_0,x_1)\rangle^{\frac{d-1}{2}}} \max_\pm \left\langle R\left|(t_0 - t_1) \pm d^\pm(x_0,x_1)\right|\right\rangle^{-K}. \tag{4.36}$$

Remark. Estimate (4.36) is a variable coefficient analogue of Lemma 3.3 of [6]. The pointwise estimate tells us that the function S is concentrated on a geodesic annulus of radius $|t_0|$ centered at x_0 and thickness $O(1/R)$. The quasi-orthogonality estimate tells us that the two functions S_0 and S_1 are only significantly correlated with one another if the two annuli on which the majority of the support of S_0 and S_1 lie are internally or externally tangent to one another, depending on whether t_0 and t_1 have the same or opposite sign respectively.

Proof of Proposition 4.5. To simplify notation, in the following proof we will suppress the use of R as an index, for instance, writing Q for Q_R. For both the pointwise and quasi-orthogonality estimates, we want to consider the operators in coordinates, so we can use the *Lax-Hörmander Parametrix* to understand the wave propagators in terms of various oscillatory integrals, though we use a slight variant of the usual parametrix which will help us in the stationary phase arguments which occur when manipulating the resulting oscillatory integrals.

Start by covering M by a finite family of suitably small open sets $\{V_\alpha\}$, such that for each α, there is a coordinate chart U_α compactly containing V_α and with $N(V_\alpha, 1.1\varepsilon_M) \subset U_\alpha$. Let $\{\eta_\alpha\}$ be a partition of unity subordinate to $\{V_\alpha\}$. It will be convenient to define $V_\alpha^* = N(V_\alpha, 0.01\varepsilon_M)$ for each α. The next lemma allows us to approximate the operator Q, and the propagators $e^{2\pi i t P}$ with operators which have more explicit representations in the coordinate system $\{U_\alpha\}$, with an error negligible to the results of Proposition 4.5.

Lemma 4.6. *Suppose ε_M is suitably small, depending on M and P. For each α, and $|t| \leq \varepsilon_M$, there exists Schwartz operators Q_α and $W_\alpha(t)$, each with kernels supported on $U_\alpha \times V_\alpha^*$, such that the following holds:*

- *For any $u : M \to \mathbb{C}$ with $\|u\|_{L^1(M)} \leq 1$ and supp$(u) \subset V_\alpha^*$,*

$$\textit{supp}(Q_\alpha u) \subset N(\textit{supp}(u), 0.01\varepsilon_M), \tag{4.37}$$

$$\textit{supp}(W_\alpha(t)u) \subset N(\textit{supp}(u), 1.01\varepsilon_M), \tag{4.38}$$

$$\|(Q - Q_\alpha)u\|_{L^\infty(M)} \lesssim_N R^{-N} \quad \textit{for all } N \geq 0, \tag{4.39}$$

  and

$$\left\|(Q_\alpha \circ (e^{2\pi i t P} - W_\alpha(t)) \circ Q_\alpha)\{u\}\right\|_{L^\infty(M)} \lesssim_N R^{-N} \quad \textit{for all } N \geq 0. \tag{4.40}$$

- *In the coordinate system of U_α, the operator Q_α is a pseudo-differential operator of order zero given by a symbol $\sigma_\alpha(x,\xi)$, where*

$$\textit{supp}(\sigma_\alpha) \subset \{\xi \in \mathbb{R}^d : R/8 \leq |\xi| \leq 8R\}, \tag{4.41}$$

  and σ_α satisfies derivative estimates of the form

$$|\partial_x^\lambda \partial_\xi^\kappa \sigma_\alpha(x,\xi)| \lesssim_{\lambda,\kappa} R^{-|\kappa|}. \tag{4.42}$$

- *In the coordinate system U_α, the operator $W_\alpha(t)$ has a kernel $W_\alpha(t,x,y)$ with an oscillatory integral representation*

$$W_\alpha(t,x,y) = \int s(t,x,y,\xi) e^{2\pi i [\phi(x,y,\xi) + t p(y,\xi)]}\, d\xi, \tag{4.43}$$

  where the function s is compactly supported in U_α, such that

$$\textit{supp}_{t,x,y}(s) \subset \{(t,x,y) : |x-y| \leq C|t|\} \tag{4.44}$$

for some constant $C > 0$, suc that

$$supp_{\xi}(s) \subset \{\xi \in \mathbb{R}^d : R/8 \leq |\xi| \leq 8R\}, \tag{4.45}$$

and such that the function s satisfies derivative estimates of the form

$$|\partial^{\lambda}_{t,x,y}\partial^{\kappa}_{\xi} s| \lesssim_{\lambda,\kappa} R^{-|\kappa|}. \tag{4.46}$$

The phase function ϕ is smooth and homogeneous of degree one in the ξ variable, and solves the eikonal equation

$$p\,(x, \nabla_x \phi(x, y, \xi)) = p(y, \xi), \tag{4.47}$$

subject to the constraint that $\phi(x, y, \xi) = 0$ if $\xi \cdot (x - y) = 0$.

We relegate the proof of Lemma 4.6 to the appendix, the proof being a technical and mostly conventional calculation involving a manipulation of oscillatory integrals using integration by parts.

Let us now proceed with the proof of the pointwise bounds in Proposition 4.5 using Lemma 4.6. Given $u : M \to \mathbb{C}$, write $u = \sum_\alpha u_\alpha$, where $u_\alpha = \eta_\alpha u$. Lemma 4.6 implies that if we define

$$S_\alpha = \int c(t)(Q_{j,\alpha} \circ W_{j,\alpha}(t) \circ Q_{j,\alpha})\{u_\alpha\}\, dt, \tag{4.48}$$

then for all $N \geq 0$,

$$\left\| S - \textstyle\sum_\alpha S_\alpha \right\|_{L^\infty(M)} \lesssim_N R^{-N}. \tag{4.49}$$

This error is negligible to the pointwise bounds we want to obtain in Proposition 4.5 if we choose $N \geq K - \frac{d+1}{2}$, since the compactness of M implies that $d_M(x, x_0) \lesssim 1$ for all $x \in M$, and so

$$R^{-N} \lesssim R^{\left(\frac{d+1}{2} - K\right)} \lesssim \frac{R^d}{\langle R d_M(x, x_0)\rangle^{\frac{d-1}{2}}} \left\langle R \left| |t_0| \pm d^{\pm}_M(x, x_0) \right| \right\rangle^{-K}. \tag{4.50}$$

We bound each of the functions $\{S_\alpha\}$ separately, combining the estimates using the triangle inequality. We continue by expanding out the implicit integrals in the definition of S_α. In the coordinate system U_α, we can write

$$\begin{aligned} S_\alpha(x) = \int c(t)\sigma(x, \eta) & e^{2\pi i \eta \cdot (x-y)} \\ & s(t, y, z, \xi) e^{2\pi i [\phi(y,z,\xi) + t p(z,\xi)]} \\ & \quad \sigma(z, \theta) e^{2\pi i \theta \cdot (z-w)} (\eta_\alpha u)(w) \\ & \qquad dt\, dy\, dz\, dw\, d\theta\, d\xi\, d\eta. \end{aligned} \tag{4.51}$$

The integral in (4.51) looks highly complicated, but can be simplified considerably by noticing that most variables are quite highly localized. In particular, oscillation in the η variable implies that the amplitude is negligible unless $|x - y| \lesssim 1/R$, oscillation in the θ

variable implies that the amplitude is negligible unless $|z - w| \lesssim 1/R$, and the support of u implies that $|w - x_0| \lesssim 1/R$. Define

$$k_1(t, x, z, \xi) = \int \sigma(x, \eta) s(t, y, z, \xi) e^{2\pi i [\eta \cdot (x-y) + \phi(y,z,\xi) - \phi(x,z,\xi)]} \, dy \, d\eta, \tag{4.52}$$

and

$$\begin{aligned} k_2(t, \xi) = \int k_1(t, x, z, \xi) \sigma(z, \theta) (\eta_\alpha u)(w) \\ e^{2\pi i [\theta \cdot (z-w) + \phi(x,z,\xi) - \phi(x,x_0,\xi) + tp(z,\xi) - tp(x_0,\xi)]} \, d\theta \, dw, \end{aligned} \tag{4.53}$$

and then set

$$a(x, \xi) = \int c(t) k_2(t, R\xi) e^{2\pi i [(t-t_0) p(x_0, R\xi)]} \, dt \, dz, \tag{4.54}$$

so that $\operatorname{supp}_\xi(a) \subset \{\xi : 1/8 \leq |\xi| \leq 8\}$, and

$$S_\alpha(x) = R^d \int a(x, \xi) e^{2\pi i R [\phi(x, x_0, \xi) + t_0 p(x_0, \xi)]} \, d\xi. \tag{4.55}$$

Integrating by parts in η and θ in (4.52) and (4.53) gives that for all multi-indices α,

$$|\partial_\xi^\alpha k_1(t, x, z, \xi)| \lesssim_\alpha R^{-|\alpha|} \quad \text{and} \quad |\partial_\xi^\alpha k_2(z, \xi)| \lesssim_\alpha R^{-|\alpha|}. \tag{4.56}$$

Using the bounds in (4.56) with the fact that $\operatorname{supp}(c)$ is contained in a $O(1/R)$ neighborhood of t_0 in (4.54) then implies $|\partial_\xi^\alpha a(x, \xi)| \lesssim_\alpha 1$ for all α.

We now account for angular oscillation of the integral by working in a kind of 'polar coordinate' system. First we find $\lambda : V_\alpha^* \times S^{d-1} \to (0, \infty)$ such that for all $|\xi| = 1$,

$$p(x_0, \lambda(x_0, \xi)\xi) = 1. \tag{4.57}$$

If $\tilde{a}(x, \rho, \eta) = a(x, \rho\lambda(x_0)\xi) \det[\lambda(x_0, \xi) I + \xi (\nabla_\xi \lambda)(x_0, \xi)^T]$, then

$$S_\alpha(x) = R^d \int_0^\infty \rho^{d-1} \int_{|\xi|=1} \tilde{a}(x, \rho, \xi) e^{2\pi i R \rho [t_0 + \phi(x, x_0, \lambda(x_0, \xi)\xi)]} \, d\xi \, d\rho. \tag{4.58}$$

Define $\Phi : S^{d-1} \to \mathbb{R}$ by setting $\Phi(\xi) = \phi(x, x_0, \lambda(x_0, \xi)\xi)$. We claim that, in the ξ variable, Φ has exactly two critical points $|\xi^+|^{-1}\xi^+$ and $|\xi^-|^{-1}\xi^-$, where $\xi^+ \in S_{x_0}^*$ is the covector corresponding to the forward geodesic from x_0 to x, and $\xi^- \in S_{x_0}^*$ is the covector corresponding to the backward geodesic from x_0 to x. Moreover,

$$\Phi(|\xi^+|^{-1}\xi^+) = d_M^+(x_0, x) \quad \text{and} \quad \Phi(|\xi^-|^{-1}\xi^-) = -d_M^-(x_0, x), \tag{4.59}$$

and the Hessian at each of these points is non-degenerate, with each eigenvalue of the Hessian having magnitude exceeding a constant multiple of $d_M^\pm(x_0, x)$. We prove that these properties hold for Φ in Proposition 4.7 of the following section, via a series of geometric arguments. It then follows from the principle of stationary phase that

$$S_\alpha(x) = \sum_\pm \frac{R^d}{\left\langle Rd_M^\pm(x_0,x)\right\rangle^{\frac{d-1}{2}}} \int_0^\infty \rho^{\frac{d-1}{2}} a_\pm(x,\rho) e^{2\pi i R\rho[t_0 \pm d_M^\pm(x_0,x)]}\, d\rho, \tag{4.60}$$

where $a_\pm$ is supported on $|\rho| \sim 1$, and for all α, $|\partial_\rho^\alpha a_\pm| \lesssim_\alpha 1$. Integrating by parts in the ρ variable if $t_0 \pm d_M^\pm(x_0,x)$ is large, we conclude that

$$|S_\alpha(x)| \lesssim \frac{R^d}{\langle Rd_M(x_0,x)\rangle^{\frac{d-1}{2}}} \sum_\pm \langle R|t_0 \pm d_M(x_0,x)|\rangle^{-K}. \tag{4.61}$$

Combining (4.49) and (4.61) completes the proof of the pointwise bounds.

The quasi-orthogonality arguments are obtained by a largely analogous method, and so we only sketch the proof. One major difference is that we can use the self-adjointness of the operators Q, and the unitary group structure of $\{e^{2\pi i tP}\}$, to write

$$\begin{aligned}
\langle S_0, S_1\rangle &= \int c_0(t)c_1(s)\langle (Q\circ e^{2\pi i tP}\circ Q)\{u_0\}, (Q\circ e^{2\pi i sP}\circ Q)\{u_1\}\rangle \\
&= \int c_0(t)c_1(s)\langle (Q^2\circ e^{2\pi i(t-s)P}\circ Q^2)\{u_0\}, u_1\rangle \\
&= \int c(t)\langle (Q^2\circ e^{2\pi i tP}\circ Q^2)\{u_0\}, u_1\rangle,
\end{aligned} \tag{4.62}$$

where $c(t) = \int c_0(u)c_1(u-t)\, du$, by Young's inequality, satisfies

$$\|c\|_{L^1(\mathbb{R})} \lesssim \|c_0\|_{L^1(\mathbb{R})}\|c_1\|_{L^1(\mathbb{R})} \leq 1 \tag{4.63}$$

and $\text{supp}(c) \subset [(t_0 - t_1) - 4/R, (t_0 - t_1) + 4/R]$. After this, one proceeds exactly as in the proof of the pointwise estimate. We write the inner product as

$$\sum\nolimits_\alpha \int c(t)\langle (Q^2\circ e^{2\pi i tP}\circ Q^2)\{\eta_\alpha u_0\}, u_1\rangle. \tag{4.64}$$

Then we use Lemma 4.6 to replace $Q^2\circ e^{2\pi i tP}\circ Q^2$ with $Q_\alpha^2\circ W_\alpha(t)\circ Q_\alpha^2$ using Lemma 4.6, modulo a negligible error. The integral

$$\sum\nolimits_\alpha \int c(t)\langle (Q_\alpha^2\circ W_\alpha(t)\circ Q_\alpha^2)\{\eta_\alpha u_0\}, u_1\rangle \tag{4.65}$$

is then only non-zero if both the supports of u_0 and u_1 are compactly contained in U_α. Thus we can switch to the coordinate system of U_α, in which we can express the inner product by oscillatory integrals of the exact same kind as those occurring in the pointwise estimate. Integrating away the highly localized variables as in the pointwise case, and then applying stationary phase in polar coordinates proves the required estimates. □

4.2.1 Some Facts About Finsler geometry

All that remains to conclude the proof of the quasi-orthogonality estimates is to verify the required properties of the critical points of the phase function Φ which occurs in the proof of Proposition 4.5. To do this we exploit the Finsler geometry on the manifold, and so before we carry out this task we take out the time to precisely introduce the concepts of Finsler geometry that occur in the proof. We will work in a particular coordinate system U_0, so we introduce the concepts in coordinates to keep things concrete.

Let $U_0 \subset \mathbb{R}^d$ be an open set. Then we may identify the tangent space TU_0 and cotangent space T^*U_0 with $U_0 \times \mathbb{R}^d$, and each fibre T_xU_0 and $T_x^*U_0$ with $\mathbb{R}^d$, though as is standard in differential we use upper indices to denote the coordinates of vectors, and lower indices to denote the coordinates of covectors. A *Finsler metric* on U_0 is a homogeneous function $F : TU_0 \to [0, \infty)$ which is smooth on $TU_0 - 0$, and strictly convex, in the sense that the Hessian matrix with entries

$$g_{ij}(x, v) = \frac{1}{2} \frac{\partial^2 F^2}{\partial v^i \partial v^j}(x, v) \tag{4.66}$$

is positive definite for all $x \in U_0$ and $v \in T_xU_0 - \{0\}$.

Given a metric F on M, we define a dual metric $F_* : T^*M \to [0, \infty)$ by setting $F_*(x, \xi) = \sup\{\xi(v) : F(x, v) = 1\}$. Then F_* is also strictly convex, so that for each $x \in U_0$ and each covector $\xi \in \mathbb{R}^d - \{0\}$, the Hessian matrix with entries

$$g^{ij}(x, \xi) = \frac{1}{2} \frac{\partial^2 F_*^2}{\partial \xi_i \partial \xi_j}(x, \xi) \tag{4.67}$$

is positive definite. For $x \in U_0$, the *Legendre transform* is then defined to the smooth map $\mathcal{L}_x : T_xU_0 - \{0\} \to T_x^*U_0 - \{0\}$ given by the formula $(\mathcal{L}_x v)_i = \sum_j g_{ij}(x, v)v^j$, defined so that $\sum (\mathcal{L}_x v)_j w^j = \sum g_{ij}(x, v)v^i w^j$ for any vector $w \in \mathbb{R}^d$. Then the matrix with entries $g_{ij}(x, v)$ is the inverse of the matrix with entries $g^{ij}(x, \mathcal{L}_x v)$, so that for a covector $\xi \in \mathbb{R}^d - \{0\}$, $(\mathcal{L}_x^{-1}\xi)^i = \sum g^{ij}(x, \xi)\xi_j$. The Legendre transform is the Finsler variant of the musical isomorphism in Riemannian geometry, though the Legendre transform is in general only *homogeneous*, rather than linear.

Let us prove in more detail that if $p : T^*M \to [0, \infty)$ is homogenous and satisfies the assumptions of Theorem **??**, the function $F(x, v) = \sup\{\xi(v) : p(x, \xi) = 1\}$ gives a Finsler metric on M. Working in a coordinate system, we may assume $M = U_0$ is an open subset of $\mathbb{R}^d$ like above. For each $x \in U_0$, the cosphere $S_x^* = \{\xi \in T_x^*M : p(x, \xi) = 1\}$ has non-vanishing Gaussian curvature. Thus all principal curvatures of S_x^* must actually be *positive*, since S_x^* is a compact hypersurface in T_x^*M. This follows from a simple modification of an argument found in Chapter 2 of [**HeinzHopf**]. Indeed, if we fix an arbitrary point $v_0 \in T_x^*M$, and consider the smallest closed ball $B \subset T_x^*M$ centered at v_0 and containing S_x^*, then the sphere ∂B must share the same tangent plane as S_x^* at some point. All principal curvatures of ∂B are positive, and at this point all principal curvatures of S_x^* must be greater than the principal curvatures of ∂B, since S_x^* curves away faster than ∂B in all directions. By continuity, we conclude that the principal curvatures are everywhere positive. Thus for each $x \in M$ and $\xi \in T_x^*M - \{0\}$, the coefficients $g^{ij}(x,\xi) = (1/2)(\partial^2 p^2/\partial\xi_i\partial\xi_j)$ form a positive-definite matrix. But inverting the procedure of the last paragraph shows that the dual norm $F(x,v) = \sup_{\xi \in S_x^*} \xi(v)$ is also strictly convex, and thus gives a Finsler metric on M with $F_* = p$.

The theory of geodesics on Finsler manifolds is similar to the Riemannian case, except for the interesting quirk that a geodesic from a point p to a point q need not necesarily be a geodesic when considered as a curve from q to p, and so we must consider *forward* and *backward* geodesics. We define the length of a curve $c : I \to U_0$ by the formula $L(c) = \int_I F(c,\dot{c})\, dt$, and use this to define the *forward distance* $d_+ : U_0 \times U_0 \to [0,\infty)$ by taking the infima of paths between points. This function is a *quasi-metric*, as it is not necessarily symmetric. We define the *backward distance* $d_- : U_0 \times U_0 \to [0,\infty)$ by setting $d_-(p,q) = d_+(q,p)$. A *constant speed speed forward geodesic* on a Finsler manifold is a curve $c : I \to U_0$ which satisfies the geodesic equation

$$\ddot{c}^a = -\sum\nolimits_{j,k} \gamma^a_{jk}(c,\dot{c})\dot{c}^j\dot{c}^k \quad \text{for } 1 \leq i \leq d, \tag{4.68}$$

where $\gamma^a_{jk} = \sum_i (g^{ai}/2)(\partial_{x_k} g_{ij} + \partial_{x_j} g_{ik} - \partial_{x_i} g_{jk})$ are the *formal Christoffel symbols*. We call the reversal of such a geodesic a *backward geodesic*. A curve c is a geodesic if and only if it's Legendre transform $(x,\xi) = \mathcal{L}(c,\dot{c})$ satisfies

$$x'(t) = (\partial_\xi F_*)(x,\xi) \quad \text{and} \quad \xi'(t) = -(\partial_x F_*)(x,\xi). \tag{4.69}$$

The equations (4.69) are first order ordinary differential equations induced by the Hamiltonian vector field $(\partial_\xi F_*, -\partial_x F_*)$ on T^*U_0. Note that in our situation, the dual norm F_* is the principal symbol p of the operator P we are studying, and so (4.69) is precisely the Hamiltonian flow alluded to in (**??**). Sufficiently short geodesics on a Finsler manifold are length minimizing. In particular, there exists $r > 0$ such that if c is a geodesic between two points x_0 and x_1, and $L(c) < r$, then $d^+_M(x_0,x_1) = L(c)$.

The last piece of technology we must consider before our analysis of critical points are *variation formulas* for arclength. Consider a function $A : (-\varepsilon,\varepsilon) \times [0,1] \to U_0$, such that for each s, the function $t \mapsto A(s,t)$ is a constant speed geodesic, and such that all such geodesics begin at the same point, i.e. so that $A(s,0)$ is independent of s. Define the vector field $V(s) = (\partial_s A)(s,1)$ and $T(s) = (\partial_t A)(s,1)$. Then if $R(s)$ gives the arclength of the geodesic $t \mapsto A(s,t)$, the first variation formula for energy tells us

$$R(s)R'(s) = \sum\nolimits_{i,j} g_{ij}(A(s,1),T(s))V^i(s)T^j(s). \tag{4.70}$$

Details can be found in Chapter 5 of [**BaoChern**]. We will consider *second variations*, i.e. estimates on R'', but we will not need the second variation formula to obtain the estimates we need, relying only on differentiating the equation (4.70) rather than explicitly working with Jacobi fields.

Remark. The second variation formula for arclength in Riemannian geometry has an exact analogue in Finsler geometry via a generalization of the theory of Jacobi fields. Sectional curvature is replaced with what is called *flag curvature*, which controls the growth of Jacobi fields. One could in principle use such quantities to obtain more explicit constants in Theorem 4.7, though we have no such need of these constants here.

4.2.2 Analysis of Critical Points

In this section, we now classify the critical points of the kinds of functions that arose in Proposition 4.5.

Proposition 4.7. *Fix a bounded open set $U_0 \subset \mathbb{R}^d$. Consider a Finsler metric $F : U_0 \times \mathbb{R}^d \to [0,\infty)$ on U_0, and it's dual metric $F_* : U_0 \times \mathbb{R}^d \to [0,\infty)$, which extends to a Finsler metric on an open set containing the closure of U_0. Fix a suitably small constant $r > 0$. Let U be an open subset of U_0 with diameter at most r which is geodesically convex (any two points are joined by a minimizing geodesic). Let $\phi : U \times U \times \mathbb{R}^d \to \mathbb{R}$ solve the eikonal equation*

$$F_*(x, \nabla_x \phi(x,y,\xi)) = F_*(y,\xi), \tag{4.71}$$

such that $\phi(x,y,\xi) = 0$ for $x \in H(y,\xi)$, where $H(y,\xi) = \{x \in U : \xi \cdot (x-y) = 0\}$. For each $x_0, x_1 \in U$, let $S^*_{x_0} = \{\xi \in \mathbb{R}^d : F_*(x_0,\xi) = 1\}$ be the cosphere at x_0, and define $\Psi : S^*_{x_0} \to \mathbb{R}$ by setting

$$\Psi(\xi) = \phi(x_1, x_0, \xi). \tag{4.72}$$

Then the function Ψ has exactly two critical points, at ξ^+ and ξ^-, where the Legendre transform of ξ^+ is the tangent vector of the forward geodesic from x_0 to x_1, and the Legendre transform of ξ^- is the tangent vector of the backward geodesic from x_1 to x_0. Moreover,

$$\Psi(\xi^+) = d_M^+(x_0,x_1) \quad \text{and} \quad \Psi(\xi^-) = -d_M^-(x_0,x_1), \tag{4.73}$$

and the Hessians H_+ and H_- of Ψ at these critical points, viewed as quadratic maps from $T_{\xi_\pm} S^*_{x_0} \to \mathbb{R}$ satisfy

$$H_+(\zeta) \geq C d_M^+(x_0,x_1)|\zeta| \quad \text{and} \quad H_-(\zeta) \leq -C d_M^-(x_0,x_1)|\zeta|, \tag{4.74}$$

where the implicit constant is uniform in x_0 and x_1.

If Ψ is as above, then the function $\Phi : S^{d-1} \to \mathbb{R}$ obtained by setting $\Phi(\xi) = \phi(x, x_0, F_*(x,\xi)^{-1}\xi)$ is precisely the kind of function that arose as a phase in Proposition 4.5, where F_* was the principal symbol p of the pseudodifferential operator we were considering. Since critical points and the Hessians of maps at critical points are stable under diffeomorphisms, and the map $\xi \mapsto F_*(x,\xi)^{-1}\xi$ is a diffeomorphism from $S^*_{x_0}$ to S^{d-1}, classifying the critical points of the map Ψ implies the required properties of the map Φ used in Proposition 4.5.

In order to prove 4.7, we rely on a geometric interpretation of Ψ following from Hamilton-Jacobi theory.

Lemma 4.8. *Consider the setup to Proposition 4.7. For any $\xi \in S^*_{x_0}$,*

$$|\Psi(\xi)| = \begin{cases} \textit{the length of the shortest curve from } H(x_0,\xi) \textit{ to } x_1 & : \textit{if } \Psi(\xi) > 0, \\ \textit{the length of the shortest curve from } x_1 \textit{ to } H(x_0,\xi) & : \textit{if } \Psi(\xi) < 0. \end{cases}$$

Proof. We rely on a construction of ϕ from Proposition 3.7 of [**Treves2**], which we briefly describe. Fix x_0 and ξ. Then there is a unique covector field $\omega : H(x_0, \xi) \to \mathbb{R}^d$ which is everywhere perpendicular to $H(x_0, \xi)$, with $\omega(x_0) = \xi$ and with $F_*(x, \omega(x)) = F_*(x_0, \xi)$ for all $x \in H(x_0, \xi)$. There exists a unique point $x(\xi) \in H(x_0, \xi)$ and a unique $t(\xi) \in \mathbb{R}$, such that the unit speed geodesic γ on M with $\gamma(0) = x(\xi)$ and $\gamma'(0) = \mathcal{L}^{-1}(x(\xi), \omega(x(\xi)))$ satisfies $\gamma(t(\xi)) = x_1$. We then have $\Psi(\xi) = t(\xi)$. If $t(\xi)$ is negative, then $\gamma|_{[t(\xi),0]}$ is a geodesic from x_1 to x_0, and if $t(\xi)$ is positive, $\gamma|_{[0,t(\xi)]}$ is a geodesic from x_0 to x_1. Because γ is a geodesic, the geometric interpretation then follows if U is a suitably small neighborhood such that geodesics are length minimizing. □

It follows immediately from Lemma 4.8 that

$$\Psi(\xi_+) = d_M^+(x_0, x_1) \quad \text{and} \quad \Psi(\xi_-) = -d_M^-(x_0, x_1). \tag{4.75}$$

A simple geometric argument also shows $\Psi(\xi^+)$ is the maximum value of Ψ on $S^*_{x_0}$, and $\Psi(\xi^-)$ is the minimum value on $S^*_{x_0}$, so that these two points are both critical. Indeed, the point x_0 lies in $H(x_0, \xi)$ for all $\xi \in \mathbb{R}^d - \{0\}$. Thus the shortest curve from x_0 to x_1 is always longer than the shortest curve from $H(x_0, \xi)$ to x_1. Similarily, the shortest curve from x_1 to x_0 is always longer than the shortest curve from x_1 to $H(x_0, \xi)$. Thus

$$-d_M^-(x_0, x_1) \leq \Psi(\xi) \leq d_M^+(x_0, x_1), \tag{4.76}$$

and so $\Psi(\xi^-) \leq \Psi(\xi) \leq \Psi(\xi^+)$ for all $\xi \in$. All that remains is to prove that ξ^+ and ξ^- are the *only* critical points of Ψ, and that these critical points are appropriately non-degenerate.

In order to simplify proofs, we employ a structural symmetry to reduce the number of cases we need to analyze. Namely, if one defines the reverse Finsler metric $F_\rho(x, v) = F(x, -v)$, then $F_\rho^*(x, \xi) = F_*(x, -\xi)$, and so the associated function Ψ^ρ which is the analogue of Ψ for F^ρ satisfies $\Psi^\rho(\xi) = -\Psi(-\xi)$. The critical points of Ψ and Ψ^ρ are thus directly related to one another, which allows us without loss of generality to study only points with $\Psi \leq 0$ (and thus only study geodesics beginning at x_1).

Proof that ξ^+ and ξ^- are the only critical points. Fix $\xi^* \in T_{x_0}M - \{\xi^\pm\}$. Using the notation defined above, let $x_* = x(\xi^*)$ and $t_* = t(\xi^*)$. Using the symmetry above, we may assume without loss of generality that $\Psi(\xi^*) \leq 0$. Since ξ^- is not perpendicular to $H(x_0, \xi^*)$ at x_0, we have $x_* \neq x_0$. If $\Psi(\xi^*) < 0$, let γ be the unique unit speed forward geodesic with $\gamma(0) = x_1$ and $\gamma(t_*) = x_*$. If $\Psi(\xi^*) = 0$, let γ be the unique unit speed forward geodesic with $\gamma'(0)$ equal to the Legendre transform of ξ^*. Pick η such that $\eta \cdot (x_* - x_0) \neq 0$, and then, for t suitably close to t_*, define a smooth map

$$\xi(t) = \frac{\xi^* + a(t)\eta}{F_*(x_0, \xi^* + a(t)\eta)}, \tag{4.77}$$

into $S^*_{x_0}$, where

$$a(t) = -\frac{\xi^* \cdot (\gamma(t) - x_0)}{\eta \cdot (\gamma(t) - x_0)} \tag{4.78}$$

is defined so that $\gamma(t) \in H(x_0, \xi(t))$. Then $\Psi(\xi(t)) = -t$, and differentiation at $t = t_*$ gives $D\Psi(\xi^*)(\xi'(t_*)) = -1$. In particular, $D\Psi(\xi^*) \neq 0$, so ξ^* is not a critical point. □

We now analyze the non-degeneracy of the critical points ξ^+ and ξ^-.

Proof that ξ^+ and ξ^- are non-degenerate. Using symmetry, it suffices without loss of generality to analyze the critical point ξ^- rather than ξ^+. Let $H_- : T_{\xi^-} S^*_{x_0} \to \mathbb{R}$ be the Hessian of Ψ at ξ^-. Let $v_0 = \mathcal{L}_{x_0}^{-1} \xi^-$, and using the notation of the last argument, let $l = t(\xi^-)$. Consider a curve $\xi(a)$ valued in $S^*_{x_0}$ with $\xi(0) = \xi^-$ and $\zeta = \xi'(0)$ for some $\zeta \in T^*_{\xi^-} S_{x_0}$. Then the second derivative of $\Psi(\xi(a))$ at $a = 0$ is $H_-(\zeta)$, and so our proof would be complete if we could show that the function $L(a) = -\Psi(\xi(a))$, which is the length of the shortest geodesic from x_1 to $H(x_0, \xi(a))$, satisfies $L''(0) \geq Cl|\zeta|$, for a constant $C > 0$ uniform in x_0, x_1, and ζ.

Consider the partial function $\operatorname{Exp}_{x_1} : \mathbb{R}^d \to U_0$ obtained from the exponential map at x_1. If r is chosen suitably small, there exists a neighborhood V of the origin in $\mathbb{R}^d$ such that $E = \operatorname{Exp}_{x_1}|_V$ is a diffeomorphism between V and U. Unlike in Riemannian manifolds, in general E is only C^1 at the origin, though smooth on $V - \{0\}$. However, for $|v| = 1$ and $t > 0$ we can write $E(tv) = (\pi \circ \varphi)(x_1, \mathcal{L}_{x_1} v, t)$, where the partial function $\varphi : (T^* U_0 - 0) \times \mathbb{R} \to (T^* U_0 - 0)$ is the flow induced by the the Hamiltonian vector field $(\partial_\xi F_*, -\partial_x F_*)$ on $T^* U_0 - 0$, and π is the projection map from $T^* U_0 - 0$ to U_0. Where defined, φ is a smooth function since the Hamiltonian vector field is smooth, and so it follows by homogeneity and the precompactness of U that the partial derivatives $(\partial^\alpha E_x)(v)$ are uniformly bounded for $v \neq 0$ and $x \in U$. It thus follows from the inverse function theorem that there exists a constant $A > 0$ such that for all x_1 and x in U with $x \neq x_1$,

$$|\partial_j G_{x_1}(x)| \leq A \quad \text{and} \quad |(\partial_j \partial_k G_{x_1})(x)| \leq A. \tag{4.79}$$

We can also pick A to be large enough that for all $x \in U$, and all $v, w \in \mathbb{R}^d - \{0\}$,

$$A^{-1} |w|^2 \leq \sum_{ij} g_{ij}(x, v) w^i w^j \leq A |w|^2. \tag{4.80}$$

and such that for all $x \in U$ and $v \in \mathbb{R}^d - \{0\}$,

$$|\partial_{x_k} g_{ij}(x, v)| \leq A. \tag{4.81}$$

Since $\zeta \in T_{\xi^-} S^*_{x_0}$, and $S^*_{x_0} = \{\xi : F_*(x_0, \xi)^2 = 1\}$, by Euler's homogeneous function theorem we have

$$\sum_{ij} g^{ij}(x_0, \xi^-) \xi'(0) \xi_j^- = \frac{1}{2} \sum_{i,j} \frac{\partial^2 F_*^2}{\partial \xi_i \partial \xi_j}(x_0, \xi^-) \zeta_i \xi_j^- = \frac{1}{2} \sum \frac{\partial F_*^2}{\partial \xi_j}(x_0, \xi^-) \zeta_j = 0. \tag{4.82}$$

Differentiating $F_*(x_0, \xi(a))^2 = 1$ twice with respect to a at $a = 0$ yields that

$$\sum_{i,j} g_{ij}(x_0, \xi^-) \xi_i''(0) \xi_j^- = -\sum_{i,j} g_{ij}(x_0, \xi^-) \zeta^i \zeta^j \tag{4.83}$$

Define vectors $n(a)$ by setting $n^i(a) = \sum g^{ij}(x_0, \xi^-) \xi_j(a)$. Then $n(a)$ is the normal vector to $H(x_0, \xi(a))$ with respect to the inner product with coefficients $g_{ij}(x_0, v_0)$. Let $u(a)$ be the orthogonal projection of v_0 onto the hyperplane $\{v : \xi(a) \cdot v = 0\}$ with respect to the inner product $g_{ij}(x_0, v_0)$. If we define

$$c(a) = \sum g_{ij}(x_0, v_0) v_0^i n^j(a) = \sum g^{ij}(x_0, \xi^-) \xi_i^- \xi_j(a), \tag{4.84}$$

then $u(a) = v_0 - c(a)n(a)$. Note that (4.82) and (4.83) imply $c(0) = 1$, $c'(0) = 0$, and $c''(0) \leq -|\zeta|^2/A$. Also $n(0) = v_0$ and $|n'(0)| \leq A|\zeta|$.

Let $x(s) = x_0 + su(a)$ and let $R(s)$ be the length of the geodesic from x_1 to $x(s)$. To control $R(s)$, define $y(s) = G(x(s))$, and consider the variation $A(s,t) = E(ty(s))$, defined so that $t \mapsto A(s,t)$ is the geodesic from x_1 to $x(s)$. The Gauss Lemma for Finsler manifolds (see Lemma 6.1.1 of [**BaoChern**]) implies that

$$A^{-1}R(s) \leq |y(s)| \leq AR(s). \tag{4.85}$$

Define $T(s) = (\partial_t A)(s,1)$ and $V(s) = (\partial_s A)(s,1)$. Then (4.70) implies

$$R(s)R'(s) = \sum_{i,j} g_{ij}(x(s), T(s)) V^i(s) T^j(s). \tag{4.86}$$

Again, the Gauss Lemma implies

$$A^{-1}R(s) \leq |T(s)| \leq AR(s). \tag{4.87}$$

We can write

$$T^i(s) = \sum_{i,j} (\partial_j E^i)(y(s)) y^j(s) \tag{4.88}$$

and

$$V^i(s) = \sum_{i,j,k} (\partial_j E^i)(y(s))(\partial_k G^j)(x(s)) u^k(a) = u^i(a). \tag{4.89}$$

In particular, $T(0) = v_0$, so $R'(0) = \sum g_{ij}(x_0, v_0) u^i(a) v_0^j = 1 - c(a)^2$. Cauchy-Schwartz applied to (4.86) also tells us that

$$|R'(s)| \lesssim_A |u(a)|. \tag{4.90}$$

Note that $u(0) = 0$, so if a is small enough, then we have $|u(a)| \leq l/2$. Taylor's theorem applied to (4.90), noting $R(0) = l$ then gives that for $|s| \leq 1$,

$$l/2 \leq R(s) \leq 2l. \tag{4.91}$$

Differentiating (4.86) tells us that

$$\begin{aligned} R'(s)^2 + R''(s)R(s) = \sum_{i,j,k} &\left[(\partial_{x_k} g_{ij})(x(s), T(s)) u^i(a) T^j(s) u^k(a) \right] \\ &+ \left[(\partial_{v_k} g_{ij})(x(s), T(s)) u^i(a) T^j(s) (\partial_s T^k)(s) \right] \\ &+ \left[g_{ij}(x(s), T(s)) u^i(a) (\partial_s T^j)(s) \right]. \end{aligned} \tag{4.92}$$

Write the right hand side as I + II + III. Using (4.81), (4.87), (4.91), and the triangle inequality gives

$$|\mathrm{I}| \lesssim R(s)|u(a)|^2 \lesssim l|u(a)|^2. \tag{4.93}$$

Since $\partial_{v_k} g_{ij} = (1/2)(\partial^3 F^2/\partial v_i \partial v_j \partial v_k)$, applying Euler's homogeneous function theorem when summing over j implies that

$$\mathrm{II} = 0. \tag{4.94}$$

Applying Cauchy-Schwarz and (4.79), we find

$$|\mathrm{III}| \lesssim |u(a)||T'(s)| \lesssim |u(a)|^2[|y(s)| + 1] \lesssim (l+1)|u(a)|^2. \tag{4.95}$$

But now combining (4.93), (4.94), (4.95), and rearranging (4.92) shows that

$$|R''(s)| \lesssim l^{-1}|u(a)|^2 + (1 + l^{-1})|u(a)|^2 \lesssim l^{-1}|u(a)|^2. \tag{4.96}$$

Taylor's theorem implies there exists $B > 0$ depending only on A and d such that

$$|R(s) - (R(0) + sR'(0))| \leq Bl^{-1}|u(a)|^2 s^2. \tag{4.97}$$

Since $R(0)$ =l, $R'(0) = 1 - c(a)^2$, $c(0) = 1$, $u(0) = 0$, $c'(0) = 0$, $|u'(0)| \leq A|\zeta|$, and $c''(0) \leq -|\zeta|^2/A$, we conclude from (4.97) that as $a \to 0$, if $s > 0$ then

$$\begin{aligned} R(-s) &\leq l - sR'(0) + Bl^{-1}|u(a)|^2 s^2 \\ &\leq l - s(A^{-1}|\zeta|^2 a^2) + A^2 B l^{-1}|\zeta|^2 s^2 a^2 + O(a^3(s + l^{-1}s^2)). \end{aligned} \tag{4.98}$$

For all s, $L(a) \leq R(s)$. Optimizing by picking $s = l/2A^3 B$ gives

$$L(a) \leq R(-s) \leq l - l|\zeta|^2 a^2/4A^4 B + O(a^3). \tag{4.99}$$

Taking $a \to 0$ and using that $L(0) = l$ gives that $L''(0) \leq -l|\zeta|^2/4A^4 B$, so setting $C = 1/4A^4 B$, we find we have proved what was required. $\square$

The functions in the sum defining $F_{k,\mu}$ are highly coupled, and it is difficult to use anything except Cauchy-Schwarz to break them apart. Since $\#(\mathcal{T}_R \cap I_{k,\mu}) \sim A^a$, if we set $F_{k,\mu} = \sum_{t \in \mathcal{T}_R \cap I_{k,\mu}} F_{k,\mu,t}$, where

$$F_{k,\mu,t} = \sum\nolimits_{(x_0,t) \in \mathcal{E}_{k,\mu}} 2^{k\frac{d-1}{2}} S_{x_0,t}. \tag{4.100}$$

Then Cauchy-Schwarz implies that

$$\|F_{k,\mu}\|_{L^2(M)}^2 \lesssim A^a \sum\nolimits_{t \in \mathcal{T}_R \cap I_{k,\mu}} \|F_{k,\mu,t}\|_{L^2(M)}^2. \tag{4.101}$$

Since the elements of $\mathcal{X}_R$ are $1/R$ separated, the functions in the sum defining $F_{k,\mu,t}$ are quite orthogonal to one another; Proposition 4.5 implies that for $x_0 \neq x_1$,

$$|\langle S_{x_0,t}, S_{x_1,t} \rangle| \lesssim R^d (R d_M(x_0, x_1))^{-K} \quad \text{for all } K \geq 0. \tag{4.102}$$

Thus

$$\|F_{k,\mu,t}\|^2_{L^2(M)} \lesssim R^d 2^{k(d-1)} \#(\mathcal{E}_k \cap (M \times \{t\})). \tag{4.103}$$

But this means that

$$A^a \sum_{t \in \mathcal{T}_R \cap I_{k,\mu}} \|F_{k,\mu,t}\|^2_{L^2(M)} \lesssim R^d 2^{k(d-1)} A^a \# \mathcal{E}_{k,\mu}. \tag{4.104}$$

Thus (**??**), (4.101), and (4.104) imply that

$$\begin{aligned} \|F_k\|^2_{L^2(M)} &\lesssim \sum_\mu \|F_{k,\mu}\|^2_{L^2(M)} + R^d \left(1 + 2^{k(d-1)} A^{1-a\left(\frac{d-3}{2}\right)}\right) \# \mathcal{E}_k \\ &\lesssim R^d \left(2^{k(d-1)} A^a + (1 + 2^{k(d-1)} A^{1-a\left(\frac{d-3}{2}\right)}\right) \# \mathcal{E}_k. \end{aligned} \tag{4.105}$$

Optimizing by picking $a = 2/(d-1)$ gives that

$$\|F_k\|^2_{L^2(M)} \lesssim R^d 2^{k(d-1)} A^{\frac{2}{d-1}} \# \mathcal{E}_k. \tag{4.106}$$

The proof is completed by combining (**??**) with (4.106).

4.2.3 L^p Estimates Via Density Decompositions

Combining the L^2 analysis of Section **??** with a density decomposition argument, we can now prove the following Lemma, which completes the analysis of the operator T^I in Proposition 4.4.

Lemma 4.9. *Using the notation of Proposition 4.4, let $T^I = \sum_{t_0 \in \mathcal{T}_R} T^I_{t_0}$, where*

$$T^I_{t_0} = b^I_{t_0}(t)(Q_R \circ e^{2\pi i t P} \circ Q_R)\, dt. \tag{4.107}$$

Then for $1 \leq p < 2(d-1)/(d+1)$,

$$\|T^I u\|_{L^p(M)} \lesssim R^{-1/p'} \left(\sum_{t_0 \in \mathcal{T}_R} \left[\|b^I_{t_0}\|_{L^p(I_0)} \langle R t_0 \rangle^{\alpha(p)} \right]^p \right)^{1/p} \|u\|_{L^p(M)}. \tag{4.108}$$

We prove Lemma 4.9 via a *density decomposition* argument, adapted from the methods of [6]. Given a function $u : M \to \mathbb{C}$, we use a partition of unity to write

$$u = \sum_{x_0 \in \mathcal{X}_R} u_{x_0}, \tag{4.109}$$

where u_{x_0} is supported on $B(x_0, 1/R)$, and

$$\left(\sum_{x_0 \in \mathcal{X}_R} \|u_{x_0}\|^p_{L^1(M)} \right)^{1/p} \lesssim R^{-d/p'} \left(\sum_{x_0 \in \mathcal{X}_R} \|u_{x_0}\|^p_{L^p(M)} \right)^{1/p} \lesssim R^{-d/p'} \|u\|_{L^p(M)}. \tag{4.110}$$

Define

$$\mathcal{X}_a = \{x_0 \in \mathcal{X}_R : 2^{a-1} < \|u_{x_0}\|_{L^1(M)} \leq 2^a\} \tag{4.111}$$

and let

$$\mathcal{T}_b = \{t_0 \in \mathcal{T}_R : 2^{b-1} < \|b^I_{t_0}\|_{L^1(M)} \leq 2^b\}. \tag{4.112}$$

Define functions $f_{x_0,t_0} = T^I_{t_0} u_{x_0}$. Lemma 4.9 follows from the following result.

Lemma 4.10. *Fix $u \in L^p(M)$, and consider $\mathcal{X}_a$, $\mathcal{T}_b$, and $\{f_{x_0,t_0}\}$ as above. For any function $c : \mathcal{X}_R \times \mathcal{T}_R \to \mathbb{C}$, and $1 < p < 2(d-1)/(d+1)$,*

$$\begin{aligned} &\Big\| \sum\nolimits_{a,b} \sum\nolimits_{(x_0,t_0)\in\mathcal{X}_a\times\mathcal{T}_b} 2^{-(a+b)} \langle Rt_0\rangle^{\frac{d-1}{2}} c(x_0,t_0) f_{x_0,t_0} \Big\|_{L^p(M)} \\ &\qquad \lesssim R^{d/p'} \Big(\sum\nolimits_{a,b} \sum\nolimits_{(x_0,t_0)\in\mathcal{X}_a\times\mathcal{T}_b} |c(x_0,t_0)|^p \langle Rt_0\rangle^{d-1} \Big)^{1/p}. \end{aligned} \tag{4.113}$$

To see how Lemma 4.10 implies Lemma 4.9, set $c(x_0,t_0) = 2^{a+b}\langle Rt_0\rangle^{-\frac{d-1}{2}}$ for $x_0 \in \mathcal{X}_a$ and $t_0 \in \mathcal{T}_b$. Then Lemma 4.10 implies that

$$\begin{aligned} &\|T^I u\|_{L^p(M)} \\ &\quad = \Big\| \sum f_{x_0,t_0} \Big\|_{L^p(M)} \\ &\quad \lesssim R^{d/p'} \Big(\sum\nolimits_{(x_0,t_0)} \Big[\|b^I_{t_0}\|_{L^1(\mathbb{R})} \|u_{x_0}\|_{L^1(M)} \langle Rt_0\rangle^{\alpha(p)} \Big]^p \Big)^{1/p} \\ &\quad \lesssim R^{-1/p'} \Big(\sum\nolimits_{t_0} \Big[\|b^I_{t_0}\|_{L^p(I_0)} \langle Rt_0\rangle^{\alpha(p)} \Big]^p \Big)^{1/p} \Big(\sum\nolimits_{x_0} \Big[\|u_{x_0}\|_{L^1(M)} R^{d/p'} \Big]^p \Big)^{1/p} \\ &\quad \lesssim R^{-1/p'} \Big(\sum\nolimits_{t_0} \Big[\|b^I_{t_0}\|_{L^p(I_0)} \langle Rt_0\rangle^{\alpha(p)} \Big]^p \Big)^{1/p} \|u\|_{L^p(M)}. \end{aligned} \tag{4.114}$$

Thus we have proved Lemma 4.9. We take the remainder of this section to prove Lemma 4.10 using a density decomposition argument.

Proof of Lemma 4.10. For $p = 1$, this inequality follows simply by applying the triangle inequality, and applying the pointwise estimates of Proposition 4.5. By methods of interpolation, to prove the result for $p > 1$, we thus only need only prove a restricted strong type version of this inequality. In other words, we can restrict c to be the indicator function of a set $\mathcal{E} \subset \mathcal{X}_R \times \mathcal{T}_R$. Write $\mathcal{E} = \bigcup_{k\geq 0} \mathcal{E}_{k,a,b}$, where

$$\mathcal{E}_{0,a,b} = \{(x,t) \in \mathcal{E} \cap (\mathcal{X}_a \times \mathcal{T}_b) : |t| \leq 1/R\} \tag{4.115}$$

and for $k > 0$, let

$$\mathcal{E}_{k,a,b} = \{(x,t) \in \mathcal{E} \cap (\mathcal{X}_a \times \mathcal{T}_b) : 2^{k-1}/R < |t| \leq 2^k/R\}. \tag{4.116}$$

Write

$$F_k = \sum\nolimits_{a,b} \sum\nolimits_{(x_0,t_0)\in\mathcal{E}_{k,a,b}} 2^{k\left(\frac{d-1}{2}\right)} 2^{-(a+b)} f_{x_0,t_0}. \tag{4.117}$$

Our proof will be completed if we can show that

$$\Big\| \sum\nolimits_k F_k \Big\|_{L^p(M)} \lesssim R^{d(1-1/p)} \Big(\sum\nolimits_k 2^{k(d-1)} \#\mathcal{E}_k \Big)^{1/p}. \tag{4.118}$$

To prove (4.118), we perform a density decomposition on the sets $\{\mathcal{E}_k\}$. For $u \geq 0$, let $\widehat{\mathcal{E}_k}(u)$ be the set of all points $(x_0,t_0) \in \mathcal{E}_k$ that are contained in a ball B with $\mathrm{rad}(B) \leq 2^k/100R$, such that $\#(\mathcal{E}_k \cap B) \geq R2^u \mathrm{rad}(B)$. Then define

$$\mathcal{E}_k(u) = \widehat{\mathcal{E}_k}(u) - \bigcup\nolimits_{u'>u} \widehat{\mathcal{E}_k}(u'). \tag{4.119}$$

Because the set $\mathcal{E}_k$ is $1/R$ discretized, we have

$$\mathcal{E}_k = \bigcup_{u \geq 0} \mathcal{E}_k(u). \tag{4.120}$$

Moreover $\mathcal{E}_k(u)$ has density type $(R2^u, 2^k/100R)$, and thus by a covering argument, also has density type $(C_d R2^u, 2^k/R)$ for $C_d = 1000^d$. Furthermore, there are disjoint balls $B_{k,u,1}, \dots, B_{k,u,N_{k,u}}$ of radius at most $2^k/100R$ such that

$$\sum_n \mathrm{rad}(B_{k,u,n}) \leq 2^{-u}/R\#\mathcal{E}_k. \tag{4.121}$$

and such that $\mathcal{E}_k(u)$ is covered by the balls $\{B^*_{k,u,n}\}$, where, for a ball B, B^* denotes the ball with the same center as B, but 5 times the radius. Now write

$$F_{k,u} = \sum_{a,b} \sum_{(x_0,t_0) \in \mathcal{E}_{k,a,b}(u)} 2^{k\left(\frac{d-1}{2}\right)} 2^{-(a+b)} f_{x_0,t_0}. \tag{4.122}$$

Using the density assumption on $\mathcal{E}_k(u)$, we can apply Lemma **??** of the last section, which implies that, with $S_{x_0,t_0} = 2^{-(a+b)} f_{x_0,t_0}$ for $(x_0, t_0) \in \mathcal{E}_{k,a,b}(u)$,

$$\Big\| \sum_k F_{k,u} \Big\|_{L^2(M)} \lesssim R^{d/2} \left(u^{1/2} 2^{u\left(\frac{1}{d-1}\right)} \right) \left(\sum_k 2^{k(d-1)} \#\mathcal{E}_k \right)^{1/2}. \tag{4.123}$$

Let $(y_{k,u,n}, t_{k,u,n})$ denote the center of $B_{k,u,n}$. Then

$$\sum_{(x_0,t_0) \in [B_{k,u,n} \cap \mathcal{E}_k(u)]} 2^{-(a+b)} f_{x_0,t_0} \tag{4.124}$$

has mass concentrated on the geodesic annulus $\mathrm{Ann}_{k,u,n} \subset M$ with center $y_{k,u,n}$, with radius $t_{k,u,n} \sim 2^k/R$, and with thickness 5 $\mathrm{rad}(B_{k,u,n})$. Thus

$$\sum_n |\mathrm{Ann}_{k,u,n}| \lesssim \sum_n (2^k/R)^{d-1} \mathrm{rad}(B_{k,u,n}) \leq (2^k/R)^{d-1} R^{-1} 2^{-u} \#\mathcal{E}_k. \tag{4.125}$$

If we set $\Lambda_u = \bigcup_k \bigcup_n \mathrm{Ann}_{k,u,n}$, then

$$|\Lambda_u| \lesssim R^{-d} 2^{-u} \sum_k 2^{k(d-1)} \#\mathcal{E}_k \tag{4.126}$$

Since $1/p - 1/2 > 1/(d-1)$, so that $\alpha(p) > 1$, Hölder's inequality implies that

$$\begin{aligned} \Big\| \sum_k F_{k,u} \Big\|_{L^p(\Lambda_u)} &\lesssim |\Lambda_u|^{1/p-1/2} \Big\| \sum_k F_{k,u} \Big\|_{L^2(\Lambda_{k,u})} \\ &\lesssim \left(R^{-d} 2^{-u} \sum_k 2^{k(d-1)} \#\mathcal{E}_k \right)^{1/p-1/2} \\ &\qquad \left(R^d \left(u 2^{u\left(\frac{2}{d-1}\right)} \right) \sum_k 2^{k(d-1)} \#\mathcal{E}_k \right)^{1/2} \\ &= R^{d(1-1/p)} \left(u^{1/2} 2^{-u\left(\frac{\alpha(p)-1}{d-1}\right)} \right) \left(\sum_k 2^{k(d-1)} \#\mathcal{E}_k \right)^{1/p} \\ &\lesssim R^{d(1-1/p)} 2^{-u\varepsilon} \left(\sum_k 2^{k(d-1)} \#\mathcal{E}_k \right)^{1/p} \end{aligned} \tag{4.127}$$

for some suitable small $\varepsilon > 0$. For each $(x_0, t_0) \in \mathcal{E}_{k,a,b}(u) \cap B_{k,u,n}$, we calculate using the pointwise bounds for the functions $\{f_{x_0,t_0}\}$ that

$$\begin{aligned} \|2^{-(a+b)} f_{x_0,t_0}\|_{L^1(\Lambda(u)^c)} &= R^d \int_{\mathrm{Ann}_R^c} \langle R d_M(x,x_0)\rangle^{-\left(\frac{d-1}{2}\right)} \langle R|t_0 - d_M(x,x_0)|\rangle^{-M}\, dx \\ &\lesssim R^{\left(\frac{d+1}{2}-M\right)} \int_{5\,\mathrm{rad}(B_{k,u,n})}^{O(1)} (t_{k,u,n}+s)^{\left(\frac{d-1}{2}\right)} s^{-M}\, ds \\ &\lesssim R^{\left(\frac{d+1}{2}-M\right)} \mathrm{rad}(B_{k,u,n})^{1-M} t_{k,u,n}^{\left(\frac{d-1}{2}\right)} \\ &\lesssim 2^{k\left(\frac{d-1}{2}\right)} (R\mathrm{rad}(B_{k,u,n}))^{1-M}. \end{aligned} \tag{4.128}$$

Thus

$$\|2^{k\left(\frac{d-1}{2}\right)} 2^{-(a+b)} f_{x_0,t_0}\|_{L^1(\mathrm{Ann}_n^c)} \lesssim 2^{k(d-1)} (R\mathrm{rad}(B_{k,u,n}))^{1-M} \tag{4.129}$$

Because the set of points in $\mathcal{E}_k$ is $1/R$ separated, there are at most $O((R\mathrm{rad}(B_{k,u,n}))^{d+1})$ points in $\mathcal{E}_k(u) \cap B_{k,u,n}$, and so the triangle inequality implies that

$$\begin{aligned} &\Big\| \sum\nolimits_{a,b} \sum\nolimits_{(x_0,t_0)\in\mathcal{E}_{k,a,b}(u)\cap B_{k,u,n}} 2^{k\left(\frac{d-1}{2}\right)} 2^{-(a+b)} f_{x_0,t_0} \Big\|_{L^1(\Lambda(u)^c)} \\ &\quad \lesssim 2^{k(d-1)} (R\mathrm{rad}(B_{k,u,n}))^{d+2-M}. \end{aligned} \tag{4.130}$$

Since $\#\mathcal{E}_k \cap B_{k,u,n} \geq R2^u\, \mathrm{rad}(B_{k,u,n})$, and $\mathcal{E}_k$ is $1/R$ discretized, we must have

$$\mathrm{rad}(B_{k,u,n}) \geq (2^u/2^d)^{\frac{1}{d-1}}\ 1/R. \tag{4.131}$$

Thus

$$\begin{aligned} \Big\| \sum\nolimits_k F_{k,u} \Big\|_{L^1(\Lambda(u)^c)} &\lesssim_M \sum\nolimits_k \sum\nolimits_n 2^{k(d-1)} (R\mathrm{rad}(B_{k,u,n}))^{d+2-M} \\ &\lesssim \sum\nolimits_k 2^{k(d-1)} \Big(R \min\nolimits_n \mathrm{rad}(B_{k,u,n})\Big)^{d+1-M} \\ &\qquad \Big(\sum\nolimits_n R\mathrm{rad}(B_{k,u,n})\Big) \\ &\lesssim \sum\nolimits_k 2^{k(d-1)} 2^{u\left(\frac{d+1-M}{d-1}\right)} (2^{-u}\#\mathcal{E}_k) \\ &\lesssim 2^{u\left(\frac{2-M}{d-1}\right)} \sum\nolimits_k 2^{k(d-1)} \#\mathcal{E}_k \end{aligned} \tag{4.132}$$

Picking $M > 2 + (1-1/p)(1/p-1/2)^{-1}$, and interpolating with the bounds on $\| \sum_k F_{k,u}\|_{L^2(M)}$ yields that

$$\begin{aligned} \Big\| \sum\nolimits_k F_{k,u} \Big\|_{L^p(\Lambda(u)^c)} &\lesssim \left(2^{u\left(\frac{2-M}{d-1}\right)}\right)^{2/p-1} \left(R^{d/2}\left(u^{1/2} 2^{u\left(\frac{1}{d-1}\right)}\right)\right)^{2(1-1/p)} \\ &\qquad \left(\sum 2^{k(d-1)} \#\mathcal{E}_k\right)^{1/p} \\ &\lesssim 2^{-u\varepsilon} R^{d(1-1/p)} \sum\nolimits_k \left(\sum 2^{k(d-1)} \#\mathcal{E}_k\right)^{1/p}. \end{aligned} \tag{4.133}$$

So now we know

$$\Big\| \sum\nolimits_k F_{k,u} \Big\|_{L^p(M)} \lesssim R^{d(1-1/p)} 2^{-u\varepsilon} \left(\sum\nolimits_k 2^{k(d-1)} \#\mathcal{E}_k\right)^{1/p}. \tag{4.134}$$

The exponential decay in u allows us to sum in u to obtain that

$$\Big\| \sum_u \sum_k F_{k,u} \Big\|_{L^p(M)} \lesssim R^{d(1-1/p)} \Big(\sum_k 2^{k(d-1)} \# \mathcal{E}_k \Big)^{1/p}. \tag{4.135}$$

This is precisely the bound we were required to prove. □

4.3 Analysis of Regime II via Local Smoothing

In this section, we bound the operators $\{T^{II}\}$, by a reduction to an endpoint local smoothing inequality, namely, the inequality that

$$\|e^{2\pi i t P} f\|_{L^{p'}(M) L_t^{p'}(I_0)} \lesssim \|f\|_{L^{p'}_{\alpha(p)-1/p'}}. \tag{4.136}$$

This inequality is proved in Corollary 1.2 of [11] for $1 < p < 2(d-1)/(d+1)$ for classical elliptic pseudodifferential operators P satisfying the cosphere assumption of Theorem **??**. The range of p here is also precisely the range of p in Theorem **??**. Alternatively, Lemma 4.10 can be used to prove (4.136) independently of [11] in the same range by a generalization of the method of Section 10 of [6].

Lemma 4.11. *Using the notation of Proposition 4.4, let*

$$T^{II} = \int b^{II}(t) (Q_R \circ e^{2\pi i t P} \circ Q_R)\, dt. \tag{4.137}$$

For $1 < p < 2(d-1)/(d+1)$*, we then have*

$$\|T^{II} u\|_{L^p(M)} \lesssim R^{\alpha(p)-1/p'} \|b^{II}\|_{L^p(I_0)} \|u\|_{L^p(M)}. \tag{4.138}$$

Proof. For each R, the *class* of operators of the form $\{T^{II}\}$ formed from a given function b^{II} is closed under taking adjoints. Indeed, if T^{II} is obtained from b^{II}, then $(T^{II})^*$ is obtained from the multiplier $\overline{b^{II}}$. Because of this self-adjointness, if we can prove that

$$\|T^{II} u\|_{L^{p'}(M)} \lesssim R^{\alpha(p)-1/p'} \|b^{II}\|_{L^p(I_0)} \|u\|_{L^{p'}(M)}, \tag{4.139}$$

then we obtain the required result by duality. We apply this duality because it is easier to exploit local smoothing inequalities in $L^{p'}(M)$ since now $p' > 2$.

We begin by noting that the operators $\{Q_R\}$, being a bounded family of order zero pseudo-differential operators, are uniformly bounded on $L^{p'}(M)$. Thus

$$\begin{aligned} \|T^{II} u\|_{L^{p'}(M)} &= \Big\| Q_R \circ \Big(\int_{I_0} b^{II}(t) e^{2\pi i t P} (Q_R u) \Big) \Big\|_{L^{p'}(M)} \\ &\lesssim \Big\| \Big(\int_{I_0} b^{II}(t) e^{2\pi i t P} (Q_R u) \Big) \Big\|_{L^{p'}(M)}. \end{aligned} \tag{4.140}$$

Applying Hölder and Minkowski's inequalities, we find that

$$\|T^{II}u\|_{L^{p'}(M)} \leq \|b^{II}\|_{L^p(\mathbb{R})} \Big\| \Big(\int_{I_0} |e^{2\pi itP}(Q_R u)|^{p'} \Big)^{1/p'} \Big\|_{L^{p'}(M)}. \tag{4.141}$$

Applying the endpoint local smoothing inequality (4.136), we conclude that

$$\begin{aligned} \|T^{II}u\|_{L^{p'}(M)} &\lesssim \|b^{II}\|_{L^p(\mathbb{R})} \|e^{2\pi iP}(Q_R u)\|_{L_t^{p'} L_x^{p'}} \\ &\lesssim \|b^{II}\|_{L^p(\mathbb{R})} \|Q_R u\|_{L^q_{\alpha(p)-1/p'}(M)}, \end{aligned} \tag{4.142}$$

Bernstein's inequality for compact manifolds (see [17], Section 3.3) gives

$$\|Q_R u\|_{L^q_{\alpha(p)-1/p'}(M)} \lesssim R^{\alpha(p)-1/p'} \|u\|_{L^p(M)}. \tag{4.143}$$

Thus we conclude that

$$\|T^{II}u\|_{L^{p'}(M)} \lesssim R^{\alpha(p)-1/p'} \|b^{II}\|_{L^p(I_0)} \|u\|_{L^{p'}(M)}, \tag{4.144}$$

which completes the proof. □

Combining Lemma 4.9 and Lemma 4.11 completes the proof of Proposition 4.4, and thus of inequality (4.4). Since (4.3) was already proven as a consequence of Lemma 4.2, this completes the proof of Theorem **??**, and thus the main results of the paper.

4.4 Appendix

In this appendix, we provide proofs of Lemmas 4.3 and 4.6.

Proof of Lemma 4.3. The intervals $\{I_{t_0} : t_0 \in \mathcal{T}_R\}$ cover $[-\varepsilon, \varepsilon]$, and so we may consider an associated partition $\mathbb{I}_{[-\varepsilon,\varepsilon]} = \sum_{t_0} \chi_{t_0}$ where $\text{supp}(\chi_{t_0}) \subset I_{t_0}$ and $|\chi_{t_0}| \leq 1$. Define $b^I_{t_0} = \chi_{t_0} b_j$ and $b^{II} = (1 - \mathbb{I}_{[-\varepsilon,\varepsilon]})b$. Then $b = \sum_{t_0} b^I_{t_0} + b^{II}$, and the support assumptions are satisfied. It remains to prove the required norm bounds for these choices. For each $n \in \mathbb{Z}$, define a function $b_n : I_0 \to \mathbb{C}$ by setting $b_n(t) = R\widehat{m}(R(t + n))$. Then $b = \sum_n b_n$. Moreover,

$$\begin{aligned} &\Bigg(\sum_{n \neq 0} \Big[\langle Rn \rangle^{\alpha(p)} \|b_n\|_{L^p(I_0)} \Big]^p \Bigg)^{1/p} \\ &\qquad \sim \Bigg(\int_{|t| \geq 1/2} \Big[\langle Rt \rangle^{\alpha(p)} |R\widehat{m}(Rt)| \Big]^p \Bigg)^{1/p} \\ &\qquad = R^{1/p'} \Bigg(\int_{|t| \geq R/2} \Big[|t|^{\alpha(p)} \widehat{m}(t) \Big]^p \Bigg)^{1/p} \leq R^{1/p'} C_p(m). \end{aligned} \tag{4.145}$$

Write $b^I_{t_0} = \sum_n b^I_{t_0,n}$ and $b^{II} = \sum_n b^{II}_n$, where $b^I_{t_0,n} = \chi_{t_0} b_n$ and $b^{II}_n = \mathbb{I}_{I_0 \setminus [-\varepsilon,\varepsilon]} b_n$. Then

$$\begin{aligned}
\|b_0^{II}\|_{L^p(I_0)} &= \left(\int_{\varepsilon \le |t| \le 1/2} |R\widehat{m}(Rt)|^p \right)^{1/p} \\
&= R^{1/p'} \left(\int_{R\varepsilon \le |t| \le R/2} |\widehat{m}(t)|^p \right)^{1/p} \lesssim R^{1/p' - \alpha(p)} C_p(m).
\end{aligned} \tag{4.146}$$

Using (4.145), (4.146), and Hölder's inequality, we conclude that

$$\begin{aligned}
\|b^{II}\|_{L^p(I_0)} &\le \sum\nolimits_n \|b_n^{II}\|_{L^p(I_0)} \\
&\le \|b_0^{II}\|_{L^p(I_0)} + \sum\nolimits_{n \neq 0} \left[|Rn|^{\alpha(p)} \|b_n^{II}\|_{L^p(I_0)} \right] \frac{1}{|Rn|^{\alpha(p)}} \\
&\le \|b_0^{II}\|_{L^p(I_0)} + R^{-\alpha(p)} \left(\sum\nolimits_{n \neq 0} \left[|Rn|^{\alpha(p)} \|b_n\|_{L^p(I_0)} \right]^p \right)^{1/p} \\
&\lesssim R^{1/p' - \alpha(p)} C_p(m).
\end{aligned} \tag{4.147}$$

A similar calculation shows that

$$\begin{aligned}
\|b^I_{t_0}\|_{L^p(I_0)} &\le \sum\nolimits_n \|b^I_{t_0,n}\|_{L^p(I_0)} \\
&= \|b^I_{t_0,0}\|_{L^p(I_0)} + \sum\nolimits_{n \neq 0} \|b^I_{t_0,n}\|_{L^p(I_0)} \\
&\lesssim \|b^I_{t_0,0}\|_{L^p(I_0)} + R^{-\alpha(p)} \Big(\sum\nolimits_{n \neq 0} |Rn|^{\alpha(p)} \|b^I_{t_0,n}\|^p_{L^p(I_0)} \Big)^{1/p} \\
&\lesssim \|b^I_{t_0,0}\|_{L^p(I_0)} + \Big(\sum\nolimits_{n \neq 0} |Rn|^{\alpha(p)} \|b^I_{t_0,n}\|^p_{L^p(I_0)} \Big)^{1/p}.
\end{aligned} \tag{4.148}$$

Using (4.148), we calculate that

$$\begin{aligned}
&\left(\sum_{t_0 \in \mathcal{T}_R} \left[\|b^I_{t_0}\|_{L^p(I_0)} \langle Rt_0 \rangle^{\alpha(p)} \right]^p \right)^{1/p} \\
&\quad \lesssim \left(\sum_{t_0 \in \mathcal{T}_R} \left[\|b^I_{t_0,0}\|_{L^p(I_0)} \langle Rt_0 \rangle^{\alpha(p)} \right]^p + \sum_{n \neq 0} \left[|Rn|^{\alpha(p)} \|b^I_{t_0,n}\|_{L^p(I_0)} \right]^p \right)^{1/p} \\
&\quad \lesssim \left(\int_{\mathbb{R}} \left[\langle Rt \rangle^{\alpha(p)} R\widehat{m}(Rt) \right]^p dt \right)^{1/p} \\
&\quad \lesssim R^{1/p'} C_p(m).
\end{aligned} \tag{4.149}$$

Since each function $b^I_{t_0}$ is supported on a length $1/R$ interval, we have

$$\|b^I_{t_0}\|_{L^1(I_0)} \lesssim R^{-1/p'} \|b^I_{t_0}\|_{L^p(I_0)}, \tag{4.150}$$

and substituting this inequality into (4.149) completes the proof. $\square$

Proof of Lemma 4.6. For each α, given our choice of ε_M, the Lax-Hörmander Parametrix construction (see Theorem 4.1.2 of [17]) guarantees that we can find operators $\tilde{W}_\alpha(t)$ and $\tilde{R}_\alpha(t)$ for $|t| \leq \varepsilon_M$, such that for $u \in L^1(M)$ with $\operatorname{supp}(u) \subset V_\alpha^*$,

$$e^{2\pi itP}u = \tilde{W}_\alpha(t)u + \tilde{R}_\alpha(t)u, \tag{4.151}$$

where $\tilde{R}_\alpha(t)$ has a smooth kernel, and the kernel of $\tilde{W}_\alpha(t)$ is given in coordinates by

$$\tilde{W}_\alpha(t)(x,y) = \int s_0(t,x,y,\xi)e^{2\pi i[\phi(x,y,\xi)+tp(y,\xi)]}\,d\xi, \tag{4.152}$$

for an order zero symbol s_0 with

$$\operatorname{supp}_{x,y,\xi}(s_0) \subset \{(x,y,\xi) \in U_\alpha \times V_\alpha^* \times \mathbb{R}^d : d_M(x,y) \leq 1.01\varepsilon_M \text{ and } |\xi| \leq 1\}, \tag{4.153}$$

and an order one symbol ϕ, homogeneous in ξ of order one, solving the Eikonal equation and vanishing for $x \in \Sigma_\alpha(y,\xi)$ as required by the lemma. The only difference here compared to Theorem 4.1.2 of [17] is that in that construction the function ϕ there is chosen to vanish for $x \in \tilde{\Sigma}_\alpha(y,\xi)$, where $\tilde{\Sigma}_\alpha(y,\xi) = \{x : \xi \cdot (x-y) = 0\}$. The only property of this choice that is used in the proof is that the perpendicular vector to $\tilde{\Sigma}_\alpha(y,\xi)$ at y is ξ, and this is also true of the hypersurfaces $\Sigma_\alpha(y,\xi)$ that we have specified, so that there is no problem making this modification.

In the remainder of the proof it will be convenient to fix a orthonormal basis $\{e_k\}$ of eigenfunctions for P, such that $\Delta e_k = \lambda_k e_k$ for a non-decreasing sequence $\{\lambda_k\}$. We fix $u \in L^1(M)$ with $\operatorname{supp}(u) \subset V_\alpha^*$ and $\|u\|_{L^1(M)} \leq 1$.

We begin by mollifying the functions Q. We proceed here with a similar approach to Theorem 4.3.1 of [17]. We fix $\rho \in C_c^\infty(\mathbb{R})$ equal to one in a neighborhood of the origin and with $\rho(t) = 0$ for $|t| \geq \varepsilon_M/2$. We write

$$\begin{aligned} Q &= \int R\widehat{q}(Rt)e^{2\pi itP}\,dt \\ &= \int R\widehat{q}(Rt)\Big\{\rho(t)\tilde{W}(t) + \rho(t)\tilde{R}(t) + (1-\rho(t))e^{2\pi itP}\Big\}\,dt \\ &= Q_I + Q_{II} + Q_{III}. \end{aligned} \tag{4.154}$$

The rapid decay of $\widehat{q}$ implies that the function $\psi(t) = R\widehat{q}(Rt)(1-\rho(t))$ satisfies $\|\partial_t^N\psi\|_{L^1(\mathbb{R})} \lesssim_M R^{-M}$, and so

$$|\widehat{\psi}(\lambda)| \lesssim_{N,M} R^{-M}\lambda^{-N}. \tag{4.155}$$

But since $Q_{III} = \widehat{\psi}(-P)$, we can write the kernel of Q_{III} as

$$Q_{III}(x,y) = \sum\nolimits_\lambda \widehat{\psi}(-\lambda_k)e_k(x)\overline{e_k(y)}, \tag{4.156}$$

Sobolev embedding and (4.155) imply that $|Q_{III}(x,y)| \lesssim_N R^{-N}$ and thus

$$\|Q_{III}u\|_{L^\infty(M)} \lesssim_N R^{-N}. \tag{4.157}$$

Integration by parts, using the fact that q vanishes near the origin, yields that

$$\left| \int R\widehat{q}(Rt)\rho(t)\tilde{R}(t,x,y) \right| \lesssim_N R^{-N}, \tag{4.158}$$

and thus

$$\|Q_{II}u\|_{L^\infty(M)} \lesssim_N R^{-N}. \tag{4.159}$$

Now we expand

$$Q_I = \iint R\widehat{q}(Rt)\rho(t)s_0(t,x,y,\xi)e^{2\pi i[\phi(x,y,\xi)+tp(y,\xi)]}\, d\xi\, dt. \tag{4.160}$$

We perform a Fourier series expansion, writing

$$c_n(x,y,\xi) = \int \rho(t)s_0(t,x,y,\xi)e^{-2\pi int}\, dt. \tag{4.161}$$

Then the symbol estimates for s_0, and the compact support of ρ imply that

$$|\partial^\alpha_{x,y}\partial^\beta_\xi c_n(x,y,\xi)| \lesssim_{\alpha,\beta,N} |n|^{-N}\langle\xi\rangle^{-\beta}. \tag{4.162}$$

Using Fourier inversion we can write

$$\begin{aligned} Q_I(x,y) &= \iint \sum\nolimits_n R\widehat{q}(Rt)c_n(x,y,\xi)e^{2\pi i[\phi(x,y,\xi)+t[n+p(y,\xi)]]}\, d\xi\, dt \\ &= \int \sum\nolimits_n q\big((n+p(y,\xi))/R\big)c_n(x,y,\xi)e^{2\pi i\phi(x,y,\xi)}\, d\xi \\ &= \int \tilde{\sigma}_\alpha(x,y,\xi)e^{2\pi i\phi(x,y,\xi)}\, d\xi, \end{aligned} \tag{4.163}$$

where

$$\tilde{\sigma}_\alpha(x,y,\xi) = \sum_{n\in\mathbb{Z}} q\left(\frac{n+p(y,\xi)}{R}\right)c_n(x,y,\xi). \tag{4.164}$$

The nth term of this sum is supported on $R/4 - n \leq p(y,\xi) \leq 4R - n$, so in particular, if $n > 4R$ then the term vanishes. For $n \leq 4R$, we have estimates of the form

$$\left|\partial^\alpha_{x,y}\partial^\beta_\xi \left\{ q\left(\frac{n+p(y,\xi)}{R}\right)c_n(x,y,\xi)\right\}\right| \lesssim_{\alpha,\beta,N} |n|^{-N}. \tag{4.165}$$

and for $-4R \leq n \leq R/8$,

$$\left|\partial^\alpha_{x,y}\partial^\beta_\xi \left\{ q\left(\frac{n+p(y,\xi)}{R}\right)c_n(x,y,\xi)\right\}\right| \lesssim_{\alpha,\beta,N} |n|^{-N}R^{-\beta}. \tag{4.166}$$

But this means that if we define

$$\sigma_\alpha(x,y,\xi) = \sum\nolimits_{-4R\leq n\leq R/8} q\left(\frac{n+p(y,\xi)}{R}\right)c_n(x,y,\xi). \tag{4.167}$$

and define

$$Q_\alpha(x,y) = \int \sigma_\alpha(x,y,\xi) e^{2\pi i \phi(x,y,\xi)}\, d\xi \tag{4.168}$$

then

$$\left|\partial_{x,y}^\alpha \partial_\xi^\beta \{\tilde{\sigma}_\alpha - \sigma\}(x,y,\xi)\right| \lesssim_{\alpha,\beta,N,M} R^{-N} \langle \xi \rangle^{-M}, \tag{4.169}$$

and so

$$\|(Q_I - Q_\alpha)u\|_{L^\infty(M)} \lesssim_N R^{-N}. \tag{4.170}$$

Combining (4.157), (4.159), and (4.170), we conclude that

$$\|(Q - Q_\alpha)u\|_{L^\infty(M)} \lesssim_N R^{-N}. \tag{4.171}$$

Since σ_α is supported on $|\xi| \sim R$, we have verified the required properties of Q_α.

Using the bounds on $Q - Q_\alpha$ obtained above, we see that

$$\left\|[(Q \circ e^{2\pi i t P} \circ Q) - (Q_\alpha \circ e^{2\pi i t P} \circ Q_\alpha)]\{u\}\right\|_{L^\infty(M)} \lesssim_N R^{-N}. \tag{4.172}$$

Since $\tilde{R}_\alpha(t)$ has a smooth kernel, we also see that

$$\begin{aligned} &\left\|(Q_\alpha \circ (e^{2\pi i t P} - \tilde{W}_\alpha(t)) \circ Q_\alpha)\{u\}\right\|_{L^\infty(M)} \\ &\quad = \left\|(Q_\alpha \circ \tilde{R}_\alpha(t) \circ Q_\alpha)\{u\}\right\|_{L^\infty(M)} \lesssim_N R^{-N}. \end{aligned} \tag{4.173}$$

We now write

$$\begin{aligned} &(Q_\alpha \circ \tilde{W}_\alpha(T))(x,y) \\ &\quad = \int \sigma_\alpha(x,\xi) s_0(t,z,y,\eta) e^{2\pi i [\phi(x,z,\xi) + \phi(z,y,\eta) + t p(y,\eta)]}\, d\xi\, d\eta\, dz. \end{aligned} \tag{4.174}$$

The phase of this equation has gradient in the z variable with magnitude $\gtrsim R$ for $|\eta| \ll R$, and $\gtrsim R|\xi|$ if $|\eta| \gg R$. Thus, if we define $s(t,x,y,\xi) = s_0(t,x,y,\xi)\chi(\xi/R)$ where $\text{supp}(\chi) \subset [1/8, 8]$, and then define

$$W_\alpha(t)(x,y) = \int s(t,x,y,\xi) e^{2\pi i [\phi(x,y,\xi) + t p(y,\xi)]}\, d\xi, \tag{4.175}$$

then we may integrate by parts in the z variable to conclude that

$$\left|(Q_R \circ (\tilde{W}_\alpha(t) - W_\alpha(t)))(x,y)\right| \lesssim_N R^{-N}, \tag{4.176}$$

and thus

$$\|(Q_R \circ (\tilde{W}_\alpha(t) - W_\alpha(t)) \circ Q_R)u\|_{L^\infty(M)} \lesssim R^{-N}. \tag{4.177}$$

This proves the required estimates for the operators W_α. $\square$

Chapter 5

Combining Scales Via Atomic Decompositions

Chapter 6

Future Work

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Appendix A

Appendix Title

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