On Molecular Gases and the Natural Numbers

An Introduction to Ergodic Theory

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July 16, 2016

Qualitative Understanding of Differential Equations

- Classic Understanding: Solve equation algebraically

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- The vector field gives us 'evolution operators' f_t on phase space, transforming phase space.

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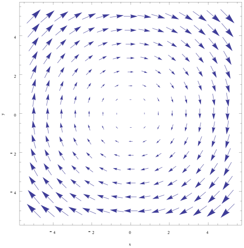

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- Let's only look at maps which 'mix well'.

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- Corresponding vector field v has zero divergence!

Who Cares About Divergence Zero?

Theorem

If a field v has divergence zero, the operators f_t preserve volume.

Proof.

Notice the inequality

$$\det(I + Mt) = 1 + \mathrm{tr}(M)t + O(t^2)$$

Apply change of variables formula on volume integral, and differentiate (notice the trace of Df_t is the divergence of v). □

Area Preserving Maps

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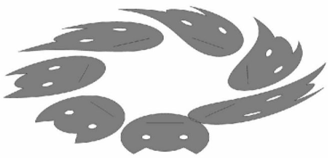

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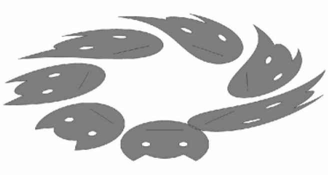

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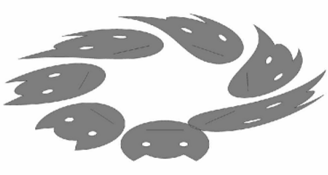
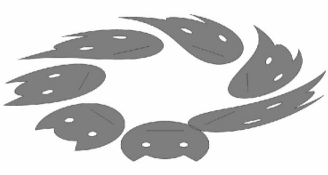
- Area preserving maps prevent space from being ‘squished’ or ‘minimized’. Cool theorems happen because of this...



Recurrence

- Consider a map $T : X \to X$ which can be seen as moving 1 second into the future ($T = f_1$).



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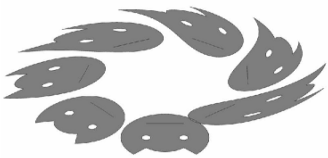

Recurrence

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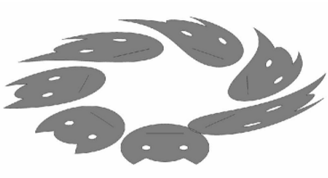
- Will $x \in X$ ever return to itself? $T^n x = x$?
- What about returning 'close' to itself, a neighbourhood of it's original position.



Poincare Recurrence Theorem

Theorem

If $T : X \to X$ is an invertible volume-preserving map, where X has bounded problem, and $U \subset X$ is a fixed set of positive volume, then almost every point of U returns to U.

Rotations of S^1

- Consider a dynamical system of S^1 which rotates the system irrationally (i.e. a map defined by $T(z) = wz$, where w is not a root of unity).

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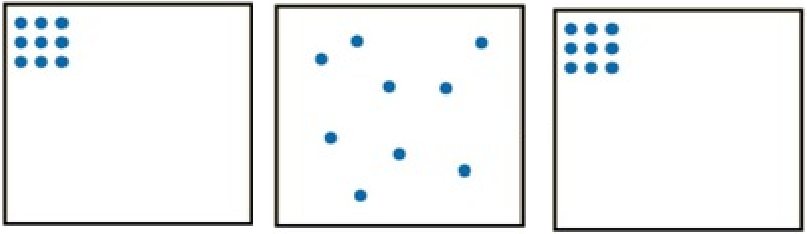
- Stronger Ergodic Theory: The orbit is uniform in S^1.

Order from Chaos



- Paradox: Gas Molecules organize themselves over time!

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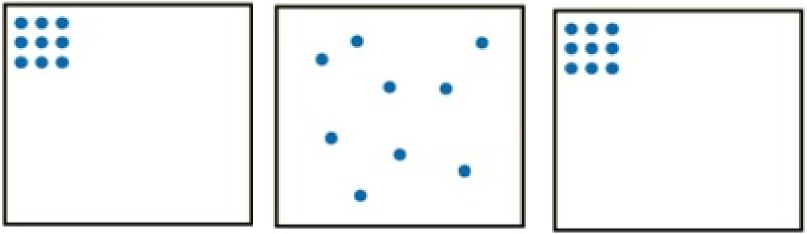

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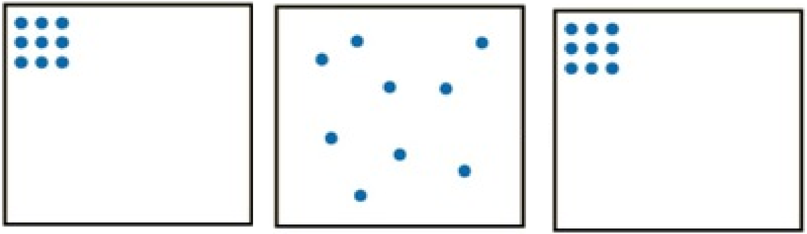

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- If willing to get an error of ε, and box has width m, then time to reccur is bounded by $(m/\varepsilon)^{3n}$, where n is the particle number. For $n = 6.02 \cdot 10^{23}$, this is a veeeerrrry long time.

Cat Map

- Consider the invertible map $T(x, y) = (2x + y, x + y)$, area preserving because it is a linear map of determinant one. Restrict the map to $[0, 1]^2$, where addition is considered modulo 1. This map still preserves area, but scrunches up space.

Powers of Two

- Consider the powers of 2: 1,2,4,8,16,32,64,128,256,512,1024,..., and just take off the first digit.

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- No regular pattern, but what about average distribution - is it uniform?

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- $\log_{10}(2^n) = n \log_{10}(2)$, so we need only look at shifts along the line by an irrational number to learn about the distribution of 2^n.
- Poincare Recurrence tells us that the uniform shifts are dense in $[0,1]$. Stronger versions of Poincare recurrence tells us it is evenly distributed, so that the probability of the first digit of a number being k is $\log(k+1) - \log(k)$.

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- We haven't even gotten to the interesting applications in number theory!