Comparison of the Weyl integration formula and the equivariant localization formula for a maximal torus of $\mathrm{Sp}(1)$

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1 Introduction

In this note, we aim to prove the Weyl integration formula for the Lie group $\mathrm{Sp}(1)$ using the Berline–Vergne/Atiyah–Bott equivariant localization formula. We do not quite succeed. The ideas are due to Loring Tu, and the execution to me. The first four sections are generalities, mostly copied from an earlier document, which should probably be skimmed, and the attempted application to $\mathrm{Sp}(1)$ follows.

2 The Weyl integral formula

Let G be a compact, connected Lie group, T its maximal torus, and W the Weyl group. Write $\mathrm{Ad}\,|_{\mathfrak{g}/\mathfrak{t}}\colon T \to \mathrm{Aut}(\mathfrak{g}/\mathfrak{t})$ for the adjoint action of T on its Lie algebra modulo the Cartan subalgebra, and $\Delta\colon T \to \mathbb{R}$ for the function

$$\Delta(t) := \det\left(\left[\mathrm{id} - \mathrm{Ad}(t^{-1})\right]\Big|_{\mathfrak{g}/\mathfrak{t}}\right).$$

If $f\colon G \to \mathbb{R}$ is a continuous function on G, and dg and dt are Haar measures on G and T respectively, and $d(gT)$ the induced probability measure on G/T, then

$$\int_G f(g)\,dg = \frac{1}{|W|}\int_T \Delta(t)\left[\int_{G/T} f(gtg^{-1})\,d(gT)\right]dt.$$

If f is a *class function*, meaning it takes one value on each conjugacy class (so that for all $g, h \in G$ we have $f(ghg^{-1}) = f(h)$), the bracketed integral reduces to $f(t)$, so that

$$\int_G f(g)\,dg = \frac{1}{|W|}\int_T \Delta(t) f(t)\,dt.$$

Note further that if Φ is the set of global roots of G, and Φ^+ the set of positive roots, then

$$\Delta = \prod_{\alpha\in\Phi}\left(1-\alpha^{-1}\right) = \prod_{\alpha\in\Phi^+}\left(1-\alpha^{-1}\right)\left(1-\alpha\right).$$

3 The equivariant localization theorem

Let T be a torus and M a T-manifold. If $X \in \mathfrak{t}$ is a vector in the Lie algebra of T and $\widetilde{X} \in \Gamma(TM)$ its associated fundamental vector field, write i_N for the inclusions into M connected components N of the zero set of $\widetilde{X}$, write ν for the normal bundle to N, and $e^T(\nu)$ for the T-equivariant Euler class of each ν. Let the orientation on N be induced by that on ν. If $\widetilde{\alpha} \in \Omega^*_T(M)$ is an equivariantly closed form, then

$$\int_M \widetilde{\alpha}(X) = \sum_N \int_N \frac{i_N^*\big(\widetilde{\alpha}(X)\big)}{e^T(\nu)(X)}.$$

It would be appropriate in this connection to state what equivariant extensions *are*. An *equivariant extension* of an m-form $\omega \in \Omega^*(M)$ is an element $\widetilde{\omega}$ of the Cartan model $\big(S(\mathfrak{t}^*) \otimes \Omega(M)\big)^T$ (basically a T-invariant polynomial in a basis $u_1, \ldots, u_n$ for $\mathfrak{t}^*$ with coefficients in $\Omega^*(M)$), of the same degree as ω, and such that the "constant term" is ω. Here the degree of elements of $\mathfrak{t}^*$ is 2, so we can write $\widetilde{\omega} = \omega + \sum_I \omega_I u^I$, where $I = (i_1, \ldots, i_n) \in \mathbb{N}^n$ is a multiindex, $u^I = u_1^{i_1} \cdots u_n^{i_n}$ has degree $2|I| = 2\sum i_k$, and $\omega_I \in \Omega^{m-2|I|}(M)$.

The integral of such a form is $\int_M \widetilde{\omega} = \sum_I u^I \int_M \omega_I$; since these summands will vanish unless $\deg \omega_I = m - 2|I| = \dim M$, this integration will return an element of degree $2|I|$ in $S(\mathfrak{t}^*)$, or in other words a homogeneous degree-$2|I|$ polynomial map $\mathfrak{t} \to \mathbb{R}$.

In particular, if $\deg \omega = \dim M$, then $\int_M \widetilde{\omega}(X) = \int_M \omega$ is a real number. It should be independent of X, subject to some genericity condition. So the equivariant localization formula, applied to equivariant extensions of top forms, gives another way to integrate functions on a manifold, assuming those functions have equivariantly closed equivariant extensions. A T-manifold M is called *equivariantly formal* if all closed forms on M have equivariantly closed extensions, and *it is known that T is equivariantly formal under this conjugation action.*

Like that $\exp(X)$ topologically generates a maximal torus of G

In the situation we are interested in, the manifold M is a connected, compact Lie group G and the group action we are concerned with is that of a maximal torus T. The fixed points under conjugation by a subgroup are the group's centralizer—in this case, the torus T itself.

Let $f\colon G \to \mathbb{R}$ be a smooth function, invariant under conjugation by T. Then $\omega = f\,dg$ is a closed T-invariant m-form on G, so since G is equivariantly formal under conjugation by T, there exists an equivariant extension $\widetilde{\omega}$. Putting together the two formulae, we have

$$\frac{1}{|W|}\int_T \Delta(t)\left[\int_{G/T} f(gtg^{-1})\,d(gT)\right] dt = \int_G f(g)\,dg = \int_G \omega = \int_G \widetilde{\omega}(X) = \int_T \frac{\widetilde{\omega}(X)|_T}{e^T(\nu)(X)},$$

where $e^T(\nu)$ is the equivariant Euler class of the normal bundle to T in G; what we would like is some way to see that the left-hand side is the same as the right without the intermediate steps.

4 The Equivariant Euler Class $e^T(\nu)$

The normal bundle ν of T in G fits in the exact sequence

$$0 \to T(T) \to (TG)|_T \to \nu \to 0,$$

and so is a vector bundle of (even) rank $m - \operatorname{rk} T$ over T, with fiber $\mathfrak{g}/\mathfrak{t}$. We claim it is a trivial bundle, which will follow if $(TG)|_T$ is a trivial bundle in such a way that $T(T)$ is a trivial subbundle. But TG is parallelizable for any Lie group G: each element of T_gG is a left translation $(\ell_g)_*X$ for precisely one $X \in \mathfrak{g}$, and by continuity $(g, X) \mapsto (\ell_g)_*X$ is a vector bundle isomorphism $G \times \mathfrak{g} \to TG$.[1]

So $\nu \cong T \times (\mathfrak{g}/\mathfrak{t})$. Now we want to see how the conjugation action acts on $T \times (\mathfrak{g}/\mathfrak{t})$ under this identification. For this purpose, we first work in $G \times \mathfrak{g}$, and see what Ad does to TG. Start with $(g, X) \in G \times \mathfrak{g}$. Write $c_h \colon g \mapsto hgh^{-1}$ for conjugation by $h \in G$, ℓ_h for left multiplication and r_h for right multiplication, so $c_h = \ell_h r_{h^{-1}} = r_{h^{-1}} \ell_h$. Then

$$\begin{aligned}
(c_h)_*(\ell_g)_*X &= (r_{h^{-1}})_*(\ell_h)_*(\ell_g)_*X \\
&= (r_{h^{-1}})_* \left[(\ell_h)_*(\ell_g)_*(\ell_{h^{-1}})_*\right] (\ell_h)_*X \\
&= (r_{h^{-1}})_*(\ell_{c_h(g)})_*(\ell_h)_*X \\
&= (\ell_{c_h(g)})_*(r_{h^{-1}})_*(\ell_h)_*X \\
&= (\ell_{c_h(g)})_*(c_h)_*X,
\end{aligned}$$

so we have the diagram

$$\begin{tikzcd}
(\ell_g)_*X \arrow[r, mapsto] \arrow[d, leftrightarrow] & (\ell_{c_h(g)})_*(c_h)_*X \arrow[d, leftrightarrow] \\
(g, X) \arrow[r, mapsto, dotted] & \big(c_h(g), (c_h)_*X\big),
\end{tikzcd}$$

Since conjugation by T on T fixes each fiber, it induces the Ad-action on each fiber of the bundle $(TG)|_T$, and so on each $\mathfrak{g}/\mathfrak{t}$ fiber of $\nu \cong T \times (\mathfrak{g}/\mathfrak{t})$.

As the conjugation action preserves each fiber, we can factor T out of the homotopy quotient:

$$\begin{aligned}
\nu_T &= (ET \times \nu)/T \\
&= \big(ET \times T \times (\mathfrak{g}/\mathfrak{t})\big)/T \\
&= T \times \big(ET \times (\mathfrak{g}/\mathfrak{t})\big)/T \\
&= T \times (\mathfrak{g}/\mathfrak{t})_T,
\end{aligned}$$

[1] To see continuity of the inverse, given a basis X_i of $\mathfrak{g}$, the left-invariant vector fields $g \mapsto (\ell_g)_*X_i$ form a frame of TG, so that $\sum a_i(\ell_g)_*X_i \mapsto \big(g, (a_1, \ldots, a_{\dim G})\big)$. The projection $TG \to G$ is continuous by definition, and this coordinate map $\sum a_i(\ell_g)_*X_i \mapsto (a_1, \ldots, a_{\dim G}) \in \mathbb{R}^{\dim G}$ is locally continuous over the trivializing neighborhoods U for TG that we already know exist since the left-invariant vector fields are continuous.

so ν_T is a product bundle over T with fiber $(\mathfrak{g}/\mathfrak{t})_T$, which we must further analyze. In particular, it is the pullback of the vector bundle $(\mathfrak{g}/\mathfrak{t})_T \to BT$ under the projection $\pi\colon T \times BT \to BT$, which induces a tensor-factor inclusion $\pi^*\colon H^*(BT) \to H^*(T) \otimes H^*(BT)$ in cohomology.

Now under the Ad-action, $\mathfrak{g}/\mathfrak{t}$ naturally decomposes as a direct sum of two-dimensional representations of T corresponding to the positive roots α of G, which we will view as one-dimensional complex representations $\mathbb{C}_\alpha$. Each homotopy quotient $S_\alpha := (\mathbb{C}_\alpha)_T$ is then a line bundle over BT, and $(\mathfrak{g}/\mathfrak{t})_T = \bigoplus_{\alpha\in\Phi^+} S_\alpha$.

Since the total Chern class is multiplicative, the Chern total classes of the line bundles S_α are all $1 + c_1(S_\alpha)$, and the top Chern class of a complex vector bundle is the Euler class of its realization, the equivariant Euler class of ν is Cite DFAT.

$$e^T(\nu) = e\big(\pi^*(\mathfrak{g}/\mathfrak{t})_T\big) = \pi^*\Big(e\big((\mathfrak{g}/\mathfrak{t})_T\big)\Big) = e\big((\mathfrak{g}/\mathfrak{t})_T\big) = c_{\frac{n-\operatorname{rk} T}{2}}\Big(\bigoplus_{\alpha\in\Phi^+} S_\alpha\Big) = \prod_{\alpha\in\Phi^+} c_1(S_\alpha) \in H^{n-\operatorname{rk} T}(BT).$$

5 Application to $\mathrm{Sp}(1)$

We want to demystify the formula

$$\frac{1}{|W|}\int_T \Delta(t)\left[\int_{\mathrm{Sp}(1)/T} f(gtg^{-1})\, d(gT)\right] dt = \int_{\mathrm{Sp}(1)} f(g)\, dg = \int_T \frac{(\widetilde{f\,dg})(X)|_T}{e^T(\nu)(X)}$$

by finding T, W, Δ, dg, dt, $d(gT)$, W, X, $\widetilde{f\,dg}$, and $e^T(\nu)$.

The compact Lie group $\mathrm{Sp}(1)$ can be seen as the multiplicative subgroup of quaternions of norm 1. The exponential formula $e^q := \sum_{n\in\mathbb{N}} q^n/n!$ defines a function on quaternions as well as on complex numbers; we will use this fact and the linearity of conjugation to simplify our identification of the adjoint representation and the Euler class.

5.1 $\mathfrak{sp}(1)$ and T and $\mathfrak{t}$ and X

The standard basis $1, i, j, k$ of the quaternions $\mathbb{H}$ defines a vector space decomposition into the direct sum of the real part $\mathbb{R}$ spanned by 1 and the purely imaginary part $\operatorname{Im}(\mathbb{H})$ spanned by i, j, k. Since i, j, k anticommute, if $u \in \operatorname{Im}(\mathbb{H})$ and $|u| = u\bar{u} = 1$, then $u^2 = -1$ and $u^3 = -u$ and $u^4 = 1$ and so on, so that $e^{tu} = \cos t + u \sin t$ for real t. Since any pure imaginary is tu for some real t and unit-norm $u \in \operatorname{Im}(\mathbb{H})$, and any quaternion of unit norm can be written as $\cos t + u \sin t$, it follows that the quaternionic exponential is a surjection $\operatorname{Im}(\mathbb{H}) \twoheadrightarrow \mathrm{Sp}(1)$, and since $\frac{d}{dt}e^{tu}\Big|_{t=0} = -\sin 0 + u\cos 0 = u$, we have a natural identification $\mathfrak{sp}(1) \leftrightarrow \operatorname{Im}(\mathbb{H})$. We will consider the conjugation action by the maximal torus $T = e^{i\mathbb{R}} = \big\{\cos t + i \sin t : t \in [0, 2\pi)\big\}$. Then $\mathfrak{t} = i\mathbb{R}$ and we should pick a nonzero $X \in \exp^{-1}\{1\}$; we will take $X = 2\pi i$.

5.2 Ad and Δ and Φ^+ and W

We can write any $q \in \mathrm{Sp}(1)$ as $z_1 + jz_2$ for $z_1, z_2 \in \mathbb{C} := \mathbb{R} + i\mathbb{R}$. We consider the effect of conjugation c_t by $t = e^{i\theta} \in T$ on these coordinates. Since i commutes with $\mathbb{C}$ and anticommutes with $j\mathbb{C}$, we have $iji^{-1} = j(-i)^2 = -j$, so for $z \in \mathbb{C}$ we have $zjz^{-1} = j\bar{z}z^{-1}$ and for $e^{i\theta} \in T$ we get $e^{i\theta}(z_1 + jz_2)e^{-i\theta} = z_1 + je^{-i2\theta}z_2$. Since multiplication on $\mathbb{H}$ is linear, the adjoint representation on $\operatorname{Im}(\mathbb{H})$ is given by

$$\mathrm{Ad}(t)Y = (c_t)_*Y = \frac{d}{ds}t(e^{sY})t^{-1}\bigg|_{s=0} = t \cdot \frac{d}{ds}(e^{sY})\bigg|_{s=0} \cdot t^{-1} = tYt^{-1} = c_t(Y).$$

In particular, taking $t = e^{i\theta}$, and conjugating the basis i, j, k of $\mathfrak{sp}(1)$, we have

$$\begin{aligned} c_t(i) &= i, \\ c_t(j) &= j(\cos 2\theta - i \sin 2t) = (\cos 2\theta)j + (\sin 2\theta)k, \\ c_t(k) &= k(\cos 2\theta - i \sin 2t) = (-\sin 2\theta)j + (\cos 2\theta)k. \end{aligned}$$

Since $i\mathbb{R} = \mathfrak{t}$, on $\mathfrak{sp}(1)/\mathfrak{t}$, with respect to the basis j, k, we have

$$\mathrm{id} - \mathrm{Ad}(t^{-1}) = \begin{pmatrix} 1 - \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & 1 - \cos 2\theta \end{pmatrix},$$

and so

$$\begin{aligned} \Delta(t) &= (1 - \cos 2\theta)^2 + (\sin 2\theta)^2 \\ &= 1 - 2\cos 2\theta + (\cos^2 2\theta + \sin^2 2\theta) \\ &= 2(1 - \cos 2\theta) \\ &= 2\big((\cos^2\theta + \sin^2\theta) - (\cos^2\theta - \sin^2\theta)\big) \\ &= 4\sin^2\theta. \end{aligned}$$

The roots Φ of the adjoint representation are the weights of the complexified representation $\big(\mathfrak{sp}(1)/\mathfrak{t}\big) \otimes \mathbb{C}$. If we take as basis vectors $v_1 = j \otimes i + k \otimes 1$ and $v_2 = j \otimes 1 + k \otimes i$, we get $\mathrm{Ad}(t)v_1 = t^2 v_1$ and $\mathrm{Ad}(t)v_2 = t^{-2}v_2$, so we have one positive root $\alpha\colon t \mapsto t^2$. So $W \cong S_2$ has two elements and $e^T(\nu) = c_1(S_\alpha) = 2u \in H^2(BT)$. Here $BT \simeq \mathbb{C}\mathrm{P}^\infty$ and u corresponds to the first Chern class of the tautological bundle on $\mathbb{C}\mathrm{P}^\infty$. Viewing u as an element of the symmetric algebra $S(\mathfrak{t}^*)$, we have $u(X) = 1$.

5.3 dg and dt and $d(gT)$

Writing $\mathbb{H} = \mathbb{C} + j\mathbb{C}$, we can decompose $q \in \mathrm{Sp}(1)$ as $z_1 + jz_2$, where $|z_1|^2 + |z_2|^2 = 1$. Then we can find $\eta \in [0, \frac{\pi}{2}]$ such that $|z_1| = \sin\eta$ and $|z_2| = \cos\eta$, and can find complex arguments ξ_1 and ξ_2 of z_1 and z_2, so that $e^{i\xi_1}\sin\eta + je^{i\xi_2}\cos\eta$. This identifies $\mathrm{Sp}(1)$ as a quotient space of $S^1 \times S^1 \times [0, \frac{\pi}{2}]$; on the subspace $\eta^{-1}(0)$, the ξ_1-coordinate is irrelevant, and on $T = \eta^{-1}(\frac{\pi}{2})$, the ξ_2-coordinate is irrelevant. Thus we can parameterize $\mathrm{Sp}(1)$ minus two circles by $E := [0, 2\pi)^2 \times (0, \frac{\pi}{2})$. We will find it helpful to write dg in this parameterization.

Writing $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ for real x_1, x_2, y_1, y_2 gives a vector space isomorphism $\mathbb{H} \leftrightarrow \mathbb{R}^4$ taking $\mathrm{Sp}(1) \leftrightarrow S^3$. The natural volume form on $\mathbb{R}^4$ is $\mathrm{vol}_{\mathbb{R}^4} = dx_1 \wedge dy_1 \wedge dx_2 \wedge dy_2$. If we write $Y = x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} + y_1\frac{\partial}{\partial y_1} + y_2\frac{\partial}{\partial y_2}$ for the radial vector field on S^3, we get

$$\mathrm{vol}_{S^3} = \iota_Y(\mathrm{vol}_{\mathbb{R}^4}) = (x_1\,dy_1 - y_1\,dx_1) \wedge dx_2 \wedge dy_2 + (x_2\,dy_2 - y_2\,dx_2) \wedge dx_1 \wedge dy_1.$$

Substituting $x_1 = \sin\eta\cos\xi_1$, $y_1 = \sin\eta\sin\xi_1$, $x_1 = \cos\eta\cos\xi_2$, $y_2 = \cos\eta\sin\xi_2$, much cancellation occurs, and after a few more lines one finds that vol_{S^3} pulls back to

$$\omega = \cos\eta\sin\eta\,d\eta \wedge d\xi_1 \wedge d\xi_2 = \frac{\sin 2\eta}{2}\,d\eta \wedge d\xi_1 \wedge d\xi_2$$

on E.

We have

$$\begin{aligned}\mathrm{vol}(S^3) = \int_E \omega &= \int_0^{2\pi} d\xi_1 \int_0^{2\pi} d\xi_2 \int_0^{\frac{\pi}{2}} \frac{\sin 2\eta}{2}\,d\eta \\ &= (2\pi)^2 \int_0^{\pi} \frac{\sin\zeta}{2} d(\zeta/2) \\ &= \pi^2(-\cos\zeta)\Big|_{\zeta=0}^{\zeta=\pi} \\ &= 2\pi^2,\end{aligned}$$

so

$$dg = \frac{1}{2\pi^2}\,\mathrm{vol}_{S^3} \leftrightarrow \frac{\omega}{2\pi^2} = \frac{\sin 2\eta}{4\pi^2}\,d\eta \wedge d\xi_1 \wedge d\xi_2.$$

Since $\mathrm{vol}(S^1) = 2\pi$, we have $dt \leftrightarrow \frac{1}{2\pi}d\xi_1$.

To find $d(gT)$, we should first understand the quotient space $\mathrm{Sp}(1)/T$. Since $(e^{i\xi_1}\sin\eta + je^{i\xi_2}\cos\eta)e^{-\xi_1} = \sin\eta + je^{i(\xi_2-\xi_1)}\cos\eta$, a fundamental domain for right translation by T is the set $\xi_1^{-1}(0) \approx S^2$ (this is the Hopf fibration of S^3). Omitting the two points $\{\eta = 0, \frac{\pi}{2}\}$ (where the arguments for the j-coordinate are arbitrary), this corresponds to $D := \{0\} \times [0, 2\pi) \times (0, \frac{\pi}{2}) \subsetneq E$. Now $\omega|_D = \frac{\sin 2\eta}{2}\,d\eta \wedge d\xi_2$, and $\int_D \omega|_D = \pi$, so

$$d(gT) = \frac{1}{\pi}\,\mathrm{vol}_{\mathrm{Sp}(1)/T} \leftrightarrow \frac{\sin 2\eta}{2\pi}\,d\eta \wedge d\xi_2.$$

5.4 $\underline{X}$ and $\widetilde{f\,dg}$

Since the fundamental vector field generated by $X = 2\pi i$ under conjugation is

$$\underline{X}_q = (c_{e^{-sX}})_{*,q}\left(\frac{d}{ds}\bigg|_{s=0}\right)$$

and

$$c_{e^{-sX}}(z_1 + jz_2) = z_1 + jz_2 e^{4\pi si} \leftrightarrow (\xi_1, \xi_2 + 4\pi s, \eta),$$

it follows that on D, we have $\underline{X} = 4\pi\frac{\partial}{\partial\xi_2}$, and a function f on $\mathrm{Sp}(1)$ is T-invariant just if $\frac{\partial f}{\partial \xi_2} = 0$.

Since dg is a top form, any form $f\,dg$ is closed, and so, if f is T-equivariant, admits an equivariantly closed extension $\widetilde{f\,dg} \in \Omega^*\big(\mathrm{Sp}(1)\big)^T[u]$. If we write $\widetilde{f\,dg} = f\,dg + u\tau$ for $\tau \in \Omega^1\big(\mathrm{Sp}(1)\big)^T$, equivariant closedness means that

$$0 = (d - u\iota_X)(f\,dg + u\tau) = d(f\,dg) + u\big(d\tau - f\iota_X(dg)\big) - u^2\iota_X\tau,$$

or that $d\tau = f\iota_X(dg)$ and $\iota_X\tau = 0$.

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$$c_{e^{-sX}}(z_1 + jz_2) = z_1 + jz_2 e^{4\pi si} \leftrightarrow (\xi_1, \xi_2 + 4\pi s, \eta),$$

it follows that on D, we have $\underline{X} = 4\pi\frac{\partial}{\partial\xi_2}$.

Thus we require that $\iota_{\frac{\partial}{\partial\xi_2}}\tau = \tau\big(\frac{\partial}{\partial\xi_2}\big) = 0$. Writing $\tau = a\,d\xi_1 + b\,d\xi_2 + c\,d\eta$, this means $b = 0$. Since τ is assumed T-invariant, it also follows that $\frac{\partial a}{\partial\xi_2} = \frac{\partial c}{\partial\xi_2} = 0$, so that

$$d\tau = \Big(\frac{\partial a}{\partial\eta} - \frac{\partial c}{\partial\xi_1}\Big)d\eta\wedge d\xi_1.$$

On the other hand,

$$f\iota_X dg = f\iota_{4\pi\frac{\partial}{\partial\xi_2}}\Big(\frac{\sin 2\eta}{2}\,\frac{d\eta\wedge d\xi_1\wedge d\xi_2}{2\pi^2}\Big) = \frac{1}{\pi}f\sin 2\eta\,d\eta\wedge d\xi_1,$$

so we want

$$\frac{\partial a}{\partial\eta} - \frac{\partial c}{\partial\xi_1} = \frac{1}{\pi}f\sin 2\eta.$$

Since f is 2π-periodic in ξ_1, the most obvious (if not particularly subtle) thing to do is to take $c = 0$. Since τ doesn't depend on ξ_1 when $\eta = 0$, it's then easiest to take $a|_{\eta^{-1}(0)} = 0$. One way to accomplish this is to set

$$a(\xi_1,\eta) = \frac{1}{\pi}\int_0^\eta f(\xi_1,\zeta)\sin 2\zeta\,d\zeta.$$

5.5 Conjugation of a T-invariant function

A function on $\mathrm{Sp}(1)$ is invariant under conjugation by T iff it is independent of ξ_2. The inner integral in the Weyl integral formula is over $f(qtq^{-1})$ for fixed $t \in T$ and $qT \in \mathrm{Sp}(1)/T$.

Given an element of $\mathrm{Sp}(1)/T$, we can find a representative q in the fundamental domain $\xi_1^{-1}(0)$. Letting $t' \in T$ and $q' = t'q(t')^{-1}$, we see that a T-invariant function will give the same value at qtq^{-1} and at $q't(q')^{-1} = t'q(t')^{-1}tt'q^{-1}(t')^{-1} = t'qtq^{-1}(t')^{-1}$, so we may as well assume $\xi_2(q) = 0$ as well, or $q = \sin\zeta + j\cos\zeta$. Writing $t = e^{i\xi}$, we get $qtq^{-1} = (\cos\xi - i\sin\xi\cos 2\zeta) + ji\sin\xi\sin 2\zeta$. Conjugating to get $\xi_2 = 0$ again, we get $\xi_1(qtq^{-1}) = \arctan\left(-\tan\xi\cos 2\zeta\right)$ and $\eta(qtq^{-1}) = \arccos(\sin\xi\sin 2\zeta)$.

So the integral, not particularly transparently, is of

$$f\big(\arctan\left(-\tan\xi_1\cos 2\eta\right), \arccos(\sin\xi_1\sin 2\eta)\big)$$

over η, for fixed ξ_1.

One wonders whether a better choice of coordinates is in order.

5.6 The desired formula, restated

Consider a real-valued function $f(\xi_1, \eta)$ on $\mathbb{R} \times [0, \frac{\pi}{2}]$ which is 2π-periodic in ξ_1 and independent of ξ_1 wherever $\eta = 0$.

The equivariant localization formula tells us

$$\int_{\mathrm{Sp}(1)} f\,dg = \int_0^{2\pi}\int_0^{2\pi}\int_0^{\pi/2} f(\xi_1,\eta)\frac{\sin 2\eta}{2}\,\frac{d\eta\,d\xi_1\,d\xi_2}{2\pi^2} = \frac{1}{2\pi}\int_0^{2\pi}\int_0^{\pi/2} f(\xi_1,\eta)\sin 2\eta\,d\eta\,d\xi_1$$

equals

$$\int_T \frac{u(X)}{2u(X)}\tau|_{\{\eta=\frac{\pi}{2}\}} = \frac{1}{2}\int_0^{2\pi} a\Big(\xi_1, \frac{\pi}{2}\Big)\,d\xi_1 = \frac{1}{2}\int_0^{2\pi}\frac{1}{\pi}\int_0^{\pi/2} f(\xi_1,\eta)\sin 2\eta\,d\eta\,d\xi_1.$$

This is at least true, but it's not informative.

As for the Weyl integration formula, it tells us that the integrals above equal

$$\frac{1}{|W|}\int_T \Delta(t)\left[\int_{\mathrm{Sp}(1)/T} f(gtg^{-1})\,d(gT)\right]dt =$$
$$\frac{1}{2}\int_0^{2\pi} 4\sin^2\xi_1\left[\int_0^{2\pi}\int_0^{\pi/2} f\big(\arctan\left(-\tan\xi_1\cos 2\eta\right), \arccos(\sin\xi_1\sin 2\eta)\big)\frac{\sin 2\eta}{2}\,\frac{d\eta\,d\xi_2}{\pi}\right]\frac{d\xi_1}{2\pi} =$$
$$\frac{1}{\pi}\int_0^{2\pi}\sin^2\xi_1\left[\int_0^{\pi/2} f\big(\arctan\left(-\tan\xi_1\cos 2\eta\right), \arccos(\sin\xi_1\sin 2\eta)\big)\sin 2\eta\,d\eta\right]d\xi_1.$$

This isn't particularly helpful.

5.7 An easier case

Suppose that f is in fact invariant under conjugation by G. Then the bracketed integral simplifies to $f(t)$, so the whole integral becomes

$$\frac{1}{2}\int_0^{2\pi} 4(\sin^2\xi_1) f\left(\xi_1, \frac{\pi}{2}\right) d\xi_1.$$

It still isn't clear to me, though, why this should have anything to do with the double integral in the equivariant localization formula.

6 Unsorted thoughts

- Would the Weyl integration formula be more what we want with different coordinates?
- Could a less stupid choice of a and c in $\tau = a\, d\xi_1 + c\, d\eta$ yield the Weyl integral formula? It's $a(\xi_1, \pi/2)\, d\xi_1$ that shows up in the equivariant localization formula. It's easy to arrange the functions be constant in ξ_1 at $\eta = 0$, say by giving them $\sin n\eta$ factors or making them integrals from 0 to η, but if we try to select a first and derive $c(\xi_1, \eta) = \int_0^{\xi_1}(\frac{\partial a}{\partial \eta} - \frac{1}{\pi} f \sin 2\eta)\, d\xi$, the resulting function will almost never be 2π-periodic in ξ_1.
- If we assume f is G-invariant, can we make some clever change of coordinates that simplifies the integral over η in the equivariant localization formula?
- The only part of either formula that involves the roots is $\Delta(t)$ in the one formula and $e^T(\nu)$ in the other. But in the Weyl integration formula, one has to integrate with respect to this function, whereas in the equivariant localization formula, it just cancels u_i's. What is the connection between these superficially similar but very different things?
- The standard proof of the Weyl integration formula starts with the function $G \times T \to G$ taking $(g, t) \mapsto gtg^{-1}$, pushes it down to a function $(G/T) \times T \to G$, cuts out the (positive-codimension) set of singular elements, and computes the integral over G by pulling it back to $(G/T) \times T$. The function $\Delta(t) = \prod_{\alpha \in \Phi}(1 - \alpha)$ comes out of the Jacobian. On the other hand G/T doesn't even show up in the equivariant localization formula. One can knock the G/T integral out of the Weyl formula, but only at the cost of assuming the function to be integrated is G-invariant, a hypothesis that isn't required for the equivariant localization. Is there a way of proving equivariant localization using G/T, or some way of pulling back both proofs (I realize this is incredibly vague) to some more complicated space, so that both formulae manifest as "quotients" of the more complicated situation?

  Given that the Weyl integral formula involves an integral over $T \times (G/T)$, is it worth considering the action of T by $t \cdot (t', gT) = (t', tgT)$, or something similar, and trying to apply equivariant localization?