Products on Tor

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Abstract

In work by Munkholm on the collapse of the Eilenberg–Moore spectral sequences corresponding to certain pullbacks, he constructs, in passing, a bilinear binary operation on Tor of a span of A_∞-algebras is a commutative graded algebra, subject to an additional homotopy commutativity criterion governed by an E_3-operad. He does not use this product and essentially disavows it. Much later, the present author was, to his own surprise, able to establish good properties for a variant of this product and as a consequence to upgrade Munkholm's main topological result from a group isomorphism to a ring isomorphism.

In unrelated work, Matthias Franz defined a weak product on the two-sided bar construction of a span of A_∞-algebras, subject to similar hypotheses, some of whose expected behavior the author verified and used to prove a topological result, without, however establishing whether the induced binary operation on Tor had any particular properties.

In this work we demonstrate that our modified product on Tor is indeed that proposed by Munkholm and that moreover is induced by the product of Franz.

At the root of homological algebra are the tensor product and its derived functors Tor^i. When $M \leftarrow A \to N$ are maps of commutative graded algebras (CGAs), the tensor product $M \otimes_A N$ is again a CGA, and the graded groups $\mathrm{Tor}^i_A(M,N)$ fit into a bigraded ring denoted simply $\mathrm{Tor}_A(M,N)$, because the multiplications $A \otimes A \longrightarrow A$, etc., are then themselves ring maps. When $M \leftarrow A \to N$ are maps of noncommutative differential graded algebras (DGAs), there is still an appropriate notion of proper projective resolution for differential graded A-modules, and accordingly a tensor product whose derived functors are again written $\mathrm{Tor}^i_A(M,N)$ (reducing to the more classical notion when the differentials are zero), but the definition from before typically no longer yields a ring structure on Tor. When however the input rings are (noncommutative) cochain algebras $C^*(X) \leftarrow C^*(B) \to C^*(E)$, there is again a ring structure on Tor, this time arising not from commutativity but from the Eilenberg–Zilber theorem, which does have to do with a sort of homotopy-commutativity property of cochain algebras. Generalizing both of these products is a product defined by Hans J. Munkholm in 1974, subject to sufficient homotopy-commutativity hypotheses on the input DGAs, and promptly abandoned. munkholm1974emss [Mun74, §9]. Munkholm seemed not to believe his product's prospects were good, because its definition depended on an arbitary choice of homotopies.

The present author, in recent work carlson2022munkholm [Car22b], revived this product and showed that subject to Munkholm's hypotheses and one additional assumption, this product rendered $\mathrm{Tor}_A(X,Y)$ a

CGA in a functorial manner, and did not in fact depend on the defining homotopies. As a side-effect, one recovers a multiplicative Eilenberg–Moore spectral sequence collapse result, refining Munkholm's additive collapse result, but the details of this result need not detain us here. The definition of the product in that work is actually a simplification of Munkholm's original, and owing to length considerations, proof that the definitions are equivalent is not actually included in that paper, the more important goal having been to demonstrate and exploit the properties of the product.

Similarly, in previous work, the author had shown a weaker collapse result using a product due to Franz on a two-sided bar construction $\mathbf{B}(X, A, Y)$ and defined under similar hypotheses. That cochain-level product also defines a product on $\operatorname{Tor}_A(X, Y)$ subject to additional mild flatness hypotheses sufficient to guarantee the hoped-for module isomorphism $H^*\mathbf{B}(X, A, Y) \cong \operatorname{Tor}_A(X, Y)$, but most of the desirable potential properties of that product went unexplored, secondary to the use of that product in the collapse result.

In the present paper, we show that the product of our previous work [Car22b] is indeed Munkholm's (Lemma 8.1), and that moreover, so long as $\mathbf{B}(X, A, Y)$ does compute Tor, Franz's product on $\mathbf{B}(X, A, Y)$ induces Munkholm's product on $\operatorname{Tor}_A(X, Y)$ (Theorem 9.10).

Outline

The plan of the work is as follows. Section 1 defines algebras and coalgebras and the bar–cobar adjunction, as well as the intermediary notion of a *twisting cochain*. Section 2 brings in the tensor product and some of its interactions with the adjunction. The one original result here is Lemma 2.8 (which may also be of interest because it can be used to prove the crucial Proposition IV.6.1 in Husemoller–Stasheff–Moore's collapse paper [HMS74, p. 179], bypassing the false Proposition IV.5.7). Section 4 discusses conditions under which maps on Tor of a span of DGAs can be defined, which are classical, and establishes their homotopy-invariance and functoriality, which are new. Section 5 recalls the notion of an SHC-algebra. Section 7 motivates and defines the product. Section 8 reformulates the product in a manner more conducive to proving its properties and establishes a CGA structure on Tor, the first clause of **??**. Section 9 shows that under standard flatness hypotheses, the product on Tor is induced by a product on the two-sided bar construction due to Franz.

These three sections involve some micromanagement of DGA homotopies and a number of diagrams, but mostly rely on formal properties of the homotopy categories of DGAs and DGCs without recourse to the cochain level.

1. Algebras, coalgebras, and twisting cochains

Prerequisites are as in the predecessor [Car22b], but we run rapidly over some highlights.

We take tensors and Homs over a fixed commutative base ring k with unity and consider cochain complexes (C, d) concentrated in nonnegative degree, with differentials d *increasing* degree by 1, writing $\mathsf{Ch}^\bullet$ for the category thereof. The Koszul sign convention is always in force. We use only augmented differential nonnegatively-graded k-algebras (henceforth DGAs) $(A, +, 0, d_A, \mu_A, \eta_A, \varepsilon_A)$ with augmentation ideal $\ker \varepsilon_A = \overline{A} \cong \operatorname{coker} \eta_A$ and coaugmented, cocomplete, differential nonnegatively-graded k-coalgebras (DGCs) $(C, +, 0, d_C, \Delta_A, \varepsilon_A, \eta_A)$ with coaugmentation coideal $\operatorname{coker} \eta_A = \overline{C} \cong \ker \varepsilon_A$. The appropriate notions of homomorphism are obvious and the corresponding categories are written as DGA and DGC respectively. A commutative DGA is a CDGA. ***Differentials*** d ***increase degree*** by 1. We use the terms DG *k-module* and *cochain complex* interchangeably. Our DGAs and DGCs are ***nonnegatively-graded***. The base ring k itself lies in both.

Given two graded k-modules C and A, we denote by $\mathsf{Mod}_n(C, A)$ the k-module of k-linear maps f sending each C_j to A_{j+n}, and set the degree $|f|$ to n for such a map. The hom-set $\mathsf{Mod}(C, A) = \bigoplus_{n \in \mathbb{Z}} \mathsf{Mod}_n(C, A)$ then becomes itself a graded k-module. If C and A are cochain complexes, then $\mathsf{Mod}(C, A)$ becomes a cochain complex under the differential $d = d_{\mathsf{Mod}(C,A)}$ given by $Df := d_A f - (-1)^{|f|} f d_C$ [Mun74, §1.1], cochain maps being described by the condition $Df = 0$. If C is a DGC and A a DGA, then $\mathsf{Mod}(C, A)$ becomes a DGA under with the ***cup product*** $f \smile g := \mu_A (f \otimes g) \Delta_C$ [Mun74, §1.8]. An element $t \in \mathsf{Mod}_1(C, A)$ satisfying the three conditions

$$\varepsilon_A t = 0 = t \eta_C, \qquad Dt = t \smile t$$

is called a ***twisting cochain*** [HMS74, §1.8][HMS74, Prop. 3.5(1)][Pro11, §§1.5, 4]. Write $\mathsf{Tw}(C, A)$ for the additive group of these.

Given $(g, t, f) \in \mathsf{DGC}(C', C) \times \mathsf{Tw}(C, A) \times \mathsf{DGA}(A, A')$, the maps ft, tg, and ftg are again twisting cochains. For each DGA A, there is a *final* twisting cochain $t^A \colon \mathbf{B}A \longrightarrow A$ defined by the property that any twisting cochain $t \colon C \longrightarrow A$ factors uniquely through a DGC map $g_t \colon C \longrightarrow \mathbf{B}A$ such that $t = t^A \circ g_t$. Here the cocomplete $\mathbf{B}A$ is the familiar ***bar construction***, which gives the object component of a functor $\mathbf{B} \colon \mathsf{DGA} \longrightarrow \mathsf{DGC}$ [Mun74, §1.6][Pro11, §2.5]. The tautological twisting cochain $t^{(-)} \colon \mathbf{B} \longrightarrow \mathrm{id}$ is a natural transformation $\mathsf{DGC} \longrightarrow \mathsf{Ch}^\bullet$. We denote this conversion in the input–output "deduction rule" format originally arising from proof theory:

$$\frac{g_t \colon C \longrightarrow \mathbf{B}A}{t \colon C \longrightarrow A.}$$

Explicitly, the bar construction is the tensor coalgebra on the desuspension $s^{-1}\overline{A}$ of $\overline{A}$, equipped with the sum of the tensor differential and the unique coderivation extending the "bar-deletion" map $(s^{-1}\overline{A})^{\otimes 2} \to \overline{A}^{\otimes 2} \xrightarrow[\mu]{} \overline{A} \to s^{-1}\overline{A}$. We will only need the following consequences of this description.

Lemma 1.1. *The bar construction $\mathbf{B}A$ admits an additive direct sum decomposition $\bigoplus_{n \geqslant 0} \mathbf{B}_n A$ with $\mathbf{B}_0 = \operatorname{im} \eta \cong k$ and such that $\bar{\Delta} \mathbf{B}_n A$ lies in the sum of submodules $\mathbf{B}_p A \otimes \mathbf{B}_q A$ with $p + q = n$ and $p, q \geqslant 1$. The filtration by the kernels $\bigoplus_{p<n} \mathbf{B}_p A$ of $\bar{\Delta}^{[n]}$ is stable under d, and in particular, $\mathbf{B}_1 A$ is a* DG *submodule. The tautological twisting cochain $t^A \colon \mathbf{B}A \longrightarrow A$ factors through a* DG *module isomorphism $\mathbf{B}_1 A \xrightarrow{\sim} \overline{A}$ of degree 1.*

For each a DGC C, there is a twisting cochain $t_C \colon C \longrightarrow \mathbf{\Omega} C$ *initial* in the sense that any twisting cochain $t \colon C \longrightarrow A$ factors uniquely through a DGA map $f^t \colon \mathbf{\Omega} C \longrightarrow A$ such that $t = f^t t_C$. The DGA $\mathbf{\Omega} C$ is referred to as the ***cobar construction***, and gives the object component of a functor $\mathbf{\Omega} \colon \mathsf{DGC} \longrightarrow \mathsf{DGA}$ [Mun74, §1.7]. The tautological twisting cochain $t_{(-)} \colon \mathrm{id} \longrightarrow \mathbf{\Omega}$ is a natural transformation $\mathsf{Ch}^\bullet \longrightarrow \mathsf{DGA}$ Thus the two functors $\mathbf{\Omega} \dashv \mathbf{B}$ form an adjoint pair [Mun74, §1.9–10]. We will have frequent recourse to the unit and counit of the adjunction $\mathbf{\Omega} \dashv \mathbf{B}$,

$$\eta\colon \mathrm{id} \longrightarrow \mathbf{B\Omega} \qquad \text{and} \qquad \varepsilon\colon \mathbf{\Omega B} \longrightarrow \mathrm{id}$$

respectively. These are both natural quasi-isomorphisms and homotopy equivalences on the level of DG modules [HMS74, Thm. II.4.4–5][Mun74, Cor. 2.15][LH02, Lem. 1.3.2.3].

The adjunction interacts with the tautological twisitng cochains as follows.

Lemma 1.2. *Given a* DGA A *and a* DGC C*, one has* $\varepsilon \circ t_{\mathbf{B}A} = t^A\colon \mathbf{B}A \longrightarrow A$ *and* $t^{\mathbf{\Omega}C} \circ \eta = t_C\colon C \longrightarrow \mathbf{\Omega}C$.

2. The tensor product

The functor $\mathbf{B}\colon \mathsf{DGA} \longrightarrow \mathsf{DGC}$ is lax monoidal with respect to the monoidal structure given on both categories by the appropriate tensor products, and $\mathbf{\Omega}\colon \mathsf{DGC} \longrightarrow \mathsf{DGA}$ is lax comonoidal.

Definition 2.1 (See Husemoller *et al.* [HMS74, Def. IV.5.3]). There exist natural transformations

$$\nabla\colon \mathbf{B}A_1 \otimes \mathbf{B}A_2 \longrightarrow \mathbf{B}(A_1 \otimes A_2), \qquad \gamma\colon \mathbf{\Omega}(C_1 \otimes C_2) \longrightarrow \mathbf{\Omega}C_1 \otimes \mathbf{\Omega}C_2$$

of functors $\mathsf{DGA} \times \mathsf{DGA} \longrightarrow \mathsf{DGC}$ and $\mathsf{DGC} \times \mathsf{DGC} \longrightarrow \mathsf{DGA}$, respectively, the ***shuffle maps***, determined by the twisting cochains

$$t^{A_1 \otimes A_2}\nabla = t^{A_1} \otimes \eta_{A_2}\varepsilon_{\mathbf{B}A_2} + \eta_{A_1}\varepsilon_{\mathbf{B}A_1} \otimes t^{A_2}, \qquad \gamma t_{C_1 \otimes C_2} = t_{C_1} \otimes \eta_{\mathbf{\Omega}C_2}\varepsilon_{C_2} + \eta_{\mathbf{\Omega}C_1}\varepsilon_{C_1} \otimes t_{C_2}.$$

These are homotopy equivalences of cochain complexes and hence quasi-isomorphisms.

Written out in terms of bar-words, $\nabla(b_1 \otimes b_2)$ is a sum of shuffle permutations of the letters of b_1 and b_2, so values of ∇ exhibit symmetry with respect to shuffles of tensor-factors. Particularly, we have the following.

Lemma 2.2. *Let* A_1 *and* A_2 *be* DGAs*. For* $b_1 \otimes b_2 \in \mathbf{B}_m A_1 \otimes \mathbf{B}_n A_2$*, the* $\nabla^{\otimes 2}$*-image of the summands of* $\Delta_{\mathbf{B}A_1 \otimes \mathbf{B}A_2}(b_1 \otimes b_2)$ *lying in* $\mathbf{B}_m A_1 \otimes k \otimes k \otimes \mathbf{B}_n A_2$ *and* $k \otimes \mathbf{B}_n A_2 \otimes \mathbf{B}_m A_1 \otimes k$ *is*

$$\nabla(b_1 \otimes 1) \otimes \nabla(1 \otimes b_2) + (-1)^{|b_1||b_2|}\nabla(1 \otimes b_2) \otimes \nabla(b_1 \otimes 1).$$

There is another important natural transformation on DGA we will rely on heavily, which we describe using a cochain-level argument.

Definition 2.3 ([Mun74, §2.1]). A ***trivialized extension***[1] is an assemblage of maps

$$\begin{tikzcd} \widetilde{A} \arrow[loop right, "h"] \arrow[d, shift right, "p"'] \\ A \arrow[u, shift right, "i"'] \end{tikzcd}$$

in which $p\colon \widetilde{A} \longrightarrow A$ is a DGA map, $i\colon A \longrightarrow \widetilde{A}$ is a DG module section, and $h \in \mathrm{Mod}_{-1}(\widetilde{A}, \widetilde{A})$ a cochain homotopy satisfying $d(h) = \mathrm{id} - ip$, and moreover, the compositions ph, hh, hi vanish.[2]

[1] We follow Munkholm in this usage. *Contraction*, *(strong) homotopy retract datum*, and *SDR-data* are all common in the literature when $\widetilde{A}$ and A are merely assumed DG modules.

[2] That $h^2 = 0$ actually follows from the other equations.

The homotopy h is the sort of data that allows us to promote i to a DGC map $\mathbf{B}A \longrightarrow \mathbf{B}\widetilde{A}$.

Lemma 2.4 (Homotopy transfer theorem for DGAs [Mun74, Prop. 2.2])**.** *Let a trivialized extension be given as in Definition 2.3. Then there exists a twisting cochain $t^i\colon \mathbf{B}A \longrightarrow \widetilde{A}$ such that $pt^i = t^A\colon \mathbf{B}A \longrightarrow A$. This $t^i\colon \mathbf{B}A \longrightarrow \widetilde{A}$ then induces a DGC map $g_{t^i}\colon \mathbf{B}A \longrightarrow \mathbf{B}\widetilde{A}$ and a DGA map $f^{t^i}\colon \boldsymbol{\Omega}\mathbf{B}A \longrightarrow \widetilde{A}$.*

Proof. We make and explain the recursive prescription $t = h(t \smile t) + it^A$. By Lemma 1.1, we may recursively define $t_n = t|_{\mathbf{B}_n A}$. Our formula sets $t_0 = 0$ and $t_1 = it^A|_{\mathbf{B}_1 A}$, and for $n \geqslant 2$ takes $t_n(b) = h(t_{<n} \smile t_{<n})(b)$, which avoids circularity because $t \smile t$ annihilates $1 \otimes b$ and $b \otimes 1$ by the "$t_0 = 0$" clause and $\bar{\Delta}_{\mathbf{B}A}$ takes $\mathbf{B}_n A$ to $\mathbf{B}_{<n}A \otimes \mathbf{B}_{<n}A$, where $t \smile t$ is already defined. Evidently $pt_n = 0$ for $n > 1$ since $ph = 0$, and $pt_0 = p0 = 0$, while $pt_1 = pit^A = t^A$.

The proof t is a twisting cochain is by induction; we will explain it because Munkholm leaves it as an exercise. That $d(t_0) = 0 = (t \smile t)|_{\mathbf{B}_0 A}$ is trivial. We have $d(t_1) = dit^A + it^A d = 0$ because i is a chain map of degree 0 and $t^A|_{\mathbf{B}_1 A}$ a chain map of degree 1, and $t \smile t$ vanishes on $\mathbf{B}_1 A$ since $t_0 = 0$ and $\bar{\Delta}$ vanishes on $\mathbf{B}_1 A \cong \ker \bar{\Delta}/\operatorname{im} \eta_{\mathbf{B}A}$.

The case $n = 2$ is the interesting case. On the one hand one has $(t \smile t)|_{\mathbf{B}_2 A} = t_1 \smile t_1$. On the other, since t^A is a twisting cochain, one has $dt^A + t^A d|_{\mathbf{B}_2 A} = t^A \smile t^A$, and using in order these facts, that $d(t_1) = 0$ and $d(t_1 \smile t_1) = 0$, that $dh + hd = \mathrm{id} - ip$, and that $pi = \mathrm{id}_A$, we find

$$\begin{aligned} d(t)|_{\mathbf{B}_2 A} &= dt_2 + t_{\leqslant 2} d \\ &= dh(t_1 \smile t_1) + dit^A + h(t_1 \smile t_1)d + it^A d \\ &= dh(t_1 \smile t_1) + hd(t_1 \smile t_1) + idt^A + it^A d \\ &= (dh + hd)(t_1 \smile t_1) + i(t^A \smile t^A) \\ &= t_1 \smile t_1 \,-\, ip(it^A \smile it^A) \,+\, it^A \smile it^A \\ &= t_1 \smile t_1. \end{aligned}$$

For $n \geqslant 3$ the proof is tautological: we have $t_n = h(t \smile t)$ since $d\mathbf{B}_n A$ lies in $\mathbf{B}_{\geqslant n-1}A$ and hence is annihilated by it^A, and so $\operatorname{im}(t \smile t)|_{\mathbf{B}_n A}$ lies in the ideal generated by $\operatorname{im} h$, which p annihilates, hence inductively

$$\begin{aligned} d(t) &= dh(t \smile t) + h(t \smile t)d \\ &= (\mathrm{id} - ip - hd)(t \smile t) + h(t \smile t)d \\ &= t \smile t \,-\, h\,d(t \smile t) \\ &= t \smile t \,-\, h\,d\big(d(t)\big) \\ &= t \smile t. \end{aligned}$$

□

There is a notion of a morphism of trivialized extensions, *viz.* a pair of DGA maps making the expected three squares for p, i, h commute [Mun74, §2.1], but we will not need it. We will however need one key example.

Example 2.5 (The universal example [Mun74, Prop. 2.14]). Given a DGA A, write s^{-1} for the DG module map $\overline{A} \xrightarrow{\sim} \mathbf{B}_1 A$ splitting the tautological twisting cochain t^A. There is a unique section $i_A\colon A \longrightarrow \boldsymbol{\Omega}\mathbf{B}A$ of $\varepsilon\colon \boldsymbol{\Omega}\mathbf{B}A \longrightarrow A$ defined to be unital and to restrict to $t_{\mathbf{B}A} \circ s^{-1}$ on $\overline{A}$. Along with a certain $h \in \mathsf{Mod}_{-1}(\boldsymbol{\Omega}\mathbf{B}A, \boldsymbol{\Omega}\mathbf{B}A)$ we can afford not to be explicit about, ε and i_A can be shown to give a trivialized extension. The cochain $t^{i_A} = h(t \smile t) + i_A t^A \colon \mathbf{B}A \longrightarrow \mathbf{\Omega B}A$ defined inductively from $t^A \colon \mathbf{B}A \longrightarrow A$ as in Lemma 2.4 works out to be the tautological $t_{\mathbf{B}A}$.

Given another trivialized extension $p \colon \widetilde{A} \to A$ with section $i \colon A \to \widetilde{A}$, by Lemma 2.4 there is an induced DGA map $f^{t^i} \colon \mathbf{\Omega B}A \longrightarrow \widetilde{A}$ satisfying $pf^{t^i} = \varepsilon \colon \mathbf{\Omega B}A \longrightarrow A$.[3]

def:psi husemollermoorestasheff1974 munkholm1974emss

Theorem 2.6 ([HMS74, Prop. IV.5.5][Mun74, k_{A_1,A_2}, p. 21, via Prop. 2.14])**.** *There exists a natural transformation*

$$\psi \colon \mathbf{\Omega B}(A_1 \otimes A_2) \longrightarrow \mathbf{\Omega B}A_1 \otimes \mathbf{\Omega B}A_2$$

of functors DGA $\times$ DGA $\longrightarrow$ DGA. *This transformation satisfies*

$$(\varepsilon_{A_1} \otimes \varepsilon_{A_2}) \circ \psi = \varepsilon_{A_1 \otimes A_2} \colon \mathbf{\Omega B}(A_1 \otimes A_2) \longrightarrow A_1 \otimes A_2$$

and reduces to the identity if A_1 or A_2 is k.

Proof. Granting the claims of Example 2.5, it is easy to check the data

$$\begin{aligned}
\varepsilon_{A_1} \otimes \varepsilon_{A_2} &\colon \mathbf{\Omega B}A_1 \otimes \mathbf{\Omega B}A_2 \longrightarrow A_1 \otimes A_2, \\
i_{A_1} \otimes i_{A_2} &\colon A_1 \otimes A_2 \longrightarrow \mathbf{\Omega B}A_1 \otimes \mathbf{\Omega B}A_2, \\
h_\psi := h_{A_1} \otimes \mathrm{id} + i_{A_1}\varepsilon_{A_1} \otimes h_{A_2} &\colon \mathbf{\Omega B}A_1 \otimes \mathbf{\Omega B}A_2 \longrightarrow \mathbf{\Omega B}A_1 \otimes \mathbf{\Omega B}A_2
\end{aligned}$$

give a trivialized extension, so Lemma 2.4 yields the required DGA map $\psi = f^{t^{i_{A_1} \otimes i_{A_2}}}$, with associated twisting cochain $t_\psi \colon \mathbf{B}(A_1 \otimes A_2) \longrightarrow \mathbf{\Omega B}A_1 \otimes \mathbf{\Omega B}A_2$. Naturality follows from the naturality of ε, i, and h.

If $A_2 = k$, then we may make the identifications $\mathbf{\Omega B}(A_1 \otimes k) = \mathbf{\Omega B}A_1$ and $\mathbf{\Omega B}A_1 \otimes \mathbf{\Omega B}A_2 = \mathbf{\Omega B}A_1 \otimes k = \mathbf{\Omega B}A_1$, so that ε_{A_2} and i_{A_2} are identified with id_k and $h_{A_2} i_{A_2} = 0$ forces $h_{A_2} = 0$, and make the identification $h_\psi = h_{A_1}$. Thus this trivialized extension reduces to the initial example $\mathbf{\Omega B}A_1 \longrightarrow A_1$ of Example 2.5. Now $\psi \colon \mathbf{\Omega B}A_1 \longrightarrow \mathbf{\Omega B}A_1$ is induced from $t_\psi \colon \mathbf{B}A_1 \longrightarrow \mathbf{\Omega B}A_1$, recursively defined by $t_\psi = h_{A_1}(t_\psi \smile t_\psi) + i_{A_1} t^{A_1}$, but then t_ψ agrees with $t^{i_{A_1}}$ from the universal Example 2.5, which we have stated is the tautological twisting cochain $t_{\mathbf{B}A_1}$ whose associated DGA map is $f^{t_{\mathbf{B}A_1}} = \mathrm{id}_{\mathbf{\Omega B}A_1}$. The proof if instead $A_1 = k$ is symmetrical. □

Remark 2.7. Munkholm's contracting homotopy h from Example 2.5 is natural in A, but not the unique such map doing the same trick. Husemoller–Moore–Stasheff's ψ is defined using a splitting result (IV.2.5), where the splitting exists because kernel is an injective object in an appropriate injective class, and hence is even less explicit. Thus, although these maps behave the same categorically, there is no reason to expect them to agree on the point-set level. Fortunately, these details are usually irrelevant. Because ψ is involved in the first cochain-level argument of this paper, the following Lemma 2.8, we use Munkholm's formulation.

thm:WDg

Lemma 2.8. *Let A_1 and A_2 be* DGAs*. Then the composition*

$$\mathbf{\Omega}(\mathbf{B}A_1 \otimes \mathbf{B}A_2) \xrightarrow{\mathbf{\Omega}\nabla} \mathbf{\Omega B}(A_1 \otimes A_2) \xrightarrow{\psi} \mathbf{\Omega B}A_1 \otimes \mathbf{\Omega B}A_2$$

agrees with γ from Definition 2.1.

[3] In fact, f^{t^i} and id_A give a map of trivialized extensions of A, so that $\varepsilon \colon \mathbf{\Omega B}A \to A$ with i_A and h can be seen as an initial object in a category of trivialized extensions of A. The identity map $A \to A$ is a final object of the same category.

Proof. We will show the twisting cochains $\mathbf{B}A_1 \otimes \mathbf{B}A_2 \longrightarrow \mathbf{\Omega B}A_1 \otimes \mathbf{\Omega B}A_2$ associated to γ and $\psi \circ \mathbf{\Omega}\nabla$ are equal. The former, $\gamma t_{\mathbf{B}A_1 \otimes \mathbf{B}A_2}$, is $t_{\mathbf{B}A_1} \otimes \eta\varepsilon + \eta\varepsilon \otimes t_{\mathbf{B}A_2}$ by Definition 2.1, whereas by naturality of $t_{(-)}$ the latter is

$$\psi \circ \mathbf{\Omega}\nabla \circ t_{\mathbf{B}A_1 \otimes \mathbf{B}A_2} = \psi \circ t_{\mathbf{B}(A_1 \otimes A_2)} \circ \nabla = t_\psi \circ \nabla.$$

Because t_ψ is given via the recursive prescription $t_\psi = h_\psi(t_\psi \smile t_\psi) + (i \otimes i)t^{A_1 \otimes A_2}$ of Lemma 2.4, the restriction of the twisting cochain $t_\psi \nabla$ to $\nabla^{-1}\big(\mathbf{B}_1(A_1 \otimes A_2)\big)$ is given by

$$(i \otimes i)(t^{A_1} \otimes \eta\varepsilon + \eta\varepsilon \otimes t^{A_2}) = t_{\mathbf{B}A_1} \otimes \eta\varepsilon + \eta\varepsilon \otimes t_{\mathbf{B}A_2},$$

agreeing with $\gamma t_{\mathbf{B}A_1 \otimes \mathbf{B}A_2}$. We must check that this holds on all of $\mathbf{B}A_1 \otimes \mathbf{B}A_2$.

We begin with $\mathbf{B}A_1 \otimes k$, on which ∇ restricts to an isomorphism $\mathbf{B}A_1 \otimes k \longrightarrow \mathbf{B}(A_1 \otimes k)$. The image of $\mathbf{B}_1(A_1 \otimes k)$ under $(i \otimes i)t^{A_1 \otimes A_2}$ lies in $\mathbf{\Omega B}A_1 \otimes k$, so one can write $(i \otimes i)t^{A_1 \otimes A_2}\nabla = it^{A_1} \otimes \eta \colon \mathbf{B}A_1 \otimes k \longrightarrow \mathbf{\Omega B}A_1 \otimes k$. The homotopy h from Example 2.5 vanishes on k (it lies in the submodule S_0, on which h is defined to vanish [Mun74, p. 17]), so one has $h_\psi = h \otimes \mathrm{id} + i\varepsilon \otimes h = h \otimes \mathrm{id}$ on $\mathbf{\Omega B}A_1 \otimes k$, and from the recursive definition $t_\psi|_{B_{\geqslant 2}(A_1 \otimes k)} = h_\psi(t_\psi \smile t_\psi)$ one sees that this process is effectively the same as that defining the cochain $t^{i_{A_1}} = t_{\mathbf{B}A_1}$ of Example 2.5, but with added inert "1" tensor factors. Thus t_ψ agrees with $t_{\mathbf{B}A} \otimes \eta_{\mathbf{\Omega B}A}$ on $\mathbf{B}A \otimes k$. The proof for $k \otimes \mathbf{B}A_2$ is symmetric, noting that $i_{A_2}\varepsilon_{A_2}(1) = 1 \in \mathbf{\Omega B}A_1$.

It remains to see $t_\psi \nabla$ vanishes on $\mathbf{B}_{\geqslant 1}A_1 \otimes \mathbf{B}_{\geqslant 1}A_2$. Let us start small by letting $b_1 \otimes b_2 \in \mathbf{B}_1 A_1 \otimes \mathbf{B}_2 A_2$ be given. We are to evaluate

$$t_\psi \nabla(b_1 \otimes b_2) = (h \otimes \mathrm{id} + i\varepsilon \otimes h)\mu_{\mathbf{\Omega B}A_1 \otimes \mathbf{\Omega B}A_2}(t_\psi \otimes t_\psi)\Delta\nabla(b_1 \otimes b_2).$$

Note that $\Delta_{\mathbf{B}(A_1 \otimes A_2)}\nabla = (\nabla \otimes \nabla)\Delta_{\mathbf{B}A_1 \otimes \mathbf{B}A_2}$. By the single computational fact Lemma 2.2 we have allowed ourselves about the comultiplication on a bar construction, we know $(\nabla \otimes \nabla)\Delta(b_1 \otimes b_2)$ is the sum of $\nabla(b_1 \otimes 1) \otimes \nabla(1 \otimes b_2)$ and $(-1)^{|b_1||b_2|}\nabla(1 \otimes b_2) \otimes \nabla(b_1 \otimes 1)$. Now t_ψ is defined to be $(i \otimes i)t^{A_1 \otimes A_2}$ on $\mathbf{B}_1(A_1 \otimes k)$ and $\mathbf{B}_1(k \otimes A_2)$, which respectively contain $\nabla(b_1 \otimes 1)$ and $\nabla(1 \otimes b_2)$, and the image of $i \colon A_j \longrightarrow \mathbf{\Omega B}A_j$ lies in the kernel of h (again S_0 [Mun74, p. 17]), so

$$\begin{aligned}
(h \otimes \mathrm{id} + i\varepsilon \otimes h)\mu\big((i \otimes i)t^{A_1 \otimes A_2}\big)^{\otimes 2}\big(\nabla(b_1 \otimes 1) \otimes \nabla(1 \otimes b_2)\big) &= (h \otimes \mathrm{id} + i\varepsilon \otimes h)\mu\big((ib_1 \otimes i1) \otimes (i1 \otimes ib_2)\big) \\
&= (h \otimes \mathrm{id} + i\varepsilon \otimes h)(ib_1 \otimes ib_2) \\
&= hib_1 \otimes ib_2 + i\varepsilon ib_1 \otimes hib_2 = 0,
\end{aligned}$$

and similarly for the other summand.

Now let $b_1 \otimes b_2$ lie in $\mathbf{B}_m A_1 \otimes \mathbf{B}_n A_2$ and suppose inductively that we know t_ψ vanishes on $\mathbf{B}_p A_1 \otimes \mathbf{B}_q A_2$ for pairs $(p, q) \neq (m, n)$ with $1 \leqslant p, q$ and $p \leqslant m$ and $q \leqslant n$. We know $t_\psi \nabla = h_\psi \mu(t_\psi \nabla \otimes t_\psi \nabla)\Delta$, where t_ψ vanishes on $\mathbf{B}_0(A_1 \otimes A_2)$ and by Lemma 1.1, Δ sends $\mathbf{B}_m A_1 \otimes \mathbf{B}_n A_2$ to the sum of terms in $(\mathbf{B}_p A_1 \otimes \mathbf{B}_q A_2) \otimes (\mathbf{B}_{p'} A_1 \otimes \mathbf{B}_{q'} A_2)$ with $p + p' = m$ and $q + q' = n$. By the induction hypothesis, the only terms of $\Delta(b_1 \otimes b_2)$ not necessarily annihilated by $h_\psi \mu(t_\psi \nabla \otimes t_\psi \nabla)$ are those lying in $(\mathbf{B}_m A_1 \otimes k) \otimes (k \otimes \mathbf{B}_n A_2)$ and $(k \otimes \mathbf{B}_n A_2) \otimes (\mathbf{B}_m A_1 \otimes k)$, to wit, $b_1 \otimes 1 \otimes 1 \otimes b_2$

and $(-1)^{|b_1||b_2|} 1 \otimes b_2 \otimes b_1 \otimes 1$. But by Lemma 2.2 and the fact t_ψ is of degree 1, we have

$$\begin{aligned}
\mu(t_\psi \otimes t_\psi)\Delta\nabla(b_1 \otimes b_2) &= \mu(t_\psi \otimes t_\psi)\big(\nabla(b_1 \otimes 1) \otimes \nabla(1 \otimes b_2) + (-1)^{|b_1||b_2|}\nabla(1 \otimes b_2) \otimes \nabla(b_1 \otimes 1)\big) \\
&= \mu\Big((-1)^{|b_1|}(t_{\mathbf{B}A_1} b_1 \otimes 1) \otimes (1 \otimes t_{\mathbf{B}A_2} b_2) \\
&\qquad\qquad + (-1)^{(|b_1|+1)|b_2|}(1 \otimes t_{\mathbf{B}A_2} b_2) \otimes (t_{\mathbf{B}A_1} b_1 \otimes 1)\Big) \\
&= \big((-1)^{|b_1|} + (-1)^{(|b_1|+1)|b_2| \,+\, (|b_1|+1)(|b_2|+1)}\big)\, \mu\big((t_{\mathbf{B}A_1} b_1 \otimes 1) \otimes (1 \otimes t_{\mathbf{B}A_2} b_2)\big) \\
&= 0.
\end{aligned}$$

□

It will frequently be the case—in fact, it is one of the central planks of Munkholm's argument—that even when there is no direct relation between DGAs A and B with isomorphic cohomology, there may nevertheless be a DGC map $\mathbf{B}A \longrightarrow \mathbf{B}B$ inducing a quasi-isomorphism $\mathbf{\Omega}\mathbf{B}A \longrightarrow \mathbf{\Omega}\mathbf{B}B$. A DGC map $\mathbf{B}A \longrightarrow \mathbf{B}B$ (inducing a quasi-isomorphism or not) is called an ***A_∞-map*** from A to B. The natural transformation ψ allows us to take tensor products of A_∞-maps.

Definition 2.9 ([Mun74, Prop. 3.3]). Let A_1, A_2, B_1, B_2 be DGAs and $g_j \colon \mathbf{B}A_j \longrightarrow \mathbf{B}B_j$ be DGC maps for $j \in \{1, 2\}$. Then we define the ***internal tensor product*** $g_1 \underline{\otimes} g_2 \colon \mathbf{B}(A_1 \otimes A_2) \longrightarrow \mathbf{B}(B_1 \otimes B_2)$ by

$$\mathbf{B}(A_1 \otimes A_2) \xrightarrow{\eta} \mathbf{B}\mathbf{\Omega}\mathbf{B}(A_1 \otimes A_2) \xrightarrow{\mathbf{B}\psi} \mathbf{B}(\mathbf{\Omega}\mathbf{B}A_1 \otimes \mathbf{\Omega}\mathbf{B}A_2) \xrightarrow{\mathbf{B}(\varepsilon\mathbf{\Omega}g_1 \otimes \varepsilon\mathbf{\Omega}g_2)} \mathbf{B}(B_1 \otimes B_2).$$

This construction exhibits only limited functoriality before passing to the homotopy category, but for our purposes the following will be enough.

Lemma 2.10 ([Mun74, Prop. 3.3(ii)]). *Given DGA maps $f_j \colon A_j \longrightarrow B_j$ for $j \in \{0, 1\}$, we have*

$$\mathbf{B}f_1 \underline{\otimes} \mathbf{B}f_2 = \mathbf{B}(f_1 \otimes f_2) \colon \mathbf{B}(A_1 \otimes A_2) \longrightarrow \mathbf{B}(B_1 \otimes B_2).$$

If A'_j and B'_j are further DGAs and $g_j \colon \mathbf{B}A'_j \longrightarrow \mathbf{B}A_j$ and $\ell_j \colon \mathbf{B}B_j \longrightarrow \mathbf{B}B'_j$ DGC maps, then

$$(\ell_1 \underline{\otimes} \ell_2) \circ \mathbf{B}(f_1 \otimes f_2) = (\ell_1 \circ \mathbf{B}f_1) \underline{\otimes} (\ell_2 \circ \mathbf{B}f_2) \quad \text{and} \quad \mathbf{B}(f_1 \otimes f_2) \circ (g_1 \underline{\otimes} g_2) = (\mathbf{B}f_1 \circ g_1) \underline{\otimes} (\mathbf{B}f_2 \circ g_2).$$

Proof. We content ourselves to render these brief diagram chases in equational form.

- For the first display, one has

  $$\mathbf{B}(\varepsilon\mathbf{\Omega}\mathbf{B}f_1 \otimes \varepsilon\mathbf{\Omega}\mathbf{B}f_2) \circ \mathbf{B}\psi \circ \eta = \mathbf{B}(f_1 \otimes f_2) \circ \mathbf{B}\big((\varepsilon \otimes \varepsilon)\psi\big)\eta = \mathbf{B}(f_1 \otimes f_2) \circ \mathbf{B}\varepsilon \circ \eta = \mathbf{B}(f_1 \otimes f_2)$$

  by naturality of ε and by Theorem 2.6.

- For the ℓ-$\mathbf{B}f$ composition, one has

  $$\begin{aligned}
  \mathbf{B}(\varepsilon\mathbf{\Omega}\ell_1 \otimes \varepsilon\mathbf{\Omega}\ell_2) \circ \mathbf{B}\psi \circ \eta \circ \mathbf{B}(f_1 \otimes f_2) &= \mathbf{B}(\varepsilon\mathbf{\Omega}\ell_1 \otimes \varepsilon\mathbf{\Omega}\ell_2) \circ \mathbf{B}\psi \circ \mathbf{B}\mathbf{\Omega}\mathbf{B}(f_1 \otimes f_2) \circ \eta \\
  &= \mathbf{B}(\varepsilon\mathbf{\Omega}\ell_1 \otimes \varepsilon\mathbf{\Omega}\ell_2) \circ \mathbf{B}(\mathbf{\Omega}\mathbf{B}f_1 \otimes \mathbf{\Omega}\mathbf{B}f_2) \circ \mathbf{B}\psi \circ \eta \\
  &= \mathbf{B}\big(\varepsilon\mathbf{\Omega}(\ell_1 \circ \mathbf{B}f_1) \otimes \varepsilon\mathbf{\Omega}(\ell_2 \circ \mathbf{B}f_2)\big) \circ \mathbf{B}\psi \circ \eta.
  \end{aligned}$$

- For the $\mathbf{B}f$-g composition, one begins with $\mathbf{B}\psi \circ \eta$ and follows with

  $$\begin{aligned}
  \mathbf{B}(f_1 \otimes f_2) \circ \mathbf{B}(\varepsilon \otimes \varepsilon) \circ \mathbf{B}(\mathbf{\Omega}g_1 \otimes \mathbf{\Omega}g_2) &= \mathbf{B}(\varepsilon \otimes \varepsilon) \circ \mathbf{B}(\mathbf{\Omega}\mathbf{B}f_1 \otimes \mathbf{\Omega}\mathbf{B}f_2) \circ \mathbf{B}(\mathbf{\Omega}g_1 \otimes \mathbf{\Omega}g_2) \\
  &= \mathbf{B}\big(\varepsilon\mathbf{\Omega}(\mathbf{B}f_1 \circ g_2) \otimes \varepsilon\mathbf{\Omega}(\mathbf{B}f_2 \circ g_2)\big).
  \end{aligned}$$

□

This new tensor product is naturally related to the old.

Lemma 2.11 ([Fra21, Lem. 4.4]). *Let A_j and B_j be* DGAs *for $j \in \{1,2\}$ and $g_j\colon \mathbf{B}A_j \longrightarrow \mathbf{B}B_j$* DGC *maps. Then $\nabla \circ (g_1 \otimes g_2) = (g_1 \underline{\otimes} g_2) \circ \nabla$.*

The existing proof is a computation, but it can also be done diagrammatically.

Proof. By naturality of $\eta\colon \mathrm{id} \longrightarrow \mathbf{B\Omega}$, we have $\eta \circ \nabla = \mathbf{B\Omega}\nabla \circ \eta$, and by Lemma 2.8 that $\mathbf{B}\psi \circ \mathbf{B\Omega}\nabla = \mathbf{B}\gamma$. Moreover η and $\mathbf{B}\gamma$ are natural, $(\varepsilon \otimes \varepsilon) \circ \psi = \varepsilon$ by Theorem 2.6, and $\mathbf{B}\varepsilon \circ \eta = \mathrm{id}$ by the unit–counit identities for the adjunction $\mathbf{\Omega} \dashv \mathbf{B}$, so the diagram

$$\begin{tikzcd}
\mathbf{B}A_1 \otimes \mathbf{B}A_2 \arrow[r, "\eta"'] \arrow[rr, bend left, "\eta \circ \nabla"] \arrow[d, "g_1 \otimes g_2"'] & \mathbf{B\Omega}(\mathbf{B}A_1 \otimes \mathbf{B}A_2) \arrow[r, "\mathbf{B\Omega}\nabla"] \arrow[rr, bend right, "\mathbf{B}\gamma"'] \arrow[d, "\mathbf{B\Omega}(g_1 \otimes g_2)"'] & \mathbf{B\Omega B}(A_1 \otimes A_2) \arrow[r, "\mathbf{B}\psi"] & \mathbf{B}(\mathbf{\Omega B}A_1 \otimes \mathbf{\Omega B}A_2) \arrow[d, "\mathbf{B}(\mathbf{\Omega}g_1 \otimes \mathbf{\Omega}g_2)"] \\
\mathbf{B}B_1 \otimes \mathbf{B}B_2 \arrow[r, "\eta"'] \arrow[dr, "\nabla"'] & \mathbf{B\Omega}(\mathbf{B}B_1 \otimes \mathbf{B}B_2) \arrow[r, "\mathbf{B\Omega}\nabla"'] \arrow[rr, bend left, "\mathbf{B}\gamma"] & \mathbf{B\Omega B}(B_1 \otimes B_2) \arrow[r, "\mathbf{B}\psi"'] \arrow[dr, "\mathbf{B}\varepsilon"] & \mathbf{B}(\mathbf{\Omega B}B_1 \otimes \mathbf{\Omega B}B_2) \arrow[d, "\mathbf{B}(\varepsilon \otimes \varepsilon)"] \\
& \mathbf{B}(B_1 \otimes B_2) \arrow[ur, "\eta"'] \arrow[rr, equal] & & \mathbf{B}(B_1 \otimes B_2)
\end{tikzcd}$$

commutes. The composite along the lower left is $\nabla \circ (g_1 \otimes g_2)$ and the composite along the upper right is $(g_1 \underline{\otimes} g_2) \circ \nabla$. □

We will need one more relation between the counit ε and the internal tensor product.

Lemma 2.12 ([Mun74, p. 49, top]). *Let A_j and B_j be* DGAs *and $g_j\colon \mathbf{B}A_j \longrightarrow \mathbf{B}B_j$ be* DGC *maps for $j \in \{1,2\}$. Then one has*

$$\varepsilon \circ \mathbf{\Omega}(g_1 \underline{\otimes} g_2) = (\varepsilon \otimes \varepsilon) \circ (\mathbf{\Omega}g_1 \otimes \mathbf{\Omega}g_2) \circ \psi\colon \mathbf{\Omega B}(A_1 \otimes A_2) \longrightarrow B_1 \otimes B_2.$$

Proof. We chase a commutative diagram.

$$\begin{tikzcd}
\mathbf{\Omega B}(A_1 \otimes A_2) \arrow[r, "\psi"] \arrow[d, "\mathbf{\Omega}\eta"'] & \mathbf{\Omega B}A_1 \otimes \mathbf{\Omega B}A_2 \arrow[r, "\mathbf{\Omega}g_1 \otimes \mathbf{\Omega}g_2"] & \mathbf{\Omega B}B_1 \otimes \mathbf{\Omega B}B_2 \arrow[r, "\varepsilon \otimes \varepsilon"] & B_1 \otimes B_2 \\
\mathbf{\Omega B\Omega B}(A_1 \otimes A_2) \arrow[r, "\mathbf{\Omega B}\psi"'] & \mathbf{\Omega B}(\mathbf{\Omega B}A_1 \otimes \mathbf{\Omega B}A_2) \arrow[u, "\varepsilon"'] \arrow[r, "\mathbf{\Omega B}(\mathbf{\Omega}g_1 \otimes \mathbf{\Omega}g_2)"'] & \mathbf{\Omega B}(\mathbf{\Omega B}B_1 \otimes \mathbf{\Omega B}B_2) \arrow[u, "\varepsilon"'] \arrow[r, "\mathbf{\Omega B}(\varepsilon \otimes \varepsilon)"'] & \mathbf{\Omega B}(B_1 \otimes B_2) \arrow[u, "\varepsilon"']
\end{tikzcd}$$

The composition along the top is the right-hand side of the display and the composition along the bottom is the left-hand side, by Definition 2.9. The left square commutes since $\varepsilon \circ \mathbf{\Omega B}\psi \circ \mathbf{\Omega}\eta = \psi \circ \varepsilon \circ \mathbf{\Omega}\eta = \psi$ by naturality of ε and the unit–counit identities for the adjunction $\mathbf{\Omega} \dashv \mathbf{B}$, and the other two squares commute by naturality of ε. □

3. Manipulation of homotopies

In this section we define the relevant notions of homotopy and discuss how to package homotopies into representing path (and path-allied) objects.

Definition 3.0.1 ([Mun74, §1.11][Mun78, §4.1])**.** Given two DGA maps $f_0, f_1\colon A' \longrightarrow A$, a DGA ***homotopy*** $f_0 \simeq f_1$ is a k-linear map $h\colon A' \longrightarrow A$ of degree -1 such that

$$\varepsilon_A h = 0, \qquad h\eta_{A'} = 0,^{4} \qquad d(h) = f_0 - f_1, \qquad h\mu_{A'} = \mu_A(f_0 \otimes h + h \otimes f_1)\ .$$

Given two twisting cochains $t_0, t_1\colon C \longrightarrow A$, a ***twisting cochain homotopy*** $t_0 \simeq t_1$ is a k-linear map $x\colon C \longrightarrow A$ of degree 0 such that

$$\varepsilon_A x = \varepsilon_C, \qquad x\eta_A = \eta_C, \qquad d(x) = t_0 \smile x - x \smile t_1.$$

Given two DGC maps $g_0, g_1\colon C \longrightarrow C'$, a DGC ***homotopy*** $g_0 \simeq g_1$ is a k-linear map $j\colon C \longrightarrow C'$ of degree -1 such that

$$\varepsilon_{C'} j = 0, \qquad j\eta_C = 0, \qquad d(j) = g_1 - g_0, \qquad \Delta_{C'} j = (g_0 \otimes j + j \otimes g_1)\Delta_C.$$

These three notions evidently each compose well with maps in the appropriate categories.

Lemma 3.0.2. *Given a homotopy $h\colon f_0 \simeq f_1\colon A' \longrightarrow A$ of DGA maps, further DGA maps $f'\colon A'' \longrightarrow A'$ and $\underline{f}\colon A \longrightarrow \underline{A}$ yield further homotopies*

$$hf'\colon f_0 f' \simeq f_1 f', \qquad \underline{f}h\colon \underline{f}f_0 \simeq \underline{f}f_1, \qquad \underline{f}hf'\colon \underline{f}f_0 f' \simeq \underline{f}f_1 f'.$$

Given a homotopy $x\colon t_0 \simeq t_1\colon C \longrightarrow A$ of twisting cochains, a DGC map $\underline{g}\colon \underline{C} \longrightarrow C$ and a DGA map $\underline{f}\colon A \longrightarrow \underline{A}$ yield further homotopies

$$x\underline{g}\colon t_0\underline{g} \simeq t_1\underline{g}, \qquad \underline{f}x\colon \underline{f}t_0 \simeq \underline{f}t_1, \qquad \underline{f}x\underline{g}\colon \underline{f}t_0\underline{g} \simeq \underline{f}t_1\underline{g}.$$

Given a homotopy $j\colon g_0 \simeq g_1\colon C \longrightarrow C'$ of DGC maps, further DGC maps $\underline{g}\colon \underline{C} \longrightarrow C$ and $g'\colon C' \longrightarrow C''$ yield further homotopies

$$j\underline{g}\colon g_0\underline{g} \simeq g_1\underline{g}, \qquad g'j\colon g'g_0 \simeq g'g_1, \qquad g'j\underline{g}\colon g'g_0\underline{g} \simeq g'g_1\underline{g}.$$

Moreover, the three notions are interchangeable under the adjunctions.

Lemma 3.0.3 ([Mun74, §1.11; Thm. 5.4, pf.])**.** *Suppose given a DGC C and a DGA A. Then there are bijections of homotopies of maps*

$$\begin{array}{c} \boldsymbol{\Omega} C \longrightarrow A \\ \hline C \longrightarrow A \\ \hline C \longrightarrow \mathbf{B}A \end{array} \tag{3.0.4}$$

[4] In the definition from our main source [Mun74], the unit and counit conditions are omitted; in later work dealing more specifically with DGA as a category, Munkholm includes them [Mun78, 4.1]. These are actually critical for the adjunction to preserve homotopy and hence later to our verification of the path object.

- *Given a* DGA *homotopy $h\colon f_0 \simeq f_1\colon \mathbf{\Omega}C \longrightarrow A$, the map $\eta_A\varepsilon_C + ht_C$ is the unique homotopy of twisting cochains $f_0t_C \simeq f_1t_C\colon C \longrightarrow A$, and a homotopy of twisting cochains $C \longrightarrow A$ arises from a unique* DGA *homotopy.*

- *Given a* DGC *homotopy $j\colon g_0 \simeq g_1\colon C \longrightarrow \mathbf{B}A$, the map $\eta_A\varepsilon_C + t^A j$ is the unique homotopy of twisting cochains $t^A g_0 \simeq t^A g_1\colon C \longrightarrow A$, and a homotopy of twisting cochains $C \longrightarrow A$ arises from a unique* DGC *homotopy.*

The adjoint functors $\mathsf{DGC}\colon \mathbf{\Omega} \dashv \mathbf{B} : \mathsf{DGA}$ *thus preserve the relation of homotopy, as follows.*

$$
\begin{array}{c}
A' \longrightarrow A \\ \hline
\mathbf{B}A' \longrightarrow A \\ \hline
\mathbf{B}A' \longrightarrow \mathbf{B}A
\end{array}
\qquad\qquad
\begin{array}{c}
C \longrightarrow C' \\ \hline
C \longrightarrow \mathbf{\Omega}C' \\ \hline
\mathbf{\Omega}C \longrightarrow \mathbf{\Omega}C'
\end{array}
$$

- *The first rule, given a homotopy $h\colon f_0 \simeq f_1\colon A' \longrightarrow A$ of* DGA *maps, produces the homotopy $\eta_A\varepsilon_{\mathbf{B}A'} + ht^{A'}$ of twisting cochains $t^A\,\mathbf{B}f_0 \simeq t^A\,\mathbf{B}f_1\colon \mathbf{B}A' \longrightarrow A$ and follows with the second bijection of (3.0.4) to produce a homotopy $\mathbf{B}f_0 \simeq \mathbf{B}f_1$.*

- *The second rule, given a homotopy $j\colon g_0 \simeq g_1\colon C \longrightarrow C'$ of* DGC *maps, produces the homotopy $\eta_{\mathbf{\Omega}C'}\varepsilon_C + t_{C'}j$ of twisting cochains $\mathbf{\Omega}g_0\,t_C \simeq \mathbf{\Omega}g_1\,t_C\colon C \longrightarrow \mathbf{\Omega}C'$ and follows with the first bijection of (3.0.4) to produce a homotopy $\mathbf{\Omega}g_0 \simeq \mathbf{\Omega}g_1$.*

Twisting cochain homotopies, despite being maps between different types of objects, are in a way more flexible than DGA or DGC homotopies, because they are composable.

Lemma 3.0.5 ([Mun74, §1.12])**.** *Let a* DGC *C and a* DGA *A be given. A homotopy $h_{0,1}\colon t_0 \simeq t_1\colon C \longrightarrow A$ of twisting cochains admits a two-sided cup-inverse $h_{0,1}^{\smile -1}$ which is a homotopy $t_1 \simeq t_0$. Given another homotopy $h_{-1,0}\colon t_{-1} \simeq t_0\colon C \longrightarrow A$, the cup product $h_{-1,0} \smile h_{0,1}$ is a homotopy $t_{-1} \simeq t_1$.*

Remark 3.0.6. A suggestive phrasing is that the twisting cochains in $\mathsf{Mod}_0(C, A)$ are the objects of a groupoid whose morphisms are the homotopies. Particularly, homotopy is an equivalence relation. The same then holds for the equivalent hom-sets $\mathsf{DGA}(\mathbf{\Omega}C, A) \longleftrightarrow \mathsf{DGC}(C, \mathbf{B}A)$, which are thus privileged over generic hom-sets $\mathsf{DGA}(A', A)$ or $\mathsf{DGC}(C, C')$, which lack this property. Note that cocompleteness is critical for the existence of inverses.

We can use Lemma 3.0.3 to exchange DGA homotopies $f_{-1} \simeq f_0 \simeq f_1$ we wish to concatenate for twisting cochains, take the cup product of these as in Lemma 3.0.5, and then move the resulting composite homotopy back to DGA using Lemma 3.0.3 to get a homotopy $f_{-1} \simeq f_1$. The next subsections attempt to describe this process wholly internally to DGA.

It is well known that the data of a homotopy $j\colon g_0 \simeq g_1\colon C \longrightarrow C'$ of maps of chain complexes can be realized as single map $C \otimes I \longrightarrow C'$, where I is the complex $k\{u_{[0,1]}\} \to k\{u_{[0]}, u_{[1]}\}$ of nondegenerate chains in the standard simplicial structure on the interval $[0, 1]$. Moreover, I admits a natural DGC structure so that homotopies of unaugmented DGCs can be realized in the same way. Dually as pointed out by Munkholm [Mun74, Thm. 5.4, pf.], the algebra of normalized cochains on the simplicial interval, with the cup product, defines a DGA $I^* = k\{v_0, v_1, e\}$ such that the data of a DGA homotopy $h\colon f_0 \simeq f_1\colon A' \longrightarrow A$ can be realized by a DGA map

$$h^P\colon A' \longrightarrow I^* \otimes A. \tag{3.0.7}$$

Since I^* has trivial cohomology $H^*(I^*) = H^0(I^*) \cong k$ generated by the class of $v_0 + v_1$, the projections

$$\pi_j\colon I^* \otimes A \longrightarrow\!\!\!\!\rightarrow k\{v_j\} \otimes A \xrightarrow{\sim} A$$

are quasi-isomorphisms such that $\pi_j \circ h^P = f_j$. Thus tensoring with I^* functorially yields what a naive path object for DGAs. Because these π_j are not augmentation-preserving and we need our DGA maps to be augmented in order to be able to apply the natural transformation ψ of Theorem 2.6 and use the $\mathbf{\Omega} \dashv \mathbf{B}$ adjunction, we replace $I^* \otimes A$ with the subalgebra PA whose unit is the isomorphism $k \longrightarrow k\{v_0 + v_1\} \otimes \operatorname{im} \eta_A$ and whose augmentation ideal to be $I^* \otimes \overline{A}$. This subalgebra is quasi-isomorphic to $I^* \otimes A$ via the inclusion. The condition $\varepsilon h = 0$ on homotopies h and the unitality condition $f_j(1) = 1$ on DGA homomorphisms guarantee the images of maps $h^P\colon A' \longrightarrow I^* \otimes A$ of (3.0.7) lie in PA.

Definition 3.0.8. Given a DGA A, we refer to this PA, equipped with the projections $\pi_0, \pi_1\colon PA \longrightarrow A$ restricted from those of $I^* \otimes A$, as the ***standard path object*** of A. Given a homotopy $h\colon f_0 \simeq f_1\colon A' \longrightarrow A$ of DGA maps, we refer to the associated DGA map $h^P\colon A' \longrightarrow PA$ of (3.0.7) as a ***right homotopy*** and the composites $\pi_j h^P = f_j$ as its ***endpoint maps***. We write $\zeta\colon A \longrightarrow PA$ for unital map defined by $a \longmapsto (v_0 + v_1) \otimes a$ on $\overline{A}$.

4. Maps on Tor

Absent topological considerations, this spectral sequence converging [ML75, XI.3.2] to differential Tor of a triple of DGAs exists more generally as a spectral sequence of Künneth type, the so-called *algebraic* EMSS, functorial in all three variables.

Lemma 4.1 ([GM74, Cor. 1.8][Mun74, Theorem 5.4])**.** *Given a DGA map $f\colon R' \longrightarrow R$, a right R'-module M', a left R'-module N', a right R-module M, a left R-module N, and DG module maps $u\colon M' \longrightarrow M$ and $v\colon N' \longrightarrow N$ making the expected squares*

$$\begin{tikzcd} M' \arrow[d] & M' \otimes R' \arrow[l] \arrow[d] \\ M & M \otimes R \arrow[l] \end{tikzcd} \qquad \begin{tikzcd} R' \otimes N' \arrow[r] \arrow[d] & N' \arrow[d] \\ R \otimes N \arrow[r] & N \end{tikzcd} \tag{4.2}$$

commute, there is induced a map of algebraic EMSSs from that of (M', R', N') to that of (M, R, N), converging to the functorial map

$$\operatorname{Tor}_f(u, v)\colon \operatorname{Tor}_{R'}(M', N') \longrightarrow \operatorname{Tor}_R(M, N)$$

of graded modules. Moreover, if the maps f, u, v are quasi-isomorphisms, then the map of spectral sequences is an isomorphism from the E_2 page on and $\operatorname{Tor}_f(u, v)$ is an isomorphism.

We will only use these considerations in the special case that M and N are DGAs and the R-module structure maps are induced by DGA homomorphisms $M \leftarrow R \rightarrow N$, so that we have the condensed compatibility diagram

$$\begin{tikzcd} M' \arrow[d, "u"'] & R' \arrow[l, "\phi_{M'}"'] \arrow[r, "\phi_{N'}"] \arrow[d, "f"] & N' \arrow[d, "v"] \\ M & R \arrow[l, "\phi_M"] \arrow[r, "\phi_N"'] & N \end{tikzcd} \tag{4.3}$$

of DGA maps. In later sections, where we will produce diagrams of Tors comprising mainly isomorphisms, we will ceaselessly apply this result. We will also need to expand the notion of a map of Tors to include squares which commute only up to homotopy, which accounts for our fixation on representing homotopies by DGA maps in Section 3.

Lemma 4.4 ([Mun74, Thm. 5.4]). *Let DGAs and DGA maps as in (4.3) be given such that the squares commute up to homotopy $h_M\colon u \circ \phi_{M'} \simeq \phi_M \circ f$ and $h_N\colon v \circ \phi_{N'} \simeq \phi_N \circ f$. Then there is induced a map*

$$\mathrm{Tor}_f(u, v; h_M, h_N)\colon \ \mathrm{Tor}_{R'}(M', N') \longrightarrow \mathrm{Tor}_R(M, N)$$

of graded modules which is a quasi-isomorphism if each of u, f, and v is.

Proof. Letting $h_M^P\colon R \longrightarrow PM'$ and $h_N^P\colon R \longrightarrow PN'$ be the respective DGA representatives, as described in Definition 3.0.8, for the homotopies h_M, h_N, the following equivalent diagrams commute by definition:

$$\begin{tikzcd} M' \arrow[d, "u"'] & R' \arrow[l, "\phi_{M'}"'] \arrow[r, "\phi_{N'}"] \arrow[d, equal] & N' \arrow[d, "v"] \\ M & R' \arrow[l] \arrow[r] \arrow[d, equal] & N \\ PM \arrow[u, "\pi_0"] \arrow[d, "\pi_1"'] & R' \arrow[l, "h_M^P"'] \arrow[r, "h_N^P"] \arrow[u, equal] \arrow[d, equal] & PN \arrow[u, "\pi_0"'] \arrow[d, "\pi_1"] \\ M \arrow[d, equal] & R' \arrow[l] \arrow[r] \arrow[d, "f"] & N \arrow[d, equal] \\ M & R \arrow[l, "\phi_M"] \arrow[r, "\phi_N"'] & N; \end{tikzcd} \qquad \begin{tikzcd} M' \arrow[d, "u"'] & R' \arrow[l, "\phi_{M'}"'] \arrow[r, "\phi_{N'}"] \arrow[d, equal] & N' \arrow[d, "v"] \\ M & R' \arrow[l] \arrow[r] & N \\ PM \arrow[u, "\pi_0"] \arrow[d, "\pi_1"'] & R' \arrow[l, "h_M^P"'] \arrow[r, "h_N^P"] \arrow[u, equal] \arrow[d, "f"] & PN \arrow[u, "\pi_0"'] \arrow[d, "\pi_1"] \\ M & R \arrow[l, "\phi_M"] \arrow[r, "\phi_N"'] & N. \end{tikzcd} \tag{4.5}$$

Since the π_j are quasi-isomorphisms, three applications of Lemma 4.1 let us set

$$\mathrm{Tor}_f(u, v; h_M, h_N) := \mathrm{Tor}_f(\pi_1, \pi_1) \circ \mathrm{Tor}_{\mathrm{id}}(\pi_0, \pi_0)^{-1} \circ \mathrm{Tor}_{\mathrm{id}}(u, v). \qquad \square$$

To make proofs to come more legible, we introduce a hopefully intuitive abbreviation convention.

Notation 4.6. Given DGA maps $X \leftarrow A \rightarrow Y$, functors $F, G, F', G'\colon \mathrm{DGA} \longrightarrow \mathrm{DGA}$, and natural transformations $F \longrightarrow G$, $F' \longrightarrow G$, $\phi\colon F \longrightarrow F'$, and $\psi\colon G \longrightarrow G'$ such that the two compositions $F \longrightarrow G'$ are equal, we make the abbreviations

$$\mathrm{Tor}_{FA} := \mathrm{Tor}_{FA}(FX, FY), \qquad \mathrm{Tor}_{FA}(GX) := \mathrm{Tor}_{FA}(GX, GY),$$

$$\mathrm{Tor}_\phi := \mathrm{Tor}_\phi(\phi,\phi)\colon \mathrm{Tor}_{FA} \longrightarrow \mathrm{Tor}_{F'A}, \qquad \mathrm{Tor}_\phi(\psi) := \mathrm{Tor}_\phi(\psi,\psi)\colon \mathrm{Tor}_{FA}(GX) \longrightarrow \mathrm{Tor}_{F'A}(G'X).$$

We will henceforth frequently employ this notation, usually without comment.

5. SHC-algebras

A commutative DGA A is one for which the multiplication $\mu\colon A \otimes A \longrightarrow A$ is itself a DGA homomorphism. Cohomology rings are of this sort, and a large part of why homotopy theory is so much more tractable over a field k of characteristic 0 is that there are functorial CDGA models for cochains. For other characteristics this is not the case [Bor51, Thm. 7.1], but we can weaken the requirement by asking only that μ extend to an A_∞-algebra map. Munkholm's product is defined in terms of such a structure, as first considered by Stasheff and Halperin.

To make sense of the following definition, recall from Lemma 1.1 that the canonical twisting cochain $t^A\colon \mathbf{B}A \longrightarrow A$ of a DGA A factors through a DG direct summand $\mathbf{B}_1 A$ on which it restricts to a cochain isomorphism of degree 1, whose inverse we dubbed $s_A^{-1}\colon \overline{A} \longrightarrow \mathbf{B}A$ in the sketch construction of Theorem 2.6. We call this map s^{-1} the ***desuspension***.

Definition 5.1 (Stasheff–Halperin [SH70, Def. 8])**.** We refer to a DGA A equipped with a DGC map $\Phi_A\colon \mathbf{B}(A\otimes A) \longrightarrow \mathbf{B}A$ such that the composition $t_A \circ \Phi \circ s^{-1}_{A\otimes A}\colon \overline{A\otimes A} \longrightarrow \overline{A}$ is the multiplication $\mu_A\colon A\otimes A \longrightarrow A$ as a ***weakly homotopy commutative*** (WHC-)***algebra***.[5] Given two WHC-algebras A and Z, a WHC-***algebra map*** from A to Z is a DGC map $g\colon \mathbf{B}A \longrightarrow \mathbf{B}Z$ such that there exists a DGC homotopy between the two paths around the square

$$\begin{tikzcd} \mathbf{B}(A\otimes A) \arrow[r, "\Phi_A"] \arrow[d, "g \,\underline{\otimes}\, g"'] & \mathbf{B}A \arrow[d, "g"] \\ \mathbf{B}(Z\otimes Z) \arrow[r, "\Phi_Z"'] & \mathbf{B}Z. \end{tikzcd} \tag{5.2}$$

The particular homotopy is not prescribed as part of the data of a WHC-algebra map. A WHC-algebra is called a SHC-algebra if additionally Φ_A satisfies three axioms ensuring unitality, commutativity, and associativity up to homotopy; we need not spell them out for our purposes in this work.

Example 5.3. If A is a CDGA, then the morphism $\Phi = \mathbf{B}\mu_A\colon \mathbf{B}(A\otimes A) \longrightarrow \mathbf{B}A$ makes A an SHC-algebra. The cohomology ring $H^*(X;)$ of a simplicial set is of this type, and will always come considered with this SHC-algebra structure. If $\rho\colon A \longrightarrow B$ is a map of CDGAs, then $\mathbf{B}\rho$ is an SHC-algebra map.

Example 5.4. If A is an SHC-algebra, then the homotopy-commutativity of Φ implies μ and $\mu\circ\chi$ are homotopic cochain maps, for if $i\colon \mathrm{id} \longrightarrow \mathbf{\Omega B}$ is the DG-module section of ε from Example 2.5, then $\varepsilon \circ \mathbf{\Omega}(\Phi \circ \mathbf{B}\chi) \circ i = \mu\circ\chi$ and $\varepsilon \circ \mathbf{\Omega}\Phi \circ i = \mu$, and so $H^*(A)$ is a CDGA (with trivial differential). Thus $\Phi = \mathbf{B}\mu_{H^*(A)}$ gives an SHC-algebra structure on $H^*(A)$ by Example 5.3.

[5] Stasheff–Halperin call this a *strongly homotopy commutative algebra* structure, but we follow Munkholm in reusing the term for a stronger notion.

Theorem 5.5 ([Mun74, Prop. 4.7]). *Let X be a simplicial set and k any ring. Then the normalized cochain algebra $C^*(X) = C^*(X;k)$ admits an* SHC*-algebra structure $\Phi_{C^*(X)}$, and this structure is* strictly natural *in the sense that given a map $f\colon Y \longrightarrow X$ of simplicial sets, the induced* DGC *map $\mathbf{B}C^*(f)\colon \mathbf{B}C^*(X) \longrightarrow \mathbf{B}C^*(Y)$ renders the square (7.1) commutative on the nose.*

This natural SHC structure on cochains is a reinterpretation of the classical Eilenberg–Zilber theorem; it is only verifying the homotopy-associativity axiom that requires substantial additional work. The class of known SHC-algebras has recently expanded significantly.

Theorem 5.6 (Franz [Fra20]). *A homotopy Gerstenhaber algebra A admits a strictly natural* WHC*-algebra structure Φ_A satifsying unitality and associativity axioms. If A is an* extended *homotopy Gerstenhaber algebra, then Φ_A also satisfies a commutativity axiom.*

Remark 5.7. A homotopy Gerstenhaber algebra is a module over the E_2-operad $F_2\mathscr{X}$, a filtrand of the so-called *surjection operad* of interval-cut operations on cochains and a quotient of the second filtrand $F_2\mathscr{E}$ of the DG-operad $\mathscr{E}$ associated to the classical Barratt–Eccles simplicial operad [MS03, BF04]. Similarly, an extended homotopy Gerstenhaber algebra is a module over a suboperad of the E_3-operad $F_3\mathscr{X}$.

6. The products on cohomology and cochains

To motivate Munkholm's product, it is easiest to first follow him in interpreting the classical products on $\operatorname{Tor}_{C^*B}(C^*X, C^*E)$ and $\operatorname{Tor}_{H^*B}(H^*X, H^*E)$ in terms of the canonical SHC-algebra structures.

The latter is the easier, so we start there. Given DGAS R_0, R_1 and right and left DG R_i-modules M_i and N_i respectively, there is a classically defined exterior product [CE56, p. 206]

$$\operatorname{Tor}_{R_0}(M_0,N_0)\otimes\operatorname{Tor}_{R_1}(M_1,N_1) \longrightarrow \operatorname{Tor}_{R_0\otimes R_1}(M_0\otimes M_1, N_0\otimes N_1),$$

functorial in all six variables in the sense that given similarly defined R_i', M_i', N_i' such that the squares (4.2) commute, then so does the square

$$\begin{tikzcd}
\operatorname{Tor}_R(M,N)\otimes\operatorname{Tor}_R(M,N) \arrow[r] \arrow[d] & \operatorname{Tor}_{R\otimes R}(M\otimes M, N\otimes N) \arrow[d] \\
\operatorname{Tor}_{R'}(M',N')\otimes\operatorname{Tor}_{R'}(M',N') \arrow[r] & \operatorname{Tor}_{R'\otimes R'}(M'\otimes M', N'\otimes N'),
\end{tikzcd}$$

and given further R_i'', M_i'', N_i'', such squares glue. If $R = R_0 = R_1$ is a *commutative* DGA, then $\mu\colon R' = R\otimes R \longrightarrow R$ is a DGA map, and if $M = M_0 = M_1$ and $N = N_0 = N_1$ are themselves DGAS, then $\mu\colon M' = M\otimes M \longrightarrow M$ and $\mu\colon N' = N\otimes N \longrightarrow N$ make the diagrams (4.2) commute, so we may follow the external product with the map

$$\operatorname{Tor}_\mu(\mu,\mu)\colon\ \operatorname{Tor}_{R\otimes R}(M\otimes M, N\otimes N) \longrightarrow \operatorname{Tor}_R(M,N)$$

to obtain the classical product on Tor. This particularly applies to $R = H^*A,\ M = H^*X,\ N = H^*Y$ for $X \leftarrow A \rightarrow Y$ maps of spaces.

Taking $C^*(X) \leftarrow C^*(B) \rightarrow C^*(E)$ as $X \leftarrow A \rightarrow Y$, instead, one can use the DGA maps

$$C^*(B) \otimes C^*(B) \xrightarrow{\iota} (C_*B \otimes C_*B)^* \xleftarrow{\nabla^*} C^*(B \times B) \xrightarrow{C^*(\Delta)} C^*(B)$$

inducing the cup product to obtain a composite

$$\begin{tikzcd}
\operatorname{Tor}_{C^*B}(C^*X, C^*E) \otimes \operatorname{Tor}_{C^*B}(C^*X, C^*E) \arrow[r, "\text{exterior}"] & \operatorname{Tor}_{C^*B \otimes C^*B}(C^*X \otimes C^*X, C^*E \otimes C^*E) \arrow[d, "\operatorname{Tor}_\iota(\iota,\iota)"'] \\
& \operatorname{Tor}_{(C_*B \otimes C_*B)^*}\big((C_*X \otimes C_*X)^*, (C_*E \otimes C_*E)^*\big) \\
\operatorname{Tor}_{C^*B}\left(C^*X, C^*E\right) & \operatorname{Tor}_{C^*(B \times B)}\big(C^*(X \times X), C^*(E \times E)\big) \arrow[l, "\operatorname{Tor}_{C^*\Delta}(C^*\Delta, C^*\Delta)"'] \arrow[u, "\operatorname{Tor}_{\nabla^*}(\nabla^*,\nabla^*)", "\wr"']
\end{tikzcd}$$

describing a product on Tor. In the situation of the Eilenberg–Moore theorem, this product can be shown to be preserved by the isomorphism with $H^*(X \times_B E)$ [mcclearyspectral][McC01, Corollary 7.18][gugenheimmay][GM74, Cor. 3.5][smith1967emss][Smi67, Prop. 3.4][carlsonfranzlong][ajwMF21, Thm. A.27].[6]

This map is not quite induced by the cup product $C^*(\Delta) \circ a^* \circ i$ because the dual Alexander–Whitney map a^* is not a DGA map, so we are forced to use the dual Eilenberg–Zilber map ∇^* instead and then take inverses. But the situation becomes better once we can use Φ from Theorem 5.5, which is defined as the composite $\mathbf{B}C^*(\Delta) \circ g_{t^a} \circ \mathbf{B}\iota$ for $g_{t^a} \colon \mathbf{B}(C_*X \otimes C_*X)^* \longrightarrow \mathbf{B}C^*(X \times X)$ the DGC map satisfying

$$t^{C^*(X \times X)} \circ g_{t^a} \circ s^{-1} = a^* \colon \ (C_*X \otimes C_*X)^* \longrightarrow C^*(X \times X).$$

But then we have

$$\begin{aligned} a^* = t^{C^*(X \times X)} \circ g_{t^a} \circ s^{-1} &= \varepsilon \circ t_{\mathbf{B}C^*(X \times X)} \circ g_{t^a} \circ s^{-1} \\ &= \varepsilon \circ \mathbf{\Omega} g_{t^a} \circ t_{\mathbf{B}(C_*X \otimes C_*X)^*} \circ s^{-1}, \end{aligned}$$

where $t_{\mathbf{B}(C_*X \otimes C_*X)^*} \circ s^{-1}$ is the section of ε from Example 2.5, so taking cohomology yields $H^*(a^*) = H^*(\varepsilon) \circ H^*(\mathbf{\Omega} g_{t^a}) \circ H^*(\varepsilon)^{-1}$.

Liberal application of Lemma 4.1 and Notation 4.6 allows us, recalling that $H^*(\nabla^*)$ and $H^*(a^*)$ are inverse, to write a commutative diagram

$$\begin{tikzcd}
\operatorname{Tor}_{C^*B \otimes C^*B} \arrow[r, "\operatorname{Tor}_\iota"] & \operatorname{Tor}_{(C_*B \otimes C_*B)^*} & \operatorname{Tor}_{C^*(B \otimes B)} \arrow[l, "\operatorname{Tor}_{\nabla^*}"'] \arrow[r, "\operatorname{Tor}_{C^*(\Delta)}"] & \operatorname{Tor}_{C^*(B)} \\
\operatorname{Tor}_{\mathbf{\Omega B}(C^*B \otimes C^*B)} \arrow[u, "\operatorname{Tor}_\varepsilon", "\wr"'] \arrow[r, "\operatorname{Tor}_{\mathbf{\Omega B}\iota}"'] & \operatorname{Tor}_{\mathbf{\Omega B}(C_*B \otimes C_*B)^*} \arrow[u, "\operatorname{Tor}_\varepsilon", "\wr"'] \arrow[r, "\operatorname{Tor}_{\mathbf{\Omega} g_{t^a}}"'] & \operatorname{Tor}_{\mathbf{\Omega B}C^*(B \otimes B)} \arrow[u, "\wr", "\operatorname{Tor}_\varepsilon"'] \arrow[r, "\operatorname{Tor}_{\mathbf{\Omega B}C^*(\Delta)}"'] & \operatorname{Tor}_{\mathbf{\Omega B}C^*(B)}, \arrow[u, "\wr", "\operatorname{Tor}_\varepsilon"']
\end{tikzcd}$$

where the composition along the bottom is $\operatorname{Tor}_{\mathbf{\Omega}\Phi}$. Precomposing with the external product $\operatorname{Tor}_{C^*B} \otimes \operatorname{Tor}_{C^*B} \longrightarrow \operatorname{Tor}_{C^*B \otimes C^*B}$, this gives us a description of the product on Tor only involving functorial DGA maps and $\Phi_{C^*(X)}$

[6] No source the author knows actually shows the product is preserved, but McCleary at least reduces it to an exercise, and Carlson–Franz spell out some of the steps to this exercise.

We would like to replace C^* with $\mathbf{\Omega B}C^*$ everywhere before performing the external product operation, and for this we need an additional step, since the external product takes $(\mathrm{Tor}_{\mathbf{\Omega B}C^*})^{\otimes 2}$ to $\mathrm{Tor}_{\mathbf{\Omega B}C^* \otimes \mathbf{\Omega B}C^*}$, and then to use $\mathrm{Tor}_{\varepsilon\otimes\varepsilon}$ to pass back to the original C^* level:

$$\begin{tikzcd}
\mathrm{Tor}_{C^*} \otimes \mathrm{Tor}_{C^*} \arrow[r] & \mathrm{Tor}_{C^* \otimes C^*} \arrow[r, equal] & \mathrm{Tor}_{C^* \otimes C^*} \arrow[r, "\text{classical}"] & \mathrm{Tor}_{C^*} \\
\mathrm{Tor}_{\mathbf{\Omega B}C^*} \otimes \mathrm{Tor}_{\mathbf{\Omega B}C^*} \arrow[r] \arrow[u, "\mathrm{Tor}_\varepsilon \otimes \mathrm{Tor}_\varepsilon", "\wr"'] & \mathrm{Tor}_{\mathbf{\Omega B}C^* \otimes \mathbf{\Omega B}C^*} \arrow[r, dotted] \arrow[u, "\mathrm{Tor}_{\varepsilon\otimes\varepsilon}", "\wr"'] & \mathrm{Tor}_{\mathbf{\Omega B}(C^* \otimes C^*)} \arrow[r, "\mathrm{Tor}_{\mathbf{\Omega}\Phi}"'] \arrow[u, "\wr", "\mathrm{Tor}_\varepsilon"'] & \mathrm{Tor}_{\mathbf{\Omega B}C^*}. \arrow[u, "\wr", "\mathrm{Tor}_\varepsilon"']
\end{tikzcd} \tag{6.1}$$

{eq:product

By Theorem 2.6, we can fill in the dotted arrow as Tor_ψ, for $\psi\colon \mathbf{\Omega B}(-\otimes -) \longrightarrow \mathbf{\Omega B}(-) \otimes \mathbf{\Omega B}(-)$.

The same construction evidently applies with H^* substituted for C^* and $\mathbf{B}\mu\colon \mathbf{B}(H^* \otimes H^*) \longrightarrow \mathbf{B}H^*$ for $\Phi\colon \mathbf{B}(C^* \otimes C^*) \longrightarrow \mathbf{B}C^*$.

7. Munkholm's product

sec:product

Munkholm now generalizes the product on Tor from these two special instances to the case of a general triple $\mathbf{B}X \leftarrow \mathbf{B}A \rightarrow \mathbf{B}Y$ of SHC-maps, so that the maps are A_∞-maps rather than DGA maps. Thus we are assuming the following homotopy-commutative squares of DGC maps.

$$\begin{tikzcd}
\mathbf{B}(X \otimes X) \arrow[d, "\Phi_X"'] & \mathbf{B}(A \otimes A) \arrow[l, "\xi \mathbin{\underline{\otimes}} \xi"'] \arrow[r, "\upsilon \mathbin{\underline{\otimes}} \upsilon"] \arrow[d, "\Phi_A"] & \mathbf{B}(Y \otimes Y) \arrow[d, "\Phi_Y"] \\
\mathbf{B}X & \mathbf{B}A \arrow[l, "\xi"] \arrow[r, "\upsilon"'] & \mathbf{B}Y
\end{tikzcd} \tag{7.1}$$

{eq:SHC-map

$$\begin{tikzcd}
\mathbf{\Omega B}(Y \otimes Y) \arrow[r, "\psi"] & \mathbf{\Omega B}Y \otimes \mathbf{\Omega B}Y \arrow[r, "\varepsilon\otimes\varepsilon"] & Y \otimes Y \\
\mathbf{\Omega B}(A \otimes A) \arrow[r, dashed, "\psi"] \arrow[u, "\mathbf{\Omega}(\upsilon \mathbin{\underline{\otimes}} \upsilon)"] \arrow[d, "\mathbf{\Omega}(\xi \mathbin{\underline{\otimes}} \xi)"'] & \mathbf{\Omega B}A \otimes \mathbf{\Omega B}A \arrow[r, equal] \arrow[u, "\mathbf{\Omega}\upsilon \otimes \mathbf{\Omega}\upsilon"] \arrow[d, "\mathbf{\Omega}\xi \otimes \mathbf{\Omega}\xi"] & \mathbf{\Omega B}A \otimes \mathbf{\Omega B}A \arrow[u, "(\varepsilon\mathbf{\Omega}\upsilon)^{\otimes 2}"'] \arrow[d, "(\varepsilon\mathbf{\Omega}\xi)^{\otimes 2}"] \\
\mathbf{\Omega B}(X \otimes X) \arrow[r, "\psi"'] & \mathbf{\Omega B}X \otimes \mathbf{\Omega B}X \arrow[r, "\varepsilon\otimes\varepsilon"'] & X \otimes X
\end{tikzcd} \tag{7.2}$$

{eq:product

Beginning with the external product, one moves from $\big(\mathrm{Tor}_{\mathbf{\Omega B}A}\big)^{\otimes 2}$ to $\mathrm{Tor}_{(\mathbf{\Omega B}A)^{\otimes 2}}$. To be in a position to apply $\mathbf{\Omega}\Phi$, one wants to arrive at $\mathrm{Tor}_{\mathbf{\Omega B}(A\otimes A)}$ using ψ, but ψ is functorial in DGA maps $X \leftarrow A \rightarrow Y$, and not necessarily in A_∞-algebra maps $\mathbf{B}X \leftarrow \mathbf{B}A \rightarrow \mathbf{B}Y$ as we are assuming. To work around this, we can apply $\mathrm{Tor}_{\mathrm{id}}(\varepsilon\otimes\varepsilon, \varepsilon\otimes\varepsilon)$ to arrive at $\mathrm{Tor}_{(\mathbf{\Omega B}A)^{\otimes 2}}(X^{\otimes 2}, Y^{\otimes 2})$, and then note that the large A-X and A-Y rectangles on the top and bottom of (7.2) do commute, do commute, for $\varepsilon^{\otimes 2} \circ (\mathbf{\Omega}\xi)^{\otimes 2} \circ \psi = \varepsilon \circ \mathbf{\Omega}(\xi \mathbin{\underline{\otimes}} \xi)$ by Lemma 2.12 and $\varepsilon^{\otimes 2} \circ \psi = \varepsilon$ by Theorem 2.6. Thus $\mathrm{Tor}_\psi\big((\varepsilon\otimes e)\psi\big)$ is a quasi-isomorphism from $\mathrm{Tor}_{\mathbf{\Omega B}(A\otimes A)}$ to $\mathrm{Tor}_{(\mathbf{\Omega B}A)^{\otimes 2}}(X^{\otimes 2}, Y^{\otimes 2})$.

Thus in place of the direct map Tor_ψ we had back when $X \leftarrow A \to Y$ were honest DGA maps, we now have the substitute

$$\operatorname*{Tor}_{(\mathbf{\Omega B}A)^{\otimes 2}}\big((\mathbf{\Omega B}X)^{\otimes 2}\big) \xrightarrow{\sim} \operatorname*{Tor}_{(\mathbf{\Omega B}A)^{\otimes 2}}(X^{\otimes 2}) \xleftarrow{\sim} \operatorname*{Tor}_{\mathbf{\Omega B}(A^{\otimes 2})}\big(\mathbf{\Omega B}(X^{\otimes 2})\big), \tag{7.3}$$

in which the first map is induced by the right $\varepsilon \otimes \varepsilon$ squares in (7.2) and the second map is induced by the large ψ-ε rectangles. (In case $X \leftarrow A \to Y$ are DGA maps, (7.3) does give the same thing as the original, because $\varepsilon^{\otimes 2} \circ \psi = \varepsilon$ means the composition $\operatorname{Tor}_{\mathrm{id}}(\varepsilon^{\otimes 2}) \circ \operatorname{Tor}_\psi(\psi)$ is $\operatorname{Tor}_\psi(\varepsilon)$.)

In the bottom right square of (6.1), we can no longer define $\operatorname{Tor}_{\mathbf{\Omega}\Phi}$ as before either, because the input data (7.1) commutes only up to homotopy. We instead have recourse to Lemma 4.4, using the assumed homotopies to factor the desired maps through $\operatorname{Tor}_{\mathbf{\Omega B}(A^{\otimes 2})}(P\mathbf{\Omega B}X)$. All told, one finally gets the following composition:

$$\Big(\operatorname*{Tor}_{\mathbf{\Omega B}A}\Big)^{\otimes 2} \xrightarrow{\text{ext}} \operatorname*{Tor}_{(\mathbf{\Omega B}A)^{\otimes 2}} \xleftarrow{\operatorname{Tor}_{\mathrm{id}}(\varepsilon^{\otimes 2})} \operatorname*{Tor}_{\mathbf{\Omega}(\mathbf{B}A)^{\otimes 2}}(X^{\otimes 2}) \xrightarrow{\operatorname{Tor}_{\psi}(\varepsilon^{\otimes 2}\psi)} \operatorname*{Tor}_{\mathbf{\Omega B}(A^{\otimes 2})} \xrightarrow{\operatorname{Tor}_{\mathrm{id}}(\mathbf{\Omega}\Phi)} \operatorname*{Tor}_{\mathbf{\Omega B}(A^{\otimes 2})}(\mathbf{\Omega B}X) \underset{\sim}{\xrightarrow{\operatorname{Tor}_{\mathrm{id}}(\pi_0)}} \operatorname*{Tor}_{\mathbf{\Omega B}(A^{\otimes 2})}(P\mathbf{\Omega B}X) \underset{\sim}{\xleftarrow{\operatorname{Tor}_{\mathrm{id}}(\pi_1)}} \operatorname*{Tor}_{\mathbf{\Omega B}A}. \tag{7.4}$$

8. The reformulation

It is useful to have a description of the substitute $\operatorname{Tor}_\psi(\varepsilon)^{-1} \circ \operatorname{Tor}_{\mathrm{id}}(\varepsilon^{\otimes 2})$ of (7.3) that behaves uniformly in the three variables of Tor. We accomplish this by replacing it with $\operatorname{Tor}_{\mathbf{\Omega}\nabla} \circ \operatorname{Tor}_\gamma^{-1}$.

Lemma 8.1. *Given* WHC*-algebra maps and homotopies as in (7.1) the product (8.2) can be equivalently expressed as the composite*

$$\Big(\operatorname*{Tor}_{\mathbf{\Omega B}A}\Big)^{\otimes 2} \xrightarrow{\text{ext}} \operatorname*{Tor}_{(\mathbf{\Omega B}A)^{\otimes 2}} \xleftarrow{\operatorname{Tor}_{\gamma}} \operatorname*{Tor}_{\mathbf{\Omega}(\mathbf{B}A)^{\otimes 2}} \underset{\sim}{\xrightarrow{\operatorname{Tor}_{\mathbf{\Omega}\nabla}}} \operatorname*{Tor}_{\mathbf{\Omega B}(A^{\otimes 2})} \xrightarrow{\operatorname{Tor}_{\mathrm{id}}(\mathbf{\Omega}\Phi)} \operatorname*{Tor}_{\mathbf{\Omega B}(A^{\otimes 2})}(\mathbf{\Omega B}X) \underset{\sim}{\xrightarrow{\operatorname{Tor}_{\mathrm{id}}(\pi_0)}} \operatorname*{Tor}_{\mathbf{\Omega B}(A^{\otimes 2})}(P\mathbf{\Omega B}X) \underset{\sim}{\xleftarrow{\operatorname{Tor}_{\mathrm{id}}(\pi_1)}} \operatorname*{Tor}_{\mathbf{\Omega B}A}. \tag{8.2}$$

This is the tractable product whose properties are explored in the predecessor [Car22b].

Proof. First note that these maps of Tors are well-defined by naturality of γ and $\mathbf{\Omega}\nabla$, and $\operatorname{Tor}_\gamma$ is invertible by Lemma 4.1 since γ is a quasi-isomorphism by Definition 2.1. Now recall from Lemma 2.8 that $\psi \circ \mathbf{\Omega}\nabla = \gamma$ and from Theorem 2.6 that $\varepsilon^{\otimes 2} \otimes \psi = \varepsilon$, so that

$$\varepsilon^{\otimes 2} \circ \gamma = \varepsilon^{\otimes 2} \circ \psi \circ \mathbf{\Omega}\nabla = \varepsilon \circ \mathbf{\Omega}\nabla$$

and the following diagram commutes, and symmetrically for the A-Y squares:

$$\begin{tikzcd}
\mathbf{\Omega}(\mathbf{B}A)^{\otimes 2} \arrow[r, "\mathbf{\Omega}\nabla"'] \arrow[rrr, bend left, "\gamma"] \arrow[d] & \mathbf{\Omega B}(A^{\otimes 2}) \arrow[r, "\psi"'] \arrow[d] & (\mathbf{\Omega B}A)^{\otimes 2} \arrow[r, equal] \arrow[d] & (\mathbf{\Omega B}A)^{\otimes 2} \arrow[d] \\
\mathbf{\Omega}(\mathbf{B}X)^{\otimes 2} \arrow[r, "\mathbf{\Omega}\nabla"] \arrow[rrr, bend right, "\gamma"'] & \mathbf{\Omega B}(X^{\otimes 2}) \arrow[r, "\varepsilon"] & X^{\otimes 2} & (\mathbf{\Omega B}X)^{\otimes 2}. \arrow[l, "\varepsilon^{\otimes 2}"']
\end{tikzcd}$$

Thus we have an equation $\mathrm{Tor}_{\mathrm{id}}(\varepsilon^{\otimes 2}) \circ \mathrm{Tor}_{\gamma} = \mathrm{Tor}_{\psi}(\varepsilon) \circ \mathrm{Tor}_{\mathbf{\Omega}\nabla}$ which on rearranging yields

$$\mathrm{Tor}_{\psi}(\varepsilon)^{-1} \circ \mathrm{Tor}_{\mathrm{id}}(\varepsilon^{\otimes 2}) = \mathrm{Tor}_{\mathbf{\Omega}\nabla} \circ \mathrm{Tor}_{\gamma}^{-1}.$$

Replacing the left-hand side with the right in (7.4) yields (8.2). □

9. The product on the two-sided bar construction

In this section we show our product on Tor is induced by a product on the bar construction due to Franz [ajwMF21, App. A] which we have used in previous work.

Definition 9.1. Given a DGC C, DGAs X and Y, and twisting cochains $\tau^X \colon C \longrightarrow X$ and $\tau^Y \colon C \longrightarrow Y$, the ***twisted tensor product*** $X \otimes_{\tau^X} C \otimes_{\tau^Y} Y$ is the complex with underlying graded k-module $X \otimes C \otimes Y$ equipped with the differential given as the the sum of (1) the tensor differential and (2) the two operations

$$\begin{aligned}(\mu_X \otimes \mathrm{id}_C)(\mathrm{id}_X \otimes \tau^X \otimes \mathrm{id}_{\mathbf{B}A})(\mathrm{id}_X \otimes \Delta_{\mathbf{B}A}) \otimes \mathrm{id}_Y \colon x \otimes c \otimes y &\longmapsto \pm x \cdot \tau^X(c_{(1)}) \otimes c_{(2)} \otimes y,\\ -\mathrm{id}_X \otimes (\mathrm{id}_{\mathbf{B}A} \otimes \mu_Y)(\mathrm{id}_{\mathbf{B}A} \otimes \tau^Y v \otimes \mathrm{id}_Y)(\Delta_{\mathbf{B}A} \otimes \mathrm{id}_Y) \colon x \otimes c \otimes y &\longmapsto \pm x \otimes c_{(1)} \otimes \tau^Y(c_{(2)}) \cdot y.\end{aligned}$$

Given a span $\mathbf{B}X \xleftarrow{\xi} \mathbf{B}A \xrightarrow{v} \mathbf{B}Y$ of DGC maps, the ***two-sided bar construction*** is the twisted tensor product $\mathbf{B}(X, A, Y) := X \underset{t^X\xi}{\otimes} \mathbf{B}A \underset{t^Y v}{\otimes} Y$.

The signs $\pm$ on the right-hand sides are determined by the definition of the map on the left by the Koszul rule. See Carlson–Franz [ajwMF21, Def. 1.16 *et seq.*] for much more detail and Carlson [Car22a] for the history of these notions.

Definition 9.2 (Wolf [Wol77, p. 322])**.** We write $\mathrm{TOR}_A(X, Y)$ for the bigraded cohomology k-module $H^*\mathbf{B}(X, A, Y)$.

This is a reasonable definition because in case the DGC maps are induced from DGA maps as $\xi = \mathbf{B}x$ and $v = \mathbf{B}y$, then the one-sided bar construction $\mathbf{B}(X, A, A)$ is a proper projective A-module resolution of X under reasonable flatness hypotheses,[7] and then $\mathbf{B}(X, A, Y)$ computes $\mathrm{Tor}_A(X, Y)$. Any more specific hypotheses guaranteeing this will complicate the statements while needlessly excluding certain cases, so we instead directly stipulate that the bar construction calculates Tor.

Definition 9.3. We say the span $\mathbf{B}X \xleftarrow{\xi} \mathbf{B}A \xrightarrow{v} \mathbf{B}Y$ of DGC maps ***satisfies sufficient flatness conditions*** if the canonical map from $\mathbf{B}(\mathbf{\Omega B}X, \mathbf{\Omega B}A, \mathbf{\Omega B}A)$ to a proper projective $\mathbf{\Omega B}A$-module resolution of $\mathbf{\Omega B}X$ induces in cohomology an isomorphism $H^*\mathbf{B}(\mathbf{\Omega B}X, \mathbf{\Omega B}A, \mathbf{\Omega B}Y) \xrightarrow{\sim} \mathrm{Tor}_A(X, Y) := \mathrm{Tor}_{\mathbf{\Omega B}A}(\mathbf{\Omega B}X, \mathbf{\Omega B}Y)$.

This is *a priori* a bit weaker than asking the bar construction to itself be a proper projective resolution, but is enough to make Wolf's TOR of A_∞-algebras agree with Munkholm's Tor. For

[7] For example, it is enough that A and X be flat over the principal ideal domain k [BMR14, after Prop. 10.19].

the proof, recall that twisted tensor products exhibit functoriality with respect to strictly commutative diagrams

$$\begin{tikzcd} X' \arrow[d, "l_X"'] & C' \arrow[l] \arrow[r] \arrow[d, "g"] & Y' \arrow[d, "l_Y"] \\ X & C \arrow[l] \arrow[r] & Y \end{tikzcd} \tag{9.4}$$

in which l_X and l_Y are DGA maps, g is a DGC map, and the horizontal maps are twisting cochains: in this case $l_X \otimes g \otimes l_Y$ is a cochain map [ajwMF21, Lem. 1.20].

Proposition 9.5. *Given a span of* $\mathbf{B}X \xleftarrow{\xi} \mathbf{B}A \xrightarrow{v} \mathbf{B}Y$ *of* DGC *maps satisfying sufficient flatness conditions, one has isomorphisms* $\mathrm{TOR}_A(X,Y) \xrightarrow{\sim} \mathrm{Tor}_A(X,Y)$ *natural in triples of* DGC *maps making the expected squares commute.*

Proof. Recall from Lemma 1.2 that the two tautological twisting cochains are connected by the natural equations $t_C = t^{\Omega C} \circ \eta \colon C^* \longrightarrow \Omega C$ and $\varepsilon \circ t_{\mathbf{B}A} = t^A \colon \mathbf{B}A \longrightarrow A$. Thus, truncating to A-X squares to suppress a symmetric A-Y subdiagram, we have a strictly commutative diagram

$$\begin{tikzcd} \mathbf{B}A \arrow[r, "\xi"] \arrow[d, equal] & \mathbf{B}X \arrow[r, "t^X"] \arrow[d, equal] & X \\ \mathbf{B}A \arrow[r, "\xi"] \arrow[d, "\eta"'] & \mathbf{B}X \arrow[r, "t_{\mathbf{B}X}"] \arrow[d, "\eta"'] & \mathbf{\Omega B}X \arrow[u, "\varepsilon"'] \arrow[d, equal] \\ \mathbf{B\Omega B}A \arrow[r, "\mathbf{B\Omega}\xi"'] & \mathbf{B\Omega B}X \arrow[r, "t^{\mathbf{\Omega B}X}"'] & \mathbf{\Omega B}X \end{tikzcd}$$

inducing maps of twisted tensor products. Note that the cohomology of the twisted tensor product associated to the top row is Wolf's $\mathrm{TOR}_A(X,Y)$ and that associated to the bottom is Munkholm's $\mathrm{Tor}_A(X,Y) := \mathrm{Tor}_{\mathbf{\Omega B}A}(\mathbf{\Omega B}X, \mathbf{\Omega B}Y)$. Since the vertical maps are quasi-isomorphisms and $\mathbf{B}(\mathbf{\Omega B}X, \mathbf{\Omega B}A, \mathbf{\Omega B}Y)$ computes Tor by assumption, the E_2 pages of the induced maps of algebraic EMSSs are isomorphisms, proving the claim. ☐

Maps of two-sided bar constructions are induced not only from diagrams of the form (9.4), but more generally by strictly commuting triples

$$\begin{tikzcd} \mathbf{B}X' \arrow[d, "\lambda_X"'] & \mathbf{B}A' \arrow[l, "\xi'"'] \arrow[r, "v'"] \arrow[d, "\lambda_A"] & \mathbf{B}Y' \arrow[d, "\lambda_Y"] \\ \mathbf{B}X & \mathbf{B}A \arrow[l, "\xi"] \arrow[r, "v"'] & \mathbf{B}Y, \end{tikzcd} \tag{9.6}$$

although we claim no functoriality for such maps on the cochain level [ajwMF21, Prop. 1.26].[8] We follow Wolf [Wol77, Thm. 7] in denoting such a map by $\mathbf{B}(\lambda_X, \lambda_A, \lambda_Y)$. If we assume the algebras

[8] They can actually be defined more generally still, using the pattern for the map Θ from the penultimate section of that paper, which resembles the maps of (4.5).

A, X, Y are WHC-algebras connected by DGA maps x, y making the squares

$$\begin{tikzcd}
\mathbf{B}(X\otimes X) \arrow[d, "\Phi_X"'] & \mathbf{B}(A\otimes A) \arrow[l, "\mathbf{B}(x\otimes x)"'] \arrow[r, "\mathbf{B}(y\otimes y)"] \arrow[d, "\Phi_A"] & \mathbf{B}(Y\otimes Y) \arrow[d, "\Phi_Y"] \\
\mathbf{B}X & \mathbf{B}A \arrow[l, "\mathbf{B}x"] \arrow[r, "\mathbf{B}y"'] & \mathbf{B}Y
\end{tikzcd} \tag{9.7}$$

commute strictly, the map $\mathbf{B}(\Phi_X, \Phi_A, \Phi_Y)$ was used by Franz [ajwMF21, Thm. A.1] to define a weak product on the two-sided bar construction, given by the composition

$$\mathbf{B}(X,A,Y)^{\otimes 2} \xrightarrow{\sim} X^{\otimes 2} \underset{t^{X^{\otimes 2}}(\mathbf{B}x)^{\otimes 2}}{\otimes} (\mathbf{B}A)^{\otimes 2} \underset{t^{Y^{\otimes 2}}(\mathbf{B}y)^{\otimes 2}}{\otimes} Y^{\otimes 2} \xrightarrow{\mathrm{id}\otimes\nabla\otimes\mathrm{id}} \mathbf{B}(X^{\otimes 2}, A^{\otimes 2}, Y^{\otimes 2}) \xrightarrow{\mathbf{B}(\Phi_X,\Phi_A,\Phi_Y)} \mathbf{B}(X,A,Y). \tag{9.8}$$

Remark 9.9. Earlier work of Franz [Fra20, Thm. 1.1, (4.2)] found natural WHC-algebra (respectively, SHC-algebra) structures on (respectively, extended) homotopy Gerstenhaber algebras A, so that a span $X \leftarrow A \rightarrow Y$ of such algebra maps puts us in the situation of (9.7). The author used these structures and the product in previous work to prove a version of the Eilenberg–Moore theorem in which the ring isomorphism is induced by a map which is already homotopy-multiplicative on the cochain level [ajwMF21, Thm. A.27].

So long as the bar constructions compute the relevant Tors in the first place, the product induced is that considered in the present paper.

Theorem 9.10. *Assuming sufficient flatness hypotheses on the DGC maps induced from the spans*

$$X \xleftarrow{x} A \xrightarrow{y} Y \qquad \text{and} \qquad X\otimes X \xleftarrow{x\otimes x} A\otimes A \xrightarrow{y\otimes y} Y\otimes Y$$

of DGA maps, the product (9.8) on the two-sided bar construction induces the product of Section 7 on Tor.

Proof. Note that since x and y are genuine DGA maps, the graded k-module $\mathrm{Tor}_A(X,Y)$ makes sense already without recourse to the counit $\varepsilon\colon \mathbf{\Omega B} \longrightarrow \mathrm{id}$. By the assumed flatness hypotheses and Proposition 9.5, we know $\mathbf{B}(X^{\otimes 2}, A^{\otimes 2}, Y^{\otimes 2})$ computes $\mathrm{Tor}_{A^{\otimes 2}}(X^{\otimes 2}, Y^{\otimes 2})$, so the composition of the first two maps of (9.8) induces the exterior product in cohomology. Since the exterior product is natural in all its arguments, we can view this map as $\mathrm{Tor}_{\varepsilon^{\otimes 2}} \circ \widetilde{\mathrm{ext}} \circ (\mathrm{Tor}_\varepsilon^{-1})^{\otimes 2}$, where $\widetilde{\mathrm{ext}}$ is the exterior product $\mathrm{Tor}_{\mathbf{\Omega B}A}^{\otimes 2} \longrightarrow \mathrm{Tor}_{(\mathbf{\Omega B}A)^{\otimes 2}}$.

Since $\varepsilon = \varepsilon^{\otimes 2} \circ \psi$ and $\psi \circ \mathbf{\Omega}\nabla = \gamma$ by Theorem 2.6 and Lemma 2.8, the diagram of isomorphisms

$$\begin{tikzcd}
& \mathrm{Tor}_{\mathbf{\Omega}(\mathbf{B}A)^{\otimes 2}} \arrow[dl, "\mathrm{Tor}_\gamma"'] \arrow[dr, "\mathrm{Tor}_{\mathbf{\Omega}\nabla}"] & \\
\mathrm{Tor}_{(\mathbf{\Omega B}A)^{\otimes 2}} \arrow[dr, "\mathrm{Tor}_{\varepsilon^{\otimes 2}}"'] & & \mathrm{Tor}_{\mathbf{\Omega B}(A^{\otimes 2})} \arrow[ll, "\mathrm{Tor}_\psi"'] \arrow[dl, "\mathrm{Tor}_\varepsilon"] \\
& \mathrm{Tor}_{A^{\otimes 2}} &
\end{tikzcd}$$

commutes as well.

Thus we have the following system of maps:

$$\begin{tikzcd}
\mathrm{Tor}^{\otimes 2}_{\mathbf{\Omega B}A} \arrow[r] \arrow[d, "\wr"'] & \mathrm{Tor}_{(\mathbf{\Omega B}A)^{\otimes 2}} & \mathrm{Tor}_{\mathbf{\Omega}(\mathbf{B}A)^{\otimes 2}} \arrow[l, "\sim"'] \arrow[r, "\sim"] & \mathrm{Tor}_{\mathbf{\Omega B}(A^{\otimes 2})} \arrow[r] \arrow[d, "\wr"] \arrow[dl, "\sim"'] & \mathrm{Tor}_{\mathbf{\Omega B}A} \arrow[d, "\wr"] \\
\mathrm{Tor}^{\otimes 2}_{A} \arrow[rr] & & \mathrm{Tor}_{A^{\otimes 2}} \arrow[from=ul, "\sim"] \arrow[r, equal] & \mathrm{TOR}_{A^{\otimes 2}} \arrow[r] & \mathrm{TOR}_{A}.
\end{tikzcd} \tag{9.11}$$

We have shown the first three interior shapes of (9.11) commute, so we will see Munkholm's product on Tor is induced by the product on the two-sided bar construction if we can show the final square also commutes.

For this, note that under our flatness hypotheses, $\mathbf{\Omega B}X \otimes_{t^{\mathbf{\Omega B}X}\mathbf{\Omega B}x} \mathbf{B\Omega B}A \otimes_{t^{\mathbf{\Omega B}Y}\mathbf{\Omega B}y} \mathbf{\Omega B}Y$ computes $\mathrm{Tor}_{\mathbf{\Omega B}A}$ and

$$\mathbf{\Omega B}(X^{\otimes 2}) \underset{t^{\mathbf{\Omega B}(X^{\otimes 2})}\mathbf{\Omega B}(x^{\otimes 2})}{\otimes} \mathbf{B\Omega B}(A^{\otimes 2}) \underset{t^{\mathbf{\Omega B}(Y^{\otimes 2})}\mathbf{\Omega B}(y^{\otimes 2})}{\otimes} \mathbf{\Omega B}(Y^{\otimes 2})$$

computes $\mathrm{Tor}_{\mathbf{\Omega B}(A^{\otimes 2})}$. Since $\mathbf{\Omega B}\Phi_Z$ is an algebra map for $Z \in \{A, X, Y\}$, we are in the situation of (9.4) and the map $\mathrm{Tor}_{\mathbf{\Omega B}(A^{\otimes 2})} \longrightarrow \mathrm{Tor}_{\mathbf{\Omega B}A}$ in the final square of (9.11) is induced by $\mathbf{\Omega}\Phi_X \otimes \mathbf{B\Omega}\Phi_A \otimes \mathbf{\Omega}\Phi_Y$. We need to find the vertical maps connecting it to $\mathrm{TOR}_{A^{\otimes 2}} \longrightarrow \mathrm{TOR}_A$.

From the naturality of the unit-counit identities $\mathbf{B}\varepsilon \circ \eta = \mathrm{id}$ and the unit and counit themselves, the diagram

$$\begin{tikzcd}
\mathbf{B}A \arrow[r, equal] \arrow[d, "\eta"'] & \mathbf{B}A \arrow[r, "\mathbf{B}x"] & \mathbf{B}X \arrow[d, equal] \\
\mathbf{B\Omega B}A \arrow[ur, "\mathbf{B}\varepsilon"] \arrow[r, "\mathbf{B\Omega B}x"'] & \mathbf{B\Omega B}X \arrow[r, "\mathbf{B}\varepsilon"'] & \mathbf{B}X
\end{tikzcd}$$

commutes, and similarly with Y and y in place of X and x, so Equation (9.6) induces a quasi-isomorphism $\mathrm{id} \otimes \eta \otimes \mathrm{id} \colon \mathbf{B}(X, A, Y) \longrightarrow \mathbf{B}(X, \mathbf{\Omega B}A, Y)$. The same argument gives a quasi-isomorphism $\mathbf{B}(X^{\otimes 2}, A^{\otimes 2}, Y^{\otimes 2}) \longrightarrow \mathbf{B}\big(X^{\otimes 2}, \mathbf{\Omega B}(A^{\otimes 2}), Y^{\otimes 2}\big)$, and the naturality of η shows $\mathrm{TOR}_{A^{\otimes 2}} \longrightarrow \mathrm{TOR}_A$ is thus given equivalently as the map induced in cohomology by

$$\mathbf{B}(\Phi_X, \mathbf{\Omega B}\Phi_A, \Phi_Y) \colon \mathbf{B}\big(X^{\otimes 2}, \mathbf{\Omega B}(A^{\otimes 2}), Y^{\otimes 2}\big) \longrightarrow \mathbf{B}(X, \mathbf{\Omega B}A, Y).$$

Similarly, the equation $\mathbf{B}\varepsilon \circ \eta = \mathrm{id}$ applied to X and Y gives a quasi-isomorphism

$$\mathbf{B}(\eta, \mathrm{id}, \eta) \colon \mathbf{B}(X, \mathbf{\Omega B}A, Y) \longrightarrow \mathbf{B}(\mathbf{\Omega B}X, \mathbf{\Omega B}A, \mathbf{\Omega B}Y)$$

as in (9.6) and a left inverse quasi-isomorphism $\varepsilon \otimes \mathrm{id} \otimes \varepsilon$ by (9.4). The same argument gives quasi-isomorphisms between $\mathbf{B}\big(X^{\otimes 2}, \mathbf{\Omega B}(A^{\otimes 2}), Y^{\otimes 2}\big)$ and $\mathbf{B}\big(\mathbf{\Omega B}(X^{\otimes 2}, \mathbf{\Omega B}(A^{\otimes 2}), \mathbf{\Omega B}(Y^{\otimes 2})\big)$. The composite

$$\mathbf{B}\big(\mathbf{\Omega B}(X^{\otimes 2}, \mathbf{\Omega B}(A^{\otimes 2}), \mathbf{\Omega B}(Y^{\otimes 2})\big) \xrightarrow{\varepsilon \otimes \mathrm{id} \otimes \varepsilon} \mathbf{B}\big(X^{\otimes 2}, \mathbf{\Omega B}(A^{\otimes 2}), Y^{\otimes 2}\big) \xrightarrow{\mathrm{id} \otimes \varepsilon \otimes \mathrm{id}} \mathbf{B}(X^{\otimes 2}, A^{\otimes 2}, Y^{\otimes 2})$$

of this last quasi-isomorphism and the obvious left inverse to the quasi-isomorphism of the previous paragraph induces the canonical isomorphism $\mathrm{Tor}_{\mathbf{\Omega B}(A^{\otimes 2})} \longrightarrow \mathrm{Tor}_{A^{\otimes 2}}$, showing the second triangle in (9.11) does commute.

In our final descent to the cochain level, we complete the square of (9.11) by showing the diagram

$$\begin{tikzcd} \mathbf{B}\big(\mathbf{\Omega B}(X^{\otimes 2}, \mathbf{\Omega B}(A^{\otimes 2}), \mathbf{\Omega B}(Y^{\otimes 2})\big) \arrow[r, "\mathbf{\Omega}\Phi_X \otimes \mathbf{B\Omega}\Phi_A \otimes \mathbf{\Omega}\Phi_Y"] & \mathbf{B}(\mathbf{\Omega B}X, \mathbf{\Omega B}A, \mathbf{\Omega B}Y) \arrow[d, "\varepsilon \otimes \mathrm{id} \otimes \varepsilon"] \\ \mathbf{B}\big(X^{\otimes 2}, \mathbf{\Omega B}(A^{\otimes 2}), Y^{\otimes 2}\big) \arrow[u, "\mathbf{B}(\eta, \mathrm{id}, \eta)"] \arrow[r, "\mathbf{B}(\Phi_X, \mathbf{B\Omega}\Phi_A, \Phi_Y)"'] & \mathbf{B}(X, \mathbf{\Omega B}A, Y) \end{tikzcd}$$

commutes on pure tensors in $\overline{X^{\otimes 2}} \otimes \mathbf{B\Omega B}(A^{\otimes 2}) \otimes \overline{Y^{\otimes 2}}$; the verification is similar but simpler if the $X^{\otimes 2}$- or $Y^{\otimes 2}$-component lies in the image of the unit. Following the definition [ajwMF21, Prop. 1.26], then, $\mathbf{B}(\Phi_X, \mathbf{B\Omega}\Phi_A, \Phi_Y)$ is given as the composition

$$\big(t^X \Phi_X(s_X^{-1} \otimes \mathbf{B}\varepsilon \circ \mathbf{B\Omega B}x^{\otimes 2}) \otimes \mathbf{B\Omega}\Phi_A \otimes t^Y \Phi_Y(\mathbf{B}\varepsilon \circ \mathbf{B\Omega B}y^{\otimes 2} \otimes s_Y^{-1})\big) \circ \big(\mathrm{id} \otimes \Delta^{[3]}_{\mathbf{B\Omega B}(A^{\otimes 2})} \otimes \mathrm{id}\big),$$

while the composition along the top is

$$\begin{aligned} &(\varepsilon \otimes \mathrm{id} \otimes \varepsilon) \circ (\mathbf{\Omega}\Phi_X \otimes \mathbf{B\Omega}\Phi_A \otimes \mathbf{\Omega}\Phi_Y) \\ &\quad \circ \big(t^{\mathbf{\Omega B}X^{\otimes 2}} \eta(s_X^{-1} \otimes \mathbf{B}\varepsilon \circ \mathbf{B\Omega B}x^{\otimes 2}) \otimes \mathrm{id} \otimes t^{\mathbf{\Omega B}Y^{\otimes 2}} \eta(\mathbf{B}\varepsilon \circ \mathbf{B\Omega B}y^{\otimes 2} \otimes s_Y^{-1})\big) \circ \big(\mathrm{id} \otimes \Delta^{[3]}_{\mathbf{B\Omega B}(A^{\otimes 2})} \otimes \mathrm{id}\big). \end{aligned}$$

The initial comultiplication is the same in both cases, and after, both maps are compositions of tensor products of maps on the X, A, and Y components we may examine separately. It is easy to see that the "A" map in both cases is $\mathbf{B\Omega}\Phi_A$. The X and Y components are symmetrical, and for variety we check the Y component and suppress the check for X this time. That the Y components are equal is the claim

$$t^Y \Phi_Y(\mathbf{B}\varepsilon \circ \mathbf{B\Omega B}y^{\otimes 2} \otimes s_Y^{-1}) \ \overset{\heartsuit}{=} \ \varepsilon \circ \mathbf{\Omega}\Phi_Y \circ t^{\mathbf{\Omega B}Y^{\otimes 2}} \eta(\mathbf{B}\varepsilon \circ \mathbf{B\Omega B}y^{\otimes 2} \otimes s_Y^{-1}).$$

It will evidently be enough to check that

$$t^Y \Phi_Y \ \overset{\heartsuit}{=} \ \varepsilon \circ \mathbf{\Omega}\Phi_Y \circ t^{\mathbf{\Omega B}Y^{\otimes 2}} \eta.$$

But this, finally, amounts to the commutativity of the following diagram, in which the parallelogram follows by naturality of $t_{(-)}$ and the triangles by Lemma 1.2.

$$\begin{tikzcd} \mathbf{B\Omega B}Y^{\otimes 2} \arrow[d, "t^{\mathbf{\Omega B}Y^{\otimes 2}}"'] & \mathbf{B}Y^{\otimes 2} \arrow[l, "\eta"'] \arrow[r, "\Phi_Y"] \arrow[dl, "t_{\mathbf{B}Y^{\otimes 2}}"] & \mathbf{B}Y \arrow[dl, "t_{\mathbf{B}Y}"'] \arrow[d, "t^Y"] \\ \mathbf{\Omega B}Y^{\otimes 2} \arrow[r, "\mathbf{\Omega}\Phi_Y"'] & \mathbf{\Omega B}Y \arrow[r, "\varepsilon"'] & Y \end{tikzcd}$$

$\square$

Remark 9.12. It thus seems even more plausible than it did in the previous paper [ajwMF21, Rmk. A.26] that the product (9.7) on the two-sided bar construction is the binary component in a sequence of operations making it an A_∞-algebra, but we have not tried to prove this. It is perhaps the most generous course to follow the spirit of Munkholm in leaving this problem for a later reader.

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