Fixed points and bordism of semifree actions

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Abstract

We apply fixed-point techniques to compute the coefficient ring of semifree circle-equivariant complex cobordism with isolated fixed points, recovering a 2004 result of Sinha through 19$^{\text{th}}$-century methods.

Homotopical equivariant complex cobordism MU^G with respect to the action of a compact Lie group G is the universal complex-oriented G-equivariant spectrum, yet concrete presentations of its coefficient ring are unknown except for certain cases when G is finite. Geometric equivariant cobordism $\Omega_*^{U:G}(-)$, whose coefficient ring is given by bordism classes of (tangentially) stably complex closed G-manifolds is similarly inaccessible in most cases as of this writing.

Restricting to the special case of *semifree* circle actions, where all orbits are free or fixed points, Sinha [Sin05] proved a number of results, including, strikingly, the following.

Theorem 0.1 (Sinha [Sin05, Thm. 1.1])**.** *Every compact, oriented, stably complex, semifree S^1-manifold with isolated fixed points is cobordant to a disjoint union of direct powers of the sphere $S^2 = \mathbb{C}\mathrm{P}^1$ with the standard complex structure and rotation action. That is, the bordism ring of such manifolds is isomorphic to the polynomial ring $\mathbb{Z}[S^2]$ on one generator.*

In the present note, we recover this result through elementary means.

1. Introduction

We begin by reviewing the definitions and existing results.[1] We will not attempt to discuss the role of equivariant complex cobordism in equivariant homotopy theory, but will try to state, briefly, what is known about it from a computational perspective. We take as a starting point Thom's result that (nonequivariant) geometric complex bordism Ω_n^U, defined in terms of bordism of stably complex n-manifolds, is also represented by the Thom spectrum MU whose $2n^{\text{th}}$ level MU_{2n} is the Thom space of the universal complex n-plane bundle, with the isomorphism induced from the Pontrjagin–Thom collapse.[2] The corresponding map for equivariant complex bordism is not an isomorphism, leading to distinct geometric and homotopical notions.

1.1. Definitions and comparison

Fix a compact Lie group G. A G-equivariant complex vector bundle (or *G-bundle*) over a G-space M is the usual thing, a complex vector bundle $E \to M$ equipped with a $\mathbb{C}$-linear G-action on E such that the

[1] The literature review in the first version of this document was deliberately sketchy, with a view toward brevity, and several correspondents requested it be expanded and clarified. The corresponding section of a second version grew to a survey of twenty-some pages, which remains unfinished. One came to realize it should be a separate paper, if it is ever finished at all; there is clearly a balance to be struck in a note of this length. Let us try again.

[2] We also assume the classic computational result, due to Milnor and Novikov, that $\Omega_*^U \cong MU_* := \pi_* MU$ is a polynomial ring $\mathbb{Z}[x_2, x_4, x_6, \ldots]$ on one generator $x_{2d} \in \pi_{2d} MU$ for each positive natural n. One has $\Omega_U^* \otimes \mathbb{Q} \cong \mathbb{Q}\big[[\mathbb{C}\mathrm{P}^1], [\mathbb{C}\mathrm{P}^2], [\mathbb{C}\mathrm{P}^3], \ldots\big]$, though not in such a way that the $[\mathbb{C}\mathrm{P}^d]$ are scalar multiples of the integral generators x_{2d}; Milnor found a natural set of manifolds giving integral generators, but the relations they satisfy are not straightforward. The famous theoretical result, due to Quillen, is that this coefficient ring MU_* carries Lazard's universal one-dimensional formal group law. It is also important, but more tautological, that $MU^*(-)$ is initial amongst complex-oriented cohomology theories.

projection is equivariant. We write $\mathbb{R}^k$ (resp. $\mathbb{C}^k$) for Euclidean space viewed as a trivial real (resp. complex) G-representation, and when V is a G-representation and we are considering some G-manifold M, we denote by $\underline{V}$ the trivial bundle $M \times V \to M$, equipped with the diagonal G-action on its total space. By a ***stably complex G-manifold*** we mean a smooth manifold M together with a complex G-bundle structure on some stabilization $TM \oplus \underline{\mathbb{R}}^k$ of its tangent G-bundle. Two G-equivariant stable complex structures on a manifold are defined to be *equivalent* if after direct-summing $\underline{\mathbb{C}}^m$ to one and $\underline{\mathbb{C}}^n$ to the other, for some m and n, the two become isomorphic as complex G-bundles via an isomorphism fixing TM. More concisely, an equivalence class is a lift of the class of $[TM]$ in reduced real equivariant topological K-theory $KO^0_G(M)/\mathbb{Z}$ along the forgetful map from reduced complex equivariant K-theory $K^0_G(M)/\mathbb{Z}$, where the two instances of $\mathbb{Z}$ are the summands respectively generated by $\underline{\mathbb{R}}$ and $\underline{\mathbb{C}}$. The ***equivariant geometric bordism*** groups $\Omega^{U:G}_n(X)$ are determined as singular G-maps of stably complex G-manifolds M^n into X, modulo the equivalence relation identifying two such maps if they together form the boundary restriction of a G-map of a stably complex G-manifold W^{n+1} into X.

Different notions of lifting and equivalence yield different groups: one might instead ask of TM that there exist a real representation W such that $TM \oplus \underline{W}$ is a complex G-bundle and define two such structures $TM \oplus \underline{W}$ and $TM \oplus \underline{W}'$ to be equivalent if there exist complex G-representations $\underline{V}, \underline{V}'$ such that $TM \oplus \underline{W} \oplus \underline{V}$ and $TM \oplus \underline{W}' \oplus \underline{V}'$ are isomorphic as complex G-bundles. In K-theoretic terms, an element is then given by a lift of $[TM] \in \widetilde{KO}^0_G(M)$ in $\widetilde{K}^0_G(M)$. Equivalently, one can ask the same of the normal bundle $\nu = s\nu_V(M)$ to M under an equivariant embedding of M in a sufficiently large real G-representation V; but as $[\nu] = -[TM]$ in $\widetilde{K}^0_G(M)$, these are equivalent notions. The resulting groups have variously been denoted $mu^G_*(X)$ and $\overline{\Omega}^{U:G}_*(X)$; see Darby's thesis [D, §§3.1–2] and Comezaña [C96, §3].[3]

Example 1.1 (Hanke [Han05, p.680]). To see these notions are distinct, consider the three-dimensional representation V of $\mathbb{Z}/2$ determined by the antipodal map. Its unit sphere SV is invariant and hence a $\mathbb{Z}/2$-space, and the normal bundle $\nu_V(SV)$ is the trivial $\mathbb{Z}/2$-bundle $\underline{\mathbb{R}}$ over SV, which can be stabilized to $\underline{\mathbb{C}}$. On the other hand, the tangent $\mathbb{Z}/2$-bundle TSV is naturally a pullback of the tangent bundle $T\mathbb{R}\mathrm{P}^2$, so a complex $\mathbb{Z}/2$-bundle structure on $TSV \oplus \underline{\mathbb{R}}^{2k}$ would amount to a complex structure on $T\mathbb{R}\mathrm{P}^2 \oplus \underline{\mathbb{R}}^{2k}$. But that in turn would induce a complex structure on $T(\mathbb{R}\mathrm{P}^2 \times \mathbb{R}^{2k})$, which is impossible as $\mathbb{R}\mathrm{P}^2 \times \mathbb{R}^{2k}$ is not orientable.

There is at least a natural map $\Omega^{U:G}_*(X) \longrightarrow mu^G_*(X)$, which Löffler asserts [L73, Lem. 2.1] and a result of Comezaña [C96, Thm. 5.4] implies is injective for all compact abelian G. Writing S^V for the one-point compactification of a complex G-representation V and $|V|$ for its real dimension, one can stabilize both $\Omega^{U:G}_*$ and mu^G_* by maps $\Omega^G_k(X) \longrightarrow \Omega^G_{k+|V|}(S^V \wedge X)$ and $mu^G_k(X) \longrightarrow mu^G_{k+|V|}(S^V \wedge X)$ given in terms of a representative $M \to X$ by "straightening the angles" of $S^V \wedge M$ to make it again a manifold, and it is a result of Comezaña–Costenoble [C96, Thm. 3.3] that the colimit of the maps $\Omega^{U:G}_{k+|V|}(S^V \wedge X) \longrightarrow mu^G_{k+|V|}(S^V \wedge X)$ is an isomorphism.[4] This colimit gives a third variant of equivariant complex bordism.

We claim this homology theory is representable, unlike $\Omega^{U:G}_*(-)$ and $mu^G_*(-)$. To see this, note that there exists a *universal* complex G–vector bundle $EU^G_{2n} \to BU^G_{2n}$ such that homotopy classes of G-maps $X \longrightarrow BU^G_{2n}$ parameterize isomorphism classes of rank-n complex G-bundles over a G-space X; the most natural model is the tautological bundle over the Grassmannian of complex n-planes in the direct sum $\mathbf{U} = \bigoplus V^\infty$ of $\aleph_0$-many copies of each irreducible complex G-representation V. The associated Thom spaces MU^G_{2n} admit a multiplication $MU^G_{2k} \wedge MU^G_{2m} \longrightarrow MU^G_{2k+2m}$ defined from the maps $BU^G_{2k} \times BU^G_{2m} \longrightarrow BU^G_{2k+2m}$ classifying the Whitney sum. For any representation V of G, the classifying map $\underline{V} \oplus EU^G_{2k} \longrightarrow EU^G_{|V|+2k}$ induces a map of Thom spaces $S^V \wedge MU^G_{2k} \longrightarrow MU^G_{|V|+2k}$ which on forgetting the G-action becomes the structure map for the nonequivariant sequential Thom spectrum MU. We take these as structure maps for the G-spectrum MU^G.[5] Once $\mathbf{U}$ is equipped with an invariant Hermitian

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[3] An analogous discussion can be made of equivariant unoriented bordism, spawning theories $\Omega^{O:G}_*$, mo^G_*, and MO^G_* which very frequently admit analogous results. Indeed, many proofs carry over with little to no change from one setting to the other. Perhaps heretically, but with the aim of bounding our exposition, we bypass any discussion of these theories.

[4] N.B. the typo on p. 392: the domain and codomain of ϕ should be transposed and similarly for Φ.

[5] One can expand this to an $R(G)$-graded spectrum in the Lewis–May–Steinberger formalism without much trouble by taking BU^G_V to be the Grassmannian of complex $\frac{1}{2}|V|$-planes in the abstract vector space $V \oplus \mathbf{U}$, basepointed at V, and one gets $MU^G_V(-) \cong$

inner product, each pair $W \leqslant V \leqslant \mathbf{U}$ defines a unique complementary representation $V/W \leqslant V$. We may then partially order G-representations and use these complements and the structure maps just defined to associate reduced (co)homology theories by

$$\widetilde{MU}{}^{2n}_G(X) := \varinjlim_V\, [S^V \wedge X, MU^G_{2n+|V|}]^G, \qquad \widetilde{MU}{}^G_{2n}(X) := \varinjlim_V\, [S^V, X \wedge MU^G_{|V|-2n}]^G,$$

where X is a space with a G-fixed basepoint and $[-,-]^G$ denotes pointed G-homotopy classes of G-maps. In odd dimensions we instead set $\widetilde{MU}{}^{2n-1}_G(X) := \widetilde{MU}{}^{2n}_G(S^1 \wedge X)$ and $\widetilde{MU}{}^G_{2n-1}(X) := \widetilde{MU}{}^G_{2n}(S^1 \wedge X)$. The unreduced versions are defined by $MU^*_G(X) := \widetilde{MU}{}^*_G(X_+)$ and $MU^G_*(X) := \widetilde{MU}{}^G_*(X_+)$ where X_+ is the disjoint union of X with a new G-fixed basepoint. This is ***homotopical equivariant complex (co)bordism***.

Geometric equivariant bordism is related to the homotopical variety via the Pontrjagin–Thom collapse: if $M \to X$ represents an element of $mu^G_n(X)$, then letting $M \hookrightarrow V$ be an equivariant embedding whose normal G-bundle ν carries a complex structure, the collapse $S^V \twoheadrightarrow \mathrm{Th}\,\nu$ followed by the projection $\nu \twoheadrightarrow M$ and the classifying map of ν induce a map $S^V \longrightarrow X_+ \wedge MU^G_{|V|-|M|}$, where $|M|$ is the real dimension of M. One checks that increasing V to $V \oplus W$ in the definition of the normal embedding gives the W-suspension of the first map, so that the image in the colimit is well-defined. The theorem of Bröcker–Hook [BH72][6] is that stabilizing along these maps induces an isomorphism

$$\varinjlim_V\, mu^G_{k+|V|}(S^V \wedge X) \xrightarrow{\sim} \widetilde{MU}{}^G_k(X).^{7}$$

Composing this isomorphism with the Comezaña–Costenoble isomorphism, one sees $\widetilde{MU}{}^G_*(-)$ is the stabilization of $\Omega^{U:G}_*(-)$ as well. That the map $\Psi\colon \Omega^{U:G}_*(-) \longrightarrow \widetilde{MU}{}^G_*(-)$ is not an isomorphism before stabilizing can be seen as a result of the failure of the Thom transversality argument in the equivariant setting: generally there is no way to *equivariantly* homotope an arbitrary G-map to be transversal.

Example 1.2 (Wasserman [W69, p. 137]). Consider $\mathbb{R}$ as a $\mathbb{Z}/2$-manifold under $x \longmapsto -x$ and the smooth equivariant map f sending $M = *$ to the $\mathbb{Z}/2$-submanifold $W = \{0\}$. There is no way to equivariantly homotope f to be transverse to W; in fact there are no other $\mathbb{Z}/2$-homotopic maps at all.

On the other hand, when the G-action on X is free, the transversality argument goes through and $\Omega^{U:G}_*(X) \longrightarrow MU^G_*(X)$ is an isomorphism [tD70, Prop. 1.3].

1.2. Comparisons and general structure

None of the theories $X \longmapsto \Omega^{U:G}_*(X)$, $mu^G_*(X)$, and $MU^G_*(X)$ is readily calculable, and each is only known for special values of G and X, but there are a number of comparison maps with other theories, very frequently injective, and in many cases identifying cobordism as a pullback. Some general structure results are also known.

$MU^G_{|V|}(-)$ and similarly for cobordism. The structure maps $S^{V/W} \wedge MU^G_W \longrightarrow MU^G_V$ for $W < V$ arise from applying the Thom construction to the map of tautological bundles taking each complex $\frac{1}{2}|W|$-plane L in $W \oplus \mathbf{U}$ to the $\frac{1}{2}|V|$-plane $V/W \oplus L$ in $V \oplus \mathbf{U}$, where V/W here denotes the orthogonal complement with respect to the inner product on $\mathbf{U}$.

It is asserted semifrequently in the literature that one can expand this to an $RO(G)$-graded spectrum, but details never seem to follow. Here are some. We index by *real* G-invariant subspaces of the complex G-universe $\mathbf{U}$. For each such space W, write $W^{\mathbb{C}} := W \cap iW$, which is a canonical maximal complex subrepresentation, and set $MU^G_W := S^{W/W^{\mathbb{C}}} \wedge MU^G_{W^{\mathbb{C}}}$, where $W/W^{\mathbb{C}}$ denotes the orthogonal complement and $MU^G_{W^{\mathbb{C}}}$ is as in the previous paragraph. Then structure maps $S^{V/W} \wedge MU^G_W \longrightarrow MU^G_V$ are defined for $W < V$ by first performing the reassociation $S^{V/W} \wedge S^{W/W^{\mathbb{C}}} \cong S^{V/W^{\mathbb{C}}} \cong S^{V/V^{\mathbb{C}}} \wedge S^{V^{\mathbb{C}}/W^{\mathbb{C}}}$ in the domain, then leaving the factor $S^{V/V^{\mathbb{C}}}$ alone and applying the structure map $S^{V^{\mathbb{C}}/W^{\mathbb{C}}} \wedge MU^G_{W^{\mathbb{C}}} \longrightarrow MU^G_{V^{\mathbb{C}}}$ from the $R(G)$-graded definition to the other factor. In words, one applies as much of the $R(G)$-graded structure as one can, benignly neglecting a "purely real" remainder sphere.

[6] proven for $mo^G_* \longrightarrow MO^G_*$, but the proof carries over *mutatis mutandis*

[7] This stabilization is in fact necessary to make equivariant cobordism representable, for the suspension maps are not generally isomorphisms before passing to the colimit.

Formal group laws

Classically, it is known that MU_* carries the universal one-dimensional formal group law. Greenlees [G01] showed that for A an abelian group, the classifying map $L_A \longrightarrow MU_*^A$ from the ring carrying the universal A-equivariant formal group law is surjective. Hanke–Wiemeler [HanW18] showed this map is an isomorphism for $A = \mathbb{Z}/2$ and Hausmann [Haus19] showed it for all compact abelian Lie groups A.

Module structure and evenness

The map from MU_* assigning a stably complex manifold M the trivial G-action is a ring map inducing an MU_*-module structure on $\Omega_*^{U:G}$, mu_*^G, and MU_G^*, and splits the augmentation map to MU_* induced by forgetting the action on a G-manifold.

Hamrick and Ossa [HamO72] showed that if G is a topologically cyclic compact Lie group, *i.e.*, a product of a finite cyclic group and a torus, $\Omega_*^{U:G}$ is a free MU_*-module on even-degree generators, after earlier work by several authors establishing special cases. Löffler [L73] announced and Comezaña [C96, §5] showed the same of $\Omega_*^{U:G}$ and MU_G^* when G is a compact abelian Lie group. He showed moreover that if X is a G-space of the form $S^W \times \prod_{j=1}^{\ell} BU_{m_j}^G$, then $\Omega_*^{U:G}(X)$ is free on even-dimensional generators and the stabilization maps

$$\Omega_*^{U:G}(BU_n^G \times X) \to \Omega_*^{U:G}(BU_{n+1}^G \times X), \qquad \Omega_*^{U:G}(X) \to \Omega_{*+2}^{U:G}(S^V \times X)\ (V \text{ irreducible}), \qquad \Omega_*^{U:G} \longrightarrow MU_G^*$$

are all split injections of MU_*-modules. The same is conjectured to hold for all compact Lie groups [U18], and Comezaña (p. 398) stated he could also prove it for the finite dihedral groups and $O(2)$, but these results appear not to have been published.

Reduction to K-theory

Okonek [O82] showed the natural transformation $MU_G^*(-) \longrightarrow K_G^*(-)$, induces a natural isomorphism

$$R(G) \underset{MU_G^*}{\otimes} MU_G^*(X) \cong K_G^*(X)$$

of $\mathbb{Z}/2$-graded rings, where the module structure map $MU_G^* \longrightarrow R(G)$ is the case $X = *$. This generalizes the Conner–Floyd isomorphism $\mathbb{Z} \otimes_{MU^*} MU^*(X) \cong K^*(X)$ [CF66]. There are many other such transformations $MU_*^G(-) \longrightarrow h_G^*(-)$ (not all so easily characterized), since $MU_*^G(-)$ is the universal G-equivariant complex-oriented cobordism theory [O82].

Injections in larger theories

tom Dieck [tD70][8] defined a natural *bundling transformation*

$$\eta\colon MU_G^*(X) \longrightarrow MU_G^*(EG \times X) \xrightarrow{\sim} MU^*\big((EG \times X)/G\big),$$

where $(EG \times X)/G =: X_G$ is the Borel construction. It is not hard to see there is a homotopy equivalence $EG_+ \wedge_G MU_k^G \simeq BG \times MU_k$ induced by the map classifying the (nonequivariant) vector bundle $(EG \times EU_k^G)/G \to (EG \times BU_k^G)/G$, and η takes a class represented by $S^V \wedge X \longrightarrow MU_{n+|V|}^G$ to the class of

$$EG_+ \underset{G}{\wedge} (S^V \wedge X) \longrightarrow EG_+ \underset{G}{\wedge} MU_{n+|V|}^G \longrightarrow MU_{n+|V|}.$$

When $X = *$, the map $MU_*^G \longrightarrow MU^*(BG)$ is known to be equivalent to the completion map $MU_G^* \longrightarrow (MU_G^*)\widehat{\ }$ with respect to the ideal $\ker(MU_*^G \to MU^*)$ due to work of Greenlees–May and La Vecchia [GM97, LV21]. When $G = T$ is a torus, the composite map $\Omega_*^{U:T} \longrightarrow MU^*(BT)$ is often referred to as the *universal toric genus* [BPR10].

[8] His prototype was Boardman's map $\Omega_*^{O:\mathbb{Z}/2} \longrightarrow MO^*(B\mathbb{Z}/2)$.

For $h \in \{H, K, MU\}$ the Atiyah–Hirzebruch spectral sequence converging to $h^*BU(n)$ collapses, so that $h^*BU(n)$ is the power series ring over h_* on classes $c_i^h \in h^{2i}BU(n)$ called the ***Conner–Floyd Chern classes***. Applying these classes to the stable normal bundle of a class $[M \to X] \in mu_*^G(X)$ judiciously and following with the relevant Gysin map (integration along the fiber for $H_G^*(X)$, the Atiyah–Singer index map for $K_G^*(X)$), one compiles them into "characteristic number" maps from $mu_G^*(X)$, factoring through $MU_*^G(X)$ and taking values in $\mathrm{Hom}(K^*BU, K_G^*X) \cong K_G^*(X)[[a_1, a_2, a_3, \ldots]] =: K_G^*(X)[[\vec{a}]]$ or the power series ring

$$\mathrm{Hom}_{h^*}(h^*BU, h^*X_G) \cong h^*(X_G) \mathop{\widehat{\otimes}}_{h_*} h_*BU \cong h^*(X_G)[[\vec{a}]].$$

These are also called "Boardman maps."[9] Most of these maps factor through the bundling map to $MU^*(X_G)$ using $MU^*(-) \longrightarrow K^*(-)$ or $MU^*(-) \longrightarrow H^*(-)$.

tom Dieck showed that $MU_*^G(X) \longrightarrow K_G^*(X)[[\vec{a}]]$ is injective when G is topologically cyclic and X is a point or the unit sphere SV in a representation V [tD74, Thms. 2&3], implying $MU_*^G \longrightarrow MU^*(BG)$ is injective in these cases as well. Composing with the map from $\Omega_*^{U:G}$, this shows manifolds are determined up to G-equivariant cobordism by their K-theoretic characteristic numbers, which Hattori [Hat74] proved independently for $G = T$ a torus. (The case $G = 1$ is due to Stong and Hattori [St65, Hat66].) Hattori [Hat74, Thm. 1.7] also proved for T a torus that $\Omega_*^{U:T} \longrightarrow MU^*(BT)$ is monic. Lü and Wang [LW18] showed the map $\Omega_*^{U:T} \longrightarrow H^*(BT)[[\vec{a}]]$ is injective as well, confirming a conjecture of Guillemin–Ginzburg–Karshon.

Variants of these maps can be defined on $\Omega_*^{U:G}(X)$ using the equivariant characteristic numbers of the tangent bundle; their values in $\mathrm{Hom}\big(H_*BU, h_G^*(X)\big)$ differ from the composition through $mu_*^G(X)$ by precomposition by the automorphism of H_*BU induced by the map $\iota\colon BU \longrightarrow BU$ classifying the "inverse" operation for the H-space structure on BU classifying the stable Whitney sum. This corresponds to the exchange of the tangent and stable normal bundles.

Localization near fixed points

The fixed point set M^G in a stably complex G-manifold M is itself a stably complex G-manifold in such a way that its normal bundle $\nu = \nu_M(M^G)$ carries a natural complex G–vector bundle structure. The compactification $\bar{\nu}$ of ν by its fiberwise visual boundary $\partial\nu$ is a stably complex G-manifold with boundary which we tacitly consider to be embedded as the closure of a regular neighborhood of M^G in M. The boundary $\partial\nu$ is itself a stably complex G-manifold whose isotropy groups $G_x < G$ for $x \in \partial\nu$ are all proper subgroups, so the pair $(\overline{\nu}, \partial\nu)$ represents an element in a bordism ring $\Omega_*^{U:G}[\mathscr{A}, \mathscr{P}]$ of G-manifolds whose isotropy groups lie in the set $\mathscr{A}$ of all closed subgroups of G and whose boundary has isotropy groups in the subset $\mathscr{P}$ of *proper* closed subgroups.[10] The pairs $(M, \varnothing)$ and $(\nu, \partial\nu)$ represent the same class in this ring, and similar reasoning shows every class is represented by a disc bundle over a G–fixed set and its boundary. This map is known to be injective for G compact abelian [HamO72, p. 173]. More generally there is a ring map $\rho_\Omega\colon \Omega_*^{U:G}(X) \longrightarrow \Omega_*^{U:G}[\mathscr{A}, \mathscr{P}](X)$ natural in G-spaces X and similarly there are maps $mu_*^G(X) \longrightarrow mu_*^G[\mathscr{A}, \mathscr{P}](X)$ and, taking colimits, $\rho_{MU}\colon MU_*^G(X) \longrightarrow MU_*^G[\mathscr{A}, \mathscr{P}]$.

There is a natural additive description for $\Omega_*^{U:G}[\mathscr{A}, \mathscr{P}]$. Let J be a set of nontrivial irreducible complex G-representations, containing each exactly once up to isomorphism. Then the interior E of a complex disk bundle $\overline{E} \to E^G$ isotypically decomposes over each component N of E^G as a direct sum over $V \in J$ of vector bundles of the form $F_V \otimes V$. Such bundles F_V are classified by maps $N \longrightarrow BU(m)$, so that $\Omega_n^{U:G}[\mathscr{A}, \mathscr{P}]$ is isomorphic to the direct sum of the groups $MU_{2m_0}\big(\prod_V BU(m_V)\big)$ over lists $(m_0, m_V)_{V \in J}$ of nonnegative numbers with $m_0 + \sum m_V|V| = m$ [tD70, §4].

It is also possible to describe $MU_*^G[\mathscr{A}, \mathscr{P}]$ in terms of fixed points, restricting a representative $S^{W+n} := S^W \wedge S^n \to MU_{|W|}^G$ of a class in MU_n^G to $S^{|W^G|+n} \to (MU_{|W|}^G)^G$. Since maps to $BU_{|W|}^G$ classify complex G-bundles, and each $|W|$-dimensional G-representation can be written uniquely up to isomorphism as

[9] The oldest version, due to Boardman, is $MO_*(-) \longrightarrow \mathrm{Hom}\big(H^*(BO;\mathbb{F}_2), H^*(-;\mathbb{F}_2)\big)$.

[10] Such a notion exists for any pair $\mathscr{F} \supsetneq \mathscr{G}$ of sets of subgroups each stable under subgroup inclusion and conjugacy, and induction along such pairs is an important proof technique. There is a tautological long exact sequence $\cdots \to \Omega_n^{U:G}[\mathscr{G}] \to \Omega_n^{U:G}[\mathscr{F}] \to \Omega_n^{U:G}[\mathscr{F}, \mathscr{G}] \xrightarrow{\partial} \Omega_{n-1}^{U:G}[\mathscr{G}] \to \cdots$ due already to Conner and Floyd [CF62] in the unoriented case and inducing other such sequences by stabilization, localization, and completion.

$\mathbb{C}^{m_0} \oplus \bigoplus_{V\in J} V^{\oplus m_V}$ for some list $(m_0, m_V)_{V\in J}$ with $m_0 + \sum_V m_V|V| = |W|$, one finds $(MU^G_{|W|})^G$ is the wedge sum over such lists of $MU(m_0) \wedge \big(\prod_{V\in J} BU(m_V)\big)_+$, and so the fixed-point map represents an element of the $\big(|W^G| + n\big)^{\text{th}}$ homotopy group of the wedge. Stabilizing, let us agree to write $BU^{\oplus J}$ for the subspace of the product BU^J containing those points all but finitely many of whose components are the basepoint. It can then be shown, after reshuffling, that the geometric fixed point spectrum $\Phi^G MU^G$ decomposes as a wedge, indexed by elements W of the augmentation ideal $I(G) \lhd R(G)$ of virtual representations of dimension 0, of spectra $S^{|W^G|} \wedge MU \wedge BU^{\oplus J}$, and that $MU^G_*[\mathscr{A}, \mathscr{P}] \cong \pi_*(\Phi^G MU^G)$ [tD70, §2][Sin01, §4].

To relate this back to MU^G_* we must introduce Euler classes. Each G-representation V is a fiber of the universal G-bundle the Thom construction converts the fiber inclusion to a G-map $u_V\colon S^V \longhookrightarrow MU^G_{|V|}$. The MU^G-homological Thom isomorphism $\widetilde{MU}{}^G_{*+|V|}(S^V) \xrightarrow{\sim} MU^G_*$ is given by applying homotopy groups to the composition of $\mathrm{id} \wedge u_V\colon MU^G_n \wedge S^V \to MU^G_n \wedge MU^G_{|V|}$ and the spectrum multiplication, which is to say u_V represents the Thom class of the bundle $V \to *$ in MU^G_*. The element of $\pi_{-|V|}(MU^G) = MU^G_{-|V|}(*)$ given by restricting u_V to $S^0 = \{0, \infty\}$ is the Euler class e_V of this bundle. Smashing, we see $u_V \cdot u_W = u_{V\oplus W}$ and $e_V \cdot e_W = e_{V\oplus W}$.

It is not obvious *a priori* that the e_V are interesting. In fact they vanish when V has nontrivial invariant subspace, for the representative above factors through $S^0 \longhookrightarrow S^{V^G}$, which is equivariantly nullhomotopic for $|V^G| > 0$. Otherwise, however, they are nonzero.[11] In fact [tD70, Lem. 2.2][Sin01, §4], carefully following through the identification of $MU^G_*[\mathscr{A}, \mathscr{P}]$ with $\pi_*(\Phi^G MU^G)$ yields graded MU_*-module isomorphisms

$$\pi_*(\Phi^G MU^G) = MU_*\Big(\bigvee_{W\in I(G)} S^{|W^G|} \wedge BU^{\oplus J}\Big) \cong \mathbb{Z}[e_V^{\pm 1}]_{V\in J} \otimes MU_*\big(BU^{\oplus J}\big)$$

such that the image of $e_V \in MU^G_*$ under $MU^G_* \to MU^G_*[\mathscr{A}, \mathscr{P}] \to \mathbb{Z}[e_V^{\pm 1}] \otimes MU_*(BU^{\oplus J})$ is $e_V \otimes 1$. The wedge summand of $\Phi^G MU^G$ indexed by a formal difference of representations $W = W' - W'' \in I(G)$ corresponds to the summand $\mathbb{Z}e_{W'/(W')^G} \cdot e^{-1}_{W''/(W'')^G} \otimes MU_*(BU^{\oplus J})$. The ring $MU_*(BU^{\oplus J})$ is $MU_*(BU)^{\otimes J}$ by the Künneth formula and the fact MU_* commutes with colimits, which in turn is isomorphic to $MU_*[X_2, X_4, X_6, \ldots]^{\otimes J}$ by the collapse of the Atiyah–Hirzebruch spectral sequence for $MU_*(BU)$. A usual choice of such $X_{2d} \in MU_{2d}(BU)$ is represented by the standard compositions $\chi_{2d}\colon \mathbb{C}\mathrm{P}^d \hookrightarrow \mathbb{C}\mathrm{P}^\infty = BU(1) \hookrightarrow BU$, corresponding in $MU^G_*[\mathscr{A}, \mathscr{P}]$ to the disc bundles of the tautological line bundles $\gamma^d \to \mathbb{C}\mathrm{P}^d$. Thus we finally have an isomorphism

$$\chi_{MU}\colon MU^G_*[\mathscr{A}, \mathscr{P}] \xrightarrow{\sim} MU_*[X_{V,2d}, e_V^{\pm 1}] \qquad (V \in J,\ d \geqslant 1),$$

where $X_{V,2d} \in MU_*(BU^{\oplus J})$ comes from the copy of X_{2d} in the V^{th} $MU_*(BU)$ factor.

The map $\Omega^{U:G}_*[\mathscr{A}, \mathscr{P}] \xrightarrow{\sim} \bigoplus MU_{2m_0}\big(\prod_V BU(m_V)\big)$ also extends to a map

$$\chi_\Omega\colon \Omega^{U:G}_*[\mathscr{A}, \mathscr{P}] \longrightarrow \mathbb{Z}[e_V^{\pm 1}] \otimes MU_*(BU^{\oplus J})$$

in a natural way. Extending the homotopy class of a map $f\colon N \longrightarrow \bigoplus_V BU(m_V)$ via the natural map $i\colon \bigoplus_V BU(m_V) \longrightarrow BU^{\oplus J}$ preserves the stable isomorphism classes of the isotopic components $E_V \otimes V$ of the corresponding disc bundle $E \to N$, but loses the dimension data $|E| = |N| + \sum_V m_V|V|$. To remember the dimensions and make the composite $\chi_W \circ \rho_\Omega\colon \Omega^{U:G}_* \to \Omega^{U:G}_*[\mathscr{A}, \mathscr{P}] \to \mathbb{Z}[e_V^{\pm 1}] \otimes MU_*(BU^{\oplus J})$ preserve the grading, we instead take $\chi_\Omega[f] := \prod_V e_V^{-m_V} \otimes [i \circ f]$.

One might suspect the composites $\chi_\Omega \circ \rho_\Omega$ and $\chi_{MU} \circ \rho_{MU} \circ \Psi\colon \Omega^{U:G}_* \longrightarrow MU_*[X_{V,2d}, e_V^{\pm 1}]$ should agree. Instead [tD70, §4], they differ by postcomposition by the involution $\tau := MU_*(\iota)^{\oplus J} \otimes \mathrm{id}$, where ι is the H-space inverse on BU. To see this, embedding a stably complex G-manifold M in some large representation W, the normal bundles $\nu_M(M^G)$ and $\nu_M(W)|_{M^G}$ are complementary in $\nu_W(M^G) = \nu_{W^G}(M^G) \oplus \nu_W(W^G)|_{M^G}$. As $\nu_{W^G}(M^G)$ has trivial G-action and $\nu_W(W^G)$ is a product bundle, the classifying maps of the V-isotypic components of the two normal bundles $\nu_M(M^G)$ and $\nu_M(W)|_{M^G}$ are stably inverse with respect to Whitney sum for each V.

[11] And since they are negative-dimensional, they do not come from mu^G_*, showing that $mu^G_* \longrightarrow MU^G_*$ is not surjective before stabilization and the stabilization maps $mu^G_*(X) \longrightarrow mu^G_{*+|W|}(S^W \wedge X)$ are not all surjective.

Further comparison is facilitated by the identification of the image of $\Omega_*^{U:G}[\mathscr{A},\mathscr{P}] \to MU_*^G[\mathscr{A},\mathscr{P}] \xrightarrow{\sim} MU_*[X_{V,2d}, e_V^{\pm 1}]$. For $d \geqslant \frac{1}{2}|V|$ write $y_{V,2d} \in \Omega_{2d}^G[\mathscr{A},\mathscr{P}]$ for the class represented by the disc bundle of $\gamma^{d-|V|/2} \otimes V \to \mathbb{C}\mathrm{P}^{d-|V|/2}$, classified by the inclusion $\mathbb{C}\mathrm{P}^{d-|V|/2} \hookrightarrow \mathbb{C}\mathrm{P}^\infty = BU(1)$ in the V^{th} coordinate. We write $Y_{V,2d} = \chi_\Omega(y_{V,2d}) = X_{V,2d-|V|} e_V^{-1}$.[12] Evidently we may exchange generators in $MU_*[X_{V,2d}, e_V^{\pm 1}] = MU_*[Y_{V,2d}, e_V^{\pm 1}]$, and then Hanke [Han05, Prop. 4] proves that χ_Ω is an isomorphism onto the subring $MU_*[V_{V,2d}, e_V^{-1}]$. Thus χ_Ω can be seen as an algebraic localization inverting the e_V^{-1}. Hanke [Han05] shows that when $G = T$ is a torus, then the following is an injective pullback square:

$$\begin{tikzcd}
\Omega_*^{U:T} \arrow[r, "\rho_\Omega"] \arrow[d, "\Psi"'] & \Omega_*^{U:T}[\mathscr{A},\mathscr{P}] \arrow[r, "\chi_\Omega", "\sim"'] \arrow[d, "{\Psi[\mathscr{A},\mathscr{P}]}"'] & MU_*[V_{V,2d}, e_V^{-1}] \arrow[d, hook] \\
MU_*^T \arrow[r, "\rho_{MU}"'] & MU_*^T[\mathscr{A},\mathscr{P}] \arrow[r, "\sim", "\tau\circ\chi_{MU}"'] & MU_*[V_{V,2d}, e_V^{\pm 1}].
\end{tikzcd} \tag{1.3}$$

That is, $\Omega_*^{U:T}$ is the intersection of MU_*^T and $MU_*[Z_{V,2d}, e_V^{-1}]$ in $MU_*[Z_{V,2d}, e_V^{\pm 1}]$.

Algebraic localization

Localization in the geometric sense of $MU_*^G \longrightarrow \pi_*(\Phi^G MU^G)$ turns out to agree with the algebraic localization inverting the Euler classes e_V. Let us write $S < MU_*^G$ for the multiplicative submonoid generated by the Euler classes e_V. As these become invertible in $MU_*^G[\mathscr{A},\mathscr{P}] \cong MU_*[Y_{V,2d}, e_V^{\pm 1}]$, there is an induced homomorphism $S^{-1}MU_*^G \longrightarrow MU_*^G[\mathscr{A},\mathscr{P}]$, which tom Dieck shows is an isomorphism [tD70, Thm. 3.1]. Sinha [Sin01, Thm. 5.1] showed that when G is abelian, multiplication by e_V is injective if and only if G acts transitively on the unit sphere SV, which is in particular the case for all representations nontrivial on the identity component T. Thus the localization map $MU_*^T \longrightarrow S^{-1}MU_*^T \cong MU_*^T[\mathscr{A},\mathscr{P}]$ is injective, but for all nontoral G the corresponding map is non-injective.

It can be shown [Sin01, Prop. 4.14][Han05, p.685] that $\tau \circ \chi_{MU} \circ \rho_{MU} \circ \Psi$ takes the class of the projectivized representation $P(\mathbb{C}^d \oplus V)$ in $\Omega_{2d+|V|}^{U:G}$ to $Y_{V,2d} + e_{V^*}^{-d}$ if $|V| = 2$, where V^* is the dual representation, and to $Y_{V,2d}$ if $|V| \geqslant 4$. Write $Z_{V,2d}$ for this class in either event. Then when $G = T$ is a torus, using $\tau \circ \chi_{MU} \circ \rho_{MU}$ to identify MU_*^T as a subring of $MU_*[Z_{V,2d}, e_V^{\pm 1}] = MU_*[Y_{V,2d}, e_V^{\pm 1}]$, Sinha [Sin01, Thm. 1.2] found inclusions

$$MU_*[Z_{V,2d}, e_V] \leqslant MU_*^T \leqslant S^{-1}MU_*^T \cong MU_*[Z_{V,2d}, e_V^{\pm 1}].$$

The bundling and characteristic number maps induce MU_*^G-module structures on their codomains, so one can localize them with respect to S as well.

$$\begin{tikzcd}
MU_*^G \arrow[r, "\eta"] \arrow[d] & MU^*(BG) \arrow[r] \arrow[d] & H^*(BG)[[\vec{a}]] \arrow[d] & MU^*(BG) \arrow[r] \arrow[d] & H^*(BG)[[\vec{a}]] \arrow[d] \\
S^{-1}MU_*^G \arrow[r] & S^{-1}MU^*(BG) \arrow[r] & S^{-1}H^*(BG)[[\vec{a}]] & S^{-1}MU^*(BG) \arrow[r] & R(G)[[\vec{a}]]
\end{tikzcd} \tag{1.4}$$

It is known that some of these squares are pullbacks. tom Dieck [tD70, §5] showed MU_*^G is the pullback of the square with lower-right corner $S^{-1}MU^*(BG)$ for $G \cong \mathbb{Z}/p$, and adopted the phrasing that the values of the bundling map are "integral" in $S^{-1}MU^*(BG)$, meaning they appear without denominator. For $G = T$ a torus, Hattori [Hat74, Thm. 1.4] showed that $\Omega_*^{U:T}$ is the pullback of the square with lower-right corner $S^{-1}RT[[\vec{a}]]$. For topologically cyclic G, tom Dieck [tD74, Thm. 1] showed that the localized Boardman map $S^{-1}MU_*^G \longrightarrow S^{-1}R(G)[\vec{a}]$ is injective but for other G, the codomain is zero [tD74, Lem. 1], putting sharp limits on this sort of pullback identification.

[12] We prefer to index generators by dimension; Hanke and Sinha have $Y_{d,V}$ for these same elements.

Again for T a torus, Darby [D15, Prop. 3.5] shows a version of the pullback square determining the subring $\Omega_*^{U:T,\mathrm{fin}}$ represented by manifolds with finite fixed point sets as the pullback of a square with lower-right corner $S^{-1}MU_*^T$. Combined with Hanke's result this gives the same ring as the pullback of a square with lower-right corner $S^{-1}MU^*(BT)$.

Although we cannot have pullback squares of the above sort when localization is noninjective, the vertical maps in the right column are frequently known to be injective as well. For example, Hattori [Hat74, Thms. 1.3, 1.7] found that for G topologically cyclic, $S^{-1}\Omega_*^{U:G} \longrightarrow S^{-1}R(G)[[\vec{t}]]$ is injective, and for T a torus, $S^{-1}\Omega_*^{U:T} \longrightarrow S^{-1}MU^*(BT)$ is injective.

1.3. Explicit computations

The most relevant explicit computations of the coefficient ring for $G = T$ a torus are those of Sinha and Gusein-Zade [Sin01, GZ71]. Sinha determined a set of constraining relations on the image of MU_*^T in the localization $MU_*^T[e_V^{-1}] \cong MU_*[Z_{2d,V}, e_V^{\pm 1}]$, but the tools available did not show this set of relations to be complete unless $T = S^1$.

Restricting to semifree circle actions, Sinha found explicit presentations [Sin05, Thms. 3.6, 3.10]. One defines Ω_*^{SF} as the bordism ring of semifree stably complex S^1-manifolds and finds the natural map to the unrestricted $\Omega_*^{U:S^1}$ is injective [Sin05, Rmk. 2.5], so we will avail ourselves of our earlier notation. A spectrum MU^{SF} is defined as MU^G was, but using only the trivial representation $\mathbb{C}$, the standard representation t, and its conjugate $\bar{t}$. Sinha finds Ω_*^{SF} is a free MU_*-module injecting in $\Omega_*^{\mathrm{SF}}[\mathscr{A},\mathscr{P}] \cong MU_*[Z_{V,2d}, e_V^{-1}]_{V \in \{t,\bar{t}\}, d \geqslant 1}$, and similarly MU_*^{SF} contains $MU_*[Z_{V,2d}, e_V]$ and injects in $MU_*^{\mathrm{SF}}[\mathscr{A},\mathscr{P}] \cong MU_*[Z_{V,2d}, e_V^{\pm 1}]$, forming an injective pullback square [Sin05, §2]. He finds explicit algebra generators for Ω_*^{SF} and MU_*^{SF} and MU_*-module bases for each, which enable him to show a certain set of geometrically defined relations is complete in each case.

Returning to the unrestricted case, Musin [Mus83] found a natural set of generators for $\Omega_*^{U:S^1}$ and the fixed point–free ring $\Omega_*^{U:S^1}[\mathscr{P}]$. Gusein-Zade [GZ71] had earlier found an equational description of $\Omega^{U:S^1}$ in terms of its image under the injection $\Omega^{U:S^1} \longrightarrow \Omega^{U:S^1}[\mathscr{A},\mathscr{P}]$ but this description is stated via equations in a sextuply-indexed array of power series dependent on several previous levels of power series, and so cannot be easily applied in practice [Mus83, Intro.] to determine if a given equivariant disc bundle actually arises as the normal bundle to the fixed point set of a manifold.

Similarly, in the finite realm, Miščenko [Miš69] equationally determined the image of $\Omega_*^{U:\mathbb{Z}/p}$ under the embedding in $\Omega_*^{U:\mathbb{Z}/p}[\mathscr{A},\mathscr{P}]$, Kosniowski [Kos76] supplied an explicit set of geometric generators, and Jack Carlisle has recently found presentations for $\Omega_*^{U:\mathbb{Z}/p}$. As for homotopical bordism, Kriz [Kriz99] found an expression for $MU_*^{\mathbb{Z}/p}$ as a pullback of a *non*injective square of maps involving a localization of a quotient of a power series ring, Strickland [Str01] then found a presentation for $MU_*^{\mathbb{Z}/2}$, Abram–Kriz [AK15] found an algebraic expression for MU_*^A for A finite abelian, and Hu has found presentations for $MU_*^{\mathbb{Z}/p^r}$. Leaving abelian groups behind, Hu–Kriz–Lu [HKL21] have also computed $MU_*^{\Sigma_3}$, where Σ_3 is the symmetric group on three letters.

2. Semifree actions

We have seen characterizations of bordism rings frequently employ homomorphic embeddings into larger rings. We adopt a variant of this approach to characterize semifree bordism with isolated fixed points.

Given a stably complex G-manifold M, for each finite list of numbers I there corresponds an equivariant Chern class $c_I^G(M)$ in the Borel cohomology $H^*(EG \otimes_G M)$, whose pushforward under the Gysin map corresponding to $M \to *$ is a so-called Borel ***equivariant Chern number*** $h_I(M) = \int_M c_I^G(M) \in H^{*-\dim M}BG$.[13] Not just any list (h_I) of elements of H^*BG arises as $\big(h_I(M)\big)$ for some M, of course; for example, when the total degree $|I|$ of c_I is less than the real dimension of M, then h_I is zero because $H^{<0}BT$ is.

[13] These can be compiled into a ring homomorphism $\Omega_*^{U:T} \longrightarrow H^*BT \widehat{\otimes} H_*BU$ if one likes, but we will not need this point of view in the present note.

When G is a torus, the Atiyah–Bott/Berline–Vergne localization formula [BeV82, AtB84] expresses the Chern numbers $h_I(M)$ entirely in terms of the image of M in $\Omega_*^{U:T}[\mathscr{A},\mathscr{P}]$. If $\nu\colon N \to M^T$ is the normal bundle, viewed as a regular neighborhood in M, then

$$\int_M c_I^G(M) = \int_{M^T} \frac{c_I^G(M)|_{M^T}}{e^T(\nu)} \tag{2.1}$$

as elements of H^*BT, where $e^T(\nu)$ is the Euler class of the induced bundle $\nu \otimes \mathrm{id}\colon N \otimes_T ET \longrightarrow M^T \otimes_T ET$. The expression on the right-hand side of (2.1), which we will call $\ell_I(\nu)$, is defined independently of whether $[\nu]$ lies in the image of the map from $\Omega_*^{U:T}$ or not, but if it does lie in that image, then we know from (2.1) that $\ell_I(\nu) = h_I(M) = 0$ for $|I| < |N|$

One can ask if these restrictions alone determine the image of $\Omega_*^{U:T} \longrightarrow \Omega_*^{U:T}[\mathscr{A},\mathscr{P}]$; that is, given a complex T-equivariant bundle $\nu\colon N \to X$ such that $\ell_I(\nu) = 0$ for $|I| < \dim N$, does it always "close up" to some closed, stably complex T-manifold M? We show the answer is yes for certain classes of actions and use this fact to recover the associated bordism rings, and particularly Sinha's Theorem 0.1.

Definition 2.2. We write t for the standard one-dimensional complex representation of $S^1 = \mathrm{U}(1)$, and $\bar{t}$ for its conjugate, and $V_j = t^{\oplus n-j} \oplus \bar{t}^{\oplus j}$. We will write u for the image of t under the standard isomorphism $\mathrm{Hom}(S^1, S^1) \xrightarrow{\sim} H^2(BS^1;\mathbb{Z})$ taking an irreducible representation V to the first Chern class of the associated complex line bundle $ES^1 \otimes_{S^1} V \to BS^1$, so that the image of $\bar{t}$ is $-u$.

Definition 2.3. ***Abstract isotropy data*** comprises a family of pairs (V_p, σ_p), indexed by a finite set P, of signs $\sigma_p \in \{\pm 1\}$, and complex S^1-representations V_p all of one common dimension. We say abstract isotropy data $(V_p, \sigma_p)_{p\in P}$ is ***semifree*** if each V_p is isomorphic to one of the representations V_j $(1 \leqslant j \leqslant n)$ for some fixed natural number $n > 1$. In this case we define $q\colon P \longrightarrow \{0, 1, \ldots, n\}$ by $V_p \cong V_{q(p)}$. The ***isotropy data*** of an oriented stably complex S^1-manifold M with isolated fixed points is the abstract isotropy data $(T_pM, \sigma_p)_{p \in M^{S^1}}$, where σ_p is 1 if the given orientation of M agrees with the orientation on T_pM induced by the stable complex structure and -1 otherwise.

If abstract isotropy data are viewed as multisets by forgetting P but remembering multiplicity, they form a semiring with addition given by disjoint union and multiplication given by $(V, \sigma) \cdot (V', \sigma') = (V \oplus V', \sigma\sigma')$. This semiring is generated by $(V, 1)$ and $(V, -1)$, where V runs over (isomorphism classes of) irreducible nontrivial representations. The ring localization of this semiring can be viewed as the ring of "abstract isotropy data up to bordism," defined by quotienting by $-(V, \sigma) \sim (V, -\sigma)$ and writing σV for the corresponding class. This ring is a polynomial $\mathbb{Z}$-algebra in the nonzero irreducible representations. Restricting to semifree abstract isotropy data, the subsemiring is generated by $(t, \pm 1), (\bar{t}, \pm 1)$ and the ring is $\mathbb{Z}[t, \bar{t}]$. We will do a bit more than characterize the semifree bordism ring, in fact computing the subsemiring of geometrically realized isotropy data within the abstract isotropy data.

Given a compact, oriented, stably complex, semifree S^1-manifold M^{2n} with isolated fixed points and isotropy data $(V_p, \sigma_p)_{p\in P}$, the formula (2.1) specializes to the identities

$$0 = \int_M c_i(TM) = \sum_{p \in M^{S^1}} \frac{c_i(\overbrace{u, \ldots, u}^{n-q(p)}, \overbrace{-u, \ldots, -u}^{q(p)})}{\sigma_p u^{n-q(p)}(-u)^{q(p)}} = \sum_{p \in M^{S^1}} \frac{c_i(\overbrace{1, \ldots, 1}^{n-q(p)}, \overbrace{-1, \ldots, -1}^{q(p)})}{\sigma_p (-1)^{q(p)} u^{n-i}} \qquad (0 \leqslant i \leqslant n-1),$$

where c_i denotes both the i^{th} Chern class and the i^{th} elementary symmetric polynomial since the symbol σ is taken. Evidently we can multiply the i^{th} identity through by u^{n-i} without changing its content, so these identities are really statements about the integers σ_p and $q(p)$.

Definition 2.4. The ***ABBV identities*** for semifree abstract isotropy data are the equations

$$I_i = \sum_{p\in P} \sigma_p (-1)^{q(p)} c_i(\overbrace{1, \ldots, 1}^{n-q(p)}, \overbrace{-1, \ldots, -1}^{q(p)}) = 0 \qquad (0 \leqslant i \leqslant n-1). \tag{2.5}$$

We will demonstrate the converse, that if abstract isotropy data satisfies the ABBV identities, then it is the isotropy data of some action, by showing that the identities prescribe the function q so rigidly that all possible q arise from disjoint unions of the following known examples.

Example 2.6. Endow S^2 with the standard action and the complex structure of $\mathbb{C}\mathrm{P}^1$, so that the isotropy representation at the north pole z is t and that at the south pole $-z$ is $\bar{t}$. Let $\varepsilon_1, \ldots, \varepsilon_n$ each be 1 or -1. Endowing the direct power $(S^2)^n$ with the product complex action and diagonal S^1-action, the isotropy representation at the fixed point $(\varepsilon_1 z, \ldots, \varepsilon_n z)$ is V_j, where $j = \big|\{k : \varepsilon_k = -1\}\big|$, so that there are precisely $\binom{n}{j}$ fixed points p such that $T_p(S^2)^n \cong V_j$. One has $\sigma_p = 1$ at each fixed point p.

Example 2.7. The real S^1-representation $V_j \oplus \mathbb{R}$ carries a natural stable complex structure given by restriction of the trivial bundle $\underline{V}_j \oplus \underline{\mathbb{R}} \oplus \underline{\mathbb{R}} = \underline{V}_j \oplus \underline{\mathbb{C}}$, inducing a stably complex S^1-manifold structure on the unit sphere $S(V_j \oplus \mathbb{R})$. The S^1-action on $S(V_j \oplus \mathbb{R})$ has two fixed points, one with isotropy data $(V_j, 1)$ and the other with $(V_j, -1)$.

Of course only the sphere-powers of Example 2.6 are necessary to generate the bordism ring, the manifolds $S(V_j \oplus \mathbb{R})$ being nullbordant by definition, but both will be necessary to generate the semiring of abstract data satisfying the ABBV identities (2.5).

Let us now derive a more transparent form of these identities. In all of the following, n is fixed. Recall that for natural numbers j, k, the expression

$$\binom{j}{k} = \frac{j(j-1)\cdots(j-k+1)}{k(k-1)\cdots 1}$$

makes sense even if $k \geqslant j$, yielding the empty product 1 in case of equality and 0 if $k > j$. Thus we have a polynomial equation

$$C_i(j) := c_i(\overbrace{1, \ldots, 1}^{n-j}, \overbrace{-1, \ldots, -1}^{j}) = \sum_{k=0}^{i} \binom{n-j}{i-k} (1)^{i-k} \binom{j}{k} (-1)^k = \sum_{k=0}^{i} a_{i,k} j^i \in \mathbb{Q}[j]$$

for some rational numbers $a_{i,k}$ with $a_{i,i} \neq 0$. Thus we can write

$$I_i = \sum_{p \in P} \sigma_p (-1)^{q(p)} C_i\big(q(p)\big) = \sum_{k=0}^{i} a_{i,k} \underbrace{\sum_{p \in P} \sigma_p (-1)^{q(p)} q(p)^k}_{S_k}. \tag{2.8}$$

Assuming the ABBV identities $I_i = 0$ all hold, since the $a_{i,i}$ are nonzero, one finds inductively that $S_i = 0$ for each i as well.

Now we define notation for the multiplicities of the different representations:

$$\begin{aligned} m_j^+ &:= \big|\{p \in P : \sigma_p = 1 \text{ and } q(p) = j\}\big|, \\ m_j^- &:= \big|\{p \in P : \sigma_p = -1 \text{ and } q(p) = j\}\big|, \\ m_j &:= m_j^+ - m_j^- \end{aligned}$$

for $0 \leqslant j \leqslant n$. Gathering terms by q-value, the identities $S_i = 0$ then become

$$\sum_{j=0}^{n} (-1)^j m_j j^i = 0 \qquad (0 \leqslant i \leqslant n-1), \tag{2.9}$$

which can be written in matrix form as follows:

$$\begin{bmatrix} 1 & -1 & 1 & \cdots & (-1)^{n-1} \\ 1 & -2 & 3 & \cdots & (-1)^{n-1} n \\ 1 & -4 & 9 & \cdots & (-1)^{n-1} n^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & -2^{n-1} & 3^{n-1} & \cdots & (-1)^{n-1} n^{n-1} \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ \vdots \\ m_n \end{bmatrix} = \begin{bmatrix} m_0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{2.10}$$

The determinant of the square matrix is up to sign a Vandermonde determinant, hence nonzero, so the matrix is invertible. Multiplying both sides by the inverse shows $m_1, \ldots, m_n$ are uniquely determined by m_0, in fact scalar multiples of m_0, and we claim that in fact $m_j = m_0\binom{n}{j}$. But clearly these multiplicities are realized by the union M of m_0 disjoint copies of the standard $(S^2)^n$ of Example 2.7, and since the isotropy data of M satisfy (2.5), they also satisfy (2.9).[14] We thus have shown any semifree abstract isotropy data satifsying (2.5) satisfies

$$m_j = \binom{n}{j} m_0.^{15} \tag{2.12}$$

To realize arbitrary abstract isotropy data satisfying (2.12), assume first that m_0 is nonnegative, and let M' be the disjoint union of m_0 copies of $(S^2)^n$ with the standard orientation, so that for each j the isotropy data contains $m_j = m_j^+ - m_j^-$ instances of $(V_j, 1)$ and none of $(V_j, -1)$. We will realize the isotropy data successfully if we can add m_j^- instances each of $(V_j, 1)$ and $(V_j, -1)$ for each j, which we can do by taking the disjoint union of M' with m_j^- copies of $S(V_j \oplus \mathbb{R})$ from Example 2.7. To instead handle the case $m_0 < 0$, one can simply reverse orientations across the board. Thus, up to renaming fixed points, $(V_p, \sigma_p)_{p \in P}$ is the isotropy data of

$$M := \begin{cases} \displaystyle\coprod_{m_0} (S^2)^n \amalg \coprod_{j=0}^{n} \coprod_{m_j^-} S(V_j \oplus \mathbb{R}), & \text{if } m_0 \geqslant 0, \\ \displaystyle\coprod_{m_0} -(S^2)^n \amalg \coprod_{j=0}^{n} \coprod_{m_j^+} S(V_j \oplus \mathbb{R}), & \text{if } m_0 \leqslant 0. \end{cases}$$

We have proved the following.

Theorem 2.13. *Any semifree abstract isotropy data $(V_p, \sigma_p)_{p \in P}$ satisfying the ABBV identities (2.5) is the isotropy data of a compact, oriented, stably complex, semifree S^1-manifold M^{2n} with isolated fixed points.*

Now recall from the discussion after Definition 2.3 that the ring of local data for geometric semifree S^1-equivariant complex bordism is $\mathbb{Z}[t, \bar{t}]$. The image of $S(V_j \oplus \mathbb{R})$ in this ring is zero, and that of S^2 is $t + \bar{t}$. Since the map from the geometric bordism ring to $\mathbb{Z}[t, \bar{t}]$ is injective, Theorem 2.13 shows its image is $\mathbb{Z}[t + \bar{t}] = \mathbb{Z}[S^2]$, yielding Sinha's Theorem 0.1.

Remark 2.14. Similar reasoning yields another result of Sinha [Sin01, Thm. 1.6], namely that any stably complex 4-dimensional S^1-manifold with precisely three fixed points is equivariantly cobordant to the projectivization $P(\mathbb{C} \oplus V \oplus W)$ for some irreducible S^1-representations V and W. Sinha mostly calculates using Euler classes in $S^{-1}MU_*^{S^1}$, but the result also follows by applying the ABBV formula (2.1) to c_0 and c_1 to determine relations amongst the six weights and three signs σ_p and comparing those for $P(\mathbb{C} \oplus V \oplus W)$. The same result also follows from theorems of Jang [J18, Thm. 7.1 (resp. Thm. 1.1)] classifying possible isotropy data for S^1-actions on compact, oriented 4-manifolds with three (resp. finitely many) fixed points.

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[14] To see this without topology, expand $(1 + x)^n$ by the binomial theorem, differentiate i times with respect to x, and and evaluate at $x = -1$ to find

$$0 = \sum_{j=0}^{n} \binom{n}{j} \frac{j!}{(j-i)!} (-1)^{j-i} \qquad (0 \leqslant i \leqslant n-1). \tag{2.11}$$

Since $j!/(j-i)!$ is a polynomial of degree i in the variable j, the right-hand side of (2.11) is a $\mathbb{Q}$-linear combination of the quantities

$$D_i = \sum_{j=0}^{n} (-1)^j \binom{n}{j} j^i.$$

That $D_0 = 0$ is (2.11) for $i = 0$, and that the other D_i vanish follows inductively from (2.11) by subtracting off multiples of $D_k = 0$ for $k < i$.

[15] Plugging this back into the expression (2.8), one finds the combinatorial identities

$$\sum_{j=0}^{n} \sum_{k=0}^{i} (-1)^{j+k} \binom{n}{j} \binom{n-j}{i-k} \binom{j}{k} = 0 \qquad (0 \leqslant i \leqslant n-1),$$

which do not seem to be otherwise obvious. It would be nice to have a combinatorial proof.

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References

[AK15] William C. Abram and Igor Kriz. The equivariant complex cobordism ring of a finite abelian group. *Math. Res. Lett.*, 22(6):1573–1588, 2015. arXiv:1509.08540, doi:10.4310/MRL.2015.v22.n6.a1.

[AtB84] Michael F. Atiyah and Raoul Bott. The moment map and equivariant cohomology. *Topology*, 23(1):1–28, 1984. doi:10.1016/0040-9383(84)90021-1.

[BeV82] Nicole Berline and Michèle Vergne. Classes caractéristiques équivariantes. Formule de localisation en cohomologie équivariante. *C. R. Acad. Sci. Paris*, 295(2):539–541, Nov 1982. http://gallica.bnf.fr/ark:/12148/bpt6k62356694/f77.

[BH72] Theodor Bröcker and Edward C. Hook. Stable equivariant bordism. *Math. Z.*, 129(3):269–277, 1972. https://www.maths.ed.ac.uk/~v1ranick/papers/hook2.pdf.

[BPR10] Victor Buchstaber, Taras Panov, and Nigel Ray. Toric genera. *Int. Math. Res. Not. IMRN*, 2010(16):3207–3262, 2010. doi:10.1093/imrn/rnp228.

[C96] Gustavo Comezaña. Some computations in equivariant complex cobordism. In J. Peter May, editor, *Equivariant homotopy and cohomology theory*, number 91. Amer. Math. Soc., 1996.

[CF62] Pierre E. Conner and Edwin Earl Floyd. Differentiable periodic maps. *Bull. Amer. Math. Soc.*, 68(2):76–86, 1962. doi:10.1090/S0002-9904-1962-10730-7.

[CF66] Pierre E. Conner and Edwin Earl Floyd. *The relation of cobordism to K-theories*, volume 28 of *Lecture Notes in Math.* Springer, 1966.

[D] Alastair Darby. *Quasitoric manifolds in equivariant complex bordism*. Dissertation, University of Manchester, Jan. 2013. https://www.research.manchester.ac.uk/portal/files/54533790/FULL_TEXT.PDF

[D15] Alastair Darby. Torus manifolds in equivariant complex bordism. *Topology Appl.*, 189:31–64, 2015. arXiv:1409.2720, doi:10.1016/j.topol.2015.03.014.

[G01] J.P.C. Greenlees. Equivariant formal group laws and complex oriented cohomology theories. *Homology Homotopy Appl.*, 3(2):225–263, 2001. doi:10.4310/HHA.2001.v3.n2.a1.

[GM97] John P.C. Greenlees and J. Peter May. Localization and completion theorems for MU-module spectra. *Ann. of Math. (2)*, 146(3):509–544, 1997. doi:10.2307/2952455.

[GZ71] Sabir Medzhidovich Gusein-Zade. U-actions of a circle and fixed points. *Izv. Math.*, 5(5):1127–1143, 1971. English transl. of *Izv. Ross. Akad. Nauk Ser. Mat.*, 35(5):1120–1136, 1971 (Russian). doi:10.1070/IM1971v005n05ABEH001209.

[HamO72] Gary Hamrick and Erich Ossa. Unitary bordism of monogenic groups and isometries. In *Proceedings of the Second Conference on Compact Transformation Groups*, pages 172–182. Springer, 1972. doi:10.1007/BFb0070041.

[Han05] Bernhard Hanke. Geometric versus homotopy theoretic equivariant bordism. *Math. Ann.*, 332(3):677–696, 2005. arXiv:math/0412550, doi:10.1007/s00208-005-0648-0.

[HanW18] Bernhard Hanke and Michael Wiemeler. An equivariant Quillen theorem. *Adv. Math.*, 340:48–75, 2018. doi:10.1016/j.aim.2018.10.009.

[Hat66] Akio Hattori. Integral characteristic numbers for weakly almost complex manifolds. *Topology*, 5(3):259–280, 1966. http://imperium.lenin.ru/~kaledin/math/hattori.pdf.

[Hat74] Akio Hattori. Equivariant characteristic numbers and integrality theorem for unitary T^n-manifolds. *Tôhoku Math. J. (2)*, 26(3):461–482, 1974. doi:10.2748/tmj/1178241139.

[Haus19] Markus Hausmann. Global group laws and equivariant bordism rings. 2019. arXiv:1912.07583.

[HKL21] Po Hu, Igor Kriz, and Yunze Lu. Coefficients of the Σ_3-equivariant complex cobordism ring, Sep 2021. arXiv:2109.00991.

[J18] Donghoon Jang. Circle actions on oriented manifolds with discrete fixed point sets and classification in dimension 4. *Journal of Geometry and Physics*, 133:181–194, 2018. arXiv:1703.95464, doi:10.1016/j.geomphys.2018.07.010.

[Kos76] Czes Kosniowski. Generators of the $\mathbb{Z}/p$ bordism ring—Serendipity. *Math. Z.*, 149(2):121–130, 1976. doi:10.1007/BF01301570.

[Kriz99] Igor Kriz. The $\mathbb{Z}/p$-equivariant complex cobordism ring. In Morava, Meyer, and Wilson, editors, *Homotopy Invariant Algebraic Structures: A Conference in Honor of J. Michael Boardman*, volume 239 of *Contemp. Math.*, pages 217–224. Amer. Math. Soc., 1999. doi:10.1090/conm/239/03603.

[L73] Peter Löffler. Characteristic numbers of unitary torus-manifolds. *Bull. Amer. Math. Soc.*, 79(6):1262–1263, 1973. doi:10.1090/S0002-9904-1973-13406-8.

[L74] Peter Löffler. Bordismengruppen unitärer torusmannigfaltigkeiten. *Manuscripta Math.*, 12(4):307–327, 1974. doi:10.1007/BF01171078.

[LV21] Marco La Vecchia. The completion and local cohomology theorems for complex cobordism for all compact Lie groups. 2021. arXiv:2107.03093.

[LW18] Zhi Lü and Wei Wang. Equivariant cohomology Chern numbers determine equivariant unitary bordism for torus groups. *Algebraic & Geometric Topology*, 18(7):4143–4160, 2018. arXiv:1408.2134, doi:10.2140/agt.2018.18.4143.

[Miš69] A.S. Miščenko. Bordisms with the action of the group Z_p and fixed points. *Math. USSR Sb.*, 9(3):291, 1969. doi:10.1070/SM1969v009n03ABEH001285.

[Mus83] Oleg Rustamovich Musin. Generators of S^1-bordism. *Math. USSR Sb.*, 44(3):325, 1983. doi:10.1070/SM1983v044n03ABEH000970.

[O82] Christian Okonek. Der Conner-Floyd-Isomorphismus für Abelsche Gruppen. *Math. Z.*, 179(2):201–212, 1982. doi:10.1007/bf01214312.

[Sin01] Dev P. Sinha. Computations of complex equivariant bordism rings. *Amer. J. Math.*, 123(4):577–605, 2001. arXiv:9910024.

[Sin05] Dev P. Sinha. Bordism of semi-free S^1-actions. *Math. Z.*, 249(2):439–454, 2005. arXiv:math/0303100, doi:10.1007/s00209-004-0707-3.

[St65] Robert E. Stong. Relations among characteristic numbers—I. *Topology*, 4(3):267–281, 1965. doi:10.1016/0040-9383(65)90011-X.

[Str01] Neil P. Strickland. Complex cobordism of involutions. *Geom. Topol.*, 5:335–345, 2001. arXiv:math/0105020, doi:10.2140/gt.2001.5.335.

[tD70] Tammo tom Dieck. Bordism of G-manifolds and integrality theorems. *Topology*, 9(4):345–358, 1970. doi:10.1016/0040-9383(70)90058-3.

[tD74] Tammo tom Dieck. Characteristic numbers of G-manifolds II. *J. Pure Appl. Algebra*, 4(1):31–39, 1974. doi:10.1016/0022-4049(74)90027-9.

[U18] Bernardo Uribe. The evenness conjecture in equivariant unitary bordism. In *Proceedings of the International Congress of Mathematicians: Rio de Janeiro 2018*, pages 1217–1239, 2018. arXiv:1807.02446, doi:10.1142/9789813272880_0094.

[W69] Arthur G. Wasserman. Equivariant differential topology. *Topology*, 8(2):127–150, 1969. doi:10.1016/0040-9383(69)90005-6.

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