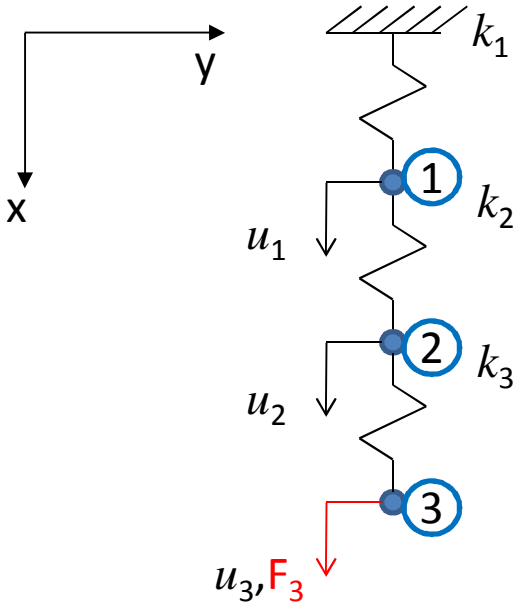
Matrix Assembly in FEA

Solving the Matrix Equations

•In Chapter 2, we spoke about how the global matrix equations are assembled in the finite element method. We now want to revisit that discussion and add some details. For example, during matrix assembly, how do we know (in general) how to associate nodal DoFs with matrix row and column entries?

•First, let's revisit the three-spring system we used as an example



•When we first looked at this problem, we weren't concerned with node numbering. With the system at the left the spring one is constrained at one of it's nodes (not shown). At that time, we simply called it node 0

•This time, we're going to be more careful and use everything we've already learned about finite element analysis to determine how this problem is actually solved in real software today

Solving the Matrix Equations

•First, we know that any general purpose code will use the numerical (rather than the canonical) spring formulation because of its generality (we don't want to be tied to any one spring orientation)

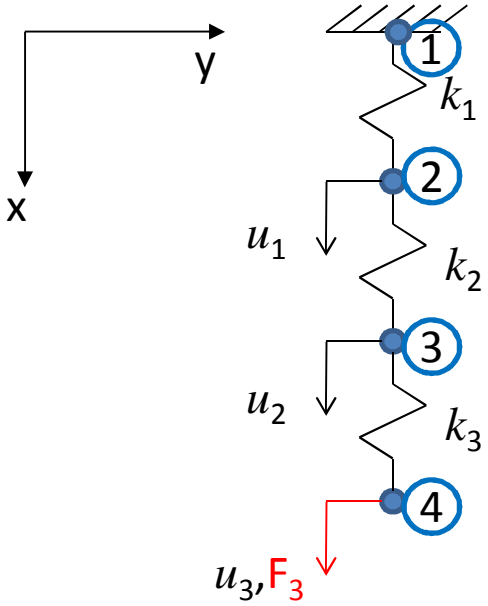
•So, the element matrices we be of the form:

$$k\begin{pmatrix} c^2 & cs & -c^2 & -cs \\ cs & s^2 & -cs & -s^2 \\ -c^2 & -cs & c^2 & cs \\ -cs & -s^2 & cs & s^2 \end{pmatrix}\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix} = \begin{Bmatrix} F_{1x} \\ F_{1y} \\ F_{2x} \\ F_{2y} \end{Bmatrix}$$

•The cosine and sine terms we be quickly evaluated early (before assembly). If we do the same we end up with (for all springs)…

Solving the Matrix Equations

$$k\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix} = \begin{Bmatrix} F_{1x} \\ F_{1y} \\ F_{2x} \\ F_{2y} \end{Bmatrix} \qquad (1)$$



•We also need to number all the nodes starting with the number 1. This will help in developing an automated scheme for matrix assembly. Specifically, what we want to do is come up with an automated means of associating a DoF with matrix rows and columns. This will be done by associating DoFs with node numbers

Solving the Matrix Equations

•A simple and common way to assocate DoFs with node numbers is to number them with a scheme such as:

$$DoF\ \# = n*i-q, \quad q=\{q \mid q \in \mathbb{N}, 0 \le q \le n\}$$

where i is the node number, n is the number of DoFs per node, and q is an integer than ranges between 0 and n (inclusive)

•Thus, for example, element 1 (k1) with nodes 1 and 2, would receive the following DoF indices (third column below):

element #	node #	q	n	DoF # (n*i-q)	description
1	1	1	2	1	ux_1
1	1	0	2	2	uy_1
1	2	1	2	3	ux_2
1	2	0	2	4	uy_2

Solving the Matrix Equations

•And, similarly for elements 2 and 3:

element #	node #	q	n	DoF # (n*i-q)	description
2	2	1	2	3	Ux_2
2	2	0	2	4	Uy_2
2	3	1	2	5	Ux_3
2	3	0	2	6	uy_3

element #	node #	q	n	DoF # (n*i-q)	description
3	3	1	2	5	Ux_3
3	3	0	2	6	Uy_3
3	4	1	2	7	Ux_4
3	4	0	2	8	uy_4

•Notice the last DoF number (8) corresponds with the size of the assembled matrix (which we already knew thanks to our estimate of number of nodes x number of DoFs per node):

Solving the Matrix Equations

•We now have everything we need to assemble the global stiffness matrix and applied load vector (the system equations). Recall from Chapter 2 that the Direct Stiffness Method (DSM) gives us the assembly algorithm:

$$\mathbf{K} = \sum_{n=1}^{N} \mathbf{k}_n \qquad (2)$$

$$\mathbf{F} = \sum_{n=1}^{N} \mathbf{f}_n$$

Where **K** and **F** are the assembled stiffness matrix and force vector, respectively, and **k**n and **f**n are the element stiffness matrix and force vector for element n in the global reference frame (this last part is important).

•And the global matrix indices for each element (and associated load vector) are given by the three tables we saw previously

Solving the Matrix Equations

•So, to keep things simple, lets assume that the k1=k2=k3=k. We can re-rewrite (1) using the global indices:

$$\mathbf{k}_1\mathbf{u}_1 = \mathbf{f}_1 = k\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \qquad (3)$$

Similarly, for the other elements:

$$\mathbf{k}_2\mathbf{u}_2 = \mathbf{f}_2 = k\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}\begin{Bmatrix} u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \qquad (4)$$

$$\mathbf{k}_3\mathbf{u}_3 = \mathbf{f}_3 = k\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}\begin{Bmatrix} u_5 \\ u_6 \\ u_7 \\ u_8 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ F \end{Bmatrix} \qquad (5)$$

Solving the Matrix Equations

•The global matrix equation is assembled by first initializing an 8 x 8 matrix and an 8 x 1 vector for K and F respectively. Then equation (2) is applied using (3), (4), and (5) to yield:

$$k\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 2 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 2 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ F \end{Bmatrix} \qquad (6)$$

•But now we have a problem. What to do about the zeros on the diagonal? These correspond to the fact that the springs have no stiffness in the x-direction. Although we don't have any loads in the x-direction, we still have a singular system. If we were solving this system by hand, we might simply eliminate the rows and columns with zeros in them. However, this is not a robust programming solution

Solving the Matrix Equations

•A simpler programming approach (which allows us to keep the same matrix size throughout) would be to use the "zero-one rule". This rule operates as follows. For each prescribed DoF, u_J:

1. Subtract $K_{iJ} \cdot u_J$ from the RHS
2. Set entries on row J equal to zero
3. Set the diagonal member on row J (JJ) equal to 1 and set $F_J = u_J$

•Application of these rules to u_2 , but also all x DoFs yields:

$$k\begin{bmatrix} 1/k & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/k & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/k & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/k & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 2 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/k & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ F \end{Bmatrix} \qquad (7)$$

*This is a common solution. Note, however that a complication arises if there are forces in the x-direction. Typical solutions to this complication can yield misleading results

Solving the Matrix Equations

•Solution of (7) yields:

$$\mathbf{u} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ F/k \\ 0 \\ 2F/k \\ 0 \\ 3F/k \end{Bmatrix}$$

•Which we immediately recognize to be the correct solution

•Now, we have applied the zero-one rule, not only to the applied zero y-displacement at node 1 (u_2), but also to all the DoFs in the x-direction. We have done this because:

- •We know we don't have DoFs in the x-direction (so setting them equal to zero shouldn't hurt anything. For exceptions, see the footnote on the previous slide). We'll revisit this concept later
- •We can't solve the system otherwise

Solving the Matrix Equations

•Equation (7) reveals another important property of finite element equations – Namely, that they result in "sparse" systems. A sparse matrix is one which is made mostly of zeroes. Another helpful feature of finite element equations is that they tend to be "banded", meaning that the majority of nonzero terms cluster around the main diagonal (or can be made to do so). This feature greatly reduces solution times in "direct" solvers. We will spend some time explaining these concepts in more detail.

•An official definition of "bandwidth" is* the following: For a square banded matrix A=aij, all matrix elements are zero outside a diagonally bordered band whose range is given by constants k1 and k2. More formally:

$$a_{ij} = 0 \text{ if } j < i - k_1 \text{ or } j > i + k_2, \text{ where } k_1, k_2 \geq 0$$

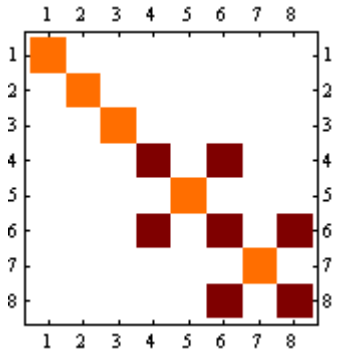
•The quantities k1 and k2 are called the left and right half-bandwidth. The matrix bandwidth is k_1+k_2+1

* http://en.wikipedia.org/wiki/Matrix_bandwidth

Solving the Matrix Equations

•In other words, the bandwidth is the smallest number of adjacent diagonals to which all nonzero elements are confined. Clearly, as the bandwidth approaches the matrix size, the matrix is no longer considered "banded"..

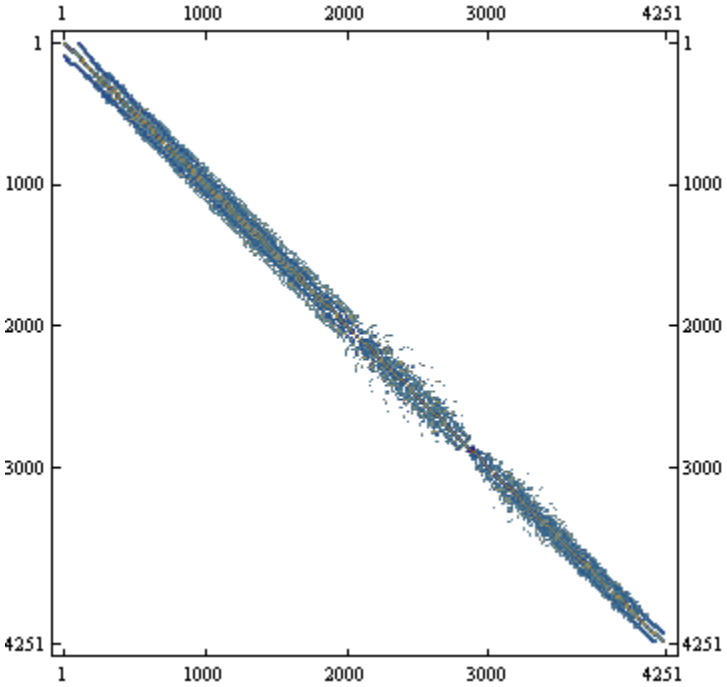
•Some examples: The K matrix of equation (7) has a half bandwidth of 2 and a bandwidth of 5. it is sometimes helpful to plot such matrices as a grid with nonzero entries represented as colored cells, and each cell of the grid representing a matrix element (symbolic math software tends to come with such functionality. We're using Mathematica). The stiffness matrix of (7) looks like this when plotted in this way:



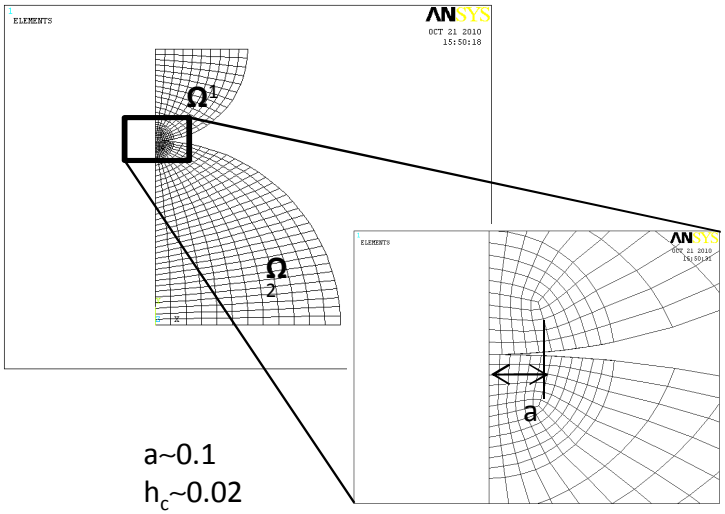
•We'll call such a plot "the sparsity pattern" of the matrix

Solving the Matrix Equations

•Below is the sparsity pattern of the stiffness matrix generated for the following plane stress problem



2100 nodes 2nd order quadrilaterals



a~0.1

h_c~0.02

Solving the Matrix Equations

•Hopefully, the student can now see the value of sparse matrices. Consider the case shown in the previous slide, for example. This matrix is 4251 x 4251. Now, if the computer is using double-precision arithmetic, each element of the matrix would get stored as a floating point number consisting of 8 bytes of memory*. This results in 4251 x 4251 x 8 bytes, or approximately 144 Mbytes of storage.

•One can see that this quickly gets out of hand, as the storage requirement grows in proportion to the square of the matrix size. For example, if the matrix size were increased by a factor of 10, the storage requirement would grow to approximately 14 Gbyte! In this case, a factor of 10 increase in matrix size corresponds to only a factor of 5 increase in model size (node count). In other words, a 10,000 node model would push the current limits on personal computing!

*on 64-bit machines using the IEEE 754 standard

Solving the Matrix Equations

•But what if we only had to store the nonzero matrix entries? In the previous case, this would result in a requirement of only around 4251 x 380 x 8 =12.9 Mbytes (almost a factor of 12 reduction!)

•Actually, this is still an over-estimate. What I did was multiply the number of rows by the bandwidth, which still contains lots of zeroes. This type of storage is called "banded" storage. Let's look at a more concrete example. Consider the following matrix (notice its asymmetric but banded):

$$\begin{bmatrix} B_{11} & B_{12} & 0 & 0 & 0 & 0 \\ B_{21} & B_{22} & B_{23} & 0 & 0 & 0 \\ 0 & 0 & B_{33} & 0 & 0 & 0 \\ 0 & 0 & B_{43} & B_{44} & B_{45} & 0 \\ 0 & 0 & 0 & 0 & B_{55} & 0 \\ 0 & 0 & 0 & 0 & B_{65} & B_{66} \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & B_{11} & B_{12} \\ B_{21} & B_{22} & B_{23} \\ 0 & B_{33} & 0 \\ B_{43} & B_{44} & B_{45} \\ 0 & B_{55} & 0 \\ B_{65} & B_{66} & 0 \end{bmatrix}$$

It can be stored much more efficiently in the 6 x 3 banded form on the right

Solving the Matrix Equations

•So, banded storage certainly helps. But one can see there's still room for improvement. In the previous example, we managed to reduce the storage requirement for a 6 x 6 down to a 6 x 3. But this is still 50 percent larger than the number of nonzero entries (12).

•So , we can see that, to the extent that not all banded matrices are "perfectly" banded (that is to say: still contain zeroes within their bands), and not all sparse matrices will be banded (for example, adding constraints tends to destroy bandedness), a more robust storage mechanism is required. Such storage mechanisms are usually referred to as "sparse"

•Solvers can be modified to operate on matrices stored in banded format without too much effort (both Cholesky and LU deocomposition methods, for example, can be easily modified to accommodate the banded format). Such banded storage schemes and solvers dominated much finite element code from the late 70's into the early 90's

Solving the Matrix Equations

•Sparse matrix storage involves storing the nonzero terms of a matrix in a vector. The row and column indices of each entry are stored in separate vectors. Several competing methods exist which are beyond the scope of this course. It is important, however, to know whether a particular linear solver operates on matrices stored in a sparse format or not.

•Because such solvers are relatively new to the marketplace (beginning in the 90's), not all commercial codes may utilize sparse matrix solvers or formats

Matrix Solvers

Solving the Matrix Equations

•There are basically two types of matrix solvers. We might call these "direct" and "iterative". Direct matrix solvers use some variant of the Gaussian elimination procedure, while iterative solvers start with an solution vector "guess", and then iterate until a residual is smaller than some threshold value.

•Depending on the software either or both of these methods might be available. Typically either or both will support either banded or sparse matrix formats.

Solving the Matrix Equations

Direct Methods

•Most direct methods use a factorization scheme (decomposition*) with back substitution**. What these means in practice is that, in order to be successful:

- •The system equations must be easily factorable (positive definite, for example)
- •You must have enough computer resources (usually RAM) in order to perform this costly step
- •The model must be fully constrained and not have material properties which vary greatly (Ratios of Young's modulus, for example, that exceed values of around 1,000)

•The last requirement is rather important. When material properties are vastly different, the matrix behaves as though the model is not properly constrained. Most solvers catch this during the "pivot" operation, in which they attempt to perform row or column operations to maximize the absolute value of numbers on the diagonal. This operation is important for stability and accuracy

see: http://en.wikipedia.org/wiki/Matrix_factorization
**: http://en.wikipedia.org/wiki/Back_substitution#Forward_and_back_substitution

Solving the Matrix Equations

Direct Methods

•We should mention here that some codes offer another variant of the direct method, which uses a Gaussian procedure to incrementally solve the model AS the matrices are being assembled! Such solvers are called “frontal” or “wavefront” solvers (this option is still available in ANSYS Mechanical APDL)

•The number of equations which are active after any element has been processed is called the “wavefront” at that time. The maximum size of this wavefront can be controlled. This is very beneficial when one has limited available RAM (this is why the procedure was once quite popular –it dates to a time when such memory was only available on the CPU itself – and then only at a fraction of what is available today)

•The total number of operations performed with this method, while still vastly smaller than if one solved the full (unbanded) system at once, is still greater than that achieved in sparse solvers. Again, the main benefit here is when available RAM is severely limited (as might be the case with an old computer)

Solving the Matrix Equations

Iterative Methods

•If a matrix fails to obey any or all of the previous requirements of direct solvers, it may be beneficial to use an iterative technique. Most modern commercial codes will have at least one such method available, which will operate either on a banded or sparse matrix format. The most popular and general method available today is the Conjugate Gradient Method (of which there many variants. This is the default solver you get when you choose an iterative solver in Workbench). Benefits of iterative methods are:

- •They tend to use less RAM than direct methods for the same problem size
- •They can still converge to reasonable solutions even when the matrices are ill-conditioned, as can occur when the model is either improperly constrained, or has considerably dissimilar material properties
- •Their performance is far less dependent on matrix properties such as symmetry. This makes them well-suited for nonlinear tasks such as contact with friction and fluid/structure interaction

Solving Structural Problems in ANSYS

Solving the Matrix Equations

•From the ANSYS 13.0 documentation (March, 2011. Performance Guide), we learn the following very important facts:

- •The amount of memory needed to assemble a matrix entirely within available RAM is approximately 1 Gbyte per million DoFs
- •The amount of memory needed to hold the factored matrix in memory is approximately 10 to 20 Gbyte per million DoFs

•The second bullet above is more relevant to our discussion. It means that if you want to use the direct solver (the default in Workbench), and you have a model with one million DoFs (not uncommon today), you need anywhere from 10 to Gbyte of RAM if you want to solve entirely within available RAM!

Solving the Matrix Equations

•The factor of 10-20 to 1 (factored matrix to assembled sparse matrix) is a direct result of the fact that decomposed triangular matrices (which is what the factorization routine tries to achieve) are necessarily dense, even if the parent matrix was very sparse!

•This represents a significant obstacle to the efficient solving of sparse systems. ANSYS offers their own rather unique solution to this restriction with an "out-of-core" solution capability. This is a proprietary means of solving the factored matrix incrementally by reading and writing pieces of it to the disk drive sequentially

•When this happens, the user is informed through the output window (Solution Information in Workbench) that an out-of-core solution is taking place. The user is also told how much RAM is required for a complete in-core solution

Solving the Matrix Equations

•By contrast the iterative family of solvers have the following basic system requirements:

- •Base amount of availabe RAM: 1 Gbyte per Million DoFs
- •Basic I/O requirement (disk read/write): 1.5 Gbyte per Million DoFs

•So one sees that the iterative solver is potentially much more efficient for large problems. Although a precise prediction of the performance can get rather complicated, it is the instructor's experience (and that of PADT generally) that the PCG solver usually produces the fastest results. In addition, we have found it to be superior in handling systems with many constraints. Students should beware, however, that this directly contradicts the ANSYS documentation