Weak Formulation of the 2D-Heat Equation

Jan Dezider Kees Koomen & Uri Cohen

January 26, 2018

1 The Time-Dependent Heat Equation

The 2D-heat equation we want to rewrite into weak form will be

$$c_1 \frac{\partial T}{\partial t} = c_2 \frac{\partial^2 T}{\partial x^2} + c_3 \frac{\partial^2 T}{\partial y^2}$$

with c_1, c_2 and c_3 as constants which will be evaluated later on. This form of the time-dependent heat equations governs only the heat conduction in the object. Thus no radiation into the surroundings and no internal heat generation. The temperature T is written as the product of the shape function matrix $\mathbf{N}(x, y)$ multiplied by the coefficient matrix $\phi^{(e)}(t)$ which contains the temperature on the nodes of the element as a function of time.

$$T^{(e)} = \mathbf{N}(x, y) \cdot \phi^{(e)}(t) \tag{1}$$

Next we apply the Galerkin Method to rewrite the partial differential equation into weak form.

$$\int \mathbf{N}^T \cdot \left(c_1 \frac{\partial T}{\partial t} - c_2 \frac{\partial^2 T}{\partial x^2} - c_3 \frac{\partial^2 T}{\partial y^2} \right) dA$$

Substituting formula 1 into equation 2 gives

$$c_1 \int \mathbf{N}^T \frac{\partial \phi^{(e)}}{\partial t} \mathbf{N} \, dA - c_2 \int \mathbf{N}^T \frac{\partial^2 \mathbf{N}}{\partial x^2} \phi^{(e)} \, dA - c_3 \int \mathbf{N}^T \frac{\partial^2 \mathbf{N}}{\partial y^2} \phi^{(e)} \, dA \tag{2}$$

In the following section we will be evaluating every term in this equation.

2 The time derivative

We start by rewriting the first term of equation 2

$$
\begin{aligned}
\mathbf{C}^{(e)}\dot{\phi}^{(e)} &= c_1 \int \mathbf{N}^T \frac{\partial \phi^{(e)}}{\partial t} \mathbf{N}\, dA \\
&= \left(c_1 \int \mathbf{N}^T \mathbf{N}\, dA \right) \frac{\partial \phi^{(e)}}{\partial t} \\
&= \left(c_1 \int \mathbf{N}^T \mathbf{N}\, dA \right) \dot{\phi}^{(e)}
\end{aligned}
$$

We will be applying the finite difference method on the $\dot{\phi}^{(e)}$ when having dealt with the spacial derivatives in equation 2.

3 The Spacial Derivatives

Since the second and third term of equation 2 are identical, we will work out the derivative with respect to x and "copy" the resulting equation for the derivative with respect to y.

We rewrite the second derivative in

$$
-c_2 \int \mathbf{N}^T \frac{\partial^2 \mathbf{N}}{\partial x^2} \phi^{(e)}\, dA \tag{3}
$$

by using the product rule of differentiation. Thus

$$
\mathbf{N}^T \frac{\partial^2 \mathbf{N}}{\partial x^2} = \frac{\partial}{\partial x}\left(\mathbf{N}^T \frac{\partial \mathbf{N}}{\partial x} \right) - \frac{\partial \mathbf{N}^T}{\partial x} \frac{\partial \mathbf{N}}{\partial x}
$$

Substituting this result in equation 3

$$
-c_2 \int \mathbf{N}^T \frac{\partial^2 \mathbf{N}}{\partial x^2} \phi^{(e)}\, dA = -c_2 \int \frac{\partial}{\partial x}\left(\mathbf{N}^T \frac{\partial \mathbf{N}}{\partial x} \right) \phi^{(e)}\, dA + c_2 \int \frac{\partial \mathbf{N}^T}{\partial x} \frac{\partial \mathbf{N}}{\partial x} \phi^{(e)}\, dA \tag{4}
$$

The first term in this expression can be rewritten using the divergence theorem as

$$
-c_2 \int \frac{\partial}{\partial x}\left(\mathbf{N}^T \frac{\partial \mathbf{N}}{\partial x} \right) \phi^{(e)}\, dA = -c_2 \int_\Gamma \mathbf{N}^T \frac{\partial \mathbf{N}}{\partial x} \phi^{(e)} \cos\theta\, d\Gamma
$$

Thus equation 4 becomes

$$
\begin{aligned}
-c_2 \int \mathbf{N}^T \frac{\partial^2 \mathbf{N}}{\partial x^2} \phi^{(e)}\, dA &= -c_2 \int \frac{\partial}{\partial x}\left(\mathbf{N}^T \frac{\partial \mathbf{N}}{\partial x} \right) \phi^{(e)}\, dA + c_2 \int \frac{\partial \mathbf{N}^T}{\partial x} \frac{\partial \mathbf{N}}{\partial x} \phi^{(e)}\, dA \\
&= c_2 \int \frac{\partial \mathbf{N}^T}{\partial x} \frac{\partial \mathbf{N}}{\partial x} \phi^{(e)}\, dA - c_2 \int_\Gamma \mathbf{N}^T \frac{\partial \mathbf{N}}{\partial x} \phi^{(e)} \cos\theta\, d\Gamma
\end{aligned}
$$

We can apply the same evaluating for the derivative with respect to y

$$
\begin{aligned}
-c_3 \int \mathbf{N}^T \frac{\partial^2 \mathbf{N}}{\partial y^2} \phi^{(e)}\, dA &= -c_3 \int \frac{\partial}{\partial y} \left(\mathbf{N}^T \frac{\partial \mathbf{N}}{\partial y} \right) \phi^{(e)}\, dA + c_3 \int \frac{\partial \mathbf{N}^T}{\partial y} \frac{\partial \mathbf{N}}{\partial y} \phi^{(e)}\, dA \\
&= c_3 \int \frac{\partial \mathbf{N}^T}{\partial y} \frac{\partial \mathbf{N}}{\partial y} \phi^{(e)}\, dA - c_3 \int_\Gamma \mathbf{N}^T \frac{\partial \mathbf{N}}{\partial y} \phi^{(e)} \sin\theta\, d\Gamma
\end{aligned}
$$

Thus the second derivative for x and y become

$$
\begin{aligned}
-c_2 \int \mathbf{N}^T \frac{\partial^2 \mathbf{N}}{\partial x^2} \phi^{(e)}\, dA - c_3 \int \mathbf{N}^T \frac{\partial^2 \mathbf{N}}{\partial y^2} \phi^{(e)}\, dA &= c_2 \int \frac{\partial \mathbf{N}^T}{\partial x} \frac{\partial \mathbf{N}}{\partial x} \phi^{(e)}\, dA - c_2 \int_\Gamma \mathbf{N}^T \frac{\partial \mathbf{N}}{\partial x} \phi^{(e)} \cos\theta\, d\Gamma \\
&\quad + c_3 \int \frac{\partial \mathbf{N}^T}{\partial y} \frac{\partial \mathbf{N}}{\partial y} \phi^{(e)}\, dA - c_3 \int_\Gamma \mathbf{N}^T \frac{\partial \mathbf{N}}{\partial y} \phi^{(e)} \sin\theta\, d\Gamma \\
&= -\int_\Gamma c_2 \mathbf{N}^T \frac{\partial \mathbf{N}}{\partial x} \phi^{(e)} \cos\theta + c_3 \mathbf{N}^T \frac{\partial \mathbf{N}}{\partial y} \phi^{(e)} \sin\theta\, d\Gamma \\
&\quad + \left(\int c_2 \frac{\partial \mathbf{N}^T}{\partial x} \frac{\partial \mathbf{N}}{\partial x} + c_3 \frac{\partial \mathbf{N}^T}{\partial y} \frac{\partial \mathbf{N}}{\partial y}\, dA \right) \phi^{(e)}
\end{aligned}
$$

Writing it more compactly

$$
\underbrace{-\int_\Gamma c_2 \mathbf{N}^T \frac{\partial \mathbf{N}}{\partial x} \phi^{(e)} \cos\theta + c_3 \mathbf{N}^T \frac{\partial \mathbf{N}}{\partial y} \phi^{(e)} \sin\theta\, d\Gamma}_{\mathbf{b}^{(e)}} + \underbrace{\left(\int c_2 \frac{\partial \mathbf{N}^T}{\partial x} \frac{\partial \mathbf{N}}{\partial x} + c_3 \frac{\partial \mathbf{N}^T}{\partial y} \frac{\partial \mathbf{N}}{\partial y}\, dA \right)}_{\mathbf{k}_D^{(e)}} \phi^{(e)}
$$

Thus the 2D time-dependent heat equation in weak form becomes

$$
\begin{aligned}
\left(c_1 \int \mathbf{N}^T \mathbf{N}\, dA \right) \dot{\phi}^{(e)} &- \int_\Gamma c_2 \mathbf{N}^T \frac{\partial \mathbf{N}}{\partial x} \phi^{(e)} \cos\theta + c_3 \mathbf{N}^T \frac{\partial \mathbf{N}}{\partial y} \phi^{(e)} \sin\theta\, d\Gamma \\
&+ \left(\int c_2 \frac{\partial \mathbf{N}^T}{\partial x} \frac{\partial \mathbf{N}}{\partial x} + c_3 \frac{\partial \mathbf{N}^T}{\partial y} \frac{\partial \mathbf{N}}{\partial y}\, dA \right) \phi^{(e)} = 0
\end{aligned}
$$

and in matrix form

$$
\mathbf{C}^{(e)} \dot{\phi}^{(e)} + \mathbf{k}_D^{(e)} \phi^{(e)} - \mathbf{b}^{(e)} = 0 \tag{5}
$$

4 The boundary

The matrix $\mathbf{b}^{(e)}$ can be written as a matrix $\mathbf{b}_I^{(e)}$ and $\mathbf{b}_b^{(e)}$

$$\mathbf{b}^{(e)} = \mathbf{b}_I^{(e)} + \mathbf{b}_b^{(e)}$$

The $\mathbf{b}_I^{(e)}$ matrix consist of elements which lying inside the object and $\mathbf{b}_b^{(e)}$ of elements lying at the boundaries of the object. We can ignore the $\mathbf{b}_I^{(e)}$ matrix because evaluating the surface integral of adjacent elements cancel each other out just like in the 1D case. We are left with the $\mathbf{b}_b^{(e)}$ matrix. Recall that $\mathbf{b}^{(e)}$

$$\mathbf{b}^{(e)} = \int_\Gamma c_2 \mathbf{N}^T \frac{\partial \mathbf{N}}{\partial x} \phi^{(e)} \cos\theta + c_3 \mathbf{N}^T \frac{\partial \mathbf{N}}{\partial y} \phi^{(e)} \sin\theta \, d\Gamma$$

using Fourier's heat convection formula, we can write this expression as

$$\begin{aligned}
\mathbf{b}^{(e)} &= \int_\Gamma c_2 \mathbf{N}^T \frac{\partial \mathbf{N}}{\partial x} \phi^{(e)} \cos\theta + c_3 \mathbf{N}^T \frac{\partial \mathbf{N}}{\partial y} \phi^{(e)} \sin\theta \, d\Gamma \\
&= \int_\Gamma -h\phi_b + hT_s \, d\Gamma
\end{aligned}$$

with ϕ_b being the temperature at the nodes of the boundary. And T_s the temperature of the surrounding. But, we will set $\mathbf{b}_b^{(e)}$ to zero since the temperature on the boundary will be set equal to the temperature of the heated metal with temperature T_m which will be placed on direct contact with the elements on the boundary.

5 Finite difference method

The global matrix which will be assembled from the element matrices will have the same form as equation 5. Recall that ϕ is a column matrix. I will write ϕ between curly brackets to avoid confusion.

$$\mathbf{C}\left\{\dot{\phi}\right\} + \mathbf{k}_D\left\{\phi\right\} - \mathbf{b} = 0$$

As stated in the previous section, we set the boundary matrix equal to zero thus

$$\mathbf{C}\left\{\dot{\phi}\right\} + \mathbf{k}_D\left\{\phi\right\} = 0 \tag{6}$$

We approximate $\dot{\phi}$ using the definition of the derivative.

$$\left\{\dot{\phi}\right\} \simeq \frac{\{\phi(t+\Delta t)\} - \{\phi(t)\}}{\Delta t}$$

Thus equation 6 becomes

$$\mathbf{C}\frac{\{\phi(t+\Delta t)\} - \{\phi(t)\}}{\Delta t} + \mathbf{k}_D\left\{\phi(t)\right\} = 0$$

Rewriting for $\{\phi(t+\Delta t)\}$

$$\mathbf{C}\{\phi(t+\Delta t)\} = -\mathbf{k}_D\{\phi(t)\}\Delta t + \mathbf{C}\{\phi(t)\}$$

We need to know $\{\phi(0)\}$ which is known for our problem. The temperature on the boundary is equal to T_m and for the elements inside the object, the temperature will be equal to $37°C$.

6 Shape functions and their derivatives

In this project we will be using triangular elements with linear interpolation functions. These are given as

$$\mathbf{N}(x, y) = \begin{bmatrix} N_1 & N_2 & N_3 \end{bmatrix}$$

with

$$\begin{aligned}
N_i &= \frac{1}{2A^{(e)}}\left[c_i + a_i x + b_i y\right] \\
N_1 &= \frac{1}{2A^{(e)}}\left[(x_2 y_3 - x_3 y_2) + (y_2 - y_3)\,x + (x_3 - x_2)\,y\right] \\
N_2 &= \frac{1}{2A^{(e)}}\left[(x_3 y_1 - x_1 y_3) + (y_3 - y_1)\,x + (x_1 - x_3)\,y\right] \\
N_3 &= \frac{1}{2A^{(e)}}\left[(x_1 y_2 - x_2 y_1) + (y_1 - y_2)\,x + (x_2 - x_1)\,y\right]
\end{aligned}$$

Thus the derivatives become

$$\begin{aligned}
\frac{\partial \mathbf{N}}{\partial x} &= \frac{1}{2A^{(e)}}\begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} \\
\frac{\partial \mathbf{N}}{\partial y} &= \frac{1}{2A^{(e)}}\begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix} \\
\frac{\partial \mathbf{N}^T}{\partial x} &= \frac{1}{2A^{(e)}}\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \\
\frac{\partial \mathbf{N}^T}{\partial y} &= \frac{1}{2A^{(e)}}\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}
\end{aligned}$$

And the products

$$\frac{\partial \mathbf{N}^T}{\partial x}\frac{\partial \mathbf{N}}{\partial x} = \frac{1}{4\left(A^{(e)}\right)^2}\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}\begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} = \frac{1}{4\left(A^{(e)}\right)^2}\begin{bmatrix} a_1^2 & a_1a_2 & a_1a_3 \\ a_2a_1 & a_2^2 & a_2a_3 \\ a_3a_1 & a_3a_2 & a_3^2 \end{bmatrix}$$

$$\frac{\partial \mathbf{N}^T}{\partial y}\frac{\partial \mathbf{N}}{\partial y} = \frac{1}{4\left(A^{(e)}\right)^2}\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}\begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix} = \frac{1}{4\left(A^{(e)}\right)^2}\begin{bmatrix} b_1^2 & b_1b_2 & b_1b_3 \\ b_2b_1 & b_2^2 & b_2b_3 \\ b_3b_1 & b_3b_2 & b_3^2 \end{bmatrix}$$

$$\mathbf{N}^T\mathbf{N} = \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix}\begin{bmatrix} N_1 & N_2 & N_3 \end{bmatrix} = \begin{bmatrix} N_1^2 & N_1N_2 & N_1N_3 \\ N_2N_1 & N_2^2 & N_2N_3 \\ N_3N_1 & N_3N_2 & N_3^2 \end{bmatrix}$$

We can evaluate $\mathbf{N}^T\mathbf{N}$ using the formula

$$\int_A N_1^m N_2^n N_3^p \, dA = \frac{m!n!p!}{(m+n+p+2)!}2A$$

thus

$$\int \mathbf{N}^T\mathbf{N}\, dA = \begin{bmatrix} N_1^2 & N_1N_2 & N_1N_3 \\ N_2N_1 & N_2^2 & N_2N_3 \\ N_3N_1 & N_3N_2 & N_3^2 \end{bmatrix} = \frac{A}{12}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

7 In conclusion

We arrived at the matrix equation

$$\mathbf{C}\left\{\phi(t+\Delta t)\right\} = -\mathbf{k}_D\left\{\phi(t)\right\}\Delta t + \mathbf{C}\left\{\phi(t)\right\}$$

With the calculated matrices

$$\begin{aligned}
\mathbf{C} &= \left(c_1 \int \mathbf{N}^T\mathbf{N}\, dA\right) = c_1\frac{A}{12}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \\
\mathbf{k}_D &= \left(\int c_2\frac{\partial \mathbf{N}^T}{\partial x}\frac{\partial \mathbf{N}}{\partial x} + c_3\frac{\partial \mathbf{N}^T}{\partial y}\frac{\partial \mathbf{N}}{\partial y}\, dA\right) \\
&= \frac{1}{4\left(A^{(e)}\right)^2}\int c_2\begin{bmatrix} a_1^2 & a_1a_2 & a_1a_3 \\ a_2a_1 & a_2^2 & a_2a_3 \\ a_3a_1 & a_3a_2 & a_3^2 \end{bmatrix} + c_3\begin{bmatrix} b_1^2 & b_1b_2 & b_1b_3 \\ b_2b_1 & b_2^2 & b_2b_3 \\ b_3b_1 & b_3b_2 & b_3^2 \end{bmatrix} dA \\
&= \frac{1}{4A^{(e)}}\left(c_2\begin{bmatrix} a_1^2 & a_1a_2 & a_1a_3 \\ a_2a_1 & a_2^2 & a_2a_3 \\ a_3a_1 & a_3a_2 & a_3^2 \end{bmatrix} + c_3\begin{bmatrix} b_1^2 & b_1b_2 & b_1b_3 \\ b_2b_1 & b_2^2 & b_2b_3 \\ b_3b_1 & b_3b_2 & b_3^2 \end{bmatrix}\right)
\end{aligned}$$

thus

$$\boxed{\begin{gathered}
\mathbf{C}\left\{\phi(t+\Delta t)\right\} = -\mathbf{k}_D\left\{\phi(t)\right\}\Delta t + \mathbf{C}\left\{\phi(t)\right\} \\
c_1\frac{A}{12}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}\left\{\phi(t+\Delta t)\right\} = -\frac{\Delta t}{4A^{(e)}}\left(c_2\begin{bmatrix} a_1^2 & a_1a_2 & a_1a_3 \\ a_2a_1 & a_2^2 & a_2a_3 \\ a_3a_1 & a_3a_2 & a_3^2 \end{bmatrix} + c_3\begin{bmatrix} b_1^2 & b_1b_2 & b_1b_3 \\ b_2b_1 & b_2^2 & b_2b_3 \\ b_3b_1 & b_3b_2 & b_3^2 \end{bmatrix}\right)\left\{\phi(t)\right\} \\
+ c_1\frac{A}{12}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}\left\{\phi(t)\right\}
\end{gathered}}$$