

Weak Formulation of the 2D-Heat Equation

Jan Dezider Kees Koomen & Uri Cohen

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1 The Time-Dependent Heat Equation

The 2D-heat equation we want to rewrite into weak form will be

$$c_1 \frac{\partial T}{\partial t} = c_2 \frac{\partial^2 T}{\partial x^2} + c_3 \frac{\partial^2 T}{\partial y^2}$$

with c_1 , c_2 and c_3 as constants which will be evaluated later on. This form of the time-dependent heat equations governs only the heat conduction in the object. Thus no radiation into the surroundings and no internal heat generation. The temperature T is written as the product of the shape function matrix $\mathbf{N}(x, y)$ multiplied by the coefficient matrix $\phi^{(e)}(t)$ which contains the temperature on the nodes of the element as a function of time.

$$T^{(e)} = \mathbf{N}(x, y) \cdot \phi^{(e)}(t) \quad (1)$$

Next we apply the Galerkin Method to rewrite the partial differential equation into weak form.

$$\int \mathbf{N}^T \cdot \left(c_1 \frac{\partial T}{\partial t} - c_2 \frac{\partial^2 T}{\partial x^2} - c_3 \frac{\partial^2 T}{\partial y^2} \right) dA$$

Substituting formula 1 into equation 2 gives

$$c_1 \int \mathbf{N}^T \frac{\partial \phi^{(e)}}{\partial t} \mathbf{N} dA - c_2 \int \mathbf{N}^T \frac{\partial^2 \mathbf{N}}{\partial x^2} \phi^{(e)} dA - c_3 \int \mathbf{N}^T \frac{\partial^2 \mathbf{N}}{\partial y^2} \phi^{(e)} dA \quad (2)$$

In the following section we will be evaluating every term in this equation.

2 The time derivative

We start by rewriting the first term of equation 2

$$\begin{aligned}\mathbf{C}^{(e)}\dot{\phi}^{(e)} &= c_1 \int \mathbf{N}^T \frac{\partial \phi^{(e)}}{\partial t} \mathbf{N} dA \\ &= \left(c_1 \int \mathbf{N}^T \mathbf{N} dA \right) \frac{\partial \phi^{(e)}}{\partial t} \\ &= \left(c_1 \int \mathbf{N}^T \mathbf{N} dA \right) \dot{\phi}^{(e)}\end{aligned}$$

We will be applying the finite difference method on the $\dot{\phi}^{(e)}$ when having dealt with the spacial derivatives in equation 2.

3 The Spacial Derivatives

Since the second and third term of equation 2 are identical, we will work out the derivative with respect to x and "copy" the resulting equation for the derivative with respect to y .

We rewrite the second derivative in

$$-c_2 \int \mathbf{N}^T \frac{\partial^2 \mathbf{N}}{\partial x^2} \phi^{(e)} dA \quad (3)$$

by using the product rule of differentiation. Thus

$$\mathbf{N}^T \frac{\partial^2 \mathbf{N}}{\partial x^2} = \frac{\partial}{\partial x} \left(\mathbf{N}^T \frac{\partial \mathbf{N}}{\partial x} \right) - \frac{\partial \mathbf{N}^T}{\partial x} \frac{\partial \mathbf{N}}{\partial x}$$

Substituting this result in equation 3

$$-c_2 \int \mathbf{N}^T \frac{\partial^2 \mathbf{N}}{\partial x^2} \phi^{(e)} dA = -c_2 \int \frac{\partial}{\partial x} \left(\mathbf{N}^T \frac{\partial \mathbf{N}}{\partial x} \right) \phi^{(e)} dA + c_2 \int \frac{\partial \mathbf{N}^T}{\partial x} \frac{\partial \mathbf{N}}{\partial x} \phi^{(e)} dA \quad (4)$$

The first term in this expression can be rewritten using the divergence theorem as

$$-c_2 \int \frac{\partial}{\partial x} \left(\mathbf{N}^T \frac{\partial \mathbf{N}}{\partial x} \right) \phi^{(e)} dA = -c_2 \int_{\Gamma} \mathbf{N}^T \frac{\partial \mathbf{N}}{\partial x} \phi^{(e)} \cos \theta d\Gamma$$

Thus equation 4 becomes

$$\begin{aligned}-c_2 \int \mathbf{N}^T \frac{\partial^2 \mathbf{N}}{\partial x^2} \phi^{(e)} dA &= -c_2 \int \frac{\partial}{\partial x} \left(\mathbf{N}^T \frac{\partial \mathbf{N}}{\partial x} \right) \phi^{(e)} dA + c_2 \int \frac{\partial \mathbf{N}^T}{\partial x} \frac{\partial \mathbf{N}}{\partial x} \phi^{(e)} dA \\ &= c_2 \int \frac{\partial \mathbf{N}^T}{\partial x} \frac{\partial \mathbf{N}}{\partial x} \phi^{(e)} dA - c_2 \int_{\Gamma} \mathbf{N}^T \frac{\partial \mathbf{N}}{\partial x} \phi^{(e)} \cos \theta d\Gamma\end{aligned}$$

We can apply the same evaluating for the derivative with respect to y

$$\begin{aligned} -c_3 \int \mathbf{N}^T \frac{\partial^2 \mathbf{N}}{\partial y^2} \phi^{(e)} dA &= -c_3 \int \frac{\partial}{\partial y} \left(\mathbf{N}^T \frac{\partial \mathbf{N}}{\partial y} \right) \phi^{(e)} dA + c_3 \int \frac{\partial \mathbf{N}^T}{\partial y} \frac{\partial \mathbf{N}}{\partial y} \phi^{(e)} dA \\ &= c_3 \int \frac{\partial \mathbf{N}^T}{\partial y} \frac{\partial \mathbf{N}}{\partial y} \phi^{(e)} dA - c_3 \int_{\Gamma} \mathbf{N}^T \frac{\partial \mathbf{N}}{\partial y} \phi^{(e)} \sin \theta d\Gamma \end{aligned}$$

Thus the second derivative for x and y become

$$\begin{aligned} -c_2 \int \mathbf{N}^T \frac{\partial^2 \mathbf{N}}{\partial x^2} \phi^{(e)} dA - c_3 \int \mathbf{N}^T \frac{\partial^2 \mathbf{N}}{\partial y^2} \phi^{(e)} dA &= c_2 \int \frac{\partial \mathbf{N}^T}{\partial x} \frac{\partial \mathbf{N}}{\partial x} \phi^{(e)} dA - c_2 \int_{\Gamma} \mathbf{N}^T \frac{\partial \mathbf{N}}{\partial x} \phi^{(e)} \cos \theta d\Gamma \\ &\quad + c_3 \int \frac{\partial \mathbf{N}^T}{\partial y} \frac{\partial \mathbf{N}}{\partial y} \phi^{(e)} dA - c_3 \int_{\Gamma} \mathbf{N}^T \frac{\partial \mathbf{N}}{\partial y} \phi^{(e)} \sin \theta d\Gamma \\ &= - \int_{\Gamma} c_2 \mathbf{N}^T \frac{\partial \mathbf{N}}{\partial x} \phi^{(e)} \cos \theta + c_3 \mathbf{N}^T \frac{\partial \mathbf{N}}{\partial y} \phi^{(e)} \sin \theta d\Gamma \\ &\quad + \left(\int c_2 \frac{\partial \mathbf{N}^T}{\partial x} \frac{\partial \mathbf{N}}{\partial x} \phi^{(e)} + c_3 \frac{\partial \mathbf{N}^T}{\partial y} \frac{\partial \mathbf{N}}{\partial y} dA \right) \phi^{(e)} \end{aligned}$$

Writing it more compactly

$$- \underbrace{\int_{\Gamma} c_2 \mathbf{N}^T \frac{\partial \mathbf{N}}{\partial x} \phi^{(e)} \cos \theta + c_3 \mathbf{N}^T \frac{\partial \mathbf{N}}{\partial y} \phi^{(e)} \sin \theta d\Gamma}_{\mathbf{b}^{(e)}} + \underbrace{\left(\int c_2 \frac{\partial \mathbf{N}^T}{\partial x} \frac{\partial \mathbf{N}}{\partial x} + c_3 \frac{\partial \mathbf{N}^T}{\partial y} \frac{\partial \mathbf{N}}{\partial y} dA \right)}_{\mathbf{k}_D^{(e)}} \phi^{(e)}$$

Thus the 2D time-dependent heat equation in weak form becomes

$$\begin{aligned} \left(c_1 \int \mathbf{N}^T \mathbf{N} dA \right) \dot{\phi}^{(e)} - \int_{\Gamma} c_2 \mathbf{N}^T \frac{\partial \mathbf{N}}{\partial x} \phi^{(e)} \cos \theta + c_3 \mathbf{N}^T \frac{\partial \mathbf{N}}{\partial y} \phi^{(e)} \sin \theta d\Gamma \\ + \left(\int c_2 \frac{\partial \mathbf{N}^T}{\partial x} \frac{\partial \mathbf{N}}{\partial x} + c_3 \frac{\partial \mathbf{N}^T}{\partial y} \frac{\partial \mathbf{N}}{\partial y} dA \right) \phi^{(e)} = 0 \end{aligned}$$

and in matrix form

$$\mathbf{C}^{(e)} \dot{\phi}^{(e)} + \mathbf{k}_D^{(e)} \phi^{(e)} - \mathbf{b}^{(e)} = 0 \quad (5)$$

4 The boundary

The matrix $\mathbf{b}^{(e)}$ can be written as a matrix $\mathbf{b}_I^{(e)}$ and $\mathbf{b}_b^{(e)}$

$$\mathbf{b}^{(e)} = \mathbf{b}_I^{(e)} + \mathbf{b}_b^{(e)}$$

The $\mathbf{b}_I^{(e)}$ matrix consist of elements which lying inside the object and $\mathbf{b}_b^{(e)}$ of elements lying at the boundaries of the object. We can ignore the $\mathbf{b}_I^{(e)}$ matrix because evaluating the surface integral of adjacent elements cancel each other out just like in the 1D case. We are left with the $\mathbf{b}_b^{(e)}$ matrix. Recall that $\mathbf{b}^{(e)}$

$$\mathbf{b}^{(e)} = \int_{\Gamma} c_2 \mathbf{N}^T \frac{\partial \mathbf{N}}{\partial x} \phi^{(e)} \cos \theta + c_3 \mathbf{N}^T \frac{\partial \mathbf{N}}{\partial y} \phi^{(e)} \sin \theta d\Gamma$$

using Fourier's heat convection formula, we can write this expression as

$$\begin{aligned} \mathbf{b}^{(e)} &= \int_{\Gamma} c_2 \mathbf{N}^T \frac{\partial \mathbf{N}}{\partial x} \phi^{(e)} \cos \theta + c_3 \mathbf{N}^T \frac{\partial \mathbf{N}}{\partial y} \phi^{(e)} \sin \theta d\Gamma \\ &= \int_{\Gamma} -h\phi_b + hT_s d\Gamma \end{aligned}$$

with ϕ_b being the temperature at the nodes of the boundary. And T_s the temperature of the surrounding. But, we will set $\mathbf{b}_b^{(e)}$ to zero since the temperature on the boundary will be set equal to the temperature of the heated metal with temperature T_m which will be placed on direct contact with the elements on the boundary.

5 Finite difference method

The global matrix which will be assembled from the element matrices will have the same form as equation 5. Recall that ϕ is a column matrix. I will write ϕ between curly brackets to avoid confusion.

$$\mathbf{C} \left\{ \dot{\phi} \right\} + \mathbf{k}_D \{ \phi \} - \mathbf{b} = 0$$

As stated in the previous section, we set the boundary matrix equal to zero thus

$$\mathbf{C} \left\{ \dot{\phi} \right\} + \mathbf{k}_D \{ \phi \} = 0 \tag{6}$$

We approximate $\dot{\phi}$ using the definition of the derivative.

$$\left\{ \dot{\phi} \right\} \cong \frac{\{ \phi(t + \Delta t) \} - \{ \phi(t) \}}{\Delta t}$$

Thus equation 6 becomes

$$\mathbf{C} \frac{\{ \phi(t + \Delta t) \} - \{ \phi(t) \}}{\Delta t} + \mathbf{k}_D \{ \phi(t) \} = 0$$

Rewriting for $\{\phi(t + \Delta t)\}$

$$\mathbf{C} \{\phi(t + \Delta t)\} = -\mathbf{k}_D \{\phi(t)\} \Delta t + \mathbf{C} \{\phi(t)\}$$

We need to know $\{\phi(0)\}$ which is known for our problem. The temperature on the boundary is equal to T_m and for the elements inside the object, the temperature will be equal to $37^\circ C$.

6 Shape functions and their derivatives

In this project we will be using triangular elements with linear interpolation functions. These are given as

$$\mathbf{N}(x, y) = \begin{bmatrix} N_1 & N_2 & N_3 \end{bmatrix}$$

with

$$\begin{aligned} N_i &= \frac{1}{2A^{(e)}} [c_i + a_i x + b_i y] \\ N_1 &= \frac{1}{2A^{(e)}} [(x_2 y_3 - x_3 y_2) + (y_2 - y_3) x + (x_3 - x_2) y] \\ N_2 &= \frac{1}{2A^{(e)}} [(x_3 y_1 - x_1 y_3) + (y_3 - y_1) x + (x_1 - x_3) y] \\ N_3 &= \frac{1}{2A^{(e)}} [(x_1 y_2 - x_2 y_1) + (y_1 - y_2) x + (x_2 - x_1) y] \end{aligned}$$

Thus the derivatives become

$$\begin{aligned} \frac{\partial \mathbf{N}}{\partial x} &= \frac{1}{2A^{(e)}} \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} \\ \frac{\partial \mathbf{N}}{\partial y} &= \frac{1}{2A^{(e)}} \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix} \\ \frac{\partial \mathbf{N}^T}{\partial x} &= \frac{1}{2A^{(e)}} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \\ \frac{\partial \mathbf{N}^T}{\partial y} &= \frac{1}{2A^{(e)}} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \end{aligned}$$

And the products

$$\begin{aligned}\frac{\partial \mathbf{N}^T}{\partial x} \frac{\partial \mathbf{N}}{\partial x} &= \frac{1}{4(A^{(e)})^2} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} = \frac{1}{4(A^{(e)})^2} \begin{bmatrix} a_1^2 & a_1 a_2 & a_1 a_3 \\ a_2 a_1 & a_2^2 & a_2 a_3 \\ a_3 a_1 & a_3 a_2 & a_3^2 \end{bmatrix} \\ \frac{\partial \mathbf{N}^T}{\partial y} \frac{\partial \mathbf{N}}{\partial y} &= \frac{1}{4(A^{(e)})^2} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix} = \frac{1}{4(A^{(e)})^2} \begin{bmatrix} b_1^2 & b_1 b_2 & b_1 b_3 \\ b_2 b_1 & b_2^2 & b_2 b_3 \\ b_3 b_1 & b_3 b_2 & b_3^2 \end{bmatrix} \\ \mathbf{N}^T \mathbf{N} &= \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix} \begin{bmatrix} N_1 & N_2 & N_3 \end{bmatrix} = \begin{bmatrix} N_1^2 & N_1 N_2 & N_1 N_3 \\ N_2 N_1 & N_2^2 & N_2 N_3 \\ N_3 N_1 & N_3 N_2 & N_3^2 \end{bmatrix}\end{aligned}$$

We can evaluate $\mathbf{N}^T \mathbf{N}$ using the formula

$$\int_A N_1^m N_2^n N_3^p dA = \frac{m!n!p!}{(m+n+p+2)!} 2A$$

thus

$$\int \mathbf{N}^T \mathbf{N} dA = \begin{bmatrix} N_1^2 & N_1 N_2 & N_1 N_3 \\ N_2 N_1 & N_2^2 & N_2 N_3 \\ N_3 N_1 & N_3 N_2 & N_3^2 \end{bmatrix} = \frac{A}{12} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

7 In conclusion

We arrived at the matrix equation

$$\mathbf{C} \{ \phi(t + \Delta t) \} = -\mathbf{k}_D \{ \phi(t) \} \Delta t + \mathbf{C} \{ \phi(t) \}$$

With the calculated matrices

$$\begin{aligned} \mathbf{C} &= \left(c_1 \int \mathbf{N}^T \mathbf{N} dA \right) = c_1 \frac{A}{12} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \\ \mathbf{k}_D &= \left(\int c_2 \frac{\partial \mathbf{N}^T}{\partial x} \frac{\partial \mathbf{N}}{\partial x} \phi^{(e)} + c_3 \frac{\partial \mathbf{N}^T}{\partial y} \frac{\partial \mathbf{N}}{\partial y} dA \right) \\ &= \frac{1}{4(A^{(e)})^2} \int c_2 \begin{bmatrix} a_1^2 & a_1 a_2 & a_1 a_3 \\ a_2 a_1 & a_2^2 & a_2 a_3 \\ a_3 a_1 & a_3 a_2 & a_3^2 \end{bmatrix} + c_3 \begin{bmatrix} b_1^2 & b_1 b_2 & b_1 b_3 \\ b_2 b_1 & b_2^2 & b_2 b_3 \\ b_3 b_1 & b_3 b_2 & b_3^2 \end{bmatrix} dA \\ &= \frac{1}{4A^{(e)}} \left(c_2 \begin{bmatrix} a_1^2 & a_1 a_2 & a_1 a_3 \\ a_2 a_1 & a_2^2 & a_2 a_3 \\ a_3 a_1 & a_3 a_2 & a_3^2 \end{bmatrix} + c_3 \begin{bmatrix} b_1^2 & b_1 b_2 & b_1 b_3 \\ b_2 b_1 & b_2^2 & b_2 b_3 \\ b_3 b_1 & b_3 b_2 & b_3^2 \end{bmatrix} \right) \end{aligned}$$

thus

$$\begin{aligned} \mathbf{C} \{ \phi(t + \Delta t) \} &= -\mathbf{k}_D \{ \phi(t) \} \Delta t + \mathbf{C} \{ \phi(t) \} \\ c_1 \frac{A}{12} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \{ \phi(t + \Delta t) \} &= -\frac{\Delta t}{4A^{(e)}} \left(c_2 \begin{bmatrix} a_1^2 & a_1 a_2 & a_1 a_3 \\ a_2 a_1 & a_2^2 & a_2 a_3 \\ a_3 a_1 & a_3 a_2 & a_3^2 \end{bmatrix} + c_3 \begin{bmatrix} b_1^2 & b_1 b_2 & b_1 b_3 \\ b_2 b_1 & b_2^2 & b_2 b_3 \\ b_3 b_1 & b_3 b_2 & b_3^2 \end{bmatrix} \right) \{ \phi(t) \} \\ &\quad + c_1 \frac{A}{12} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \{ \phi(t) \} \end{aligned}$$