EN3057: Coursework #1

Due on Friday, December 9^{th}, 2016

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Contents

Note: I have used the symbol $\times$ (multiplication sign) to denote the multiplication of non-vector/scalar quantities - despite its prohibition in the introduction. Hopefully, you agree that the use of the sign does not cause confusion in the places that I have used it. Equations 1.1.3 and 1.1.4 are much more readable with a multiplication sign.

Note: I have included the feedback form at the end of this document, I hope you don't mind it being in a different style!

1 Section 1 - Control System Design using Frequency Response

1.1 Exercise #1

Consider a closed loop system (as shown in Figure 1). The controller provides a proportional control action $K(s) = k$. If the process transfer function is given by $G(s)$ as:

$$G(s) = \frac{4(s+3)}{(s+1)(s+2)(s+6)} \tag{1.1.1}$$

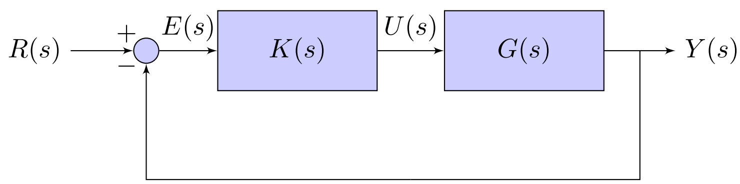


Figure 1: Feedback system for section 1.

1.1.1 (a)

Calculate the steady-state error for a unit step input.

In order to find the steady-state error, it is not entirely necessary to find the closed-loop transfer function, but it can be helpful and serve as a sanity check because it will provide us with information about the type of the system. We therefore find the closed-loop transfer function. This involves some simple block diagram reduction and then some algebraic manipulation using $G(s)$, as in Equation 1.1.1.

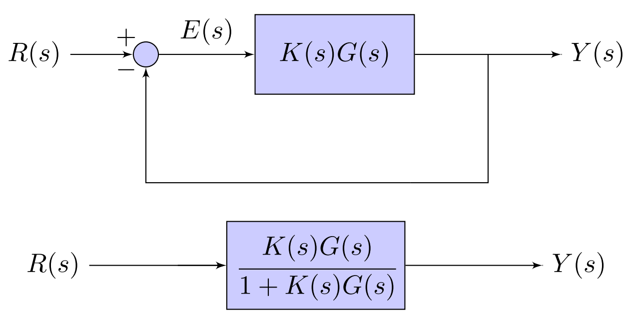


Figure 2: Block diagram simplification

(a)

After further simplification, a transfer function, $P(s)$, is arrived at, as shown in Equation 1.1.2. Note that I have related the inputs and outputs to their Laplacian equivalencies ($\mathcal{L}[y(t)] = Y(s)$ and $\mathcal{L}[r(t)] = R(s)$).

$$P(s) \equiv \frac{Y(s)}{R(s)} = \frac{K(s)G(s)}{1+K(s)G(s)} \tag{1.1.2}$$

Substituting in given expressions for $K(s)$ and $G(s)$:

$$\begin{aligned} Y(s) &= \frac{k \times \frac{4(s+3)}{(s+1)(s+2)(s+6)}}{1+k \times \frac{4(s+3)}{(s+1)(s+2)(s+6)}} \times R(s) \\ Y(s) &= \frac{4k(s+3)}{(s+1)(s+2)(s+6)+4k(s+3)} \times R(s) \equiv \frac{4k(s+3)}{4ks+12k+s^3+9s^2+20s+12} \times R(s) \end{aligned} \tag{1.1.3}$$

From here, it is clear that we are dealing with a type 0 system, so we know our steady-state error formula will be something like, $\frac{1}{1+k_p}$, where K_p is a Static Position Error Constant. This is because there is a pole of multiplicity $N = 0$ in the denominator of our closed-loop transfer function, $P(s)$ i.e. the denominator is all multiplied by s^0. It should be noted that for a step function: $R(s) = 1/s$. With all of this in mind, we are able to proceed with our working.

$$\begin{aligned} E_c(s) &= \frac{1}{1+K(s)G(s)} \times R(s) \\ &= \frac{1}{1+k \times \frac{4(s+3)}{(s+1)(s+2)(s+6)}} \times \frac{1}{s} \\ &= \frac{(s+1)(s+2)(s+6)}{s \times [(s+1)(s+2)(s+6)+4k(s+3)]} \end{aligned} \tag{1.1.4}$$

(a)

It is relatively easy to find the limit of this equation (as derived in Equation 1.1.4) using MATLAB (shown below Equation 1.1.5).

$$\begin{aligned} e_c(\infty) = \lim_{s\to 0} sE_c(s) &= \lim_{s\to 0} s\left[\frac{(s+1)(s+2)(s+6)}{s \times [(s+1)(s+2)(s+6)+4k(s+3)]}\right] \\ &= \lim_{s\to 0}\left[\frac{(s+1)(s+2)(s+6)}{[(s+1)(s+2)(s+6)+4k(s+3)]}\right] \\ &= \frac{(2)(6)}{(2)(6)+4k(3)} = \frac{12}{12+12k} = \frac{1}{1+k} \end{aligned} \tag{1.1.5}$$

```
>> syms s k;
t=limit(((s+1)*(s+2)*(s+6))/((s+1)*(s+2)*(s+6)+4*k*(s+3)),s,0);
simplify(t) % output

ans =

1/(k + 1)
```

This finally leaves us with our assumption that the e_{ss} for a type 0 system is:

$$e_{ss} = \frac{1}{1+k} \tag{1.1.6}$$

1.1.2 (b)

Modify controller $K(s)$ to achieve a steady-state error of zero for a unit step input.

There are a number of ways to do this. Clearly, by making k a scalar gain proportional controller the response is not really satisfactory for most applications due to the huge oscillations between the first second or so. This is really observable in Figure 3.

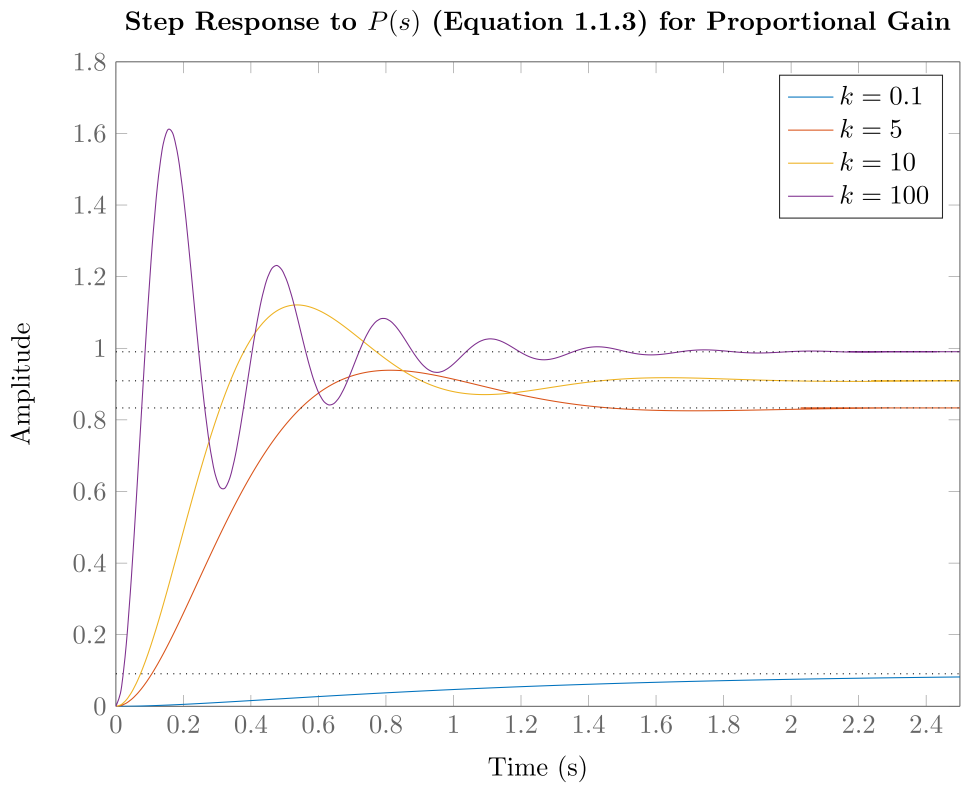


Figure 3: Step Response for $P(s)$ Plot with k as Scalar

(b)

This problem is overcome by introducing a k/s term or an integral action (with proportional gain) term for $K(s)$. You can see that for the $k = 1/4s$ plot of Figure 4, the response time is greatly increased but the desired steady-state value is reached and there is almost no over-shoot. Graphically, it seems that an integral action seems to stabilise the system but we should verify this algebraically before settling on a solution. An algebraic solution for the stability of the system is quite difficult to come to for all of k though, but it is possible to confirm that the steady state error, e_{ss} is indeed equal to 0. First, a derivation of $E_c(s)$ is made as in Equation 1.1.8, this is then used to calculate the steady state error as in Equation 1.1.9.

$$E_c(s) = \frac{1}{1 + K(s)G(s)} \times R(s) \tag{1.1.7}$$

$$
\begin{aligned}
E_c(s) &= \frac{1}{1 + {}^{k}/_{s} \times \frac{4(s+3)}{(s+1)(s+2)(s+6)}} \times \frac{1}{s} \\
&= \frac{s \times [(s+1)(s+2)(s+6)]}{s \times [(s+1)(s+2)(s+6)] + 4k \times (s+3)}
\end{aligned} \tag{1.1.8}
$$

$$
\begin{aligned}
E_c(s) &= \frac{s \times [(s+1)(s+2)(s+6)]}{s \times [(s+1)(s+2)(s+6)] + 4k \times (s+3)} \\
e_c(\infty) = \lim_{s \to 0} sE_c(s) &= \lim_{s \to 0} s \left[\frac{s \times [(s+1)(s+2)(s+6)]}{s \times [(s+1)(s+2)(s+6)] + 4k \times (s+3)} \right] \\
&= \lim_{s \to 0} \left[\frac{s^2 \times [(s+1)(s+2)(s+6)]}{s \times [(s+1)(s+2)(s+6)] + 4k \times (s+3)} \right] \\
&= \frac{0 \times [(1)(2)(6)]}{0 \times [(1)(2)(6)] + 12k} = \frac{0}{0 + 12k} \\
&\therefore e_{ss} = 0
\end{aligned} \tag{1.1.9}
$$

Graphically, this algebraic solution holds, as for varrying values of k, the steady state error is observably equal to 0.
It is also worth mentioning that there could be an improvement or an optimisation upon this system by possibly introducing a P-I controller of the type: $K(s) = k_1 + {}^{k_2}/_{s}$.

Step Response to $P(s)$ (Equation 1.1.3) for an Integral Action with Proportional Gain (b)

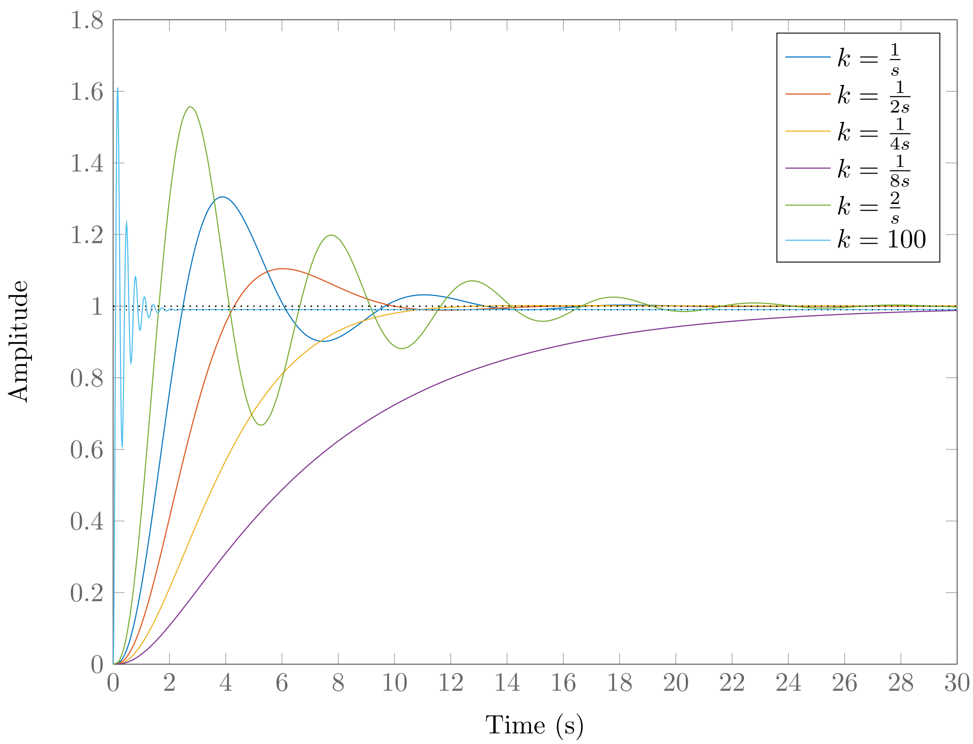


Figure 4: Step Response for $P(s)$ Plot with k as Integral Action

1.1.3 (c)

Propose a controller that would ensure a steady-state error of 0.25 for a unit ramp input.

Procedurally, this is very similar to part (b). However, one should observe that for a ramp input $R(s) = \frac{1}{s^2}$. We are asked to find a value of k which gives us an $e_{ss} = 0.25$.
Using Equation 1.1.3 and keeping the value of $R(s)$ needed in mind, a closed loop representation of the system for a ramp input can be found, which is only here for completeness:

$$Y(s) = \frac{4k(s+3)}{4ks^3 + 12ks^2 + s^5 + 9s^4 + 20s^3 + 12s^2} \tag{1.1.10}$$

Reusing Equation 1.1.7:

$$E_c(s) = \frac{1}{1 + \frac{4k(s+3)}{(s+1)(s+2)(s+6)}} \times \frac{1}{s^2}$$
$$E_c(s) = \frac{(s+1)(s+2)(s+6)}{4k(s+3) + s^3 + 9s^2 + 20s + 12} \times \frac{1}{s^2}$$

Further work shows,

$$e_c(\infty) = \lim_{s\to 0} sE_c(s) = \lim_{s\to 0} s\left[\frac{(s+1)(s+2)(s+6)}{(4k(s+3) + s^3 + 9s^2 + 20s + 12) \times s^2}\right]$$
$$= \lim_{s\to 0}\left[\frac{(s+1)(s+2)(s+6)}{(4k(s+3) + s^3 + 9s^2 + 20s + 12) \times s}\right]$$

At this point, it's clear that the only thing that will bring the evaluation of the denominator back from zero, is $k = \frac{k_1}{s}$. (c)

$$0.25 = \lim_{s\to 0}\left[\frac{(s+1)(s+2)(s+6)}{({}^{4k_1}/_s(s+3) + (s+1)(s+2)(s+6)) \times s}\right]$$
$$0.25 = \lim_{s\to 0}\left[\frac{(s+1)(s+2)(s+6)}{4k_1(s+3) + (s)(s+1)(s+2)(s+6))}\right]$$
$$0.25 = \frac{(1)(2)(6)}{4k_1(3) + (0)}$$
$$\frac{1}{4} = \frac{1}{k_1}$$
$$\therefore k_1 = 4$$

As a sanity check:

```
>> syms k s;
k=4/s;
t = ((s+1)*(s+2)*(s+6))/((4*k*(s+3)+(s+1)*(s+2)*(s+6))*s);
% t is an equivelant form of E_c(s)
limit(t,s,0) % limit of t as s goes to 0

ans =

1/4
```

Therefore, it can be concluded that $K(s) = {}^4/_s$ provides a $e_{ss} = 0.25$.

1.1.4 (d)

Obtain the closed loop transfer function of the system using the controller obtained in part (c)

Using Equation 1.1.10 and substituting in $k = \frac{4}{s}$ and recalling that this is the representation of the system when a ramp input is used:

$$\begin{aligned}\frac{Y(s)}{R(s)} &= \frac{4(\frac{4}{s})(s+3)\times s^2}{4\frac{4}{s}s^3 + 12\frac{4}{s}s^2 + s^5 + 9s^4 + 20s^3 + 12s^2} \\ &= \frac{16s+48}{s^4+9s^3+20s^2+28s+48}\end{aligned} \tag{1.1.11}$$

(d)

1.1.5 (e)

Using the Routh-Hurwitz stability criterion, show that the characteristic polynomial of the closed loop system is Hurwitz.

From Equation 1.1.11 it is clear to see that the transfer functions is of the same form as given in Equation 1.1.12.

$$G(s) = \frac{N(s)}{D(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_{m-1}s + b_m}{a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n} \tag{1.1.12}$$

The characteristic equation $\Delta(s)$ is obtained by equation $D(s)$ to zero.

$$\Delta(s) = D(s) = s^4 + 9s^3 + 20s^2 + 28s + 48 = 0$$

All coefficients are positive and the quadrinomial is complete i.e. there are no missing terms. The roots of the characteristic polynomial, as obtained form MATLAB are as given in Equation 1.1.13.

$$\begin{aligned} s_1 &= -6.3615 \\ s_2 &= -2.5415 \\ s_{3,4} &= -0.0485 \pm 1.7224j \end{aligned} \tag{1.1.13}$$

(e)

The roots all lie in the LHP because their respective real parts are less than zero. The closed loop system is therefore Hurwitz.

For completeness, the Routh array could be constructed. It is expected to find that there are no changes in sign in the elements of the first column of the Routh array, thus verifying that the closed loop system is truly Hurwitz.

s^4	1	20	48
s^3	9	28	0
s^2	$^{152}/_{9}$	48	0
s^1	$^{46}/_{19}$	0	0
s^0	48	0	0

Table 1: Routh array for a characteristic polynomial obtained in (d)

1.1.6 (f)

Use MATLAB to plot the closed loop step response of the system with the controller obtained for part (c).

Exercise #1 [(f)] continued on next page. . .

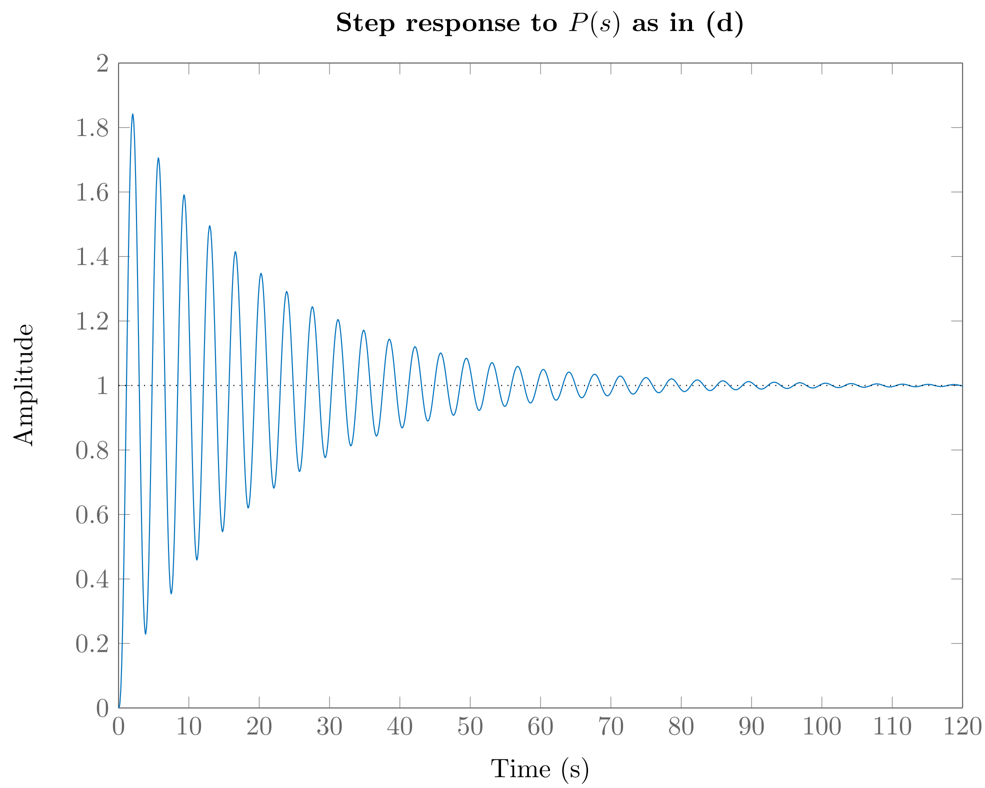


Figure 5: Step Response for $P(s)$ Plot with $k = {}^{4}\!/_{s}$ - Integral Action

```
>> n = [16 48]; d = [1 9 20 28 48];
T = tf(n, d);
step(T)
matlab2tikz('fig1.3') % used to create a tikz file for use with LaTeX
```

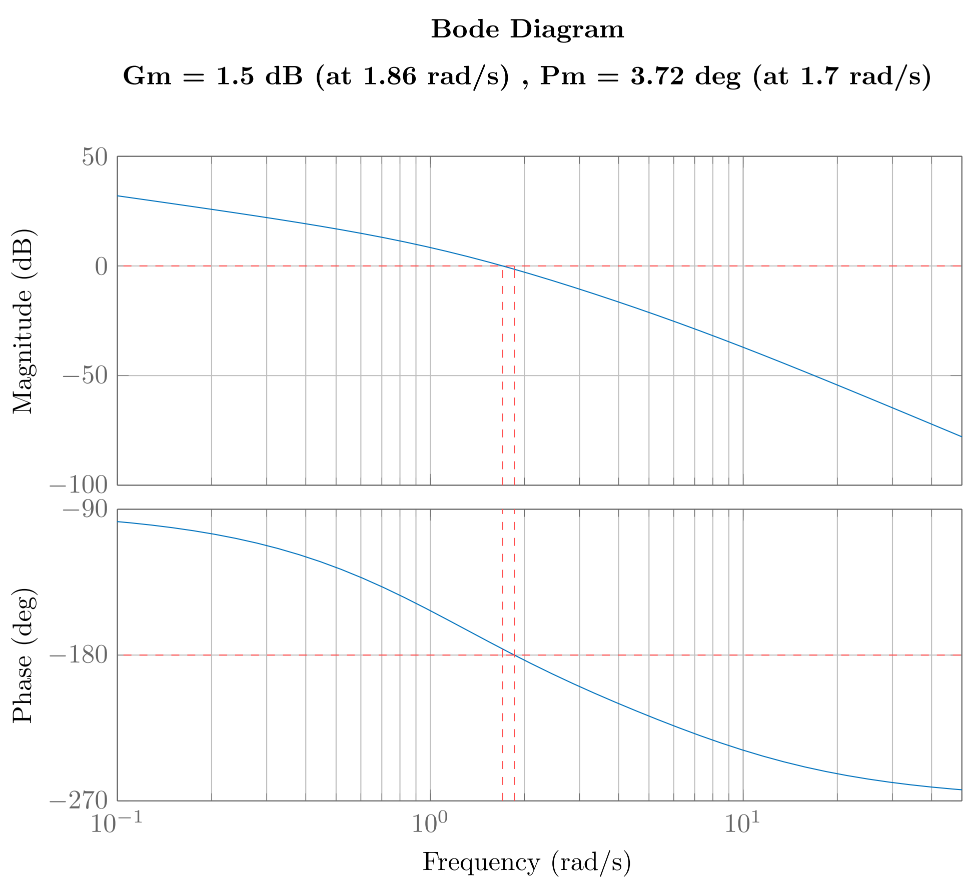
(f)

1.2 Exercise #2

Although the controller obtained for Exercise 1(c) provied a zero steady-state error for a unit step input, the response is not acceptable for a practical control system. Such a poor response is a consequense of the low stability margins of the system. It is desired to upgrade the design with an additional compensator that achieves a better performance.

1.2.1 (a)

Use MATLAB to obtain the gain and phase margins of the open loop transfer function $K(s)G(s)$ with the controller obtained in Exercise 1(c). Show these results in a Bode plot.



(a)

Figure 6: Bode plot of an open loop transfer function, obtained by multiplying $K(s)$ and $G(s)$. Including Gain and Phase margins

```
>> s = tf('s');
OL = (4*(s+3)*4)/((s+1)*(s+2)*(s+6)*s); % here, G(s) is just multiplied by 4/s
bode(OL)
margin(OL)
matlab2tikz('fig1.4') % used to create a tikz file for use with LaTeX
```

1.2.2 (b)

Design a phase lag compensator to increase the phase margin to at least 40°.

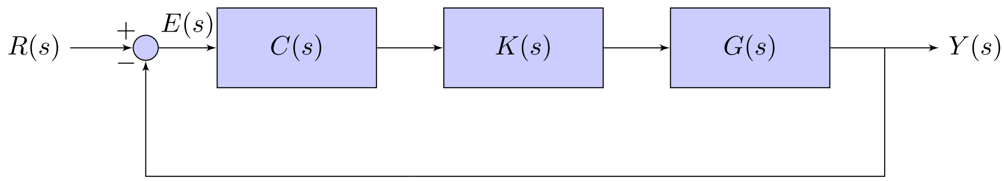


Figure 7: Block diagram - addition to Exercise 1(a) with a phase-lag compensator ($C(s)$)

A phase-lag compensator can be represented by $C(s)$ as in Figure 7 or by Equation 1.2.3.

$$C(s) = \frac{1+\tau s}{1+\alpha\tau s} \text{ for } \alpha > 1 \tag{1.2.1}$$

P_m has already been given in Figure 6 as 3.72°. A small safety margin should be added to the desired phase margin to ensure the requirement is satisfied. Therefore, in actuality, a phase margin of 55° is required. The phase response of the compensated system at the cross over frequency should therefore be $-125°$.

$$\phi(\omega_c) = -128° \text{ where } \omega_c = 0.538 \text{ rad/s} \tag{1.2.2}$$

At this frequency, the gain is 16.1 dB. Therefore, the compensator attenuation should be $\kappa = -16.1$ dB. Knowing this, the value of α can be determined from the equation for the attenuation factor, as in Equation 1.2.3

$$\kappa = 20\log_{10}\left(\frac{1}{\alpha}\right) \Leftrightarrow \alpha = 10^{-\kappa/20} \tag{1.2.3}$$ (b)

This leaves $\alpha = 6.38263$. To ensure that the effects of the phase-lag compensator are negligible, it is advisable to place the corner frequency of the compensator at least a decade below the crossover frequency obtained in Equation 1.2.2. This leaves a $\omega_{\text{zero}} = 0.0538$ rad/s.

$$\frac{1}{\tau} = \omega_{\text{zero}} \Leftrightarrow \tau = \frac{1}{\omega_{\text{zero}}} \tag{1.2.4}$$

Using the relationship between ω_{zero} and τ, it is possible to arrive at a value of $\tau = 18.59$. With the values of α and τ in mind, it is now possible to find the pole of the compensator using Equation 1.2.3.

$$C(s) = \frac{1+18.59s}{1+(6.38)(18.59)s} = \frac{1+18.59s}{1+118.6s}$$

Now, simplifying our block diagram:

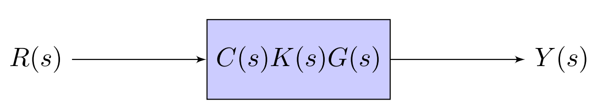
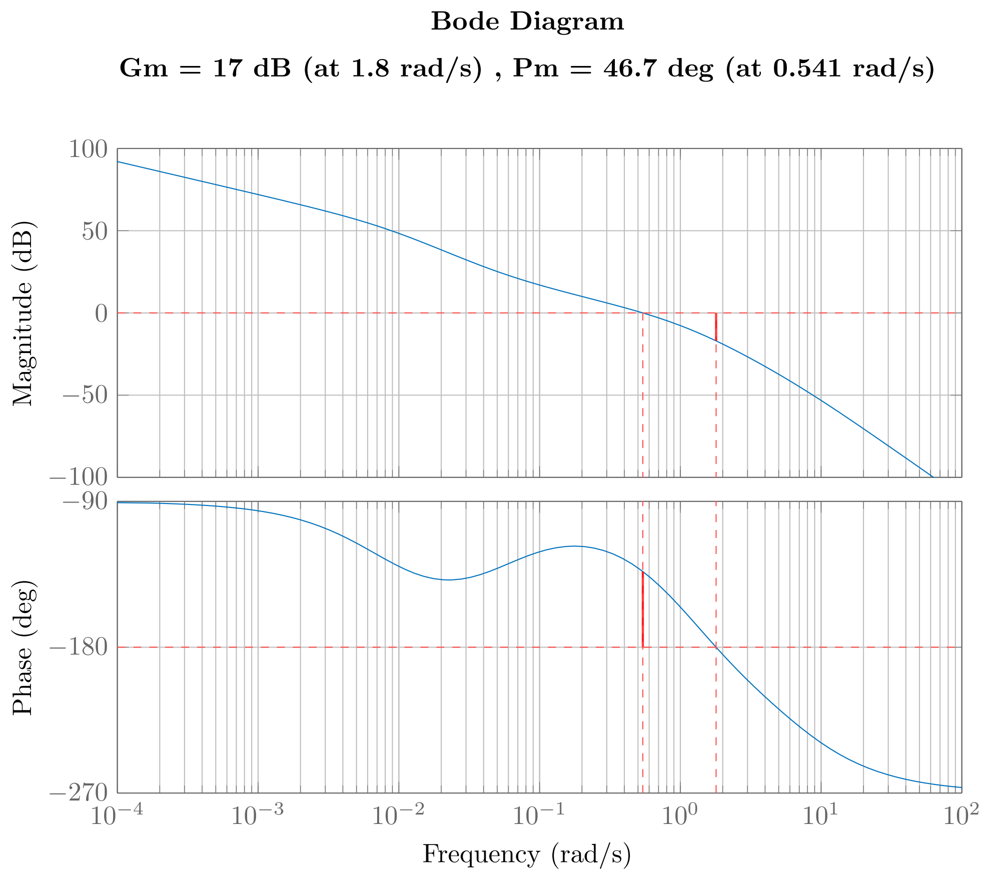


Figure 8: Open-loop simplification of block diagram - addition to Exercise 1(a) with a phase-lag compensator ($C(s)$)

We now have an open-loop transfer function as given in Equation 1.2.5.

$$P(s)_{\text{ol}} = \frac{2.5079(s+3)(s+0.05379)}{s(s+6)(s+2)(s+1)(s+0.008432)} \tag{1.2.5}$$

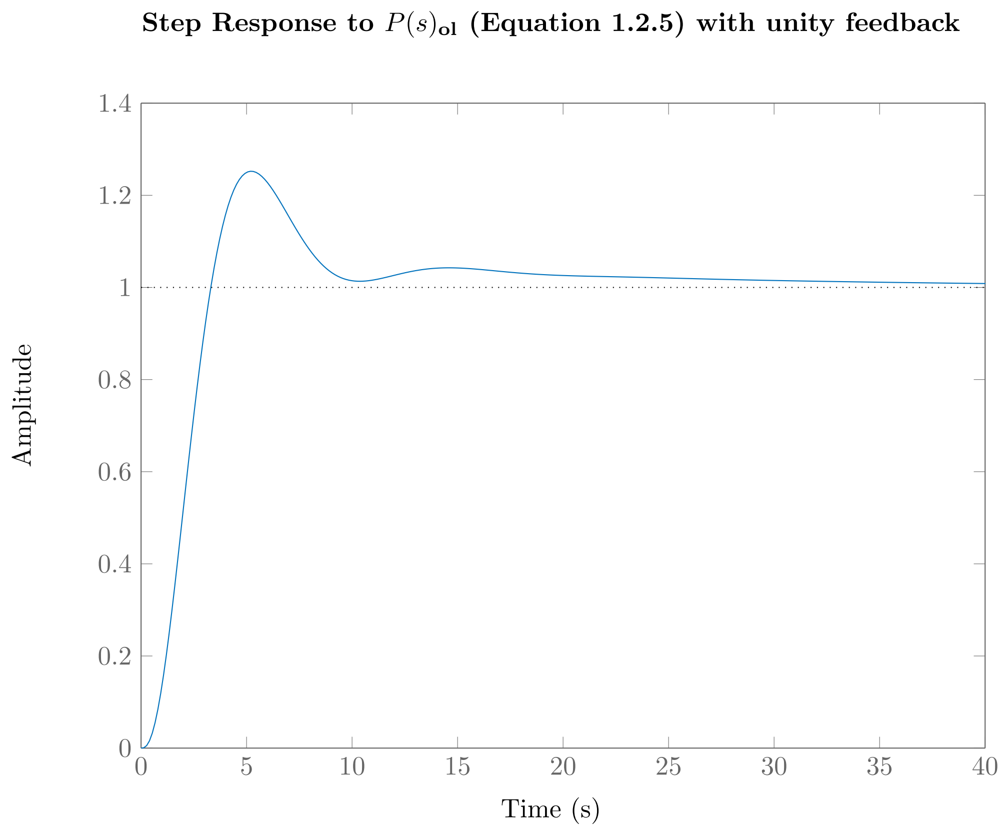


(b)

Figure 9: Open-loop Bode plot of system with a phase-lag compensator, obtained by multiplying $C(s)K(s)G(s)$. Including the Gain and Phase margins

```
s = tf('s');
OL = (4*(s+3)*4)/((s+1)*(s+2)*(s+6)*s);
OL = OL * ((1+18.59*s)/(1+118.6*s)); % OL x C(s)
bode(OL)
margin(OL)
matlab2tikz('fig1.5') % used to create a tikz file for use with LaTeX
```

So, Figure 9 shows that $P_m \geq 40°$. Out of interest, why not see what a closed-loop step response now looks like? See Figure 10.



(b)

Figure 10: Step plot for a closed loop system obtained from the open-loop transfer function in Equation 1.2.5 with unity feedback

```
s = tf('s');
OL = (4*(s+3)*4)/((s+1)*(s+2)*(s+6)*s);
OL = OL * ((1+18.59*s)/(1+118.6*s));
CL = feedback(OL, 1); % closed-loop of C(s)K(s)G(s)
step(CL)
matlab2tikz(`fig1.6') % used to create a tikz file for use with LaTeX
```

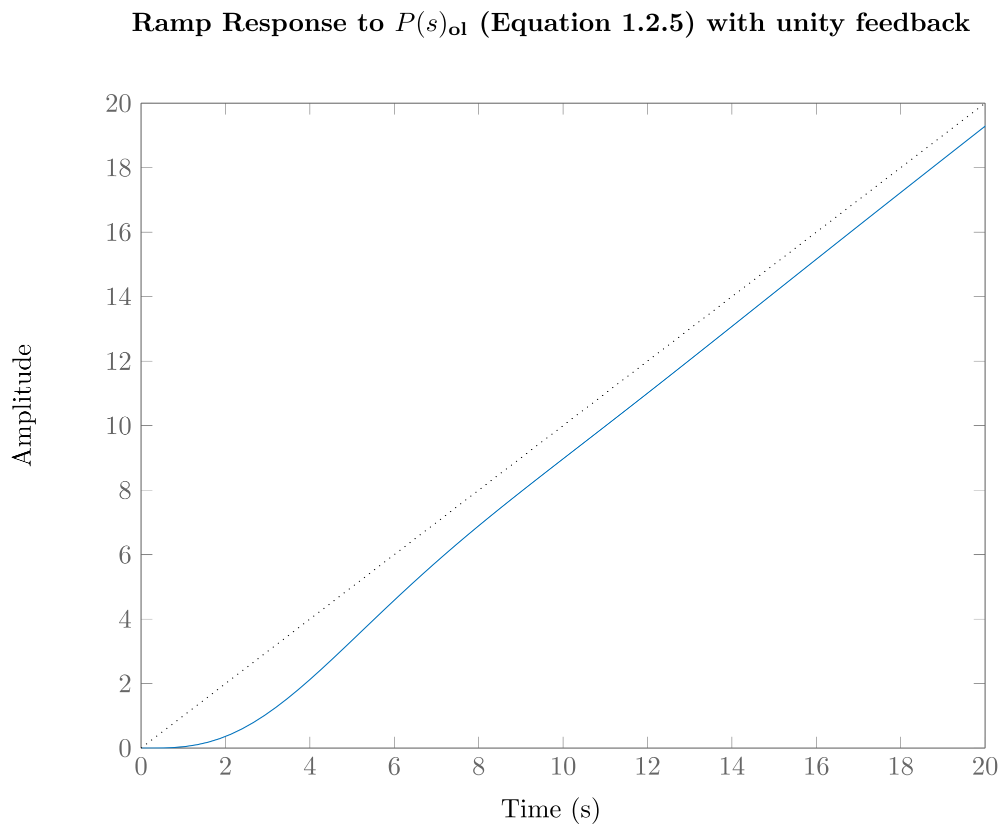


Figure 11: Ramp response plot for a closed loop system obtained from the open-loop transfer function in Equation 1.2.5 with unity feedback

```
s = tf('s');
OL = (4*(s+3)*4)/((s+1)*(s+2)*(s+6)*s);
OL = OL * ((1+18.59*s)/(1+118.6*s));
CL = feedback(OL, 1); % closed-loop of C(s)K(s)G(s)
step(CL/s) % shortcut to obtain ramp... Step(s) = 1/s and 1/s * 1/s = 1/s^2 = Ramp(s)
matlab2tikz(`fig1.7') % used to create a tikz file for use with LaTeX
```

(b)

1.2.3 (c)

Design a phase-lead compensator to increase the phase margin to at least 30°

A phase-lead compensator can be represented again by $C(s)$ as in Figure 7 or by Equation 1.2.6, where the is an important distinction to be made from .

$$C(s) = \frac{1+\tau s}{1+\alpha\tau s} \text{ for } \alpha < 1 \tag{1.2.6}$$

(c)

P_m is again given as 3.72° as given in Figure 6. It is necessary to determine the phase lead, $\phi_{\max}$.

$$\phi_{\max} = (P_{m,\text{req}} - P_m) + 15^\circ \tag{1.2.7}$$

Using Equation 1.2.7 it is possible to see that $\phi_{\max} = 41.28°$, where 15° is a safety margin applied to counter the shift effect of the compensator. α can be equated to $\phi_{\max}$ by the relationship shown in Equation 1.2.8.

$$\alpha = \frac{1 - \sin\phi_{\max}}{1 + \sin\phi_{\max}} \tag{1.2.8}$$

This relationship gives us an $\alpha = 0.205$. Finding the compensator gain, $|C(j\omega_{\max})|$ at $\omega_{\max}$ is given by a relationship shown in Equation 1.2.9.

$$|C(j\omega_{\max})| = \frac{1}{\sqrt{\alpha}} \tag{1.2.9}$$

For dB values, Equation 1.2.9 turns into Equation 1.2.10.

$$|C(j\omega_{\max})| = 20\log_{10}\left(\frac{1}{\sqrt{\alpha}}\right) \tag{1.2.10}$$

(c)

This leaves $|C(j\omega_{\max})| = 15.8$ dB. The frequency at this value of magnitude is ($\omega_{\max}$ =) 3.875 rad/s. This frequency will be the new crossover frequency for the compensated network. To find the frequency of the maximum phase point, Equation 1.2.11

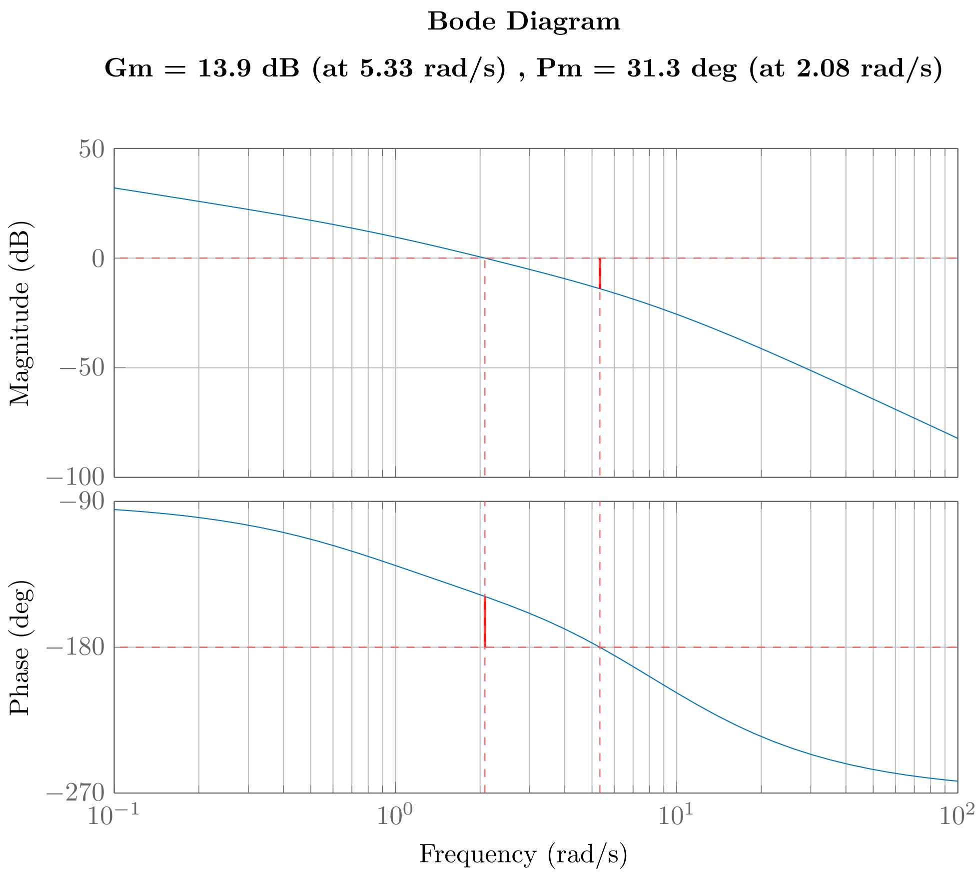
$$\omega_{\max} = \frac{1}{\tau\sqrt{\alpha}} \Leftrightarrow \tau = \frac{1}{\omega_{\max}\sqrt{\alpha}} \tag{1.2.11}$$

From Equation 1.2.11 a value of $\tau = 0.57$ s. With the above parameters in mind, it is possible to derive an equation to represent the phase-lead compensator using Equation 1.2.6.

$$C(s) = \frac{1 + 0.57s}{1 + 0.117s}$$

This leaves an open loop transfer function as shown in Equation 1.2.12.

$$P(s)_{\text{ol}} = \frac{77.949(s+3)(s+1.754)}{s(s+8.547)(s+6)(s+2)(s+1)} \tag{1.2.12}$$



(c)

Figure 12: Open-loop Bode plot of system with a phase-lead compensator, obtained by multiplying $C(s)K(s)G(s)$. Including the Gain and Phase margins

```
s = tf('s');
OL = (4*(s+3)*4)/((s+1)*(s+2)*(s+6)*s);
OL = OL * (1+0.57*s)/(1+0.117*s); % OL x C(s)
bode(OL)
margin(OL)
matlab2tikz('fig1.8') % used to create a tikz file for use with LaTeX
```

So, Figure 12 shows that $P_m \geq 30°$.

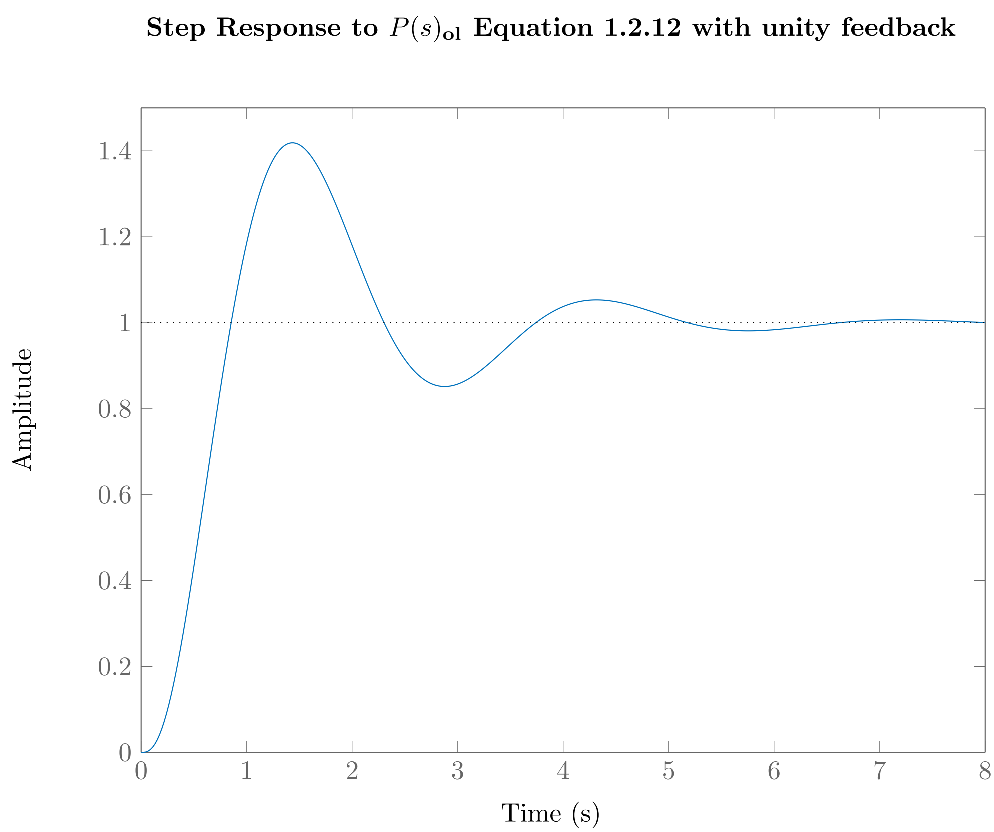


Figure 13: Step plot for a closed loop system obtained from $C(s)K(s)G(s)$ with unity feedback

```
s = tf('s');
OL = (4*(s+3)*4)/((s+1)*(s+2)*(s+6)*s);
OL = OL * (1+0.57*s)/(1+0.117*s); % OL x C(s)
CL= feedback(OL,1);
step(CL)
matlab2tikz('fig1.9') % used to create a tikz file for use with LaTeX
```

(c)

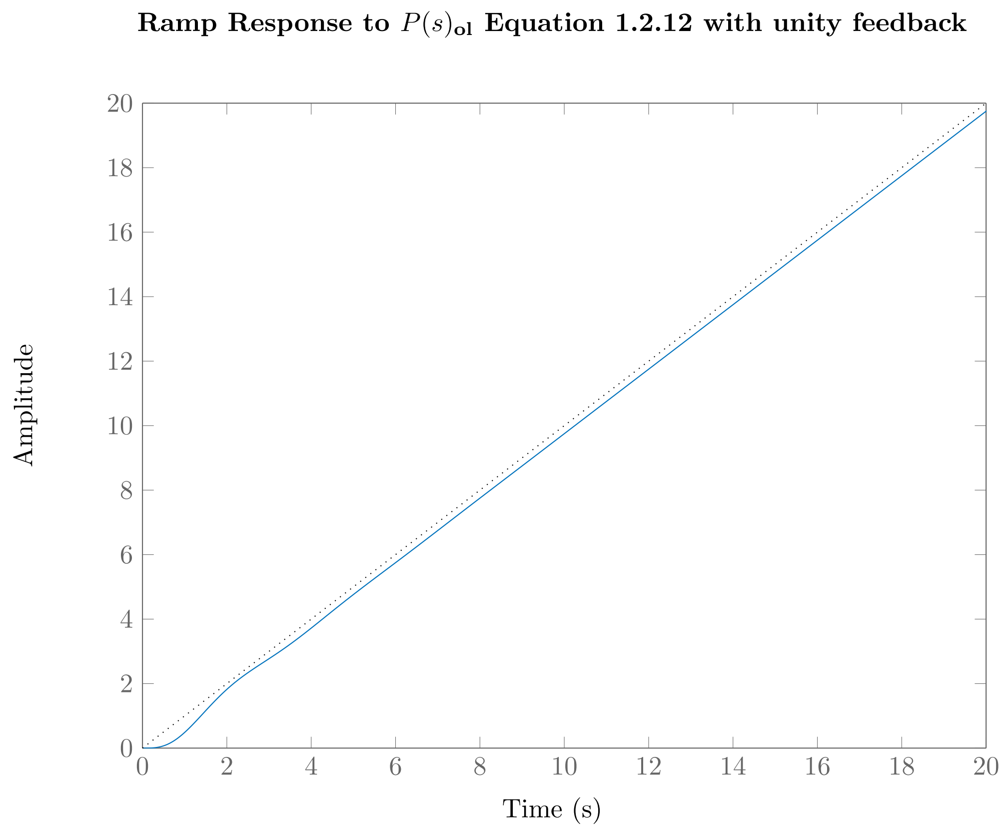


Figure 14: Ramp response plot for a closed loop system obtained from $C(s)K(s)G(s)$ with unity feedback

```
s = tf('s');
OL = (4*(s+3)*4)/((s+1)*(s+2)*(s+6)*s);
OL = OL * (1+0.57*s)/(1+0.117*s); % OL x C(s)
CL= feedback(OL,1);
step(CL/s)
matlab2tikz('fig2.0') % used to create a tikz file for use with LaTeX
```

(c)

1.2.4 (d)

Use MATLAB to plot the open loop frequency response (Bode plot) showing the effects of the lag and lead compensators. Include the gain and phase margins.

See Figures 9 and 12. (d)

1.2.5 (e)

Use MATLAB to plot the step response of the compensated closed loop systems.

See Figures 10 and 13. (e)

1.2.6 (f)

Use MATLAB to plot the ramp response of the compensated closed loop systems. Restrict your plots to the first 20 seconds only (or less).

See Figures 11 and 14. (f)

2 Section 2 - State-space Modelling

2.1 Exercise #3

Consider a block of mass m mounted on a massles cart, as shown in Figure 15. The system is modelled through the differential equation shown in Equation 2.1.1.

$$m\frac{d^2y(t)}{dt^2} + b\frac{dy(t)}{dt} + ky(t) = b\frac{du(t)}{dt} + ku(t) \tag{2.1.1}$$

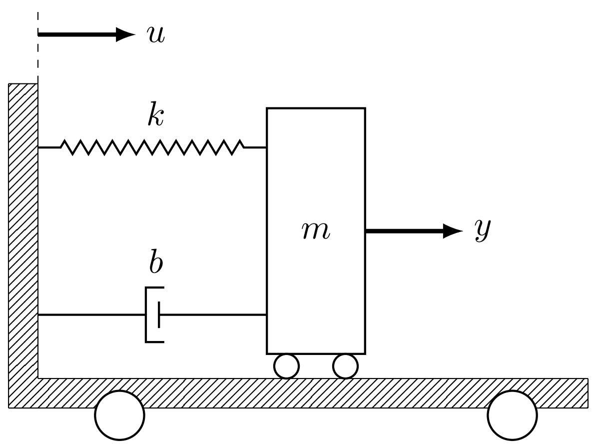


Figure 15: Block diagram of a simple mechanical system, analagous to a mass on a cart. Represented mathematically by Equation 2.1.1

2.1.1 (a)

Determine the transfer function $G(s) = {}^{Y(s)}/_{U(s)}$.

It is not stated, but an assumption will be made that all initial conditions are zero. Taking the Laplace transform of the equation modelling the system ($\mathcal{L}$[Equation 2.1.1]) yields Equation 2.1.2.

$$\begin{aligned} ms^2Y(s) + bsY(s) + ks &= bsU(s) + kU(s) \\ Y(s)[ms^2 + bs + k] &= U(s)[bs + k] \\ G(s) = \frac{Y(s)}{U(s)} &= \frac{bs + k}{ms^2 + bs + k} \end{aligned} \tag{2.1.2}$$

(a)

2.1.2 (b)

Use the transfer function obtained in 3(a) to obtain a state-space representation of the system.

The transfer function for the system was obtained in Equation 2.1.2. Comparing Equation 2.1.2 to the transfer function standard form (Equation 1.1.12 but for $n = m$) it can be observed that:

$$a_n = {}^k\!/\!_m \text{ and } a_{n-1} = {}^b\!/\!_m \tag{2.1.3}$$

With this small amount of knowledge it is possible to obtain part of the controllable Canonical form, as in Equation 2.1.4.

$$\text{Modified standard form: } \begin{bmatrix} \dot{x_1}(t) \\ \dot{x_n}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a_n & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_n(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$\begin{bmatrix} \dot{x_1}(t) \\ \dot{x_2}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -{}^k\!/\!_m & -{}^b\!/\!_m \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \tag{2.1.4}$$

$$a_n = {}^k\!/\!_m,\ b_n = {}^k\!/\!_m,\ a_1 = {}^b\!/\!_m,\ b_1 = {}^b\!/\!_m \text{ and } b_0 = 0 \text{ } (\because \text{ G(s) is strictly proper})$$ (b)

With this information in mind, it is possible to complete the controllable Canonical form.

$$\text{Modified standard form: } y(t) = \begin{bmatrix} (b_n - a_n b_0) & (b_1 - a_1 b_0) \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_n(t) \end{bmatrix} + b_0 u(t)$$

$$y(t) = \begin{bmatrix} {}^k\!/\!_m & {}^b\!/\!_m \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_n(t) \end{bmatrix} \tag{2.1.5}$$

Taking Equations 2.1.4 and 2.1.5 into consideration, **one** state-space representation of $G(s)$ is complete, in controllable Canonical form. There are an infinite number of these. Matrices **A** (system), **B** (control), **C** (output) and **D** (feedforward) are given below.

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -{}^k\!/\!_m & -{}^b\!/\!_m \end{bmatrix},\ \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ \mathbf{C} = \begin{bmatrix} {}^k\!/\!_m & {}^b\!/\!_m \end{bmatrix},\ \mathbf{D} = 0$$

2.2 Exercise #4

Consider a system represented by the following differential equations:

$$Ri_1(t) + L_1 \frac{di_1(t)}{dt} + v(t) = v_a(t)$$

$$L_2 \frac{di_2(t)}{dt} + v(t) = v_b(t)$$

$$i_1(t) + i_2(t) = C \frac{dv(t)}{dt}$$

In this system, R, L_1, L_2 and C are given constants, and $v_a(t)$ and $v_b(t)$ are inputs. Let the state variables be defined as $x_1(t) = i_1(t)$, $x_2(t) = i_2(t)$, and $x_3(t) = v(t)$. Obtain a state-space representation of the system where the output is $y(t) = x_3(t)$. Notice the system has two inputs and one output.

The equations given in the question can be rewritten as in the below equations.

$$Rx_1(t) + L_1\frac{dx_1(t)}{dt} + x_3(t) = v_a(t)$$

$$L_2\frac{dx_2(t)}{dt} + x_3(t) = v_b(t)$$

$$x_1(t) + x_2(t) = C\frac{dx_3(t)}{dt}$$

In terms of the required states, the above equations can be rewritten as below.

$$\frac{dx_1(t)}{dt} = \frac{1}{L_1}v_a(t) - \frac{R}{L_1}x_1(t) - \frac{1}{L_1}x_3(t)$$

$$\frac{dx_2(t)}{dt} = \frac{1}{L_2}v_b(t) - \frac{1}{L_2}x_3(t)$$

$$\frac{dx_3(t)}{dt} = \frac{1}{C}x_1(t) + \frac{1}{C}x_2(t)$$

From here, it is very easy to obtain **one** state-space representation of the system.

$$\begin{bmatrix}\dot{x_1}(t)\\ \dot{x_2}(t)\\ \dot{x_3}(t)\end{bmatrix} = \begin{bmatrix}-R/L_1 & 0 & -1/L_1\\ 0 & 0 & -1/L_2\\ 1/C & 1/C & 0\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\\ x_3(t)\end{bmatrix} + \begin{bmatrix}1/L_1 & 0\\ 0 & 1/L_2\\ 0 & 0\end{bmatrix}\begin{bmatrix}v_a\\ v_b\end{bmatrix}$$

$$y = \begin{bmatrix}0 & 0 & 1\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\\ x_3(t)\end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix}-R/L_1 & 0 & -1/L_1\\ 0 & 0 & -1/L_2\\ 1/C & 1/C & 0\end{bmatrix}, \mathbf{B} = \begin{bmatrix}1/L_1 & 0\\ 0 & 1/L_2\\ 0 & 0\end{bmatrix}, \mathbf{C} = \begin{bmatrix}0 & 0 & 1\end{bmatrix}, \mathbf{D} = 0$$

□

2.3 Exercise #5

A system is described by:

$$\frac{d\mathbf{x}}{dt} = \begin{bmatrix}2 & -1\\ 3 & -5\end{bmatrix}\mathbf{x} + \begin{bmatrix}1\\ 2\end{bmatrix}u$$

$$y = \begin{bmatrix}1 & -1\end{bmatrix}\mathbf{x}$$

Determine the transfer function $G(s) = {}^{Y(s)}/_{U(s)}$.

It is possible to represent the matrix components of the state-space representation individually to simplify things.

$$\mathbf{A} = \begin{bmatrix}2 & -1\\ 3 & -5\end{bmatrix}, \mathbf{B} = \begin{bmatrix}1\\ 2\end{bmatrix}, \mathbf{C} = \begin{bmatrix}1 & -1\end{bmatrix}, \mathbf{D} = 0 \tag{2.3.1}$$

□

The transfer function of a system can be represented in terms of these matrices, **A**, **B**, **C** and **D**.

$$G(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D} \tag{2.3.2}$$

Using the values of **A**, **B**, **C** and **D** obtained in Equation 2.3.1 and substituting them into Equation 2.3.2, it is possible to obtain a system representation as in Equation 2.3.3a.

$$G(s) = \begin{bmatrix} 1 & -1 \end{bmatrix} \left(s \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & -1 \\ 3 & -5 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix} \tag{2.3.3a}$$

$$= \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} s-2 & 1 \\ -3 & s+5 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \tag{2.3.3b}$$

$$= \begin{bmatrix} 1 & -1 \end{bmatrix} \left(\frac{1}{(s-2)(s+5)+3} \begin{bmatrix} s+5 & -1 \\ 3 & s-2 \end{bmatrix} \right) \begin{bmatrix} 1 \\ 2 \end{bmatrix} \tag{2.3.3c}$$

$$= \begin{bmatrix} 1 & -1 \end{bmatrix} \frac{1}{(s-2)(s+5)+3} \begin{bmatrix} s+5 & -1 \\ 3 & s-2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \tag{2.3.3d}$$

$$= \begin{bmatrix} 1 & -1 \end{bmatrix} \frac{1}{(s-2)(s+5)+3} \begin{bmatrix} (s+5)+(2)(-1) \\ 3+2(s-2) \end{bmatrix} \tag{2.3.3e}$$

$$= \frac{1}{(s-2)(s+5)+3} \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} s+3 \\ 2s-1 \end{bmatrix} \tag{2.3.3f}$$

$$= \frac{1}{(s-2)(s+5)+3} \begin{bmatrix} 4-s \end{bmatrix} \tag{2.3.3g}$$

$$= \frac{-s+4}{s^2+3s-7} \tag{2.3.3h}$$

□

2.4 Exercise #6

A system has the following transfer function representation:

$$Q(s) = \frac{s^2+2s}{s^2(s^2+3s+4)}$$

Derive a state-space model for the system.

$Q(s)$ can be alternatively represented as in Equation 2.4.1.

$$Q(s) = \frac{s^2+2s+0}{s^4+3s^3+4s^2+0s+0} \tag{2.4.1}$$

Compare this transfer function to the universal transfer function, which is reprinted in Equation 2.4.2.

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1}s + b_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n} \tag{2.4.2}$$

□

The controllable Canonical state-space representation takes a form shown in Equation 2.4.3.

$$\begin{bmatrix} \dot{x_1}(t) \\ \dot{x_2}(t) \\ \vdots \\ \dot{x_{n-1}}(t) \\ \dot{x_n}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_{n-1}(t) \\ x_n(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u(t) \tag{2.4.3a}$$

$$y(t) = \begin{bmatrix} b_n - a_n b_0 & b_{n-1} - a_{n-1} b_0 & \cdots & b_1 - a_1 b_0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} + b_0 u(t) \tag{2.4.3b}$$

Comparing the transfer function given in Equation 2.4.1 with Equation 2.4.2, the coefficients of the various orders of s can be observed and used to populated the matrices given in Equation 2.4.3. This leaves one state-space representation for $Q(s)$ as in Equation 2.4.4.

$$\begin{bmatrix} \dot{x_1}(t) \\ \dot{x_2}(t) \\ \dot{x_3}(t) \\ \dot{x_4}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -4 & -3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u(t) \tag{2.4.4a}$$

$$y(t) = \begin{bmatrix} 0 & 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} \tag{2.4.4b}$$

Just for fun, this can be tested using MATLAB.

```
>> A = [0 1 0 0; 0 0 1 0; 0 0 0 1; 0 0 -4 -3];
B = [0;0;0;1];
C = [0 2 1 0];
D = [0];
[n,d] = ss2tf(A,B,C,D)


n =

     0     0     1     2     0


d =

    1.0000    3.0000    4.0000         0         0
```

And, the coefficients of the polynomial given as `n` and `d` match the Equation of $Q(s)$ given in Equation 2.4.1.

□

2.5 Exercise #7

Consider the system shown in Figure 16. The equations modelling the system are shown below, as in Equations 2.5.1.

$$
\begin{aligned}
m_1 \frac{d^2 y_1(t)}{dt^2} + b\frac{dy_1(t)}{dt} + k[y_1(t) - y_2(t)] &= 0 \\
m_2 \frac{d^2 y_2(t)}{dt^2} + k[y_2(t) - y_1(t)] &= u(t)
\end{aligned}
\tag{2.5.1}
$$

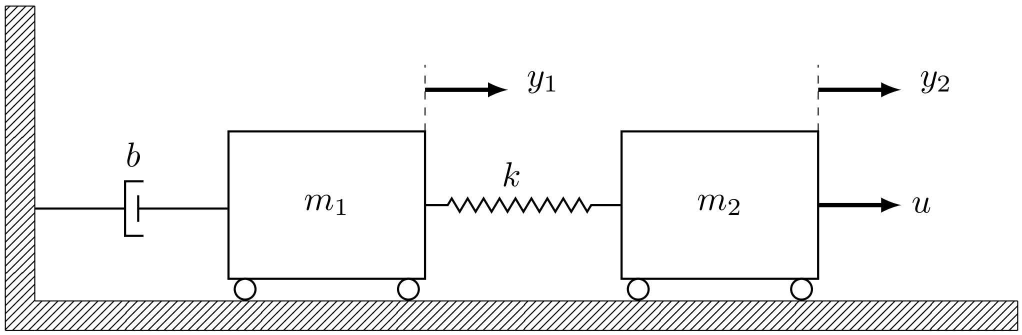


Figure 16: Masses-damper-spring system for Exercise 7. Represented mathematically by Equation 2.5.1

Obtain a state-space representation if the output variables are $y_1(t)$ and $y_2(t)$. Notice that the system has one input and two outputs.

The state variables can be defined as in Equation 2.5.2. By rearranging Equations 2.5.1 for the derivatives and substituting in the state variables, the set of equations shown below (Equations 2.5.3) can be obtained.

$$
\begin{aligned}
x_1 &= y_1 \\
x_2 &= {}^{dy_1}\!/_{dt} \\
x_3 &= y_2 \\
x_4 &= {}^{dy_2}\!/_{dt}
\end{aligned}
\tag{2.5.2}
$$

$$
\begin{aligned}
\frac{dx_1}{dt} &= x_2 \\
\frac{dx_2}{dt} &= -\frac{b}{m_1}\frac{dy_1}{dt} - \frac{k}{m_1}(y_1 - y_2) \\
\frac{dx_3}{dt} &= x_4 \\
\frac{dx_4}{dt} &= -\frac{k}{m_2}(y_2 - y_1) + \frac{u}{m_2}
\end{aligned}
\tag{2.5.3}
$$

$\square$

It is possible to further simplify Equations 2.5.3. The final form of these equations is shown in 2.5.4. From 2.5.4 the equations can then be expressed in matrix form, as in 2.5.5.

$$
\begin{aligned}
\frac{dx_1}{dt} &= x_2 \\
\frac{dx_2}{dt} &= -\frac{k}{m_1}x_1 - \frac{b}{m_1}x_2 + \frac{k}{m_1}x_3 \\
\frac{dx_3}{dt} &= x_4 \\
\frac{dx_4}{dt} &= \frac{1}{m_2}u + \frac{k}{m_2}x_1 - \frac{k}{m_2}x_3
\end{aligned}
\tag{2.5.4}
$$

$$\begin{bmatrix} \dot{x_1}(t) \\ \dot{x_2}(t) \\ \dot{x_3}(t) \\ \dot{x_4}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -k/m_1 & -b/m_1 & k/m_1 & 0 \\ 0 & 0 & 0 & 1 \\ k/m_2 & 0 & -k/m_2 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/m_2 \end{bmatrix} u(t) \tag{2.5.5}$$

From Equations 2.5.3, particularly paying attention to $x_1 = y_1$ and $x_3 = y_2$, it is possible to now derive the output equation as in Equation 2.5.6

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \tag{2.5.6}$$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -k/m_1 & -b/m_1 & k/m_1 & 0 \\ 0 & 0 & 0 & 1 \\ k/m_2 & 0 & -k/m_2 & 0 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/m_2 \end{bmatrix}, \mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \mathbf{D} = 0 \tag{2.5.7}$$

$\square$

3 Section 3 - Design in State-Space

3.1 Exercise #8

The system shown in Figure 17 has a transfer function $G(s)$ given as in Equation 3.1.1.

$$G(s) = \frac{4}{(s+1)(s+2)^2} \tag{3.1.1}$$

This SISO system has a state-space representation, as given below in Equation 3.1.2.

$$\begin{aligned} \frac{d\mathbf{x}(t)}{dt} &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t) \\ y(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{D}u(t) \end{aligned} \tag{3.1.2}$$

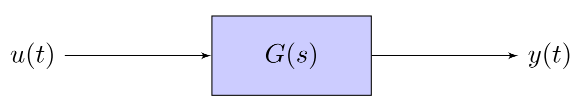


Figure 17: Simple system for Exercise 8

3.1.1 (a)

Find matrices **A**, **B**, **C**, and **D**. Construct the characteristic polynomial of **A**. Verify that the eigenvalues of **A** are the poles of G(s).

Using Equations 2.4.3 it is possible to derive the Canonical state-space representation for the system defined by the transfer function, as in Equation 3.1.1.

$$\begin{bmatrix} \dot{x_1}(t) \\ \dot{x_2}(t) \\ \dot{x_3}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -8 & -5 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t) \tag{3.1.3a}$$

$$y(t) = \begin{bmatrix} 4 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} \tag{3.1.3b}$$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -8 & -5 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \mathbf{C} = \begin{bmatrix} 4 & 0 & 0 \end{bmatrix}, \mathbf{D} = 0 \tag{3.1.4}$$

The characteristic polynomial of $\mathbf{A}$ is given by Equation 3.1.5.

$$\Delta(s) = \det(s\mathbf{I} - \mathbf{A}) = |s\mathbf{I} - \mathbf{A}| = 0 \tag{3.1.5}$$

This characteristic polynomial is given in Equation 3.1.6.

(a)

$$\begin{aligned} \Delta(s) &= \left| s \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -8 & -5 \end{bmatrix} \right| = 0 \\ &= \begin{vmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 4 & 8 & s+5 \end{vmatrix} = 0 \\ &= s\left[(s)(s+5) - (-1)(8)\right] - (-1)\left[-(-1)(4)\right] + (0)\left[(\dots)\right] = 0 \\ &= s^3 + 5s^2 + 8s + 4 = 0 \end{aligned} \tag{3.1.6}$$

The Eigenvalues of matrix $\mathbf{A}$ will be given by the roots of the characteristic polynomial, $\Delta(s)$ given in Equation 3.1.6. To verify that the poles of $G(s)$ (i.e. the roots of $(s+1)(s+2)^2 = 0$) are the same as the Eigenvalues of $\mathbf{A}$, the two polynomials should be equal. This is shown in Equation 3.1.7. No further work is needed here, clearly the roots of these two polynomials will be the same, because they ***are the same*** polynomial.

$$\begin{aligned} (s+1)(s+2)^2 &\stackrel{?}{=} s^3 + 5s^2 + 8s + 4 \\ (s+1)(s+2)^2 &\stackrel{?}{=} (s+1)(s^2 + 4x + 4) \\ (s+1)(s+2)^2 &= (s+1)(s+2)^2 \end{aligned} \tag{3.1.7}$$

3.1.2 (b)

Verify that the system is controllable.

The system's controllability is analysed through the controllability matrix, $\mathbf{M}_c$. When this matrix has full-rank, that is to say, $\det(\mathbf{M}_c) \neq 0$, the system can be said to be controllable.

(b)

The controllability matrix, $\mathbf{M}_c$, is given by Equation 3.1.8.

$$\mathbf{M}_c = \begin{bmatrix}\mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B}\end{bmatrix} \tag{3.1.8}$$

So, some matrix calculations are needed to simplify the calculation of $\mathbf{M}_c$.

$$\mathbf{AB} = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -4 & -8 & -5\end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}0\\1\\-5\end{bmatrix}$$

$$\begin{aligned}\mathbf{A}^2\mathbf{B} = \mathbf{A}(\mathbf{AB}) &= \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -4 & -8 & -5\end{bmatrix}\left(\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -4 & -8 & -5\end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix}\right)\\ &= \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -4 & -8 & -5\end{bmatrix}\begin{bmatrix}0\\1\\-5\end{bmatrix}\\ &= \begin{bmatrix}1\\-5\\17\end{bmatrix}\end{aligned}$$

(b)

Therefore, $\mathbf{M}_c$ becomes obtainable, as in Equation 3.1.9. Next, the determinant of $\mathbf{M}_c$ can be found, as in Equation 3.1.10. From this result, the system is determined to be controllable.

$$\mathbf{M}_c = \begin{bmatrix}0 & 0 & 1\\ 0 & 1 & -5\\ 1 & -5 & 17\end{bmatrix} \tag{3.1.9}$$

$$\det(\mathbf{M}_c) = (1)\begin{vmatrix}0 & 1\\ 1 & -5\end{vmatrix} = -1 \tag{3.1.10}$$

3.1.3 (c)

Calculate a state feedback vector $\mathbf{K}$ that will move the poles of the system to $s_1 = -2$, $s_2 = -4$ and $s_3 = -5$. Calculate the amplification factor for the input pre-scaler.

Introducing state feedback into the system will modify Equation 3.1.2 to become Equation 3.1.11, where $\mathbf{K} = \begin{bmatrix}k_1 & k_2 & k_3\end{bmatrix}$.

$$\begin{aligned}\frac{d\mathbf{x}(t)}{dt} &= \mathbf{Ax}(t) + \mathbf{B}\left[u(t) - \mathbf{Kx}(t)\right]\\ &= (\mathbf{A} - \mathbf{BK})\mathbf{x}(t) + \mathbf{B}u(t)\end{aligned} \tag{3.1.11}$$

(c)

The poles, s_1, s_2 and s_3 will now be the Eigenvalues of $\mathbf{A} - \mathbf{BK}$. Hence, the characteristic equation given by Equation 3.1.6 becomes as in Equation 3.1.12.

$$\Delta_K(s) = \det\left[s\mathbf{I} - (\mathbf{A} - \mathbf{BK})\right] = |s\mathbf{I} - \mathbf{A} + \mathbf{BK}| = 0 \tag{3.1.12}$$

Working this through numerically,

$$
\begin{aligned}
\Delta_K(s) &= \left| s\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} - \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -4 & -8 & -5\end{bmatrix} + \begin{bmatrix}0\\ 0\\ 1\end{bmatrix}\begin{bmatrix}k_1 & k_2 & k_3\end{bmatrix} \right| \\
&= \left| \begin{bmatrix}s & -1 & 0\\ 0 & s & -1\\ 4 & 8 & s+5\end{bmatrix} + \begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ k_1 & k_2 & k_3\end{bmatrix} \right| \\
&= \begin{vmatrix}s & -1 & 0\\ 0 & s & -1\\ k_1+4 & k_2+8 & k_3+s+5\end{vmatrix} \\
&= s[s(k_3+s+5)+(k_2+8)]+(k_1+4) \\
&= s^3+5s^2+k_3s^2+8s+k_2s+4+k_1 \\
&= s^3+s^2(5+k_3)+s(8+k_2)+4+k_1
\end{aligned}
$$

Note that the poles can be represented as a factorised product, $(s_1+2)(s_2+4)(s_3+5)=0$. Therefore, in order to place the poles at these places, an equality is given as in Equation 3.1.13.

$$
\begin{aligned}
s^3+s^2(5+k_3)+s(8+k_2)+4+k_1 &= (s_1+2)(s_2+4)(s_3+5) \\
s^3+s^2(5+k_3)+s(8+k_2)+4+k_1 &= s^3+11s+38s+40
\end{aligned}
\tag{3.1.13}
$$

Comparing coefficients, as in Equation 3.1.14, it is possible to obtains solutions for k_1, k_2 and k_3.

(c)

$$5+k_3=11 \tag{3.1.14a}$$

$$\boxed{\therefore k_3=6} \tag{3.1.14b}$$

$$8+k_2=38 \tag{3.1.14c}$$

$$\boxed{\therefore k_2=30} \tag{3.1.14d}$$

$$4+k_1=40 \tag{3.1.14e}$$

$$\boxed{\therefore k_1=36} \tag{3.1.14f}$$

With this knowledge, it is now possible to populate the state feedback gain matrix, **K** with its respective values.

$$\mathbf{K} = \begin{bmatrix}36 & 30 & 6\end{bmatrix}$$

There are now essentially, two systems. One system is defined by **A**, **B**, **C**, and **D** and another is defined by $\mathbf{A}^*$, **B**, **C**, and **D**, where $\mathbf{A}^* = \mathbf{A} - \mathbf{BK}$. $\mathbf{A}^*$ is calculated in Equation 3.1.15.

$$\mathbf{A}^* = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -40 & -38 & -11\end{bmatrix} \tag{3.1.15}$$

With this new matrix in mind, it is possible to obtain the transfer function of this new state-space represented system. The transfer function of a system can be represented in terms of these matrices, $\mathbf{A}^*$, **B**, **C**, and **D**, as in Equation 2.3.2. Noticing of course, that for this example, **A** of Equation 2.3.2 is substituted as $\mathbf{A}^*$, as in Equation 3.1.15.

Representing $\mathbf{A}^*$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$ as a transfer function, will provide Equation 3.1.16.

$$G^*(s) = \frac{4}{s^3 + 11s^2 + 38s + 40} \tag{3.1.16}$$

It is now possible to apply the final value theorem (as in Equation 3.1.17) to $G^*(s)$. For a step input, the final value of $G^*(s)$ can be shown to be 0.1.

$$\lim_{t\to\infty} g^*(t) = \lim_{s\to 0} sG^*(s) \tag{3.1.17}$$

Assuming the input to $G^*(s)$ is a step input i.e. $U(s) = {}^1\!/_s$, yields:

$$G^*(s) = \frac{4}{(s^3 + 11s^2 + 38s + 40)(s)}$$

Applying the final value theorem:

$$\begin{aligned}
\lim_{s\to 0} sG^*(s) &= \lim_{s\to 0} \frac{4(s)}{(s^3 + 11s^2 + 38s + 40)(s)} \\
&= \lim_{s\to 0} \frac{4}{(s^3 + 11s^2 + 38s + 40)} \\
&= \frac{4}{((0)^3 + 11(0)^2 + 38(0) + 40)} \\
&= \frac{4}{40} \\
&= 0.1
\end{aligned}$$

(c)

This is verifiable with MATLAB.

```
>> syms s;
limit(((4*s)/((s^3 + 11*s^2 + 38*s + 40)*s)),s,0,'right')

ans =

1/10
```

Clearly, a final value of 0.1 is not what is required from the system. Therefore, an amplification factor is necessary. This amplification factor is 10. Let $G^*(s)$ now be the SISO transfer function for the system with state feedback, but also including an amplification factor. It can now be represented in state space.

$$\mathbf{A}^* = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -40 & -38 & -11 \end{bmatrix}, \mathbf{B}^* = 10 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \mathbf{C} = \begin{bmatrix} 4 & 0 & 0 \end{bmatrix}, \mathbf{D} = 0$$

3.1.4 (d)

Use MATLAB to plot the superimposed graphs of the step responses before and after applying the state feedback $\mathbf{K}$.

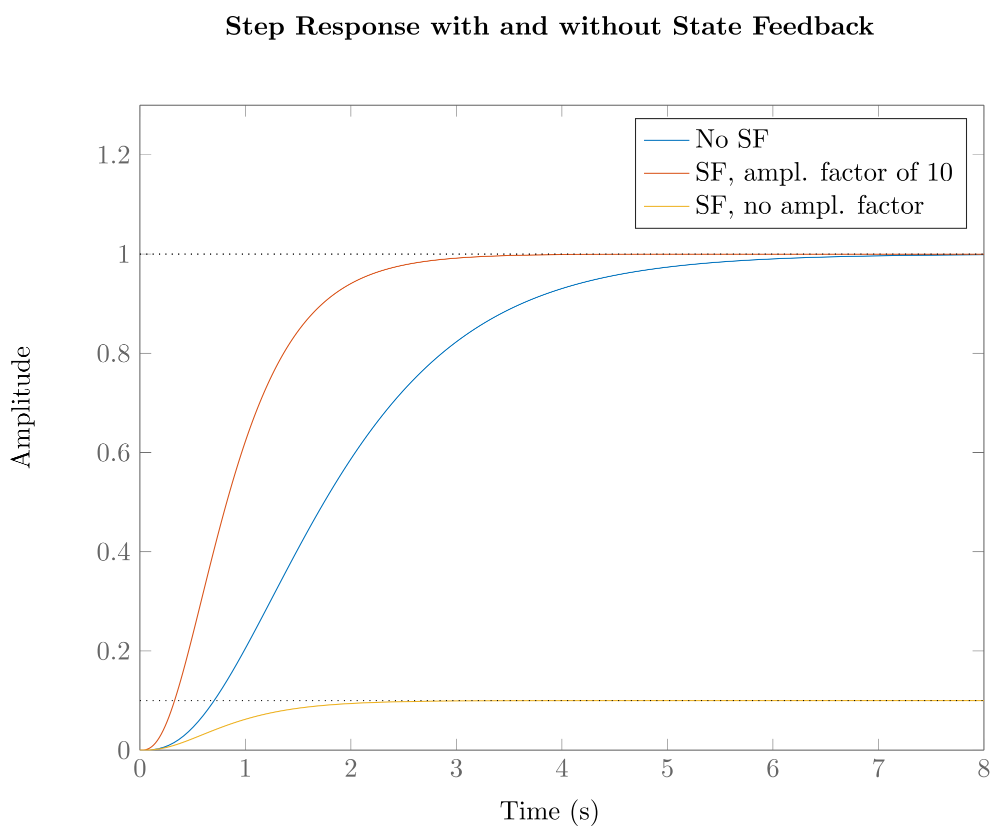


Figure 18: Step Response for $P(s)$ Plot with k as Scalar

```
>> A = [0 1 0; 0 0 1; -4 -8 -5];
B = [0; 0; 1];
C = [4 0 0]; D = [0];
K = [36 30 6];
sysnoK = ss(A, B, C, D);
sysK = ss(A-B*K,B,C,D);
sysK_gain = ss(A-B*K,10*B,C,D);
step(sysnoK)
hold % assuming you have not already got a figure held
step(sysK)
step(sysK_gain)
```

(d)

3.1.5 (e)

Verify that the system is observable.

The system will be observable if the observability matrix, $\mathbf{M}_O$ has full-rank. That is to say, $\det(\mathbf{M_O}) \neq 0$. For this system, an observability matrix is derived as in Equation 3.1.18.

$$\mathbf{M}_O = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \mathbf{CA}^2 \end{bmatrix} \tag{3.1.18}$$

(e)

So, some matrix calculations are needed to simplify the calculation of $\mathbf{M}_O$.

$$\begin{aligned}\mathbf{CA} &= \begin{bmatrix}4 & 0 & 0\end{bmatrix}\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -4 & -8 & -5\end{bmatrix}\\ &= \begin{bmatrix}0 & 4 & 0\end{bmatrix}\\ \mathbf{CA}^2 = (\mathbf{CA})\mathbf{A} &= \begin{bmatrix}0 & 4 & 0\end{bmatrix}\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -4 & -8 & -5\end{bmatrix}\\ &= \begin{bmatrix}0 & 0 & 4\end{bmatrix}\end{aligned}$$

(e)

Therefore, $\mathbf{M}_O$ becomes obtainable, as in Equation 3.1.19. Next the determinant of $\mathbf{M}_O$ can be found, as in Equation 3.1.20. From this result, the system is determined to be observable.

$$\mathbf{M}_O = \begin{bmatrix}4 & 0 & 0\\ 0 & 4 & 0\\ 0 & 0 & 4\end{bmatrix} \tag{3.1.19}$$

$$\det(\mathbf{M}_O) = (4)\begin{vmatrix}4 & 0\\ 0 & 4\end{vmatrix} = 64 \tag{3.1.20}$$

3.1.6 (f)

Design an observer to track the states of the system. Place all the poles of this observer -4 rad/s.

The system has already been determined to be observable. The observer error equation is shown in Equation 3.1.21, where $\mathbf{K}_e = \begin{bmatrix}k_{e1}\\ k_{e2}\\ k_{e3}\end{bmatrix}$.

$$\frac{d\mathbf{e}(t)}{dt} = (\mathbf{A} - \mathbf{K}_e\mathbf{C})\mathbf{e}(t) \tag{3.1.21}$$

Much like the state feedback vector, the observer poles that are specified should be the Eigenvalues of $\mathbf{A} - \mathbf{K}_e\mathbf{C}$. The characteristic equation for the observer is shown in Equation 3.1.22.

$$\Delta_{\mathbf{K}_e}(s) = \det[s\mathbf{I} - (\mathbf{A} - \mathbf{K}_e\mathbf{C})] = |s\mathbf{I} - \mathbf{A} + \mathbf{K}_e\mathbf{C}| = 0 \tag{3.1.22}$$

Substituting the required matrices into Equation 3.1.22 provides the values for $k_{1,\ 2,\ 3}$.

(f)

$$\begin{aligned}\Delta_{\mathbf{K}_e}(s) &= \left| s\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} - \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -4 & -8 & -5\end{bmatrix} + \begin{bmatrix}k_{e1}\\ k_{e2}\\ k_{e3}\end{bmatrix}\begin{bmatrix}4 & 0 & 0\end{bmatrix}\right|\\ &= \left|\begin{bmatrix}s & -1 & 0\\ 0 & s & -1\\ 4 & 8 & s+5\end{bmatrix} + \begin{bmatrix}4k_{e1} & 0 & 0\\ 4k_{e2} & 0 & 0\\ 4k_{e3} & 0 & 0\end{bmatrix}\right|\\ &= \begin{vmatrix}4k_{e1}+s & -1 & 0\\ 4k_{e2} & s & -1\\ 4k_{e3}+4 & 8 & s+5\end{vmatrix}\\ &= s^3 + 5s^2 + 4k_{e1}s^2 + 8s + 20k_{e1}s + 4k_{e2}s + 32k_{e1} + 20k_{e2} + 4k_{e3} + 4\\ &= s^3 + s^2(5 + 4k_{e1}) + s(8 + 20k_{e1} + 4k_{e2}) + 32k_{e1} + 20k_{e2} + 4k_{e3} + 4\end{aligned}$$

In order to have all of the poles at -4 rad/s, $s_{1,\ 2,\ 3} = -4$. With this in mind, coefficients can begin to be equated and solved for, as in Equations 3.1.23 and 3.1.24.

$$\begin{aligned} s^3 + s^2(5 + 4k_{e1}) + s(8 + 20k_{e1} + 4k_{e2}) + 32k_{e1} + 20k_{e2} + 4k_{e3} + 4 &= (s_1 + 4)(s_2 + 4)(s_3 + 4) \\ s^3 + s^2(5 + 4k_{e1}) + s(8 + 20k_{e1} + 4k_{e2}) + 32k_{e1} + 20k_{e2} + 4k_{e3} + 4 &= s^3 + 12s^2 + 48s + 64 \end{aligned} \tag{3.1.23}$$

$$5 + 4k_{e1} = 12 \tag{3.1.24a}$$

$$\boxed{\therefore k_{e1} = 7/4} \tag{3.1.24b}$$

(f)

$$8 + 20k_{e1} + 4k_{e2} = 48 \tag{3.1.24c}$$

$$\boxed{\therefore k_{e2} = 5} \tag{3.1.24d}$$

$$4 + 32k_{e1} + 20k_{e2} + 4k_{e3} = 64 \tag{3.1.24e}$$

$$\boxed{\therefore k_{e3} = -24} \tag{3.1.24f}$$

Therefore, the observer gain matrix is shown to be Equation 3.1.25.

$$\mathbf{K}_e = \begin{bmatrix} 7/4 \\ 5 \\ -24 \end{bmatrix} \tag{3.1.25}$$

3.2 Exercise #9

Consider a linear system of the form:

$$\begin{aligned} \frac{d\mathbf{x}(t)}{dt} &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t) \\ y(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{D}u(t) \end{aligned}$$

Where,

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -1 & -2 \end{bmatrix},\ \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},\ \mathbf{C} = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \text{ and } \mathbf{D} = \begin{bmatrix} 0 \end{bmatrix} \tag{3.2.1}$$

3.2.1 (a)

Verify the system is controllable and observable.

The controllability matrix, $\mathbf{M}_c$, is given by Equation 3.1.8. So, some matrix calculations are needed to simplify the calculation of $\mathbf{M}_c$.

$$\mathbf{AB} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -1 & -2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$$

$$\begin{aligned} \mathbf{A}^2\mathbf{B} = \mathbf{A}(\mathbf{AB}) &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -1 & -2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} \end{aligned}$$

Therefore, $\mathbf{M}_c$ becomes obtainable, as in Equation 3.2.2. Next, the determinant of $\mathbf{M}_c$ can be found, as in Equation 3.2.3. From this result, the system is determined to be controllable.

$$\mathbf{M}_c = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -2 \\ 1 & -2 & 3 \end{bmatrix} \tag{3.2.2}$$

$$\det(\mathbf{M}_c) = (1) \begin{vmatrix} 0 & 1 \\ 1 & -2 \end{vmatrix} = -1 \tag{3.2.3}$$

(a)

The system will be observable if the observability matrix, $\mathbf{M}_O$ has full-rank. That is to say, $\det(\mathbf{M_O}) \neq 0$. For this system, an observability matrix is derived as in Equation 3.1.18.
So, some matrix calculations are needed to simplify the calculation of $\mathbf{M}_O$.

$$\begin{aligned} \mathbf{CA} &= \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -1 & -2 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & 1 \end{bmatrix} \\ \mathbf{CA}^2 = (\mathbf{CA})\mathbf{A} &= \begin{bmatrix} 0 & 4 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -8 & -5 \end{bmatrix} \\ &= \begin{bmatrix} -4 & -1 & -1 \end{bmatrix} \end{aligned}$$

Therefore, $\mathbf{M}_O$ becomes obtainable, as in Equation 3.2.4. Next the determinant of $\mathbf{M}_O$ can be found, as in Equation 3.2.5. From this result, the system is determined to be observable.

$$\mathbf{M}_O = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -4 & -1 & -1 \end{bmatrix} \tag{3.2.4}$$

$$\det(\mathbf{M}_O) = (1) \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} - (1) \begin{vmatrix} 0 & 1 \\ -4 & -1 \end{vmatrix} = -4 \tag{3.2.5}$$

Exercise #9 continued on next page...

3.2.2 (b)

Design a state feedback vector to place the poles of the system at $s_1 = -2$ and $s_{2,\,3} = -1 \pm j1$.

First, it would be sensible to obtain a polynomial for the desired roots. This can be done by equating $s_{1,\,2,\,3}$ to 0 and finding the product of these equations.

$$\begin{aligned}\Delta_r(s) &= (s+2)(s+1+i)(s+1-i) \\ &= s^3 + 4s^2 + 6s + 4\end{aligned} \tag{3.2.6}$$

This can again be verified with MATLAB.

```
>> r = [-2 -1+i -1-i];
poly(r)

ans =

     1     4     6     4
```

The system has already been shown to be controllable, so next, the characteristic polynomial of $\mathbf{A}$ can be found by using Equation 3.1.12.

$$\begin{aligned}\Delta_K(s) &= \left| s\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} - \begin{bmatrix}0 & 1 & 0\\0 & 0 & 1\\-4 & -1 & -2\end{bmatrix} + \begin{bmatrix}0\\0\\1\end{bmatrix}\begin{bmatrix}k_1 & k_2 & k_3\end{bmatrix}\right| \\ &= \left|\begin{bmatrix}s & -1 & 0\\0 & s & -1\\4 & 1 & s+2\end{bmatrix} + \begin{bmatrix}0 & 0 & 0\\0 & 0 & 0\\k_1 & k_2 & k_3\end{bmatrix}\right| \\ &= \begin{vmatrix}s & -1 & 0\\0 & s & -1\\k_1+4 & k_2+1 & k_3+s+2\end{vmatrix} \\ &= s[s(k_3+s+2)+(k_2+1)]+(k_1+4) \\ &= s^3 + k_3s^2 + 2s^2 + k_2s + s + k_1 + 4 \\ &= s^3 + s^2(2+k_3) + s(1+k_2) + 4 + k_1\end{aligned}$$

(b)

This equation can now be equated to Equation 3.2.6 as in Equation 3.2.7.

$$s^3 + s^2(2+k_3) + s(1+k_2) + 4 + k_1 = s^3 + 4s^2 + 6s + 4 \tag{3.2.7}$$

It is now possible to solve for k_1, k_2 and k_3 as in Equation 3.2.8.

$$2 + k_3 = 4 \tag{3.2.8a}$$
$$\boxed{\therefore k_3 = 2} \tag{3.2.8b}$$
$$1 + k_2 = 6 \tag{3.2.8c}$$
$$\boxed{\therefore k_2 = 5} \tag{3.2.8d}$$
$$4 + k_1 = 4 \tag{3.2.8e}$$
$$\boxed{\therefore k_1 = 0} \tag{3.2.8f}$$

Hence, $\mathbf{K}$ can be populated with its respective values of k, as obtained in Equation 3.2.8.

$$\mathbf{K} = \begin{bmatrix}0 & 5 & 2\end{bmatrix}$$

3.2.3 (c)

Design an observer to track the states of the system. Place all the poles of the observer at $s_{1,\ 2,\ 3} = -1$.

Again, the system has already been determined to be observable. The observer equation is shown in Equation 3.1.21, where $\mathbf{K}_e = \begin{bmatrix} k_{e1} \\ k_{e2} \\ k_{e3} \end{bmatrix}$. Much like the state feedback vector, the observer poles that are specified to us (i.e. $s_{1,\ 2,\ 3} = -1$) should be the Eigenvalues of $\mathbf{A} - \mathbf{K}_e\mathbf{C}$. The characteristic equation for the observer is shown in Equation 3.1.22.

$$\begin{aligned}
\Delta_{\mathbf{K}_e}(s) &= \left| s \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -1 & -2 \end{bmatrix} + \begin{bmatrix} k_{e1} \\ k_{e2} \\ k_{e3} \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \right| \\
&= \left| \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 4 & 1 & s+2 \end{bmatrix} + \begin{bmatrix} k_{e1} & k_{e1} & 0 \\ k_{e2} & k_{e2} & 0 \\ k_{e3} & k_{e3} & 0 \end{bmatrix} \right| \\
&= \begin{vmatrix} k_{e1}+s & k_{e1}-1 & 0 \\ k_{e2} & k_{e2}+s & -1 \\ k_{e3}+4 & k_{e3}+1 & s+2 \end{vmatrix} \\
&= s^3 + 2s^2 + k_{e1}s^2 + k_{e2}s^2 + s + 2k_{e1}s + 3k_{e2}s + k_{e3}s - 3k_{e1} + 2k_{e2} + k_{e3} + 4 \\
&= s^3 + s^2(2 + k_{e1} + k_{e2}) + s(1 + 2k_{e1} + 3k_{e2} + k_{e3}) + k_{e3} + 2k_{e2} - 3k_{e1} + 4
\end{aligned}$$

In order to have all of the observer poles at -1, one must equate the above equation with $(s+1)(s+1)(s+1) = s^3 + 3s^2 + 3s + 1$.

$$s^3 + s^2(2 + k_{e1} + k_{e2}) + s(1 + 2k_{e1} + 3k_{e2} + k_{e3}) + k_{e3} + 2k_{e2} - 3k_{e1} + 4 = s^3 + 3s^2 + 3s + 1$$

(c)

From here, it is simple to compare the coefficients and find a value for k_{e1}, k_{e2} and k_{e3} as in Equation 3.2.9.

$$2 + k_{e1} + k_{e2} = 3 \tag{3.2.9a}$$
$$\boxed{\therefore k_{e1} = 1 - k_{e2}} \tag{3.2.9b}$$
$$1 + 2 - 2k_{e2} + 3k_{e2} + k_{e3} = 3 \tag{3.2.9c}$$
$$3 + k_{e2} + k_{e3} = 3 \tag{3.2.9d}$$
$$\boxed{\therefore k_{e2} = -k_{e3}} \tag{3.2.9e}$$
$$4 - 3(1 - k_{e2}) - 2k_{e3} + k_{e3} = 1 \tag{3.2.9f}$$
$$4 - 3(1 + k_{e3}) - 2k_{e3} + k_{e3} = 1 \tag{3.2.9g}$$
$$1 - 4k_{e3} = 1 \tag{3.2.9h}$$
$$\boxed{\therefore k_{e3} = 0,\ k_{e2} = 0,\ k_{e1} = 1} \tag{3.2.9i}$$

Hence, the observer gain matrix can be populated with its respective values of k from Equation 3.2.9, leaving a rather disappointing result (for the work required...) as in Equation 3.2.10.

$$\mathbf{K}_e = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \tag{3.2.10}$$