Free-rotation shell theory

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GitHub project

1 Introduction

2 Methods

2.1 Vitual work and linearization

$$\Delta\delta\mathcal{W}(\boldsymbol{x},\delta\boldsymbol{u})\cdot\Delta\boldsymbol{u}+\delta\mathcal{W}(\boldsymbol{x},\delta\boldsymbol{u})\approx 0. \tag{1}$$

$$\delta\mathcal{W}(\boldsymbol{x},\delta\boldsymbol{u})=\underbrace{\int_{\Omega_R}\varrho_R\ddot{\boldsymbol{u}}\cdot\delta\boldsymbol{u}dV}_{\delta\mathcal{W}_{kin}}+\underbrace{\int_{\Omega_R}\boldsymbol{S}:\delta\boldsymbol{E}dV}_{\delta\mathcal{W}_{int}}-\underbrace{\int_{\Omega_R}\boldsymbol{f}\cdot\delta\boldsymbol{u}dV-\int_{\Gamma}\boldsymbol{t}\cdot\delta\boldsymbol{u}d\Gamma}_{\delta\mathcal{W}_{ext}}=0 \tag{2}$$

$$\delta\mathcal{W}_{int}=\int_{\Omega_R}\boldsymbol{S}:\delta\boldsymbol{E}dV=\int_A\int_H\boldsymbol{S}:\delta\boldsymbol{E}d\xi_3dA=\int_A\left[\boldsymbol{N}:\delta\boldsymbol{E}^m+\boldsymbol{M}:\delta\boldsymbol{\chi}\right]dA \tag{3}$$

$$\boldsymbol{N}=\int_H\boldsymbol{S}d\xi_3 \text{ and } \boldsymbol{M}=\int_H\xi_3\boldsymbol{S}d\xi_3 \tag{4}$$

$$\boldsymbol{E}=\boldsymbol{E}^m+\xi_3\boldsymbol{\chi} \tag{5}$$

$$\begin{aligned}\Delta\delta\mathcal{W}_{int}\cdot\Delta\boldsymbol{u} &= \int_A\int_H\left[\delta\boldsymbol{E}:\mathbb{C}:\Delta\boldsymbol{E}+\boldsymbol{S}:\Delta\delta\boldsymbol{E}\right]d\xi_3dA\\ &= \int_A\left[\delta\boldsymbol{E}^m:\mathbb{C}^m:\Delta\boldsymbol{E}^m+\delta\boldsymbol{E}^m:\mathbb{C}^{mb}:\Delta\boldsymbol{\chi}+\delta\boldsymbol{\chi}:\mathbb{C}^{mb}:\Delta\boldsymbol{E}^m+\delta\boldsymbol{\chi}:\mathbb{C}^b:\Delta\boldsymbol{\chi}\right]dA\\ &\quad+\int_A\left[\boldsymbol{N}:\Delta\delta\boldsymbol{E}^m+\boldsymbol{M}:\Delta\delta\boldsymbol{\chi}\right]dA\end{aligned} \tag{6}$$

$$\mathbb{C}^m=\int_H\mathbb{C}d\xi_3 \text{ and } \mathbb{C}^{mb}=\int_H\xi_3\mathbb{C}d\xi_3 \text{ and } \mathbb{C}^b=\int_H\xi_3^2\mathbb{C}d\xi_3 \tag{7}$$

2.2 Strain definition

$$\boldsymbol{E}^m = E^m_{\alpha\beta} = \frac{1}{2}(a_{\alpha\beta} - A_{\alpha\beta}) \tag{8}$$

$$A_{\alpha\beta} = \vec{A}_\alpha \cdot \vec{A}_\beta \text{ and } a_{\alpha\beta} = \vec{a}_\alpha \cdot \vec{a}_\beta \tag{9}$$

$$\boldsymbol{\chi} = \chi_{\alpha\beta} = \kappa_{\alpha\beta} - \kappa^R_{\alpha\beta} \tag{10}$$

$$\kappa^R_{\alpha\beta} = \frac{1}{2}(\vec{A}_\alpha \cdot \vec{A}_{3,\beta} + \vec{A}_\beta \cdot \vec{A}_{3,\alpha}) \text{ and } \kappa_{\alpha\beta} = \frac{1}{2}(\vec{a}_\alpha \cdot \vec{a}_{3,\beta} + \vec{a}_\beta \cdot \vec{a}_{3,\alpha}) \tag{11}$$

$$\kappa^R_{\alpha\beta} = -\vec{A}_{\beta,\alpha} \cdot \vec{A}_3 \text{ and } \kappa_{\alpha\beta} = -\vec{a}_{\beta,\alpha} \cdot \vec{a}_3 \tag{12}$$

$$\kappa_{\alpha\beta} = -\frac{1}{A^M} \int_{A^M} \vec{a}_{\beta,\alpha} \cdot \vec{a}_3 dA^M = -\frac{1}{A^M} \oint_{\Gamma^M} n_\alpha \vec{a}_\beta \cdot \vec{a}_3 d\Gamma^M = \vec{h}_{\alpha\beta} \cdot \vec{a}_3 \tag{13}$$

2.3 Kinematics

$$\vec{A}_\alpha = \vec{\varphi},^R_\alpha = \frac{\partial \vec{\varphi}^R}{\partial \xi_\alpha} \text{ and } \vec{a}_\alpha = \vec{\varphi},_\alpha = \frac{\partial \vec{\varphi}}{\partial \xi_\alpha} \tag{14}$$

$$\vec{X}(\xi_1, \xi_2, \xi_3) = \vec{\varphi}^R(\xi_1, \xi_2) + \xi_3 \vec{A}_3(\xi_1, \xi_2) \tag{15}$$

$$\vec{x}(\xi_1, \xi_2, \xi_3) = \vec{\varphi}(\xi_1, \xi_2) + \xi_3 \vec{a}_3(\xi_1, \xi_2) \tag{16}$$

$$\vec{G}_i = \frac{\partial \vec{X}}{\partial \xi_i} = \vec{G}_\alpha + \vec{G}_3 \text{ where } \vec{G}_\alpha = \vec{A}_\alpha + \xi_3 \vec{A}_{3,\alpha} \text{ and } \vec{G}_3 = \vec{A}_3 \tag{17}$$

$$\vec{g}_i = \frac{\partial \vec{g}}{\partial \xi_i} = \vec{g}_\alpha + \vec{g}_3 \text{ where } \vec{g}_\alpha = \vec{a}_\alpha + \xi_3 \vec{a}_{3,\alpha} \text{ and } \vec{g}_3 = \vec{a}_3 \tag{18}$$

$$G_{ij} = \vec{G}_i \cdot \vec{G}_j = \begin{pmatrix} A_{\alpha\beta} + 2\xi_3 \kappa^R_{\alpha\beta} + \xi_3^2 \vec{A}_{3,\alpha} \cdot \vec{A}_{3,\beta} & (\vec{A}_\alpha + \xi_3 \vec{A}_{3,\alpha}) \cdot \vec{A}_3 \\ \vec{A}_3 \cdot (\vec{A}_\beta + \xi_3 \vec{A}_{3,\beta}) & \vec{A}_3 \cdot \vec{A}_3 \end{pmatrix} = \begin{pmatrix} A_{\alpha\beta} + 2\xi_3 \kappa^R_{\alpha\beta} & 0 \\ 0 & 1 \end{pmatrix} \tag{19}$$

$$g_{ij} = \vec{g}_i \cdot \vec{g}_j = \begin{pmatrix} a_{\alpha\beta} + 2\xi_3 \kappa_{\alpha\beta} + \xi_3^2 \vec{a}_{3,\alpha} \cdot \vec{a}_{3,\beta} & (\vec{a}_\alpha + \xi_3 \vec{a}_{3,\alpha}) \cdot \vec{a}_3 \\ \vec{a}_3 \cdot (\vec{a}_\beta + \xi_3 \vec{a}_{3,\beta}) & \vec{a}_3 \cdot \vec{a}_3 \end{pmatrix} = \begin{pmatrix} a_{\alpha\beta} + 2\xi_3 \kappa_{\alpha\beta} & 0 \\ 0 & a_{33} \end{pmatrix} \tag{20}$$

$$G_{ij} = \begin{pmatrix} A_{\alpha\beta} & 0 \\ 0 & 1 \end{pmatrix} \tag{21}$$

$$g_{ij} = \begin{pmatrix} a_{\alpha\beta} + 2\xi_3\chi_{\alpha\beta} & 0 \\ 0 & a_{33} \end{pmatrix} \tag{22}$$

$$\boldsymbol{F} = \frac{\partial \vec{x}}{\partial \vec{X}} = \frac{\partial \vec{x}}{\partial \xi_i} \otimes \frac{\partial \xi_i}{\partial \vec{X}} = \vec{g}_i \otimes \vec{G}^i \tag{23}$$

$$\boldsymbol{C} = \boldsymbol{F}^T \boldsymbol{F} = g_{ij} \vec{G}^i \otimes \vec{G}^j \Rightarrow a_{33} = C_{33} \tag{24}$$

$$J^2 = C_{33} \frac{\det(g_{\alpha\beta})}{\det(G_{\alpha\beta})} \tag{25}$$

$$E_{ij} = \frac{1}{2}(g_{ij} - G_{ij}) = \frac{1}{2} \begin{pmatrix} a_{\alpha\beta} - A_{\alpha\beta} + 2\xi_3\chi_{\alpha\beta} & 0 \\ 0 & a_{33} - 1 \end{pmatrix} = \begin{pmatrix} E^m_{\alpha\beta} + \xi_3\chi_{\alpha\beta} & 0 \\ 0 & E_{33} \end{pmatrix} \tag{26}$$

2.4 Discretization

$$\begin{array}{ll} \text{node 1: } (\eta_1, \eta_2) = (0, 0) & \text{node 4: } (\eta_1, \eta_2) = (1, 1) \\ \text{node 2: } (\eta_1, \eta_2) = (1, 0) & \text{node 5: } (\eta_1, \eta_2) = (-1, 1) \\ \text{node 3: } (\eta_1, \eta_2) = (0, 1) & \text{node 6: } (\eta_1, \eta_2) = (1, -1) \end{array} \tag{27}$$

$$L^1 = \eta_1 \quad L^2 = \eta_2 \quad L^3 = 1 - \eta_1 - \eta_2 \tag{28}$$

$$\begin{array}{ll} N^1 = \eta_3 + \eta_1\eta_2 & N^4 = \frac{1}{2}\eta_3(\eta_3 - 1) \\ N^2 = \eta_1 + \eta_2\eta_3 & N^5 = \frac{1}{2}\eta_1(\eta_1 - 1) \\ N^3 = \eta_2 + \eta_3\eta_1 & N^6 = \frac{1}{2}\eta_2(\eta_2 - 1) \end{array} \tag{29}$$

$$\eta_3 = 1 - \eta_1 - \eta_2 \tag{30}$$

$$\vec{\varphi} = \sum_{a=1}^{6} N^a(\eta_1, \eta_2)\vec{\varphi}^a \text{ and } \vec{\varphi},_\alpha = \sum_{a=1}^{6} N,^a_\alpha \vec{\varphi}^a \tag{31}$$

$$\vec{\varphi},^I_\alpha = \sum_{a=1}^{3} N,^a_\alpha \vec{\varphi}^a + N,^{I+3}_\alpha \vec{\varphi}^{I+3} \tag{32}$$

$$a_{\alpha\beta} = \sum_{I=1}^{3} L^I a^I_{\alpha\beta} = \sum_{I=1}^{3} L^I \vec{\varphi},^I_\alpha \cdot \vec{\varphi},^I_\beta \tag{33}$$

$$\vec{h}_{\alpha\beta} = \sum_{I=1}^{3} (L,^I_\alpha \vec{\varphi},^I_\beta + L,^I_\beta \vec{\varphi},^I_\alpha) \tag{34}$$

$$\vec{a}_3 = \frac{\vec{\varphi},^M_1 \times \vec{\varphi},^M_2}{|\vec{\varphi},^M_1 \times \vec{\varphi},^M_2|} \text{ with } \vec{\varphi},^M_\alpha = \sum_{I=1}^{3} L,^I_\alpha \vec{\varphi}^I \tag{35}$$

2.5 First variation of strain

$$\begin{aligned}
\delta \boldsymbol{E}^m &= \delta E^m_{\alpha\beta} = \frac{1}{2}\delta a_{\alpha\beta} = \frac{1}{2}\sum_{I=1}^{3} L^I(\delta\vec{\varphi},^I_\alpha \cdot\vec{\varphi},^I_\beta + \vec{\varphi},^I_\alpha \cdot\delta\vec{\varphi},^I_\beta) \\
&= \frac{1}{2}\sum_{I=1}^{3} L^I \left[\sum_{a=1}^{3}(N,^a_\alpha \cdot\vec{\varphi},^I_\beta + \vec{\varphi},^I_\alpha \cdot N,^a_\beta)\delta\vec{\varphi}^a + (N,^{I+3}_\alpha \cdot\vec{\varphi},^I_\beta + \vec{\varphi},^I_\alpha \cdot N,^{I+3}_\beta)\delta\vec{\varphi}^{I+3}\right] \quad (36)
\end{aligned}$$

$$\delta\boldsymbol{\chi} = \delta\chi_{\alpha\beta} = \delta\kappa_{\alpha\beta} = \delta\vec{h}_{\alpha\beta}\cdot\vec{a}_3 + \vec{h}_{\alpha\beta}\cdot\delta\vec{a}_3 \quad (37)$$

$$\begin{aligned}
\delta\vec{h}_{\alpha\beta}\cdot\vec{a}_3 &= \sum_{I=1}^{3}(L,^I_\alpha \delta\vec{\varphi},^I_\beta + L,^I_\beta \delta\vec{\varphi},^I_\alpha)\cdot\vec{a}_3 \\
&= \sum_{I=1}^{3}\left[\sum_{a=1}^{3}(L,^I_\alpha N,^a_\beta + L,^I_\beta N,^a_\alpha)\delta\vec{\varphi}^a + (L,^I_\alpha N,^{I+3}_\beta + L,^I_\beta N,^{I+3}_\alpha)\delta\vec{\varphi}^{I+3}\right]\cdot\vec{a}_3 \quad (38)
\end{aligned}$$

$$\begin{aligned}
\vec{h}_{\alpha\beta}\cdot\delta\vec{a}_3 &= \vec{h}_{\alpha\beta}\cdot(\delta a_{31}\vec{a}_1^\diamond + \delta a_{32}\vec{a}_2^\diamond) = -(\vec{a}_3\cdot\delta\vec{\varphi},^M_1 \vec{a}_1^\diamond + \vec{a}_3\cdot\delta\vec{\varphi},^M_2 \vec{a}_2^\diamond)\cdot\vec{h}_{\alpha\beta} \\
&= -\left[\sum_{I=1}^{3}\left(L,^I_1 \vec{a}_1^\diamond\cdot\vec{h}_{\alpha\beta} + L,^I_2 \vec{a}_2^\diamond\cdot\vec{h}_{\alpha\beta}\right)\right](\vec{a}_3\cdot\delta\vec{\varphi}^I) \quad (39) \\
& \quad (40)
\end{aligned}$$

$$\vec{a}_1^\diamond = \frac{\vec{\varphi},^M_2 \times\vec{a}_3}{|\vec{\varphi},^M_1 \times\vec{\varphi},^M_2|} \text{ and } \vec{a}_2^\diamond = -\frac{\vec{\varphi},^M_1 \times\vec{a}_3}{|\vec{\varphi},^M_1 \times\vec{\varphi},^M_2|} \quad (41)$$

2.6 Second variation of strain

$$\Delta\delta\boldsymbol{E}^m = \Delta\delta E^m_{\alpha\beta} = \frac{1}{2}\Delta\delta a_{\alpha\beta} = \frac{1}{2}\sum_{I=1}^{3} L^I(\delta\vec{\varphi},^I_\alpha \cdot\Delta\vec{\varphi},^I_\beta + \Delta\vec{\varphi},^I_\alpha \cdot\delta\vec{\varphi},^I_\beta) \quad (42)$$

$$\Delta\delta\boldsymbol{\chi} = \Delta\delta\chi_{\alpha\beta} = \Delta\delta\kappa_{\alpha\beta} = \delta\vec{h}_{\alpha\beta}\cdot\Delta\vec{a}_3 + \Delta\vec{h}_{\alpha\beta}\cdot\delta\vec{a}_3 \quad (43)$$

3 Results

4 Discussion

5 Conclusions