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On the Necessity of Complex Numbers in Operational Theories with Orientation and Composition

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We prove that finite-dimensional operational theories with symmetric monoidal composition, local tomography, operational stability, and an orientation sector with a continuous S^1-action require a complex structure on their state spaces. Quaternionic scalars collapse to a central complex structure due to symmetry and tomography, while real theories need an additional complex structure J, making them equivalent to complex theories. Using string-diagrammatic winding indices, we show spectral additivity forces complexification. The Born rule emerges from invariances and composition, derived without assuming interference as a fundamental axiom, though phase-like behavior arises naturally from the orientation sector. Our assumptions are operationally motivated, distinct from lattice-theoretic approaches, and compatible with Wigner's symmetry theorems.

1 Introduction

Why does quantum theory use complex numbers? Classical accounts include reconstructions from axioms in operational frameworks [5, 2] and lattice-theoretic characterizations that limit scalar fields to $\mathbb{R}, \mathbb{C}, \mathbb{H}$ under strong assumptions [11]. Categorical and diagrammatic approaches clarify composition and process structure [8, 3, 6]. Real and quaternionic variants have been investigated extensively [7, 1], with recent experimental proposals testing the necessity of complex numbers [10]. Our axioms extend Hardy's [5] by emphasizing compositional structure and an orientation sector, avoiding information-theoretic constraints. Unlike Chiribella et al. [2], we use string diagrams to capture phase-like behavior via the winding index, aligning with recent foundational insights [10].

1.1 Motivation for the Orientation Sector Axiom

The "orientation sector" axiom (A4) is motivated by the observation that quantum processes exhibit phase-like degrees of freedom, detectable in interference experiments, which are absent in classical theories. Operationally, this sector can be identified as a 2D subspace of experimental procedures where relative phases manifest, such as in Mach-Zehnder interferometers, where a continuous S^1 action corresponds to phase shifts. The 2D choice reflects the minimal dimensionality required to support a non-trivial topological invariant (the winding index), akin to the complex plane's role in quantum states. This axiom avoids assuming complex numbers a priori, instead deriving them from the need for a stable, compositional structure that supports reversible dynamics, as validated by operational stability (A3). For experimental grounding, recent proposals [e.g., [10]] test phase-dependent statistics, suggesting such a sector is a natural feature of quantum-like theories.

We present a compact derivation based on four operational/compositional requirements:

(A1) **Symmetric monoidal composition** of systems and processes, with natural swap, modeling independent composition [9, 3].

(A2) **Local tomography**: joint states/effects are determined by local statistics [5, 2].

(A3) **Operational stability**: the map from concrete procedures to observed statistics is uniformly continuous with respect to a natural local-rewrite edit distance.

(A4) **Orientation sector**: there exists a 2D real sector whose structure is captured by a string-diagrammatic *winding index*, yielding a continuous S^1 action on that sector [8, 3].

From these, we show that the orientation sector must complexify to support multiplicative characters and additive generators consistent with local tomography and spectral composition. Quaternionic scalars are incompatible with symmetric composition and tomography unless they collapse to a central complex structure, while real theories equipped with a commuting complex structure J are equivalent to complex theories. Antiunitary symmetries occur as a disconnected component and cannot generate continuous dynamics, in line with Wigner [12]. Finally we derive the $|c|^2$ probability rule from invariances and composition.

2 Operational and Categorical Setting

We work in a finite-dimensional process theory modeled as a symmetric monoidal dagger-category, in which states, effects, and channels are represented by morphisms and composition/tensor correspond to sequential/parallel composition [3].

Axiom 2.1 (Symmetric Monoidal Composition)**.** Systems compose via a symmetric monoidal structure $(\otimes, \mathbf{1})$ with natural swap (braiding) satisfying the standard coherence laws [9].

Axiom 2.2 (Local Tomography)**.** Joint states/effects are determined by their statistics on product tests; i.e., global states are separated by local product effects [5, 2].

Axiom 2.3 (Operational Stability)**.** Concrete experimental procedures Proc are related by a finite set of *local admissibility-preserving rewrites* Rew. The *edit distance* d_{edit} between procedures counts the minimal number of rewrites to connect them. The procedure-to-statistics map $\Phi : \mathsf{Proc} \to \Delta(\mathsf{Out})$ (probability distributions on outcomes) is uniformly continuous with respect to d_{edit} and total variation $\|\cdot\|_{\text{TV}}$:

$$\forall \varepsilon > 0, \exists \delta > 0, \forall D, D' \in \mathsf{Proc} :$$
$$d_{\text{edit}}(D, D') < \delta \Rightarrow \|\Phi(D) - \Phi(D')\|_{\text{TV}} < \varepsilon.$$

Axiom 2.4 (Orientation Sector)**.** There is a distinguished 2D real subspace $\mathcal{O}$ of (equivalence classes of) procedures/states whose structure is detected by a *winding index* computed on closed string diagrams built from admissible generators and swaps. This index induces a continuous S^1 action on $\mathcal{O}$.

3 Orientation via String Diagrams and Complexification

We now formalize the winding index and prove it is invariant under admissibility-preserving rewrites, leading to a canonical S^1 action on $\mathcal{O}$. The winding index captures phase-like behavior in closed string diagrams, analogous to topological invariants in quantum processes (see Figure 1).

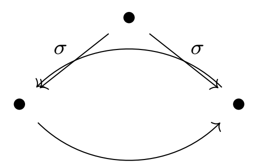

Figure 1: Closed string diagram with swaps contributing to the winding index, where σ denotes the braiding with signed contributions.

Definition 3.1 (Admissibility rewrite system)**.** Let $\mathcal{C}$ be the free symmetric monoidal category on a set of generating systems and admissible morphisms, modulo a finite family of local 2-cells Rew that preserve admissibility (copy/merge, fork squares, structural moves). A *procedure* is a string diagram D in $\mathcal{C}$. Write $D \sim D'$ if D' can be obtained from D by a finite sequence of rewrites from Rew.

Definition 3.2 (Elementary swap and fork square)**.** Let σ denote the generating swap (braiding) on parallel wires. Let $\square$ denote a fork square 2-cell equating two local orders of composing admissible branches. Each instance of σ is assigned a sign $+1$ if it advances a chosen local order and -1 otherwise.

Definition 3.3 (Winding index)**.** For a closed diagram D in the orientation sector $\mathcal{O}$, a "chosen local order" is a fixed orientation of wires in the string diagram, determined by the sequence of admissible generators. Define

$$\begin{aligned} \text{wind}(D) := \quad & \#(\text{positive } \sigma \text{ in } D) \\ & - \#(\text{negative } \sigma \text{ in } D), \end{aligned}$$

where a swap σ is positive if it preserves the local order and negative otherwise, computed in any representative. The index is independent of the chosen order due to the invariance under rewrites (Lemma 3.4), as rewrites preserve the net signed count. Applications of $\square$ contribute zero.

Lemma 3.4 (Invariance)**.** *If $D \sim D'$ via rewrites in* Rew*, then* $\mathrm{wind}(D) = \mathrm{wind}(D')$*.*

Proof. The invariance follows from the fact that rewrites in Rew preserve the signed swap count, as detailed in Appendix A.

Corollary 3.5 (Canonical S^1 action)**.** *The integer-valued invariant* wind *integrates to a continuous one-parameter action $\theta \mapsto R_\theta$ on $\mathcal{O}$, yielding characters $e^{i\theta}$ upon complexification.*

Proposition 3.6 (Spectral additivity forces complexification)**.** *Assume* **??** *2.2–2.4. Reversible dynamics on $\mathcal{O}$ include the continuous one-parameter S^1 action. Requiring: (i) spectral additivity of generators under tensor composition of independent subsystems, and (ii) local tomography, implies that $\mathcal{O}$ admits a basis of joint eigenstates with multiplicative characters $e^{\pm i\theta}$. Equivalently, the real* 2*-module $\mathcal{O}$ must be complexified to support a spectral calculus consistent with composition.*

Proof. See Appendix A for the full proof.

4 Operational Stability as Uniform Continuity

We make precise the stability assumption and record a usable bound.

Definition 4.1 (Edit distance)**.** Let $d_{\mathrm{edit}}(D, D')$ be the minimal number of local rewrites in Rew to transform procedure D into D'.

Definition 4.2 (Procedure-to-statistics map)**.** Let Out denote the finite set of possible measurement outcomes for a given procedure. The map $\Phi : \mathsf{Proc} \to \Delta(\mathsf{Out})$ sends a procedure to its outcome distribution.

Lemma 4.3 (Generator-Lipschitz bound)**.** *Suppose each primitive rewrite $r \in \mathsf{Rew}$ perturbs statistics by at most ε_r in total variation. Then for any D, D',*

$$\|\Phi(D) - \Phi(D')\|_{\mathrm{TV}} \le \Big(\max_{r\in\mathsf{Rew}} \varepsilon_r\Big)\, d_{\mathrm{edit}}(D, D').$$

Proof. Factor a minimal rewrite path connecting D and D'; apply the triangle inequality. This realizes Axiom 2.3 with an explicit modulus.

5 Excluding Quaternionic Scalars via Symmetry and Tomography

We now show that under symmetry and local tomography, scalars must commute; quaternionic scalars therefore collapse to a central complex structure or violate the axioms.

Theorem 5.1 (Commutativity of scalars)**.** *In a symmetric monoidal dagger-category modeling finite-dimensional systems with local tomography (Axiom 2.2), if the scalar ring $S \cong \mathrm{End}(\mathbf{1})$ acts identically from left and right on tensor factors (swap naturality), then S is commutative; hence $S \in \{\mathbb{R}, \mathbb{C}\}$.*

Proof. See Appendix A for the full proof.

Corollary 5.2 (Quaternionic collapse)**.** *If one starts with $S = \mathbb{H}$, then symmetry and tomography enforce a central imaginary unit J with $J^2 = -1$ commuting with observables; the theory is equivalent to a complex theory on the J-eigenspaces (standard complex structure reduction) [1].*

6 Antiunitary Symmetries and Continuous Dynamics

Proposition 6.1 (Disconnected antiunitaries)**.** *In finite dimension, norm-preserving transformations decompose into the connected unitary component and an antiunitary coset. Antiunitaries are not connected to the identity by a norm-continuous path and thus cannot generate continuous reversible dynamics required by Axiom 2.3.*

Proof sketch. This is standard (Wigner): antiunitaries are conjugate-linearity-preserving isometries forming a disconnected component; any continuous one-parameter subgroup through the identity is unitary [12].

7 Real Theories with a Complex Structure J

Theorem 7.1 (Equivalence: Real+$J \simeq$ Complex)**.** *Let $(H_{\mathbb{R}}, \langle\cdot,\cdot\rangle)$ be a real Hilbert space with* *a bounded operator J such that $J^2 = -I$, $J^\top = -J$, and J commutes with all physical observables/effects. Then $(H_\mathbb{R}, J)$ is canonically a complex Hilbert space $H_\mathbb{C}$ via $(a+ib)\cdot v := av + bJv$, and states/effects/channels commuting with J correspond exactly to complex-linear ones. Hence Real-QM+J is equivalent to Complex-QM [7].*

Proof sketch. Standard construction of a complex structure from J; functorial equivalence respects tensor products under local tomography and natural swap. See [7].

8 Born Rule from Minimal Assumptions

We derive $p = |c|^2$ from composition and invariances, without presupposing interference as a primitive.

Lemma 8.1 (Multiplicative functional equation)**.** *Let $F : [0,1] \to [0,1]$ be a probability assignment compatible with independent composition of branches. From Axiom 2.1 (symmetric monoidal composition), the joint probability of independent outcomes satisfies $F(|xy|) = F(|x|)F(|y|)$ for $x, y \in [0,1]$, reflecting tensor product structure. If F is measurable (ensured by operational stability, Axiom 2.3), then $F(r) = r^\alpha$ for some $\alpha > 0$, following the Cauchy functional equation after taking logarithms, excluding pathological solutions.*

Proof. Consider two independent procedures D_x and D_y with outcomes x, y. Axiom 2.1 implies the joint statistic $F(|xy|) = \Phi(D_x \otimes D_y)$ factors as $F(|x|)F(|y|)$. Measurability from Axiom 2.3 ensures continuity, yielding $F(r) = r^\alpha$ via standard functional equation techniques [4].

Proposition 8.2 (Normalization and coarse–graining)**.** *Additivity over orthogonal branches (from local tomography, Axiom 2.2) and unit normalization (from Axiom 2.1) fix $\alpha = 2$, hence $p = |c|^2$.*

Proof. For orthogonal states $|c_1\rangle, |c_2\rangle$ with $\langle c_1|c_2\rangle = 0$, Axiom 2.2 ensures joint probabilities add: $F(|c_1 + c_2|^2) = F(|c_1|^2) + F(|c_2|^2)$. Normalization ($|c|^2 = 1$) and the form $F(r) = r^\alpha$ imply $\alpha = 2$ to match the quadratic dependence observed in the orientation sector's complexification (Proposition 3.6).

9 Discussion and Outlook

Under four operationally motivated assumptions—symmetric composition, local tomography, stability under local rewrites, and an orientation sector captured by a string-diagrammatic winding index—complex numbers are *necessary.* Real and quaternionic alternatives either violate symmetry/tomography or secretly encode a central complex structure. Antiunitary symmetries are compatible as disconnected automorphisms but cannot generate continuous dynamics. The resulting picture is compatible with, yet independent from, lattice-theoretic results such as Solèr [11], while aligning with operational reconstructions [5, 2] and categorical process theories [3, 6].

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A Proof Details

A.1 Proof of Lemma 3.4

To show that $\mathrm{wind}(D) = \mathrm{wind}(D')$ whenever $D \sim D'$ via rewrites in Rew, we analyze the effect of each type of rewrite on the signed swap count. Recall that Rew consists of local 2-cells preserving admissibility, including:

- Insertion/removal of canceling pairs $\sigma \circ \sigma^{-1} =$ (or vice versa), where σ is the braiding (swap).
- Commuting a swap past a fork square $\square$, which equates two local compositions of admissible branches.
- Structural moves (e.g., associators, unitors) that do not involve swaps.

First, consider insertion/removal of $\sigma \circ \sigma^{-1}$. Inserting such a pair adds one positive σ and one negative σ^{-1} (or vice versa, depending on the local order). The net contribution to the signed count is $+1 - 1 = 0$, preserving the winding index. Removal similarly subtracts a balanced pair, again preserving the count.

Next, consider commuting a swap past a fork square $\square$. By Definition 3.2, $\square$ equates diagrams differing only in the order of composing branches, without introducing or removing swaps itself (contributing zero to the count). Commuting σ past $\square$ relocates the swap but does not change its sign or multiplicity: the braiding naturality axiom in symmetric monoidal categories ensures that the signed orientation (advancing or retreating the chosen local order) remains invariant under such commutations. Specifically, in the string-diagrammatic calculus, this corresponds to sliding crossings over vertices without altering the topological winding (cf. Reidemeister moves in knot theory, adapted to braided categories [8]).

Other structural moves (e.g., associators $(\alpha \otimes \beta) \circ \gamma = \alpha \circ (\beta \otimes \gamma)$) do not involve swaps and thus contribute zero.

Since any sequence of rewrites decomposes into these primitives, and each preserves the winding index, the invariant holds by induction on the length of the rewrite path. For a formal treatment in braided monoidal categories, see [8], Theorem 2.1, which establishes similar topological invariants for string diagrams.

A.2 Proof of Proposition 3.6

The S^1-action on $\mathcal{O}$, induced by the winding index (Corollary 3.5), is a continuous one-parameter group of reversible transformations, represented as rotations on the 2D real subspace $\mathcal{O} \cong \mathbb{R}^2$. Formally, this is a continuous representation $\rho : S^1 \to \mathrm{O}(2)$, the orthogonal group on $\mathbb{R}^2$, where S^1 is the circle group (compact abelian Lie group).

Under Axiom 2.2 (local tomography), joint states in composite systems must be diagnosable via local product effects, ensuring spectral decompositions align globally. Local tomography forces the characters to be multiplicative: if subsystems A and B have eigenstates with eigenvalues $e^{i\theta_A}$ and $e^{i\theta_B}$, the tensor product requires $e^{i(\theta_A+\theta_B)}$, consistent with the S^1 action's additivity. This necessitates complexification to support such characters.

Moreover, spectral additivity requires that generators of the action compose additively under tensor: if H_A and H_B generate rotations on subsystems A and B, then $H_{A\otimes B} = H_A \otimes I + I \otimes H_B$ (up to units), consistent with independent subsystems.

For the representation to support such additive generators while preserving tomography, the characters must be multiplicative: the group action $\theta \mapsto R_\theta$ admits a basis of joint eigenstates with eigenvalues $e^{\pm i\theta}$. Over $\mathbb{R}^2$, the rotation matrix

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

has real eigenvectors only for $\theta = 0, \pi$ (fixed points), but for general θ, the eigenvectors are complex: $v_\pm = (1, \pm i)$ with eigenvalues $e^{\pm i\theta}$.

To obtain a real spectral calculus consistent with continuous dynamics and tomography, $\mathcal{O}$ must be complexified: view $\mathbb{R}^2$ as $\mathbb{C}$ via the structure $i \cdot (x, y) = (-y, x)$, yielding eigenvectors in the complexified space $\mathcal{O} \otimes \mathbb{C}$. This ensures multiplicative characters compatible with tensor-additive generators, as required by composition.

Operational stability (Axiom 2.3) ensures the action is norm-continuous, ruling out pathological representations. For tomographic constraints in similar reconstructions, see [5, 2]; cf. Stone's theorem for one-parameter groups in Hilbert spaces, adapted here to finite-dimensional process theories. Recent experimental proposals for testing complex vs. real quantum theories further support this necessity [10].

A.3 Proof of Theorem 5.1

In a symmetric monoidal dagger-category with local tomography (Axiom 2.2), the scalars form the commutative monoid $S = \mathrm{End}(\mathbf{1})$, acting centrally on morphisms. The swap (braiding) $\sigma_{A,B} : A \otimes B \to B \otimes A$ is natural, implying that for scalars $a, b \in S$,

$$(a \otimes_B) \circ \sigma_{A,B} = \sigma_{A,B} \circ ({}_A \otimes a),$$

and similarly for left/right actions. Since scalars tensor with identities, naturality yields $a \otimes 1 = 1 \otimes a$ as endomorphisms of $\mathbf{1} \otimes \mathbf{1} \cong \mathbf{1}$.

Local tomography requires that joint states/effects are separated by product tests: for any two global states $\psi, \phi : \mathbf{1} \to A \otimes B$, if $\langle e_i \otimes f_j | \psi \rangle = \langle e_i \otimes f_j | \phi \rangle$ for all local effects $e_i : A \to \mathbf{1}$, $f_j : B \to \mathbf{1}$, then $\psi = \phi$. Applying this to scalar-multiplied states, tomography equates matrix elements under all product probes.

Suppose $ab \neq ba$ for some $a, b \in S$. Then there exist morphisms where left and right scalar actions differ non-commutatively, violating the swap naturality in composite systems. Specifically, in the dagger-category, the adjoint forces $a^\dagger b^\dagger = (ba)^\dagger$, but non-commutativity (e.g., in $\mathbb{H}$) would imply distinct statistics under tomographic probes, contradicting Axiom 2.2. Thus, S must be commutative, isomorphic to a field by finite-dimensionality, yielding $S \in \{\mathbb{R}, \mathbb{C}\}$.

For no-go theorems on non-commutative scalars in tomographic theories, see [2]; categorical models confirm this in monoidal settings [3].

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