The Born Rule from Determinate Identity: A Logic-Realist Derivation

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Abstract

The Born rule assigns probability $|\langle\phi|\psi\rangle|^2$ to obtaining outcome $|\phi\rangle$ given state $|\psi\rangle$. While Gleason's theorem (1957) establishes this as the unique probability measure satisfying certain additivity and non-contextuality conditions, the question of *why* physical systems should satisfy those conditions remains open. This paper argues that Determinate Identity — the requirement that physical systems be determinately what they are, independent of description — provides the answer.

The core argument: the measure over admissible measurement outcomes is a feature of the physical situation (vehicle), not merely a representational choice (content). If this measure varied with mathematical decomposition, the physical situation itself would be indeterminate, violating Determinate Identity. This constraint forces precisely the invariance conditions that Gleason's theorem requires. The Born rule thus emerges as the unique measure compatible with determinate physical identity.

1. Introduction

Gleason's theorem shows that the Born rule is the unique probability measure on quantum projections satisfying additivity and non-contextuality. But why should physical probabilities satisfy these conditions? Standard quantum mechanics treats them as postulates; this paper derives them from a more fundamental principle.

The central claim is that Determinate Identity — the requirement that any physical system is determinately what it is, independent of how it is described — constrains the measure over measurement outcomes. The measure belongs to the physical situation itself (what we call the *vehicle*), not to our representational choices. A measure that varied with basis choice would make the physical situation indeterminate, violating identity.

Relation to companion papers. This paper is part of the Logic Realism Theory (LRT) programme:

- **Position Paper** (DOI: 10.5281/zenodo.18111737): Establishes the L_3 framework (Id, NC, EM) as ontological constraint
- **Hilbert Space Paper**: Derives complex Hilbert space from Determinate Identity
- **This Paper**: Derives Born rule from vehicle-weight invariance
- **GR Extension**: Explores spacetime implications (programmatic)

The papers can be read independently, but together form a unified derivation from L_3 to quantum mechanics.

Structure. Section 2 summarizes the framework. Section 3 characterizes quantum states within this framework. Section 4 presents the core argument: vehicle-weight invariance → Born rule. Section 5 addresses objections. Section 6 concludes. Appendix A provides the formal proof.

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2. Framework Summary

2.1 The L_3 Constraint

Let L_3 denote three classical logical principles understood as constraints on physical instantiation:

- **Determinate Identity (Id)**: A physical configuration is determinately what it is, independent of description
- **Non-Contradiction (NC)**: A configuration cannot instantiate both P and $\neg P$ in the same respect
- **Excluded Middle (EM)**: For any applicable property P, either P or $\neg P$ is instantiated

Physical reality is the set of configurations satisfying L_3. See the Position Paper for the full metaphysical development.

2.2 Vehicle and Content

A representation has: - **Vehicle**: The physical structure doing the representing (brain state, quantum state, measurement apparatus) - **Content**: What is represented (outcome possibilities, measurement alternatives)

Vehicles must satisfy L_3. Contents need not — we can represent contradictions without instantiating them.

The key insight: The measure over measurement outcomes is part of the *vehicle*, not the content. It characterizes how the physical situation is objectively poised toward outcomes, not how we choose to describe it.

2.3 Readings of Determinate Identity

> **Id-Strong**: To lack determinate identity is to be nothing. There is no ontological category of "indeterminate things."
>
> **Id-Weak**: Any physical reality that admits determinate measurement and record must satisfy Determinate Identity at the level of those measurements and records.

The Born rule derivation requires only Id-Weak. The measure over macroscopic outcomes must be determinate because macroscopic records have determinate identity.

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3. Quantum States and Admissible Structure

3.1 Superposition as Representational

A quantum state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ does not represent a system that is "both 0 and 1." The state describes admissible structure — what outcomes are possible given the preparation. The system has the determinate property of *being in state* $|\psi\rangle$. This is not a violation of Determinate Identity; it is a specific physical configuration.

3.2 Admissible Continuations

When a measurement couples a quantum system to a macroscopic pointer:

1. **Decoherence** selects a stable pointer basis $\{|U\rangle, |D\rangle\}$
2. L_3 requires exactly one pointer reading to be instantiated
3. The mutually exclusive decohered outcomes are called *admissible continuations*

The question: with what weights are the admissible continuations selected?

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4. Vehicle-Weight Invariance and the Born Rule

4.1 The Core Constraint

Given a quantum state $|\psi\rangle$ and admissible outcomes $\{|i\rangle\}$, we seek a measure $\mu(i|\psi)$ weighting the admissible continuations.

The identity stability requirement: For a physical system to satisfy Determinate Identity, its total measure cannot depend on how it is decomposed into components. If the total varied with basis choice, the system would not be determinately what it is.

This forces: 1. **Additivity**: Measure over mutually exclusive outcomes sums to fixed total 2. **Non-contextuality**: Weight assigned to an outcome cannot depend on which alternatives it is grouped with

4.2 Vehicle-Weight Objectivity

A skeptic might claim: "The system is determinate, but its probabilistic description need not be."

This escape is blocked by the vehicle/content distinction. The measure is not content (what the state represents). It characterizes the physical situation's disposition toward outcomes — part of the vehicle, not the content.

If the measure were merely representational, it could vary with decomposition. But then the physical situation would be "70% poised toward outcome A" in one decomposition and "50% poised toward outcome A" in another. The system would fail to be determinately poised toward any distribution.

This violates Determinate Identity at the vehicle level.

4.3 Record-Robustness Argument

Lemma (Dispositional continuity): If macroscopic records are objective and counterfactually stable, then pre-record physical states must have objective modal structure toward record-alternatives.

Proof sketch: Records satisfy Id (they are determinately what they are). The transition from pre-record state S to record R either: - Introduces determinacy ex nihilo (violates grounding requirements) - Preserves determinacy that was already in S

The latter requires S to have objective dispositions toward outcomes. Vehicle-weight objectivity follows.

4.4 The Derivation

Theorem (Vehicle-invariance → Born rule): Any finitely additive measure μ on projections in $\mathcal{H}$ ($\dim \geq 3$) satisfying vehicle-weight invariance is of the form $\mu(P) = \mathrm{Tr}(\rho P)$. For pure states, this yields $|\langle\phi|\psi\rangle|^2$.

Proof: See Appendix A. The chain: 1. Vehicle-weight invariance → total weight constant across all decompositions 2. → Unitary invariance (bases related by unitaries) 3. → Trace form (representation theory) 4. → Born rule for pure states

4.5 No Circularity

The argument does not assume basis-independence to derive basis-independence:

- **Which outcomes are admissible?** Fixed by decoherence (physics, not representation)
- **How must the measure behave over those outcomes?** Constrained by Determinate Identity

These operate at different levels. Decoherence fixes outcomes; Id constrains the measure.

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5. Objections and Replies

5.1 "You assumed the measure exists"

Objection: The argument assumes there is a well-defined measure over outcomes. Isn't that already assuming something like the Born rule?

Reply: We assume only that macroscopic records have determinate relative frequencies in long runs. This is an empirical fact, not a theoretical postulate. The question is what measure generates these frequencies. Id constrains it to be the Born rule.

5.2 "What about dimension 2?"

Objection: Gleason's theorem fails in dimension 2. Does your argument?

Reply: Yes, but this is physically irrelevant. Any system capable of measurement and stable records must have dimension ≥ 3 (see Hilbert Space Paper, §4.4). Dimension-2 systems cannot support the decoherence required for pointer stability.

5.3 "This is just Gleason's theorem"

Objection: You've just motivated Gleason's premises. The real work is done by Gleason.

Reply: Correct — and this is the point. Gleason's theorem is mathematical. The question was: *why* should physical probabilities satisfy Gleason's premises? We answer: because Determinate Identity requires vehicle-weight invariance, which forces additivity and non-contextuality. The "why" is metaphysical; the derivation is mathematical.

5.4 "QBism avoids this"

Objection: QBism treats probabilities as agent credences, not physical features. No vehicle-weight objectivity needed.

Reply: QBism abandons the project of giving an ontological account of quantum probabilities. For those who seek such an account, the measure must be objective. Additionally, QBism cannot explain intersubjective convergence: why do independent agents with different priors converge on the same frequencies?

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6. Conclusion

The Born rule is not a brute postulate. It is the unique measure over measurement outcomes compatible with Determinate Identity.

The derivation: 1. Macroscopic records have determinate identity (Id-Weak) 2. Pre-record states must have objective dispositions toward outcomes (record-robustness) 3. These dispositions (vehicle-weights) cannot vary with decomposition (Id at vehicle level) 4. Vehicle-weight invariance $\rightarrow$ additivity + non-contextuality 5. Gleason's theorem $\rightarrow p_i = |\langle\phi_i|\psi\rangle|^2$

Combined with the Hilbert Space Paper (deriving complex Hilbert space from Id), this completes the derivation of quantum probability structure from L_3 constraints.

What remains empirical: Which systems exist, their Hamiltonians, initial conditions. The *form* of the probability rule is derived; its *application* to specific systems requires physics.

Appendix A: Formal Proof

A.1 Setup

Let $\mathcal{H}$ be a complex separable Hilbert space with $\dim(\mathcal{H}) \geq 3$. Let $\mathcal{P}(\mathcal{H})$ be the lattice of orthogonal projections. Let $\mu : \mathcal{P}(\mathcal{H}) \to [0,1]$ be a finitely additive probability measure on projections.

A.2 Vehicle-Weight Invariance (Formal)

Definition A.1: A measure μ satisfies *vehicle-weight invariance* iff for every maximal orthonormal resolution $\{P_i\}$ of the identity ($\sum P_i = I$, orthogonal, rank-1):

$$\sum_i \mu(P_i) = 1$$

(constant across all decompositions).

A.3 Key Lemmas

Lemma A.1 (Transitivity): Any two maximal orthonormal decompositions are related by a unitary U.

Proof: The unitary group acts transitively on ordered orthonormal bases (Stiefel manifold). □

Lemma A.2 (Unitary invariance): If μ satisfies Definition A.1, then $\mu(UPU^\dagger) = \mu(P)$ for every unitary U.

Proof: Fix rank-1 projection P. Extend to maximal decomposition $D_1 = \{P, R_2, ...\}$. For any unitary U, let $Q = UPU^\dagger$. By transitivity, decompositions containing P and Q are related by a unitary. Invariance of total weight forces $\mu(Q) = \mu(P)$. □

Lemma A.3 (Trace form): Any unitarily invariant finitely additive measure on $\mathcal{P}(\mathcal{H})$ ($\dim \geq 3$) has form $\mu(P) = \mathrm{Tr}(\rho P)$ for unique density operator ρ.

Proof: Follows from irreducibility of adjoint representation of $U(\mathcal{H})$ on trace-class operators (Dixmier 1981). □

A.4 Main Theorem

Theorem A.1: Any finitely additive measure μ on $\mathcal{P}(\mathcal{H})$ ($\dim \geq 3$) satisfying vehicle-weight invariance yields the Born rule for pure states:

$$\mu(|\phi\rangle\langle\phi|) = |\langle\phi|\psi\rangle|^2$$

Proof: 1. Vehicle-weight invariance (Def A.1) 2. $\to$ Unitary invariance (Lemma A.2) 3. $\to$ Trace form $\mu(P) = \mathrm{Tr}(\rho P)$ (Lemma A.3) 4. For pure states $\rho = |\psi\rangle\langle\psi|$, on rank-1 projectors:

$$\mu(|\phi\rangle\langle\phi|) = \mathrm{Tr}(|\psi\rangle\langle\psi| \cdot |\phi\rangle\langle\phi|) = |\langle\phi|\psi\rangle|^2$$

□

A.5 Circularity Audit

- **Input**: Vehicle-weight invariance (from Determinate Identity)
- **Mathematical machinery**: Gleason-Busch theorem, representation theory
- **Output**: Born rule

No premise contains the conclusion. The derivation is sound.

References

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