A Primer on Propositional Modal Logic

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1 The Language of Propositional Logic

1. The primitive vocabulary of propositional logic is:

   (a) Proposition letters: $P, Q, R, P_1, Q_1, R_1, \ldots$

   (b) Logical operators: $\neg, \rightarrow$

   (c) Parenthases: (,)

2. The rules for well-formed formulae ('wffs') are as follows:

   (a) Any proposition letter is a wff.

   (b) If $\ulcorner \phi \urcorner$ is a wff, then $\ulcorner \neg\phi \urcorner$ is a wff.

   (c) If $\ulcorner \phi \urcorner$ and $\ulcorner \psi \urcorner$ are wffs, then $\ulcorner (\phi \rightarrow \psi) \urcorner$ are wffs.

   (d) Nothing else is a wff.

3. The following are introduced as stipulative definitions:

$$(\phi \vee \psi) \stackrel{\text{def}}{=} ((\phi \rightarrow \psi) \rightarrow \psi)$$
$$(\phi \wedge \psi) \stackrel{\text{def}}{=} \neg(\phi \rightarrow \neg\psi)$$
$$(\phi \leftrightarrow \psi) \stackrel{\text{def}}{=} \neg((\phi \rightarrow \psi) \rightarrow \neg(\psi \rightarrow \phi))$$

2 The Axiomatic System PL

4. Here is a simple axiomatization of classical propositional logic:

$$P \rightarrow (Q \rightarrow P) \tag{A1}$$
$$(P \rightarrow (Q \rightarrow R)) \rightarrow ((P \rightarrow Q) \rightarrow (P \rightarrow R)) \tag{A2}$$
$$(\neg Q \rightarrow \neg P) \rightarrow ((\neg Q \rightarrow P) \rightarrow Q) \tag{A3}$$

   To accept these as axioms means that, at any point in an axiomatic proof, you may write down (A1), (A2), or (A3).

   In this axiomatic proof system, in addition to writing down (A1), (A2), or (A3), we are also allowed to write down anything permitted by the following two *rules of inference*:

   Uniform Substitution (*US*): You may uniformly replace any propositional letter, $\ulcorner \alpha \urcorner$, occurring in a theorem of PL with another wff of PL, $\ulcorner \psi \urcorner$.

$$\text{from } \vdash_{PL} \phi[\alpha_1, \alpha_2, \ldots, \alpha_N], \text{ infer } \vdash_{PL} \phi[\psi_1/\alpha_1, \psi_2/\alpha_2, \ldots, \psi_N/\alpha_N]$$

   Modus Ponens (*MP*): From a wff of the form $\ulcorner \phi \rightarrow \psi \urcorner$ and a wff of the form $\ulcorner \phi \urcorner$, you may infer $\ulcorner \psi \urcorner$.

5. If, by writing down the axioms (A1), (A2), and (A3), together with all of the sentences in Γ, and successively applying the rules of inference, we can eventually write down $\ulcorner \phi \urcorner$, then we will write

$$\Gamma \vdash_{PL} \phi$$

and we will say that $\ulcorner\phi\urcorner$ is PL*-provable* from the wffs in Γ. If $\ulcorner\phi\urcorner$ is provable in the system PL from no premises, then we say that $\ulcorner\phi\urcorner$ is a *theorem* of PL, and we write

$$\vdash_{PL} \phi$$

6. Here's a sample axiomatic proof establishing that $\vdash_{PL} P \to P$:

1.	$\vdash_{PL} (P \to (Q \to R)) \to ((P \to Q) \to (P \to R))$	(A2)
2.	$\vdash_{PL} (P \to ((P \to P) \to P)) \to ((P \to (P \to P)) \to (P \to P))$	1 (US)
3.	$\vdash_{PL} P \to (Q \to P)$	(A1)
4.	$\vdash_{PL} P \to ((P \to P) \to P)$	3 (US)
5.	$\vdash_{PL} (P \to (P \to P)) \to (P \to P)$	2, 4 (MP)
6.	$\vdash_{PL} P \to (P \to P)$	3 (US)
7.	$\vdash_{PL} P \to P$	5, 6 (MP)

2.1 Semantics For PL

7. An *interpretation function* $[\![\ ]\!]$ is a function from wffs of PL to $\{0, 1\}$.

   (a) We require that an interpretation satisfy the following conditions:

   ($\neg$) $[\![\neg\phi]\!] = 1$ iff $[\![\phi]\!] = 0$.

   ($\to$) $[\![\phi \to \psi]\!] = 1$ iff $[\![\phi]\!] = 0$ or $[\![\psi]\!] = 1$.

8. We say that $\ulcorner\phi\urcorner$ is a PL-consequence of a set of wffs Γ, or that the argument from Γ to $\ulcorner\phi\urcorner$ is PL-valid, written

$$\Gamma \vDash_{PL} \phi$$

   iff there is no PL-interpretation $[\![\ ]\!]$ such that $[\![\gamma]\!] = 1$ for every $\gamma \in \Gamma$, yet $[\![\phi]\!] = 0$. Or, equivalently, iff, for every interpretation on which all of the premises in Γ are true, $\ulcorner\phi\urcorner$ is true as well.

9. Here's an interesting and unexpected and fantastic fact: $\ulcorner\phi\urcorner$ is PL*-provable* from Γ iff the argument from Γ to $\ulcorner\phi\urcorner$ is PL-valid:

$$\Gamma \vdash_{PL} \phi \qquad \text{if and only if} \qquad \Gamma \vDash_{PL} \phi$$

3 The Language of Propositional Modal Logic

10. The primitive vocabulary of propositional modal logic is exactly the language of propositional logic, plus one additional logical operator, $\Box$.

11. We add the following to our rules for well-formed formulae from propositional logic:

    (a) If $\ulcorner\phi\urcorner$ is a wff, then $\ulcorner\Box\phi\urcorner$ is a wff.

12. And we introduce the additional stipulative definition:

$$\Diamond\phi \stackrel{\text{def}}{=} \neg\Box\neg\phi$$

4 The System K

13. The axiomatic system K is characterized by one additional axiom, K (the distribution axiom), and one additional rule of inference, N (the rule of necessitation).

$$K : \Box(P \to Q) \to (\Box P \to \Box Q)$$
$$N : \text{ from } \vdash_K \phi, \text{ infer } \vdash_K \Box\phi$$

    If we add K and N to the axioms and rules of inference for PL, we get the axiomatic system K. Since we know that every theorem of PL can be proven from the axioms and rules of inference for PL, we can make our axiomatic system K a bit easier to work with by allowing ourselves to write down, as an axiom, *any* theorem of PL, and allowing ourselves to appeal to *any* valid PL rule of inference. Then, the axiomatic system K will have the following axioms:

$$\vdash_K \phi, \text{ for all theorems of } PL, \ulcorner\phi\urcorner \qquad (PLT)$$
$$\vdash_K \Box(P \to Q) \to (\Box P \to \Box Q) \qquad (K)$$

    And the following rules of inference:

all valid PL inferences	(PLR)
from $\vdash_K \phi[\alpha_1, \alpha_2, \ldots, \alpha_N]$, infer $\vdash_K \phi[\psi_1/\alpha_1, \psi_2/\alpha_2, \ldots, \psi_N/\alpha_N]$	(US)
from $\vdash_K \phi$, infer $\vdash_K \Box\phi$	(N)

(PLR) allows you to appeal to *any* PL-valid rule of inference.

14. If $\ulcorner \phi \urcorner$ is provable in the system K from the premises in Γ, then we write

$$\Gamma \vdash_K \phi$$

If $\ulcorner \phi \urcorner$ is provable in the system K from no premises, then we say that $\ulcorner \phi \urcorner$ is a *theorem* of K, and we write

$$\vdash_K \phi$$

15. Here is a proof schema establishing that a wff of the form $\ulcorner \neg\Diamond\phi \urcorner$ is equivalent to a wff of the form $\ulcorner \Box\neg\phi \urcorner$:

1)	$\vdash_K \neg\neg P \leftrightarrow P$	(PL)
2)	$\vdash_K \neg\neg\Box\neg\phi \leftrightarrow \Box\neg\phi$	1 (US)
3)	$\vdash_K \neg\Diamond\phi \leftrightarrow \Box\neg\phi$	2 *def.* '$\Diamond$'

We may similarly establish that a wff of the form $\ulcorner \neg\Box\phi \urcorner$ is equivalent to a wff of the form $\ulcorner \Diamond\neg\phi \urcorner$.[1] Let's introduce these as *derived* rules of inference, which we may call 'MN', for *Modal Negation*:

$$\begin{array}{l} \text{from } \zeta[\neg\Box\phi], \text{ infer } \zeta[\Diamond\neg\phi] \\ \text{from } \zeta[\neg\Diamond\phi], \text{ infer } \zeta[\Box\neg\phi] \\ \text{from } \zeta[\Box\neg\phi], \text{ infer } \zeta[\neg\Diamond\phi] \\ \text{from } \zeta[\Diamond\neg\phi], \text{ infer } \zeta[\neg\Box\phi] \end{array} \qquad (MN)$$

(Here $\ulcorner \zeta[\neg\Box\phi] \urcorner$ is just any wff, $\ulcorner \zeta \urcorner$, which contains the wff $\ulcorner \neg\Box\phi \urcorner$ as a sub-wff, and similarly for $\ulcorner \zeta[\Diamond\neg\phi] \urcorner$, $\ulcorner \zeta[\neg\Diamond\phi] \urcorner$, and $\ulcorner \zeta[\Box\neg\phi] \urcorner$.)

4.1 Semantics for K

16. A *K-frame* is a pair $< \mathcal{W}, R >$ consisting of a set of worlds $\mathcal{W}$, a binary relation $R \subseteq \mathcal{W} \times \mathcal{W}$ (known as the 'accessibility relation'—though 'Rww^*' is colloquially glossed as 'w sees w^*').

[1] We may also prove that, if $\vdash_K \phi \leftrightarrow \psi$, then we may replace $\ulcorner \phi \urcorner$ with $\ulcorner \psi \urcorner$ wherever it appears.

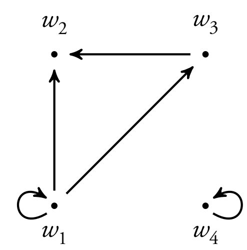

Figure 1: A *K-frame* consisting of the worlds w_1, w_2, w_3, and w_4, and the accessibility relation R such that Rw_1w_1, Rw_1w_2, Rw_1w_3, Rw_3w_2, and Rw_4w_4.

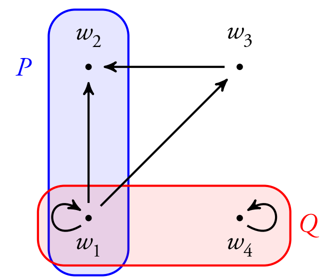

Figure 2: A *K-model* consisting of the frame from figure 1 together with an an interpretation $[\![\ ]\!]$ such that P is true in w_1 and w_2 and Q is true in w_1 and w_4.

17. A *K-model* is a triple $< \mathcal{W}, R, [\![\ ]\!] >$ consisting of a K-frame and an interpretation function $[\![\ ]\!]$, from pairs of wffs and worlds $w \in \mathcal{W}$ to $\{1, 0\}$.

    (a) We require that our interpretation function $[\![\ ]\!]$ satisfy the following conditions:

    ($\neg$) $[\![\neg\phi]\!]^w = 1$ iff $[\![\phi]\!]^w = 0$.
    ($\rightarrow$) $[\![\phi \rightarrow \psi]\!]^w = 1$ iff $[\![\phi]\!]^w = 0$ or $[\![\psi]\!]^w = 1$.
    ($\Box$) $[\![\Box\phi]\!]^w = 1$ iff, for every $w^* \in \mathcal{W}$, if Rww^*, then $[\![\phi]\!]^{w^*} = 1$.

    Given this semantics for '$\neg$' and '$\Box$', it is possible to show that, given our definition of $\ulcorner \Diamond\phi \urcorner$,

    ($\Diamond$) $[\![\Diamond\phi]\!]^w = 1$ iff there is some w^* such that Rww^* and $[\![\phi]\!]^{w^*} = 1$.

18. We will say that $\ulcorner\phi\urcorner$ is a K-consequence of a set of wffs Γ, or that the argument from Γ to $\ulcorner\phi\urcorner$ is K-valid, written

$$\Gamma \models_K \phi$$

iff there is no K-model $<\mathcal{W}, R, [\![\ ]\!]>$, with some $w \in \mathcal{W}$ such that $[\![\gamma]\!]^w = 1$ for every $\gamma \in \Gamma$, yet $[\![\phi]\!]^w = 0$. Or, equivalently: iff for every world in every K-model at which all the premises in Γ are true, $\ulcorner\phi\urcorner$ is true as well.

19. Here's an interesting and unexpected and fantastic fact: $\ulcorner\phi\urcorner$ is K-provable from the premises in Γ iff the argument from Γ to $\ulcorner\phi\urcorner$ is K-valid.

$$\Gamma \vdash_K \phi \quad \text{if and only if} \quad \Gamma \models_K \phi$$

20. Notice that, in the K-model shown in figure 2, the wff '$\Box P \to \Diamond P$' is false at world w_2. (All of the none of the worlds which w_2 sees are P-worlds, so $[\![\Box P]\!]^{w_2} = 1$. Yet there is no world which w_2 sees which is a P-world, so $[\![\Diamond P]\!]^{w_2} = 0$.)

5 The System D

21. Suppose we add to our axiomatic system K the following axiom

$$D : \Box P \to \Diamond P$$

This gives us the system D. The system D will have the following axioms:

$$\vdash_D \phi, \text{ for all theorems of } PL, \ulcorner\phi\urcorner \qquad (PL)$$

$$\vdash_D \Box(P \to Q) \to (\Box P \to \Box Q) \qquad (K)$$

$$\vdash_D \Box P \to \Diamond Q \qquad (D)$$

And the following rules of inference:

all PL valid inferences (PLR)

from $\vdash_D \phi[\alpha_1, \alpha_2, \ldots, \alpha_N]$, infer $\vdash_D \phi[\psi_1/\alpha_1, \psi_2/\alpha_2, \ldots, \psi_N/\alpha_N]$ (US)

from $\vdash_D \phi$, infer $\vdash_D \Box\phi$ (N)

(We will still retain the *derived* rule of inference, MN.)

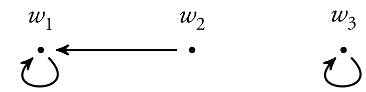

Figure 3: A *D-frame* consisting of the worlds w_1, w_2, and w_3, and an accessibility relation R such that Rw_1w_1, Rw_2w_1, and Rw_3w_3.

22. Here is an axiomatic proof showing that $\vdash_D \neg\Diamond P \to \Diamond\neg P$ (this is not a theorem in K).

1.	$\vdash_D \Box P \to \Diamond P$	(D)
2.	$\vdash_D \Box\neg P \to \Diamond\neg P$	1 (US)
3.	$\vdash_D \neg\Diamond P \to \Diamond\neg P$	2 (MN)

5.1 Semantics For D

Our semantics for K imposed absolutely no constraints on the accessibility relation R. If we require that the accessibility relation R be *serial*—that is, that every world sees at least one (not necessarily distinct) world—then we get a semantics for the system D.

23. A *D-frame* is a pair $<\mathcal{W}, R>$ of a set of worlds $\mathcal{W}$ and a *serial* binary relation $R \subseteq \mathcal{W} \times \mathcal{W}$. A binary relation R over $\mathcal{W}$ is *serial* iff every $w \in \mathcal{W}$ bears R to *something*.

Seriality
A binary relation $R \subseteq A \times A$ is serial iff, for all $a \in A$, there is some $b \in A$ such that Rab.

$$\forall a \exists b \;\; Rab$$

24. A D *model* is a triple $<\mathcal{W}, R, [\![\ ]\!]>$ consisting of a D-frame $<\mathcal{W}, R>$ and an interpretation function $[\![\ ]\!]$, from pairs of wffs and worlds $w \in \mathcal{W}$ to $\{1, 0\}$.

    (a) As with a K-model, we require that the interpretation function $[\![\ ]\!]$ satisfy these constraints:

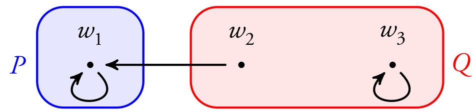

Figure 4: A D-*model* consisting of the frame from figure 3 together with an interpretation $[\![\ ]\!]$ such that P is true in w_1 and Q is true in w_2 and w_3.

$$
\begin{aligned}
&(\neg)\ \ [\![\neg\phi]\!]^w = 1 \text{ iff } [\![\phi]\!]^w = 0 \\
&(\rightarrow)\ \ [\![\phi \rightarrow \psi]\!]^w = 1 \text{ iff } [\![\phi]\!]^w = 0 \text{ or } [\![\psi]\!]^w = 1 \\
&(\Box)\ \ [\![\Box\phi]\!]^w = 1 \text{ iff } [\![\phi]\!]^{w^*} = 1 \text{ for all } w^* \text{ such that } Rww^*.
\end{aligned}
$$

25. We will say that $\ulcorner\phi\urcorner$ is a D-consequence of a set of wffs Γ, or that the argument from Γ to $\ulcorner\phi\urcorner$ is D-valid,

$$\Gamma \vDash_D \phi$$

iff there is no D-model $< \mathcal{W}, R, [\![\ ]\!] >$, with some $w \in \mathcal{W}$, such that $[\![\gamma]\!]^w = 1$ for every $\gamma \in \Gamma$, yet $[\![\phi]\!]^w = 0$. Or, equivalently: iff for every world in every D-model at which all the premises in Γ are true, $\ulcorner\phi\urcorner$ is true as well.

26. Here's an interesting and unexpected and fantastic fact: $\ulcorner\phi\urcorner$ is D-provable from the premises in Γ iff the argument from Γ to $\ulcorner\phi\urcorner$ is D-valid.

$$\Gamma \vdash_D \phi \quad \text{if and only if} \quad \Gamma \vDash_D \phi$$

27. Notice that, in the D-model from figure 4, '$\Box P \rightarrow P$' is false at world w_2. (For, at w_2, '$\Box P$' is true—since 'P' is true at every world that w_2 sees. However, at w_2, 'P' is false. So '$\Box P \rightarrow P$' is false.)

6 The System T

28. Suppose we add to our axiomatic system K the following axiom

$$T : \Box P \rightarrow P$$

or, equivalently,

$$T^* : P \rightarrow \Diamond P$$

29. This gives us the axiomatic system T. T will have the following axioms:

$$
\begin{aligned}
&\vdash_T \phi, \text{ for all theorems of } PL, \ulcorner\phi\urcorner && (PL)\\
&\vdash_T \Box(P \rightarrow Q) \rightarrow (\Box P \rightarrow \Box Q) && (K)\\
&\vdash_T \Box P \rightarrow P && (T)
\end{aligned}
$$

And the following rules of inference:

$$
\begin{aligned}
&\text{all } PL \text{ valid inferences} && (PLR)\\
&\text{from } \vdash_T \phi[\alpha_1, \alpha_2, \ldots, \alpha_N], \text{ infer } \vdash_T \phi[\psi_1/\alpha_1, \psi_2/\alpha_2, \ldots, \psi_N/\alpha_N] && (US)\\
&\text{from } \vdash_T \phi, \text{ infer } \vdash_T \Box\phi && (N)
\end{aligned}
$$

Since all the axioms and rules of inference of K are axioms and rules of inference of T, we retain the derived rule (MN).

30. I say that adding '$P \rightarrow \Diamond P$' as an axiom is equivalent to adding '$\Box P \rightarrow P$' as an axiom. That's because, given this axiomatic framework, we can derive '$\Box P \rightarrow P$' as a theorem if we take '$P \rightarrow \Diamond P$' as an axiom, like so:

1.	$\vdash_{T^*} P \rightarrow \Diamond P$	(T^*)
2.	$\vdash_{T^*} \neg P \rightarrow \Diamond\neg P$	1 (US)
3.	$\vdash_{T^*} \neg P \rightarrow \neg\Box P$	2 (MN)
4.	$\vdash_{T^*} \Box P \rightarrow P$	3 (PLR)

And we can derive '$P \rightarrow \Diamond P$' as a theorem if we take '$\Box P \rightarrow P$' as an axiom, like so:

1.	$\vdash_T \Box P \rightarrow P$	(T)
2.	$\vdash_T \Box\neg P \rightarrow \neg P$	1 (US)
3.	$\vdash_T \neg\Diamond P \rightarrow \neg P$	2 (MN)
4.	$\vdash_T P \rightarrow \Diamond P$	3, (PLR)

31. Notice that I did not carry over the axiom D, '$\Box P \rightarrow \Diamond P$'. That is: ($D$) is not among the axioms for the system T. The reason for this is that, given (T), (D) is redundant. We can derive (D) as a theorem within T. To see this, just extend the second axiomatic proof from 30 above with an application of *hypothetical syllogism*.

5.	$\vdash_T \Box P \rightarrow \Diamond P$	1, 4 (PLR)

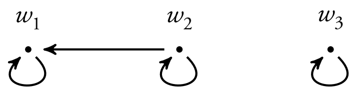

Figure 5: A T-*frame* consisting of the worlds w_1, w_2, and w_3, and an accessibility relation R such that Rw_1w_1, Rw_2w_2, Rw_2w_1, and Rw_3w_3.

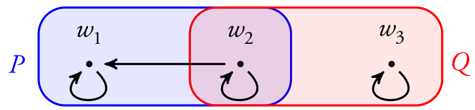

Figure 6: A T-*model* consisting of the T-frame from figure 5, together with an interpretation, $[\![\ ]\!]$, such that P is true in w_1 and w_2, and Q is true in w_2 and w_3.

6.1 Semantics For T

If we require that the accessibility relation R be *reflexive*—that is, that every world sees itself—then we get a semantics for the system T.

32. A T-*frame* is a pair $< \mathcal{W}, R >$ of a set of worlds $\mathcal{W}$ and a *reflexive* binary relation $R \subseteq \mathcal{W} \times \mathcal{W}$. A relation R on $\mathcal{W}$ is *reflexive* iff every $w \in \mathcal{W}$ bears R to itself. More generally,

    REFLEXIVITY
    A binary relation $R \subseteq \mathrm{A} \times \mathrm{A}$ is REFLEXIVE iff, for all $a \in \mathrm{A}$, Raa.

    $$\forall a \ \ Raa$$

33. A T-*model* is a triple $< \mathcal{W}, R, [\![\ ]\!] >$ consisting of a T-frame $< \mathcal{W}, R >$ and an interpretation function $[\![\ ]\!]$, from pairs of wffs and worlds $w \in \mathcal{W}$ to $\{1, 0\}$.

    (a) As usual, we require that $[\![\ ]\!]$ satisfy the following constraints.

    ($\neg$) $[\![\neg\phi]\!]^w = 1$ iff $[\![\phi]\!]^w = 0$.
    ($\rightarrow$) $[\![\phi \rightarrow \psi]\!]^w = 1$ iff $[\![\phi]\!]^w = 0$ or $[\![\psi]\!]^w = 1$.
    ($\Box$) $[\![\Box\phi]\!]^w = 1$ iff, for every w^* such that Rww^*, $[\![\phi]\!]^{w^*} = 1$.

34. We will say that $\ulcorner\phi\urcorner$ is a T-consequence of a set of wffs Γ, or that the argument from Γ to $\ulcorner\phi\urcorner$ is T-valid,

    $$\Gamma \models_T \phi$$

    iff there is no T-model $< \mathcal{W}, R, [\![\ ]\!] >$, with some $w \in \mathcal{W}$, such that $[\![\gamma]\!]^w = 1$ for every $\gamma \in \Gamma$, yet $[\![\phi]\!]^w = 0$. Or, equivalently: iff for every world in every T-model at which all the premises in Γ are true, $\ulcorner\phi\urcorner$ is true as well.

35. Here's an interesting and unexpected and fantastic fact: $\ulcorner\phi\urcorner$ is T-provable from the premises in Γ iff the argument from Γ to $\ulcorner\phi\urcorner$ is T-valid.

    $$\Gamma \vdash_T \phi \quad \text{if and only if} \quad \Gamma \models_T \phi$$

7 The System B

36. Suppose we add to our axiomatic system T the following axiom

    $$B \ : \ P \rightarrow \Box\Diamond P$$

    or, equivalently,

    $$B^* \ : \ \Diamond\Box P \rightarrow P$$

    This gives us the axiomatic system B. B will have the following axioms:

    $$\vdash_B \phi, \text{ for all theorems of } PL, \ulcorner\phi\urcorner \qquad (PLT)$$
    $$\vdash_B \Box(P \rightarrow Q) \rightarrow (\Box P \rightarrow \Box Q) \qquad (K)$$
    $$\vdash_B \Box P \rightarrow P \qquad (T)$$
    $$\vdash_B P \rightarrow \Box\Diamond P \qquad (B)$$

    And the following rules of inference:

    all valid PL inferences (PLR)
    from $\vdash_B \phi[\alpha_1, \alpha_2, \ldots, \alpha_N]$, infer $\vdash_B \phi[\psi_1/\alpha_1, \psi_2/\alpha_2, \ldots, \psi_N/\alpha_N]$ (US)
    from $\vdash_B \phi$, infer $\vdash_B \Box\phi$ (N)

    Since all the axioms and rules of inference of K are axioms and rules of inference of B, we retain the derived rule (MN).

37. I say that adding '$\Diamond\Box P \rightarrow P$' as an axiom is equivalent to adding '$P \rightarrow \Box\Diamond P$' as an axiom. That's because, taking '$P \rightarrow \Box\Diamond P$' as an axiom, we can derive '$\Diamond\Box P \rightarrow P$', like so:

1.	$\vdash_B P \rightarrow \Box\Diamond P$	(B)
2.	$\vdash_B \neg P \rightarrow \Box\Diamond\neg P$	1 (US)
3.	$\vdash_B \neg P \rightarrow \Box\neg\Box P$	2 (MN)
4.	$\vdash_B \neg P \rightarrow \neg\Diamond\Box P$	3 (MN)
5.	$\vdash_B \Diamond\Box P \rightarrow P$	4 (PLR)

And we can derive '$P \rightarrow \Box\Diamond P$' as a theorem if we take '$\Diamond\Box P \rightarrow P$' as an axiom, like so:

1.	$\vdash_{B^*} \Diamond\Box P \rightarrow P$	(B')
2.	$\vdash_{B^*} \Diamond\Box\neg P \rightarrow \neg P$	1 (US)
3.	$\vdash_{B^*} \Diamond\neg\Diamond P \rightarrow \neg P$	2 (MN)
4.	$\vdash_{B^*} \neg\Box\Diamond P \rightarrow \neg P$	3 (MN)
5.	$\vdash_{B^*} P \rightarrow \Box\Diamond P$	4 (PLR)

7.1 Semantics For B

If we require that the accessibility relation R be *reflexive* and *symmetric*—that is, that every world sees itself and every world sees all the worlds that see it—then we get a semantics for the system B.

38. A *B-frame* is a pair $< \mathcal{W}, R >$ of a set of worlds $\mathcal{W}$ and a *reflexive* and *symmetric* binary relation $R \subseteq \mathcal{W} \times \mathcal{W}$. Requiring that R is reflexive and symmetric means requiring that every world see itself and that, if w sees w^*, then w^* must also see w. More generally,

REFLEXIVITY
A binary relation $R \subseteq \mathrm{A} \times \mathrm{A}$ is REFLEXIVE iff, for all $a \in \mathrm{A}$, Raa.

$$\forall a \;\; Raa$$

SYMMETRY
A binary relation $R \subseteq \mathrm{A} \times \mathrm{A}$ is SYMMETRIC iff, for all $a, b \in \mathrm{A}$, if Rab, then Rba.

$$\forall a \, \forall b \;\; (Rab \rightarrow Rba)$$

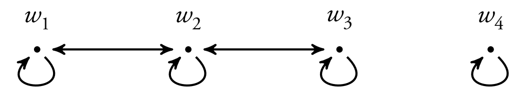

Figure 7: A *B-frame* consisting of the worlds w_1, w_2,, w_3, and w_4, and an accessibility relation R such that Rw_1w_1, Rw_1w_2, Rw_2w_2, Rw_2w_1, Rw_2w_3, Rw_3w_3, Rw_3w_2, and Rw_4w_4.

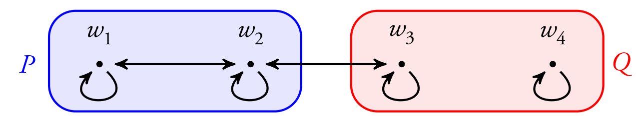

Figure 8: A *B-model* consisting of the frame from figure 7, together with an interpretation, $[\![\;]\!]$, according to which P is true at w_1 and w_2, and Q is true at w_3 and w_4.

39. A *B-model* is a triple $< \mathcal{W}, R, [\![\;]\!] >$ consisting of a B-frame $< \mathcal{W}, R >$ and an interpretation function, $[\![\;]\!]$, from pairs of wffs and worlds $w \in \mathcal{W}$ to $\{1, 0\}$.

    (a) As usual, we require that $[\![\;]\!]$ satisfy the following constraints.

    ($\neg$) $[\![\neg\phi]\!]^w = 1$ iff $[\![\phi]\!]^w = 0$.
    ($\rightarrow$) $[\![\phi \rightarrow \psi]\!]^w = 1$ iff $[\![\phi]\!]^w = 0$ or $[\![\psi]\!]^w = 1$.
    ($\Box$) $[\![\Box\phi]\!]^w = 1$ iff, for every w^* such that Rww^*, $[\![\phi]\!]^{w^*} = 1$.

40. We will say that $\ulcorner\phi\urcorner$ is a B-consequence of a set of wffs Γ, or that the argument from Γ to $\ulcorner\phi\urcorner$ is B-valid,

$$\Gamma \models_B \phi$$

iff there is no B-model $< \mathcal{W}, R, [\![\;]\!] >$, with some $w \in \mathcal{W}$, such that $[\![\gamma]\!]^w = 1$ for every $\gamma \in \Gamma$, yet $[\![\phi]\!]^w = 0$. Or, equivalently: iff for every world in every B-model at which all the premises in Γ are true, $\ulcorner\phi\urcorner$ is true as well.

41. Here's an interesting and unexpected and fantastic fact: $\ulcorner\phi\urcorner$ is B-provable from the premises in Γ iff the argument from Γ to $\ulcorner\phi\urcorner$ is B-valid.

$$\Gamma \vdash_B \phi \quad \text{if and only if} \quad \Gamma \models_B \phi$$

42. Notice that, in the B-model from figure 8, '$\Box P \rightarrow \Box\Box P$' is false at w_1. (Since 'P' is true at ever world that w_1 sees, '$\Box P$' is true there. However, '$\Box P$' is false

at w_2, since 'P' is false at w_3, and w_2 sees w_3. So '$\Box P$' is not true at w_2, and is therefore not true at every world that w_1 sees. So '$\Box\Box P$' is false at w_1. So, at w_1, '$\Box P$' is true while '$\Box\Box P$' is false, so '$\Box P \to \Box\Box P$' is false at w_1.)

(a) Also note that, in the same B-model, '$\Diamond\Diamond Q \to \Diamond Q$' is false at w_1. (Since w_2 sees a Q-world, '$\Diamond Q$' is true at w_2. And since w_1 sees w_2, '$\Diamond\Diamond Q$' is true at w_1. However, w_1 does not see any Q-worlds, so '$\Diamond Q$' is false at w_1. So '$\Diamond\Diamond Q \to \Diamond Q$' is false at w_1.)

8 The System S4

43. Suppose we add to our axiomatic system T the following axiom

$$\text{S4} : \Box P \to \Box\Box P$$

or, equivalently,

$$\text{S4}^* : \Diamond\Diamond P \to \Diamond P$$

This gives us the axiomatic system S4. S4 will have the following axioms:

$$\vdash_{S4} \phi, \text{ for all theorems of } PL, \ulcorner\phi\urcorner \qquad (PL)$$

$$\vdash_{S4} \Box(P \to Q) \to (\Box P \to \Box Q) \qquad (K)$$

$$\vdash_{S4} \Box P \to P \qquad (T)$$

$$\vdash_{S4} \Box P \to \Box\Box P \qquad (\text{S4})$$

And the following rules of inference:

all PL-valid inferences (PLR)

from $\vdash_{S4} \phi[\alpha_1, \alpha_2, \ldots, \alpha_N]$, infer $\vdash_{S4} \phi[\psi_1/\alpha_1, \psi_2/\alpha_2, \ldots, \psi_N/\alpha_N]$ (US)

from $\vdash_{S4} \phi$, infer $\vdash_{S4} \Box\phi$ (N)

Since all the axioms and rules of inference of K are axioms and rules of inference of S4, we retain the derived rule (MN).

(a) Notice that we *did not* include the axiom (B). Nor is (B) a theorem of this system. For each of the previous axiomatic systems, we have been *enlarging* the number of theorems. That is, for any $\ulcorner\phi\urcorner$ of PML, if $\ulcorner\phi\urcorner$ is a theorem of K, then it is a theorem of D; if $\ulcorner\phi\urcorner$ is a theorem of D, then it is a theorem of T; and if $\ulcorner\phi\urcorner$ is a theorem of T, then it is a theorem of B.

$$\vdash_K \phi \Rightarrow \vdash_D \phi \Rightarrow \vdash_T \phi \Rightarrow \vdash_B \phi$$

This is not true of B and S4. Not every theorem of B is a theorem of S4,

$$\vdash_B \phi \not\Rightarrow \vdash_{S4} \phi$$

and not every theorem of S4 is a theorem of B

$$\vdash_{S4} \phi \not\Rightarrow \vdash_B \phi$$

Nevertheless, if $\ulcorner\phi\urcorner$ is a theorem of T, then it is a theorem of S4:

$$\vdash_K \phi \Rightarrow \vdash_D \phi \Rightarrow \vdash_T \phi \Rightarrow \vdash_{S4} \phi$$

44. I say that adding '$\Diamond\Diamond P \to \Diamond P$' as an axiom is equivalent to adding '$\Box P \to \Box\Box P$' as an axiom. That's because, given this axiomatic framework, we can derive '$\Diamond\Diamond P \to \Diamond P$' as a theorem if we take '$\Box P \to \Box\Box P$' as an axiom, as follows:

1.	$\vdash_{S4} \Box P \to \Box\Box P$	(S4)
2.	$\vdash_{S4} \Box\neg P \to \Box\Box\neg P$	1 (US)
3.	$\vdash_{S4} \neg\Diamond P \to \Box\neg\Diamond P$	2 (MN)
4.	$\vdash_{S4} \neg\Diamond P \to \neg\Diamond\Diamond P$	3 (MN)
5.	$\vdash_{S4} \Diamond\Diamond P \to \Diamond P$	4 (PLR)

And we can derive '$\Box P \to \Box\Box P$' as a theorem if we take '$\Diamond\Diamond P \to \Diamond P$' as an axiom:

1.	$\vdash_{S4^*} \Diamond\Diamond P \to \Diamond P$	(S4*)
2.	$\vdash_{S4^*} \Diamond\Diamond\neg P \to \Diamond\neg P$	1 (US)
3.	$\vdash_{S4^*} \Diamond\neg\Box P \to \neg\Box P$	2 (MN)
4.	$\vdash_{S4^*} \neg\Box\Box P \to \neg\Box P$	3 (MN)
5.	$\vdash_{S4^*} \Box P \to \Box\Box P$	4 (PLR)

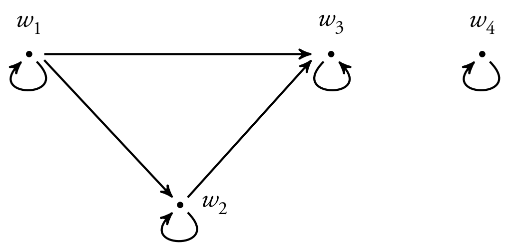

Figure 9: An *S4-frame* consisting of the worlds $w_1, w_2,$, w_3, and w_4, and an accessibility relation R such that Rw_1w_1, Rw_1w_2, Rw_2w_2, Rw_2w_3, Rw_1w_3, Rw_3w_3, and Rw_4w_4.

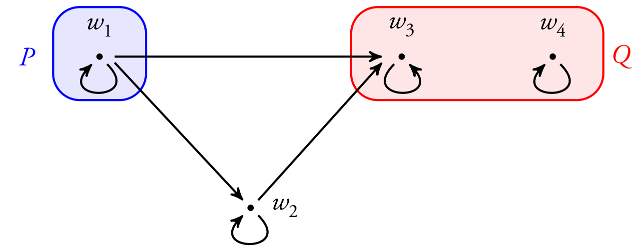

Figure 10: An *S4-model* consisting of the frame from figure 9, together with an interpretation on which P is true at w_1 and Q is true at w_3 and w_4.

8.1 Semantics For S4

If we require that the accessibility relation R be *reflexive* and *transitive*—that is, that every world sees itself and every world sees the worlds seen by the worlds it sees—then we get a semantics for the system S4.

45. An *S4-frame* is a pair $< \mathcal{W}, R >$ of a set of worlds $\mathcal{W}$ and a *reflexive* and *transitive* binary relation $R \subseteq \mathcal{W} \times \mathcal{W}$. Requiring that R is reflexive and transitive means requiring that every world see itself and that, if w sees w^*, and w^* sees w^{**}, then w must also see w^{**}. More generally,

    REFLEXIVITY
    A binary relation $R \subseteq \mathrm{A} \times \mathrm{A}$ is REFLEXIVE iff, for all $a \in \mathrm{A}$, Raa.

    $$\forall a \ \ Raa$$

    TRANSITIVITY
    A binary relation $R \subseteq \mathrm{A} \times \mathrm{A}$ is TRANSITIVE iff, for all $a, b, c \in \mathrm{A}$, if Rab and Rbc, then Rac.

    $$\forall a \, \forall b \, \forall c \ \ ((Rab \wedge Rbc) \rightarrow Rac)$$

46. An *S4-model* is a triple $< \mathcal{W}, R, [\![\ ]\!] >$ consisting of an S4-frame $< \mathcal{W}, R >$ and an interpretation function $[\![\ ]\!]$, from pairs of wffs and worlds $w \in \mathcal{W}$ to $\{1, 0\}$.

    (a) As usual, we require that $[\![\ ]\!]$ satisfy the following constraints.

    ($\neg$) $[\![\neg\phi]\!]^w = 1$ iff $[\![\phi]\!]^w = 0$.

    ($\rightarrow$) $[\![\phi \rightarrow \psi]\!]^w = 1$ iff $[\![\phi]\!]^w = 0$ or $[\![\psi]\!]^w = 1$.

    ($\Box$) $[\![\Box\phi]\!]^w = 1$ iff, for every w^* such that Rww^*, $[\![\phi]\!]^{w^*} = 1$.

47. We will say that $\ulcorner \phi \urcorner$ is an S4-consequence of a set of wffs Γ, or that the argument from Γ to $\ulcorner \phi \urcorner$ is S4-valid,

    $$\Gamma \models_{\text{S4}} \phi$$

    iff there is no S4-model $< \mathcal{W}, R, [\![\ ]\!] >$, with some $w \in \mathcal{W}$, such that $[\![\gamma]\!]^w = 1$ for every $\gamma \in \Gamma$, yet $[\![\phi]\!]^w = 0$. Or, equivalently: iff for every world in every S4-model at which all the premises in Γ are true, $\ulcorner \phi \urcorner$ is true as well.

48. Here's an interesting and unexpected and fantastic fact: $\ulcorner \phi \urcorner$ is S4-provable from the premises in Γ iff the argument from Γ to $\ulcorner \phi \urcorner$ is S4-valid.

    $$\Gamma \vdash_{\text{S4}} \phi \quad \text{if and only if} \quad \Gamma \models_{\text{S4}} \phi$$

49. (a) Note that the B-axiom, '$P \rightarrow \Box\Diamond P$', is false at world w_1 of the S4-model shown in figure 10. ('P' is true at w_1. At w_3, '$\Diamond P$' is false, since w_3 does not see any P-worlds. Since w_1 sees w_3, this means that '$\Box\Diamond P$' is false at w_1. So '$P \rightarrow \Box\Diamond P$' is false at w_1.)

    (b) Note also that, for similar reasons, '$\Diamond P \rightarrow \Box\Diamond P$' is false at w_1.

9 The System S5

50. Suppose we add to our axiomatic system T the following axiom

$$\text{S5} : \Diamond P \rightarrow \Box\Diamond P$$

or, equivalently,

$$\text{S5}^* : \Diamond\Box P \rightarrow \Box P$$

This gives us the axiomatic system S5. S5 will have the following axioms:

$$\vdash_{S5} \phi, \text{ for all theorems of } PL, \ulcorner\phi\urcorner \quad (PL)$$

$$\vdash_{S5} \Box(P \rightarrow Q) \rightarrow (\Box P \rightarrow \Box Q) \quad (K)$$

$$\vdash_{S5} \Box P \rightarrow P \quad (T)$$

$$\vdash_{S5} \Diamond P \rightarrow \Box\Diamond P \quad (\text{S5})$$

And the following rules of inference:

all PL-valid inferences (PLR)

from $\vdash_{S5} \phi[\alpha_1, \alpha_2, \ldots, \alpha_N]$, infer $\vdash_{S5} \phi[\psi_1/\alpha_1, \psi_2/\alpha_2, \ldots, \psi_N/\alpha_N]$ (US)

from $\vdash_{S5} \phi$, infer $\vdash_{S5} \Box\phi$ (N)

Since all the axioms and rules of inference of K are axioms and rules of inference of S5, we retain all the derived rule (MN).

51. I say that adding the axiom '$\Diamond\Box P \rightarrow \Box P$' to T is equivalent to adding '$\Diamond P \rightarrow \Box\Diamond P$' to T. That's because we can derive '$\Diamond\Box P \rightarrow \Box P$' as a theorem in S5:

1.	$\vdash_{S5} \Diamond P \rightarrow \Box\Diamond P$	(S5)
2.	$\vdash_{S5} \Diamond\neg P \rightarrow \Box\Diamond\neg P$	1 (US)
3.	$\vdash_{S5} \neg\Box P \rightarrow \Box\neg\Box P$	2 (MN)
4.	$\vdash_{S5} \neg\Box P \rightarrow \neg\Diamond\Box P$	3 (MN)
5.	$\vdash_{S5} \Diamond\Box P \rightarrow \Box P$	4 (PLR)

And, if we take '$\Diamond\Box P \rightarrow \Box P$' as an axiom, then we may derive '$\Diamond P \rightarrow \Box\Diamond P$' as a theorem:

1.	$\vdash_{S5^*} \Diamond\Box P \rightarrow \Box P$	(S5*)
2.	$\vdash_{S5^*} \Diamond\Box\neg P \rightarrow \Box\neg P$	1 (US)
3.	$\vdash_{S5^*} \Diamond\neg\Diamond P \rightarrow \neg\Diamond P$	2 (MN)
4.	$\vdash_{S5^*} \neg\Box\Diamond P \rightarrow \neg\Diamond P$	3 (MN)
5.	$\vdash_{S5^*} \Diamond P \rightarrow \Box\Diamond P$	4 (PLR)

52. Note that we didn't include either the B axiom or the S4 axiom in the system S5. However, both of these axioms are theorems of S5. Here is a proof showing that the B axiom is a theorem in S5:

1.	$\vdash_{S5} P \rightarrow \Diamond P$	(T^*)
2.	$\vdash_{S5} \Diamond P \rightarrow \Box\Diamond P$	(S5)
3.	$\vdash_{S5} P \rightarrow \Box\Diamond P$	1, 2 (PLR)

If we extend the above proof, we may derive the S4 axiom in S5:

4.	$\vdash_{S5} \Box P \rightarrow \Diamond\Box P$	1 (US)
5.	$\vdash_{S5} \Diamond\Box P \rightarrow \Box P$	(S5*)
6.	$\vdash_{S5} \Box P \leftrightarrow \Diamond\Box P$	4, 5 (PLR)
7.	$\vdash_{S5} \Box P \rightarrow \Box\Diamond\Box P$	3 (US)
8.	$\vdash_{S5} \Box P \rightarrow \Box\Box P$	6, 7 (SE)

(On line 8, I used a rule 'SE', for *substitution of equivalents*. It says that, if $\ulcorner\phi \leftrightarrow \psi\urcorner$ is a theorem, then $\ulcorner\psi\urcorner$ may be replaced for $\ulcorner\phi\urcorner$ wherever $\ulcorner\phi\urcorner$ appears as a subwff, and *vice versa*. This rule is derivable in K.)

(a) So both S4 and B are redundant axioms for the system S5. In fact, adding the S5 axiom to the system T is *equivalent* to adding *both* the S4 axiom *and* the B axiom to T. We have already shown that adding S5 to T brings along both S4 and B as theorems. We will now show that adding to T both the axiom S4 and the axiom B brings along S5 as a theorem:

1. $\vdash_{TBS4} \Diamond\Diamond P \to \Diamond P$ (S4)
2. $\vdash_{TBS4} P \to \Diamond P$ (T^*)
3. $\vdash_{TBS4} \Diamond P \to \Diamond\Diamond P$ 2 (US)
4. $\vdash_{TBS4} \Diamond P \leftrightarrow \Diamond\Diamond P$ 1, 3 (PLR)
5. $\vdash_{TBS4} P \to \Box\Diamond P$ (B)
6. $\vdash_{TBS4} \Diamond P \to \Box\Diamond\Diamond P$ 5 (US)
7. $\vdash_{TBS4} \Diamond P \to \Box\Diamond P$ 4, 6 (SE)

9.1 Semantics For S5

If we require that the accessibility relation R be *reflexive* and *euclidean*—that is, that every world sees itself and all the worlds that it sees see each other—then we get a semantics for the system S5.

53. An *S5-frame* is a pair $< \mathcal{W}, R >$ of a set of worlds $\mathcal{W}$ and a *reflexive* and *euclidean* binary relation $R \subseteq \mathcal{W} \times \mathcal{W}$. Requiring that R is reflexive and Euclidean means requiring that every world see itself and that, if w sees both w^* and w^{**}, then w^* sees w^{**}. More generally,

    REFLEXIVITY
    A binary relation $R \subseteq A \times A$ is REFLEXIVE iff, for all $a \in A$, Raa.

    $$\forall a \;\; Raa$$

    EUCLIDEANESS
    A binary relation $R \subseteq A \times A$ is EUCLIDEAN iff, for all $a, b, c \in A$, if Rab and Rac, then Rbc.

    $$\forall a \, \forall b \, \forall c \;\; ((Rab \wedge Rac) \to Rbc)$$

54. A binary relation is reflexive and Euclidean if and only if it is reflexive, symmetric, and transitive. So another, equivalent, definition of an S5-frame is this: An *S5-frame* is a pair $< \mathcal{W}, R >$ of a set of worlds $\mathcal{W}$ and a *reflexive*, *symmetric*, and *transitive* binary relation $R \subseteq \mathcal{W} \times \mathcal{W}$.

    REFLEXIVITY
    A binary relation $R \subseteq A \times A$ is REFLEXIVE iff, for all $a \in A$, Raa.

    $$\forall a \;\; Raa$$

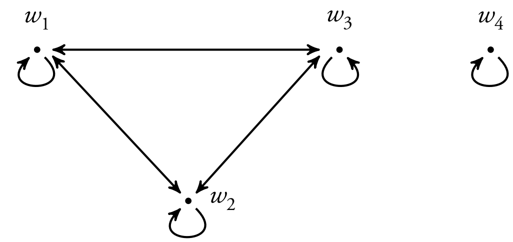

Figure 11: An *S5-frame* consisting of the worlds $w_1, w_2,$, w_3, and w_4, and an accessibility relation R such that Rw_1w_1, Rw_1w_2, Rw_1w_3, Rw_2w_1, Rw_2w_2, Rw_2w_3, Rw_3w_1, Rw_3w_2, Rw_3w_3, and Rw_4w_4.

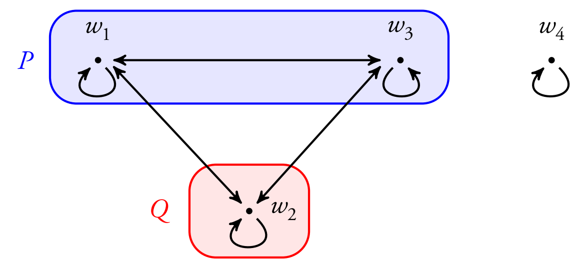

Figure 12: An *S5-model* consisting of the frame from figure 11, together with an interpretation, $[\![\;]\!]$, according to which P is true at w_1 and w_3, and Q is true at w_2.

SYMMETRY
A binary relation $R \subseteq A \times A$ is SYMMETRIC iff, for all $a, b \in A$, if Rab, then Rba.

$$\forall a \, \forall b \;\; (Rab \to Rba)$$

TRANSITIVITY
A binary relation $R \subseteq A \times A$ is TRANSITIVE iff, for all $a, b, c \in A$, if Rab and Rbc, then Rac.

$$\forall a \, \forall b \, \forall c \;\; ((Rab \wedge Rbc) \to Rac)$$

55. An *S5-model* is a triple $< \mathcal{W}, R, [\![\;]\!] >$ consisting of an S5-frame $< \mathcal{W}, R >$ and an interpretation function $[\![\;]\!]$, from pairs of wffs and worlds $w \in \mathcal{W}$ to $\{1, 0\}$.

(a) As usual, we require that $[\![\ ]\!]$ satisfy the following constraints.

($\neg$) $[\![\neg\phi]\!]^w = 1$ iff $[\![\phi]\!]^w = 0$.

($\to$) $[\![\phi \to \psi]\!]^w = 1$ iff $[\![\phi]\!]^w = 0$ or $[\![\psi]\!]^w = 1$.

($\Box$) $[\![\Box\phi]\!]^w = 1$ iff, for every w^* such that Rww^*, $[\![\phi]\!]^{w^*} = 1$.

56. We will say that $\ulcorner\phi\urcorner$ is an S5-consequence of a set of wffs Γ, or that the argument from Γ to $\ulcorner\phi\urcorner$ is S5-valid,

$$\Gamma \models_{S5} \phi$$

iff there is no S4-model $< \mathcal{W}, R, [\![\ ]\!] >$, with some $w \in \mathcal{W}$, such that $[\![\gamma]\!]^w = 1$ for every $\gamma \in \Gamma$, yet $[\![\phi]\!]^w = 0$. Or, equivalently: iff for every world in every S5-model at which all the premises in Γ are true, $\ulcorner\phi\urcorner$ is true as well.

57. Here's an interesting and unexpected and fantastic fact: $\ulcorner\phi\urcorner$ is S5-provable from the premises in Γ iff the argument from Γ to $\ulcorner\phi\urcorner$ is S5-valid.

$$\Gamma \vdash_{S5} \phi \quad \text{if and only if} \quad \Gamma \models_{S5} \phi$$

10 Summary

58. In sum, here are the systems we've discussed characterized by the axioms they add to the system K, along with the constraints on the accessibility relation R those axioms imply.

System	**Additional Axioms**	**Constraints on R**
D	$\Box P \to \Diamond P$	R is serial
T	$\Box P \to P$	R is reflexive
B	$\Box P \to P$ $P \to \Box\Diamond P$ (or: $\Diamond\Box P \to P$)	R is reflexive & symmetric
S4	$\Box P \to P$ $\Box P \to \Box\Box P$ (or: $\Diamond\Diamond P \to \Diamond P$)	R is reflexive & transitive
S5	$\Box P \to P$ $\Diamond P \to \Box\Diamond P$ (or: $\Diamond\Box P \to \Box P$)	R is reflexive & Euclidean

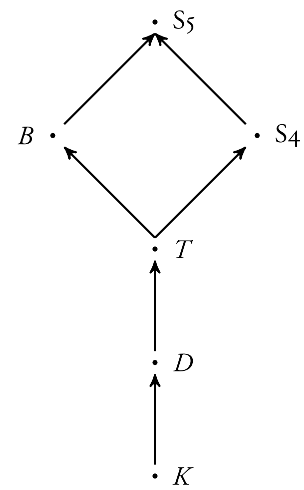

Figure 13: Relationship between the systems of propositional modal logic.

11 Relationships Between The Systems

59. We may visualize the relationship between the logical systems we have learned with the graph from figure 13. There, the arrows correspond to relations of *validity preservation*. The graph tells us that, if an argument or a wff of PML is valid in K, then it will be valid in D. If it is valid in D, then it will be valid in T. If it is valid in T, then it will be valid in B, and it will be valid in S4. And, if an argument or a wff of PML is valid in either B or S4, then it will be valid in S5.

Validity-preservation is transitive, so the graph also tells us, for instance, that if an argument or wff of PML is valid in K, then it is valid in B; and that, if it is valid in D, then it is valid in S5.

60. There are other modal systems out there. Some of these are *weaker* than the system K. That is, there are arguments or wffs that are valid in K which are *not* valid in these modal logics. Such modal logics are known as *non-normal* modal logics. Any modal logic which is at least as strong as K is known as

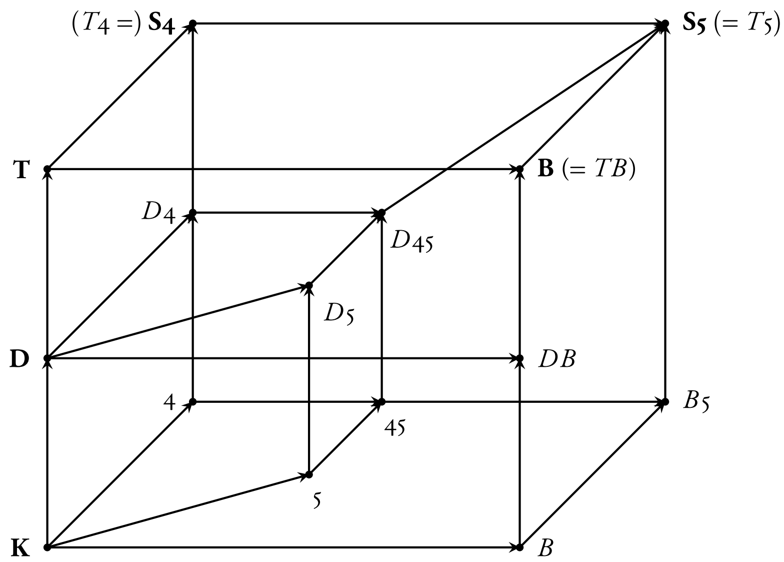

Figure 14: Some other normal modal logics.

a *normal* modal logic. That is: any modal logic L which is such that all the arguments or wffs which are valid in K are valid in L is a normal modal logic.

(a) Even amongst the normal modal logics, there are a great many which we have not explored here. For a taste, figure 14 shows some of the possible modal logics which we can get just from mixing and matching the modal axioms we have already seen—namely, D, T, B, S4, and S5. Each of the systems shown below has the axioms and rules of system K, plus some combination of the axioms D, T, B, S4, and S5. In the figure, they are ordered in terms of validity preservation, or strength.

(b) In that figure, The boldfaced logics are the ones we have studied. '$D4$' is the logic that you get if you add D and S4 to the system K; '$B5$' is the logic that you get if you add B and S5 to K; and so on. For each of these normal modal logics, you may get a possible worlds semantics for the logic which is sound and complete by imposing constraints on the accessibility relation corresponding to those axioms. So, for instance, to get a semantics for the logic 45, you simply define a 45-frame to be a pair $< \mathcal{W}, R >$ of a set of worlds and a binary relation R which is *transitive* and *Euclidean*; and a 45-model is defined in the usual way.