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SENSITIVITY ANALYSIS IN INTEGER PROGRAMMING*

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This paper uses an IP duality Theory recently developed by the authors and others to derive sensitivity analysis tests for IP problems. Results are obtained for cost, right hand side and matrix coefficient variation.

1. Introduction

A major reason for the widespread use of LP models is the existence of simple procedures for performing sensitivity analyses. These procedures rely heavily on LP duality theory and the interpretation it provides of the simplex method. Recent research has provided a finitely convergent IP duality theory which can be used to derive similar procedures for IP sensitivity analyses (Bell and Shapiro [3]; see also Bell [1], Bell and Fisher [2], Fisher and Shapiro [6], Fisher, Northup and Shapiro [7], Shapiro [18]). The IP duality theory is a constructive method for generating a sequence of increasingly strong dual problems to a given IP problem terminating with a dual producing an optimal solution to the given IP problem. Preliminary computational experience with the IP dual methods has been promising and is reported in [7]. From a practical point of view, however, it may not be possible when trying to solve a given IP problem to pursue the constructive procedure as far as the IP dual problem which solves the given problem. The practical solution to this difficulty is to imbed the use of IP duality theory in a branch and bound approach (see [7]).

The IP problem we will study is

$$\begin{aligned} & v = \min cx \\ (1) \qquad & \text{s.t. } Ax + Is = b \\ & x_j = 0 \text{ or } 1, \; s_i = 0, 1, 2, \ldots, U_i, \end{aligned}$$

where A is an $m \times n$ integer matrix with coefficients a_{ij} and columns a_j, b is an $m \times 1$ integer vector with components b_i, and c is a $1 \times n$ real vector with components c_j. For future reference, let $F = \{x^p, s^p\}_{p=1}^{P}$ denote the set of all feasible solutions to (1).

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We have chosen to add the slack variables explicitly to (1) because they behave in a somewhat unusual manner unlike the behavior of slack variables in LP. Suppose for the moment that we relax the integrality constraints in problem (1); that is, we allow $0 \leq x_j \leq 1$ and $0 \leq s_i \leq U_i$. Let u_i^* denote an optimal dual variable for the ith constraint in this LP, and let s_i^* denote an optimal value of the slack. By LP complementary slackness, we have $u_i^* < 0$ implies $s_i^* = 0$ and $u_i^* > 0$ implies $s_i^* = U_i$. In the LP relaxation of (1), it is possible that $0 < s_i^* < U_i$ only if $u_i^* = 0$. On the other hand, in IP we may have a non-zero price u_i^* and $0 < s_i^* < U_i$ because the discrete nature of the IP problem makes it impossible for scarce resources to be exactly consumed. Specific mathematical results about this phenomenon will be given in Section 2.

2. Review of IP duality theory

A dual problem to (1) is constructed by reformulating it as follows. Let G be any finite abelian group with the representation

$$G = Z_{q_1} \oplus Z_{q_2} \oplus \cdots \oplus Z_{q_r}$$

where the positive integers q_i satisfy $q_i \geq 2$, $q_i \mid q_{i+1}$, $i = 1, \ldots, r-1$, and Z_{q_i} is the cyclic group of order q_i. Let g denote the order of G; clearly $g = \prod_{i=1}^{r} q_i$ and we enumerate the elements as $\sigma_0, \sigma_1, \ldots, \sigma_{g-1}$ with $\sigma_0 = 0$. Let $\varepsilon_1, \ldots, \varepsilon_m$ be any elements of this group and for any n-vector f, define the element $\phi(f) = \sigma \in G$ by

$$\phi(f) = \sum_{i=1}^{m} f_i \varepsilon_i .$$

The mapping ϕ naturally partitions the space of integer m-vectors into g equivalence classes $S_0, S_1, \ldots, S_{g-1}$ where $f^1, f^2 \in S_K$ if and only if $\sigma_K = \phi(f^1) = \phi(f^2)$. The element σ_K of G is associated with the set S_K; that is, $\phi(f) = \sigma_K$ for all integer m-vectors $f \in S_K$.

It can easily be shown that (1) is equivalent to (has the same feasible region as)

(2a) $\qquad v = \min cx,$

(2b) $\qquad \text{s.t. } Ax + Is = b,$

(2c) $\qquad \displaystyle\sum_{j=1}^{n} \alpha_j x_j + \sum_{i=1}^{m} \varepsilon_i s_i = \beta,$

$\qquad x_j = 0 \text{ or } 1,$

(2d) $\qquad s_i = 0, 1, 2, \ldots, U_i,$

where $\alpha_j = \phi(a_j)$ and $\beta = \phi(b)$. The group equations (2c) are a system of r congruences and they can be viewed as an aggregation of the linear system $Ax + Is = b$. Hence the equivalence of (1) and (2). For future reference, let Y be the set of (x, s) solutions satisfying (2c) and (2d).Note that $F \subseteq Y$.

The IP dual problem induced by G is constructed by dualizing with respect to the constraints $Ax + Is = b$. Specifically, for each u define

(3) $$L(u) = ub + \min_{(x,s)\in Y} \{(c - uA)x - us\}.$$

The Langrangean minimization (3) can be performed in a matter of a few seconds or less for g up to 5000; see Glover [10], Gorry, Northup and Shapiro [11]. The ability to do this calculation quickly is essential to the efficacy of the IP dual methods. If g is larger than 5000, methods are available to try to circumvent the resulting numerical difficulty (Gorry, Shapiro and Wolsey [12]). However, there is no guarantee that these methods will work, and computational experience has shown that the best overall strategy is to combine these methods with branch and bound.

Sensitivity analysis on IP problem (1) depends to a large extent on sensitivity analysis with respect to the group G and the Langrangean L. Let

(4)
$$\begin{aligned} g(\sigma; u) = \min &\sum_{j=1}^{n} (c_j - ua_j)x_j + \sum_{i=1}^{m} - u_i s_i \\ \text{s.t.} &\sum_{j=1}^{n} \alpha_j x_j + \sum_{i=1}^{m} \varepsilon_i s_i = \sigma, \\ & x_j = 0 \text{ or } 1, \\ & s_i = 0, 1, 2, \ldots, U_i. \end{aligned}$$

Then $L(u) = ub + g(\beta; u)$. Moreover, the algorithms in [10] and [11] can be used to compute $g(\sigma; u)$ for all $\sigma \in g$ without a significant increase in computation time.

It is well known and easily shown that the function L is concave, continuous and a lower bound on v. The IP dual problem is to find the greatest lower bound

(5)
$$\begin{aligned} w &= \max L(u) \\ \text{s.t. } & u \in \mathbf{R}^m. \end{aligned}$$

If $w = +\infty$, then the IP problem (1) is infeasible.

The desired relation of the IP dual problem (5) to the primal IP problem (1) is summarized by the following:

Optimality Conditions: The pair of solutions $(x^*, s^*) \in Y$ and $u^* \in \mathbf{R}^m$ is said to satisfy the optimality conditions if

(i) $L(u^*) = u^*b + (c - u^*A)x^* - u^*s$

(ii) $Ax^* + Is^* = b$.

It can easily be shown that a pair satisfying these conditions is optimal in the respective primal and dual problems. For a given IP dual problem, there is no guarantee that the optimality conditions can be established, but attention can be restricted to optimal dual solutions for which we try to find a complementary optimal primal solution. If the dual IP problem cannot be used to solve the primal problem, then $v > w$ and we say there is a *duality gap*; in this case, a stronger IP dual problem is constructed.

Specifically, solution of the IP problem (1) by dual methods is constructively achieved by generating a finite sequence of groups $\{G^k\}_{k=0}^K$, sets $\{Y^k\}_{k=0}^K$, and IP dual problems analogous to (5) with maximal objective function value w^k. The group $G^0 = Z_1$, $Y^0 = \{(x, s) \mid x_j = 0 \text{ or } 1,\ s_i = 0, 1, 2, \ldots, U_i\}$ and the corresponding IP dual problem can be shown to be the linear programming relaxation of (1). The groups here have the property that G^k is a subgroup of G^{k+1}, implying directly that $Y^{k+1} \subseteq Y^k$ and therefore that $v \geq w^{k+1} \geq w^k$. Sometimes we will refer to G^{k+1} as a supergroup of G^k.

The critical step in this approach to solving the IP problem (1) is that if an optimal solution to the k^{th} dual does not yield an optimal integer solution, then we are able to construct the supergroup G^{k+1} so that $Y^{k+1} \subsetneq Y^k$. Moreover, the construction eliminates the infeasible IP solutions $(x, s) \in Y^k$ which are used in combination by the IP dual problem to produce a fractional solution to the optimality conditions. Since the set Y^0 is finite, the process must converge in a finite number of IP dual problem constructions to an IP dual problem yielding an optimal solution to (1) by the optimality conditions, or prove that (1) has no feasible solution. Details are given in [3].

The following theorem exposes how this IP duality theory extends the notion of complementary slackness to IP.

Theorem 1. *Suppose that $(x^*, s^*) \in Y$ and $u^* \in \mathbf{R}^m$ satisfy the optimality conditions. Then*

(i) *$u_i^* < 0$ and $s_i^* > 0$ implies $\varepsilon_i \neq 0$.*

(ii) *$u_i^* > 0$ and $s_i^* < U_i$ implies $\varepsilon_i \neq 0$.*

Proof. Suppose $u_i^* < 0$ and $s_i^* > 0$ but $\varepsilon_i = 0$. Recall that $(x^*, s^*) \in Y$ implies that $\sum_{j=1}^n \alpha_j x_j^* + \sum_{i=1}^m \varepsilon_i s_i^* = \beta$ and $L(u^*) = u^*b + (c - u^*A)x^* - u^*s$. Since $\varepsilon_i = 0$, we can reduce the value of s_i to 0 and still have a solution in Y. But this new solution in the Lagrangean has a cost of $L(u^*) + u_i^* s^*{}_i{}^* < L(u^*)$ contradicting the optimality of (x^*, s^*). The proof of case (ii) is similar. □

The IP dual problem (5) is actually a large scale linear programming problem. Let $Y = \{x^t, s^t\}_{t=1}^T$ be an enumeration of Y. The LP formulation of (5) is

$$
\begin{aligned}
& w = \max v \\
(6) \qquad & v \leq ub + (c - uA)x^t - us^t \\
& t = 1, \ldots, T.
\end{aligned}
$$

The linear programming dual to (6) is

$$w = \min \sum_{t=1}^{T} (cx^t)\omega_t,$$

$$\text{s.t.} \sum_{t=1}^{T} (Ax^t + Is^t)\,\omega_t = b, \tag{7}$$

$$\sum_{t=1}^{T} \omega_t = 1,$$

$$\omega_t \geq 0\,.$$

The number of rows T in (6), or columns in (7), is enormous. The solution methods given in Fisher and Shapiro [6] generate columns as needed by ascent algorithms for solving (6) and (7) as a primal-dual pair. The columns are generated by solving the Lagrangean problem (3).

The formulation (7) of the IP dual has a convex analysis interpretation. Specifically, the feasible region in (7) corresponds to

$$\{(x, s) \mid Ax + Is = b, 0 \leq x_j \leq 1, 0 \leq s_i \leq U_i\} \cap [Y]$$

where the left hand set is the feasible region of the LP relaxation of the IP problem (1) and "[ ]" denotes convex hull. Thus, in effect, the dual approach approximates the convex hull of the set of feasible integer points by the intersection of the LP feasible region with the polyhedron $[Y]$. When the IP dual problem (5) solves the IP problem (1), then $[Y]$ has cut away enough of the LP feasible region to approximate the convex hull of feasible integer solutions in a neighborhood of an optimal IP solution.

3. Sensitivity analysis of cost coefficients

Sensitivity analysis of cost coefficients is easier than sensitivity analysis of right hand side coefficients because the set F of feasible solutions remains unchanged. As described in the previous section, suppose we have constructed an IP dual problem for which the optimality conditions are satisfied by some pair $(x^*, s^*) \in Y$ and u^*.

The first question we wish to answer is

> In what range of values can c_l vary without changing the value of the zero-one variable x_l in the optimal solution (x^*, s^*)?

We answer this question by studying the effect of changing c_l on the Lagrangean.

Theorem 2. *Let (x^*, s^*) and u^* denote optimal solutions to the primal and dual IP problems, respectively, satisfying the optimality conditions. Suppose the zero-one variable $x_l^* = 0$ and we consider varying its cost coefficient c_l to $c_l + \Delta c_l$. Then (x^*, s^*) remains optimal if*

(8) $$\Delta c_l \geq \min\{0, g(\beta; u^*) - (c_l - u^* a_l) - g(\beta - \alpha_l; u^*)\},$$

where $g(\cdot, u^*)$ *is defined in* (4).

Proof. Clearly, if $x_l^* = 0$ and (8) indicates that $\Delta c_l \geq 0$, or c_l increases, then x^* remains optimal with $x_l^* = 0$. Thus we need consider only the case when $g(\beta; u^*) - (c_l - u^* a_l) - g(\beta - \alpha_l; u^*) < 0$. Let $g(\sigma; u^* \mid x_l = k; \Delta c_l)$ denote the minimal cost in (4) if we are constrained to set $x_l = k$ when the change in the l^{th} cost coefficient is Δc_l. If Δc_l satisfies (8), then

(9)
$$\begin{aligned} g(\beta; u^* \mid x_l = 1; \Delta c_l) &= c_l + \Delta c_l - u^* a_l + g(\beta - \alpha_l; u^* \mid x_l = 0; \Delta c_l) \\ &= c_l + \Delta c_l - u^* a_l + g(\beta - \alpha_l; u^* \mid x_l = 0; 0) \\ &\geq c_l + \Delta c_l - u^* a_l + g(\beta - \alpha_l; u^*) \\ &\geq g(\beta; u^*) \\ &= g(\beta; u^* \mid x_l = 0; 0) \\ &= g(\beta; u^* \mid x_l = 0; \Delta c_l), \end{aligned}$$

where the first equality follows from the definition of $g(\beta; u^* \mid x = 1; \Delta c_l)$, the second equality because the value of Δc_l is of no consequence if $x_l = 0$, the first inequality because $g(\beta - \alpha_l; u^*)$ may or may not be achieved with $x_l = 0$, the second inequality by our assumption that (8) holds, the third equality because $g(\beta; u^*) = (c - u^* A)x^* - u^* s$ and $x_l^* = 0$, and the final equality by the same reasoning as the second equality. Thus, as long as Δc_l satisfies (8), it is less costly to set $x_l = 0$ rather than $x_l = 1$. □

On the other hand, marginal analysis when $x_l^* = 1$ is not as easy because the variable is used in achieving the minimal value $g(\beta; u^*)$. Clearly x^* remains optimal if c_l is decreased. As c_l is increased, x_l should eventually be set to zero unless it is uniquely required for feasibility.

Theorem 3. *Let* (x^*, s^*) *and* u^* *denote optimal solutions to the primal and dual IP problems, respectively, satisfying the optimality conditions. Suppose the zero-one variable* $x_l^* = 1$ *and we consider varying its cost coefficient* c_l *to* $c_l + \Delta c_l$. *Then* (x^*, s^*) *is not optimal in the Lagrangean if*

(10) $$\Delta c_l > \min\{c_j - u^* a_j \mid j \in J(\alpha_l) \text{ and } x_j^* = 0\} - (c_l - u^* a_l)$$

where

$$J(\alpha_l) = \{j \mid \phi(a_j) = \phi(a_l) = \alpha_l\}.$$

We assume there is at least one $x_j^* = 0$ *for* $j \in J(\alpha_l)$ *because otherwise the result is meaningless.*

Proof. Note that we can order the elements $j_1, j_2, \ldots, j_V$ in $J(\alpha_l)$ by increasing cost $c_j - x^*a_j$ with respect to u^* such that $x^*_1 = 1$, $j = 1, \ldots, j_v$, $x^*_j = 0$, $j = j_v + 1, \ldots, j_V$. This is because all these variables x_j have the same effect in the constraints in (4). By assumption, there is an $x^*_j = 0$, and if $c_l + \Delta c_l - u^*a_l > c_j - u^*a_j$, then $x_j = 1$ will be preferred to $x_l = 1$ in (4). In this case, (x^*, s^*) is no longer optimal in the Lagrangean and the optimality conditions are destroyed. □

The inequality (10) can be a gross overstatement of when (x^*, s^*) ceases to be optimal in the Lagrangean. Systematic solution of $g(\beta; u^*)$ for increasing values of c_l is possible by the parametric methods we discuss next.

A more general question about cost coefficient variation in the IP problem (1) is the following

> How does the optimal solution change as the objective function c varies in the interval $[c^0, c^1]$?

Parametric IP analysis of this type has been studied by Nauss [15], but without the IP duality theory, and by the author in [21] in the context of multicriterion IP. We give some of the relevant results here. The work required to do parametric IP analysis is greater than the sensitivity analysis described above which is effectively marginal analysis.

For $\theta \in [0, 1]$ define the function

$$
\begin{aligned}
(11) \qquad & v(\theta) = \min((1-\theta)c^0 + \theta c^1)x, \\
& Ax + Is = b, \\
& x_j = 0 \text{ or } 1, \\
& s_i = 0, 1, 2, \ldots, U_i.
\end{aligned}
$$

It is easy to show that $v(\theta)$ is a piecewise linear concave function of θ. The IP dual objective function can be used to approximate $v(\theta)$ from below. Specifically, suppose (11) is solved by the IP dual at $\theta = 0$ and we consider increasing it. From (7), we have

$$
\begin{aligned}
& w(\theta) = \min \sum_{t=1}^{T} ((1-\theta)c^0 + \theta c^1)x^t w_t \\
(12) \qquad & \text{s.t.} \sum_{t=1}^{T} (Ax^t + Is^t) w_t = b \\
& \sum_{t=1}^{T} w_t = 1 \\
& w_t \geq 0
\end{aligned}
$$

where $w(\theta)$ is also a piecewise linear concave function of θ, and $w(0) = v(0)$ because we assume an IP dual has been constructed which solves the primal.

Without loss of generality, assume (x^1, s^1) is the optimal IP solution for $\theta = 0$. Then, $w_1 = 1$, $w_t = 0$, $t \geq 1$ is the optimal solution in the LP (12), and we can do parametric variation of $\theta \geq 0$ to find the maximal value, say θ^*, such that $v(\theta) = w(\theta)$ for all $\theta \in [0, \theta^*]$. The difficulty is that the number of columns in (12) is enormous. For this purpose, generalized linear programming can be used to generate columns as needed for the parametric analysis. Included is the possibility that when $w_1 = 1$, $w_t = 0$, $t \geq 1$, becomes non-optimal, another feasible IP solution will become the optimal solution in (12).

When a sufficiently large θ is reached such that $v(\theta) - w(\theta) > 0$, then we can use the iterative IP dual analysis described in section two to strengthen the dual and ultimately eliminate the gap. Numerical excesses may make this impractical. However, any IP dual can be used to reduce the work of branch and bound in the parametric analysis of the interval $[c^0, c^1]$ being studied. These IP duals give stronger lower bounds than the LP and related lower bounds used in [15].

4. Sensitivity analysis of right-hand side coefficients

This is a rich area of research which needs continuing investigation. Nevertheless, we can report on some results already obtained. Again we suppose that the IP duality theory has yielded an IP dual problem for which the optimality conditions hold for $(x^*, s^*) \in Y$, $u^* \in \mathbf{R}^m$. As in LP, constraint i is not binding if $u_i^* = 0$. Specifically, it can easily be shown that x^* is optimal in IP(1) with the right hand side b_i equal to any of the numbers

$$\sum_{j=1}^{n} a_{ij}x_j^*, \quad \sum_{j=1}^{n} a_{ij}x_j^* + 1, \ldots, \sum_{j=1}^{n} a_{ij}x_j^* + U_i,$$

for any row i with $u_i^* = 0$. The optimal value of the slack variable on such a row is $b_i - \sum_{j=1}^{n} a_{ij}x_j^*$.

To study further the effects of varying b, we define the perturbation function for b in a finite set B

$$\begin{aligned} & v(b) = \min cx \\ (13) \qquad & \text{s.t. } Ax + Is = b \\ & x_j = 0 \text{ or } 1 \\ & s_i = 0, 1, 2, \ldots, U_i. \end{aligned}$$

Attention is limited to a finite set rather than all integer vectors in $\mathbf{R}^m$ because a finite set is more likely to be the type of interest, and also because it avoids troublesome technical difficulties. For a finite set of integer right hand sides, universal upper bounds on the slacks can be found and used. The function $v(b)$ is poorly behaved, except for the property that $b' \geq b$ implies $v(b') \leq v(b)$, which makes it difficult to study. Note also that v is defined only on the integers and it is not differentiable, unlike perturbation functions in nonlinear programming.

For each right hand side $b \in B$, the finitely convergent duality theory given in [3] produces an IP dual problem which solves (13) by establishing the optimality conditions. These IP duals are related but specific results about their relationship are difficult to obtain. Instead, we consider the IP dual which solves (13) with the given right hand side b^0, and investigate its properties with respect to the other $b \in B$.

Let G denote the group used in the construction of the IP dual solving (13) with $b = b^0$. This group induces a family of g related dual problems defined on each of the g equivalence classes of right hand sides $S_0, S_1, \ldots, S_{g-1}$. For $b \in S_\sigma$, we have

$$\text{(14)} \qquad \begin{aligned} w_\sigma(b) &= \max L_\sigma(u) \\ &\quad \text{s.t. } u \in \mathbf{R}^m, \end{aligned}$$

where

$$L_\sigma(u) = ub + g(\sigma; u).$$

By assumption

$$v(b^0) = cx^* = w_{\beta^0}(b^0) = L_{\beta^0}(u^*),$$

where $\beta^0 = \phi(b^0)$. The solution of problem (4) for all right hand sides σ gives us optimal solutions to a number of other IP problems (13). Let $(x(\sigma), s(\sigma))$ denote the optimal solution to (4) with right hand side σ when $u = u^*$. It is easy to show by direct appeal to the optimality conditions that $(x(\sigma), s(\sigma))$ is optimal in (13) with

$$b_i = \sum_{j=1}^{n} a_{ij}x_j(\sigma) + s_i(\sigma), \quad i = 1, \ldots, m.$$

Moreover, for i such that $u_i^* = 0$, $x(\sigma)$ and the corresponding slack values are optimal in (13) with

$$b_i = \sum_{j=1}^{n} a_{ij}x_j(\sigma), \ldots, \sum_{j=1}^{n} a_{ij}x_j(\sigma) + U_i.$$

Thus, we may immediately have optimal solutions to some of the IP problems (13) for $b \in B$. In addition, $x(\sigma)$ is a feasible but possibly non-optimal solution in (13) with $b \in B$ if $Ax(\sigma) \leq b$ and

$$b_i - \sum_{j=1}^{n} a_{ij}x_j(\sigma) \in \{0, 1, \ldots, U_i\}.$$

Note, however, that not all constraints of an IP problem can be allowed to vary parametrically. For example, a constraint of the form $x_{11} + x_{12} \leq 1$ indicating that project 1 can be started in period 1 or period 2, but not both, makes no sense when extended to the constraint $x_{11} + x_{12} \leq 2$. Some constraints of this type can be included in the Lagrangean calculation.

Marsten and Morin [14] have devised schemes for parametric analysis of the right hand side of IP problems. The duality results here can be integrated with their approach to provide tighter lower bounds for the branch and bound procedure. They consider b to be a real vector and observe jumps in the function $v(b)$. The selection of integer data for A and b in effect limits attention to the points where $v(b)$ might change value.

5. Sensitivity analysis of matrix coefficients

This analysis is similar to the cost coefficient analysis since the dual approach is to convert constraints to costs. The question we address is:

> In what range of values can a coefficient a_{il} vary without changing the value of the zero-one variables x_l in the optimal solution (x^*, s^*)?

As before, the answer to this question is easier if $x_l^* = 0$.

Theorem 4. *Let (x^*, s^*) and u^* denote optimal solutions to the primal and dual IP problems, respectively, satisfying the optimality conditions. Suppose the zero-one variable $x_l^* = 0$ and we consider varying the coefficient a_{il} to $a_{il} + \Delta a_{il}$ where Δa_{il} is integer. Then x^* remains optimal if Δa_{il} satisfies*

$$(15)\qquad -u_i^* \Delta a_{il} \geqslant \min\{0, g(\beta; u^*) - (c_l - u^* a_l) - g(\beta - \alpha_l - \Delta a_{il}\varepsilon_i; u^*)\}.$$

There is no restriction on Δa_{il} if $u_i^* = 0$.

Proof. The proof is identical to the proof of Theorem 2. The change Δa_{il} causes the change $-u_i^* \Delta a_{il}$ in the cost coefficient analogous to the change Δc_l in Theorem 2, and the group identity of a_l is changed to $\alpha_l + \Delta a_{il}\varepsilon_i$. □

A result similar to Theorem 3 for the case when $x_l^* = 1$ can be obtained. We omit further details. A more general type of IP matrix coefficient variation is the problem of IP column generation. Such a problem would arise, for example, if there were a subproblem to be solved whose solution provided a candidate column a with cost coefficient c to be added to IP(1). A construction to do this using the IP duality theory appears possible but will not be developed here.

6. Conclusions

We have presented some results for performing IP sensitivity analyses using IP duality theory. More research into these methods is needed, particularly a more extensive study of the family of IP dual problems which result as the right hand side in (1) is varied. Computational experience with these methods, in conjunction with branch and bound, is crucial and will suggest important areas of research. We mention that the work of Burdet and Johnson [5] appears to provide an analytic formalism for combining duality and branch and bound. Finally, the IP duality theory has been extended to mixed IP in [20] indicating that the results here can be readily extended to sensitivity analysis for mixed IP.

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