0/1 - Matrices

Alexandra Lassota

Consider the problem of deciding whether the polytope $P = \{x \mid Ax = b, 0 \leq x \leq 1, x \in \mathbb{N}^{2^m}\}$ is non-empty where A is the complete 0/1–matrix if size $m \times 2^m$.

We denote by x^i the i–th entry of a vector x. Let a_i be a column of A for $i \in \{1, \ldots, 2^m\}$. We define *pairs* as tuples (a_i, a_j) of columns such that $a_i + a_j = \mathbf{1}$. Call a_j the *complement* of a_i and vice versa. Clearly, we can partition all columns in A into disjoint pairs and this partition is unique.

Consider the decomposition $b = k \cdot \mathbf{1} + u$ of the right-hand side b such that $Ax = u$ is feasible for some $0 \leq x \leq 1, x \in \mathbb{N}^{2^m}$ and k is maximal. Call such a decomposition *maximal*.

Theorem 1. *Given a maximal decomposition $b = k \cdot \mathbf{1} + u$. If $Ax = b$ is feasible, then we can extend each support minimal solution x_u for $Ax = u$ to a solution $x^* = x_u + x_k$ such that $x_u + x_k \leq \mathbf{1}$.*

Proof. Suppose the opposite, i.e., there exists a solution $x' = x_{u'} + x'_k$ such that $Ax' = b$ but $x_{u'}$ is not support minimal or k is not maximal w.r.t. the definition of a maximal decomposition, and a support minimal solution cannot be extended to solve the original problem.

Define X'_1 as the set of columns taken by $x_{u'}$, i.e., take the i–th column if $x^i_{u'} = 1$. Let X'_2 contain all complements of the columns chosen for X'_1. Denote by X'_R the remaining columns of A, i.e., all columns which are not present in X'_1 nor X'_2. Note that X'_R can be partitioned into $|X'_R|/2$ many pairs.

If $X'_1 \cup X'_2 \neq \emptyset$, this means that X'_1 contains a pair (a_i, a_j). In that case, we can reduce u' by $\mathbf{1}$ and increase k' by 1 and obtain u'' and k''. There exists a solution to the new decomposition, in particular, obtain $x_{u''}$ by setting $x^i_{u'} = x^j_{u'} = 0$. Similarly, x''_k is obtained by setting the i–th and j–th entry to 1.

So, let us assume that $X'_1 \cup X'_2 = \emptyset$. If $k' \leq |X'_R|/2$, meaning columns present in X'_R are sufficient to define a solution for $Ax = k' \cdot \mathbf{1}$. In that case, some support minimal solution x would yield that $|X_R| > |X'_R|$ and thus, we can extend to a solution for $Ax = b$.

Thus, we can additionally assume that $k > |X'_R|/2$. Thus, we necessarily have to take some column a_i in X'_2 for our solution x'. For that column, there exists a complement a_j in X'_1. Now, we can show that u' is not minimal by defining a solution with smaller support to solve a smaller u. To do so, set $x^i_{u'} = 0$. Meaning, we delete a_i and a_j from the sets X'_1 and X'_2, respectively, and add them to X'_R. This way, we expressed the same solution but with a decomposition for a smaller $u < u'$ using less columns. We can repeat this step for every column in X'_2. This way, we obtain a solution using only columns in X'_R to satisfy the scaled $\mathbf{1}$–vector, and obtain a sufficient small u''. Indeed, the u'' obtained this way might not be minimal w.r.t. the maximal decomposition. However, if we can solve the problem for some u'' larger than a minimal u such that the remaining solution only uses columns in the set X'_R, we already argued above that this results also holds for the maximal decomposition. $\square$