CIL Series 7

The Nonstandard Deviations

May 6, 2015

1 Problem 1 (1D Compression and Orthonormal Basis):

Proof. Let

$$\hat{\mathbf{x}}_{-\sigma} = \mathbf{x} - \hat{\mathbf{x}}_{\sigma} = \sum_{k=\tilde{K}+1}^{K} z_{\sigma(k)} \mathbf{u}_{\sigma(k)},$$

then $\|\hat{\mathbf{x}}_{-\sigma}\|_2^2$ denotes the error term of our approximation.

$$\begin{aligned}
\|\hat{\mathbf{x}}_{-\sigma}\|_2^2 &= \langle \hat{\mathbf{x}}_{-\sigma}, \hat{\mathbf{x}}_{-\sigma} \rangle && \text{rewrite as inner product} \\
&= \left\langle \sum_{k=\tilde{K}+1}^{K} z_{\sigma(k)} \mathbf{u}_{\sigma(k)}, \sum_{k=\tilde{K}+1}^{K} z_{\sigma(k)} \mathbf{u}_{\sigma(k)} \right\rangle && \text{definition of } \hat{\mathbf{x}}_{-\sigma} \\
&= \sum_{n=1}^{N} \left\{ \left(\sum_{k=\tilde{K}+1}^{K} z_{\sigma(k)} u_{\sigma(k),n} \right) \left(\sum_{k=\tilde{K}+1}^{K} z_{\sigma(k)} u_{\sigma(k),n} \right) \right\} && \text{definition of inner product} \\
&= \sum_{n=1}^{N} \left\{ \sum_{k=\tilde{K}+1}^{K} \left[z_{\sigma(k)} u_{\sigma(k),n} \left(\sum_{l=\tilde{K}+1}^{K} z_{\sigma(l)} u_{\sigma(l),n} \right) \right] \right\} && \text{distributivity} \\
&= \sum_{n=1}^{N} \sum_{k=\tilde{K}+1}^{K} \sum_{l=\tilde{K}+1}^{K} z_{\sigma(k)} u_{\sigma(k),n} z_{\sigma(l)} u_{\sigma(l),n} && \text{pull out sum} \\
&= \sum_{k=\tilde{K}+1}^{K} \sum_{l=\tilde{K}+1}^{K} \sum_{n=1}^{N} z_{\sigma(k)} u_{\sigma(k),n} z_{\sigma(l)} u_{\sigma(l),n} && \text{reorder sums} \\
&= \sum_{k=\tilde{K}+1}^{K} z_{\sigma(k)} \sum_{l=\tilde{K}+1}^{K} z_{\sigma(l)} \sum_{n=1}^{N} u_{\sigma(k),n} u_{\sigma(l),n} && \text{pull out terms} \\
&= \sum_{k=\tilde{K}+1}^{K} z_{\sigma(k)} \sum_{l=\tilde{K}+1}^{K} z_{\sigma(l)} \left\langle \mathbf{u}_{\sigma(k)}, \mathbf{u}_{\sigma(l)} \right\rangle && \text{definition of inner product} \\
&= \sum_{k=\tilde{K}+1}^{K} z_{\sigma(k)} \sum_{l=\tilde{K}+1}^{K} z_{\sigma(l)} [k = l] && \text{orthonormal} \\
&= \sum_{k=\tilde{K}+1}^{K} z_{\sigma(k)}^2
\end{aligned}$$

It follows that in order to minimize the error term we choose the indices of the $\tilde{K}$ largest z_k for our permutation. □