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Assignment 3

CS/CSYS 302
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1

1.a

1. Every edge is from a node a to node b where $a > b$
2. A cycle exist if there is a node in the graph that can be reached by following edges starting from that node e.g. $a \rightarrow b \rightarrow ... \rightarrow a$
3. Given 1, index of nodes that are being visited following any path always decreases
4. Given 2 and 3, a node cannot be revisited. Therefore the described network is acyclic.

1.b

Average in-degree of vertex i equals to

$$(N - i) \times p,$$

since for vertex i, there are $N - i$ possible vertex to have an edge from with probability p.

Average out-degree of vertex i equals to

$$(i - 1) \times p,$$

since for vertex i, there are $i - 1$ possible vertex to have an edge to with probability p.

1.c

The description divides the vertices into two set: (1) vertices bigger than i, (2) vertices smaller or equal to i. Set (1) contains $N - i$ vertices and set (2) contains i vertices. Therefore the expected number of edges from set (1) to set (2) equals to

$$(N - i) \times i \times p = (Ni - i^2)p.$$

1.d

Smallest value is 0 and occurs when $i = N$ or $i = 0$, which creates an empty set (1) and empty set (2), respectively.

Largest value is $(\frac{N}{2})^2$ occur when $i = \frac{N}{2}$ which maximizes $(N - i) \times i$.

2

2.a

$$\frac{d\tilde{I}_k}{dt} = \beta k(\frac{N_k - I_k}{N})\theta - \alpha \tilde{I}_k$$

$$= \beta k(p_k - \tilde{I}_k)\theta - \alpha \tilde{I}_k,$$

where $\theta = \frac{\sum_k k\tilde{I}_k}{\sum_k kp_k}$.

2.b

$$\frac{d\tilde{I}_k^u}{dt} = \beta k(p_k^u - \tilde{I}_k^u)\theta^{uu} + (1-\rho)\beta k(p_k^u - \tilde{I}_k^u)\theta^{uv} - \alpha \tilde{I}_k^u$$

$$\frac{d\tilde{I}_k^v}{dt} = (1-\rho)\beta k(p_k^v - \tilde{I}_k^v)\theta^{vx} - \alpha \tilde{I}_k^v$$

where $p_k^u = \frac{I_k^u + S_k^u}{N}$, $p_k^v = \frac{I_k^v + S_k^v}{N}$, $\theta^{uu} = \frac{\sum_k k\tilde{I}_k^u}{\sum_k k(p_k^u + p_k^v)}$, $\theta^{uv} = \frac{\sum_k k\tilde{I}_k^v}{\sum_k k(p_k^u + p_k^v)}$, and $\theta^{vx} = \frac{\sum_k k(\tilde{I}_k^u + \tilde{I}_k^v)}{\sum_k k(p_k^u + p_k^v)}$.

θ^{uu} represents the probability that a random edge around an unvaccinated susceptible node leads to an unvaccinated infectious node.

θ^{uv} represents the probability that a random edge around an unvaccinated susceptible node leads to a vaccinated infectious node.

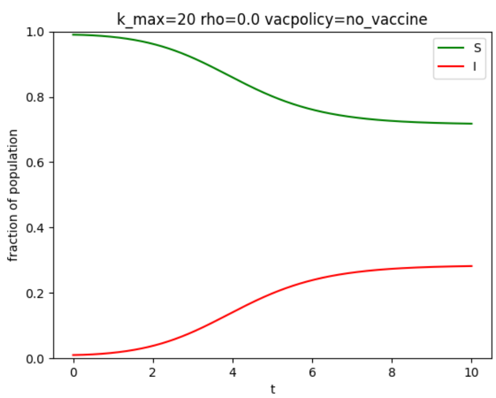
θ^{vx} represents the probability that a random edge around a vaccinated susceptible node leads to any infectious node.

2.c

For question 2.c and 2.d, we use Euler forward method with step size of 0.01 and integrate the system for 1000 timesteps. In both experiments, we choose k_{max} as 20 and 1% of the population starts infected.

Before demonstrating our results, we show the result of an experiment we conducted to check whether the system works as expected. In Figure 1, we show the time series plots of population when there is no vaccination (left) and 40% of population gets vaccinated with a vaccine with $\rho = 0$ (right). The results are identical as expected.

Next, we experiment with vaccinating 40% of the population randomly with vaccines of varying degrees of effectiveness ($\rho \in \{0.2, 0.4, 0.6, 0.8, 1.0\}$). Since individuals get vaccinated randomly, we assume that 40% of each compartment would get vaccinated. As Figure 2 demonstrates, even when the vaccine is fully effective ($\rho = 1.0$), the epidemic cannot be fully eradicated.



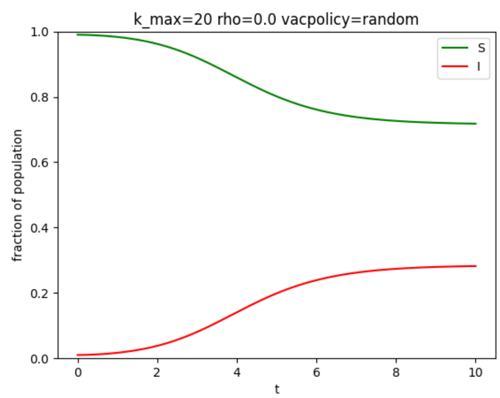


Figure 1: Sanity check for our system for $\rho = 0$, i.e. vaccine has no effect. (left) no vaccination (right) random vaccination. The plots are identical as expected.

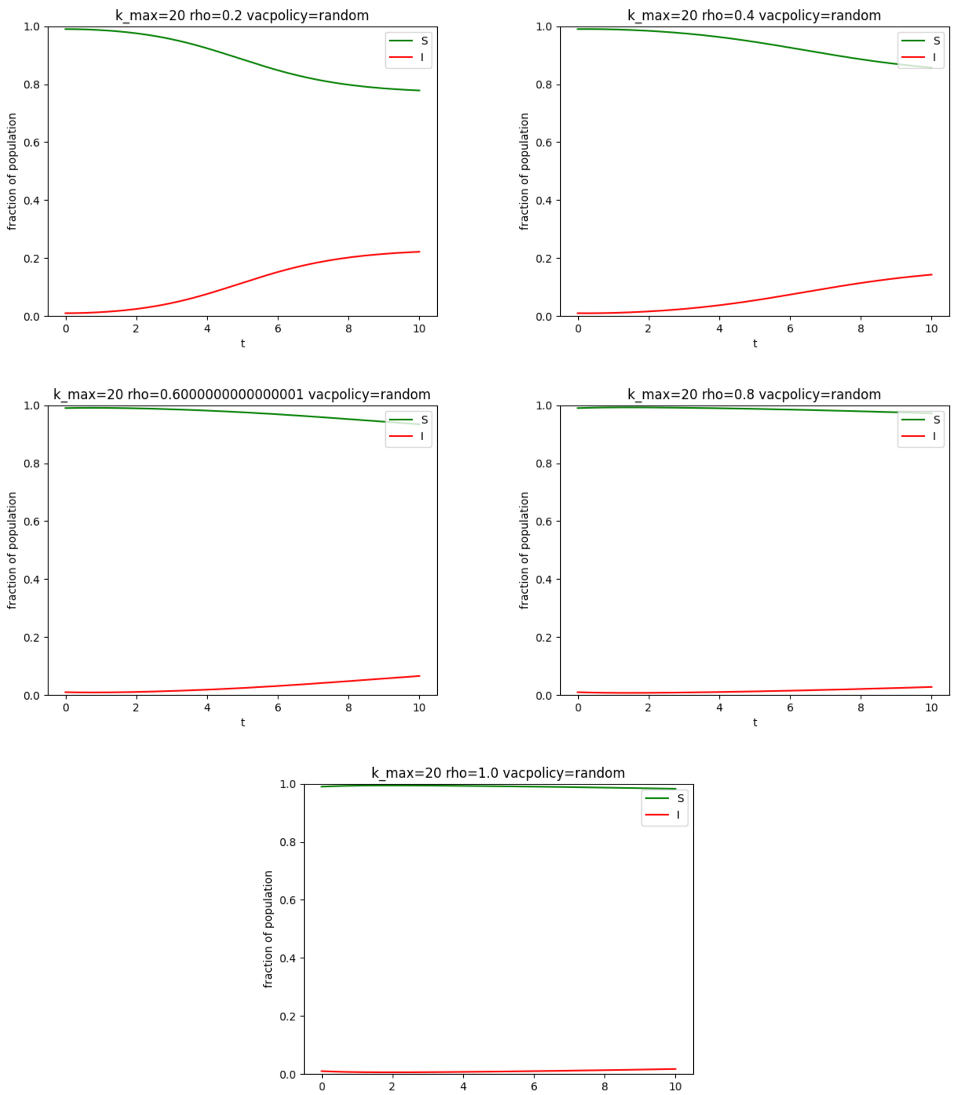
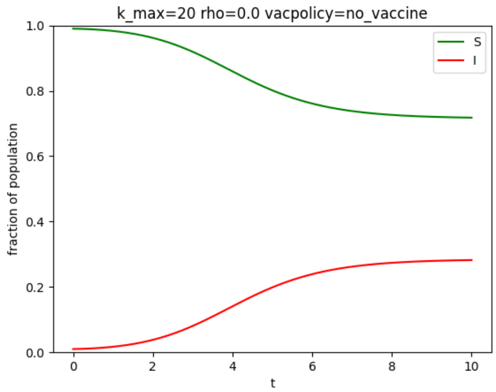


Figure 2: Results of random vaccination.



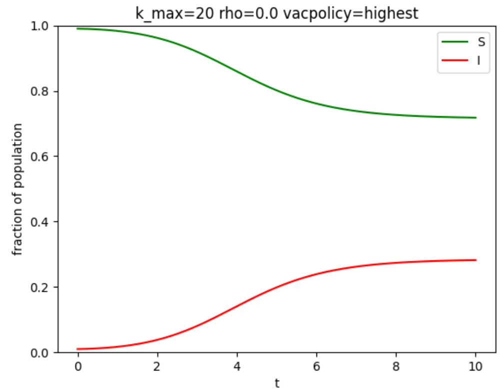


Figure 3: Sanity check for our system for $\rho = 0$, i.e. vaccine has no effect. (left) no vaccination (right) highest degree vaccination. The plots are identical as expected.

2.d

In this experiment, 40% of the population with highest degree get vaccinated. Similar to previous experiments, we perform a sanity check for this vaccination policy as well. As Figure 3 demonstrates, the systems performs the same when the vaccine is not effective.

Figure 4 shows the results of vaccinating individuals with highest degree. The epidemic can be eradicated with a vaccine with $\rho = 0.6$.

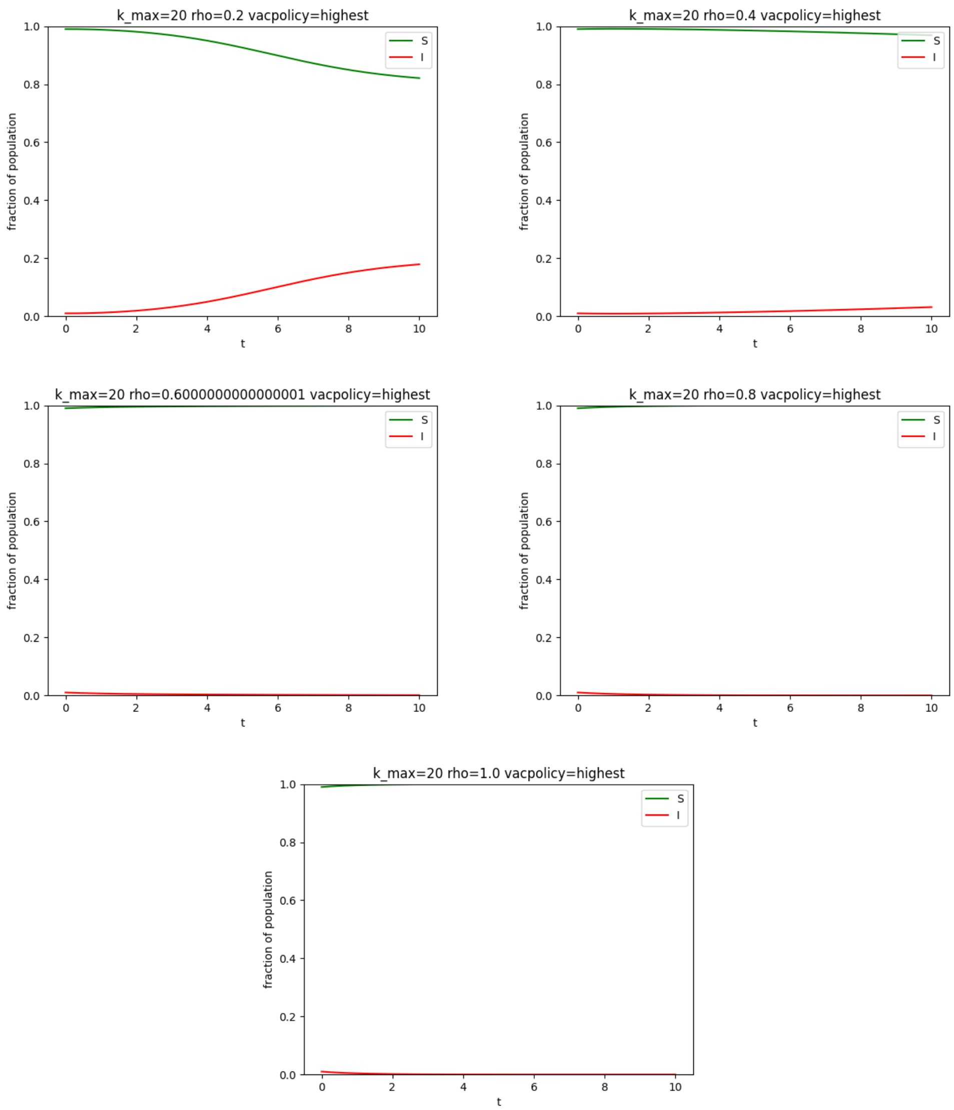


Figure 4: Results of vaccinating individuals with highest degree.

3

The network we chose from the Colorado Index of Complex Networks is one that represents the neural connections of the Caenorhabditis elegans nematode. Upon visualizing the network using two separate layouts, we can see that some nodes have a much higher degree than others, and that overall, the network is highly connected. The network, as downloaded, is a directed multigraph with an average degree of approximately 15.89 and a density of approximately 0.027. The nodes in the graph are neurons and the edges are synaptic links between neurons. See visualizations (5 and 6) below. We also wanted to see if there were any neurons that seemed to be "central" in the communication of the graph, so we looked at betweenness centrality of the graph. We also wanted to look at the average clustering coeffecient to see how much nodes in this graph cluster together. To look at both of these we had to convert the multigraph to a simple graph. We did so by adding the weights on the duplicate edges. We then converted this directed simple graph to an undirected graph, since networkx does not have a method of measuring these two metrics on directed graphs. Once we did this the nodes with the highest and lowest betweenness centrality can be seen in 7, and the average clustering coeffecient was approximately 0.29. This points to some mild clustering but maybe not as much as we were expecting to see as well at least one node being fairly "central" (with value at approximately 0.3) to communications while others are not essential in any communicative channels (with values at 0). We see the betweenness centrality drop off fairly quickly from 0.3 to 0.058 and by the last value in the table, it is down to 0.026.

4

We now apply the voter model to the simple undirected version of our graph.

4.a

We initialize with a 40% red (or infected) and 60% blue (or susceptible) population and we complete 10 runs for as long as it takes to reach a consensus. In this case, we define consensus as all of the nodes in the graph having the same state. We plotted these 10 outcomes for visualization in 8. It looks like it takes less than 40K iterations for consensus most of the time. The majority of the runs ending in a fully blue (susceptible) population with only one ending with fully red (infected) population. We then ran the voter model with an initialization of 60% red and 40% blue to see how that compared. See 9 for this comparison. It looks like convergence takes slightly longer in the case of the 60% red initialization, but it looks like there is more convergence to red with 6 compared to only 1 in the previous case.

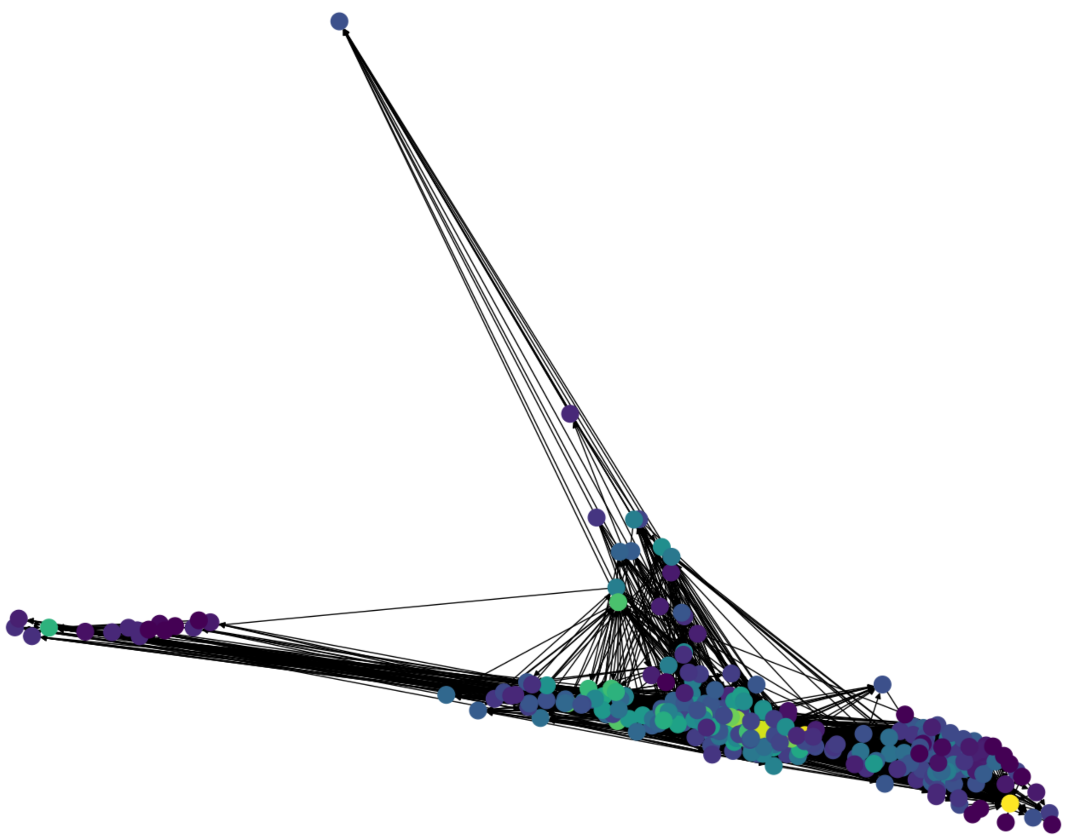


Figure 5: Visualization using the spring layout of Networkx of the neural connections of the Caenorhabditis elegans nematode. Node colors are related to the degree.

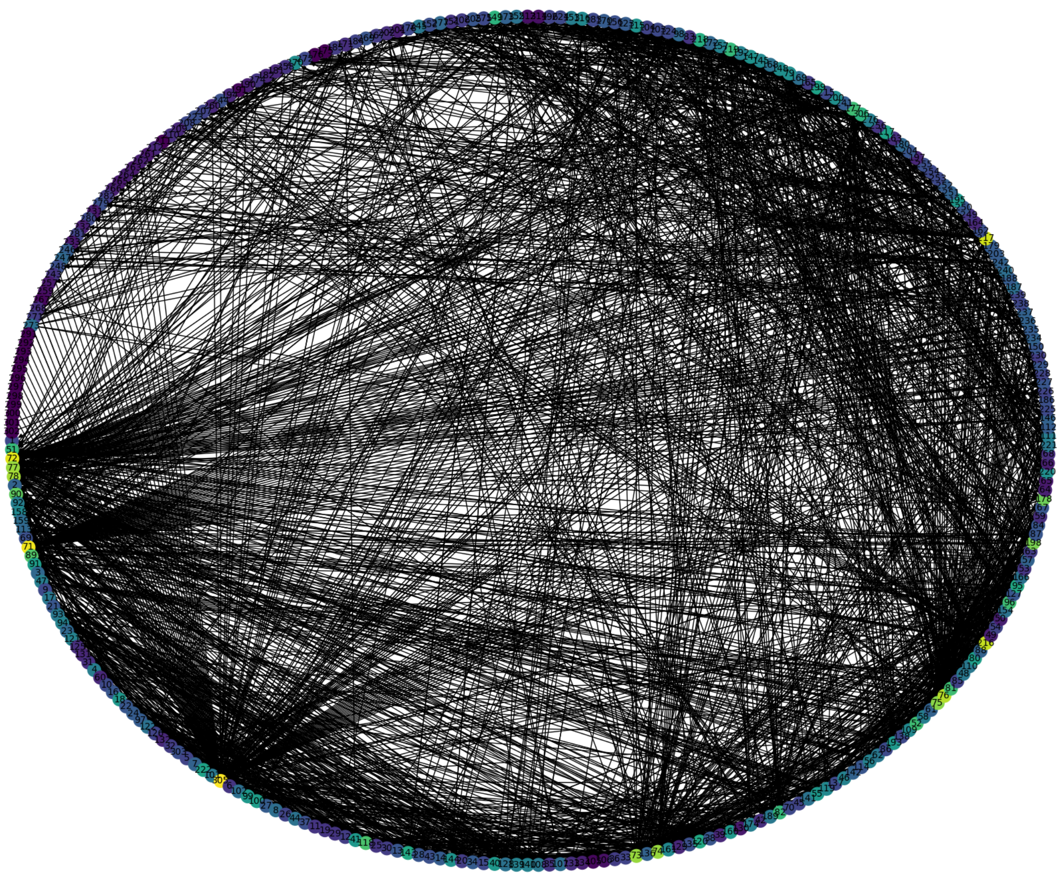


Figure 6: Visualization using the shell layout of Networkx of the neural connections of the Caenorhabditis elegans nematode. Node colors are related to the degree.

Highest Betweenness Centrality		Lowest Betweeness Centrality	
Node	Betweeness Centrality	Node	Betweeness Centrality
'305'	0.303	'65'	0.0
'71'	0.058	'175'	0.0
'72'	0.048	'176'	0.0
'306'	0.044	'191'	0.0
'78'	0.031	'211'	0.0
'198'	0.031	'267'	0.0
'77'	0.029	'291'	0.0
'74'	0.027	'292'	0.0
'177'	0.026	'293'	0.0
'217'	0.026	'294'	0.0

Figure 7: The 10 highest and lowest betweenness centralities for the graph in 5

4.b

The next thing we performed was to recreate our graph using the configuration model. We did so by applying the configuration model with the same original degree distribution for all of the nodes in the original graph. We then ran both of the above experiments on this graph. See 10 and 11. It seems that it converges much more quickly than the original graph on the voter model. All cases converge well before 80,000 iterations in all runs and both cases. It seems that with a starting red percentage of 40 seems to lead to less convergence to a red consensus (3) than starting with a percentage of 60 red (8). Overall, there is more convergence to red than to blue in this graph than the original. We wanted to visualize this graph to see why it might not be fully converging (12 and 13). It seems like there may be slightly more larger clusters in this graph than the original. I think this has a major impact on the voter model. The effect of creating this graph with the configuration model is that it is now a multigraph with some edges connecting the same nodes. This could lead to those nodes having a strong influence on one another in the voter model. I think it also has an interesting effect on clustering that I'm not sure I fully understand. Either way, it is a very interesting result that we observe here.

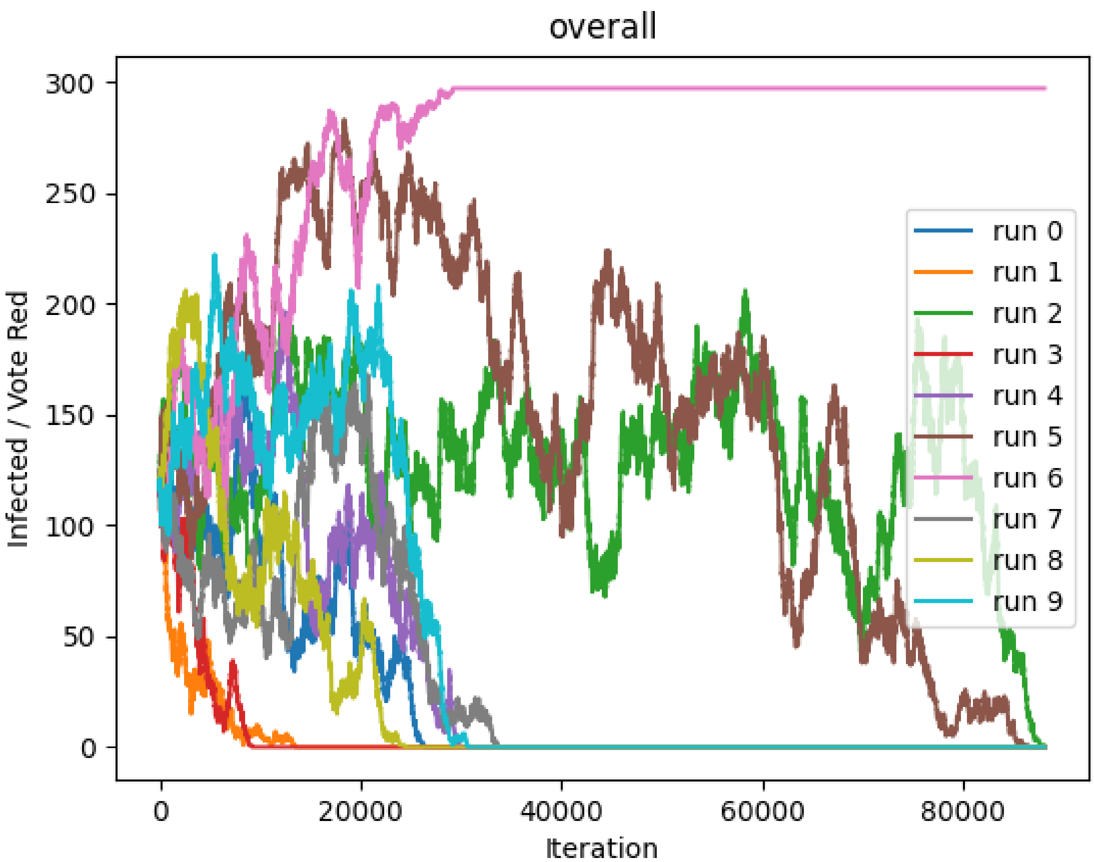


Figure 8: Plot across iterations of the voter model on the graph from 5 with a 40% initialization of red (infected) nodes.

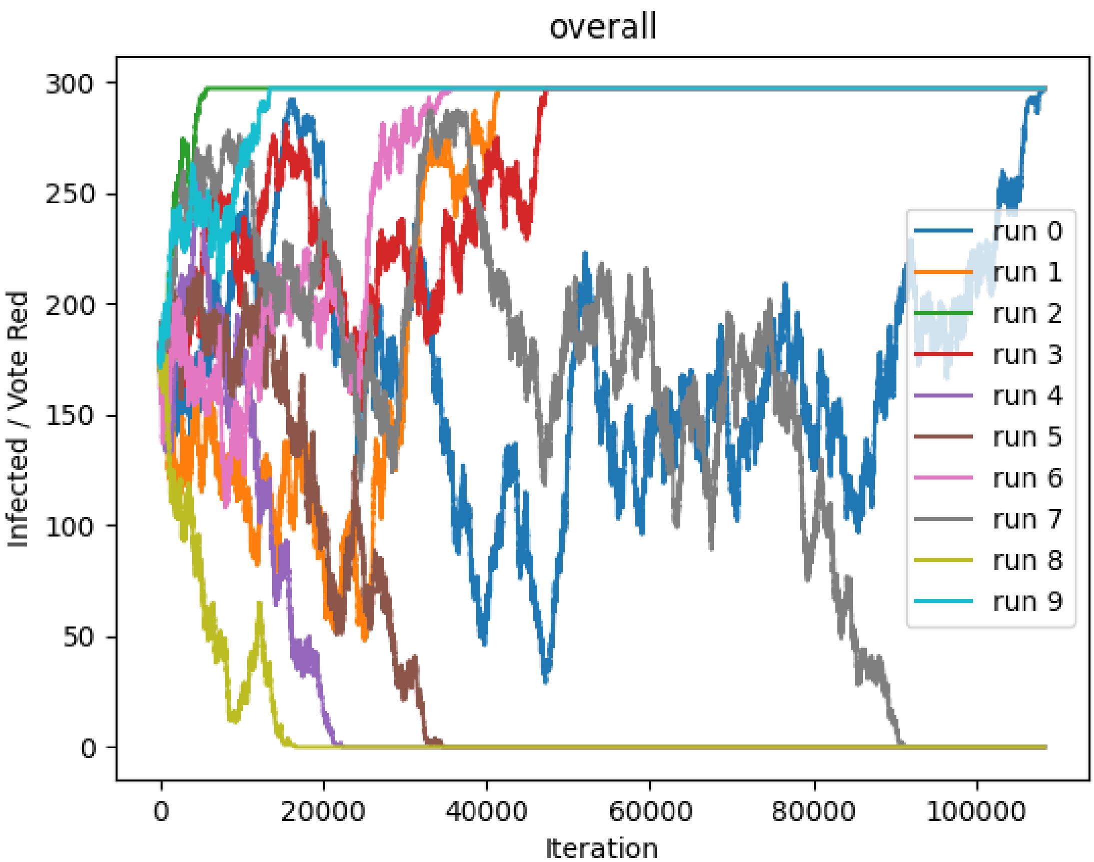


Figure 9: Plot across iterations of the voter model on the graph from 5 with a 60% initialization of red (infected) nodes.

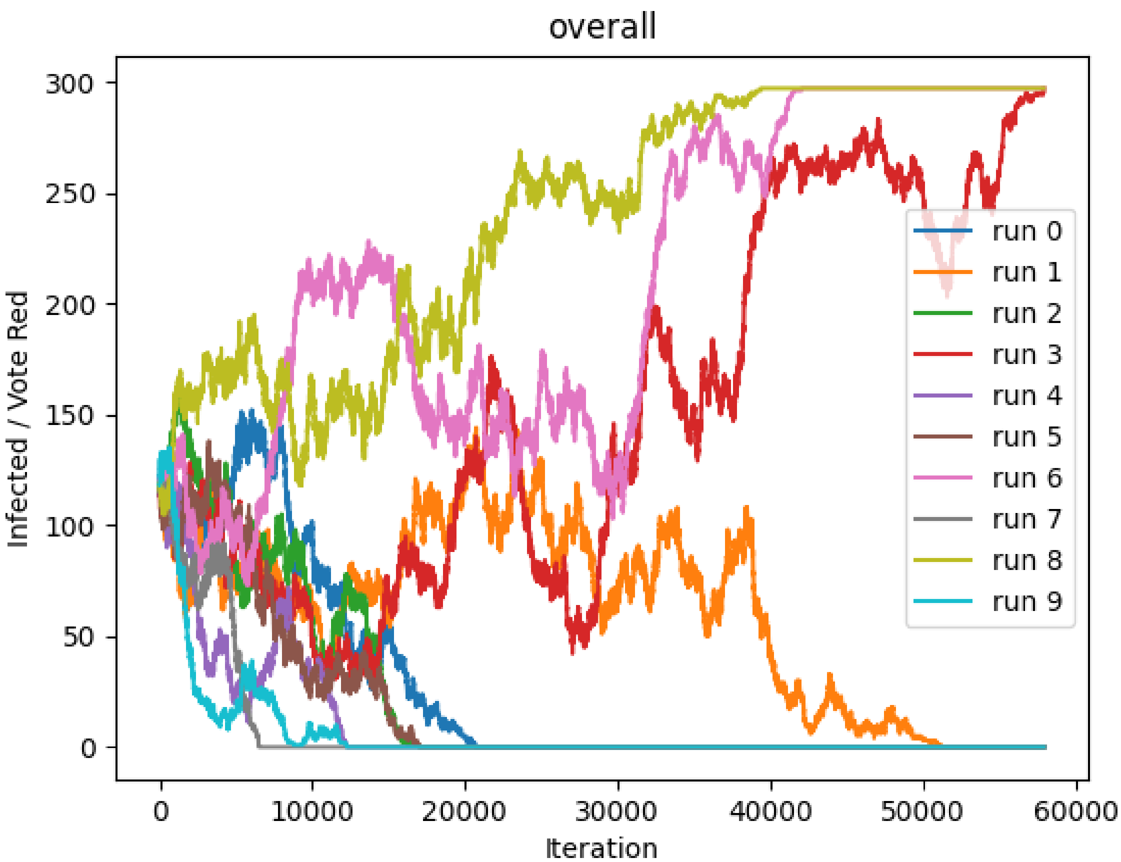


Figure 10: Plot across iterations of the voter model on the graph from 12 with a 40% initialization of red (infected) nodes.

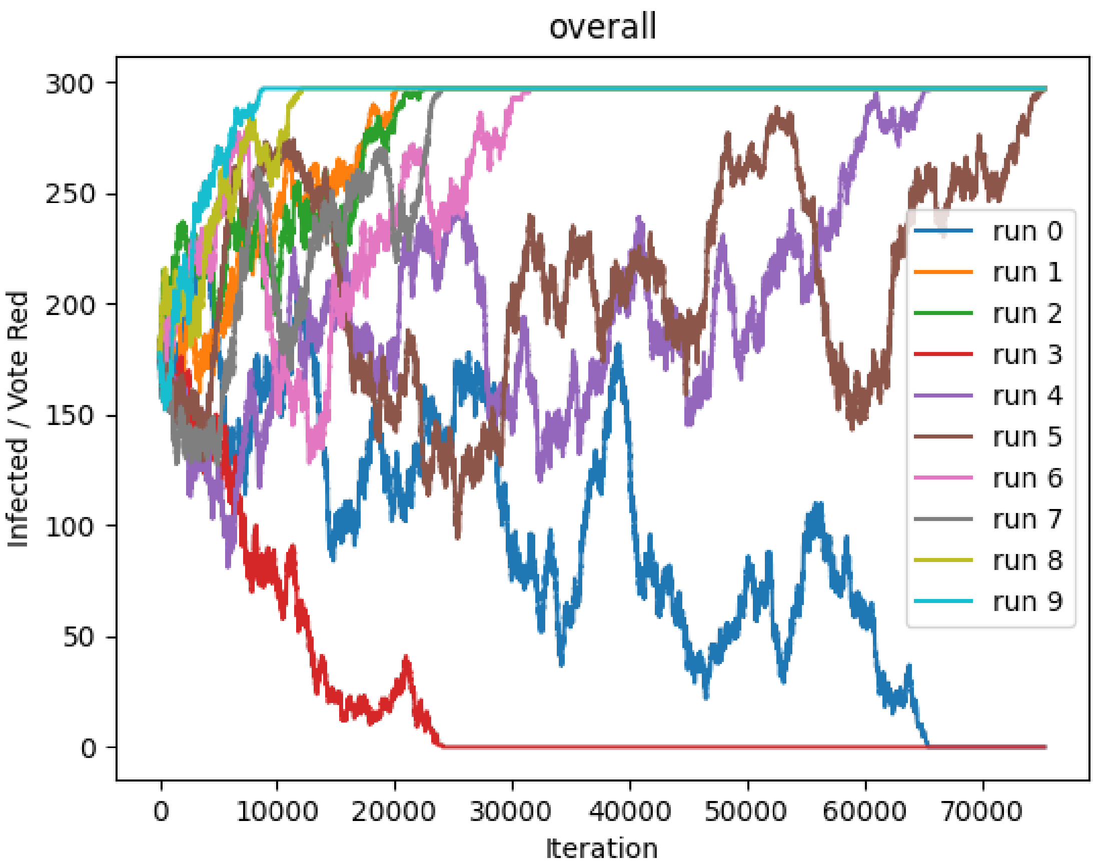


Figure 11: Plot across iterations of the voter model on the graph from 12 with a 60% initialization of red (infected) nodes.

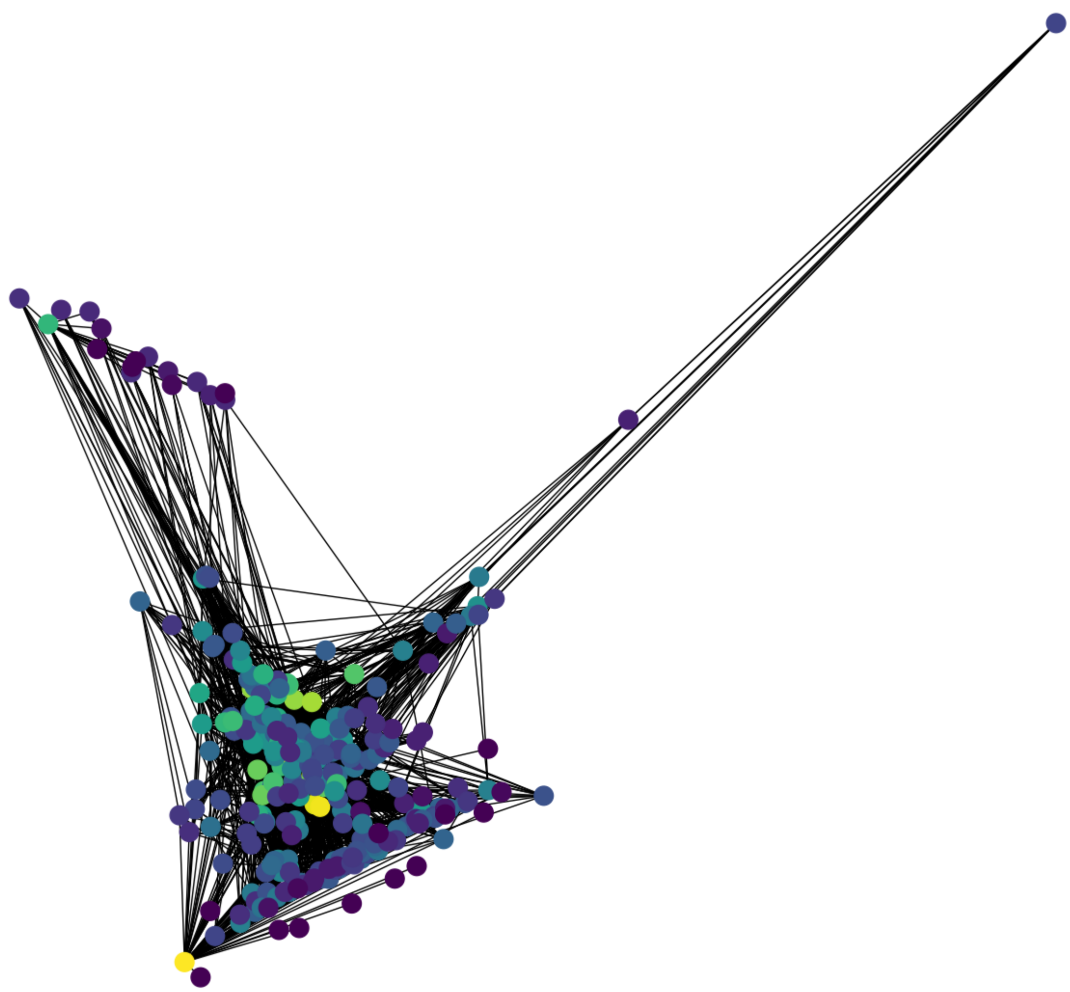


Figure 12: Visualization using the spring layout of Networkx of the configuration technique generated graph with the degree distribution and nodes from the graph in 5.

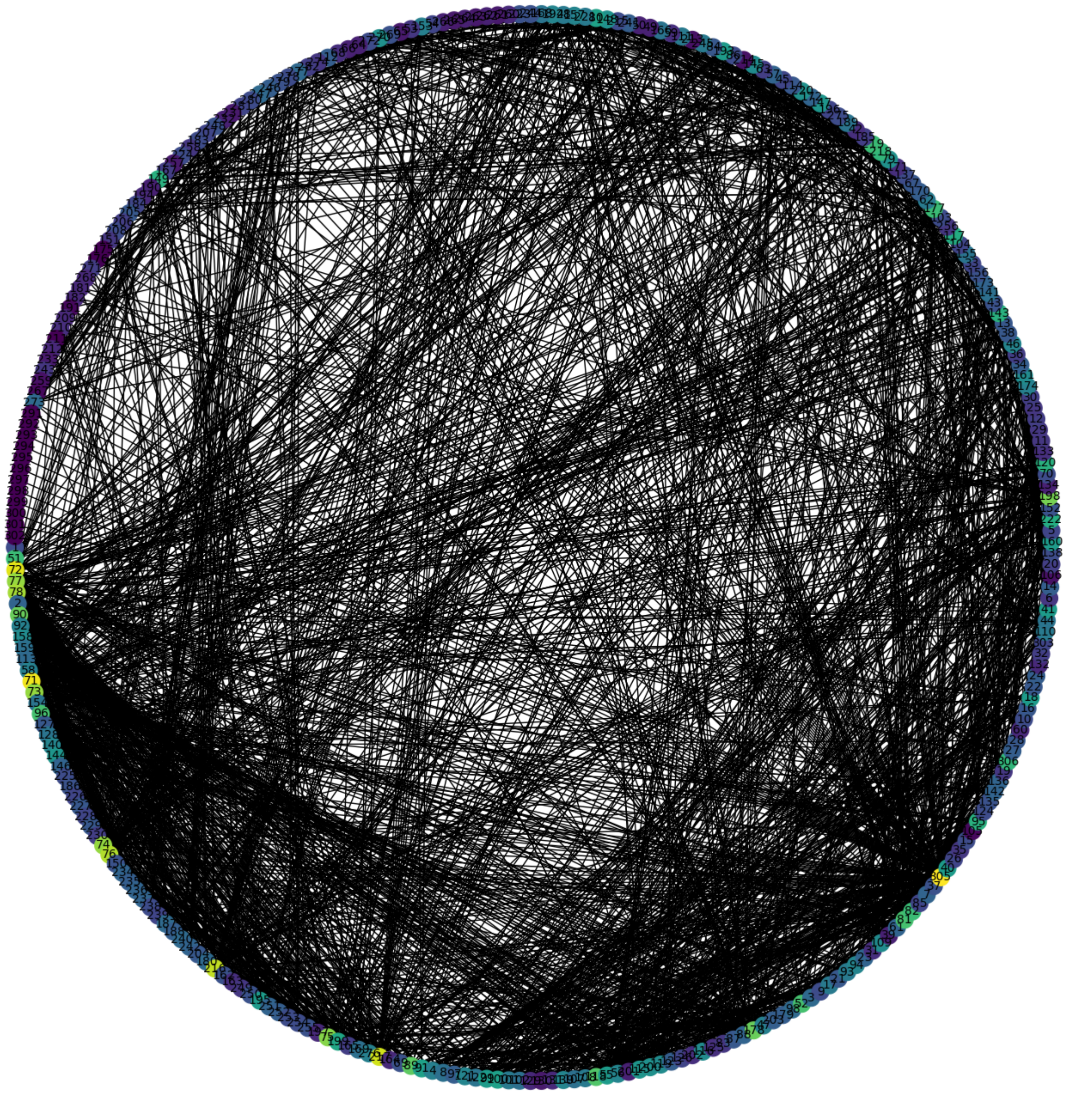


Figure 13: Visualization using the shell layout of Networkx of the configuration technique generated graph with the degree distribution and nodes from the graph in 5.