Lecture

Intro to Bayesian Inference – How to check if a coin is fair?

Everything starts with coins (Bernoulli Random Variable):

Coin flip: $X = \begin{cases} \text{"Heads"} & \text{w/ prob. } p \\ \text{"Tails"} & \text{w/ prob. } q = 1-p \end{cases}$

Prob of getting k heads out of n flips if the probability of 1 head is p is:

$$\Pr(k \text{ heads} \mid p) = \binom{n}{k} p^k (1-p)^{n-k}$$

is $p = \frac{1}{2}$?

We can answer this w/ a hypothesis test:

Assuming coin is fair ($r = \frac{1}{2}$), what is the probability of seeing $\geq k$ heads on n flips:

Ex: $n = 100$, $k = 60$ $\quad$ $p\text{-value} = \sum_{k=60}^{100} \binom{n}{k} \underbrace{r^k (1-r)^{n-k}}_{(\frac{1}{2})^n} = 0.0284$!

Significant! coin is biased...

$n = 1000$, $k = 600 \Rightarrow p\text{-value} = 1.36 \times 10^{-10}$

Let's tackle this problem from the Bayesian perspective.

Bayes Thm:

$P(A, B)$ joint prob. $\quad$ $P(A \mid B)$ conditional prob.

$P(A \mid B) = P(A)$ if A and B are independent.

$P(A) = \sum_B P(A, B)$ marginal prob.

relationship: $P(A, B) = P(A \mid B) P(B)$

$$P(A,B) = P(B,A)$$

$$P(A|B)P(B) = P(B|A)P(A)$$

Bayes Thm: $$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

$$P(A|B) = \frac{P(B|A)P(A)}{\sum_{A'} P(B|A')P(A')}$$

<u>Bayesian Inference</u> is the application of Bayes Thm. to probabilistically link <u>data</u> and <u>models</u>.

"likelihood" "prior probability"

$$P(\text{model}|\text{data}) = \frac{P(\text{data}|\text{model})\,P(\text{model})}{P(\text{data})}$$

"Posterior Prob."

"evidence"
"normalizing constant"
uncommon { "partition function" (physics)
Z zustandssumme (sum over states)

Let's apply this to our coin problem:

prior: $\boxed{f(p)=1}$ → uniform prior, all values of p equally likely

$$Pr(p\,|\,k \text{ heads}) = \frac{Pr(k \text{ heads}\,|\,p)\, f(p)}{\int_0^1 Pr(k \text{ heads}\,|\,r)\, f(r)\, dr}$$

"model" "data"

$$= \frac{\binom{n}{k} p^k (1-p)^{n-k}}{\int_0^1 \binom{n}{k} r^k (1-r)^{n-k}\, dr}$$

$$= \frac{p^k (1-p)^{n-k}}{\int_0^1 r^k (1-p)^{n-k}\, dr} \leftarrow \text{integrate this?}$$

Beta distribution { $$= \frac{p^k (1-p)^{n-k}}{B(k+1, n-k+1)}$$ ← Beta function ↔ or use binomial theorem b/c n,k are integers

$$= \frac{(n+1)!}{k!(n-k)!} p^k (1-p)^{n-k} = (n+1)\binom{n}{k} p^k (1-p)^{n-k}$$

So that's the posterior:

$$\Pr(p \mid k \text{ heads}) = (n+1)\binom{n}{k} p^k (1-p)^{n-k}$$

$\boxed{\Rightarrow \text{Let's plot this} \Rightarrow \text{slides!}}$