Poisson as limit of Binomial

Binomial: $B(n,k) = \binom{n}{k} p^k (1-p)^{n-k}$ (not to be confused w/ Beta function)

If $n \to \infty$, $p \to 0$ s.t. $np \to \lambda$, then $\binom{n}{k} p^k (1-p)^{n-k} \to \frac{\lambda^k}{k!} e^{-\lambda}$.

Proof (weak, assuming $np = \lambda$)

$p = \frac{\lambda}{n} \to B(n,k) = \binom{n}{k} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k}$ now take limit.

$$\lim_{n\to\infty} B(n,k) = \lim_{n\to\infty} \frac{n!}{k!(n-k)!} \frac{\lambda^k}{n^k} \left(1 - \frac{\lambda}{n}\right)^{n-k}$$

$$= \lim_{n\to\infty} \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!\, n^k} \lambda^k \left(1 - \frac{\lambda}{n}\right)^{n-k}$$

The largest term in $n(n-1)(n-2)\cdots(n-k+1)$ is n^k: $n(n-1)\cdots(n-k+1) = n^k + O(n^{k-1})$

The ratio of this and n^k goes to 1 as $n \to \infty$: $\lim_{n\to\infty} \frac{n^k + O(n^{k-1})}{n^k} = 1$

$$\lim_{n\to\infty} B(n,k) = \frac{\lambda^k}{k!} \lim_{n\to\infty} \left(1 - \frac{\lambda}{n}\right)^{n-k} \quad \text{(almost there!)}$$

$$= \frac{\lambda^k}{k!} \underbrace{\lim_{n\to\infty} \left(1 - \frac{\lambda}{n}\right)^{n}}_{= e^{-\lambda}} \cdot \underbrace{\lim_{n\to\infty} \left(1 - \frac{\lambda}{n}\right)^{-k}}_{=1}$$

(This limit is essentially the definition of exponential functions, but see next page for derivations)

$$\therefore \lim_{n\to\infty} B(n,k) = \frac{\lambda^k}{k!} e^{-\lambda}$$

$$e^{-\lambda} = \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{n!} = 1 - \lambda + \frac{\lambda^2}{2!} - \frac{\lambda^3}{3!} + \dots$$

$$\left(1-\frac{\lambda}{n}\right)^n = \sum_{k=0}^{n} \binom{n}{k}\left(\frac{-\lambda}{n}\right)^k = \underbrace{\binom{n}{0}}_{1} - \underbrace{\binom{n}{1}}_{n}\frac{\lambda}{n} + \underbrace{\binom{n}{2}}_{\frac{n(n-1)}{2}}\frac{\lambda^2}{n^2} - \underbrace{\binom{n}{3}}_{\frac{n(n-1)(n-2)}{6}}\frac{\lambda^3}{n^3} + \dots$$

$$= 1 - \lambda + \underbrace{\frac{n^2-n}{2n^2}}_{\lim_{n\to\infty} = \frac{1}{2}}\lambda^2 - \underbrace{\frac{n^3+O(n^2)}{3!\,n^3}}_{\lim_{n\to\infty} = \frac{1}{3!}}\lambda^3 + \dots$$

Series are term-by-term equal when $n \to \infty$

$$\therefore \boxed{\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n = e^{-\lambda}}$$

$$\ln\left(\left(1-\frac{\lambda}{n}\right)^n\right) = n\ln(1-\lambda n^{-1})$$ take log (Another derivation)

$$= \frac{\lambda \ln(1-\lambda n^{-1})}{\lambda n^{-1}}$$ mult. by $\frac{\lambda/n}{\lambda/n}$

note that:
$$\lim_{x\to 0}\frac{\ln(1-x)}{x} = -1$$ and: $\lambda n^{-1} \to 0$ as $n \to \infty$.

Use λn^{-1} instead of x : $$\lim_{n\to\infty}\frac{\lambda \ln(1-\lambda n^{-1})}{\lambda n^{-1}} = \lambda \lim_{n\to\infty}\frac{\ln(1-\lambda n^{-1})}{\lambda n^{-1}} = \lambda(-1) = -\lambda$$

$$\Rightarrow \lim_{n\to\infty}\ln\left(\left(1-\frac{\lambda}{n}\right)^n\right) = -\lambda$$

Last, exponentiate both sides:

$$\boxed{\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n = e^{-\lambda}}$$