# Matching in Multi-Agent Pathfinding using M*

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## Abstract

Multi-agent pathfinding (*MAPF*) is the process of finding collisison-free paths for multiple agents. (*MAPF*) can be extended by assigning agents to teams. In a team, agents need to be assigned (or matched) to a team goal such that the *sum of individual costs* is minimised. This extension is called *MAPFM*. *M** is a complete and optimal algorithm to solve *MAPF*. In this paper, a modification is proposed to *M** to allow it to solve *MAPFM* problems as well. To accomplish this, two strategies are proposed, called *inmatching* and *prematching*. In this paper, it is shown that *prematching* is generally preferable to *inmatching*, the benefits of different optimisations for *M** are compared, and it is shown how well *M** stacks up against other *MAPF* solvers extended to perform matching.

## 1. Introduction

A large number of real-world situations require the planning of collision-free routes for multiple agents.[1] For example, the routing of trains over a rail network [1], directing robots in warehouses [2] or making sure autonomous cars do not collide on the road [3]. Problems of this nature are called *Multi-agent pathfinding* problems, which will hereafter be abbreviated to *MAPF*.

In *MAPF*, each agent has a starting position and a goal position. For every agent, a route needs to be found from their start to their goal, without two agents colliding. Finding these collision-free routes has been proven to be *NP-hard* [4].

One algorithm to solve *MAPF* is called *M** [5]. *M** is derived from the *A** algorithm [6] as described by Standley [7]. *A** generally plans the paths of agents together. This means that in each timestep, the number of possible next states grows exponentially with the number of agents. Contrasting that, in *M**, agents follow an individually optimal path whenever possible. In each timestep, only the subset of colliding agents is jointly planned.

When directing robots through warehouses, there may be different models of robots with different capabilities. Some robots may be able to restock shelves while others collect orders. The *MAPF* problem can be generalised to represent such a problem with teams of agents. In contrast to *MAPF*, an agent does not have a single goal in this generalisation. Instead, agents can use any goal associated with the team it is in, but no two agents can end at the same goal. An algorithm solving this problem must assign each agent to a goal. Such an assignment is called a matching. This problem is therefore named multi-agent pathfinding with matching (hereafter abbreviated to *MAPFM*).

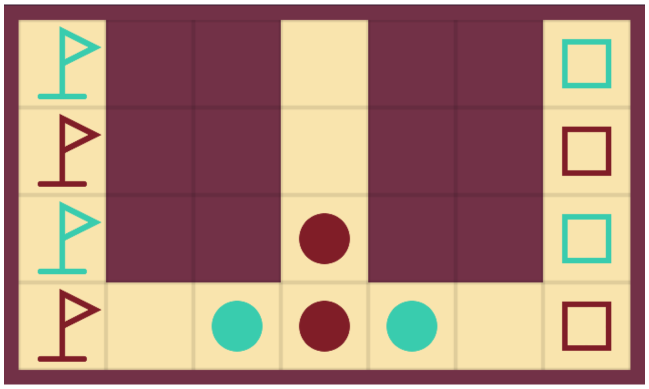
Figure 1: A snapshot of agents solving a *MAPFM*. Two teams of agents are moving from their starts to their goals. Squares, flags and circles are starts, goals and agents respectively. One red agent (at 4 squares from the left, 3 squares from the top) waits to let the three other agents pass.

In this paper, *MAPFM* is first defined, after which two

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[1]An agent is the collective name for a single unit for which a path is calculated. The word 'agent' could refer to a single robot, a train or a car, but can also refer to an abstract or simulated entity. In MAPF, paths are planned for multiple agents.

techniques are discussed which extend *M** to also solve *MAPFM* problems. These two techniques are then compared, both to each other, and to a number of other algorithms which also solve *MAPFM* problems. Apart from this comparison, a number of extensions to *M** are introduced which improve *M**'s performance. Some of these extensions can also improve the performance of *M** on plain *MAPF* problems.

## 2. Prior work

Before this research, a separate study has been performed [8] on a problem called the target-assignment and pathfinding (*TAPF*) problem. The difference between *TAPF* and *MAPFM* is that in *TAPF*, the *makespan* (the cost of the longest path) instead of the *sum of individual costs* (the cost of all paths combined) is minimised. To solve *TAPF*, [8] uses conflict based search (*CBS*) [9], using a min-cost flow based algorithm for the lower level. The min-cost flow is used to solve *TAPF* for a single team in polynomial time. The higher level of *CBS* combines single team solutions to find a solution for all teams. In parallel research to this paper, [10] shows that such a min-cost flow based algorithm can also be used to solve *MAPFM* problems.

To the best of the author's knowledge, solving *TAPF* or *MAPFM* with other algorithms than *CBS* (such as *M**) has not yet been explored.

In contrast, a lot of research has been done finding and improving algorithms which solve *MAPF* [5, 7, 9, 11–14]. The current state-of-the-art *MAPF* algorithms are *Conflict Based Search* (*CBS*) and *Branch-and-Cut-and-Price* (*BCP*). However, which of the existing algorithms for *MAPF* can be adapted such that they also perform well at solving *MAPFM* problems is yet unclear.

## 3. Definitions of *MAPF* and *MAPFM*

The definition of multi-agent pathfinding used in this paper is based on Stern's definition given in [15]. The definition is given below.

A *MAPF* problem $P$ consists of the following elements:

$$P = \langle G, s, g \rangle$$

- $G$ is an undirected graph $\langle V, E \rangle$
  - $V$ is a set of vertices
  - $E$ is a set of unweighted edges between vertices
- $s$ is a list of $k$ vertices where every $s_i$ is a starting position for an agent $a_i$
- $g$ is a list of $k$ vertices where every $g_i$ is a goal position for an agent $a_i$

The goal of *MAPF* is to find a path for all agents $a_i$ from $s_i$ to $g_i$, without vertex and edge conflicts. This means that in a timestep $t$, two agents may not be on the same vertex, and between two timesteps, two agents may not travel over the same edge.

The cost $c_i$ of a path $\pi_i$ is the number of timesteps until the last time that agent $a_i$ arrives at its goal $g_i$. [15] defines two primary approaches to define the cost $c$ of a solution to a *MAPF* instance. The *sum of individual costs* is the sum of the cost of all paths ($c = \sum_{n=0}^{k} c_n$). Alternatively, the *makespan* is equal to the maximum cost of all paths ($c = \max_{n=0}^{k} c_n$). An optimal solution to a *MAPF* instance is a set of paths $\pi$ with the smallest possible cost $c$.

Although the algorithms presented in this paper would work on any graph $G$, in examples and experiments, $G$ is simplified to a 4-connected grid. In this paper, this grid is sometimes called a 'map', on which agents move from their start positions to their goal positions.

### *MAPFM*

Now, the concept of matching is added to the definition of *MAPF*, to create multi-agent pathfinding with matching (*MAPFM*). The new *MAPFM* problem adds two elements to $P$, as shown below:

$$P' = \langle G, s, g, sc, gc \rangle$$

- $sc$ is a list of colours. Each starting vertex $s_i$ is assigned a colour $sc_i$.
- $gc$ is a list of colours. Each starting vertex $s_i$ is assigned a colour $sc_i$.

This definition divides all agents into teams. An agent $a_i$'s team colour is the colour of its starting location $sc_i$. In total there are $K$ teams $k_1 \dots k_n$. $K$ is equal to the number of different colours in $sc$ and $gc$.

In *MAPFM*, agents do not have a single goal. Instead, teams have several goals all with the same colour $gc_i$. A team has as many goals as there are agents. A solution to *MAPFM* is a set of paths for each agent, in which every agent travels to a goal with a colour equal to the colour of that agent.

*MAPFM* uses the *sum of individual costs* as an optimisation criterion.

## 4. A description of *M**

*M** [5] is a complete and optimal pathfinding algorithm, similar to *A**. However, *M** is specifically designed for *MAPF*. In *M**, the search space consists of the combination of positions of all the agents on a grid. To use *A** for *MAPF*, the planning of paths for all agents needs to be coupled to make sure no states are expanded in which collisions occur. Coupled planning means that the states used in *A** contain the positions of all agents at the same time, and every expansion expands new states for all agents at the same time.

Unlike in *A**, in *M** the planning of agents is initially not coupled. Instead, *M** assumes that the optimal path for an agent to their goal does not contain any collisions with other agents. As long as this is true, the planning of agents is separated.

However, the assumption that all optimal paths are collision-free obviously does not always hold. Therefore, each state in *M** also holds, apart from the positions of each agent, two sets called the *collision set* and the *backpropagation set*. When *M** detects that a state contains collisions, the algorithm does not continue expanding this state. It instead uses information stored in these two sets to backtrack and find the shortest route around these collisions. Agents associated with the collision temporarily plan routes in a coupled fashion.

After collisions have been circumvented, *M** plans agents independently again according to their individually optimal path. It does, however, record information about the previous collisions (in these *backtracking-* and *collision sets*). This is to make sure that when in the future another collision occurs, it can either resolve this collision locally or backtrack back to the previous collision to circumvent this old collision differently, potentially avoiding the new collision altogether.

Just like *A**, *M** uses a priority queue and a heuristic to prioritise the exploration of states. This finds optimal solutions as long as an admissible heuristic is used such as the Manhattan- or Euclidean distance to the closest goal location for each agent.

## 5. *M** and matching

To add matching to *M**, this paper proposes two options which are called "inmatching" and "prematching". In this section, both are explained and their advantages and disadvantages are discussed.

### 5.1. Inmatching

*Inmatching* is the process of performing matching as a part of the pathfinding algorithm that is used. To understand it, it is useful to first look at inmatching in *A**. With *A**, the expansion of a state consists of all possible moves for all agents. *A** searches through the search space, until the goal state has been removed from the frontier. With an admissible heuristic, *A** guarantees that backtracking from this first goal state gives a shortest path between the start and goal location.

With *inmatching*, there is not one goal state. Instead, any state in which all agents stand on a goal of their own colour is considered a goal state. This means there is more than one goal state. An admissible heuristic for *inmatching M** is the distance to the nearest goal state.

#### Inmatching applied to *M**

*Inmatching* can similarly be used with *M**. However, there is something that makes it much less efficient in *M** than in *A**, the implications of which will be evaluated in Section 8.1.

One of the core ideas of *M** is that agents, when not colliding, follow an individually optimal path to their goal. An individually optimal path is a path for an agent to a goal with minimal cost. However, an individually optimal path ignores other agents. Therefore the actual path which *M** finds for that agent may become longer when collisions occur.

In *MAPF*, an agent has a single individually optimal path because an agent only has one goal it can travel to. Therefore, as long as all agents follow this individually optimal path (and do not collide), the branching factor [2] of *M** is 1.

However, in *MAPF*, agents commonly have multiple goals positions. With *inmatching*, agents need to consider paths to every goal position to ensure optimality. Consequently agents also have multiple individually optimal paths, one to travel to each goal position of the agent's team. Thus, with *inmatching*, the branching factor of *M** may be larger than one even when there are no collisions.

The upper bound of this branching factor can be expressed as follows:

$$\prod_{i=1}^{k} \text{goals}(a_i)$$

Here, the *goals* function gives the number of goals in team of agent $a_i$. In *M** states, the position of all agents is combined. The final branching factor is thus the cross product of paths for each agent.

*inmatching M** will hereafter be abbreviated to *imM**

### 5.2. Prematching

**Algorithm 1:** prematch *M**

```
Result: Find the matching with smallest cost
matchings ← find all matchings($starts, goals$)
foreach $m \in matchings$ do
    $S(i) \leftarrow \text{mstar}(starts, m)$ {Evaluate with M*}
end
return min(S) {by calculated cost}
```

As an alternative to *inmatching*, there is *prematching*. With *prematching* the *MAPFM* problem is transformed in a number of *MAPF* problems. Each possible matching is calculated in advance, and normal *M** as described by [5] is performed on each matching exhaustively. This exhaustive search is shown in algorithm 1.

In Section 6.3 an extension to *prematching M** is proposed which uses a heuristic to prune some of the matchings which otherwise needed to be evaluated.

*prematching M** will hereafter be abbreviated to *pmM**

## 6. Extensions to *M**

*M** can be improved upon in various ways. Some of these extensions are only applicable to *M** with match-

[2]The branching factor is the number of child states which are created from one parent state. A branching factor of 1 means that each parent only creates one child state. In this text, a state refers to an *M** state as described in Section 4.

ing, while others improve *M** itself.

In this section these extensions are described in detail.

### 6.1. Precomputed paths and heuristics

A part of solving *M** is finding the individually optimal path for each agent. This can be done with conventional pathfinding algorithms such as *A**. But to avoid repeated calculation, it is important to first create a lookup table of the minimal distance to each goal, for every open square on the map. With this table, finding in which way an agent should move to reach the goal in an individually optimal manner becomes a constant time operation.

When using *prematching*, the same goal locations and start locations are (although in a different permutation) reused multiple times. The lookup table can remain the same for all evaluations of matchings.

Using this lookup table, the heuristic of *M** can be improved as well. Instead of using the *euclidean-* or *Manhattan distance* to the goal, the lookup table can be used to find the exact distance. The heuristic found with this approach also takes the obstacles on the map into account.

### 6.2. Operator decomposition

In [7], Standley describes a technique called operator decomposition (*OD*). Operator decomposition is designed to be an extension to *A** to improve the runtime of solving of *MAPF* problems.

In *MAPF*, when an agent moves, it can perform one of five actions (assuming a grid map is used): wait on the same square, or move in one of the four cardinal directions. With *A**, the movement of all agents is coupled. When a state is expanded, every child state contains the new position of every agent. This means that if there are 5 agents, which may all perform one of 5 possible actions, $5^5 = 3125$ child states are created and added to the frontier. This is called a full expansion.

Operator decomposition can lower this large branching factor by introducing the concept of partial states. As the name implies, when a state is removed from the frontier, it is not fully expanded. Instead, only the 5 possible actions of a single agent are evaluated. These 5 partial states are added to the frontier. When a partial state is removed from the frontier, the actions of the next agent are evaluated too until the actions of all agents are evaluated and a full expansion is reached again.

So in the worst case, *OD* still performs a full expansion. However, the advantage of *OD* is that partial states can also be prioritised in the frontier. When the expansion of a subset of agents is found to result in a low heuristic, the partial state is given a high priority to be further expanded. This may lead to finding solutions to *MAPF* instances more quickly.

Even though *M** tries to avoid the coupled planning of agents as much as possible, when collisions occur, branching factors can still grow large. [16] showed that as a result *M** can also benefit from *OD*. In Section 8.2 the benefits *OD* has on *M** when solving matching problems is evaluated.

### 6.3. Pruning of matchings

**Algorithm 2:** prematch *M** with pruning

```
Result: Find the matching with smallest cost
        without evaluating every matching.
$m_{best} \leftarrow \varnothing$

$M \leftarrow$ find all matchings($starts, goals$);
sort($M$) {on heuristic; ascending}
foreach $m \leftarrow M$ do
    if $heuristic(m) < cost(m_{best})$ then
        $s \leftarrow$ mstar($starts, m$);
        $m_{best} \leftarrow min(m_{best}, s)$;
    end
end
```

*pmM**, as previously described, has to evaluate every matching of every team. However, it is possible to discard some matchings without evaluating them with *M** at all, by using heuristics.

To do this, the heuristic (i.e. the sum of distances between start and goal locations) of the initial state of every matching is calculated. Because this heuristic is admissible, it represents a lower bound for the cost of this matching.

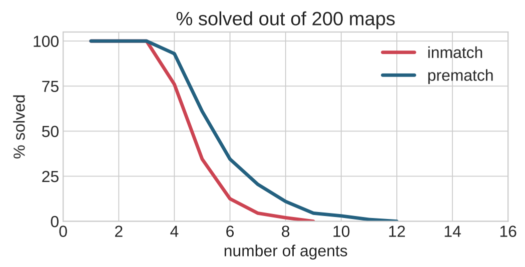
Then, when *prematching* evaluates every matching $m$, it keeps track of the best matching $m_{best}$ which is the matching with best cost so far. However, when heuristic($m$) $\geq$ $cost$($m_{best}$), $m$ can immediately be pruned, because this matching $m$ can never yield a solution which has a lower cost than $m_{best}$.

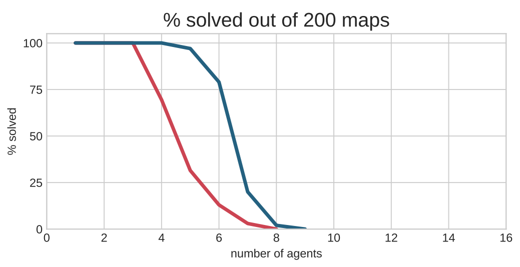
**Sorting**
To take maximum advantage of pruning, matchings can be sorted based on their heuristic. Making sure the matching with the lowest heuristic is evaluated first can increase chances that later matchings are pruned. Algorithm 2 shows how pruning with sorting works.

## 7. Experimental setup

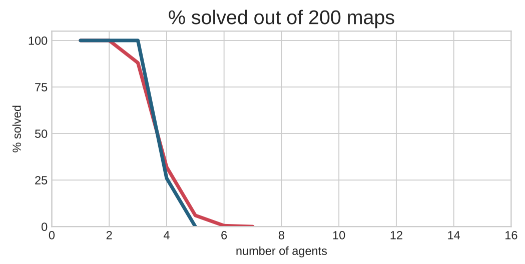
To evaluate the performance of matching *M**, a number of experiments were performed. For each of these experiments, the algorithm used is written in Python 3.9 and benchmarks were run on a virtualized system with a 12 core Intel Xeon E5-2683 running at 2GHz, which has 8Gib of RAM.

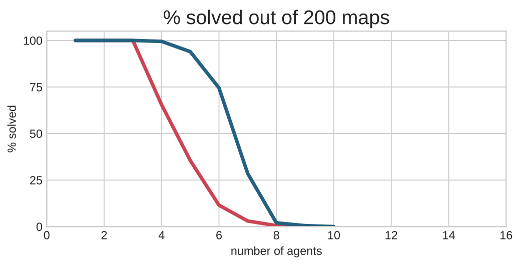
There are various factors deciding how a *MAPF* solving algorithm performs in benchmarks. Three important factors are the total number of agents for which paths need to be found, the number of teams over which the





(a) Agents split over 3 teams. Left: 75% wall, right: 25%wall





(b) All agents in 1 team. Left: 75% wall, right: 25%wall

Figure 2: Percentage of maps solved in 2 minutes out of 200 random $20 \times 20$ maps.

agents are distributed, and the layout of the maps on which benchmarks are run.

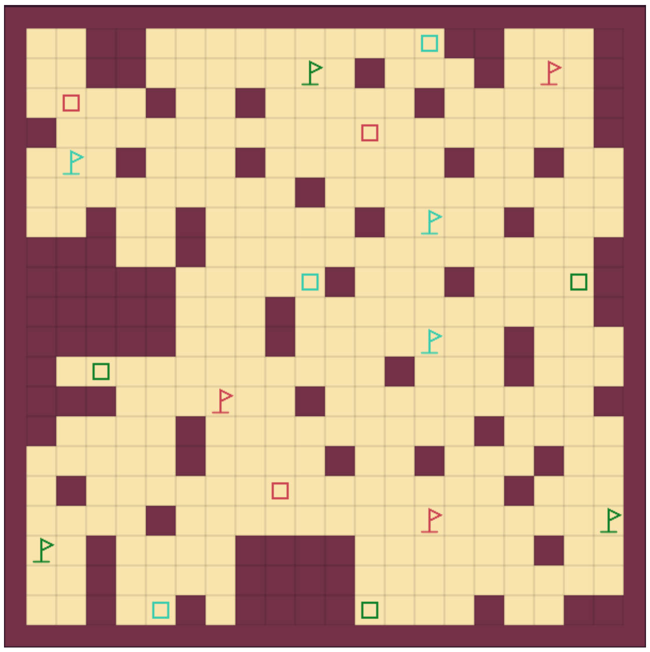


Figure 3: A map which is 25% obstacle, with agents distributed in three teams. This map was one of many used in the performed experiments.

The experiments are grouped into sets of four *scenarios*, to show the differences in performance as these parameters vary. Performance is assessed on maps where either 25% or 75% of the grid is an obstacle, and in situations where agents are grouped into either 1 or 3 teams.

For these benchmarks, $20 \times 20$ grid maps[3] are randomly generated such that they are guaranteed to be solvable (a proof of which can be found in A). Four sets of maps were generated for each experiment with the previously discussed parameters. Agents and goals are uniformly distributed on this map and teams are assigned randomly. When the team size does not divide the number of agents, there will be one smaller team. An example of such a randomly generated map can be found in Figure 3.

Because *MAPFM* is an *NP-Hard* problem, it is possible that finding the solution to an instance of the problem takes an unreasonable amount of time. Therefore, in each experiment, a timeout of 2 minutes is used. Experimental results show the percentage of maps (out of a set of 200) which the algorithm manages to solve within this timeout.

## 8. Results and Discussion

In this section, the results of a number of experiments is shown. The purpose of this is firstly to show which of the two matching strategies is superior, and why this is the case. Secondly, the experiments quantify the benefits of different extensions to *pmM**. Lastly, the results of the experiments are put into context by comparing *M** to other *MAPFM* algorithms performing the same experiments.

### 8.1. Matching strategies

In Section 5, two strategies were proposed for adding matching to *M**. Both strategies were tested following the experimental setup described in Section 7. In Figure

[3]Maps of different sizes will likely also show differences in performance. However, experiments with other map sizes are not included in this paper.

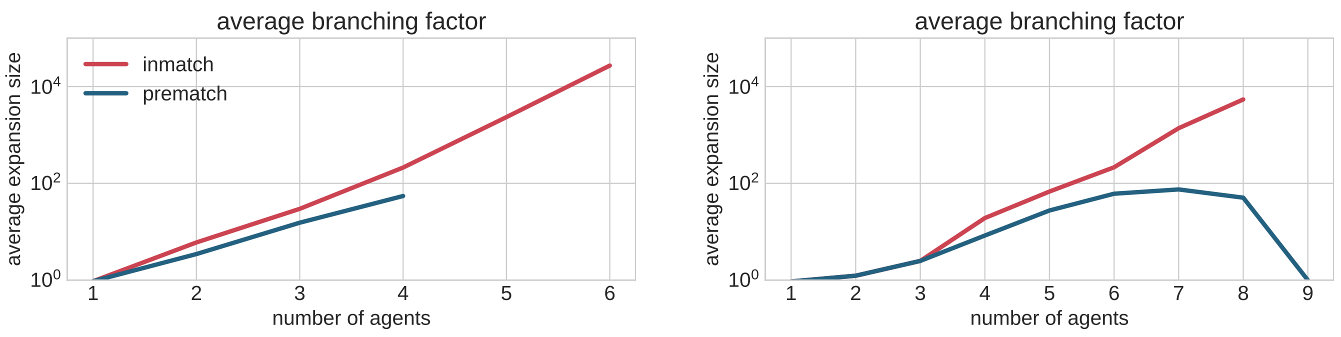


Figure 4: Average branching factor on the same maps as used in figure 2b (i.e. all agents in one team). Left: 75% wall, right: 25% wall. The blue line in the left figure stops earlier because *prematching* was not able to solve maps with more agents.

2 the results of this experiment are shown.

From these graphs, it can be seen that *prematching* is generally superior to *inmatching* in all but one of the scenarios.

**Branching factor**

In Section 5.1, it was hypothesised that this difference in performance could occur, because of the larger branching factor.

To demonstrate this is indeed the case, another, separate, experiment was performed. Again on 200 random maps as described in Section 7, but this time showing the average number of states expanded each expansion.

In Figure 4, on a logarithmic scale, the outcomes of this experiment are shown. Indeed, on average, inmatching expands many more states.

**When *inmatching* is better**

However, in one of the scenarios in Figure 2b *inmatching* does seem to perform slightly better than *prematching*. In this benchmark, all agents are in a single team, and the map is 75% filled with obstacles.

When agents are all in one team, there are more matchings compared to when agents are separated into multiple teams. With *prematching*, each matching needs to be exhaustively searched, which is slow.

In contrast, the problem of *inmatching* expanding so many states is reduced in this scenario. There are fewer directions for agents to go because of the large number of obstacles.

So in general, when there are many matchings, but few options for agents to move in, *inmatching* can perform better.

## 8.2. Extensions to $M*$

Various extensions have been proposed in Section 6. To show the benefits of each extension, they were compared following the experimental setup described in Section 7.

In Figure 5, the performance of these extensions is shown. Each line in the graph shows a new extension added to all of the previous extensions. For example the line displaying operator decomposition also uses precomputed heuristics and all other previous extensions.

Because in Section 8.1, *prematching* was observed to generally perform better, only the *prematching* matching strategy is used in this experiment.

**Pruning and sorting**

In the graphs where 25% the maps is filled with obstacles, pruning and sorting have a large impact on performance. However, in graphs where 75% is a wall there is barely a difference at all. The heuristic used to prune, is the Manhattan distance between the agent start locations and the closest goal. This does not take into account the obstacles in the map.

Pruning is only possible when the heuristic is larger than the best matching found so far (as described in Section 6.3). This does not happen often with this inaccurate heuristic. Therefore, using the precomputed heuristic (also shown in Figure 5) makes a large difference.

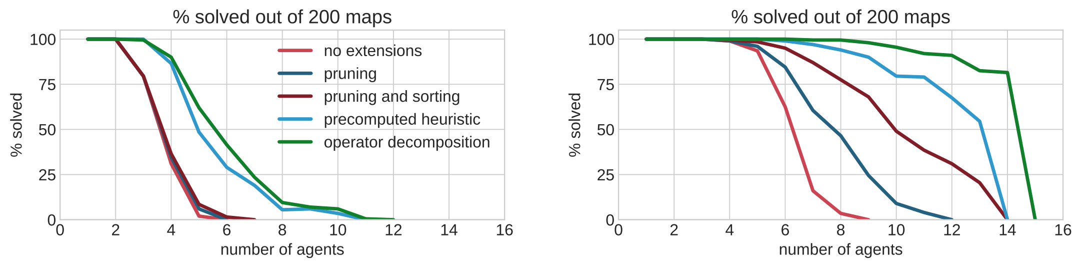
When there are fewer obstacles in the map, the probability of pruning is much higher, explaining the difference in performance of pruning and sorting between these two scenarios.

**Memory usage**

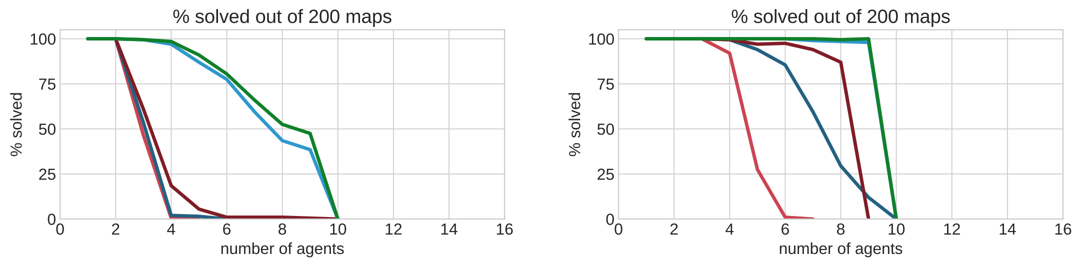
The best variant of $M*$ can be seen solving maps with 14 agents in Figure 5a. After this, the percentage abruptly drops to 0. This is not caused by the complexity of the problems, but rather by the memory that $M*$ uses here. Attempts with maps with 15 agents saw memory usages of over 8GiB to solve a single instance, which caused attempts to be stopped by the operating system.

## 8.3. Comparison with other algorithms

This research is part of a set of parallel studies on how to extend a collection of *MAPF* algorithms with matching. All these studies used exactly the same problem definition but using one of the following base algorithms:



(a) 3 teams. Left: 75% wall, right: 25%wall



(b) 1 team. Left: 75% wall, right: 25%wall

Figure 5: Percentage of maps solved in 2 minutes out of 200 random $20 \times 20$ maps. Each line and extension to *M** is graphed in combination with all previous extensions. For example, the line displaying *operator decomposition* also uses precomputed heuristics and all other previous extensions.

- Extended partial expansion *A** (*EPEA**) [12] (implementation and research by [17])
- *A** with operator decomposition and independence detection (*A*+OD+ID*) [7] (implementation and research by [18])
- Increasing cost tree search (ICTS) [14] (implementation and research by [19])
- Conflict-Based Min-Cost-Flow (CBM) [8] (implementation and research by [10])

Experiments were performed following the experimental setup described in Section 7. All algorithms were benchmarked on the same computer to ensure a fair comparison. Still, it is hard to compare algorithms well because, for example, *CBM* made use of external libraries written in *C++*. But, compared to the exponential nature of *MAPF*, such a strategy providing a linear speed-up is relatively insignificant.

The results of this comparison can be found in Figure 6.

In Figure 6b, it may look like an error was made. *CBM* solves 100% of the maps up to maps with 25 agents. In fact, *CBM* is able to solve maps with many more agents. *CBM* makes use of min-cost flow to solve *MAPFM* for single teams in polynomial time. Solutions for single teams are checked, and modified when conflicts are found between agents in different teams.

But, this means that in cases where there *is* only one team, *CBM* is able to solve *MAPFM* in polynomal time, scaling linearly with the number of agents for which a route needs to be found.

In the experiments where multiple teams were used (as shown in Figure 6a), *CBM* performs more comparably with other algorithms. Still, when there are few obstacles, *CBM* excels.

Apart from *CBM*, all other algorithms (including *M**) show very similar performance characteristics. All these algorithms use a form of prematching, where all matchings are exhaustively searched. This seems to be a factor which limits the capabilities of all these algorithms.

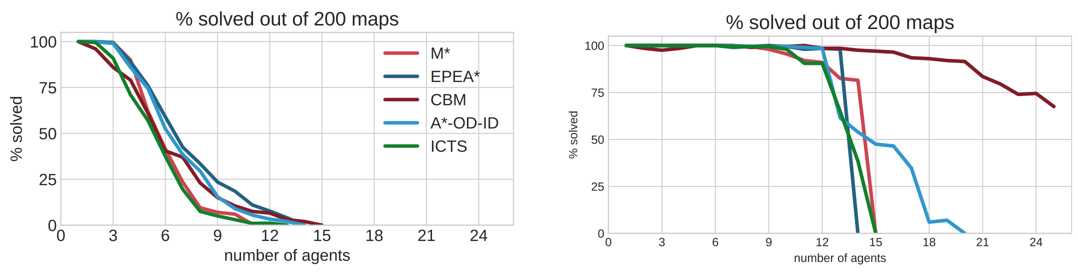
Only *A*+OD+ID* performs exceptionally well on maps 25% filled with obstacles, and with 3 teams. The reason for this is likely due to the way independence detection works in the implementation described in [18].

It must however be said, that this comparison is **not** complete. For example, testing with agents split over more teams, on larger maps, or with specifically crafted obstacles such as long corridors may show very different results.

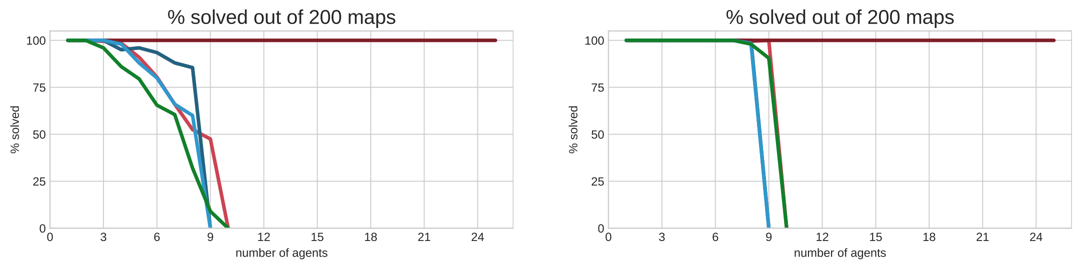
## 9. Conclusion

In this paper, a generalisation of multi agent pathfinding (*MAPF*) was introduced, called matching. This new problem is called *MAPFM*. *M** an algorithm for *MAPF*, was modified to solve these *MAPFM* problems. To do this, two strategies were explored called *inmatching* and *prematchings*. Experimental results showed that in many cases, *prematching* is superior to *inmatching*.

Subsequently, several improvements to *prematching M** were considered and their benefits were experimentally evaluated. Pruning and sorting of matchings, using pre-



(a) 3 teams. Left: 75% wall, right: 25%wall



(b) 1 team. Left: 75% wall, right: 25%wall. In the right graph, the lines for *EPEA** and *A*+OD+ID* overlap, and so do the lines for *ICTS* and *M**.

Figure 6: Percentage of maps solved in 2 minutes out of 200 random $20 \times 20$ maps.

computed heuristics, and independence detection were shown to have a large effect on performance.

Finally, it was shown that *pmM** has very similar performance characteristics as *ICTS*, *EPEA** and *A*+OD+ID*. *CBM* performed notably better than *M** in experiments performed in this paper.

## 10. Future work

### Partial Expansion

One problem with the *inmatching* strategy presented in Section 5.1 is that the branching factor is very large. However, a lot of the expanded states are never needed or accessed again. Even with *prematching* this was a problem. For example, while creating the graphs in Figure 5, the data points where 14 or 15 agents were involved could use as much as 8GiB of memory.

*Partial-* or *enhanced partial expansion* as presented in [12] and [13] only expands states when they are needed to reduce memory usage. It is likely that partial expansion can also be applied to *M** to improve its memory performance.

### Conflict based search

In this paper, matching *M** was compared to several other algorithms. One of those was CBS [9]. However, this implementation used a min-cost flow based approach similar to the one described in [8], but altered to find the *sum of individual costs* instead of the *makespan*.

*CBS*, without matching and these min-cost flow extensions, is one of the most studied algorithms for *MAPF* ([9, 20–23]). Comparing *M** with *inmatching* and *prematching* to *CBS* using *inmatching* and *prematching* as well, could provide useful data.

### Waypoints

In previous research [24–27], *MAPF* was extended with waypoints instead of matching. It may be possible to combine these two extensions. For example, each team may have a set of waypoints. It does not matter which agent visits which waypoint as long as all waypoints are visited. Alternatively, each agent has their own waypoints, but the goals are still shared with a team of other agents.

### Agents and goals

In the definition of matching presented in this paper, in each team the number of goals was always equal to the number of agents. However, in real world scenarios this may not always be the case.

For example, there may be a number of robots who have a number of tasks to do. Some robots cannot perform certain tasks so robots are teamed. But there may be more tasks than robots in each team, and robots will need to prioritise. Some research has already been done in this area, but this uses *TAPF* as a basis [28].

### Recursive M*

In [5], an extension to *M** is described called recursive *M**. It could increase the performance of *M** on regular *MAPF* problems. Whether recursive *M** makes a difference combined with matching, has not yet been verified.

## 11. Reproducibility

Results in this paper have been generated using an implementation of $M*$ made in Python specifically for this research. The code for this, together with the maps and raw experimental results, are publicly available on Github at https://github.com/jonay2000/research-project, doubly licensed under the Apache 2.0 and MIT licenses. Reports of bugs and new additions to this repository are always welcome.

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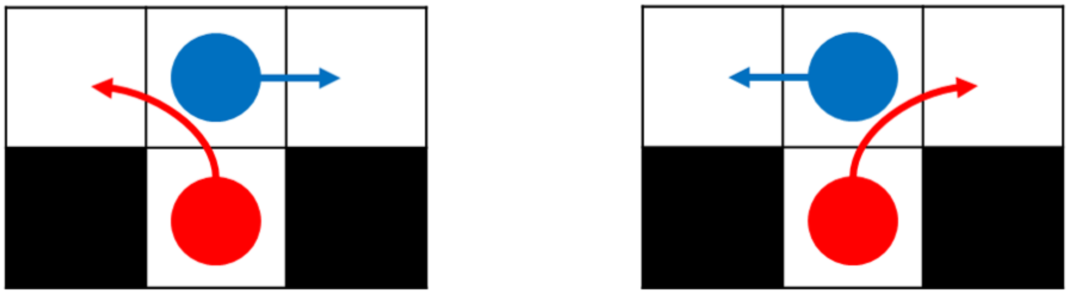

Figure 7: Agents passing each other on three-neighbour nodes

## A. Generating solvable maps

By Jonathan Dönszelmann and Jaap de Jong

For experiments in this paper, *MAPFM* instances (sometimes called maps) are randomly generated. To generate these random maps, we created a program called the Multi-Agent Pathfinding instance generator (*MPIG*) which creates these maps. We designed *MPIG* to always create maps which are solvable. In this section, we show how *MPIG* works, and how *MPIG* can guarantee that each map is solvable by giving a generic process which can be used to solve any map generated by *MPIG*.

To simplify the explanation, we first show the process of generating solvable *MAPF* instances.

### A.1. Properties of a solvable *MAPF* instance

*MAPF* instances are solvable, whenever it is possible for every agent to reach their goal. *MPIG* ensures that this is always possible, by guaranteeing that every generated map has the following two properties:

1. Every map is connected. There are no disconnected subgraphs.
2. Maps with $k$ agents and $k$ goal locations contain at least $k$ squares with three or more neighbours (i.e. squares where at least three adjacent squares are traversable). Squares with three or more neighbours are important because at these points, agents can pass each other as can be seen in Figure 7.

To guarantee the first property is held, *MPIG* starts generation of maps at a single square, and neighbours of this single square are recursively expanded (by either adding obstacles or traversable squares) to generate the rest of the map. Obstacles are not created when this would cause a disconnected subgraph to be created. The second property is guaranteed by simply discarding maps when whenever there are fewer than $k$ squares with three or more neighbours. A new map is randomly generated. This technique is used because chances are quite high that random maps contain more than $k$ squares with three or more neighbours.

### A.2. Proof of solvability

In this section, it is proven that when maps are connected, and there are at least $k$ three-neighbour squares, they are solvable. This proof consists of the following three parts which will be proven separately:

1. Every agent can always travel to a three-neighbour square from their starting location
2. When every agent is on a three-neighbour square, they can move to reorder themselves such that every agent can be on any of the three-neighbour squares.
3. There is always a configuration of agents on three-neighbour squares that allows all agents to go to their goal.

#### Part 1

**Theorem A.1.** *There is always at least one agent which can reach a three-neighbour square without collisions.*

*Proof of Theorem A.1.* The map grid is connected. Therefore all three-neighbour squares are in the same subgraph as all the agents, starting positions and goal positions. An agent can be blocked from reaching a three-neighbour square by another agent. But then this other agent can move to the three-neighbour square and an agent can still reach a three-neighbour square. □

**Theorem A.2.** *All agents can reach a three-neighbour square without collisions.*

*Proof of Theorem A.2.* There are $n$ three-neighbour squares. Thus, there are enough three-neighbour squares to accommodate all agents. The process for every agent to reach this three-neighbour square is as follows:

Step 1: a single agent moves to a three-neighbour square, which is possible according to Theorem A.1.

Step 2: an attempt is made to move another agent $a_i$ to a three-neighbour square. Two situations may occur:

1. Agent $a_i$ can move to a three-neighbour square $u$ without obstruction.

2. Another agent $a_j$ which previously moved to a three-neighbour square $v$ obstructs $a_i$ from reaching a three-neighbour square $u$.

In the second situation, agent $a_j$ can instead move to square $u$, freeing square $v$ for agent $a_i$.

Step 2 can be repeated until all agents reach a three-neighbour square thus proving Theorem A.2.

□

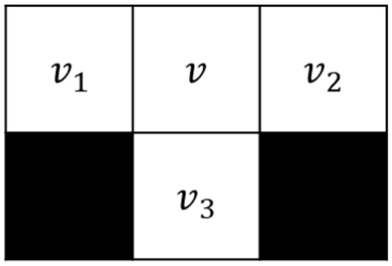

Figure 8: The trivial map with a single three-neighbour square

**Part 2**

**Theorem A.3.** *In every map, from every neighbour of a square $s$ with three or more neighbours it is possible to pathfind to a square with fewer than three neighbours, without visiting $s$ again.*

*Proof of Theorem A.3.* A proof by construction follows:

In the trivial map with a single three-neighbour square (see figure 8), Theorem A.3 holds, since each neighbour is a square with fewer than three neighbours itself.

Every possible map with at least one three-neighbour square, can be derived from the trivial map, by adding more open squares around it.

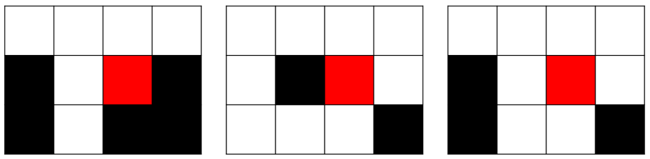

Figure 9: Three different ways of connecting squares. Red squares are added to maps.

Adding a new square $u$ can have one of three effects on each neighbour $v$:

1. $v$ has a single neighbour. Connecting to $v$ makes $v$ a two-neighbour square. Theorem A.3 trivially holds for $v$, because $v$ is not a three-neighbour square.
2. $v$ has two neighbours $v_1$ and $v_2$. Connecting to $v$ makes $v$ a three-neighbour square. If $v_1$ or $v_2$ have fewer than three neighbours, then Theorem A.3 trivially holds. If $v_1$ or $v_2$ have three or more neighbours, then they must already be part of the map and Theorem A.3 holds for $v_1$ and $v_2$. Since $u$ is a three-neighbour square, it is possible to pathfind to one of the remaining neighbours of $u$, which are directly or indirectly connected to a square with fewer than three neighbours. Since Theorem A.3 holds for all neighbours $v_1, v_2$ and $u$ of $v$, it must now also hold for $v$.
3. $v$ has three neighbours, $v_1$, $v_2$ and $v_3$. Connecting to $v$ makes $v$ a four-neighbour square. The same reasoning used in effect 2 can be used to show that Theorem A.3 still holds for $v$.

Adding a new square $u$ can also have one of the following effects on $u$ itself:

1. It can create a new two-neighbour square $u$ by connecting two squares (shown figure 9). Since $u$ has fewer than three neighbours, Theorem A.3 still holds for $u$.
2. It can create a new three-neighbour square $u$ by connecting three squares (shown figure 9). A square with which a connection is made (called $v$), can be in one of three possible configurations for which Theorem A.3 holds, as explained in the previous part of the proof. Since Theorem A.3 holds for all neighbours of $u$, it also holds for $u$ itself since every neighbour is always directly or indirectly connected to a one- or two-neighbour square.
3. It can create a new four-neighbour square $u$ by connecting four squares (shown in figure 9). A square with which a connection is made (called $v$), can be in one of three possible configurations for which Theorem A.3 holds. The reasoning from the previous effect can be used to show that Theorem A.3 also holds in this effect.

For the trivial map from Figure 8, Theorem A.3 trivially holds. Every map with one or more three-neighbour squares can be constructed from the trivial map by adding squares to it. Adding squares to a map for which Theorem A.3 holds, was shown to exclusively create new maps for which the Theorem still holds. If a map cannot be derived from the trivial map, then it does not contain three-neighbour squares. Theorem A.3 holds for maps without any three-neighbour squares.

Therefore, Theorem A.3 holds for all maps.

□

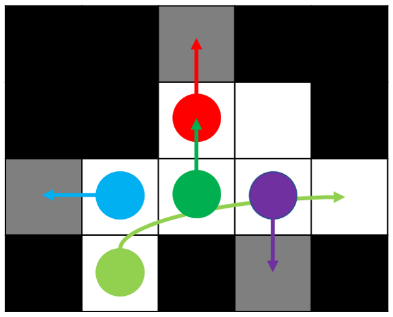

Figure 10: An example of how an agent can pass other agents even if there is no space between three-neighbour squares

Consider the scenario where each agent is positioned on a three-neighbour square. As a result of theorem A.3, each neighbour of a three-neighbour square $u$ has a so-called *diversion square*, which is the one- or two-neighbour square that is reachable from the neighbour without visiting $u$. The existence of *diversion square* is proven by Proof A.2.

**Theorem A.4.** *An agent $a_i$ on a three-neighbour $v$ square can always be passed by another agent $a_j$*

*Proof of Theorem A.4.* The three-neighbour node $v$ has three neighbours $v_1$, $v_2$ and $v_3$ (as shown in Figure 8). $a_j$ passing $v$ means that it is coming from one of the neighbours of $v$ (say $v_1$) and wants to travel to another one of the neighbours (say $v_2$). For $a_j$ to travel from $v_1$ to $v_2$, $a_i$ must move out of the way to $v_3$. $v_3$ can either be:

- A square with one or two neighbours. It is therefore empty because all agents are on three-neighbour squares. $a_i$ can simply move to $v_3$ and let $a_j$ pass.
- A square with three or more neighbours. In this case, there may be an agent $a_k$ on $v_3$. If there is an agent on $v_3$, it must also move out of the way. Theorem A.3 shows that it is always possible to pathfind to a square with fewer than three agents from neighbours of three agent squares. Since squares with fewer than three neighbours are empty, this provides a place for agents to move in to make room for passing agents. Therefore, $a_k$ must move either onto an empty square, or move onto a square with another agent which after possible repetitions will always find an empty square to move onto. This square to move in is from now on called a *diversion square*. Figure 10 shows how all agents move out of the way to empty squares to allow the lime agent to pass.

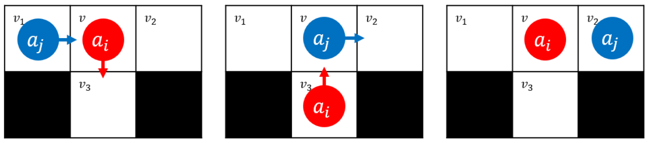

Figure 11: An example of how an agent can pass another agent.

After having encountered one of these two scenarios, agent $a_j$ has moved to $v$, and $a_i$ has moved out of the way to $v_3$. For agent $a_j$ to now completely pass $a_i$, $a_j$ must continue to $v_3$ (these steps are show in Figure 11). However, if $v_3$ is another three-neighbour square, another agent $a_k$ may be on it. Two situations can now occur:

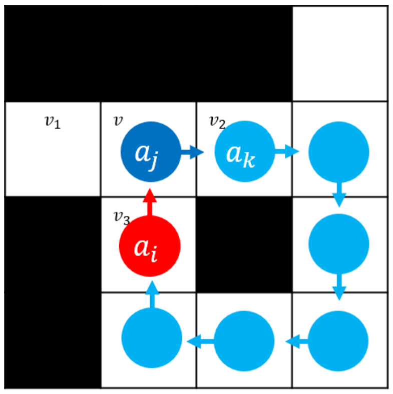

Figure 12: An example of how agents can pass with a single diversion square.

- $a_k$ can move out of the way just like $a_j$ did. Theorem A.3 shows that this is always possible to find a diversion square. $a_i$ can now also move back to $v$.
- $a_k$ can not move out of the way. Even though Theorem A.3 shows that there is always a diversion square to move out of the way, $a_j$ moving out of the way may have taken up this diversion square. However, if both $v_2$ and $v_3$ have the same diversion square, a connection must exist between $v_2$ and $v_3$. Because the definition of *MAPF* allows following, it is now possible for $a_k$ to move out of the way, following agents in front of $a_k$ in a sort of chain. The head of the chain is $a_i$. $a_u$ moves back to $v$, in a way making $v$ the diversion square. This motion can be seen in Figure 12

After this process, $a_i$ is back on $v$ and $a_j$ has passed to $v_3$ □

**Theorem A.5.** *Any two agents on adjacent three-neighbour squares (i.e. directly connected or connected with a corridor) can swap places, both moving to the three-neighbour square where the other agent was standing.*

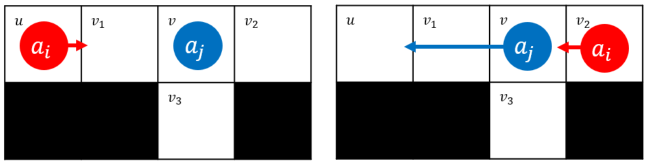

Figure 13: An illustration of two agents swapping by passing each other.

*Proof of Theorem A.5.*

**Lemma A.6.** *The swapping of two agents $a_i$ and $a_j$, positioned on three-neighbour squares $u$ and $v$ respectively, is equivalent to $a_i$ passing $a_j$ (or vice versa). After the passing, both agents can move to the square the other agent used to be without conflict.*

The proof of Theorem A.4 shows that an agent coming from one neighbour of a three-neighbour square (called $v_1$) can always pass the three-neighbour square

($v$) to move to another neighbour of the three-neighbour square (called $v_3$).

$a_i$ and $a_j$ both start on a three-neighbour square ($u$ and $v$ respectively). $u$ is adjacent to $v$, it may for example be connected to $v_1$. Because $v$ and $u$ are adjacent, there can not be any three-neighbour square $w$ between $u$ and $v$, because then $w$ and $v$ would be adjacent instead. Because there are no three-neighbour squares between them, there are also no agents to block $a_i$ from moving to $v_1$ to pass $a_j$ (this process can be seen in 13).

When $a_i$ has passed $a_j$, $a_i$ stands on $v_3$. $a_j$ is now free to move in the direction of $v_1$ because this is where $v_i$ came from. Therefore the path must be clear for $a_j$ to move to $u$. At the same time, $a_i$ can follow $a_j$, and stop on $v$ (this is show in Figure 13).

Therefore, given that two agents can pass each other which is proven in Theorem A.4, two agents on adjacent three-neighbour squares can also swap places. □

**Theorem A.7.** *If all agents are assigned to and located on a three-neighbour square, they can move to create every other assignment of agents to three-neighbour squares.*

*Proof of Theorem A.7.* Any permutation of a set of elements can be created using only pairwise swaps by using the Steinhaus–Johnson–Trotter algorithm. [29] The proof of theorem A.5 showed that pairwise swaps of agents on adjacent three-neighbour squares are possible on any map. □

**Part 3**

**Theorem A.8.** *Every connected map with $n$ agents on $n$ three-neighbour squares is directly solvable from at least one assignment $p$ of agents to three-neighbour squares.*

*Proof of Theorem A.8.* Consider the scenario where every agent is positioned on its corresponding goal. By theorem A.2, the agents can all travel to three-neighbour nodes without collision. This results in a assignment $p$ of agents to three-neighbour nodes.

If all agents are positioned on the three-neighbour square that corresponds to them with the previously found assignment, then the paths can be reversed and executed in the reverse order to move every agent to its corresponding goal, thus solving the map.

Because this assignment $p$ can be created for every map, each map is directly solvable from at least one assignment of agents to three-neighbour squares. □

**Theorem A.9.** *Every connected map with $n$ agents and at least $n$ three-neighbour squares is solvable.*

*Proof of Theorem A.9.* By Theorem A.2, it is possible for every agent to reach a three-neighbour node. By Theorem A.7, every assignment of agents to three-neighbour squares can be created. Theorem A.8 shows that there is always an assignment for which the map is solvable. Therefore, every map with at least $n$ three-neighbour squares is solvable. □

## A.3. Solving *MAPFM* instances

**Theorem A.10.** *Every connected MAPFM map with $n$ agents and at least $n$ three-neighbour squares is solvable.*

*Proof of Theorem A.10.* A *MAPFM* instance can be decomposed into many *MAPF* instances by considering all possible assignments of agents to goals, which can exhaustively be searched. Theorem A.9 shows that every *MAPF* instance is solvable. As a result, every possible assignment of agents to goals of a *MAPFM* instance is also solvable. Therefore, all *MAPFM* instances with $n$ agents and $n$ three-neighbour squares are solvable as well. □