CONVOLUTION METHOD FOR TIME-DEPENDENT BEACH-PROFILE RESPONSE

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ABSTRACT: A simple analytical solution is presented for approximating the time-dependent beach-profile response to severe storms. This solution is in the form of a convolution integral involving a time-varying erosion-forcing function and an exponential erosion-response function. The erosion-forcing function reflects changes in the nearshore water level and breaking wave height. In this paper, an idealized storm-surge hydrograph is considered from which an analytic solution is obtained for beach and dune erosion associated with severe storms such as hurricanes or northeasters. It is shown that for a given initial beach geometry and sediment size, the peak water level and the incipient breaking wave height determine the maximum erosion potential that would be achieved if the beach were allowed to respond to equilibrium. Because of the assumed exponential erosion rate, beach response obtained from the convolution method is found to lag the erosion forcing in time, and is damped relative to the maximum erosion potential such that only a fraction of the equilibrium response actually occurs.

INTRODUCTION

Despite the recent development of numerical models for predicting beach-profile response, there remains a need for simple methods of analyzing beach erosion or accretion due to variable wave and water-level conditions. For example, the Federal Emergency Management Agency (FEMA) ("National" 1988) recently adopted the so-called 540 Rule to determine the adequacy of coastal dunes, based on limited empirical data that suggest that on average a 100-year storm will erode 50.3 m^3/m (540 cu ft/ft) of sand from above the peak storm-surge level and seaward of the dune crest. This simple method was selected over available numerical erosion models due, in part at least, to the perceived complexity of numerical models for widespread application. Further illustrating the need for simple methods of estimating beach-profile response is the lack of any recommended method for predicting cross-shore beach-profile change in the *Shore Protection Manual* (1984), the primary design handbook for coastal erosion and shore protection published by the U.S. Army Corps of Engineers.

In this paper, we present a simple analytical method suitable for preliminary calculations of the time-dependent beach and dune erosion due to severe coastal storms. In the first part of this paper, the development of this method is described based on the observation that beaches subjected to steady-state erosion-forcing conditions respond toward a stable or equilibrium form in approximately an exponential manner. For general application, we then model the beach as a linear system such that this exponential beach response is convolved with a time-dependent erosion-forcing function

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to obtain the time-dependent erosion response. The forcing mechanisms considered here are storm-induced variations in water level, and the present paper assumes a simplified storm-surge hydrograph from which analytic results are obtained for the resulting beach erosion. The magnitude of the erosion response is then shown to be determined by two parameters: (1) The equilibrium or maximum potential response, R_∞, that would occur if the beach were allowed to respond completely to a new equilibrium; and (2) the characteristic erosion time scale, T_s, that governs the exponential rate at which the profile responds toward this new equilibrium.

In the second part of the paper, we develop closed-form solutions for the maximum potential beach response, R_∞. For arbitrary beach-profile forms, this may be found by a variety of existing geometrical methods, such as that of Brunn (1962) for long-term sea-level rise or of Edelman (1972) for storm-induced dune erosion. For idealized beach profiles, Dean (1976, 1991) has presented analytical solutions for R_∞ based on the concave equilibrium beach-profile form given originally by Bruun (1954) and later by Dean (1976, 1977). In the present paper, these methods are further generalized to include the effect of a sloping beach face along with idealized dune geometries. In the third section of this paper, a method for estimating the exponential erosion time scale, T_S, is developed. In the past, several investigators have shown that laboratory beach profiles respond toward equilibrium with approximately an exponential dependence over time. Numerical models based on equilibrium beach-profile concepts also give approximately exponential response to steady-state erosion-forcing conditions. At present, however, no general method is available for predicting the exponential time scale or rate parameter, and empirical data are limited. In the present paper, we therefore investigate this time scale through numerical experiments using the calibrated numerical erosion model of Kriebel (1986).

In the last part of the paper, we illustrate the practical application of the convolution method through two example problems. As noted, the convolution method is developed for idealized storm-surge hydrographs to obtain analytical results for beach and dune response to storm-surge conditions. This solution is then applied to: (1) A short-duration hurricane; and (2) a long-duration extratropical storm or northeaster. These cases illustrate the ability of the method to identify the appropriate response to different duration storm events and to different initial beach geometries, sediment sizes, wave heights, and water-level conditions.

Theoretical Development

The basis for the convolution method is the observation that beach response to steady-state forcing conditions is approximately exponential in time. For laboratory conditions, where an initially plane beach is subjected to a fixed water level and steady wave action, the response of any depth contour as a function of time, $R(t)$, may be aproximated by the form

$$R(t) = R_\infty(1 - e^{-t/T_S}) \quad (1)$$

where R_∞ = the maximum response o the contour (advance or retreat) that occurs after the system reaches equilibrium; and where T_S = the characteristic time scale of the exponential response. The response suggested by (1) has been observed in small-scale laboratory experiments as reported by Swart (1974), and in large-scale wave-tank experiments as reported by Dette and Uliczka (1987), Sunamura and Maruyama (1987), and Larson and Kraus

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(1989). Eq. (1) also seems to describe the response of an equilibrium beach profile to a sudden increase in water level with constant breaking-wave conditions, as noted by Kriebel and Dean (1985) and Larson and Kraus (1989) based on numerical simulation.

According to (1), the rate of response of a beach profile must then be proportional to the difference between the instantaneous profile form and its final or equilibrium form. As a result, the differential equation governing the exponential beach response to steady-state forcing conditions must be of the form

$$\frac{dR(t)}{dt} = \frac{1}{T_S}[R_\infty - R(t)] \quad \text{(2)}$$

As noted, R_∞ is generally defined as the maximum potential advance or retreat of any elevation contour if the beach were allowed to reach a new equilibrium relative to the water level and breaking wave conditions. As illustrated in Fig. 1, however, the variables $R(t)$ and R_∞ are most conveniently defined as the erosion or retreat of the top of the berm. The rate of the beach response is then governed by the exponential time scale, T_S. In this case, a beach subjected to steady-state forcing conditions would reach 63% of the equilibrium response after time T_S while over 99% of the equilibrium response would be achieved after a time of $5T_S$.

A more general form of the differential equation governing the macroscale profile response may be obtained by assuming a time-varying erosion-forcing function on the right-hand side of (2). In earlier sensitivity studies with the numerical erosion model of Kriebel and Dean (1985), it was found that the equilibrium response R_∞ varied almost linearly with changes in water level, while the water level did not effect the rate or time scale of profile response. As a result, it is assumed that for a time-varying water level, the maximum potential response may be determined for the peak water level while the erosion-forcing function may then be expressed as R_∞ times a unit-amplitude function of time, $f(t)$. Thus, a linear differential equation governing the profile response to variations in water level is assumed to have the form

$$\frac{dR(t)}{dt} = \frac{1}{T_S}[R_\infty f(t) - R(t)] \quad \text{(3)}$$

or

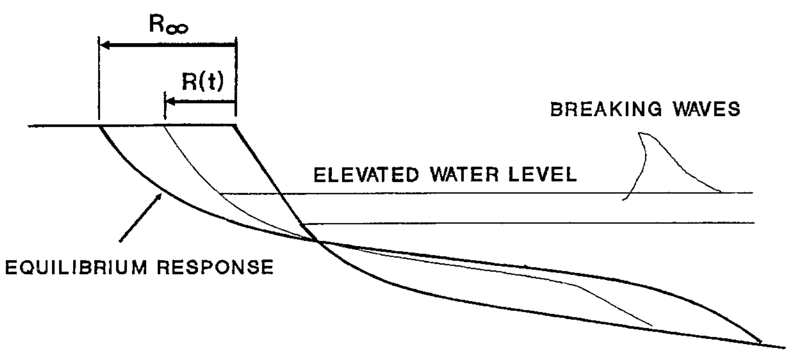

FIG. 1. Definition Sketch for Beach-Profile Response

$$\frac{dR(t)}{dt} + \alpha R(t) = \alpha R_\infty f(t) \quad (4)$$

where α = the characteristic rate parameter of the system, defined as

$$\alpha = \frac{1}{T_S} \quad (5)$$

The general solution to (3) or (4) may be obtained formally by the method of Laplace transforms. However, it is well known that the solution to differential equations of this kind may be given by the convolution of the time-dependent forcing and the characteristic solution for steady forcing, in this case an exponential function. As a result, the time-dependent solution for macroscale profile respone may be expressed in the form of a convolution integral as

$$R(t) = \frac{R_\infty}{T_S}\int_0^t f(\tau)e^{-(t-\tau)/T_S}\,d\tau \quad (6a)$$

$$R(t) = \alpha R_\infty \int_0^t f(\tau)e^{-\alpha(t-\tau)}\,d\tau \quad (6b)$$

where τ = a time lag. This response is typical of many linear dynamical systems and has other analogies in civil engineering, for example in the unit hydrograph methods applied to the prediction of streamflow. In the present case, the system output (time-dependent beach response) is governed by the input (time-dependent erosion-forcing function) and by the basic filtering characteristics (exponential response) of the beach system to any step-type forcing. As a result, (6) suggests two important and general characteristics of beach-profile response: that the beach profile response will *lag* behind the erosion forcing, and that it will be *damped* relative to the maximum erosion potential of the system. Furthermore, it is evident that a beach has a certain memory so that the response at any one time is dependent on the forcing conditions that have occurred over some preceding period of time.

In addition to the recession or retreat of the berm, the convolution method may be used to approximate the time-dependent volume of sand eroded from the beach face. In this case, it is assumed that all elevation contours erode at the same relative rate so that the dimensionless erosion, $R(t)/R_\infty$, is the same everywhere. The relative volume eroded is then the same as the relative berm retreat such that

$$\frac{V(t)}{V_\infty} = \frac{R(t)}{R_\infty} \quad (7)$$

where $V(t)$ = the time-dependent volume eroded above some reference datum while V_∞ = the equilibrium eroded volume above that same datum. It is noted that the convolution method automatically satisfies continuity or cross-shore conservation of sand since the equilibrium responses R_∞ and V_∞ are determined based on this requirement, as will be shown.

The convolution method given here differs from the more rigorously derived solution for time-dependent profile response obtained by Kobayashi (1987). Kobayashi's method, based on a solution of a diffusion-type equation, is more closely related to sediment-transport mechanisms since it is based on wave-energy dissipation throughout the surf zone. His solution

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then yields a time-dependent profile response to steady-state forcing in terms of error functions rather than in terms of exponential functions. These error functions probably describe the beach response more accurately than the exponential functions adopted here; however, they are considerably more complicated and are not as amenable to further analytical work. Swart (1974), on the other other hand, developed a time-dependent model for profile development based on observations of small-scale wave-tank tests and used the same type of exponential adopted here with good results.

Beach Response to Idealized Storm Surge

The convolution method may now be applied to any time-varying forcing function. In this paper, we consider an idealized yet fairly realistic storm-surge hydrograph in order to obtain closed-form solutions for beach response to hurricanes or extratropical storms. In this case, the water-level rise and fall are approximated by the sine-squared function

$$f(t) = \sin^2(\sigma t), \qquad \text{for } 0 < t < T_D \quad \text{(8)}$$

where $\sigma = \pi/T_D$, and where T_D = the total storm-surge duration defined as the time from the beginning to the end of the water-level rise. The actual water level would be given by $S \sin^2(\sigma t)$, where S is the peak storm surge. The maximum potential erosion, R_∞, would then be determined from this peak water level and from the maximum breaking wave height during the storm. For simplicity, the breaking wave height is assumed to remain constant over the duration of the storm. As shown by Kriebel (1986), such an assumption is conservative and gives erosion estimates that are about 10% larger than would be achieved by use of a time-varying wave height.

Substituting the time-dependent forcing in (8) into the convolution integral in (6), gives the following time-dependent erosion response

$$R(t) = \alpha R_\infty \int_0^t \sin^2\sigma(\tau) e^{-\alpha(t-\tau)}\, d\tau \quad \text{(9)}$$

This may be integrated directly to obtain a closed-form solution for the time-dependent erosion response. In dimensionless form, this solution is then a function of just one parameter, β, as

$$\frac{R(t)}{R_\infty} = \frac{1}{2}\left\{1 - \frac{\beta^2}{1+\beta^2}\exp\left(-\frac{2\sigma t}{\beta}\right) - \frac{1}{1+\beta^2}[\cos(2\sigma t) + \beta \sin(2\sigma t)]\right\} \quad \text{(10)}$$

where β = the ratio of the erosion time scale to the storm duration, as

$$\beta = \frac{2\sigma}{\alpha} = 2\pi\frac{T_S}{T_D} \quad \text{(11)}$$

The predicted time-dependent beach response from the convolution method in (10) is shown in Figs. 2(*a*) and (*b*). These figures depict both the erosion-forcing function and the predicted profile response for two different values of β. If the profile responded instantly to the water-level change, the response would follow the $\sin^2(\sigma t)$ curve and would give a dimensionless recession of unity at time $T_D/2$. Because of the relatively slow morphologic response typical of natural beaches however, the predicted response based on the convolution integral lags the storm-surge hydrograph and only a fraction of the maximum potential erosion is realized before recovery is

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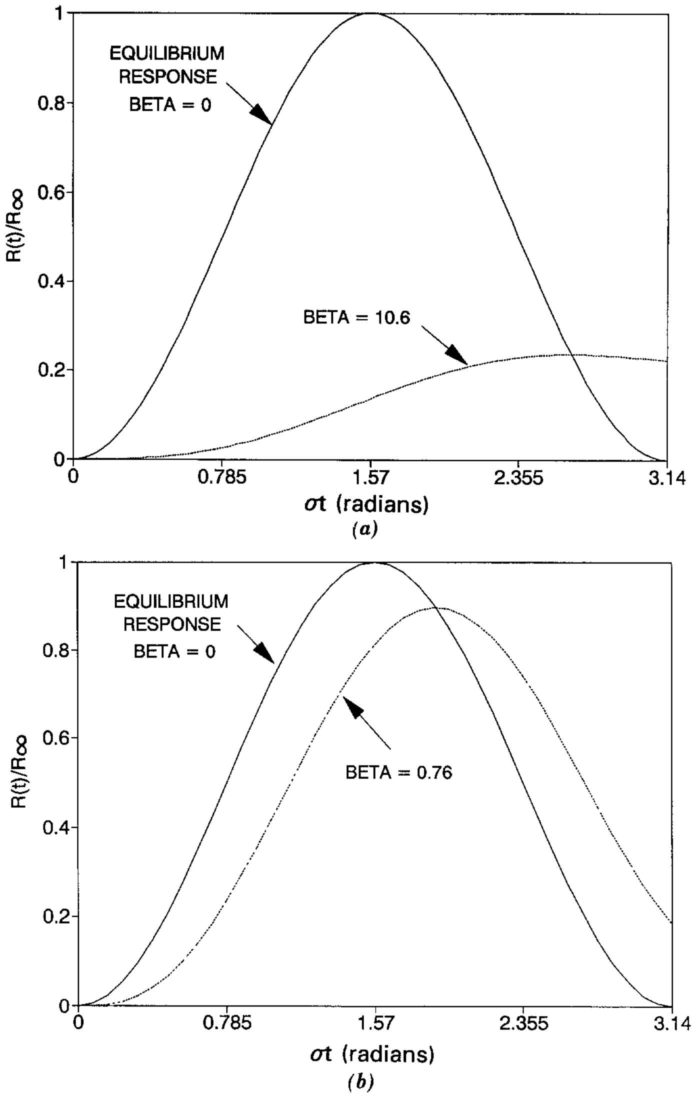

FIG. 2. Examples of Profile Response to Idealized Sine-Squared Storm Surge: (*a*) Short-Duration Hurricane; and (*b*) Long-Duration Northeaster

initiated. The two response curves shown in Fig. 2 correspond approximately to a short-duration hurricane ($\beta = 10.6$) in Fig. 2(*a*), and to a long-duration northeaster ($\beta = 0.76$) in Fig. 2(*b*). Both of these conditions will be used in case studies to follow and illustrate the role that storm duration plays in determining the maximum erosion response. Qualitatively, the results in Fig. 2 agree remarkably with those given by Kriebel and Dean (1985) based on a finite difference numerical solution of the same storm-surge hydrograph but with different governing differential equations.

Because the beach-response process is modeled here as a linear system, the solution for storm-induced beach erosion in (10) can also be expressed in terms of an amplitude and phase lag of the maximum erosion response. Referring to Fig. 3, we first find the time of maximum erosion, t_m, by setting the time derivative of (10) equal to zero, which gives

$$\exp\left(-\frac{2\sigma t_m}{\beta}\right) = \cos(2\sigma t_m) - \frac{1}{\beta}\sin(2\sigma t_m) \quad \text{(12)}$$

This may be solved iteratively for the phase, σt_m, which is between $\pi/2$ and π. The phase lag of the maximum response is then given by $\sigma t_m - \pi/2$. The magnitude of the peak response, $R_{\max}$, may then be found by substituting (12) back into (10), which gives

$$\frac{R_{\max}}{R_\infty} = \frac{1}{2}[1 - \cos(2\sigma t_m)] \quad \text{(13)}$$

The results of (12) and (13) for the phase lag and amplitude of the maximum beach response to storm surge are shown graphically in Fig. 4, where both the amplitude and phase lag are functions of the single parameter β. As will be shown, short-duration hurricanes generally fall to the right on each curve such that the maximum erosion would occur near the end of the storm and may be only 20–40% of the erosion potential of the storm. For long-duration extratropical storms, on the other hand, the maximum erosion may be from 40% to 90% of the erosion potential and may occur closer to the time of the peak water level.

The solution in (10) also gives a rough indication of the behavior of the beach profile at the end of the storm where, following the time of maximum erosion, recovery begins to occur as the water level returns to normal. Eq.

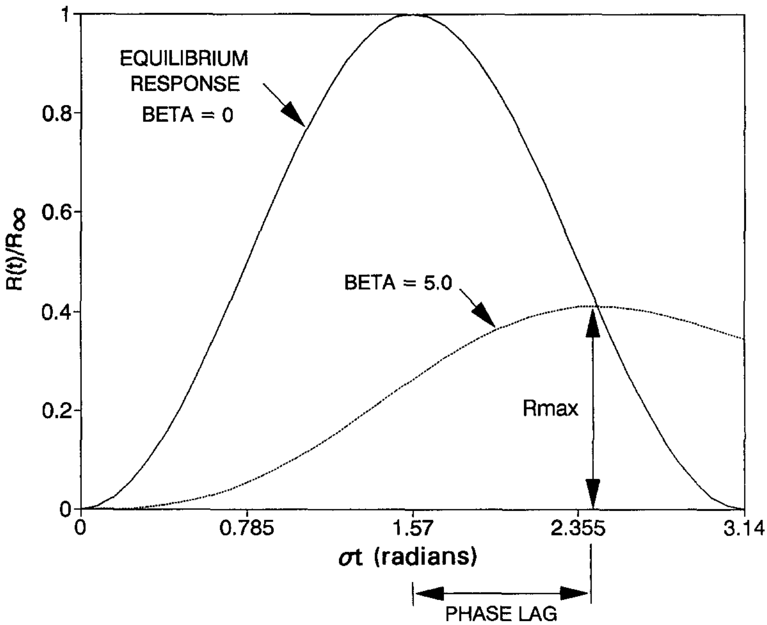

FIG. 3. Definition of Amplitude and Phase Lag of Maximum Erosion

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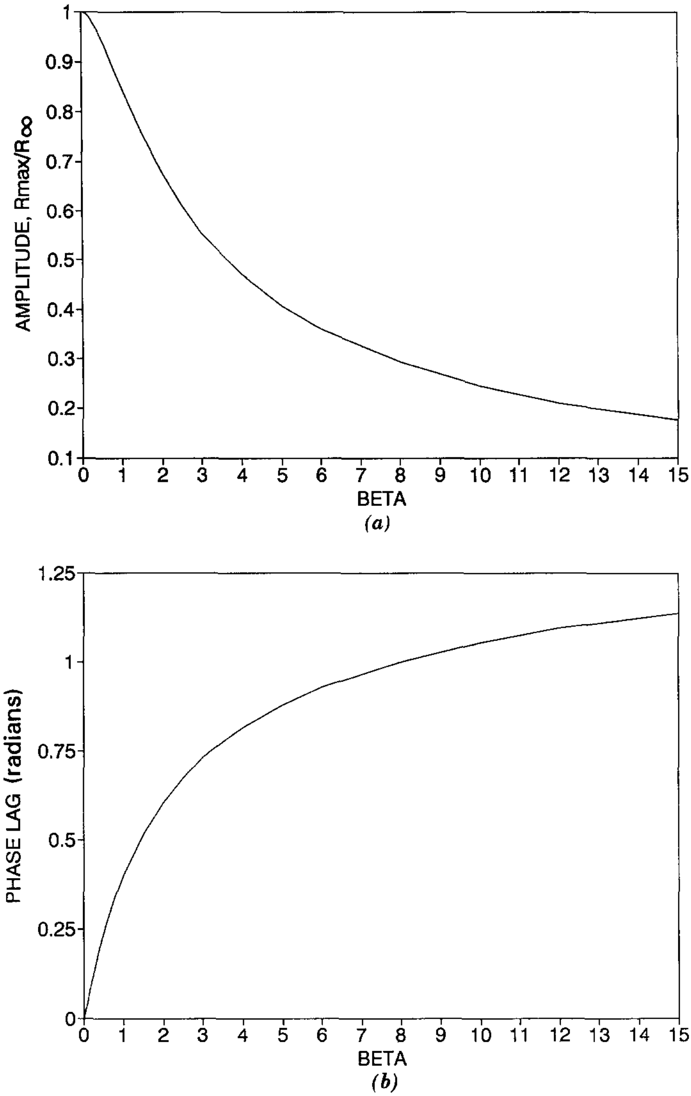

FIG. 4. Erosion Response Functions for Sine-Squared Storm Surge: (*a*) Amplitude; and (*b*) Phase Lag

(10) may be evaluated at time $2\sigma t = 2\pi$ in order to find the net erosion at the end of the storm. The convolution method then suggests that this is followed by a period of recovery at an exponential rate until the response returns to zero. The predicted sequence of storm erosion and poststorm recovery is shown in Fig. 5, based on the assumptions that all material eroded during the storm is eventually recovered and that the recovery-rate parameter is identical to the erosion-rate parameter. As shown, the predicted poststorm recovery rate is greatest just at the end of the storm, in accordance with field observations.

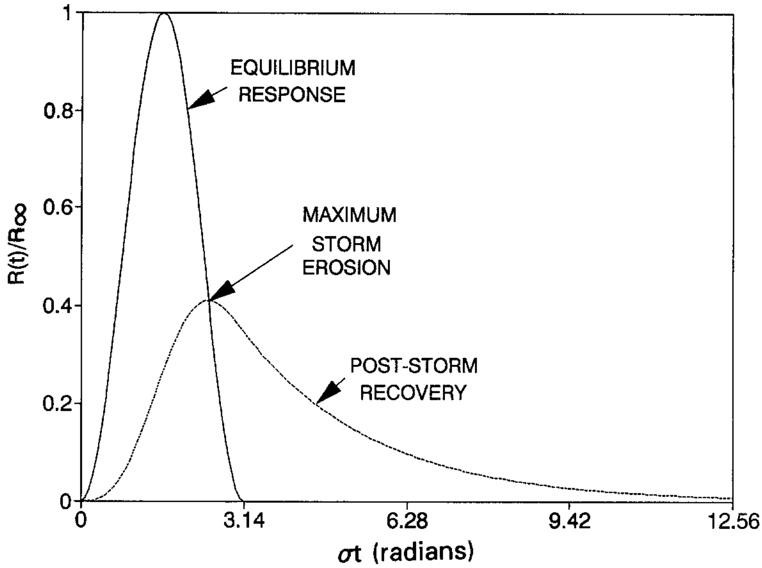

FIG. 5. Example of Storm-Erosion and Poststorm Beach Recovery Predicted by Convolution Method

Based on Fig. 5, it is interesting to note that even without changing the rate parameter α (or time scale T_S), recovery appears to occur more slowly than erosion. This is due to the fact that the beach profile is not as far out of equilibrium at the end of the storm when the water level drops as it is during the peak of the storm when the water level is elevated. In nature, however, the smaller waves during the recovery period would likely impose a new rate parameter on the system that would further slow the beach recovery. In addition, these smaller waves may not be able to activate sand to a deep enough depth to allow full recovery to occur. Since neither of these effects are accounted for, the convolution method overestimates post-storm beach recovery and is therefore not recommended at the present time for quantitatively estimating the extent of recovery.

Equilibrium Profile Response

The maximum potential erosion of a beach profile, R_∞, can be established geometrically for many beach-profile conditions based on some simplifying assumptions. The classic geometric solution for beach response is that of Bruun (1962) for erosion due to long-term sea-level rise. In this method, the existing beach profile is assumed to be in statistical equilibrium with respect to the prevailing mean water level and wave climate. This profile is then shifted upward by an amount equal to the water-level rise and landward by an amount R_∞ until a volume balance is achieved between the volume of sand eroded from the upper portion of the beach and the volume of sand deposited offshore in deeper water. Methods discussed in the present paper are similar in principle but are based on the equilibrium beach-profile theory proposed by Dean (1976, 1977). This theory gives an analytic description of the equilibrium beach-profile form that is then amenable to closed-form solutions of the profile response to specified water-level and wave conditions.

Equilibrium Beach-Profile Forms

As shown by Dean (1976, 1977), a simplified yet realistic equilibrium form for open-coast beach profiles is given by a power-law curve as $h = Ax^{2/3}$ or equivalently as

$$x = \left(\frac{h}{A}\right)^{3/2} \qquad (14)$$

where h = the water depth at a distance x offshore from the still-water level and A = a parameter that governs the overall steepness of the profile. The so-called A parameter varies primarily with sediment grain size or fall velocity and has been discussed by several investigators [see Dean (1987) or (1991) for a review]. Based on dimensional arguments, A may be related empirically to the sediment fall velocity, ω, in the form $A \approx (\omega^2/g)^{1/3}$. By fitting an expression of this form through the empirical data presented by Dean (1987), a useful relationship for the A parameter is obtained as

$$A = 2.25 \left(\frac{\omega^2}{g}\right)^{1/3} \qquad (15)$$

This is valid for the range of typical sand grain sizes from 0.1–0.4 mm and for water temperatures of about 20°C.

The profile form in (14) formally gives an infinite slope at the shoreline. If a horizontal berm is specified at elevation B above mean sea level, then the resulting profile has a square-berm form as shown in Fig. 6(a). While this is clearly not realistic, it may be interpreted as a first-order approximation of natural beach-profile forms. An improved approximation of the equilbrium profile form should include a linearly sloping beach face from the berm crest to a point where the beach slope is tangent to the concave equilibrium profile form as shown in Fig. 6(b). The equilibrium profile from (14) may be adopted in the surf zone and replaced by a linear beach-face slope near and above the still-water level, as

$$x = \frac{h}{m}, \qquad h < h_T \qquad (16a)$$

$$x = x_0 + \left(\frac{h}{A}\right)^{3/2}, \qquad h > h_T \qquad (16b)$$

where m = the linear beach-face slope, and x_0 = the distance from the still-water shoreline to the virtual origin of the concave equilibrium profile form, given by $x_0 = h_T/3m$, where h_T is the depth at which the linear slope is tangent to the concave profile, which may be shown to equal $4A^3/9m^2$.

Response of Equilibrium Profile with Square Berm

The equilibrium-profile response for the so-called square-berm profile may be determined geometrically for a given water-level rise, S, and a given breaking depth h_b. For this derivation, it is assumed that the breaking depth remains constant and is related linearly to the breaking wave height as $h_b = H_b/\gamma$, where γ is the breaker index, usually taken to be 0.78–1.0. The

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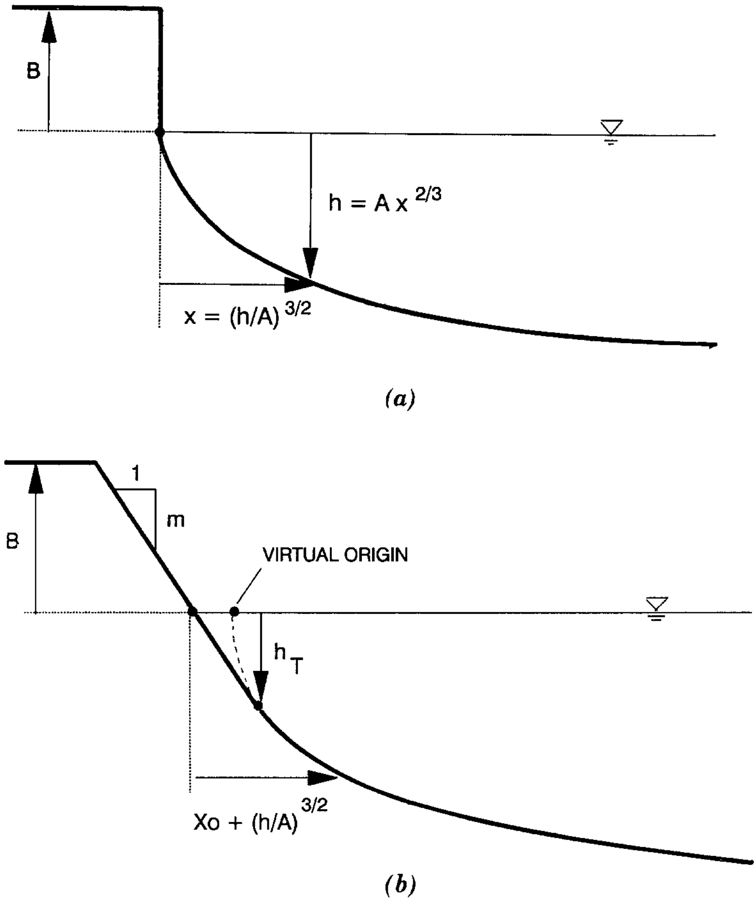

FIG. 6. Idealized Equilibrium Profile Forms: (*a*) Square-Berm Profile with Infinite Foreshore Slope; and (*b*) Equilibrium Profile with Linear Foreshore Slope

solution for R_∞ is then obtained by shifting the equilibrium profile in (14) vertically through distance S, and then landward through distance R_∞ such that a mass balance is achieved between the eroded and deposited sand volumes. Dean (1976, 1991) performed this mass balance by truncating the new equilibrium beach profile vertically at the breaking depth and by formally integrating the $Ax^{2/3}$ beach-profile form. A transcendental equation for the equilibrium berm recession was then obtained, and the result was expressed in graphical form.

By adopting a different assumption for the offshore boundary, the mass balance can be performed without actually integrating the $Ax^{2/3}$ equilibrium profile form such that the solution may be obtained in closed form. As depicted in Fig. 7, the solution for the equilibrium beach recession is obtained by equating the volume eroded from the beach face, $V_1 + V_2$, with the volume deposited offshore, $V_7 + V_8$. The present solution differs from the earlier solution of Dean (1976) in that a wedge of sand with slope S/R_∞ (volume V_8) is assumed to form offshore between the original and the new

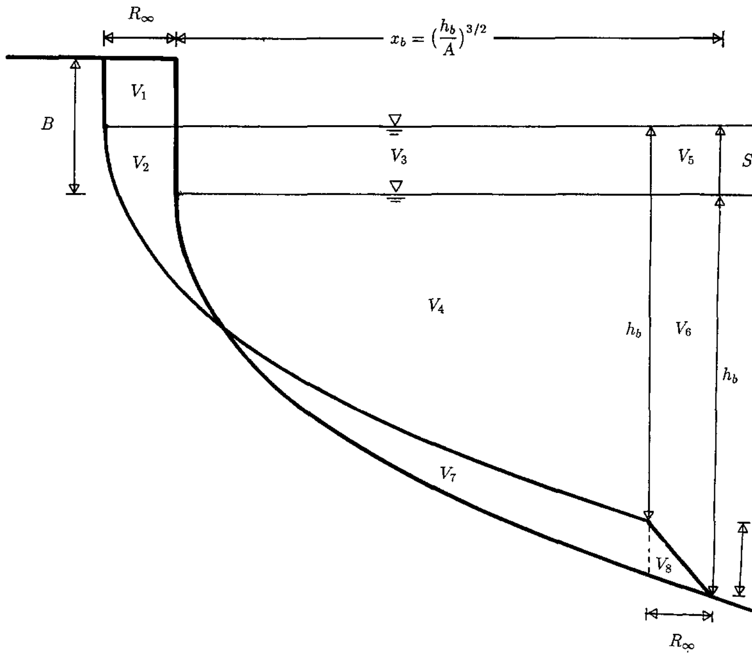

FIG. 7. Maximum Potential Response of Square-Berm Equilibrium Profile to Water-Level Rise

breaking depths. With this assumption, the balance of eroded and deposited sand volumes is simplified if the water volume in the surf zone is also included. As a result, the volume balance requires that the volumes $V_1 + (V_2 + V_3 + V_4) + V_5 + V_6$ equal the volumes $V_3 + V_5 + (V_4 + V_6 + V_7 + V_8)$. Terms in the parentheses may be given formally by intergration of the $Ax^{2/3}$ profile form over the surf zone out to depth h_b. However, because the profile forms are identical before and after the water-level rise, these terms in parentheses are identical and cancel each other. This avoids integration of the $Ax^{2/3}$ profile and avoids the previous transcendental solution.

Following these arguments, conservation of sand across the beach profile actually requires that

$$R_\infty(B - S) + (V_2 + V_3 + V_4) + R_\infty\left(h_b + \frac{1}{2}S\right)$$

$$= Sx_b + (V_4 + V_6 + V_7 + V_8) \quad \text{(17)}$$

Canceling common terms involving the volume of water in the surf zone then yields the closed-form solution for the maximum erosion potential as

$$R_\infty = \frac{Sx_b}{B + h_b - \dfrac{S}{2}} \quad \text{(18)}$$

where the surf zone width, x_b, is a function of the breaking wave height and the sediment size as

$$x_b = \left(\frac{h_b}{A}\right)^{3/2} \qquad (19)$$

The result in (18) is similar in form to the Bruun rule (Bruun 1962) and, in fact, reduces to the Bruun rule if the water-level rise S is small relative to the total active profile height $B + h_B$. Under these conditions, this solution is also equivalent to Dean's previous solution since the volume of sand in the offshore wedge is then negligible. As S increases, however, (18) predicts somewhat more erosion than Dean's solution due to the offshore deposition of sand assumed here. For most conditions of interest, where h_b is much greater than S, (18) gives results that are up to 10–20% larger. It should be noted that the assumption of a wedge of sand offshore of the break point both simplifies the mathematics and provides a closer approximation to nature where some sand will always be deposited just seaward of the break point.

In addition to the horizontal retreat of the berm, the equilibrium volume eroded from the beach face may also be easily determined. Referring to Fig. 7, we find that the maximum potential volume eroded from above the peak storm-surge level is given by

$$V_{S\infty} = R_\infty(B - S) \qquad (20)$$

While this volume is often used as a measure of storm erosion, it is sometimes problematic in that it may be zero is $S = B$ even though a large amount of sand may actually have eroded from the foreshore. An alternate measure of the storm erosion is the volume eroded from above the original mean sea level given by

$$V_{M\infty} = R_\infty B - \frac{2}{5}\frac{S^{5/2}}{A^{3/2}} \qquad (21)$$

Response of Equilibrium Profile with Sloping Beach Face

To introduce more realism into the solution for the maximum erosion potential, a linear beach slope may be assumed that joins the $Ax^{2/3}$ profile as described in (16). The initial and final profile forms may then be depicted in Fig. 8, where it is assumed that the beach face erodes without changing slope. Conservation of sand again requires that $V_1 + V_2 = V_7 + V_8$. However, by once again including the common water volume in the surf zone and by including a sloping wedge of sand offshore between the initial and final break points with slope $S(R_\infty + S/m)$, the volume balance may be simplified as described. The resulting equilibrium response is found to be given by

$$R_\infty = \frac{S\left(x_b - \dfrac{h_b}{m}\right)}{B + h_b - \dfrac{S}{2}} \qquad (22)$$

with a new definition of the surf-zone width

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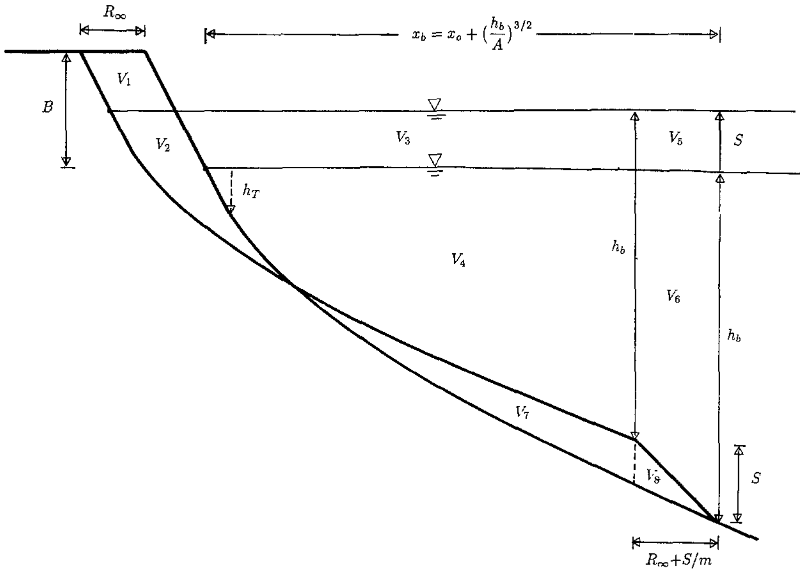

FIG. 8. Maximum Potential Response of Modified Equilibrium Profile to Water-Level Rise

$$x_b = x_0 + \left(\frac{h_b}{A}\right)^{3/2} \quad \text{(23)}$$

For severe storm conditions where h_b is large, x_0 is generally small in comparison to the second term and may usually be neglected for simplicity.

Referring to (22), it may be seen that the solution for the square-berm profile in (18) is recovered as the beach face steepens and approaches an infinite slope. For realistic beach-face slopes, however, the value R_∞ will be smaller than that predicted using the square-berm profile. This is in agreement with the findings of Chiu (1977), Kriebel and Dean (1985), and others that beaches with steeper beach-face slopes (for a given offshore profile) have a greater erosion potential than beaches with milder beach slopes. Comparison of results from (22) to numerical results for the equilibrium response from the Kriebel and Dean (1985) numerical model show that the two are in very good agreement. Some 60 senstitivity tests have been performed with numerous values of A, B, h_b, S, and m. On average, (22) gives results that agree to within about 4% of the numerical model results.

Once again, the maximum potential volume eroded from the beach face may be determined for the equilibrium profile with a sloping beach face. Referring to Fig. 8, we find that the volume eroded from above the storm-surge level at equilibrium is again given by (20), although R_∞ must now be given by (22). The volume eroded from above the original mean sea level at equilibrium is then slightly different and is given by

$$V_{M\infty} = R_\infty B + \frac{S^2}{2m} - \frac{2}{5}\frac{S^{5/2}}{A^{3/2}} \quad \text{(24)}$$

Response of Equilibrium Profile with Dunes

For beach profiles backed by sand dunes, the procedures described may be repeated to obtain first-order estimates of the equilibrium dune response. Two idealized cases will be considered. First, if the profile is backed by high dunes with no backshore, as in Fig. 9(*a*), the solution may be obtained most easily under the assumptions that the water level, S, does not rise above elevation B and that the entire dune face erodes uniformly without

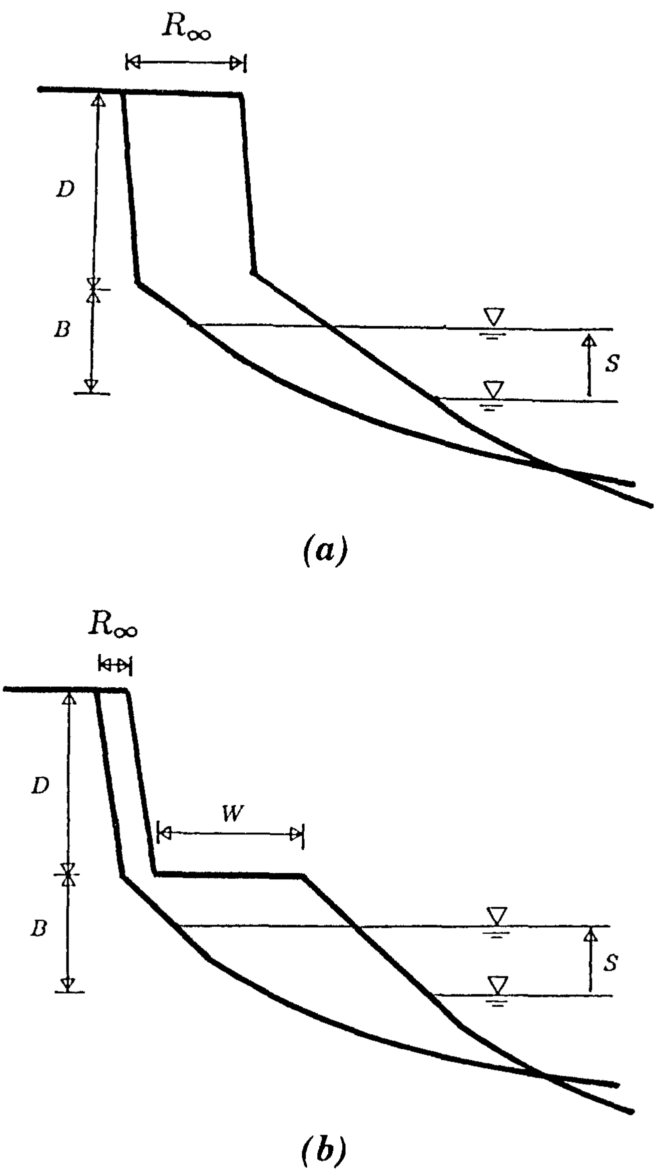

FIG. 9. Special Cases of Equilibrium Profile Response: (*a*) Dune with No Backshore; and (*b*) Dune with Wide Backshore

formation of a distinct erosion scarp. Following the same methods used, the solution for the retreat of the dune crest is

$$R_\infty = \frac{S\left(x_b - \dfrac{h_b}{m}\right)}{B + D + h_b - \dfrac{S}{2}} \qquad (25)$$

Thus, the dune height D enters the demonimator such that the dune acts as a reservoir of sand to reduce the potential dune retreat R_∞. In the second case, where the profile s backed by a dune and has a wide backshore of width W, as depicted in Fig. 9(b), the solution for dune erosion may be found under the same assumptions stated previously as

$$R_\infty = \frac{S\left(x_b - \dfrac{h_b}{m}\right)}{B + D + h_b - \dfrac{S}{2}} - \frac{W\left(B + h_b - \dfrac{S}{2}\right)}{B + D + h_b - \dfrac{S}{2}} \qquad (26)$$

In this case, both the height of the dune and the presence of the wide backshore are beneficial in limiting the retreat of the dune face. In fact, the backshore may be so wide that (26) gives $R_\infty < 0$, indicating that no dune retreat will occur and that only the berm will erode. A potential application of (26) for beach fill design might involve setting R_∞ equal to zero and solving for the width of th backshore, W, that would be required to eliminate the chance of significant dune retreat.

The equilibrium volume of sand eroded from the beach and dune may then be determined for each case shown in Fig. 9. In general, the volume above the peak storm-surge level is

$$V_{S\infty} = R_\infty D + (R_\infty + W)(B - S) \qquad (27)$$

The volume eroded above the original mean sea level is then

$$V_{M\infty} = R_\infty D + (R_\infty + W)B + \frac{S^2}{2m} - \frac{2}{5}\frac{S^{5/2}}{A^{3/2}} \qquad (28)$$

It is interesting to interpret (27) under the simplest case where $W = 0$. In this case, if the result from (25) for R_∞ is used, it is found that the equilibrium volume eroded from above the peak storm-surge level may eventually be expressed in the form

$$V_{S\infty} = S\left(x_b - \frac{h_b}{m}\right)\left[1 + \frac{h_b + \dfrac{S}{2}}{D + B - S}\right]^{-1} \qquad (29)$$

As a result, it is found from (25) that R_∞ is reduced as the dune height increases while from (29) the volume eroded from above the peak surge level is increased as the height of the dune increases. For beach fill design, where artificial dunes may be constructed, there is therefore a design trade-off between the dune height and dune width in order to minimize both the horizontal dune retreat and the volume of sand required for construction.

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TIME SCALE OF PROFILE RESPONSE

The time scale of profile response, T_S (or rate parameter α), unlike the equilibrium recession, R_∞, seemingly cannot be found from simple geometrical comparisons of pre- and poststorm equilibrium profile forms. An analytical estimate of the time scale would be useful but, at present, the only available results for the erosion time scale are based on either calibrated numerical models or laboratory data. Field data presently are too limited to allow meaningful empirical relationships to be developed between the erosion time scale and beach geometry, sediment, wave, and water-level conditions. Likewise, however, laboratory data are also of limited use in practice, either because of scale effects associated with the model-to-prototype scale ratio or because the planar initial beach profiles used in most laboratory tests do not represent natural beaches.

As a result, erosion time scales are investigated here based on numerical experiments using the Kriebel and Dean (1985) numerical erosion model as recalibrated by Kriebel (1986). Initial profiles were defined according to (14), and two values of the A parameter were tested, $A = 0.083$ m$^{1/3}$ (0.124 ft$^{1/3}$) and $A = 0.137$ m$^{1/3}$ (0.204 ft$^{1/3}$), corresponding to median grain sizes of 0.2 mm and 0.3 mm, respectively. Four beach-face slopes were tested, from $m = 1/30$ to $m = 1/10$, along with three different berm heights, $B =$ 1.52, 2.29, and 3.05 m (5, 7.5, and 10 ft, respectively). For each beach configuration, five different breaking wave heights of 0.76, 1.52, 2.29, 3.05, and 4.57 m (2.5, 5.0, 7.5, 10.0, and 15.0 ft, respectively) were evaluated.

Numerical tests were performed by raising the water level by a fixed amount, S, and by allowing the beach to respond to a new equilibrium form. From the resulting time-dependent berm erosion, best-fit values of the erosion time scale T_S (or rate parameter α) were determined. From preliminary tests, it was found that the time scale was independent of the water level such that the dimensionless response, $R(t)/R_\infty$, was essentially identical for each water level tested. As a result, a standard 0.30-m (1.0-ft) water-level rise was adopted, and numerical tests were carried out on 60 permutations of the remaining variables: A, m, B, and H_b (or h_b since h_b equal $H_b/0.78$). It is noted that the numerical results are independent of wave period, since the numerical erosion model relies on shallow-water wave theory and does not itself depend on wave period.

Results of these numerical tests are presented in Figs. 10(*a*) and (*b*) where the time scale, T_S, is plotted as a function of the variable, H_b/ω, as suggested by simple dimensional arguments. It is found that the time scale is more strongly dependent on the breaking wave height and on the sediment size. The time scale varies by about one order of magnitude from the smallest wave height to the largest, such that the smaller wave heights have much smaller time scales (larger rate parameters) than larger wave heights. The reason for this is that the larger wave heights define a wider surf zone in which sand must be moved farther offshore, thus requiring a much longer time for the profile to equilibrate. For example, from Fig. 10(*a*) for the 0.2-mm sand, the beach may equilibrate in about $5T_S$ or in about 25 hours for the smallest wave heights (smallest value of H_b/ω), while for the largest wave height (largest value of H_b/ω) it may require 150–400 hours of wave action. In addition, it is found that values of T_S are about four times smaller for the 0.3-mm sediment (fall velocity, ω, of 4.70 cm/s) than for the 0.2-mm sediment (fall velocity of 2.23 cm/s), such that the time scale is inversely proportional to ω^2.

Based on further dimensional arguments, an empirical relationship that

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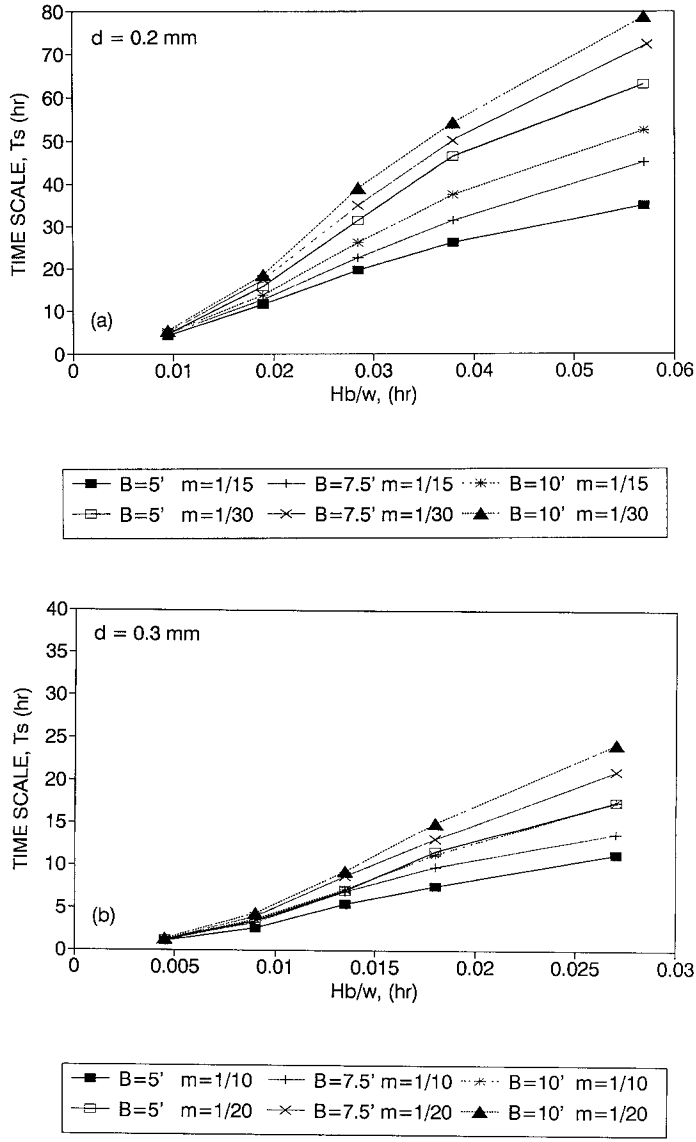

FIG. 10. Time Scale for Erosion Based on Numerical Model Results: (*a*) Sediment Grain Size of 0.2 mm; and (*b*) Sediment Grain Size of 0.3 mm

seems to describe the results of the numerical models tests has been investigated. In this case, the dimensionless time scale is found to be a function of three dimensionless parameters as

$$\frac{T_S\omega}{H_b} = f\left(\frac{(gH_b)^{1/2}}{\omega}, \frac{B}{h_b}, \frac{mx_b}{h_b}\right) \quad (30)$$

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where the breaking height H_b or breaking depth h_b can be used interchangeably since H_b is assumed to equal $0.78h_b$. In (30), the first parameter represents a Froude number based on the shallow-water wave celerity (or group velocity) and the sediment fall speed, the second parameter represents the height of the available sand reservoir in the berm relative to the vertical depth of the active surf zone, and the third parameter represents the ratio of the beach-face slope to the average surf-zone slope.

Following the relationships suggested, an empirical expression that seems to describe the erosion time scale as determined from the numerical model tests has been found in the form

$$T_S = C_1 \frac{H_b^{3/2}}{g^{1/2}A^3}\left(1 + \frac{h_b}{B} + \frac{mx_b}{h_b}\right)^{-1} \quad (31)$$

where (15) has been used to relate the fall velocity to the A parameter for an equilibrium profile. In Fig. 11, the numerically generated values of the erosion time scale from the 60 numerical tests are plotted as a function of the expression on the right-hand side of (31). As may be seen, the numerical data collapse to very nearly a straight line such that the dimensionless coefficient C_1 may be determined from the slope of the data as $C_1 = 320$. Eq. (31), with this value of the empirical coefficient is then recommended at the present time for practical determination of the erosion time scale. For cases of dune erosion, the berm height B should be replaced by the total berm plus dune height $B + D$.

Application of Convolution Method

To illustrate the practical application of the convolution method, two case studies are now presented with input parameters summarized in Table 1.

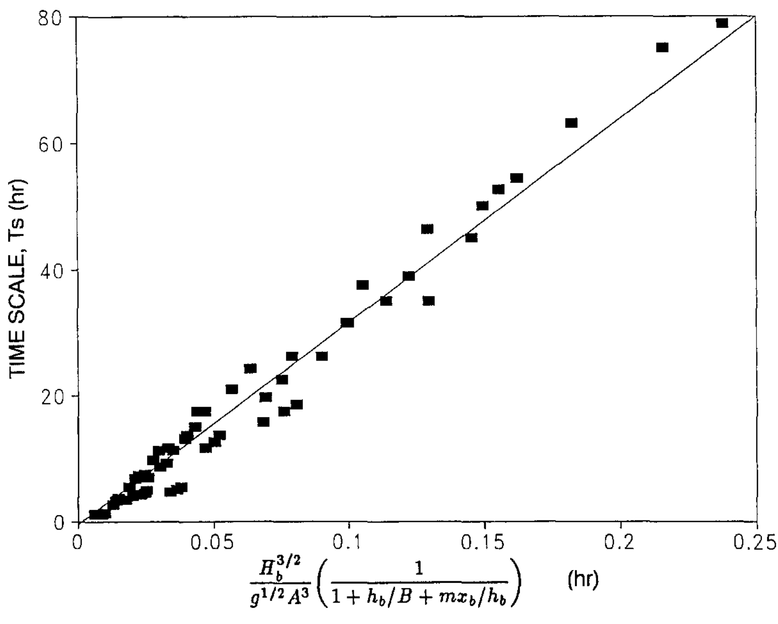

FIG. 11. Empirical Dependence of Erosion Time Scale on Wave and Sediment Properties Based on Numerical Model Results

TABLE 1. Case Studies of Equilibrium Response and Convolution Method

Variable (1)	Hurricane Eloise (2)	March 1984 northeaster (3)
	(*a*) Profile	
A ($m^{1/3}$)	0.135	0.175
$B + D$ (m)	5.5	3.4
m	0.10	0.12
	(*b*) Storm	
S (m)	2.74	1.83
T_D (hours)	8	66
h_b (m)	4.9	6.1

First, a short-duration hurricane is considered using average conditions from hurricane Eloise on the Florida Gulf Coast based on information from Chiu (1977). Second, a long-duration northeast storm is considered using average conditions from a March 1984 storm along the New Jersey coast based on data from the Philadelphia District of the Corps of Engineers. These cases are interesting because the hurricane had a higher peak storm surge, while the northeaster had a much longer storm duration.

Based on the initial profile forms, the peak surge level, and the estimated breaking depth, analytical solutions are first obtained for the maximum erosion potential, R_∞. For the hurricane, where the initial profile had high dunes with no backshore, the maximum potential dune erosion is estimated to be about 51.8 m (170 ft) based on (25). For the northeaster, where no dunes were present, the maximum potential erosion of the berm is estimated to be about 33.5 m (110 ft) based on (22). In terms of the potential volume eroded above the peak storm-surge level, the example hurricane has an erosion potential of 142.2 m^3/m (1530 cu ft/ft) based on (27), while the northeaster has an erosion potential of 51.1 m^3/m (550 cu ft/ft) according to (24).Thus, the short-duration hurricane is potentially more severe due mainly to its higher peak surge level, but also due to the finer sediment size and steeper beach slope at the Gulf coast site.

Next, time-dependent erosion effects are considered. First, erosion time scales were estimated using numerical results from (31) and Fig. 11. Based on the profile and wave parameters given in Table 1, the erosion time scales were found to be about 13.5 hours for the hurricane and about eight hours for the northeaster, such that a rate parameter of 0.07 hr^{-1} is applied to the hurricane while a rate parameter of 0.13 hr^{-1} is applied to the northeaster. As a result, the ratio β of the erosion time scale to the storm duration is found to differ by more than an order of magnitude: 10.6 for the short-duration hurricane and 0.76 for the long-duration northeaster. Based on (10) as shown in Figs. 2(*a*) and (*b*) [or Fig. 4(*a*)], it is then found that the time-dependent dimensionless maximum erosion, R_{max}/R_∞, is only 0.24 for the hurricane but is 0.91 for the northeaster. Thus, the longer storm duration allows the northeaster to achieve a greater percentage of its erosion potential.

In both cases, the convolution method provides reasonable estimates of the actual erosion response. It is estimated, for example, that the hurricane achieves only 24% of the maximum erosion potential, or about 12.4 m (41 ft) of dune erosion. According the Chiu (1977), average dune erosion [erosion of the 3.05-m (10-ft) elevation contour] during hurricane Eloise varied from about 12.2–15.2 m (40–50 ft) near the location of storm landfall. The time-dependent volumetric response above the peak surge level is then predicted to be about $0.24 \times 142.2 = 34.1$ m³/m (367 cu ft/ft). Chiu (1977) gives an average eroded volume of about 18.6 m³/m (200 cu ft/ft) with maximum observed values of up to 73.5 m³/m (791 cu ft/ft). However, these reported values are the volume eroded from above mean sea level. If (28) is applied to this case, the time-dependent estimate of the volume eroded from above mean sea level is then 53.0 m³/m (570 cu ft/ft), which is in the range of measured values but much larger than the average value reported by Chiu (1977). Part of this discrepancy is due, no doubt, to the approximate nature of the convolution method, however, a significant amount of beach recovery had also occurred by the time of poststorm beach survey so that the reported erosion volumes are also lower than those that actually occurred.

For the northeast storm, it is estimated from the convolution method that about 91% of the maximum erosion potential, or 30.5 m (100 ft) of berm erosion, is achieved during the storm. Based on data from the Philadelphia District of the Corps of Engineers, the actual berm erosion on six beach profiles at Point Pleasant, New Jersey during the storm varied from about 30.5 m (100 ft) to more than 42.7 m (140 ft), with an average over the six profiles of about 35.1 m (115 ft). In terms of the volume eroded from above the peak surge level, it is estimated by the convolution method that about $0.91 \times 51.1 = 46.5$ m³/m (501 cu ft/ft) of sand would be eroded for this storm. An analysis of the six profiles suggests that the volumetric erosion varied from about 42.8–53.4 m³/m (460–575 cu ft/ft) with an average of 50.2 m³/m (540 ct ft/ft).

Summary and Conclusions

The convolution method provides a simplified procedure for computing cross-shore beach-profile response to time-varying water-level and wave conditions. The method is based on the assumption that the beach is a linear-dynamic system. As a result, the system output (the beach erosion response) is determined as a function of the system input (the erosion forcing due to variable water level and breaking waves) and the characteristic exponential beach response to any step-type forcing function. This is formalized through a convolution integral in which the erosion-forcing function is convolved with the basic exponential response function to produce a lagged and damped time-dependent erosion response.

One attractive feature of the convolution method is that it may be applied either analytically or numerically to estimate the beach response to storm-surge conditions. The emphasis in this paper has been on an analytical solution for an idealized storm-surge hydrograph from which the erosion response may be obtained in closed form. For a measured (surveyed) beach profile, however, the maximum potential beach response may be estimated numerically by shifting the measured profile upward and landward until a mass balance is achieved. Once this is accomplished, the equilibrium response of each elevation contour, denoted by index i, can be determined as $R_{i\infty}$. The time-dependent recession of each contour can then be approximated by the convolution method as

$$R_i(t_j) = \alpha R_{i\infty} \int_0^{t_j} f(\tau) e^{-\alpha(t_j - \tau)} \, d\tau \quad \text{(32)}$$

where $f(t_j)$ could now represent a measured storm-surge hydrograph at times t_j, and where the convolution integral would be solved numerically. This procedure would allow all contours to respond exponentially with the same percentage of the maximum potential erosion. An unrealistic feature of this approach, and in fact of the entire convolution method, is that the upper contours of the beach profile would erode throughout the storm event, whereas in nature they would be activated only during the extreme high water levels.

Despite these inherent limitations, we believe that the convolution method can be a useful analysis tool for problems where preliminary calculations of beach-profile response are required. The method is also well suited to problems where a sensitivity analysis is required to determine the erosion potential as a function of numerous combinations of the initial beach geometry, the sediment size, the incident breaking wave height, and the peak storm-surge level. The two case studies presented in this paper illustrate the utility of the method in practice. Based on comparisons of predicted beach response to measured values, the convolution method seems to include time-dependent erosion effects in a reasonable way such that both storm-surge elevation and duration are important in defining the storm-erosion response.

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