Quantified statements worksheet

Part One

Assume:

- the domain consists of integers
- O(x) is "x is odd"
- L(x) is "$x < 10$"
- G(x) is "$x>9$"

What is the truth value of the following statements?

1. $\exists x\ [O(x)]$ True, e.g., take x=1
2. $\forall x\ [L(x) \rightarrow O(x)]$ False, e.g., take x=2
3. $\forall x\ [L(x) \rightarrow \neg G(x)]$ True. If an integer is <10 (9 or less), then it is not >9.
4. $\exists x\ [L(x) \wedge G(x)]$ False. There is no integer that is <10 and >9.
5. $\forall x\ [L(x) \vee G(x)]$ True.
6. $\exists x\ [L(x) \longrightarrow G(x)]$ True, e.g., take x= 20.

Assume:

- the domain consists of integers
- A(x) is "$x<5$"
- B(x) is "$x<7$"

What is the truth value of the following statements?

1. $\exists x\ [A(x)]$ True, e.g., take x=4.
2. $\exists x\ [A(x) \wedge B(x)]$ True, e.g., take x=4.
3. $\forall x\ [A(x) \wedge B(x)]$ False, e.g., take x=6.
4. $\forall x\ [A(x) \rightarrow B(x)]$ True. If x<5 then x<7.
5. $\forall x\ [B(x) \rightarrow A(x)]$ False. E.g., take x=6, then B(x) is true but A(x) is false.
6. $\exists x\ [A(x) \longrightarrow F]$ True. Take x=6. Then, A(x) is False and $A(x) \longrightarrow F$ is True.

Part Two

How do you write the negation of the following statements (use De Morgan's laws for quantified statements):

- All Americans eat cheeseburgers

  Negation: It is not the case that all Americans eat cheeseburgers.

  Use De Morgan's law: There exists an American that does not eat cheeseburgers.

- There is a smart student at NMSU. Note: The original statement is True; therefore, its negation will be False.

  Solution 1: Let domain D be students at NMSU. Let smart(x) mean "x is smart". Then, the original statement can be written as $\exists$ x smart(x). Negation is $\neg$ ($\exists$ x smart(x)). Using De Morgan's law we get: $\forall$x $\neg$smart(x), which is "All NMSU students are not smart".

  Solution 2. Let domain D be students. Let smart(x) mean "x is smart". Let atNMSU(x) mean "x is at NMSU". Then, the original statement can be written as $\exists$ x (smart(x) $\wedge$ atNMSU(x)). Negation is $\neg$($\exists$ x (smart(x) $\wedge$ atNMSU(x))). Using De Morgan's law we get: $\forall$x $\neg$(smart(x) $\wedge$ atNMSU(x)) $\equiv$ $\forall$x ($\neg$smart(x) $\vee$ $\neg$atNMSU(x)), which is "Every student either is not smart or is not at NMSU".

Part Three

How do you write the negation of the following statements?

- $\forall$ real numbers x, if $x^2 >= 1$ then $x > 0$

Original statement: $\forall$ real numbers x, $(x^2 >= 1) \rightarrow (x>0)$

Negation: $\neg(\forall$ real numbers x, $(x^2 >= 1) \rightarrow (x>0))$

$\equiv \exists$ real number x, $\neg((x^2 >= 1) \rightarrow (x>0))$

$\equiv \exists$ real number x, $\neg(\neg(x^2 >= 1) \vee (x>0))$

$\equiv \exists$ real number x, $(\neg\neg(x^2 >= 1) \wedge \neg(x>0))$

$\equiv \exists$ real number x, $(x^2 >= 1) \wedge \neg(x>0)$

- For every student at NMSU if they have been at NMSU for at least two years then they are classified as a Junior.

Original statement: $\forall$ student at NMSU x, beenAtNMSUforAtLeast2Years(x) $\rightarrow$ Junior(x)

Similar, to the previous example, negation is the following:

$\exists$ student at NMSU x, beenAtNMSUforAtLeast2Years(x) $\wedge$ $\neg$ Junior(x)

That is, "There is a student at NMSU that has been at NMSU for at least 2 years but is not a Junior".

Part Four

How would you write these in English? Assume that the domain for x is all humans.

($\forall$ x) [GoesToNMSU(x) $\rightarrow$ Smart(x)]

Everyone who goes to NMSU is smart. (Note: the statement does not mean that everyone goes to NMSU).

$(\forall x)$ $[$ GoesToNMSU$(x) \wedge$ Smart(x) $]$

Everyone goes to NMSU and is smart. (Note: the statement means that everybody goes to NMSU and everybody is smart).

$(\exists x)$ $[$ GoesToNMSU$(x) \rightarrow$ Smart(x) $]$

Someone either does not go to NMSU or is smart.

$(\exists x)$ $[$ GoesToNMSU$(x) \wedge$ Smart(x) $]$

Someone goes to NMSU and is smart.