HW 4

1) 6.1.1 (h) The n^{th} term is $2^{\lceil \log n \rceil}$

$2^{\lceil \log 1 \rceil}, 2^{\lceil \log 2 \rceil}, 2^{\lceil \log 3 \rceil}, 2^{\lceil \log 4 \rceil}, 2^{\lceil \log 5 \rceil}, 2^{\lceil \log 6 \rceil}, 2^{\lceil \log 7 \rceil}, 2^{\lceil \log 8 \rceil}, 2^{\lceil \log 9 \rceil}, 2^{\lceil \log 10 \rceil}$

$= 1, 2, 2, 2, 2, 2, 2, 2, 2, 2$

$\boxed{\text{Non-decreasing. Not increasing, decreasing, or non-increasing}}$

2) 6.1.4 (b) $\{a, ar, ar^2, ar^3 \ldots\}$ $\{x_n\}$ $x_n = ar^{n-1}$

$\forall n \in \mathbb{N} = \{1, 2, 3 \ldots\}$

if $a \neq 0$, $\{x_n\}$ is zero sequence,

$\frac{x_{n+1}}{x_n} = \frac{ar^n}{ar^{n-1}} = r$ $\forall n = 1, 2, 3.$

if $x_{n+1} < x_n$

$\frac{x_{n+1}}{x_n} < 1$

$r < 1$

$\therefore \boxed{a \neq 0, \text{ and } r < 1}$

3) 6.2.1 (e) $b_1 = 1$, $b_2 = 3$, and $b_n = b_{n-1} - 7 \cdot b_{n-2}$ for $n \geq 3$

$b_1 = 1$, $b_2 = 3$, $b_3 = b_2 - 7 \cdot b_1 = -4$, $b_4 = b_3 - 7 \cdot b_2 = -4 - 21 = -25$,

$b_5 = b_4 - 7 \cdot b_3 = -25 + 28 = 3$, $b_6 = b_5 - 7 \cdot b_4 = 3 + 175 = 178$,

$\boxed{1, 3, -4, -25, 3, 178}$

4) 6.3.1 (c) $\sum_{k=-3}^{2} k^3 = (-3)^3 + (-2)^3 + (-1)^3 + 0^3 + 1^3 + 2^3$

$= (-27) + (-8) + (-1) + 0 + 1 + 8$

$= \boxed{-27}$

(g) $\sum_{k=0}^{100} (3+5k) = \sum_{k=0}^{100} 3 + \sum_{k=0}^{100} 5k = 3(101) + \frac{5 \cdot 100 \cdot 101}{2} = 303 + 25250 = \boxed{25553}$

(h) $\sum_{k=0}^{100} 3 \cdot (1.1)^k = 3\left(\frac{1.1^{101} - 1}{0.1}\right) = 30(1.1^{101} - 1) \approx \boxed{454730.2072}$

5) 6.3.3 (b) $\sum_{j=-2}^{100} (j^2+2^j+\frac{j}{2}) = \sum_{j=-2}^{99} (j^2+2^j+\frac{j}{2}) + (100^2+2^{100}+\frac{100}{2})$

$= \boxed{\sum_{j=-2}^{99} (j^2+2^j+\frac{j}{2}) + (10000+2^{100}+50)}$

(d) $\sum_{k=0}^{m+2} (k^2-4k+1) = \boxed{\sum_{k=0}^{m+1} (k^2-4k+1) + (m+2)^2 - 4(m+2)+1}$

6) 6.3.4 (a) $\sum_{j=0}^{n} (j+2)$, i for j, $i=j+2$

For $j=0$, $i=j+2=0+2=2$

For $j=n$, $i=j+2=n+2$

If $i=j+2$, then $j=i-2$, and $(j+2)=((i-2)+2)=i$

$\sum_{j=0}^{n} (j+2) = \boxed{\sum_{i=2}^{n+2} i}$

(e) $\sum_{k=10}^{20} (6k-4)$, j for k, $j=k+5$

For $k=10$, $j=k+5=10+5=15$

For $i=20$, $j=k+5=20+5=25$

If $j=k+5$, then $k=j-5$, and $6k-4=6(j-5)-4=6j-34$

$\sum_{k=10}^{20} (6k-4) = \boxed{\sum_{j=15}^{25} (6j-34)}$

7) 6.4.2 (c) for any positive integer n, $\sum_{j=1}^{n} j(j-1) = \frac{n(n^2-1)}{3}$

Proof.

By induction on n,

Base case:

$n=1$. $\sum_{j=1}^{1} j(j-1) = 1\cdot(1-1) = 0$ and $\frac{n(n^2-1)}{3} = \frac{1\cdot(1-1)}{3} = \frac{0}{3} = 0$, so equality holds when $n=1$.

Inductive step:

Assume that k is a positive integer and $\sum_{j=1}^{k} j\cdot(j-1) = \frac{k(k^2-1)}{3}$.

Show that $\sum_{j=1}^{k+1} j(j-1) = \frac{(k+1)((k+1)^2-1)}{3} = \frac{(k+1)(k^2+2k)}{3} = \frac{k(k+1)(k+2)}{3}$

$$\sum_{j=1}^{k+1} j(j-1) = \sum_{j=1}^{k} j(j-1) + (k+1)(k+1-1)$$

$$= \frac{k(k^2-1)}{3} + (k+1)(k+1-1) \quad \text{by the inductive hypothesis}$$

$$= \frac{k(k^2-1)}{3} + (k+1)k$$

$$= \frac{k(k+1)(k-1)}{3} + (k+1)k$$

$$= (k+1)k\left[\frac{k-1}{3}+1\right]$$

$$= (k+1)k\cdot\frac{k+2}{3}$$

$$= \frac{k(k+1)(k+2)}{3}$$

Therefore, $\sum_{j=1}^{k+1} j(j-1) = \frac{k(k+1)(k+2)}{3}$. ■

(e) For any non-negative integer n, $\sum_{j=0}^{n} j\cdot 3^j = \frac{3}{4}[n\cdot 3^{n+1} - (n+1)3^n + 1]$

Proof.

By induction on n.

Base case:

$n=0$, $\sum_{j=0}^{0} j\cdot 3^j = 0$ and $\frac{3}{4}[0-1\cdot 1+1] = 0$, so equality holds when $n=0$.

Inductive step:

Assume that k is a non-negative integer and $\sum_{j=0}^{k} j\cdot 3^j = \frac{3}{4}[k\cdot 3^{k+1} - (k+1)3^k + 1]$

Show that $\sum_{j=0}^{k+1} j\cdot 3^j = \frac{3}{4}[(k+1)\cdot 3^{k+2} - (k+2)3^{k+1} + 1]$.

$$\sum_{j=0}^{k+1} j\cdot 3^j = \sum_{j=0}^{k} j\cdot 3^j + (k+1)\cdot 3^{(k+1)}$$

$$= \frac{3}{4}[k\cdot 3^{k+1} - (k+1)3^k + 1] + (k+1)\cdot 3^{(k+1)}$$

$$= \frac{3}{4}[k\cdot 3^{k+1} - (k+1)3^k + 1 + \frac{4}{3}(k+1)\, 3^{k+1}]$$

$$= \frac{3}{4}[k\cdot 3^{k+1} - (k+1)3^k + 1 + 4(k+1)\, 3^k]$$

$$= \frac{3}{4}[k\cdot 3^{k+1} - (k+1)3^k + 1 + 6(k+1)3^k - 2(k+1)3^k]$$

$$= \frac{3}{4}[k\cdot 3^{k+1} - (k+1)3^k - 2(k+1)3^k + 2(k+1)3^{k+1} + 1]$$

$$= \frac{3}{4}[k\cdot 3^{k+1} + 3^{k+1} - 3^{k+1} - 3(k+1)\cdot 3^k + 2(k+1)3^{k+1} + 1]$$

$$= \frac{3}{4}[(k+1)3^{k+1} - 3^{k+1} - (k+1)3^{k+1} + 2(k+1)3^{k+1} + 1]$$

$$= \frac{3}{4}[3(k+1)3^{k+1} - (k+2)3^{k+1} + 1]$$

$$= \frac{3}{4}[(k+1)\cdot 3^{k+2} - (k+2)3^{k+1} + 1]$$

Therefore, $\sum_{j=0}^{k+1} j\cdot 3^j = \frac{3}{4}[(k+1)\cdot 3^{k+2} - (k+2)3^{k+1} + 1]$. ∎

holds for $n=4$ ✓

6.4.3 (f) Proof. By induction on n.

Base case: Show that for $n \geq 4$, $3^n \leq (n+1)!$. $3^4 = 81 \leq 120 = (4+1)!$, so the inequality.

Inductive step: Assume that k is positive integer that $k \geq 4$, and $3^k \leq (k+1)!$. Show that

$3^{k+1} \leq (k+2)!$. we will prove that $n = k+1$

$$3^{k+1} = 3^k \cdot 3^1$$

$\leq (k+1)! \cdot 3$ by inductive hypothesis ← since $n \geq 4$, $k \geq n$, $(k+2)$ is always

$\leq (k+1)!\,(k+2)$ ← greater than 3, plug in $(k+2)$ in 3.

$\leq (k+2)(k+2-1)!$ ← $(k+2)! = (k+2)(k+2-1)!$

$\leq (k+2)(k+1)!$ $\qquad = (k+2)(k+1)!$

Therefore, $3^{k+1} \leq (k+2)!$. ∎