HW 2

1. 3.2.2 Prove each statement using a proof by exhaustion.

(c) For all positive integers $n \leq 4$, $(n+1)^3 \geq 3^n$

Proof.

The only integers n such that $n \leq 4$, are $n=0, n=1, n=2, n=3$, and $n=4$.

- when $n=0$, $(n+1)^3 = 1$ and $3^n = 1$. $1=1$
- when $n=1$, $(n+1)^3 = 8$ and $3^1 = 3$. $8>3$
- when $n=2$, $(n+1)^3 = 27$ and $3^2 = 9$. $27>9$
- when $n=3$, $(n+1)^3 = 64$ and $3^n = 27$. $64>27$
- when $n=4$, $(n+1)^3 = 125$ and $3^n = 81$. $125>81$. ■

2. 3.2.3 Find a counterexample to show that each of the statements is false.

(c) For every positive integer x, $x^3 < 2^x$

if $x=3$, $x^3=27$, and $2^3=8$, $27<8$ is false.

3. 3.2.5 Prove each existential statement given below

(h) For every pair of real numbers, x and y, there exists a real number z such that $x-z=z-y$

Let x and y be real numbers, let $z = \frac{x+y}{2}$ is a real numbers. Then

$$x - z = x - \frac{x+y}{2} = \frac{2x-(x+y)}{2} = \frac{2x-x-y}{2} = \frac{x-y}{2}$$

$$z - y = \frac{x+y}{2} - y = \frac{x+y-2y}{2} = \frac{x-y}{2}$$

Because both $x-z$ and $z-y$ equals $\frac{x-y}{2}$

Therefore, $x-z = z-y$.

4. 3.2.6 For each statement below state what needs to be proven in order to show that the existential statement is false, response should be universal statement

(d) There are positive integers m and n such that $\sqrt{m+n} = \sqrt{m} + \sqrt{n}$

$\boxed{\text{For every positive integers } m \text{ and } n,\ \sqrt{m+n} \neq \sqrt{m} + \sqrt{n}}$

prove: $(\sqrt{m+n})^2 = (\sqrt{m} + \sqrt{n})^2$

$m+n = m + 2\sqrt{m}\sqrt{n} + n$

$2\sqrt{mn} = 0$

$mn = 0$

either $m=0$ or $n=0$, since $m>0$ $n>0$

a contradiction occured.

Therefore, $\sqrt{m+n} \neq \sqrt{m} + \sqrt{n}$

5. 3.3.3 Theorem: If n and m are odd integers, then n^2+m^2 is even. For each "proof" of the theorem, explain where the proof uses invalid resoning or skips essential steps.

(a) Invalid, for general case.
Can't prove a theorem by putting n and m, by a specific number.

Second and third line

(b) The proof is correct, no mistake.

(c) contradiction, There can not be two line suggest $n=2k+1$ and $n^2=n\cdot n$, wipe out the
Only to show that product of two odd integers are odd, and the sum of two odd integer is even.

(d) skip steps, in the proof, need to break down the sum of the squares
$n^2+m^2=(2k+1)^2+(2j+1)^2$ as $4k^2+4k+1+4j^2+4j+1=2(2k^2+2k+2j^2+2j+1)$
Then to show that the sum is two times an integer

(e) Invalid, the values of m and n can not be the same, must different variables
for example, the $n=m=2j+1$ have to replace as $n=2j+1$ and $m=2k+1$
where k and j are any integers.

6. 3.4.1 (e) if x is an even integer and y is an odd integer, then x^2+y^2 is odd.
Proof.
Assume that both x is an even integer and y is an odd integer. We will show that x^2+y^2 is odd.
Since x is even, $x=2m$, for some integer m, Since y is odd, $y=2n+1$ for some integer n.

$$x^2+y^2=(2m)^2+(2n+1)^2=4m^2+4n^2+4n+1=4(m^2+n^2+n)+1$$

Since both m and n are integers, m^2+n^2+n is also an integers. $4=2^2$. Since x^2+y^2 can be expressed as 4 times an integer, which equal 2 times an integer, $4(m^2+n^2+n)$ will be even, and add 1 at the end which makes $4(m^2+n^2+n)+1$ is an odd integer, which x^2+y^2 is an odd integer. ∎

7. 3.4.2 Prove each statement using a direct proof

(f) The average of two rational numbers is also rational.

Proof.

Assume x and y are rational numbers. Since x and y are rational, $x = \frac{a}{b}$ and $y = \frac{c}{d}$, where a, b, c and d are integers, and $b \neq 0$ and $d \neq 0$. We will show that $\frac{x+y}{2}$ is a rational number.

$$\frac{x+y}{2} = \frac{\frac{a}{b}+\frac{c}{d}}{2} = \frac{ad+bc}{2bd}$$

Since a, b, c, and d are all integers, $ad+bc$ and $2bd$ are also integers. Since $b \neq 0$, and $d \neq 0$, then $bd \neq 0$, then $2bd \neq 0$. Therefore, $\frac{x+y}{2}$, (which is the average of two rational numbers) is equal to the ratio of two integers in which the denominator is not zero which implies that $\frac{x+y}{2}$ is rational. ∎

8. 3.5.1 Prove statement by contrapositive

(d) For every integer n, if n^2-2n+7 is even, then n is odd.

Proof.

We will assume that n is an even integer and show that n^2-2n+7 is an odd integer.

Since n is even, then $n = 2k$ for some integer k. Plugging in the expression 2k for n in n^2-2n+7 gives

$$(2k)^2-2(2k)+7 = 4k^2-4k+7 = 2(2k^2+2k+3)+1$$

Since k is an integer, $2k^2+2k+3$ is also an integer, n^2-2n+7 is equal to two times an integer plus 1, and therefore n^2-2n+7 is an odd integer. ∎

9. 3.5.4 Prove statement by contrapositive

(a) For every pair of real numbers x and y, if $x^3+xy^2 \leq x^2y+y^3$, then $x \leq y$.

Proof.

We assume that for positive real numbers x and y, it is not the case that $x \leq y$ and prove that $x^3+xy^2 \leq x^2y+y^3$

Since it is not true that $x \leq y$, by De Morgan's law, the inequalities $x<y$ and $x=y$ are both false, then $x>y$ must be true. Because $x>y$, then show that $x-y>0$, The equation says:

$$(x^3+xy^2)-(x^2y+y^3) = x(x^2+y^2)-y(x^2+y^2) = (x-y)(x^2+y^2) > 0$$

then $x^3+xy^2 > x^2y+y^3$ must greater than 0.

$(x-y)>0$ and $(x^2+y^2)>0$

For every pair of x, y real numbers with $x \neq 0, y \neq 0$ and $(x^2+y^2) \geq 0$, which is $x=0$, $y=0$, but the $x>y$, so $x \neq 0$, and $y \neq 0$,

$\therefore x>y$

Therefore, $x^3+xy^2 > x^2y+y^3$. ■

10. 3.6.6 Give a proof.

(b) If a person buys at least 400 cups of coffee in a year, then there is at least one day in which the person has bought at least two cups of coffee

Proof.

Proof by contradiction → Suppose that person has less than 2 cups of coffee a day. Total cups of coffee will equal to 400. The person has bought at least one cup of coffee each day which mean that total number of cups of coffee $= 1 \times 366 = 366$, which contradicts with the argument: the person buys at least 400 cups of coffee in a year.

11. 3.6.6. Give a proof.

(e) There is no largest odd integer.

Proof.

Proof by contradiction. Suppose that there is a largest odd integer x. If x is an odd integer, then $2x+1$ is also an odd integer. Futhermore, $2x+1 > x$. Therefore, $2x+1$ is an odd integer that is greater then x, which contradicts the assumption that x is the largest odd integer. Therefore, the assumption that there exists a largest odd integer is false. ■

12. 3.7.1 Prove statement

(d) If x is a real number such that $x^2-3x-10<0$, then $-2<x<5$.

Proof.

We will show that if $x \le -2$ or $x \ge 5$, then $x^2-3x-10 \ge 0$. Note that $x^2-3x-10 = (x+2)(x-5)$, so we must show that $(x+2)(x-5) \ge 0$.

Case 1 = $x \le -2$. Adding 2 to both sides of the inequality gives that $x+2 \le 0$. Since $x-5 \le x+2$, this implies that $x-5 \le 0$. Multiplying both sides of the inequality $x+2 \le 0$ by $x-5$ means that $(x+2)(x-5) \ge 0$. Note that the inequality is reversed since the multiplier $x-5$ is less then or equal to 0.

Case 2 = $x \ge 5$. Subtracting 5 from both sides of the inequality gives that $x-5 \ge 0$. Since $x-5 \le x+2$, this implies that $x+2 \ge 0$. Multiplying both sides of the inequality $x-5 \ge 0$ by $x+2$ means that $(x+2)(x-5) \ge 0$. ■

13. 3.7.2 Prove statement

(f) Let x and y be two integers. If xy is not an integer multiple of 5, then neither x nor y is an integer multiple of 5.

Proof.

We will show that if x or y is an integer multiple of 5, then xy is an integer multiple of 5.

We consider two cases. Case 1 is that x is an integer multiple of 5, Case 2 is that y is integer multiple of 5.

$x=5k$

$xy=5ky$

$y=5x$

$xy=5xy$

Case 1: x is integer multiple of 5. Since x is an integer multiple of 5 $x=5k$ for some integer k. Plugging in $5k$ for x in the expression xy gives

$$xy=(5k)y=5ky.$$

Since k and y are integers, $5ky$ is also integer. ky times 5 must be an integer multiple of 5. Therefore, xy is integer multiple of 5.

Case 2: y is integer multiple of 5, Since y is an integer multiple of 5, $y=5j$ for some integer j. Plugging in $5j$ for y in the expression xy gives

$$xy=x(5j)=5jx=5xj.$$

Since j and x are integers, $5xj$ is also integer. xj times 5 must be an integer multiple of 5. Therefore, xy is integer multiple of 5. ∎