Proof by Contrapositive

Proof by Contradiction

How would you prove

Prove: For all integers n,

if n^2 is even then n is even.

Remember the Definition of Contrapositive

- Original Statement

  $p \to q$

- Contrapositive

  $\neg q \to \neg p$

The original statement is logically equivalent to its contrapositive. If contrapositive is true, so is the original statement.

Directions for Writing Proofs by Contrapositive

1. Express the original statement to be proved in the form:

   For all x, if A(x) is true then B(x) is true.

2. Rewrite this statement in the contrapositive.

   For all x, if B(x) is false then A(x) is false.

3. Prove the contrapositive by direct proof.

Examples of proof by contrapositive

Prove: For all integers n if n^2 is even then n is even

Examples of proof by contrapositive

Prove: If a product of two positive real numbers is greater than 100, then at least one of the numbers is greater than 10.

Examples of proof by contrapositive

Prove: For all integers n if n^2 is odd then n is odd.

Prove: If the sum of two real numbers is less than 50, then at least one of the numbers is less than 25.

Proof by Contradiction

- Another tool is Proof by Contradiction.

- Introduction:

  If A is true, what do you know about $\neg A$?

  If A is false, what do you know about $\neg A$?

  If $\neg A$ is false, what do you know about A?

Proof by Contradiction

1. Suppose that the statement is false. That is, suppose the negation of the statement is true
2. Show that this new supposition leads to a contradiction.
3. Conclude that the original statement must be true.

Examples of proof by contradiction

- There is no greatest integer

Examples of proof by contradiction

- If a is a rational number and b is an irrational number, then a+b is an irrational number.

Examples of proof by contradiction

- There is no integer that is both even and odd

Section 2.5

Proof by cases

Proof by Cases

- Sometimes a general proof over ALL cases is nearly impossible to do.
  - But, the proof can be clearly divided into several distinct ***cases*** which cover all possible situations.
  - The proof then consists of several smaller proofs where you cover each case individually.

Example of proof by cases

- The product of any two consecutive integers is even.

Proof.

Let n and $n+1$ be two consecutive integers.

Case 1: n is even.

…

Case 2: n is odd.

…

Example of proof by cases

- Prove that if a and b are real numbers, where neither is equal to zero, then

$$\left| \frac{a}{b} \right| = \frac{|a|}{|b|}$$

Proof.

Case 1: a≥0 and b ≥0

Case 2: a≥0 and b <0

Case 1: a<0 and b ≥0

Case 2: a<0 and b <0

Example of proof by cases

- For all real numbers x,

$$x + |x - 7| \geq 7$$

Proof.

Case 1: $x \geq 7$

Case 2: $x < 7$

Example of proof by cases

- Let n be an integer. If 3 does not divide n, then 3 divides $n^2 - 1$.

Proof.

Case 1: $n=3k+1$ for some integer k

Case 2: $n=3k+2$ for some integer k