Introduction to proofs

Part One

Define the following vocabulary words.

- theorem

A theorem is a statement that can be proven to be true.

- proof

A proof consists of a series of steps, each of which follows logically from assumptions, or from previously proven statements, whose final step should result in the statement of the theorem being proven.

- axioms

Axioms are statements assumed to be true.

- proof by exhaustion

If the domain of a universal statement is small, the statement can be proved by checking each element individually. A proof of this kind is called a proof by exhaustion.

- counterexample
  A counterexample is an assignment of values to variables that shows that a universal statement is false.

Part Two

For each of the following theorems either prove by direct proof or disprove by counterexample

1. There is a perfect square that can be written as the sum of two other perfect squares.
   Proof. We can take 9, 16, and 25. All of the numbers are perfect squares and $25=16+9$.

2. There is an integer n such that $2n^2 - 5n + 2$ is prime.
   Proof. If $n=3$, then $2n^2 - 5n + 2 = 5$ which is prime.

3. For all real numbers a and b, if $a<b$ then $a^2 < b^2$
   The statement is False. Counterexample: Let $a=-5$, $b=2$. Then, $a^2 = 25$, $b^2 = 4$, and 25 is not less than 4.

4. For all integers n, if n is odd then $(n-1)/2$ is odd.
   The statement is False. Counterexample: Let $n=5$. Then, n is odd but $(n-1)/2 = 2$ is not odd.

5. For all integers m and n, if $2m + n$ is odd then m and n are both odd.
   The statement is False. Counterexample: Let $m=2$ and $n=1$. Then, $2m+n = 5$ which is odd, but m and n are not both odd since m is even.

Part Three

Find the mistake in the proof. Explain where the proof uses invalid reasoning or skips essential steps.

Theorem: If w, x, y, z are integers where w divides x and y divides z, then wy divides xz.

(a) Proof.

Since, by assumption, w divides x, then x = kw for some integer k. Since, by assumption, y divides z, then z = ky for some integer k. Plug in the expression kw for x and ky for z in the expression xz to get

$xz=(kw)(ky)=(k^2)(wy)$

Since k is an integer, then k^2 is also an integer. Since xz equals wy times an integer, then wy divides xz.

Mistake: The same variable name k is used in the definition of "divides" to establish that w divides x and y divides z. If x = kw and z = ky, then x/w = z/y which is not necessarily true.

(b) **Proof.**

Since, by assumption, w divides x, then x = kw for some integer k.
Since, by assumption, y divides z, then z = jy for some integer j.
Let m be an integer such that xz = m⋅wy. Since xz equals wy times an integer, then wy divides xz.

Mistake: This "proof" assumes what is to be proven (that wy divides xz) before it is proven.

(c) **Proof.**

Since w divides x, then x = kw. Since, by assumption, y divides z, then z = jy. Plug in the expression kw for x and jy for z in the expression xz to get

$xz=(kw)(jy)=(kj)(wy)$

Since k and j are integers, then kj is also an integer. Since xz equals wy times an integer, then wy divides xz.

Mistake: This proof does not introduce the variable names k and j and indicate that the variables represent integers.