Species in HoTT

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Abstract

Combinatorial species were developed by Joyal (1981) as an abstract treatment of enumerative combinatorics, especially problems of counting the number of ways of putting some structure on a finite set. Many of the results of species theory are special cases of more general properties of homotopy types, making homotopy type theory (HoTT) a useful tool for dealing with species. These tools become even more apposite when one generalizes species to higher groupoids, as Baez and Dolan (2001) do. What follows are notes I wrote while learning about species. They're mainly summary of the notes Derek Wise took during John Baez's "Quantization and Categorification" seminar in AY2004 (Baez and Wise, 2003, 2004b,a), with some reference to Bergeron et al. (2013), Baez and Dolan (2001), and Aguiar and Mahajan (2010).

1 Defining species

As originally defined by Joyal (1981), a species is an endofunctor of the groupoid FinSet of finite sets and bijections between them. Baez and Dolan (2001) generalized species to stuff types, which are functors $F : X \to \mathsf{FinSet}$ for any groupoid X. This formulation admits a natural generalization to ∞-species, or homotopical species, by replacing the groupoid X with an ∞-groupoid—in other words, by working with homotopy types. So we can formulate species in HoTT as follows.

Definition 1.1. The canonical finite type with n elements is defined by induction on $\mathbb{N}$ by

$$\begin{aligned} \mathsf{Fin}(0) &:\equiv \mathbf{0} \\ \mathsf{Fin}(\mathsf{succ}(n)) &:\equiv \mathsf{Fin}(n) + \mathbf{1} \end{aligned}$$

Definition 1.2. The type of finite sets is

$$\mathsf{FinSet} :\equiv \sum_{(A:\mathcal{U})} \sum_{(n:\mathbb{N})} \|A = \mathsf{Fin}(n)\|$$

Definition 1.3. A species is a type $X : \mathcal{U}$ equipped with a function $f : X \to \mathsf{FinSet}$.

The definition of Fin won't be unfolded, so you can use your favorite. Its role is to provide a standard set with n elements, allowing us to define a predicate $\mathsf{isFinSet}$ such that $\mathsf{FinSet} :\equiv \{A : \mathcal{U} \mid \mathsf{isFinSet}(A)\}$. Since $\mathsf{isFinSet}$ is a family of mere propositions, we'll often ignore it and abusively write A for both a term of type FinSet and its first component, which is of type $\mathcal{U}$, eliding reference to the inhabitant of $\mathsf{isFinSet}(A)$.

Now, the intended interpretation of a species $f : X \to \mathsf{FinSet}$ is that for some finite set A, an element of $\mathsf{fib}_f(A)$ is a way of equipping A with "f-stuff". This gives us a natural way to come up with examples of species. For some type family $\Phi : \mathsf{FinSet} \to \mathcal{U}$, there is a corresponding species $\mathsf{pr}_1 : \sum_{(A:\mathsf{FinSet})} \Phi(A) \to \mathsf{FinSet}$. By Lemma 4.8.2 of the HoTT Book (UFP, 2013), any species is equal to one of this form. So we can specify a species up to equality by giving a family $\Phi : \mathsf{FinSet} \to \mathcal{U}$ and taking the first projection to be the map.

All of this is just setup for the main application of species: generating functions. Given a species X, we want to cook up a formal power series

$$|X|(z) = \sum_{n \geq 0} X_n \, z^n$$

where $X_n : [0, \infty)$. Intuitively, X_n should represent the "number of ways of putting X stuff on an n-element set". For a species $f : X \to \mathsf{FinSet}$, this description picks out something like the cardinality of $\mathsf{fib}_f(A)$, where A is an n-element set. There are two problems with this: (i) we need a notion of cardinality, and (ii) we need some way to pick an A from all of the n-element sets. To solve the first problem we use groupoid cardinality $|X|$ of X.

Definition 1.4. For any $X : \mathcal{U}$, if $\|X = \mathsf{Fin}(n)\|$ for some $n : \mathbb{N}$, then $|X| = n$.

Definition 1.5. The groupoid cardinality $|X|$ of a type X is, if it converges, the sum

$$|X| = \sum_{[x] \in \pi_0(X)} \prod_{k \geq 1} |\pi_k(X, x)|^{(-1)^k} = \sum_{[x] \in \pi_0(X)} \frac{1}{|\pi_1(X,x)|} \, |\pi_2(X,x)| \, \frac{1}{|\pi_3(X,x)|} \, |\pi_4(X,x)| \cdots$$

Via the homotopy hypothesis, this is equivalently the Euler characteristic of a topological space presenting the type X.

This solves the problem of finding some notion of cardinality. To solve problem (ii), we avoid an arbitrary choice of A by decategorifying and regarding $f : X \to \mathsf{FinSet}$ as a fibration over $\mathbb{N}$, rather than FinSet. Specifically, we consider the sequence

$$\begin{tikzcd} X \arrow[r, "f"] & \mathsf{FinSet} \arrow[r, "|-|_0"] \arrow[rr, bend right, "\mathsf{card}"'] & \|\mathsf{FinSet}\|_0 \arrow[r, "\mathsf{ext}(\mathsf{card})"] & \mathbb{N} \end{tikzcd}$$

where $\mathsf{card} :\equiv \mathsf{pr}_1 \circ \mathsf{pr}_2$, given our definition of FinSet, and ext is the extension along $|-|_0$, the 0-truncation map. The reason this counts as decategorification is that we're squashing FinSet into a set, meaning that we're identifying all of the equinumerous sets. This produces a type $\|\mathsf{FinSet}\|_0 \simeq \mathbb{N}$. So, finally, we have the

Definition 1.6. The generating function $|X|(z)$ of a species $f : X \to \mathsf{FinSet}$ is given by

$$|X|(z) = \sum_{n \geq 0} \left| \mathsf{fib}_{\mathsf{card} \circ f}(n) \right| \, z^n = \sum_{n \geq 0} X_n \, z^n$$

where this expression defines X_n.

We'll variously write $|f|(z)$, $|X|(z)$, or $|\Phi|(z)$, as appropriate, for species $f : X \to \mathsf{FinSet}$ or $\Phi : \mathsf{FinSet} \to \mathcal{U}$.

2 Computing cardinalities

There are two kinds of results that simplify the computation of generating functions. First, one can compute the cardinality of composite types in terms of the cardinalities of their components. Second, the definition of the generating function can be simplified to eliminate reference to card. I'm going to be a bit sloppy about the scope of this section's discussion. Everything works for essentially finite 1-types, which are going to be the ones I'm mainly dealing with.

Four properties of the groupoid cardinality follow from the definition above. In fact, these four axiomatically characterize the groupoid cardinality:

Claim 2.1. $|\mathbf{1}| = 1$.

Claim 2.2. If $\|X \simeq Y\|$, then $|X| = |Y|$.

Claim 2.3. $|X + Y| = |X| + |Y|$

Claim 2.4. If there is a map $f : X \to Y$ and $|\mathsf{fib}_f(y)| = |\mathsf{fib}_f(y')|$ for all $y, y' : Y$, then $|X| = |\mathsf{fib}_f(y)| \cdot |Y|$ for any $y : Y$.

The first three of these are immediate. The fourth follows from the homotopy hypothesis. If $p : A \to B$ is a fibration with fiber F and B is path-connected, then the Euler characteristic $\chi(E)$ of E is equal to $\chi(F) \cdot \chi(B)$. Since any function is equivalent to a fibration, for any two points y, y' connected by a path we have $\mathsf{fib}_f(y) \simeq \mathsf{fib}_f(y')$, translating this result into groupoid-speak and then into HoTT, then summing over the connected components gives Claim 2.4.

Similar properties for Σ-types and products follow from these axioms. For Σ-types we have the

Claim 2.5. For any type X and family $P : X \to \mathcal{U}$, if $|P(x)| = |P(x')|$ for all $x, x' : X$ then

$$\left| \sum_{y:X} P(y) \right| = |P(x)| \cdot |X|$$

for any $x : X$.

This is a combination of the sum and fibration claims 2.3 and 2.4. If X is 0-connected, then we automatically get $\prod_{(x,x':X)} (P(x) \simeq P(x'))$ by transport. If, more generally, X is essentially finite, then the further hypothesis is required. For products, we have the following

Claim 2.6. The cardinality of the product $X \times Y$ is

$$|X \times Y| = |X| \cdot |Y|$$

which follows from the definition $X \times Y :\equiv \sum_{(x:X)} Y$ and the property for Σ-types.

Now consider the definition of the generating function of some $f : X \to \mathsf{FinSet}$. Note first that, because $\mathsf{card}(\mathsf{Fin}(n)) \equiv n$, we can write

$$\left|\mathsf{fib}_{\mathsf{card}\circ f}(n)\right| \equiv \left|\mathsf{fib}_{\mathsf{card}\circ f}(\mathsf{card}(\mathsf{Fin}(n)))\right| = \left| \sum_{w:\mathsf{fib}_{\mathsf{card}}(\mathsf{card}(\mathsf{Fin}(n)))} \mathsf{fib}_f(\mathsf{pr}_1 w) \right| = \left| \sum_{w:\sum_{(A:\mathcal{U})} \|A = \mathsf{Fin}(n)\|} \mathsf{fib}_f(\mathsf{pr}_1 w) \right|$$

by Exercise 4.4 of the HoTT Book. Since the type of w is 0-connected, we can replace $\mathsf{pr}_1(w)$ with $\mathsf{Fin}(n)$:

$$X_n = \left| \sum_{w:\sum_{(A:\mathcal{U})} \|A = \mathsf{Fin}(n)\|} \mathsf{fib}_f(\mathsf{Fin}(n)) \right|$$

so we can always calculate X_n using the fiber of f over $\mathsf{Fin}(n)$.

This expression can be simplified further by considering the type $\sum_{(A:\mathcal{U})} \|A = \mathsf{Fin}(n)\|$. This is our representative of the type of n-element sets, but it has a number of other descriptions. In particular, it is the delooping of the automorphism group of $\mathsf{Fin}(n)$. We have the following definitions:

Definition 2.1. The automorphism group of the type $X : \mathcal{U}$ is $\mathrm{Aut}(X) :\equiv (X = X)$.

Definition 2.2. The delooping of an automorphism group $\mathrm{Aut}(X)$ is

$$B\mathrm{Aut}(X) :\equiv \sum_{A:\mathcal{U}} \|A = X\|$$

Definition 2.3. The action groupoid for the action of $\mathrm{Aut}(X)$ on itself is

$$E\mathrm{Aut}(X) :\equiv \sum_{w:B\mathrm{Aut}(X)} (\mathsf{pr}_1(w) = X)$$

Giving the canonical fibration

$$\mathsf{pr}_1 : E\mathrm{Aut}(X) \to B\mathrm{Aut}(X)$$

By Theorem 7.3.9 of the HoTT book, $B\mathrm{Aut}(C)$ is 0-connected, so we can write

$$X_n = |B\mathrm{Aut}(\mathsf{Fin}(n))| \cdot \left|\mathsf{fib}_f(\mathsf{Fin}(n))\right|$$

which can be further simplified by computing $B\mathrm{Aut}(\mathsf{Fin}(n))$ once and for all. Note that $E\mathrm{Aut}(X)$ is contractible for all X, since it's a type of based paths, so we have

$$1 = |E\mathrm{Aut}(X)| = |B\mathrm{Aut}(X)| \cdot |\mathrm{Aut}(X)|$$

reducing our problem to computing $|\mathrm{Aut}(\mathsf{Fin}(n))|$, the number of permutations of an n-element set. This is $n!$, so, finally,

$$X_n = \frac{1}{n!} \left|\mathsf{fib}_f(\mathsf{Fin}(n))\right|$$

This suggests the notation $1/\!/n! :\equiv B\mathrm{Aut}(\mathsf{Fin}(n))$, which I will sometimes use in the following.

Plugging this back into the definition of $|X|(z)$, we have

$$|X|(z) = \sum_{n\geq 0} \left|\mathsf{fib}_f(\mathsf{Fin}(n))\right| \frac{z^n}{n!}$$

So, we could have defined $|X|(z)$ as an exponential generating function, without first thinking of X as a fibration over $\mathbb{N}$. Doing it our way is informative, however. A common rule of thumb is that one should use ordinary generating functions when working with labeled structures and exponential generating functions when working with unlabeled structures. In our setup, a labeling of some $A : \mathsf{FinSet}$ is a path $A = \mathsf{Fin}(\mathsf{card}(A))$. In other words, we replace the type of finite sets with the type of labeled finite sets:

$$\sum_{(A:\mathcal{U})} \sum_{(n:\mathbb{N})} \|A = \mathsf{Fin}(n)\| \qquad \Longrightarrow \qquad \sum_{(A:\mathcal{U})} \sum_{(n:\mathbb{N})} (A = \mathsf{Fin}(n))$$

and then proceed as before. Equivalently, we could write our species as a fibration $\sum_{(A:\mathsf{FinSet})} \Phi(A)$ for some $\Phi : \mathsf{FinSet} \to \mathcal{U}$ and then work with the altered species $\Phi(A) \times (A = \mathsf{Fin}(\mathsf{card}(A)))$. So "labeled Φ-stuff on a finite set" is the same as "Φ-stuff on a finite set and a labeling of the set". The cardinality of the nth fiber is then

$$\frac{1}{n!} |\Phi(\mathsf{Fin}(n)) \times (\mathsf{Fin}(n) = \mathsf{Fin}(n))| = |\Phi(\mathsf{Fin}(n))|$$

as the $|\mathrm{Aut}(\mathsf{Fin}(n))|$ cancels out the $1/n!$. The idea, then, is that a species generating function gives an exponential generating function when one computes out the dependence on card, and then a labeling turns it into an ordinary generating function.

Another reason to define $|X|(z)$ as we do is that it allows the labeled and unlabeled cases to be treated at once. Working directly with the reduced generating functions requires one to insert various factorials by hand to make some equations hold, and similar bookkeeping is needed for working with labeled and unlabeled structures differently. Taking the nth generating function coefficient to be the groupoid cardinality of the fiber over n allows us to define each operation on a species once, and the modifications required for labels take care of themselves.

3 Speciation

One power of the species approach is that it allows one to combine structures in a natural way. As an example in the previous section, we took a structure type $\Phi : \mathsf{FinSet} \to \mathcal{U}$ and created a labeled version $\Phi(A) \times (A = \mathsf{Fin}(\mathsf{card}(A)))$, naturally read as "the species of finite sets equipped with Φ-stuff and a labeling", using the propositions-as-types interpretation. The other type formers allow for similar combinations.

Throughout this section, we suppose that $f : X \to \mathsf{FinSet}$ and $g : Y \to \mathsf{FinSet}$ are two species, and that $\Phi, \Psi : \mathsf{FinSet} \to \mathcal{U}$ are two families of stuff on FinSet.

3.1 Coproduct

The simplest example is the coproduct. The recursion principle for the coproduct gives us a species with type $X + Y \to \mathsf{FinSet}$.

Definition 3.1. The coproduct species $(f+g) : X + Y \to \mathsf{FinSet}$ is given by

$$\begin{aligned}
&(f+g) : X + Y \to \mathsf{FinSet} \\
&(f+g) :\equiv \mathsf{rec}_{X+Y}(\mathsf{FinSet}, f, g)
\end{aligned}$$

For structure types $\Phi, \Psi : \mathsf{FinSet} \to \mathcal{U}$, we have

$$\left(\sum_{A:\mathsf{FinSet}} \Phi(A)\right) + \left(\sum_{B:\mathsf{FinSet}} \Psi(B)\right) \simeq \sum_{A:\mathsf{FinSet}} (\Phi(A) + \Psi(A))$$

and so we can think of the species $X + Y$ as "the species of finite sets equipped with X-stuff or Y stuff", though "or" here is un-truncated.

The generating function $|X + Y|(z)$ can be simply expressed in terms of $|X|(z)$ and $|Y|(z)$. We have

$$\begin{aligned}
(X+Y)_n &= \frac{1}{n!}\left|\mathsf{fib}_{f+g}(\mathsf{Fin}(n))\right| \\
&= \frac{1}{n!}\left|\sum_{x:X}(f(x) = \mathsf{Fin}(n)) + \sum_{y:Y}(g(y) = \mathsf{Fin}(n))\right| \\
&= \frac{1}{n!}\left|\mathsf{fib}_f(n) + \mathsf{fib}_g(n)\right| \\
&= X_n + Y_n
\end{aligned}$$

since $|X + Y| = |X| + |Y|$ for all groupoids X and Y. So, the generating function $|X + Y|(z)$ is the sum of the generating functions $|X|(z) + |Y|(z)$.

3.2 Hadamard product

We can tell a similar story with $\times$ instead of $+$. Where $X + Y$ is the species of "X stuff or Y stuff on a finite set", the product should give "X stuff and Y stuff on a finite set". But this is ambiguous: there are two ways to put two kinds of stuff on a finite set. The problem, essentially, is that while the recursion principle for the coproduct takes two species and gives a new one, the recursion principle for the product needs something of type $X \to Y \to \mathsf{FinSet}$ to produce a species. The two kinds of products on species correspond to different ways of doing this.

One way is to "superpose" the X stuff and Y stuff. That is, we might take the fiber over $\mathsf{Fin}(n)$ to be

$$\begin{aligned}
\mathsf{fib}_f(\mathsf{Fin}(n)) \times \mathsf{fib}_g(\mathsf{Fin}(n)) &\equiv \left(\sum_{x:X} (f(x) = \mathsf{Fin}(n))\right) \times \left(\sum_{y:Y} (g(y) = \mathsf{Fin}(n))\right) \\
&\simeq \sum_{(x:X)} \sum_{(y:Y)} ((f(x) = g(y)) \times (f(x) = \mathsf{Fin}(n))) \\
&\simeq \sum_{z:X \times_{\mathsf{FinSet}} Y} (f(\mathsf{pr}_1(z)) = \mathsf{Fin}(n)) \\
&\equiv \mathsf{fib}_{\langle f,g \rangle}(\mathsf{Fin}(n))
\end{aligned}$$

This leads us to

Definition 3.2. The Hadamard product species $\langle f, g \rangle : X \times_{\mathsf{FinSet}} Y \to \mathsf{FinSet}$ is given by

$$\begin{aligned}
&\langle f, g \rangle : X \times_{\mathsf{FinSet}} Y \to \mathsf{FinSet} \\
&\langle f, g \rangle : (x, y, p) \mapsto f(x)
\end{aligned}$$

For our structure types, we have

$$\left(\sum_{A:\mathsf{FinSet}} \Phi(A)\right) \times_{\mathsf{FinSet}} \left(\sum_{B:\mathsf{FinSet}} \Psi(B)\right) \simeq \sum_{A:\mathsf{FinSet}} (\Phi(A) \times \Psi(A))$$

which is naturally understood as "a finite set equipped with Φ stuff on top of Ψ stuff", as we wanted.

The generating function for the Hadamard product is

$$(X \times_{\mathsf{FinSet}} Y)_n = \frac{1}{n!} \left|\mathsf{fib}_{\langle f,g \rangle}(\mathsf{Fin}(n))\right| = \frac{1}{n!} \left|\mathsf{fib}_f(\mathsf{Fin}(n))\right| \cdot \left|\mathsf{fib}_g(\mathsf{Fin}(n))\right| = n! X_n \cdot Y_n$$

This formula motivates another name for this species: the inner product. When the ring of formal power series is treated as Fock space, the Hadamard product of two formal power series gives the inner product on Fock space. The relationship between the generating functions themselves is messy.

3.3 Cauchy product

A more natural relationship between generating functions would be some species $X \cdot Y$ such that

$$|X \cdot Y|(z) = |X|(z) \cdot |Y|(z) = \left(\sum_{n \geq 0} X_n z^n\right) \left(\sum_{m \geq 0} Y_m z^m\right) = \sum_{n \geq 0} \left(\sum_{k=0}^{n} X_k Y_{n-k}\right) z^n$$

For this to obtain, we should have

$$\left|\mathsf{fib}_{f \cdot g}(\mathsf{Fin}(n))\right| = \sum_{k=0}^{n} \binom{n}{k} \left|\mathsf{fib}_f(\mathsf{Fin}(k))\right| \cdot \left|\mathsf{fib}_g(\mathsf{Fin}(n-k))\right|$$

Decategorifying this expression gives the

Definition 3.3. The Cauchy product species $(f \cdot g) : X \cdot Y \to \mathsf{FinSet}$ is

$$\begin{aligned}
&(f \cdot g) : X \times Y \to \mathsf{FinSet} \\
&(f \cdot g) : (x, y) \mapsto f(x) + g(y)
\end{aligned}$$

And for our structure types,

$$\left(\sum_{A:\mathsf{FinSet}} \Phi(A)\right) \cdot \left(\sum_{B:\mathsf{FinSet}} \Psi(B)\right) \simeq \sum_{(A:\mathsf{FinSet})} \sum_{(U,V:\mathsf{FinSet})} \sum_{(p:U+V=A)} (\Phi(U) \times \Psi(V))$$

So, we can interpret the Cauchy product species as "a finite set chopped in two with Φ stuff on one part and Ψ stuff on the other".

The coefficients of the generating function are

$$\begin{aligned}
(X \cdot Y)_n &\equiv \frac{1}{n!} \left| \sum_{(x:X)} \sum_{(y:Y)} (f(x) + g(y) = \mathsf{Fin}(n)) \right| \\
&= \frac{1}{n!} \left| \sum_{r,s:\mathbb{N}} \left(\sum_{w:1/\!/r!} \mathsf{fib}_f(\mathsf{Fin}(r)) \right) \times \left(\sum_{w:1/\!/s!} \mathsf{fib}_g(\mathsf{Fin}(s)) \right) \times (\mathsf{Fin}(r) + \mathsf{Fin}(s) = \mathsf{Fin}(n)) \right| \\
&= \sum_{k=0}^{n} \frac{1}{k!} \left| \mathsf{fib}_f(\mathsf{Fin}(k)) \right| \cdot \frac{1}{(n-k)!} \left| \mathsf{fib}_g(\mathsf{Fin}(n-k)) \right| \\
&\equiv \sum_{k=0}^{n} X_k Y_{n-k}
\end{aligned}$$

and so we obtain the relation $|X \cdot Y|(z) = |X|(z) \cdot |Y|(z)$ between the generating functions.

3.4 Composition

Continuing to take inspiration from nice relations between generating functions, we look for some species $X \circ Y$ such that

$$|X \circ Y|(z) = |X|(|Y|(z)) = \sum_{n \geq 0} \left(\sum_{k=0}^{n} \sum_{n_1 + \cdots + n_k = n} X_k \cdot \prod_{i=1}^{k} Y_{n_i} \right) z^n$$

To categorify this equation, we replace the partition of n with a partition of the set $\mathsf{Fin}(n)$, and we replace the product $\prod_i Y_{n_i}$ with a dependent product type. To do this, we first define partitions of a finite set.

Definition 3.4. For any $n : \mathbb{N}$ and $S : \mathsf{Fin}(n) \to \mathsf{FinSet}$, the $\mathsf{Fin}(n)$-indexed sum $\bigoplus_{i=1}^{n} S_i$ of the family S is the type defined by induction on the naturals by

$$\bigoplus_{i=1}^{0} S_i :\equiv \mathbf{0} \qquad \bigoplus_{i=1}^{\mathsf{succ}(n)} S_i :\equiv S_n + \bigoplus_{i=1}^{n} S_i$$

Definition 3.5. A partition of a finite set $A : \mathsf{FinSet}$ into $n : \mathbb{N}$ parts by a family $P : \mathsf{Fin}(n) \to \mathsf{FinSet}$ is a term of type

$$P \vdash_n A :\equiv \left(A = \bigoplus_{i=1}^{n} P_i \right)$$

That is, it is a family $P : \mathsf{Fin}(n) \to \mathsf{FinSet}$ such that A is equal to the $\mathsf{Fin}(n)$-indexed sum of P.

With these, we can now define

Definition 3.6. For species $f : X \to \mathsf{FinSet}$ and $g : Y \to \mathsf{FinSet}$, the composite species is given by

$$\begin{aligned}
(f \circ g) &: \left(\sum_{x:X} (\mathsf{Fin}(\mathsf{card}(f(x))) \to Y) \right) \to \mathsf{FinSet} \\
(f \circ g) &: (x, P, p) \mapsto \bigoplus_{i=1}^{\mathsf{card}(f(x))} (g(P(i)))
\end{aligned}$$

For structure types, this gives

$$\left(\sum_{A:\mathsf{FinSet}} \Phi(A)\right) \circ \left(\sum_{B:\mathsf{FinSet}} \Psi(B)\right)$$
$$\simeq \sum_{(A:\mathsf{FinSet})} \sum_{(C:\mathsf{FinSet})} \sum_{B:\mathsf{Fin}(\mathsf{card}(C))\to\mathsf{FinSet}} \left((B \vdash_{\mathsf{card}(C)} A) \times \Phi(C) \times \prod_{k:\mathsf{Fin}(\mathsf{card}(C))} \Psi(B_k)\right)$$

So, the composition of the species Φ and Ψ is the species of "a partitioned finite set, with Φ stuff on the partition and Ψ stuff on each element of the partition".

The nth coefficient of the generating function is

$$\begin{aligned}
(X \circ Y)_n &= \frac{1}{n!}\left| \sum_{(x:X)} \sum_{(P:\mathsf{Fin}(\mathsf{card}(f(x)))\to Y)} \left(\mathsf{Fin}(n) = \bigoplus_{i=1}^{\mathsf{card}(f(x))} g(P(i)) \right)\right| \\
&= \left| \sum_{(k:\mathbb{N})} \sum_{(m:\mathsf{Fin}(k)\to\mathbb{N})} \sum_{(p:m_1+\cdots+m_k=n)} \left(\sum_{w:1/\!/k!} \mathsf{fib}_f(\mathsf{Fin}(k)) \right) \times \prod_{(i:\mathsf{Fin}(k))} \sum_{(w:1/\!/m_i!)} \mathsf{fib}_g(\mathsf{Fin}(m_i)) \right| \\
&= \sum_{k=0}^{n} \sum_{m_1+\cdots+m_k=n} \frac{1}{k!}\left|\mathsf{fib}_f(\mathsf{Fin}(k))\right| \cdot \prod_{i=1}^{k} \frac{1}{m_i!}\left|\mathsf{fib}_g(\mathsf{Fin}(m_i))\right| \\
&= \sum_{k=0}^{n} \sum_{m_1+\cdots+m_k=n} X_k \cdot \prod_{i=1}^{k} Y_{m_i}
\end{aligned}$$

And so $|X \circ Y|(z) = |X|(|Y|(z))$.

3.5 Differentiation

Again, we would like to construct a species whose generating function has a nice property. Since

$$\frac{d}{dz}\sum_{n\geq 0} X_n z^n = \sum_{n\geq 1} n\, X_n z^{n-1} = \sum_{n\geq 0} (n+1) X_{n+1} z^n$$

we want to define a species $f' : X' \to \mathsf{FinSet}$ such that $|X'|(z) = \frac{d}{dz}|X|(z)$. Categorifying this equation gives the

Definition 3.7. The derivative of a species $f : X \to \mathsf{FinSet}$ is

$$f' : \sum_{(A:\mathsf{FinSet})} \sum_{(x:X)} (f(x) = A + \mathbf{1}) \to \mathsf{FinSet}$$
$$f' : (A, x, p) \mapsto A$$

For structure types, we have

$$\left(\sum_{A:\mathsf{FinSet}} \Phi(A)\right)' = \sum_{A,B:\mathsf{FinSet}} (\Phi(B) \times (B = A + \mathbf{1}))$$

So Φ' stuff on A is "Φ stuff on $A + \mathbf{1}$".

The nth coefficient of the generating function is

$$\begin{aligned}
X'_n &= \frac{1}{n!} \left| \sum_{(A:\mathsf{FinSet})} \sum_{(x:X)} \left((f(x) = A + \mathbf{1}) \times (A = \mathsf{Fin}(n)) \right) \right| \\
&= \frac{1}{n!} \left| \sum_{x:X} \left((f(x) = \mathsf{Fin}(n) + \mathbf{1}) \right) \right| \\
&= \frac{1}{n!} \left| \mathsf{fib}_f(\mathsf{Fin}(n+1)) \right| \\
&= (n+1) X_{n+1}
\end{aligned}$$

giving

$$|X'|(z) = \frac{d}{dz} |X|(z)$$

3.6 Pointing

From generating function theory, we know that for any polynomial P, the generating function of the sequence $P(n)\, X_n$ is given by

$$P(z\partial_z)\, |X|(z) = \sum_{n \geq 0} P(n)\, X_n\, z^n$$

It suffices to consider the case $P(n) = n$, for which we have by categorification the

Definition 3.8. For any species $f : X \to \mathsf{FinSet}$, the pointing $f_\bullet : X_\bullet \to \mathsf{FinSet}$ is

$$\begin{aligned}
f_\bullet &: \left(\sum_{x:X} f(x) \right) \to \mathsf{FinSet} \\
f_\bullet &: (x, w) \mapsto f(x)
\end{aligned}$$

For a structure type, we have

$$\left(\sum_{A:\mathsf{FinSet}} \Phi(A) \right)_\bullet \simeq \sum_{A:\mathsf{FinSet}} (\Phi(A) \times A)$$

So $\Phi_\bullet$ stuff on A is "Φ stuff on A and a distinguished point of A".

The nth coefficient of the generating function is

$$(X_\bullet)_n = \frac{1}{n!} \left| \sum_{(x:X)} \sum_{(w:f(x))} (f(x) = \mathsf{Fin}(n)) \right| = \frac{1}{n!} \left| \mathsf{fib}_f(\mathsf{Fin}(n)) \right| \cdot |\mathsf{Fin}(n)| = n\, X_n$$

And so $|X_\bullet|(z) = (z\, \partial_z)\, |X|(z)$.

3.7 Inhabiting

It's often useful to have a version of the species that only exists on inhabited sets. This is the easy

Definition 3.9. For any species $f : X \to \mathsf{FinSet}$, the inhabited version of the species $f_+ : X_+ \to \mathsf{FinSet}$ is

$$\begin{aligned}
f_+ &: \left(\sum_{x:X} \| f(x) \| \right) \to \mathsf{FinSet} \\
f_+ &: (x, p) \mapsto f(x)
\end{aligned}$$

For a structure type,

$$\left(\sum_{A \mathsf{FinSet}} \Phi(A)\right)_+ \simeq \sum_{A:\mathsf{FinSet}} (\Phi(A) \times \|A\|)$$

So, as intended, Φ_+ stuff on a set is "being an inhabited set equipped with Φ stuff".

The nth coefficient of the generating function is

$$(X_+)_n = \frac{1}{n!}\left|\mathsf{fib}_f(\mathsf{Fin}(n))\right| \cdot |\|\mathsf{Fin}(n)\|| = \begin{cases} 0 & n = 0 \\ X_n & n > 0 \end{cases}$$

since $\mathsf{Fin}(n)$ is decidable and $\|\mathsf{Fin}(n)\|$ is a mere proposition. So $|X_+|(z) = |X|(z) - X_0$.

4 Examples

This section lists a handful of examples.

4.1 (-2)-stuff

The simplest structure to put on a finite set is a contractible one. For example, we know that $\mathsf{isFinSet}(A)$ is a mere proposition, and it is inhabited for all $A : \mathsf{FinSet}$, so it's contractible. So the total space of the type is

$$\sum_{A:\mathsf{FinSet}} \mathsf{isFinSet}(A) \simeq \sum_{A:\mathsf{FinSet}} \mathbf{1} \simeq \mathsf{FinSet}$$

So any species of contractible stuff on a finite set is equal to $\mathsf{id} : \mathsf{FinSet} \to \mathsf{FinSet}$, the species of finite sets. Call this species $E :\equiv \mathsf{id}_{\mathsf{FinSet}}$, for the species of ensembles. The fiber over n is

$$\sum_{A:\mathcal{U}} \|A = \mathsf{Fin}(n)\| \equiv B\mathrm{Aut}(\mathsf{Fin}(n))$$

so the generating function of the species of ensembles is

$$|E|(z) = \sum_{n \geq 0} \frac{z^n}{n!} = e^z$$

4.2 (-1)-stuff

Moving up the hierarchy, a family of mere propositions can produce more interesting examples. Considered as an exponential generating function, the generating function obtained from (-2)-stuff has coefficients in $\{1\}$. For (-1)-stuff, we can choose coefficients from $\{0, 1\}$. So for example, we have

Empty stuff A finite set equipped with the stuff $\Phi : A \mapsto \mathbf{0}$ gives the species

$$0 :\equiv \mathsf{rec}_{\mathbf{0}}(\mathsf{FinSet}) : \mathbf{0} \to \mathsf{FinSet}$$

which has the generating function

$$|0|(z) = 0$$

***n*-element sets** The stuff of "being an n-element set" gives the species

$$(Z^n /\!\!/ n!) :\equiv \mathsf{pr}_1 : \left(\sum_{A:\mathsf{FinSet}} (\mathsf{card}(A) = n) \right) \to \mathsf{FinSet}$$

and it has the generating function

$$|Z^n /\!\!/ n!|(z) = \frac{z^n}{n}$$

As a special case, for $n = 0$ we have $|Z^n /\!\!/ n!|(z) = 1$, so we'll also write

$$1 :\equiv (Z^0 /\!\!/ 0!)$$

Inhabited finite sets This is the species E_+, and we clearly have

$$|E_+|(z) = \sum_{n \geq 1} \frac{z^n}{n!} = e^z - 1$$

Even and odd sets The stuff Φ of "being an even set" gives us the species

$$\text{COSH} :\equiv \mathsf{pr}_1 : \left(\sum_{(A:\mathsf{FinSet})} \sum_{(n:\mathbb{N})} \|A = \mathsf{Fin}(2n)\| \right) \to \mathsf{FinSet}$$

So the fiber over n is

$$\sum_{(w:1 /\!\!/ n!)} \sum_{(m:\mathbb{N})} (2m = n)$$

giving the generating function

$$|\text{COSH}|(z) = \sum_{n \text{ even}} \frac{z^n}{n!} = \cosh(z)$$

Similarly, "being an odd set" is the species SINH with generating function

$$|\text{SINH}|(z) = \sum_{n \text{ odd}} \frac{z^n}{n!} = \sinh(z)$$

4.3 0-stuff

On the next rung of the ladder we have families of sets. Now we obtain exponential generating functions with coefficients in $\mathbb{N}$. For example,

Labeled *n*-element sets The stuff of "being a labeled n-element set" gives the species

$$Z^n :\equiv \mathsf{pr}_1 : \left(\sum_{A:\mathsf{FinSet}} (A = \mathsf{Fin}(n)) \right) \to \mathsf{FinSet}$$

with generating function

$$|Z^n|(z) = z^n$$

Labeled finite sets This species is

$$\mathsf{pr}_1 : \left(\sum_{(A:\mathsf{FinSet})} \sum_{(n:\mathbb{N})} (A = \mathsf{Fin}(n)) \right) \to \mathsf{FinSet}$$

and it has generating function

$$\sum_{n \geq 0} z^n = \frac{1}{1-z}$$

Linear order A linear order on a set A is a mere relation $- < - : A \to A \to \mathsf{Prop}$ that's transitive, antisymmetric, and total. We have the species

$$L :\equiv \mathsf{pr}_1 : \left(\sum_{(A:\mathsf{FinSet})} \sum_{(<:A \to A \to \mathsf{Prop})} (\mathsf{Transitive}(<) \times \mathsf{Antisymmetric}(<) \times \mathsf{Total}(<)) \right) \to \mathsf{FinSet}$$

It's not too hard to see that this species is equal to the species of labeled finite sets; there is a natural total order on $\mathsf{Fin}(\mathsf{card}(A))$, and the elements of A can be matched up with elements of $\mathsf{Fin}(\mathsf{card}(A))$ by matching the orders. So the generating function of L is also

$$|L|(z) = \sum_{n \geq 0} z^n = \frac{1}{1-z}$$

Permutations A permutation on A is a bijection $\sigma : A \to A$. Since A is a set, being a bijection is equivalent to being an equivalence, so we have by univalence the species

$$\mathrm{Per} :\equiv \mathsf{pr}_1 : \left(\sum_{A:\mathsf{FinSet}} (A = A) \right) \to \mathsf{FinSet}$$

This has the same fiber over n as the species of finite labeled sets, so we again have

$$|\mathrm{Per}|(z) = \sum_{n \geq 0} z^n = \frac{1}{1-z}$$

FinSet**-coloring** For K a finite set, a K-coloring of A is a map $\phi : A \to K$. This gives the species

$$\mathsf{pr}_1 : \left(\sum_{A:\mathsf{FinSet}} (A \to K) \right) \to \mathsf{FinSet}$$

which has generating function

$$|A \to K|(z) = \sum_{n \geq 0} |K|^n \frac{z^n}{n!} = e^{|K| z}$$

Consider the property of being a K-colored, 1-element set. This is the species given by the stuff

$$\Phi(A) :\equiv \|A = \mathsf{Fin}(1)\| \times (A \to K)$$

which is equivalent to the species equipped with stuff

$$\tilde{\Phi}(A) :\equiv K \times \|A = \mathsf{Fin}(1)\|$$

and hence to the Cauchy product $K \cdot Z$. Indeed, the generating function is given by

$$|\Phi|(z) = |K|\, z$$

and so we can write nZ for the species "being a $\mathsf{Fin}(n)$-colored 1-element set". As the notation suggests,

$$|E \circ nZ|(z) = |E|(|nZ|(z)) = e^{nz}$$

4.4 Fock space

Consider the operators $a, a^* : \mathbb{C}[z] \to \mathbb{C}[z]$ defined by

$$a\psi = \psi'$$
$$a^*\psi = z\psi$$

for all $\psi \in \mathbb{C}[z]$. We equip $\mathbb{C}[z]$ with an inner product by demanding that

$$\langle a^*\psi, \phi\rangle = \langle \psi, a\phi\rangle \qquad \|1\| = 1$$

for all $\psi, \phi \in \mathbb{C}[z]$. Completing $\mathbb{C}[z]$ in this inner product gives Fock space $K[z]$.

5 Cayley's Formula

We recount Joyal's proof of the following

Theorem 1 (Cayley's Formula)**.** *The number of labeled trees on n vertices is n^{n-2}.*

Recall that a tree is a connected graph such that any two edges are connected by exactly one path. A graph on a finite set A is a mere relation $R : A \to A \to \mathsf{Prop}$, so we're looking at the species

$$T :\equiv \sum_{(A:\mathsf{FinSet})} \sum_{(R:A\to A\to\mathsf{Prop})} (\mathsf{isTree}(R) \times (A = \mathsf{Fin}(\mathsf{card}(A))))$$

Pointing this species twice gives a species $V :\equiv T_{\bullet\bullet}$ of vertebrates, so-called because we can think of the path connecting the two special points as a vertebral column, with a rooted tree at each vertebra. By the calculation above for pointed species, we have $V_n = n^2\, T_n$ so Cayley's formula follows if we can show that $V_n = n^n$.

To show this, note that V is equivalent to the species $L_+ \circ T_\bullet$, which are inhabited linear orders of rooted trees. Given a vertebrate $v : V$, we disconnect it by forgetting the links between vertebrae, then we induce a linear order on the connected components using the linear order on the vertebral column. Since $L_n = \mathrm{Per}_n$, it suffices to consider $S_+ \circ T_\bullet$, which is equal to End_+. Since $\mathrm{End}_n = n^n$, for $n > 0$ we have $V_n = n^n$, from which follows Cayley's formula.

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