Jane Downer

A.

$$a = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad b = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \quad c = \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}$$

1. $2a - b$

$$= 2 \times \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 2 \times 1 - 4 \\ 2 \times 2 - 5 \\ 2 \times 3 - 6 \end{bmatrix} = \boxed{\begin{bmatrix} -2 \\ -1 \\ 0 \end{bmatrix}}$$

2. $\hat{a}$

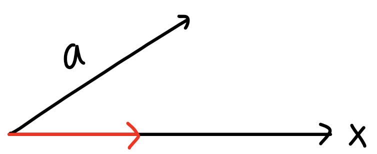
$$= \frac{1}{\|a\|} \times a$$

$$\|a\| = \sqrt{\overset{1}{1^2} + \overset{4}{2^2} + \overset{9}{3^2}} = \sqrt{14}$$

$$= (1/\sqrt{14}) \times \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} 1/\sqrt{14} \\ 2/\sqrt{14} \\ 3/\sqrt{14} \end{bmatrix}}$$

3. $\|a\| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$



$$\hat{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\cos(\theta) = \left(\frac{a \cdot \hat{i}}{\|a\| \cdot \|\hat{i}\|} \right)$$

$$\theta = \cos^{-1}\left(\frac{a \cdot \hat{i}}{\|a\| \cdot \|\hat{i}\|}\right)$$

$$a \cdot \hat{i} = (1)(1) + (2)(0) + (3)(0) = 1$$

$$= \cos^{-1}\left(\frac{1}{\sqrt{14} \times 1}\right)$$

$$\|a\| = \sqrt{14}$$

$$\|\hat{i}\| = 1$$

$$\approx 74.5^\circ$$

$$\boxed{\begin{array}{c} \|a\| = \sqrt{14} \\ \theta = 74.5^\circ \end{array}}$$

4.

$$\cos(\theta_x) = \frac{a \cdot \hat{i}}{\|a\| \cdot \|\hat{i}\|} \qquad \cos(\theta_y) = \frac{a \cdot \hat{j}}{\|a\| \cdot \|\hat{j}\|} \qquad \cos(\theta_z) = \frac{a \cdot \hat{k}}{\|a\| \cdot \|\hat{k}\|}$$

$$\|\hat{i}\| = \|\hat{j}\| = \|k\| = 1$$

$$\|a\| = \sqrt{14}$$

$$a \cdot \hat{i} = (1)(1) + (2)(0) + (3)(0) = 1$$

$$a \cdot \hat{j} = (1)(0) + (2)(1) + (3)(0) = 2$$

$$a \cdot \hat{k} = (1)(0) + (2)(0) + (3)(1) = 3$$

$$\boxed{\cos(\theta_x) = 1/\sqrt{14} \qquad \cos(\theta_y) = 2/\sqrt{14} \qquad \cos(\theta_z) = 3/\sqrt{14}}$$

5. angle between a and b

$$\theta = \cos^{-1}\left(\frac{a \cdot b}{\|a\| \cdot \|b\|}\right)$$

$$a \cdot b = \underbrace{(1)(4)}_{4} + \underbrace{(2)(5)}_{10} + \underbrace{(3)(6)}_{18} = 32$$

$$= \cos^{-1}\left(\frac{32}{\sqrt{14} \times \sqrt{77}}\right)$$

$$\|a\| = \sqrt{14}$$

$$\|b\| = \sqrt{4^2 + 5^2 + 6^2} = \sqrt{77}$$

$$\approx \boxed{12.93^\circ}$$

6. $$\boxed{\begin{aligned} a \bullet b &= (1)(4) + (2)(5) + (3)(6) = 32 \\ b \bullet a &= (4)(1) + (5)(2) + (6)(3) = 32 \end{aligned}}$$

7. $$\cos(\theta_{a,b}) = \left(\frac{a \bullet b}{\|a\| \cdot \|b\|}\right)$$

$$a \bullet b = \cos(\theta_{a,b}) \|a\| \cdot \|b\|$$

$$= \cos(12.93°) \times \sqrt{14} \times \sqrt{77}$$

$$= \boxed{32}$$

8. scalar projection of b onto a = $\|b\| \cdot \cos(\theta_{a,b})$

$$= \sqrt{77} \cdot \cos(12.93°)$$

$$\approx \boxed{8.55}$$

9. perpendicular vector, when dotted with a, equals $\emptyset$.

$$p = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}$$

$$a \bullet p = 1 \cdot p_1 + 2 \cdot p_2 + 3 \cdot p_3 = 0$$

one example that satisfies this: $\boxed{p = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}}$

10.

$$a \times b = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= \hat{i}(\overbrace{a_2 b_3 - b_2 a_3}^{-3}) - \hat{j}(\overbrace{a_1 b_3 - b_1 a_3}^{-6}) + \hat{k}(\overbrace{a_1 b_2 - b_1 a_2}^{-3})$$

$$= \begin{bmatrix} (1)(-3) - 0 + 0 \\ 0 - (1)(-6) + 0 \\ 0 - 0 + (1)(-3) \end{bmatrix} = \begin{bmatrix} -3 \\ 6 \\ -3 \end{bmatrix}$$

$$b \times a = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix}$$

$$= \hat{i}(b_1 a_2 - a_1 b_2) - \hat{j}(b_1 a_3 - a_1 b_3) + \hat{k}(b_1 a_2 - a_1 b_2)$$

$$= \hat{i}(\overbrace{4 \cdot 2 - 1 \cdot 5}^{3}) - \hat{j}(\overbrace{4 \cdot 3 - 1 \cdot 6}^{6}) + \hat{k}(\overbrace{4 \cdot 2 - 1 \cdot 5}^{3})$$

$$= \begin{bmatrix} 1 \cdot 3 - 0 + 0 \\ 0 - 1 \cdot 6 + 0 \\ 0 - 0 + 1 \cdot 3 \end{bmatrix} = \begin{bmatrix} 3 \\ -6 \\ 3 \end{bmatrix}$$

$$\boxed{a \times b = \begin{bmatrix} -3 \\ 6 \\ -3 \end{bmatrix} \quad b \times a = \begin{bmatrix} 3 \\ -6 \\ 3 \end{bmatrix}}$$

11. $\boxed{\text{both } a \times b \text{ and } b \times a \text{ are perpendicular to } a \text{ and } b.}$

$$\begin{bmatrix} -3 \\ 6 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = -3 + 12 - 9 = 0 \checkmark \qquad \begin{bmatrix} 3 \\ -6 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 3 - 12 + 9 = 0 \checkmark$$

$$\begin{bmatrix} -3 \\ 6 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = -12 + 30 - 18 = 0 \checkmark \qquad \begin{bmatrix} 3 \\ -6 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \; 12 - 30 + 18 = 0 \checkmark$$

12.

$$\begin{array}{ccc} a & b & c \end{array}$$

$$\begin{bmatrix} 1 & 4 & -1 \\ 2 & 5 & 1 \\ 3 & 6 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 4 & -1 & 0 \\ 2 & 5 & 1 & 0 \\ 3 & 6 & 3 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 4 & -1 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & -6 & 6 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 4 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{cases} x + 4y - z = 0 \\ y = z \\ 0 = 0 \end{cases}$$

$z = 1 \longrightarrow y = 1, \; x = -3$

$$-3 \underset{a}{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}} + \underset{b}{\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}} + \underset{c}{\begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\hookrightarrow \boxed{-3a + b + c = 0}$$

13.

$a = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad a^T = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$

$b = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \quad b^T = \begin{bmatrix} 4 & 5 & 6 \end{bmatrix}$

$$a^T b = \underset{1\times 3}{\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}} \underset{3\times 1}{\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}} = \begin{bmatrix} 1\cdot 4 + 2\cdot 5 + 3\cdot 6 \end{bmatrix} = \underset{1\times 1}{\boxed{\begin{bmatrix} 32 \end{bmatrix}}}$$

$$ab^T = \underset{3\times 1}{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}} \underset{1\times 3}{\begin{bmatrix} 4 & 5 & 6 \end{bmatrix}} = \underset{3\times 3}{\boxed{\begin{bmatrix} 4 & 5 & 6 \\ 8 & 10 & 12 \\ 12 & 15 & 18 \end{bmatrix}}}$$

B.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & -2 & 3 \\ 0 & 5 & -1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & -4 \\ 3 & -2 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ -1 & 1 & 3 \end{bmatrix} \quad d = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

1.

$$2A - B = \begin{bmatrix} 2 & 4 & 6 \\ 8 & -4 & 6 \\ 0 & 10 & -2 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & -4 \\ 3 & -2 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} 1 & 2 & 5 \\ 6 & -5 & 10 \\ -3 & 12 & -3 \end{bmatrix}}$$

2.

$$AB = \begin{bmatrix} (1)(1) + \overset{14}{(2)(2)} + (3)(3) & (1)(2) + \overset{-2}{(2)(1)} + (3)(-2) & (1)(1) + \overset{-4}{(2)(-4)} + (3)(1) \\ (4)(1) + \overset{9}{(-2)(2)} + (3)(3) & (4)(2) + \overset{0}{(-2)(1)} + (3)(-2) & (4)(1) + \overset{15}{(-2)(-4)} + (3)(1) \\ (0)(1) + \overset{7}{(5)(2)} + (-1)(3) & (0)(2) + \overset{7}{(5)(1)} + (-1)(-2) & (0)(1) + \overset{-21}{(5)(-4)} + (-1)(1) \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} 14 & -2 & -4 \\ 9 & 0 & 15 \\ 7 & 7 & -21 \end{bmatrix}}$$

$$= \begin{bmatrix} (1)(1) + \overset{9}{(2)(4)} + (1)(0) & (1)(2) + \overset{3}{(2)(-2)} + (1)(5) & (1)(3) + \overset{8}{(2)(3)} + (1)(-1) \\ (2)(1) + \overset{6}{(1)(4)} + (-4)(0) & (2)(2) + \overset{-18}{(1)(-2)} + (-4)(5) & (2)(3) + \overset{13}{(1)(3)} + (-4)(-1) \\ (3)(1) + \overset{-5}{(-2)(4)} + (1)(0) & (3)(2) + \overset{15}{(-2)(-2)} + (1)(5) & (3)(3) + \overset{2}{(-2)(3)} + (1)(-1) \end{bmatrix}$$

$$BA = \boxed{\begin{bmatrix} 9 & 3 & 8 \\ 6 & -18 & 13 \\ -5 & 15 & 2 \end{bmatrix}}$$

3.

$$AB = \begin{bmatrix} 14 & -2 & -4 \\ 9 & 0 & 15 \\ 7 & 7 & -21 \end{bmatrix} \longrightarrow \boxed{(AB)^T = \begin{bmatrix} 14 & 9 & 7 \\ -2 & 0 & 7 \\ -4 & 15 & -21 \end{bmatrix}}$$

$$B^T = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & -2 \\ 1 & -4 & 1 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 4 & 0 \\ 2 & -2 & 5 \\ 3 & 3 & -1 \end{bmatrix}$$

$$B^TA^T = \begin{bmatrix} (1)(1) + \overset{14}{(2)(2)} + (3)(3) & (1)(4) + \overset{9}{(2)(-2)} + (3)(3) & (1)(0) + \overset{7}{(2)(5)} + (3)(-1) \\ (2)(1) + \overset{-2}{(1)(2)} + (-2)(3) & (2)(4) + \overset{0}{(1)(-2)} + (-2)(3) & (2)(0) + \overset{7}{(1)(5)} + (-2)(-1) \\ (1)(1) + \overset{-4}{(-4)(2)} + (1)(3) & (1)(4) + \overset{15}{(-4)(-2)} + (1)(3) & (1)(0) + \overset{-21}{(-4)(5)} + (1)(-1) \end{bmatrix}$$

$$\boxed{B^TA^T = \begin{bmatrix} 14 & 9 & 7 \\ -2 & 0 & 7 \\ -4 & 15 & -21 \end{bmatrix}}$$

4.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & -2 & 3 \\ 0 & 5 & -1 \end{bmatrix}$$

$$|A| = \overbrace{1\left[(-2)(-1) - (5)(3)\right]}^{-13} - \overbrace{2\left[(4)(-1) - (0)(3)\right]}^{8} + \overbrace{3\left[(4)(5) - (0)(-2)\right]}^{60}$$

$$= -13 + 8 + 60$$

$$= \boxed{55}$$

$$C = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ -1 & 1 & 3 \end{bmatrix}$$

$$|C| = \overbrace{1\left[(5)(3) - (1)(6)\right]}^{9} - \overbrace{2\left[(4)(3) - (-1)(6)\right]}^{36} + \overbrace{3\left[(4)(1) - (-1)(5)\right]}^{27}$$

$$= 9 - 36 + 27$$

$$= \boxed{0}$$

Ø determinant $\Longrightarrow$ linearly <u>dependent</u> rows (demonstrated in A-12)

5. $B: \; r_1 \cdot r_2 = (1)(2) + (2)(1) + (1)(-4) = 0$

$r_1 \cdot r_3 = (1)(3) + (2)(-2) + (1)(1) = 0$

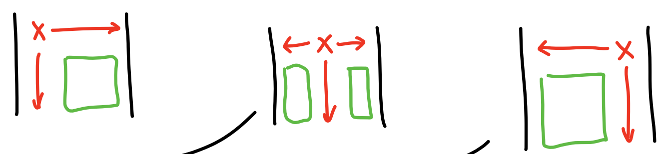
$r_2 \cdot r_3 = (2)(3) + (1)(-2) + (-4)(1) = 0$

The rows of B form an orthogonal set. (i.e. dot products between each vector pair $= \emptyset$).

6.

Inverse of 3×3 matrix: $M^{-1} = \text{adjoint}(M) / \text{determinant}(M)$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & -2 & 3 \\ 0 & 5 & -1 \end{bmatrix}$$



$$\text{adj}(A) = \begin{bmatrix} \overset{-13}{\begin{vmatrix} -2 & 3 \\ 5 & -1 \end{vmatrix}} & -\overset{4}{\begin{vmatrix} 4 & 3 \\ 0 & -1 \end{vmatrix}} & \overset{20}{\begin{vmatrix} 4 & 0 \\ -2 & 5 \end{vmatrix}} \\ -\overset{17}{\begin{vmatrix} 2 & 3 \\ 5 & -1 \end{vmatrix}} & \overset{1}{\begin{vmatrix} 1 & 3 \\ 0 & -1 \end{vmatrix}} & -\overset{-5}{\begin{vmatrix} 1 & 2 \\ 0 & 5 \end{vmatrix}} \\ \overset{-10}{\begin{vmatrix} 1 & 2 \\ 4 & -2 \end{vmatrix}} & -\overset{9}{\begin{vmatrix} 1 & 3 \\ 4 & 3 \end{vmatrix}} & \overset{-10}{\begin{vmatrix} 1 & 2 \\ 4 & -2 \end{vmatrix}} \end{bmatrix}^T = \begin{bmatrix} -13 & 17 & -10 \\ 4 & 1 & 9 \\ 20 & -5 & -10 \end{bmatrix}$$

$\det(A)$: $|A| = 55$ (found earlier)

$$\boxed{A^{-1} = \begin{bmatrix} -13 & 17 & -10 \\ 4 & 1 & 9 \\ 20 & -5 & -10 \end{bmatrix} \times \frac{1}{55}}$$

$$B = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & -4 \\ 3 & -2 & 1 \end{bmatrix}$$

$$\operatorname{adj}(B) = \begin{bmatrix} \overset{-7}{\begin{vmatrix} 1 & -4 \\ 2 & 1 \end{vmatrix}} & \overset{10}{-\begin{vmatrix} 2 & -4 \\ 3 & 1 \end{vmatrix}} & \overset{-7}{\begin{vmatrix} 2 & 1 \\ 3 & -2 \end{vmatrix}} \\ \overset{-4}{-\begin{vmatrix} 2 & 1 \\ -2 & 1 \end{vmatrix}} & \overset{-2}{\begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix}} & \overset{8}{-\begin{vmatrix} 1 & 2 \\ 3 & -2 \end{vmatrix}} \\ \overset{-9}{\begin{vmatrix} 2 & 1 \\ 1 & -4 \end{vmatrix}} & \overset{6}{-\begin{vmatrix} 1 & 1 \\ 2 & -4 \end{vmatrix}} & \overset{-3}{\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix}} \end{bmatrix}^{T} = \begin{bmatrix} -7 & -4 & -9 \\ 10 & -2 & 6 \\ -7 & 8 & -3 \end{bmatrix}$$

$$\det(B) = \overbrace{1[(1)(1)-(-2)(-4)]}^{-7} \overbrace{-\,2[(2)(1)-(3)(-4)]}^{-28} \overbrace{+\,1[(2)(-2)-(3)(1)]}^{-7}$$

$$= -42$$

$$\boxed{B^{-1} = \begin{bmatrix} -7 & -4 & -9 \\ 10 & -2 & 6 \\ -7 & 8 & -3 \end{bmatrix} \times \left(-\frac{1}{42}\right)}$$

7.

$$C = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ -1 & 1 & 3 \end{bmatrix}$$

$$\text{adjoint } C = \begin{bmatrix} \overset{9}{\begin{vmatrix} 5 & 6 \\ 1 & 3 \end{vmatrix}} & -\overset{-18}{\begin{vmatrix} 4 & 6 \\ -1 & 3 \end{vmatrix}} & \overset{9}{\begin{vmatrix} 4 & 5 \\ -1 & 1 \end{vmatrix}} \\ -\overset{-3}{\begin{vmatrix} 2 & 3 \\ 1 & 3 \end{vmatrix}} & \overset{6}{\begin{vmatrix} 1 & 3 \\ -1 & 3 \end{vmatrix}} & -\overset{-3}{\begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix}} \\ \overset{-3}{\begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix}} & -\overset{6}{\begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix}} & \overset{-3}{\begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix}} \end{bmatrix}^T = \begin{bmatrix} 9 & -3 & -3 \\ -18 & 6 & 6 \\ 9 & -3 & -3 \end{bmatrix}$$

$$\text{determinant } C = 1 \times \begin{vmatrix} 5 & 6 \\ 1 & 3 \end{vmatrix} - 2 \times \begin{vmatrix} 4 & 6 \\ -1 & 3 \end{vmatrix} + 3 \times \begin{vmatrix} 4 & 5 \\ -1 & 1 \end{vmatrix}$$

$$= 1(15-6) - 2(12+6) + 3(4+5)$$

$$= 9 - 36 + 27$$

$$= 0$$

$$C^{-1} = \frac{\text{adj } C}{\det C} = \boxed{\text{undefined}}, \text{ since determinant} = 0$$

Connection to B-4:
matrices w/ linearly dependent
rows or columns are not invertible

8.

$$Ad = \begin{bmatrix} (1)(1) + (2)(2) + (3)(3) \\ (4)(1) + (-2)(2) + (3)(3) \\ (0)(1) + (5)(2) + (-1)(3) \end{bmatrix} = \boxed{\begin{bmatrix} 14 \\ 9 \\ 7 \end{bmatrix}}$$

10. vector projection of v onto u: $\left(\frac{v \cdot u}{\|u\|^2}\right) u$

↳ for each row of A, vector projection onto d is:

$$\left(\frac{row \cdot d}{\|d\|^2}\right) d = \frac{row \cdot d}{14} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & -2 & 3 \\ 0 & 5 & -1 \end{bmatrix}$$

row 1: $(1)(1) + (2)(2) + (3)(3) = 14$

↳ $\frac{14}{14} d = \boxed{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}$

row 2: $(4)(1) + (-2)(2) + (3)(3) = 9$

$\frac{9}{14} d = \boxed{\frac{9}{14}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}$

row 3: $(0)(1) + (5)(2) + (-1)(3) = 7$

↳ $\frac{7}{14} d = \boxed{\frac{1}{2}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}$

reversed

9. scalar projections = lengths of vector projections

row 1: $\sqrt{1^2 + 2^2 + 3^2} = \boxed{\sqrt{14}}$

row 2: $\sqrt{\left(\frac{9}{14}\right)^2 [1^2 + 2^2 + 3^2]} = \frac{9}{14} \cdot \sqrt{14} = \boxed{\frac{9}{\sqrt{14}}}$

row 3: $\sqrt{\left(\frac{1}{2}\right)^2 (1^2 + 2^2 + 3^2)} = \boxed{\frac{1}{2}\sqrt{14}}$

11.

linear combination = column 1 $\cdot d_1$ + column 2 $\cdot d_2$ + column 3 $\cdot d_3$

$$\boxed{1\begin{bmatrix}1\\4\\0\end{bmatrix} + 2\begin{bmatrix}2\\-2\\5\end{bmatrix} + 3\begin{bmatrix}3\\3\\-1\end{bmatrix}}$$

12. $Bx = d$

$$\begin{bmatrix}1 & 2 & 1\\2 & 1 & -4\\3 & -2 & 1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}1\\2\\3\end{bmatrix}$$

(1) $x_1 + 2x_2 + x_3 = 1$

(2) $2x_1 + x_2 - 4x_3 = 2$

(3) $3x_1 - 2x_2 + x_3 = 3 \longrightarrow x_3 = 3 - 3x_1 + 2x_2$

rewritten: (1) $x_1 + 2x_2 + 3 - 3x_1 + 2x_2 = 1$

$-2x_1 + 4x_2 = -2$

$x_1 - 2x_2 = 1$

$x_1 = 1 + 2x_2$

(2) $2x_1 + x_2 - 4[3 - 3x_1 + 2x_2] = 2$

$2x_1 + 12x_1 + x_2 - 8x_2 - 12 = 2$

$14x_1 - 7x_2 = 14$

$x_1 - \frac{1}{2}x_2 = 1$

$x_1 = 1 + \frac{1}{2}x_2$

$x_2 = 0$

$x_1 = 1$

$x_3 = 3 - 3 + 0 = 0$

$$\boxed{x = \begin{bmatrix}1\\0\\0\end{bmatrix}}$$

13. $Cx = d$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ -1 & 1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 4 & 5 & 6 & 2 \\ -1 & 1 & 3 & 3 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -3 & -6 & -2 \\ -1 & 1 & 3 & 3 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -3 & -6 & -2 \\ 0 & 3 & 6 & 4 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -3 & -6 & -2 \\ 0 & 0 & 0 & 2 \end{array}\right] \longrightarrow$$

impossible - no solution

relationship to B-4 and B-7:
$\emptyset$ determinant implies the matrix is (1) non-invertible, and (2), has either no solution or infinitely many solutions.

C.

$$D = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \quad E = \begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix} \quad F = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

1. $$\begin{bmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 0$$

from first row: $a(1-\lambda) + 2b = 0$

$$b = \frac{(\lambda-1)}{2} a$$

from second row: $3a + (2-\lambda)b = 0$

$$3a + (2-\lambda)(\lambda-1)\tfrac{1}{2} a = 0$$

$$a(6 + 2\lambda - 2 - \lambda^2 + \lambda) = 0$$

$$-\lambda^2 + 3\lambda + 4 = 0$$

$$\lambda^2 - 3\lambda - 4 = 0$$

$$(\lambda-4)(\lambda+1) = 0 \longrightarrow \lambda = 4 \text{ or } \lambda = -1$$

$\boxed{\lambda = 4:}$

$$\left[\begin{array}{cc|c} 1-4 & 2 & 0 \\ 3 & 2-4 & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} -3 & 2 & 0 \\ 3 & -2 & 0 \end{array}\right] \longrightarrow \text{eigenvector: } \boxed{\begin{bmatrix} 2 \\ 3 \end{bmatrix}}$$

$\boxed{\lambda = -1:}$

$$\left[\begin{array}{cc|c} 1-(-1) & 2 & 0 \\ 3 & 2-(-1) & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 2 & 2 & 0 \\ 3 & 3 & 0 \end{array}\right] \longrightarrow \text{eigenvector: } \boxed{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}$$

2. $\boxed{V_1 \cdot V_2 = 2\cdot 1 - 3\cdot 1 = -1}$

3. $\begin{bmatrix} 2-\lambda & -2 \\ -2 & 5-\lambda \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = 0$

i. $(2-\lambda)u - 2v = 0$

$v = \frac{1}{2}u(2-\lambda)$

ii $-2u + (5-\lambda)v = 0$

$-2u + (5-\lambda)(2-\lambda)\cdot\frac{1}{2}u = 0$

$-4u + (5-\lambda)(2-\lambda)u = 0$

$u(-4 + 10 - 5\lambda - 2\lambda + \lambda^2) = 0$

$6 - 7\lambda + \lambda^2 = 0$

$(\lambda - 1)(\lambda - 6) = 0$

$\lambda = 1, 6$

$\lambda = 1$: $\begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = 0$

$u - 2v = 0$

$u = 2v$

$-2u + 4v = 0$

$2u = 4v$

$u = 2v$

$V_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\lambda = 6$: $\begin{bmatrix} -4 & -2 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = 0$

$-2u - v = 0$

$v = -2u$

$V_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$V_1 \cdot V_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 2 \end{bmatrix} = -2 + 2 = 0$

$\boxed{V_1 \cdot V_2 = 0}$

4. Dot product is $\emptyset$ because E is symmetrical, meaning its eigenvectors are orthogonal.

5. $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \longrightarrow x_1 + 2x_2 = 0$ $\qquad \boxed{x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}}$

6. $\begin{bmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$(1-\lambda)v_1 + 2v_2 = 0$

$v_2 = \frac{1}{2}(\lambda - 1)$

$2v_1 + (4-\lambda)v_2 = 0$

$2v_1 + \frac{1}{2}(4-\lambda)(\lambda-1)v_1 = 0$

$v_1(4 + 4\lambda - 4 + \lambda^2 + \lambda) = 0$

$\lambda^2 + 5\lambda = 0 \longrightarrow \lambda = 0$ or $\lambda = 5$

$\lambda = 0$: $\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 2 & 4 & 0 \end{array}\right] \longrightarrow$ eigenvector: $c * \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$\lambda = -5$: $\left[\begin{array}{cc|c} -4 & 2 & 0 \\ 2 & -1 & 0 \end{array}\right] \longrightarrow$ eigenvector: $c * \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}$

7. $\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 3 & 2 & 0 \end{array}\right]$

$\hookrightarrow$ can only be true when multiplied by $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$. i.e., only one solution. This checks out because the matrix is full-rank (i.e. linearly independent row vectors/column vectors).

D. $f(x) = x^2 + 3$ $g(x) = x^2$ $q(x,y) = x^2 + y^2$

1. $\boxed{\begin{aligned} f'(x) &= 2x \\ f''(x) &= 2 \end{aligned}}$

2. $\boxed{\begin{aligned} \partial q / \partial x &= 2x \\ \partial q / \partial y &= 2y \end{aligned}}$

3. $\nabla q(x,y) = \begin{bmatrix} \partial q/\partial x \\ \partial q/\partial y \end{bmatrix} = \boxed{\begin{bmatrix} 2x \\ 2y \end{bmatrix}}$

4. $\frac{d}{dx} f(g(x)) \longrightarrow$ outer: $f(g(x)) = g(x)^2 + 3$
   inner: $g(x) = x^2$

chain rule:

$$\frac{d}{dx} f(g(x)) = \frac{df}{dg} \cdot \frac{dg}{dx}$$
$$= \frac{d}{dg}\left[g(x)^2 + 3\right] \cdot \frac{d}{dx}\left[x^2\right]$$
$$= 2g(x) \cdot 2x$$
$$= 2x^2 \cdot 2x$$
$$= \boxed{4x^3}$$

without chain rule:

$$\frac{d}{dx} f(g(x)) = \frac{d}{dx}\left[g(x)^2 + 3\right]$$
$$= \frac{d}{dx}\left[(x^2)^2 + 3\right]$$
$$= \frac{d}{dx}\left[x^4 + 3\right]$$
$$= \boxed{4x^3}$$

E. repeated in Python
(attached separately)