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## Homework 0

A.

$$a = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad b = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \quad c = \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}$$

1.  $2a - b$

$$= 2 \times \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 2 \times 1 - 4 \\ 2 \times 2 - 5 \\ 2 \times 3 - 6 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ 0 \end{bmatrix}$$

2.  $\hat{a}$

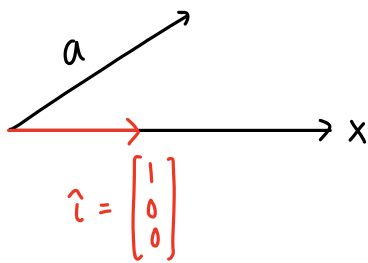
$$= \frac{1}{\|a\|} \times a$$

$$\|a\| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$$

$$= (1/\sqrt{14}) \times \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1/\sqrt{14} \\ 2/\sqrt{14} \\ 3/\sqrt{14} \end{bmatrix}$$

$$3. \quad \|a\| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$$



$$\cos(\theta) = \left( \frac{a \cdot \hat{i}}{\|a\| \cdot \|\hat{i}\|} \right)$$

$$\theta = \cos^{-1} \left( \frac{a \cdot \hat{i}}{\|a\| \cdot \|\hat{i}\|} \right)$$

$$a \cdot \hat{i} = (1)(1) + (2)(0) + (3)(0) = 1$$

$$= \cos^{-1} \left( \frac{1}{\sqrt{14} \times 1} \right)$$

$$\|a\| = \sqrt{14}$$

$$\|\hat{i}\| = 1$$

$$\approx 74.5^\circ$$

$$\|a\| = \sqrt{14}$$

$$\theta = 74.5^\circ$$

4.

$$\cos(\theta_x) = \frac{a \cdot \hat{i}}{\|a\| \cdot \|\hat{i}\|}$$

$$\cos(\theta_y) = \frac{a \cdot \hat{j}}{\|a\| \cdot \|\hat{j}\|}$$

$$\cos(\theta_z) = \frac{a \cdot \hat{k}}{\|a\| \cdot \|\hat{k}\|}$$

$$\|\hat{i}\| = \|\hat{j}\| = \|\hat{k}\| = 1$$

$$a \cdot \hat{i} = (1)(1) + (2)(0) + (3)(0) = 1$$

$$a \cdot \hat{j} = (1)(0) + (2)(1) + (3)(0) = 2$$

$$a \cdot \hat{k} = (1)(0) + (2)(0) + (3)(1) = 3$$

$$\|a\| = \sqrt{14}$$

$$\cos(\theta_x) = 1/\sqrt{14}$$

$$\cos(\theta_y) = 2/\sqrt{14}$$

$$\cos(\theta_z) = 3/\sqrt{14}$$

5. angle between  $a$  and  $b$

$$\theta = \cos^{-1} \left( \frac{a \cdot b}{\|a\| \cdot \|b\|} \right)$$

$$= \cos^{-1} \left( \frac{32}{\sqrt{14} \times \sqrt{77}} \right)$$

$$\approx 12.93^\circ$$

$$a \cdot b = (1)(4) + (2)(5) + (3)(6) = 32$$

$$\|a\| = \sqrt{14}$$

$$\|b\| = \sqrt{4^2 + 5^2 + 6^2} = \sqrt{77}$$

6.

$$a \cdot b = (1)(4) + (2)(5) + (3)(6) = 32$$

$$b \cdot a = (4)(1) + (5)(2) + (6)(3) = 32$$

7.  $\cos(\theta_{a,b}) = \left( \frac{a \cdot b}{\|a\| \cdot \|b\|} \right)$

$$a \cdot b = \cos(\theta_{a,b}) \|a\| \cdot \|b\|$$

$$= \cos(12.93^\circ) \times \sqrt{14} \times \sqrt{77}$$

$$= 32$$

8. scalar projection of  $b$  onto  $a = \|b\| \cdot \cos(\theta_{a,b})$

$$= \sqrt{77} \cdot \cos(12.93^\circ)$$

$$\approx 8.55$$

9. perpendicular vector, when dotted with  $a$ , equals 0.

$$p = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}$$

$$a \cdot p = 1 \cdot p_1 + 2 \cdot p_2 + 3 \cdot p_3 = 0$$

(one example that satisfies this:

$$p = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

10.

$$a \times b = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= \hat{i}(\overbrace{a_2 b_3 - b_2 a_3}^{-3}) - \hat{j}(\overbrace{a_1 b_3 - b_1 a_3}^{-6}) + \hat{k}(\overbrace{a_1 b_2 - b_1 a_2}^{-3})$$

$$= \begin{bmatrix} (1)(-3) - 0 & + & 0 \\ 0 & - (1)(-6) & + & 0 \\ 0 & - 0 & + & (1)(-3) \end{bmatrix} = \begin{bmatrix} -3 \\ 6 \\ -3 \end{bmatrix}$$

$$b \times a = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix}$$

$$= \hat{i}(b_1 a_2 - a_1 b_2) - \hat{j}(b_1 a_3 - a_1 b_3) + \hat{k}(b_1 a_2 - a_1 b_2)$$

$$= \hat{i}(\overbrace{4 \cdot 2 - 1 \cdot 5}^3) - \hat{j}(\overbrace{4 \cdot 3 - 1 \cdot 6}^6) + \hat{k}(\overbrace{4 \cdot 2 - 1 \cdot 5}^3)$$

$$= \begin{bmatrix} 1 \cdot 3 - 0 & + & 0 \\ 0 & - 1 \cdot 6 & + & 0 \\ 0 & - 0 & + & 1 \cdot 3 \end{bmatrix} = \begin{bmatrix} 3 \\ -6 \\ 3 \end{bmatrix}$$

$$a \times b = \begin{bmatrix} -3 \\ 6 \\ -3 \end{bmatrix}$$

$$b \times a = \begin{bmatrix} 3 \\ -6 \\ 3 \end{bmatrix}$$

11.

both  $a \times b$  and  $b \times a$  are perpendicular to  $a$  and  $b$ .

$$\begin{bmatrix} -3 \\ 6 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = -3 + 12 - 9 = 0 \quad \checkmark \quad \begin{bmatrix} 3 \\ -6 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 3 - 12 + 9 = 0 \quad \checkmark$$

$$\begin{bmatrix} -3 \\ 6 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = -12 + 30 - 18 = 0 \quad \checkmark \quad \begin{bmatrix} 3 \\ -6 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = 12 - 30 + 18 = 0 \quad \checkmark$$

12.

$$\begin{array}{ccc} a & b & c \\ \begin{bmatrix} 1 & 4 & -1 \\ 2 & 5 & 1 \\ 3 & 6 & 3 \end{bmatrix} & \begin{bmatrix} x \\ y \\ z \end{bmatrix} & = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \end{array}$$

$$\left[ \begin{array}{ccc|c} 1 & 4 & -1 & 0 \\ 2 & 5 & 1 & 0 \\ 3 & 6 & 3 & 0 \end{array} \right] \longrightarrow \left[ \begin{array}{ccc|c} 1 & 4 & -1 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & -6 & 6 & 0 \end{array} \right] \longrightarrow \left[ \begin{array}{ccc|c} 1 & 4 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$\begin{cases} x + 4y - z = 0 \\ y = z \\ 0 = 0 \end{cases}$$

$$z = 1 \longrightarrow y = 1, x = -3$$

$$-3 \begin{array}{c} a \\ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \end{array} + \begin{array}{c} b \\ \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \end{array} + \begin{array}{c} c \\ \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} \end{array} = \begin{array}{c} \\ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \end{array}$$

$$\hookrightarrow -3a + b + c = 0$$

$$13. \quad a = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad a^T = [1 \ 2 \ 3]$$

$$b = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \quad b^T = [4 \ 5 \ 6]$$

$$a^T b = \underset{1 \times 3}{[1 \ 2 \ 3]} \underset{3 \times 1}{\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}} = [1 \cdot 4 + 2 \cdot 5 + 3 \cdot 6] = \boxed{[32]} \quad 1 \times 1$$

$$a b^T = \underset{3 \times 1}{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}} \underset{1 \times 3}{[4 \ 5 \ 6]} = \underset{3 \times 3}{\begin{bmatrix} 4 & 5 & 6 \\ 8 & 10 & 12 \\ 12 & 15 & 18 \end{bmatrix}}$$

B.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & -2 & 3 \\ 0 & 5 & -1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & -4 \\ 3 & -2 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ -1 & 1 & 3 \end{bmatrix} \quad d = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

1.

$$2A - B = \begin{bmatrix} 2 & 4 & 6 \\ 8 & -4 & 6 \\ 0 & 10 & -2 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & -4 \\ 3 & -2 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} 1 & 2 & 5 \\ 6 & -5 & 10 \\ -3 & 12 & -3 \end{bmatrix}}$$

2.

$$AB = \begin{bmatrix} (1)(1) + \overset{14}{(2)(2)} + (3)(3) & (1)(2) + \overset{-2}{(2)(1)} + (3)(-2) & (1)(1) + \overset{-4}{(2)(-4)} + (3)(1) \\ (4)(1) + \overset{9}{(-2)(2)} + (3)(3) & (4)(2) + \overset{0}{(-2)(1)} + (3)(-2) & (4)(1) + \overset{15}{(-2)(-4)} + (3)(1) \\ (0)(1) + \overset{7}{(5)(2)} + (-1)(3) & (0)(2) + \overset{7}{(5)(1)} + (-1)(-2) & (0)(1) + \overset{-21}{(5)(-4)} + (-1)(1) \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} 14 & -2 & -4 \\ 9 & 0 & 15 \\ 7 & 7 & -21 \end{bmatrix}}$$

continued

$$BA = \begin{bmatrix} (1)(1) + (2)(4) + (1)(0) & (1)(2) + (2)(-2) + (1)(5) & (1)(3) + (2)(3) + (1)(1) \\ (2)(1) + (1)(4) + (-4)(0) & (2)(2) + (1)(-2) + (-4)(5) & (2)(3) + (1)(3) + (-4)(1) \\ (3)(1) + (-2)(4) + (1)(0) & (3)(2) + (-2)(-2) + (1)(5) & (3)(3) + (-2)(3) + (1)(1) \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 3 & 8 \\ 6 & -18 & 13 \\ -5 & 15 & 1 \end{bmatrix}$$

$$3. \quad AB = \begin{bmatrix} 14 & -2 & -4 \\ 9 & 0 & 15 \\ 7 & 7 & -21 \end{bmatrix} \rightarrow (AB)^T = \begin{bmatrix} 14 & 9 & 7 \\ -2 & 0 & 7 \\ -4 & 15 & -21 \end{bmatrix}$$

$$B^T = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & -2 \\ 1 & -4 & 1 \end{bmatrix} \quad A^T = \begin{bmatrix} 1 & 4 & 0 \\ 2 & -2 & 5 \\ 3 & 3 & -1 \end{bmatrix}$$

$$B^T A^T = \begin{bmatrix} (1)(1) + (2)(2) + (3)(3) & (1)(4) + (2)(-2) + (3)(3) & (1)(0) + (2)(5) + (3)(-1) \\ (2)(1) + (1)(2) + (-2)(3) & (2)(4) + (1)(-2) + (-2)(3) & (2)(0) + (1)(5) + (-2)(-1) \\ (1)(1) + (-4)(2) + (1)(3) & (1)(4) + (-4)(-2) + (1)(3) & (1)(0) + (-4)(5) + (1)(-1) \end{bmatrix}$$

$$B^T A^T = \begin{bmatrix} 14 & 9 & 7 \\ -2 & 0 & 7 \\ -4 & 15 & -21 \end{bmatrix}$$

4.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & -2 & 3 \\ 0 & 5 & -1 \end{bmatrix}$$

$$\begin{aligned} |A| &= 1 \overbrace{[(-2)(-1) - (5)(3)]}^{-13} - 2 \overbrace{[(4)(-1) - (0)(3)]}^8 + 3 \overbrace{[(4)(5) - (0)(-2)]}^{60} \\ &= -13 + 8 + 60 \\ &= \boxed{55} \end{aligned}$$

$$C = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ -1 & 1 & 3 \end{bmatrix}$$

$$\begin{aligned} |C| &= 1 \overbrace{[(5)(3) - (-1)(6)]}^9 - 2 \overbrace{[(4)(3) - (-1)(6)]}^{36} + 3 \overbrace{[(4)(1) - (-1)(5)]}^{27} \\ &= 9 - 36 + 27 \\ &= \boxed{0} \end{aligned}$$

$\Rightarrow$  0 determinant  $\Rightarrow$  linearly dependent rows  
(demonstrated in A-12)

5.

$$B: r_1 \cdot r_2 = (1)(2) + (2)(1) + (1)(-4) = 0$$

$$r_1 \cdot r_3 = (1)(3) + (2)(-2) + (1)(1) = 0$$

$$r_2 \cdot r_3 = (2)(3) + (1)(-2) + (-4)(1) = 0$$

The rows of  $B$  form an orthogonal set.  
(i.e. dot products between each vector pair = 0).



6. Inverse of  $3 \times 3$  matrix:  $M^{-1} = \text{adjoint}(M) / \text{determinant}(M)$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & -2 & 3 \\ 0 & 5 & -1 \end{bmatrix}$$

$$\text{adj}(A) = \begin{bmatrix} \begin{vmatrix} -2 & 3 \\ 5 & -1 \end{vmatrix} & -\begin{vmatrix} 4 & 3 \\ 0 & -1 \end{vmatrix} & \begin{vmatrix} 4 & 0 \\ -2 & 5 \end{vmatrix} \\ -\begin{vmatrix} 2 & 3 \\ 5 & -1 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 0 & -1 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 0 & 5 \end{vmatrix} \\ \begin{vmatrix} 1 & 2 \\ 4 & -2 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ 4 & 3 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 4 & -2 \end{vmatrix} \end{bmatrix}^T = \begin{bmatrix} -13 & 17 & -10 \\ 4 & 1 & 9 \\ 20 & -5 & -10 \end{bmatrix}$$

$\det(A): |A| = 55$  (found earlier)

$$B = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & -4 \\ 3 & -2 & 1 \end{bmatrix}$$

$$\text{adj}(B) = \begin{bmatrix} \begin{vmatrix} 2 & -4 \\ 3 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & -4 \\ 3 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 3 & -2 \end{vmatrix} \\ -\begin{vmatrix} 2 & 1 \\ -2 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 3 & -2 \end{vmatrix} \\ \begin{vmatrix} 2 & 1 \\ 1 & -4 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 2 & -4 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} \end{bmatrix}^T = \begin{bmatrix} -7 & -4 & -9 \\ 10 & -2 & 6 \\ -7 & 8 & -3 \end{bmatrix}$$

$$\det(B) = 1 \overbrace{[(1)(1) - (-2)(-4)]}^{-7} - 2 \overbrace{[(2)(1) - (3)(-4)]}^{-28} + 1 \overbrace{[(2)(-2) - (3)(1)]}^{-7} = -42$$

$$A^{-1} = \begin{bmatrix} -13 & 17 & -10 \\ 4 & 1 & 9 \\ 20 & -5 & -10 \end{bmatrix} \times \frac{1}{55}$$

$$B^{-1} = \begin{bmatrix} -7 & -4 & -9 \\ 10 & -2 & 6 \\ -7 & 8 & -3 \end{bmatrix} \times \left(-\frac{1}{42}\right)$$

7.

$$C = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ -1 & 1 & 3 \end{bmatrix}$$

$$\text{adjoint } C = \begin{bmatrix} \overset{9}{|5 \ 6|} - \overset{-18}{|4 \ 6|} & \overset{9}{|4 \ 5|} \\ -\overset{-3}{|2 \ 3|} & \overset{6}{|1 \ 3|} - \overset{-3}{|1 \ 2|} \\ \overset{-3}{|2 \ 3|} & -\overset{6}{|4 \ 6|} & \overset{-3}{|4 \ 5|} \end{bmatrix}^T = \begin{bmatrix} 9 & -3 & -3 \\ -18 & 6 & 6 \\ 9 & -3 & -3 \end{bmatrix}$$

$$\begin{aligned} \text{determinant } C &= 1 \times \begin{vmatrix} 5 & 6 \\ 1 & 3 \end{vmatrix} - 2 \times \begin{vmatrix} 4 & 6 \\ -1 & 3 \end{vmatrix} + 3 \times \begin{vmatrix} 4 & 5 \\ -1 & 1 \end{vmatrix} \\ &= 1(15-6) - 2(12+6) + 3(4+5) \\ &= 9 - 36 + 27 \\ &= 0 \end{aligned}$$

$$C^{-1} = \frac{\text{adj } C}{\det C} = \text{undefined, since determinant} = 0$$

↪ correction to B-4:  
matrices w/ linearly dependent  
rows or columns are not invertible

8.

$$Ad = \begin{bmatrix} (1)(1) + (2)(2) + (3)(3) \\ (4)(1) + (-2)(2) + (3)(3) \\ (0)(1) + (5)(2) + (-1)(3) \end{bmatrix} = \begin{bmatrix} 14 \\ 9 \\ 7 \end{bmatrix}$$

10. vector projection of  $v$  onto  $u$ :  $\left( \frac{v \cdot u}{\|u\|^2} \right) u$

↳ for each row of  $A$ , vector projection onto  $d$  is:

$$\left( \frac{\text{row} \cdot d}{\|d\|^2} \right) d = \frac{\text{row} \cdot d}{14} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & -2 & 3 \\ 0 & 5 & -1 \end{bmatrix}$$

row 1:  $(1)(1) + (2)(2) + (3)(3) = 14$

↳  $\frac{14}{14} d = \boxed{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}$

row 2:  $(4)(1) + (-2)(2) + (3)(3) = 9$

$\frac{9}{14} d = \boxed{\frac{9}{14} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}$

row 3:  $(0)(1) + (5)(2) + (-1)(3) = 7$

↳  $\frac{7}{14} d = \boxed{\frac{1}{2} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}$

reversed

9. scalar projections = lengths of vector projections

row 1:  $\sqrt{1^2 + 2^2 + 3^2} = \boxed{\sqrt{14}}$

row 2:  $\sqrt{\left(\frac{9}{14}\right)^2 [1^2 + 2^2 + 3^2]} = \frac{9}{14} \cdot \sqrt{14} = \boxed{\frac{9}{\sqrt{14}}}$

row 3:  $\sqrt{\left(\frac{1}{2}\right)^2 (1^2 + 2^2 + 3^2)} = \boxed{\frac{1}{2} \sqrt{14}}$

11.

linear combination = column 1 ·  $d_1$  + column 2 ·  $d_2$  + column 3 ·  $d_3$

$$\hookrightarrow 1 \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 2 \\ -2 \\ 5 \end{bmatrix} + 3 \begin{bmatrix} 3 \\ 3 \\ -1 \end{bmatrix}$$

12.  $Bx = d$

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & -4 \\ 3 & -2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

(1)  $x_1 + 2x_2 + x_3 = 1$

(2)  $2x_1 + x_2 - 4x_3 = 2$

(3)  $3x_1 - 2x_2 + x_3 = 3 \rightarrow x_3 = 3 - 3x_1 + 2x_2$

rewritten: (1)  $x_1 + 2x_2 + 3 - 3x_1 + 2x_2 = 1$

$$-2x_1 + 4x_2 = -2$$

$$x_1 - 2x_2 = 1$$

$$x_1 = 1 + 2x_2$$

(2)  $2x_1 + x_2 - 4[3 - 3x_1 + 2x_2] = 2$

$$2x_1 + 12x_1 + x_2 - 8x_2 - 12 = 2$$

$$14x_1 - 7x_2 = 14$$

$$x_1 - \frac{1}{2}x_2 = 1$$

$$x_1 = 1 + \frac{1}{2}x_2$$

$$x_2 = 0$$

$$x_1 = 1$$

$$x_3 = 3 - 3 + 0 = 0$$

$$x = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

13.  $Cx = d$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ -1 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\left[ \begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 4 & 5 & 6 & 2 \\ -1 & 1 & 3 & 3 \end{array} \right]$$

$$\left[ \begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -3 & -6 & -2 \\ -1 & 1 & 3 & 3 \end{array} \right]$$

$$\left[ \begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -3 & -6 & -2 \\ 0 & 3 & 6 & 4 \end{array} \right]$$

$$\left[ \begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -3 & -6 & -2 \\ 0 & 0 & 0 & 2 \end{array} \right]$$

impossible - no solution



relationship to B-4 and B-7:  
 $\emptyset$  determinant implies the matrix is  
 (1) non-invertible, and (2), has either no  
 solution or infinitely many solutions.

C.

$$D = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \quad E = \begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix} \quad F = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

$$1. \quad \begin{bmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 0$$

$$\text{from first row: } a(1-\lambda) + 2b = 0$$

$$b = \frac{(\lambda-1)}{2} a$$

$$\text{from second row: } 3a + (2-\lambda)b = 0$$

$$3a + (2-\lambda)\frac{\lambda-1}{2}a = 0$$

$$a(6 + 2\lambda - 2 - \lambda^2 + \lambda) = 0$$

$$-\lambda^2 + 3\lambda + 4 = 0$$

$$\lambda^2 - 3\lambda - 4 = 0$$

$$(\lambda-4)(\lambda+1) = 0 \rightarrow \lambda = 4 \text{ or } \lambda = -1$$

$$\boxed{\lambda = 4:} \quad \begin{bmatrix} 1-4 & 2 & | & 0 \\ 3 & 2-4 & | & 0 \\ -3 & 2 & | & 0 \\ 3 & -2 & | & 0 \end{bmatrix} \rightarrow \text{eigenvector: } \boxed{\begin{bmatrix} 2 \\ 3 \end{bmatrix}}$$

$$\boxed{\lambda = -1:} \quad \begin{bmatrix} 1-(-1) & 2 & | & 0 \\ 3 & 2-(-1) & | & 0 \\ 2 & 2 & | & 0 \\ 3 & 3 & | & 0 \end{bmatrix} \rightarrow \text{eigenvector: } \boxed{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}$$

2.

$$V_1 \cdot V_2 = 2 \cdot 1 - 3 \cdot 1 = -1$$

3.

$$\begin{bmatrix} 2-\lambda & -2 \\ -2 & 5-\lambda \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = 0$$

 $\lambda = 1:$ 

$$\begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = 0$$

$$\text{i. } (2-\lambda)u - 2v = 0$$

$$v = \frac{1}{2}u(2-\lambda)$$

$$u - 2v = 0$$

$$u = 2v$$

$$-2u + 4v = 0$$

$$2u = 4v$$

$$u = 2v$$

$$V_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\text{ii. } -2u + (5-\lambda)v = 0$$

$$-2u + (5-\lambda)(2-\lambda) \cdot \frac{1}{2}u = 0$$

$$-4u + (5-\lambda)(2-\lambda)u = 0$$

$$u(-4 + 10 - 5\lambda - 2\lambda + \lambda^2) = 0$$

$$6 - 7\lambda + \lambda^2 = 0$$

$$(\lambda - 1)(\lambda - 6) = 0$$

$$\lambda = 1, 6$$

 $\lambda = 6:$ 

$$\begin{bmatrix} -4 & -2 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = 0$$

$$V_1 \cdot V_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 2 \end{bmatrix} = -2 + 2 = 0$$

$$-2u - v = 0$$

$$v = -2u$$

$$V_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$V_1 \cdot V_2 = 0$$

4.

Dot product is 0 because  $E$  is symmetrical, meaning its eigenvectors are orthogonal.

$$5. \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \rightarrow x_1 + 2x_2 = 0$$

$$x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$6. \begin{bmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$(1-\lambda)v_1 + 2v_2 = 0$$

$$v_2 = \frac{1}{2}(\lambda-1)$$

$$2v_1 + (4-\lambda)v_2 = 0$$

$$2v_1 + \frac{1}{2}(4-\lambda)(\lambda-1)v_1 = 0$$

$$v_1(4 + 4\lambda - 4 + \lambda^2 + \lambda) = 0$$

$$\lambda^2 + 5\lambda = 0 \rightarrow \lambda = 0 \text{ or } \lambda = -5$$

$$\left[ \begin{array}{l} \lambda = 0: \left[ \begin{array}{cc|c} 1 & 2 & 0 \\ 2 & 4 & 0 \end{array} \right] \rightarrow \text{eigenvector: } c * \begin{bmatrix} -2 \\ 1 \end{bmatrix} \\ \lambda = -5: \left[ \begin{array}{cc|c} -4 & 2 & 0 \\ 2 & -1 & 0 \end{array} \right] \rightarrow \text{eigenvector: } c * \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} \end{array} \right]$$

$$7. \left[ \begin{array}{cc|c} 1 & 2 & 0 \\ 3 & 2 & 0 \end{array} \right]$$

↳ can only be true when multiplied by  $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ .  
 i.e., only one solution. This checks out because the matrix is full-rank (i.e. linearly independent row vectors / column vectors).



D.

$$f(x) = x^2 + 3$$

$$g(x) = x^2$$

$$q(x, y) = x^2 + y^2$$

1.

$$\begin{aligned} f'(x) &= 2x \\ f''(x) &= 2 \end{aligned}$$

2.

$$\begin{aligned} \partial q / \partial x &= 2x \\ \partial q / \partial y &= 2y \end{aligned}$$

$$3. \nabla q(x, y) = \begin{bmatrix} \partial q / \partial x \\ \partial q / \partial y \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$$

$$4. \frac{d}{dx} f(g(x)) \longrightarrow \text{outer: } f(g(x)) = g(x)^2 + 3$$

inner:  $g(x) = x^2$

chain rule:

$$\begin{aligned} \frac{d}{dx} f(g(x)) &= \frac{df}{dg} \cdot \frac{dg}{dx} \\ &= \frac{d}{dg} [g(x)^2 + 3] \cdot \frac{d}{dx} [x^2] \\ &= 2g(x) \cdot 2x \\ &= 2x^2 \cdot 2x \\ &= 4x^3 \end{aligned}$$

without chain rule:

$$\begin{aligned} \frac{d}{dx} f(g(x)) &= \frac{d}{dx} [g(x)^2 + 3] \\ &= \frac{d}{dx} [(x^2)^2 + 3] \\ &= \frac{d}{dx} [x^4 + 3] \\ &= 4x^3 \end{aligned}$$

E.

repeated in Python  
(attached separately)