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CS 584

Assignment 4

Question 1

1.

$$A = \begin{bmatrix} 4 & 6 & 5 \\ 1 & 2 & 3 \\ 7 & 10 & 7 \\ 6 & 8 & 4 \\ 6 & 10 & 11 \end{bmatrix} = \underset{m\times k}{\overset{W}{\begin{bmatrix} \; \end{bmatrix}}} \underset{k\times N}{\overset{H}{\begin{bmatrix} \; \end{bmatrix}}}$$

Step 1: initialize random, non-negative W

generated random entries in numpy: $W = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}$ $(k=2)$

Step 2: repeat:

2.1 $H \leftarrow \left[(W^TW)^{-1}W^TA\right]_+$

$$W^TW^{-1} = \left(\begin{bmatrix} 7 & 3 & 2 & 8 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}\right)^{-1}$$

calculated in Python

$$= \begin{bmatrix} 0.667 & -0.333 \\ -0.333 & 0.667 \end{bmatrix}$$

$$W^TA = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 4 & 6 & 5 \\ 1 & 2 & 3 \\ 7 & 10 & 7 \\ 6 & 8 & 4 \\ 6 & 10 & 11 \end{bmatrix} = \begin{bmatrix} (4+6) & (6+8) & (5+4) \\ (4+1) & (6+2) & (5+3) \end{bmatrix} = \begin{bmatrix} 10 & 14 & 9 \\ 5 & 8 & 8 \end{bmatrix}$$

$$\rightarrow H = (W^TW)^{-1}W^TA = \begin{bmatrix} 0.667 & -0.333 \\ -0.333 & 0.667 \end{bmatrix}\begin{bmatrix} 10 & 14 & 9 \\ 5 & 8 & 8 \end{bmatrix} = \begin{bmatrix} 5 & 6.667 & 3.333 \\ 0 & 0.667 & 2.333 \end{bmatrix}$$

2.2 $W \leftarrow \left[AH^T(HH^T)^{-1}\right]_+$

$$AH^T = \begin{bmatrix} 4 & 6 & 5 \\ 1 & 2 & 3 \\ 7 & 10 & 7 \\ 6 & 8 & 4 \\ 6 & 10 & 11 \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 6.667 & 3.333 \\ 0.667 & 2.333 \end{bmatrix} = \begin{bmatrix} 76.667 & 15.667 \\ 28.333 & 8.333 \\ 125 & 23 \\ 96.667 & 14.667 \\ 133.333 & 32.333 \end{bmatrix}$$

$$(HH^T)^{-1} = \left(\begin{bmatrix} 5 & 6.667 & 3.333 \\ 0 & 0.667 & 2.333 \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 6.667 & 3.333 \\ 0.667 & 2.333 \end{bmatrix}\right)^{-1}$$

calculated in Python

$$= \begin{bmatrix} 0.018 & -0.038 \\ -0.038 & 0.248 \end{bmatrix}$$

$$\rightarrow W = \left[AH^T(HH^T)^{-1}\right]_+ = \begin{bmatrix} 76.667 & 15.667 \\ 28.333 & 8.333 \\ 125 & 23 \\ 96.667 & 14.667 \\ 133.333 & 32.333 \end{bmatrix}\begin{bmatrix} 0.018 & -0.038 \\ -0.038 & 0.248 \end{bmatrix} = \begin{bmatrix} (0.018\cdot 76.667+(-0.038\cdot 15.667)) & (-0.038\cdot 76.667+0.248\cdot 15.667) \\ (0.018\cdot 28.333+(-0.038\cdot 8.333)) & (-0.038\cdot 28.333+0.248\cdot 8.333) \\ (0.018\cdot 125.0+(-0.038\cdot 23.000)) & (-0.038\cdot 125.0+0.248\cdot 23.000) \\ (0.018\cdot 96.667+(-0.038\cdot 14.667)) & (-0.038\cdot 96.667+0.248\cdot 14.667) \\ (0.018\cdot 133.333+(-0.038\cdot 32.333)) & (-0.038\cdot 133.333+0.248\cdot 32.333) \end{bmatrix} = \begin{bmatrix} 0.8 & 1.0 \\ 0.2 & 1.0 \\ 1.4 & 1.0 \\ 1.2 & 0.0 \\ 1.2 & 3.0 \end{bmatrix}$$

Step 3: check for convergence

$$\rightarrow WH = \begin{bmatrix} 0.8 & 1.0 \\ 0.2 & 1.0 \\ 1.4 & 1.0 \\ 1.2 & 0.0 \\ 1.2 & 3.0 \end{bmatrix}\begin{bmatrix} 5 & 6.667 & 3.333 \\ 0 & 0.667 & 2.333 \end{bmatrix} = \begin{bmatrix} 0.8\cdot 5+1.0\cdot 0 & 0.8\cdot 6.667+1.0\cdot 0.667 & 0.8\cdot 3.333+1.0\cdot 2.333 \\ 0.2\cdot 5+1.0\cdot 0 & 0.2\cdot 6.667+1.0\cdot 0.667 & 0.2\cdot 3.333+1.0\cdot 2.333 \\ 1.4\cdot 5+1.0\cdot 0 & 1.4\cdot 6.667+1.0\cdot 0.667 & 1.4\cdot 3.333+1.0\cdot 2.333 \\ 1.2\cdot 5+0.0\cdot 0 & 1.2\cdot 6.667+0.0\cdot 0.667 & 1.2\cdot 3.333+0.0\cdot 2.333 \\ 1.2\cdot 5+3.0\cdot 0 & 1.2\cdot 6.667+3.0\cdot 0.667 & 1.2\cdot 3.333+3.0\cdot 2.333 \end{bmatrix} = \begin{bmatrix} 4.000 & 6.000 & 5.000 \\ 1.000 & 2.000 & 3.000 \\ 7.000 & 10.000 & 7.000 \\ 6.000 & 8.000 & 4.000 \\ 6.000 & 10.000 & 11.000 \end{bmatrix}$$

compare to $A = \begin{bmatrix} 4 & 6 & 5 \\ 1 & 2 & 3 \\ 7 & 10 & 7 \\ 6 & 8 & 4 \\ 6 & 10 & 11 \end{bmatrix} \rightarrow$ converged!

Not necessary to repeat steps 2.1 and 2.2.

$$\boxed{W = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad H = \begin{bmatrix} 5 & 6.667 & 3.333 \\ 0 & 0.667 & 2.333 \end{bmatrix}}$$

2.

$$A = \begin{bmatrix} 12 & 22 & 41 & 35 \\ 19 & 20 & 13 & 48 \\ 11 & 14 & 16 & 29 \\ 14 & 16 & 14 & 36 \end{bmatrix} \begin{matrix} 1_0 \\ 2_0 \\ 3_0 \\ 4_0 \end{matrix}$$

row

	1	2	3	4
0	1_0	2_0	3_0	4_0
1	1_0-3_0	2_0-3_0	$2*3_0-2_0$	4_0-3_0
2	$1_0-2\cdot3_0+2\cdot2_0-2\cdot4_0$	$2_0+3_0-2\cdot4_0$	$3_0-2\cdot2_0+2\cdot4_0$	$3\cdot4_0-2\cdot3_0-2_0$
3	$\frac{3}{2}1_0-\frac{9}{2}3_0+\frac{7}{2}2_0-\frac{5}{2}4_0$	$5\cdot3_0-3\cdot4_0-1_0$	$\frac{5}{2}\cdot3_0-\frac{3}{2}\cdot2_0+\frac{1}{2}\cdot4_0-\frac{1}{2}\cdot1_0$	$4_0-4\cdot3_0+2_0+1_0$
4				$4\cdot4_0-9\cdot3_0+2_0+2\cdot1_0$
5				$\frac{7}{2}4_0-\frac{23}{2}3_0+\frac{5}{2}2_0+\frac{5}{2}1_0$

iteration

$[0\ 0\ 0\ 0]$

$$1-3: \begin{bmatrix} 1 & 8 & 25 & 6 \\ 19 & 20 & 13 & 48 \\ 11 & 14 & 16 & 29 \\ 14 & 16 & 14 & 36 \end{bmatrix} \begin{matrix} 1_1 = 1_0-3_0 \\ 2_0 \\ 3_0 \\ 4_0 \end{matrix}$$

$$2-3: \begin{bmatrix} 1 & 8 & 25 & 6 \\ 8 & 6 & -3 & 19 \\ 11 & 14 & 16 & 29 \\ 14 & 16 & 14 & 36 \end{bmatrix} \begin{matrix} 1_1 = 1_0-3_0 \\ 2_1 = 2_0-3_0 \\ 3_0 \\ 4_0 \end{matrix}$$

$$4-3: \begin{bmatrix} 1 & 8 & 25 & 6 \\ 8 & 6 & -3 & 19 \\ 11 & 14 & 16 & 29 \\ 3 & 2 & -2 & 7 \end{bmatrix} \begin{matrix} 1_1 = 1_0-3_0 \\ 2_1 = 2_0-3_0 \\ 3_0 \\ 4_1 = 4_0-3_0 \end{matrix}$$

$$3-2: \begin{bmatrix} 1 & 8 & 25 & 6 \\ 8 & 6 & -3 & 19 \\ 3 & 8 & 19 & 10 \\ 3 & 2 & -2 & 7 \end{bmatrix} \begin{matrix} 1_1 = 1_0-3_0 \\ 2_1 = 2_0-3_0 \\ 3_1 = 2\cdot3_0-2_0 \\ 4_1 = 4_0-3_0 \end{matrix}$$

$$2-4 \text{ twice}: \begin{bmatrix} 1 & 8 & 25 & 6 \\ 2 & 2 & 1 & 5 \\ 3 & 8 & 19 & 10 \\ 3 & 2 & -2 & 7 \end{bmatrix} \begin{matrix} 1_1 = 1_0-3_0 \\ 2_2 = 2_0+3_0-2\cdot4_0 \\ 3_1 = 2\cdot3_0-2_0 \\ 4_1 = 4_0-3_0 \end{matrix}$$

$$3-2: \begin{bmatrix} 1 & 8 & 25 & 6 \\ 2 & 2 & 1 & 5 \\ 1 & 6 & 18 & 5 \\ 3 & 2 & -2 & 7 \end{bmatrix} \begin{matrix} 1_1 = 1_0-3_0 \\ 2_2 = 2_0+3_0-2\cdot4_0 \\ 3_2 = 3_0-2\cdot2_0+2\cdot4_0 \\ 4_1 = 4_0-3_0 \end{matrix}$$

$$4-2: \begin{bmatrix} 1 & 8 & 25 & 6 \\ 2 & 2 & 1 & 5 \\ 1 & 6 & 18 & 5 \\ 1 & 0 & -3 & 2 \end{bmatrix} \begin{matrix} 1_1 = 1_0-3_0 \\ 2_2 = 2_0+3_0-2\cdot4_0 \\ 3_2 = 3_0-2\cdot2_0+2\cdot4_0 \\ 4_2 = 3\cdot4_0-2\cdot3_0-2_0 \end{matrix}$$

$$1-3: \begin{bmatrix} 0 & 2 & 7 & 1 \\ 2 & 2 & 1 & 5 \\ 1 & 6 & 18 & 5 \\ 1 & 0 & -3 & 2 \end{bmatrix} \begin{matrix} 1_2 = 1_0-2\cdot3_0+2\cdot2_0-2\cdot4_0 \\ 2_2 = 2_0+3_0-2\cdot4_0 \\ 3_2 = 3_0-2\cdot2_0+2\cdot4_0 \\ 4_2 = 3\cdot4_0-2\cdot3_0-2_0 \end{matrix}$$

$$4+1: \begin{bmatrix} 0 & 2 & 7 & 1 \\ 2 & 2 & 1 & 5 \\ 1 & 6 & 18 & 5 \\ 1 & 2 & 4 & 3 \end{bmatrix} \begin{matrix} 1_2 = 1_0-2\cdot3_0+2\cdot2_0-2\cdot4_0 \\ 2_2 = 2_0+3_0-2\cdot4_0 \\ 3_2 = 3_0-2\cdot2_0+2\cdot4_0 \\ 4_3 = 4_0-4\cdot3_0+2_0+1 \end{matrix}$$

$$3-4, \text{divided by } 2: \begin{bmatrix} 0 & 2 & 7 & 1 \\ 2 & 2 & 1 & 5 \\ 0 & 2 & 7 & 1 \\ 1 & 2 & 4 & 3 \end{bmatrix} \begin{matrix} 1_2 = 1_0-2\cdot3_0+2\cdot2_0-2\cdot4_0 \\ 2_2 = 2_0+3_0-2\cdot4_0 \\ 3_3 = \frac{5}{2}\cdot3_0-\frac{3}{2}\cdot2_0+\frac{1}{2}\cdot4_0-\frac{1}{2}\cdot1_0 \\ 4_3 = 4_0-4\cdot3_0+2_0+1 \end{matrix}$$

$$1-3: \begin{bmatrix} 0 & 0 & 0 & 0 \\ 2 & 2 & 1 & 5 \\ 0 & 2 & 7 & 1 \\ 1 & 2 & 4 & 3 \end{bmatrix} \begin{matrix} 1_3 = \frac{3}{2}1_0-\frac{9}{2}3_0+\frac{7}{2}2_0-\frac{5}{2}4_0 \\ 2_2 = 2_0+3_0-2\cdot4_0 \\ 3_3 = \frac{5}{2}\cdot3_0-\frac{3}{2}\cdot2_0+\frac{1}{2}\cdot4_0-\frac{1}{2}\cdot1_0 \\ 4_3 = 4_0-4\cdot3_0+2_0+1 \end{matrix}$$

$$2-4: \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & -3 & 2 \\ 0 & 2 & 7 & 1 \\ 1 & 2 & 4 & 3 \end{bmatrix} \begin{matrix} 1_3 = \frac{3}{2}1_0-\frac{9}{2}3_0+\frac{7}{2}2_0-\frac{5}{2}4_0 \\ 2_3 = 5\cdot3_0-3\cdot4_0-1_0 \\ 3_3 = \frac{5}{2}\cdot3_0-\frac{3}{2}\cdot2_0+\frac{1}{2}\cdot4_0-\frac{1}{2}\cdot1_0 \\ 4_3 = 4_0-4\cdot3_0+2_0+1 \end{matrix}$$

$$4-2: \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & -3 & 2 \\ 0 & 2 & 7 & 1 \\ 0 & 2 & 7 & 1 \end{bmatrix} \begin{matrix} 1_3 = \frac{3}{2}1_0-\frac{9}{2}3_0+\frac{7}{2}2_0-\frac{5}{2}4_0 \\ 2_3 = 5\cdot3_0-3\cdot4_0-1_0 \\ 3_3 = \frac{5}{2}\cdot3_0-\frac{3}{2}\cdot2_0+\frac{1}{2}\cdot4_0-\frac{1}{2}\cdot1_0 \\ 4_4 = 4\cdot4_0-9\cdot3_0+2_0+2\cdot1_0 \end{matrix}$$

$$4-3: \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & -3 & 2 \\ 0 & 2 & 7 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} 1_3 = \frac{3}{2}1_0-\frac{9}{2}3_0+\frac{7}{2}2_0-\frac{5}{2}4_0 \\ 2_3 = 5\cdot3_0-3\cdot4_0-1_0 \\ 3_3 = \frac{5}{2}\cdot3_0-\frac{3}{2}\cdot2_0+\frac{1}{2}\cdot4_0-\frac{1}{2}\cdot1_0 \\ 4_5 = \frac{7}{2}4_0-\frac{23}{2}3_0+\frac{5}{2}2_0+\frac{5}{2}1_0 \end{matrix}$$

$$\begin{cases} \frac{3}{2}1_0-\frac{9}{2}3_0+\frac{7}{2}2_0-\frac{5}{2}4_0 = [0\ 0\ 0\ 0] \\ \frac{7}{2}4_0-\frac{23}{2}3_0+\frac{5}{2}2_0+\frac{5}{2}1_0 = [0\ 0\ 0\ 0] \end{cases}$$

Row reduction shows that, via these 2 equations, each row can be written as a linear combination of the others.