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Math 476
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Homework 2

(1)

Suppose that the random variable Y has a gamma distribution with parameters $\alpha = 2$ and an unknown β. In Exercise 6.46, you used the method of moment-generating functions to prove a general result implying that $2Y/\beta$ has a χ^2 distribution with 4 degrees of freedom (df). Using $2Y/\beta$ as a pivotal quantity, derive a 90% confidence interval for β.

$Y \sim \Gamma(\alpha, \beta) \longrightarrow \alpha = 2,\ \beta$ unknown

$U = 2Y/\beta \sim \chi^2_4 \longrightarrow \beta = 2Y/U$

$\longrightarrow Y = \frac{1}{2}\cdot\beta\cdot u$

$\longrightarrow \left|\frac{dy}{du}\right| = \frac{1}{2}\cdot\beta$

i. Finding $f_u(u)$: 2 methods

① $f_u(u) = f_y(y)\cdot\left|\frac{dy}{du}\right|$

$$= \left[\frac{1}{\Gamma(\alpha)\cdot\beta^\alpha}\right] y^{\alpha-1} e^{-y/\beta} \times \frac{1}{2}\beta$$

$$= \frac{1}{\Gamma(2)\,\beta}\cdot y\cdot e^{-y/\beta}\cdot\frac{1}{2}$$

$$= \frac{1}{\Gamma(2)\beta}\cdot\frac{\beta\cdot u}{2}\cdot\frac{1}{2}\cdot e^{-u/2}$$

$$= \frac{u}{4}\cdot e^{-u/2}$$

② p.d.f. for χ^2_v

$$= \frac{u^{(v/2)-1}\, e^{-u/2}}{2^{v/2}\cdot\Gamma(v/2)}$$

$$= \frac{u\cdot e^{-u/2}}{2^2\,\Gamma(2)}$$

$$= \frac{u}{4}\cdot e^{-u/2}$$

$\Gamma(2) = ?$

$\Gamma(x+1) = x\cdot\Gamma(x)$

$x + 1 = 2$

$x = 1$

$= 1\cdot\Gamma(1)$

$= 1$

ii. We need: $P(a \le U \le b) = 0.9$ (i.e., $P(U < a) = P(U > b) = 0.05$)

$$\int f_u(u)\cdot du = \int \frac{u e^{-u/2}}{4}\cdot du$$

$$= \int \frac{1}{4}\cdot(-2v)\cdot e^{v}\cdot(-2)\,dv$$

$$= \int v\cdot e^{v}\cdot dv$$

$$= v\cdot e^{v} - e^{v}\Big|_{?}^{?}$$

$$= -\frac{u}{2}\cdot e^{-u/2} - e^{-u/2}\Big|_{?}^{?}$$

It will be hard to solve for a and b directly from this

instead, from table:

$\chi^2_{0.05,4} = 9.48773$

$\chi^2_{0.95,4} = 0.710721$

$(0.710721 \le 2Y/\beta \le 9.48773)$

$$\boxed{(2Y/9.4877 \le \beta \le 2Y/0.7107)}$$

(2)

The administrators for a hospital wished to estimate the average number of days required for inpatient treatment of patients between the ages of 25 and 34. A random sample of 500 hospital patients between these ages produced a mean and standard deviation equal to 5.4 and 3.1 days, respectively. Construct a 95% confidence interval for the mean length of stay for the population of patients from which the sample was drawn.

$n = 500$ hospitals — large enough
$\bar{x} = 5.4$ days
$s = 3.1$ days
$\alpha = 0.05, \quad \alpha/2 = 0.025$
$z_{\alpha/2} = 1.96$

$$-1.96 \le \frac{5.4 - \mu}{\frac{3.1}{\sqrt{500}}} \le 1.96$$

$$(-1.96)\frac{3.1}{\sqrt{500}} - 5.4 \le -\mu \le 1.96\frac{3.1}{\sqrt{500}} - 5.4$$

$$(1.96)\frac{3.1}{\sqrt{500}} + 5.4 \ge -\mu \ge -1.96\frac{3.1}{\sqrt{500}} + 5.4$$

$$(-1.96)\frac{3.1}{\sqrt{500}} + 5.4 \le \mu \le 1.96\frac{3.1}{\sqrt{500}} + 5.4$$

$$\boxed{(5.1283, 5.6717)}$$

(3)

For a comparison of the rates of defectives produced by two assembly lines, independent random samples of 100 items were selected from each line. Line A yielded 18 defectives in the sample, and line B yielded 12 defectives.

a Find a 98% confidence interval for the true difference in proportions of defectives for the two lines.

b Is there evidence here to suggest that one line produces a higher proportion of defectives than the other?

$n_A = 100 \qquad p_A = 18/100 = 0.18$

$n_B = 100 \qquad p_B = 12/100 = 0.12$

(a) $\alpha = 0.02 \qquad s_A = (\hat{p}_A \hat{q}_A)/n_A = (0.18)(0.82)/100 = 0.001476$

$\alpha/2 = 0.01 \qquad s_B = (\hat{p}_B \hat{q}_B)/n_B = (0.12)(0.88)/100 = 0.001056$

$$-z_{0.01} \le \frac{(\hat{p}_A - \hat{p}_B) - (p_A - p_B)}{\sqrt{(\hat{p}_A \hat{q}_A)/n_A + (\hat{p}_B \hat{q}_B)/n_B}} \le z_{0.01}$$

$$-2.326 \le \frac{(0.18 - 0.12) - (p_A - p_B)}{\sqrt{0.001476 + 0.001056}} \le 2.326$$

C.I.: $0.06 \pm (2.326)(0.050319)$

$\boxed{(-0.057, 0.177)}$

(b) Since the confidence interval includes zero, we are unable to rule out the possibility that the difference is zero.

No, there is no evidence.

(4) **Problem 4** We sample n observations $Y_1, \ldots, Y_n$ from a uniform distribution on $[0, \theta]$, and θ is unknown. Use the pivotal method to find a 90% lower confidence bound for θ.

pivotal quantity: $U = \frac{Y_{(n)}}{\theta} \longrightarrow \theta = Y_{(n)}/U$, $Y_{(n)} = \theta \cdot U$, and $\left|\frac{dy}{du}\right| = \theta$

$$
\begin{aligned}
F(Y_{(n)}) &= P(Y_{(n)} \le y) \\
&= P(Y_1 \le y, \ldots, Y_n \le y) \\
&= \left(\frac{y}{\theta}\right)^n
\end{aligned}
$$

$$
f_{Y_{(n)}}(y) = \begin{cases} 1 & y > \theta \\ n \cdot \left(\frac{y}{\theta}\right)^{n-1} \cdot \frac{1}{\theta} & 0 \le y \le \theta \\ 0 & \text{otherwise} \end{cases}
$$

$$
f_u(u) = f_{Y_{(n)}}(y) \cdot \left|\frac{dy}{du}\right|
$$

$$
= \begin{cases} 1 & \theta u > \theta \\ n \cdot \left(\frac{y}{\theta}\right)^{n-1} \cdot \frac{1}{\theta} \cdot \theta & 0 \le \theta \cdot u \le \theta \\ 0 & \text{otherwise} \end{cases}
$$

$$
= \begin{cases} 1 & u > 1 \\ nu^{n-1} & 0 \le u \le 1 \\ 0 & \text{otherwise} \end{cases}
$$

$P(\theta \ge L) = 0.9$

equivalent: $P(Y_{(n)}/u \ge L)$

$= P(u \le Y_{(n)}/L)$

$= \int_0^{Y_{(n)}/L} n \cdot u^{n-1} \, du$

$= u^n \cdot \theta \Big|_0^{Y_{(n)}/L}$

$= \left(\frac{Y_{(n)}}{L}\right)^n = 0.9$

$Y_{(n)}/L = 0.9^{1/n}$

$L = \boxed{Y_{(n)}/0.9^{1/n}}$

=

(5) **Problem 5** We sample one observation from a normal population $N(0, \sigma^2)$ and σ^2 is unknown.

1) Find a 95% confidence interval for σ^2.
2) Find a 95% confidence interval for σ.

$$W = \frac{(n-1)\cdot S^2}{\sigma^2} \sim \chi^2_{n-1}$$

$$P\left(\chi^2_{n-1,1-\alpha/2} \le W \le \chi^2_{n-1,\alpha}\right) = P\left(\chi^2_{n-1,1-\alpha/2} \le \frac{(n-1)\cdot S^2}{\sigma^2} \le \chi^2_{n-1,\alpha}\right)$$

$$= P\left(\frac{1}{\chi^2_{n-1,1-\alpha/2}} \ge \frac{\sigma^2}{(n-1)\cdot S^2} \ge \frac{1}{\chi^2_{n-1,\alpha/2}}\right)$$

$$= P\left(\frac{(n-1)\cdot S^2}{\chi^2_{n-1,\alpha/2}} \le \sigma^2 \le \frac{(n-1)\cdot S^2}{\chi^2_{n-1,1-\alpha/2}}\right)$$

$$= P\left(\frac{(1-1)\cdot S^2}{\chi^2_{n-1,\alpha/2}} \le \sigma^2 \le \frac{(1-1)\cdot S^2}{\chi^2_{n-1,1-\alpha/2}}\right)$$

$$= P\left(0 \le \sigma^2 \le 0\right) = P\left(\sqrt{0} \le \sqrt{\sigma^2} \le \sqrt{0}\right) = P\left(0 \le \sigma \le 0\right) = 0.95$$

(1) $\boxed{(0,0)}$

(2) $\boxed{(0,0)}$

(6) **Problem 6** Suppose that we are to obtain a single observation Y from an exponential distribution with mean θ. Use the single observation to form a 90% confidence interval for θ by using two different pivotal quantities:

1) Use Y/θ as a pivotal quantity.
2) Use $2Y/\theta$ as a pivotal quantity.

$Y_1, \dots, Y_n \sim \text{exponential}\left(\lambda = \frac{1}{\theta}\right)$

$U = cY/\theta$
$\theta = cY/U$
$Y = U\cdot\theta/c$
$\left|\frac{dy}{du}\right| = \theta/c$

C.I: Find A and B such that $(A \le U \le B) = 0.9$

$f_Y(y) = \lambda e^{-\lambda y}$
$= \frac{1}{\theta} e^{-y/\theta}$

$\therefore\ P(U > B) = P(U < A) = 0.05$

$P(U > B) = 1 - F_Y(B)$, $P(U < A) = F_Y(A)$

$f_U(u) = f_Y(y)\cdot\left|\frac{dy}{du}\right|$
$= \frac{1}{\theta} e^{-\left(\frac{u\theta}{c}\right)/\theta} \times \left|\frac{\theta}{c}\right|$
$= e^{-u/c} \times \left|\frac{1}{c}\right|$

$F_U(u) = \int_A^B e^{-u/c} \times \left|\frac{1}{c}\right| du$
$= -e^{-u/c}\Big|_A^B$

(1) $P(U > B) = 1 - F_Y(B)$
$= 1 - \left[-e^{-u/1}\Big|_0^B\right]$
$= 1 - \left(-e^{-B} - (-1)\right)$
$= e^{-B} = 0.05$
$\hookrightarrow B = -\ln(0.05) = 2.996$

$P(U < A) = F_Y(A)$
$= -e^{-u}\Big|_0^A$
$= -e^{-A} - (-1)$
$= -e^{-A} + 1 = 0.05$
$\hookrightarrow e^{-A} = 0.95$
$\hookrightarrow A = -\ln(0.95) = 0.0513$

$0.0513 \le U \le 2.996$
$0.0513 \le Y/\theta \le 2.996$
$\frac{Y}{2.996} \le \theta \le \frac{Y}{0.0513}$

$\boxed{\left(Y/2.996,\ Y/0.0513\right)}$

(2) $P(U > B) = 1 - F_Y(B)$
$= 1 - \left[-e^{-u/2}\Big|_0^B\right]$
$= 1 - \left(-e^{-B/2} + 1\right)$
$\hookrightarrow e^{-B/2} = 0.05$
$\hookrightarrow B/2 = -\ln(0.05) = 2.996$
$\hookrightarrow B = 2 \times 2.996$

$P(U < A) = F_Y(A)$
$= -e^{-u/2}\Big|_0^A$
$= -e^{-\frac{1}{2}\cdot A} - (-e^{0})$
$= -e^{-A/2} + 1 = 0.05$
$e^{-A/2} = 0.95$
$A/2 = -\ln(0.95)$
$A = 2 \times 0.0513$

$2\cdot 0.0513 \le U \le 2\cdot 2.996$
$2\cdot 0.0513 \le 2Y/\theta \le 2\cdot 2.996$
$\frac{Y}{2.996} \le \theta \le \frac{Y}{0.0513}$

$\boxed{\left(Y/2.996,\ Y/0.0513\right)}$