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Homework 4

Problem 1 An experimenter has prepared a drug dosage level that she claims will induce sleep for 85% of people suffering from insomnia. After examining the dosage, we feel that her claims regarding the effectiveness of the dosage are inflated. In an attempt to disprove her claim, we administer her prescribed dosage to 25 insomniacs and we observe Y , the number for whom the drug dose induces sleep. We wish to test the hypothesis

$$\begin{cases} H_0: & p = 0.85 \\ & vs \\ H_a: & p < 0.85 \end{cases}$$

Assume that the rejection region $RR = \{y : y \leq 16\}$ is used.

1) Find α, the type I error probability.

2) Find β, the type II error probability when $p = 0.8$.

$n = 25$

(1)

$$\begin{aligned}
\alpha &= P(\text{reject } H_0 \mid H_0 \text{ is true}) \\
&= P(y \leq 16 \mid p = 0.85) \\
&= \sum_{y=0}^{16} \binom{25}{y} \cdot 0.85^y (1 - 0.85)^{25-y} \\
&= 0.00797 \longrightarrow \boxed{\alpha = 0.00797}
\end{aligned}$$

(2)

$$\begin{aligned}
\beta &= P(\text{accept } H_0 \mid H_a \text{ is true}) \\
&= P(Y > 16 \mid p = 0.8) \\
&= \sum_{y=17}^{25} \binom{25}{y} 0.8^y (1-0.8)^{25-y} \\
&= 0.953 \longrightarrow \boxed{\beta = 0.953}
\end{aligned}$$

Problem 2 Suppose that $Y_1, \ldots, Y_n$ is a random sample from a normal distribution $N(\mu, \sigma^2)$, where σ^2 is known to be 100. We would like to test the following hypotheses:

$$\begin{cases} H_0: & \mu = 75 \\ & vs \\ H_a: & \mu > 75 \end{cases}$$

Perform the test by using the sample mean $\overline{Y}$ as the test statistic.

1) Decide the rejection region (RR) such that the test has significance level $\alpha = 0.05$.
2) Suppose $\mu = 80$, i.e. the H_a is true. What is the probability of Type II error using this RR?

$Y_1, \ldots, Y_n \sim N(\mu, \sigma^2)$

$\theta^2 = 100$

$n = 16$

(1) $RR = ?$

$$z_{0.05} = 1.645$$

$$\frac{\hat{\theta} - \theta_0}{\sigma_{\hat{\theta}}} > z_\alpha$$

$$\sqrt{\frac{100}{16}} = \frac{10}{4} = \frac{5}{2} = 2.5$$

$$\frac{\bar{Y} - 75}{\sqrt{100/16}} > 1.645$$

$$\bar{Y} > (1.645)(2.5) + 75$$

$$\bar{Y} > 79.1125 \longrightarrow \boxed{RR = \{\bar{y} : \bar{y} \le 79.1125\}}$$

(2) $\beta = P(\text{accept } H_0 \mid H_a \text{ is true})$

$$= P(\bar{y} \le 79.1125 \mid \mu = 80)$$

$$= P\left(z \le \frac{79.1125 - 80}{2.5}\right)$$

$$= P(z \le -0.355)$$

$$= 0.3613 \longrightarrow \boxed{\beta = 0.3613}$$

10.18 The hourly wages in a particular industry are normally distributed with mean \$13.20 and standard deviation \$2.50. A company in this industry employs 40 workers, paying them an average of \$12.20 per hour. Can this company be accused of paying substandard wages? Use an $\alpha = .01$ level test.

$\mu = 13.20$
$\sigma = 2.5$
$n = 40$
$\bar{y} = 12.20$
$\alpha = 0.01$

$$\begin{cases} H_0: \mu = 13.20 \\ H_a: \mu < 13.20 \end{cases}$$

$$z = \frac{12.20 - 13.20}{2.5/\sqrt{40}}$$

$$= -2.53 \rightarrow P(z \leq -2.53) = 0.0057 < 0.01$$

Yes, this company can be accused of paying substandard wages at the significance level $\alpha = 0.01$.

10.20 The Rockwell hardness index for steel is determined by pressing a diamond point into the steel and measuring the depth of penetration. For 50 specimens of an alloy of steel, the Rockwell hardness index averaged 62 with standard deviation 8. The manufacturer claims that this alloy has an average hardness index of at least 64. Is there sufficient evidence to refute the manufacturer's claim at the 1% significance level?

$n = 50$
$\bar{y} = 62$
$s = 8$
$\alpha = 0.01$

$$\begin{cases} H_0: \mu = 64 \\ H_a: \mu < 64 \end{cases}$$

$$z = \frac{62 - 64}{8/\sqrt{50}}$$

$$= -1.77 \rightarrow P(z < -1.77) = 0.0384 > 0.01$$

No, there is not sufficient evidence to refute the manufacturer's claim at the $\alpha = 0.01$ significance level

Problem 5 A random sample of 36 second graders who participated in sports had manual dexterity scores with mean 32.19 and standard deviation 4.34. An independent sample of 49 second graders who did not participate in sports had manual dexterity scores with mean 31.68 and standard deviation 4.56.

1) Test to see whether sufficient evidence exists to indicate that second graders who participate in sports have a higher mean dexterity score. Use $\alpha = 0.05$.
2) For the rejection region used in part 1), calculate β when $\mu_1 - \mu_2 = 2$.
3) Assume $n_1 = n_2 = n$. Find the sample size n that gives $\alpha = .05$ and $\beta = .05$ when $\mu_1 - \mu_2 = 2$.

$n_1 = 36 \quad n_2 = 49$

$\bar{y}_1 = 32.19 \quad \bar{y}_2 = 31.68$

$s_1 = 4.34 \quad s_2 = 4.56$

$$\begin{cases} H_0: \mu_1 - \mu_2 = 0 \\ H_a: \mu_1 - \mu_2 > 0 \end{cases}$$

(1) $z_{0.05} = 1.645 \longrightarrow RR: \{z : z \geq 1.645\}$

$$z = \frac{(\bar{y}_1 - \bar{y}_2) - (\mu_1 - \mu_2)}{\sqrt{(s_1^2/n_1) + (s_2^2/n_2)}}$$

$$= \frac{(32.19 - 31.68) - 0}{\sqrt{(4.34^2/36) + (4.56^2/49)}}$$

$$= 0.5239 \not\geq 1.645$$

No, sufficient evidence does not exist to indicate that the students who played sports had higher dexterity scores.

(2) $\beta = P(\text{accept } H_0 \mid H_a \text{ is true})$

$\overline{RR}: \{z : z < 1.645\}$

$$z = \frac{\bar{y}_1 - \bar{y}_2}{\sqrt{(4.34^2/36) + (4.56^2/49)}} < 1.645$$

$$(1.645)\left(\sqrt{(4.34^2/36) + (4.56^2/49)}\right) = 1.589 \longrightarrow \text{do not reject } H_0 \text{ if } (\bar{y}_1 - \bar{y}_2) < 1.589$$

$$P\left((\bar{y}_1 - \bar{y}_2) < 1.589 \mid \mu_1 - \mu_2 = 2\right) = P\left(z < \frac{1.589 - 2}{\sqrt{(4.34^2/36) + (4.56^2/49)}}\right)$$

$$= P(z < -0.422)$$

$$= 0.3365 \rightarrow \boxed{\beta = 0.3365}$$

$$\begin{cases} H_0: \mu_1 - \mu_2 = 0 \\ H_a: \mu_1 - \mu_2 > 0 \end{cases}$$

(3)

$$\begin{aligned}
\alpha &= P(\text{reject } H_0 \mid H_0 \text{ is true}) \\
&= P\left(\bar{Y}_1 - \bar{Y}_2 \geq k \text{ when } \mu_1 - \mu_2 = (\mu_1 - \mu_2)_0\right) \\
&= P\left(\frac{(\bar{Y}_1 - \bar{Y}_2) - (\mu_1 - \mu_2)_0}{\sqrt{(s_1^2/n) + (s_2^2/n)}} \geq \frac{k - (\mu_1 - \mu_2)_0}{\sqrt{(s_1^2/n) + (s_2^2/n)}}\right) \\
&= P(z \geq Z_\alpha)
\end{aligned}$$

$$\begin{aligned}
\beta &= P(\text{fail to reject } H_0 \mid H_a \text{ is true}) \\
&= P\left(\bar{Y}_1 - \bar{Y}_2 < k \text{ when } \mu_1 - \mu_2 = (\mu_1 - \mu_2)_a\right) \\
&= P\left(\frac{(\bar{Y}_1 - \bar{Y}_2) - (\mu_1 - \mu_2)_a}{\sqrt{(s_1^2/n) + (s_2^2/n)}} < \frac{k - (\mu_1 - \mu_2)_a}{\sqrt{(s_1^2/n) + (s_2^2/n)}}\right) \\
&= P(z < -Z_\beta)
\end{aligned}$$

$$Z_\alpha + Z_\beta = \frac{\left(k - (\mu_1 - \mu_2)_0\right) + \left((\mu_1 - \mu_2)_a - k\right)}{\sqrt{(s_1^2/n) + (s_2^2/n)}}$$

denominator $= \dfrac{\sqrt{s_1^2 + s_2^2}}{\sqrt{n}}$

$$= \frac{\left[(\mu_1 - \mu_2)_a - (\mu_1 - \mu_2)_0\right] \cdot \sqrt{n}}{\sqrt{s_1^2 + s_2^2}}$$

$$n = \frac{(Z_\alpha + Z_\beta)^2 \cdot (s_1^2 + s_2^2)}{\left[(\mu_1 - \mu_2)_a - (\mu_1 - \mu_2)_0\right]^2} = \frac{(1.645 + 0.422)^2 \cdot (4.34^2 + 4.56^2)}{(2 - 0)^2} = 42.32$$

round up: 43

$$\boxed{n = 43}$$

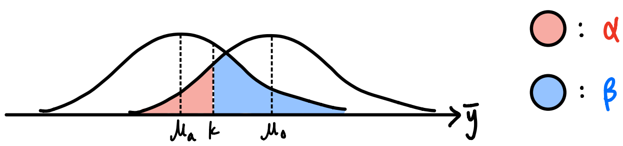
Problem 6 Suppose that you want to test

$$\begin{cases} H_0: & \mu = \mu_0 \\ & vs \\ H_a: & \mu < \mu_0 \end{cases}$$

Derive the required sample size n, given the desired values of α and β.

critical value

$$RR: \{\bar{y} : \bar{y} < k\}$$



$$\begin{aligned} \alpha &= P(\text{reject } H_0 \mid H_0 \text{ is true}) \\ &= P(\bar{Y} < k \text{ when } \mu = \mu_0) \\ &= P\left(\frac{\bar{Y} - \mu_0}{\sigma/\sqrt{n}} < \frac{k - \mu_0}{\sigma/\sqrt{n}} \text{ when } \mu = \mu_0\right) \\ &= P(z < -z_\alpha) \end{aligned}$$

$$\begin{aligned} \beta &= P(\text{fail to reject } H_0 \mid H_0 \text{ is false}) \\ &= P(\bar{Y} \geq k \text{ when } \mu = \mu_a) \\ &= P\left(\frac{\bar{Y} - \mu_a}{\sigma/\sqrt{n}} \geq \frac{k - \mu_a}{\sigma/\sqrt{n}} \text{ when } \mu = \mu_a\right) \\ &= P(z \geq z_\beta) \end{aligned}$$

$$\begin{cases} z_\alpha = \dfrac{\mu_0 - k}{\sigma/\sqrt{n}} \\ z_\beta = \dfrac{k - \mu_a}{\sigma/\sqrt{n}} \end{cases}$$

$$z_\alpha + z_\beta = \frac{(\mu_0 - k) + (k - \mu_a)}{\sigma/\sqrt{n}}$$

$$\sqrt{n} = \frac{(z_\alpha + z_\beta) \cdot \sigma}{\mu_0 - \mu_a}$$

$$\boxed{n = \frac{(z_\alpha + z_\beta)^2 \cdot \sigma^2}{(\mu_0 - \mu_a)^2}}$$