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Math 476

Homework 1

7.42 The fracture strength of tempered glass averages 14 (measured in thousands of pounds per square inch) and has standard deviation 2.

a What is the probability that the average fracture strength of 100 randomly selected pieces of this glass exceeds 14.5?

b Find an interval that includes, with probability 0.95, the average fracture strength of 100 randomly selected pieces of this glass.

$\mu = 14$
$\sigma = 2$

(a) $n = 100 \longrightarrow$ CLT, $\sim$ normal
$P(\bar{x} > 14.5)$

$$z\text{-score} = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}}$$

$$= \frac{14.5 - 14}{2/\sqrt{100}}$$

$$\approx 2.50$$

table takes uppermost/right-hand tail $\longrightarrow P(z \geq 2.50) = \boxed{0.0062}$

(b) $\alpha = \frac{1 - 0.95}{2} = 0.025$

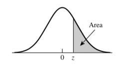
$z_{\alpha/2}$ occurs where $P(z > z_{\alpha/2} = 0.025) \longrightarrow 1.96$

interval bounds $= \bar{x} \pm z_{\alpha/2} \cdot (\sigma/\sqrt{n})$

$= 14.5 \pm 1.96 \cdot (2/\sqrt{100})$

$\boxed{\text{13.608 to 14.392 thousands of pounds per square inch}}$

Normal Curve Areas
Standard normal probability in right-hand tail (for negative values of z, areas are found by symmetry)



	Second decimal place of z									
z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.4960	.4920	.4880	.4840	.4801	.4761	.4721	.4681	.4641
0.1	.4602	.4562	.4522	.4483	.4443	.4404	.4364	.4325	.4286	.4247
0.2	.4207	.4168	.4129	.4090	.4052	.4013	.3974	.3936	.3897	.3859
0.3	.3821	.3783	.3745	.3707	.3669	.3632	.3594	.3557	.3520	.3483
0.4	.3446	.3409	.3372	.3336	.3300	.3264	.3228	.3192	.3156	.3121
0.5	.3085	.3050	.3015	.2981	.2946	.2912	.2877	.2843	.2810	.2776
0.6	.2743	.2709	.2676	.2643	.2611	.2578	.2546	.2514	.2483	.2451
0.7	.2420	.2389	.2358	.2327	.2296	.2266	.2236	.2206	.2177	.2148
0.8	.2119	.2090	.2061	.2033	.2005	.1977	.1949	.1922	.1894	.1867
0.9	.1841	.1814	.1788	.1762	.1736	.1711	.1685	.1660	.1635	.1611
1.0	.1587	.1562	.1539	.1515	.1492	.1469	.1446	.1423	.1401	.1379
1.1	.1357	.1335	.1314	.1292	.1271	.1251	.1230	.1210	.1190	.1170
1.2	.1151	.1131	.1112	.1093	.1075	.1056	.1038	.1020	.1003	.0985
1.3	.0968	.0951	.0934	.0918	.0901	.0885	.0869	.0853	.0838	.0823
1.4	.0808	.0793	.0778	.0764	.0749	.0735	.0722	.0708	.0694	.0681
1.5	.0668	.0655	.0643	.0630	.0618	.0606	.0594	.0582	.0571	.0559
1.6	.0548	.0537	.0526	.0516	.0505	.0495	.0485	.0475	.0465	.0455
1.7	.0446	.0436	.0427	.0418	.0409	.0401	.0392	.0384	.0375	.0367
1.8	.0359	.0352	.0344	.0336	.0329	.0322	.0314	.0307	.0301	.0294
1.9	.0287	.0281	.0274	.0268	.0262	.0256	.0250	.0244	.0239	.0233
2.0	.0228	.0222	.0217	.0212	.0207	.0202	.0197	.0192	.0188	.0183
2.1	.0179	.0174	.0170	.0166	.0162	.0158	.0154	.0150	.0146	.0143
2.2	.0139	.0136	.0132	.0129	.0125	.0122	.0119	.0116	.0113	.0110
2.3	.0107	.0104	.0102	.0099	.0096	.0094	.0091	.0089	.0087	.0084
2.4	.0082	.0080	.0078	.0075	.0073	.0071	.0069	.0068	.0066	.0064
2.5	.0062	.0060	.0059	.0057	.0055	.0054	.0052	.0051	.0049	.0048
2.6	.0047	.0045	.0044	.0043	.0041	.0040	.0039	.0038	.0037	.0036
2.7	.0035	.0034	.0033	.0032	.0031	.0030	.0029	.0028	.0027	.0026
2.8	.0026	.0025	.0024	.0023	.0023	.0022	.0021	.0021	.0020	.0019
2.9	.0019	.0018	.0017	.0017	.0016	.0016	.0015	.0015	.0014	.0014
3.0	.00135									
3.5	.000 233									
4.0	.000 031 7									
4.5	.000 003 40									
5.0	.000 000 287									

From R. E. Walpole, *Introduction to Statistics* (New York: Macmillan, 1968).

8.6 Suppose that $E(\hat{\theta}_1) = E(\hat{\theta}_2) = \theta$, $V(\hat{\theta}_1) = \sigma_1^2$, and $V(\hat{\theta}_2) = \sigma_2^2$. Consider the estimator $\hat{\theta}_3 = a\hat{\theta}_1 + (1-a)\hat{\theta}_2$.

a Show that $\hat{\theta}_3$ is an unbiased estimator for θ.

b If $\hat{\theta}_1$ and $\hat{\theta}_2$ are independent, how should the constant a be chosen in order to minimize the variance of $\hat{\theta}_3$?

(a) $E(\hat{\theta}_3) = E\left[a\hat{\theta}_1 + (1-a)\hat{\theta}_2\right]$

$= a \cdot E(\hat{\theta}_1) + (1-a) \cdot E(\hat{\theta}_2)$

$= a \cdot \theta + (1-a) \cdot \theta$

$= a\theta + \theta - a\theta$

$= \theta$ ✓✓ unbiased

(b) $\text{var}(\hat{\theta}_3) = \text{var}\left(a \cdot \hat{\theta}_1 + (1-a) \cdot \hat{\theta}_2\right)$

$= a^2 \cdot \text{var}(\hat{\theta}_1) + (1-a)^2 \cdot \text{var}(\hat{\theta}_2)$

$= a^2 \cdot \sigma_1^2 + (1-2a+a^2) \cdot \sigma_2^2$

$= a^2 \cdot \sigma_1^2 + \sigma_2^2 - 2a \cdot \sigma_2^2 + a^2 \cdot \sigma_2^2$

minimize:

$$\frac{\partial \cdot \text{var}(\hat{\theta}_3)}{\partial a} = 2a \cdot \sigma_1^2 + 0 - 2 \cdot \sigma_2^2 + 2a \cdot \sigma_2^2 = 0$$

$$a(\cancel{2} \cdot \sigma_1^2 + \cancel{2} \cdot \sigma_2^2) = \cancel{2} \cdot \sigma_2^2$$

$$a = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2}$$

$$\boxed{\text{var}(\hat{\theta}_3) \text{ is minimized when } a = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2}}$$

8.8 Suppose that Y_1, Y_2, Y_3 denote a random sample from an exponential distribution with density function

$$f(y) = \begin{cases} \left(\frac{1}{\theta}\right) e^{-y/\theta}, & y > 0, \\ 0, & \text{elsewhere.} \end{cases}$$

Consider the following five estimators of θ:

$$\hat{\theta}_1 = Y_1, \quad \hat{\theta}_2 = \frac{Y_1 + Y_2}{2}, \quad \hat{\theta}_3 = \frac{Y_1 + 2Y_2}{3}, \quad \hat{\theta}_4 = \min(Y_1, Y_2, Y_3), \quad \hat{\theta}_5 = \overline{Y}.$$

a Which of these estimators are unbiased?

b Among the unbiased estimators, which has the smallest variance?

$\lambda = \frac{1}{\theta}$ $\quad E(Y) = \frac{1}{\lambda} = \theta$ (definition of exponential distribution)

(a) $\hat{\theta}_1$: $E(\hat{\theta}_1) = E(Y_1)$

$= \theta$ ✓✓

$\hat{\theta}_2$: $E(\hat{\theta}_2) = E\left[\frac{Y_1 + Y_2}{2}\right]$

$= \frac{1}{2} \cdot E(Y_1) + \frac{1}{2} \cdot E(Y_2)$

$= \frac{1}{2} \cdot \theta + \frac{1}{2} \cdot \theta$

$= \theta$ ✓✓

$\hat{\theta}_3$: $E(\hat{\theta}_3) = E\left[\frac{Y_1 + 2Y_2}{3}\right]$

$= \frac{1}{3} \cdot E(Y_1) + \frac{2}{3} \cdot E(Y_2)$

$= \frac{1}{3} \cdot \theta + \frac{2}{3} \cdot \theta$

$= \theta$ ✓✓

$\hat{\theta}_4$: $E(\hat{\theta}_4) = ?$ $\qquad F(y) = \begin{cases} 1 - e^{-y/\theta} & y > 0 \\ 0 & \text{elsewhere} \end{cases}$

$f_{Y(1)} = n\left(1 - F(y)\right)^{n-1} \cdot f(y)$

$= n\left(1 - (1 - e^{-y/\theta})\right)^{n-1} \cdot \left(\frac{1}{\theta}\right) e^{-y/\theta}$

$= n\left(e^{-y/\theta}\right)^{n-1} \cdot \frac{1}{\theta} \cdot e^{-y/\theta}$

$= n \cdot e^{-yn/\theta} \cdot \frac{1}{\theta}$

$= \frac{n}{\theta} \cdot e^{-yn/\theta}$

the exponential distribution with mean θ/n

$\therefore E(\hat{\theta}_4) = E(Y_{(1)}) = \theta/3$

biased!

$\hat{\theta}_5$: $E(\hat{\theta}_5) = E(\bar{Y})$

$$= E\left(\frac{Y_1 + Y_2 + Y_3}{3}\right)$$

$$= \frac{1}{3}\cdot E(Y_1) + \frac{1}{3}\cdot E(Y_2) + \frac{1}{3}\cdot E(Y_3)$$

$$= \frac{1}{3}\cdot\theta + \frac{1}{3}\cdot\theta + \frac{1}{3}\cdot\theta$$

$$= \theta \quad //$$

$$\boxed{\begin{array}{l}\text{unbiased: } \hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3, \hat{\theta}_5 \\ \text{biased: } \hat{\theta}_4\end{array}}$$

(b) definition of exponential distribution: $var(Y) = \frac{1}{\lambda^2} = \theta^2$

$$var(\hat{\theta}_1) = var(Y_1) = \theta^2$$

$$var(\hat{\theta}_2) = var\left(\frac{Y_1 + Y_2}{2}\right)$$

$$= var\left(\frac{1}{2}\cdot Y_1 + \frac{1}{2}\cdot Y_2\right)$$

$$= \left(\frac{1}{2}\right)^2\cdot var(Y_1) + \left(\frac{1}{2}\right)^2\cdot Y_2 \rightarrow \text{variance of linear combinations}$$

$$= \frac{\theta^2}{2}$$

$$var(\hat{\theta}_3) = var\left(\frac{Y_1 + 2Y_2}{3}\right)$$

$$= var\left(\frac{1}{3}\cdot Y_1 + \frac{2}{3}\cdot Y_2\right)$$

$$= \left(\frac{1}{3}\right)^2\cdot var(Y_1) + \left(\frac{2}{3}\right)^2\cdot var(Y_2)$$

$$= \frac{1}{9}\cdot\theta^2 + \frac{4}{9}\cdot\theta^2$$

$$= \frac{5\cdot\theta^2}{9}$$

$$var(\hat{\theta}_5) = var\left(\frac{Y_1 + Y_2 + Y_3}{3}\right)$$

$$= var\left(\frac{1}{3}\cdot Y_1 + \frac{1}{3}\cdot Y_2 + \frac{1}{3}\cdot Y_3\right)$$

$$= \left(\frac{1}{3}\right)^2\cdot var(Y_1) + \left(\frac{1}{3}\right)^2\cdot var(Y_2) + \left(\frac{1}{3}\right)^2\cdot var(Y_3)$$

$$= \frac{1}{9}\cdot\theta^2 + \frac{1}{9}\cdot\theta^2 + \frac{1}{9}\cdot\theta^2$$

$$= \frac{\theta^2}{3}$$

$$\boxed{\hat{\theta}_5 \text{ has the smallest variance.}}$$

8.9 Suppose that $Y_1, Y_2, \ldots, Y_n$ constitute a random sample from a population with probability density function

$$f(y) = \begin{cases} \left(\frac{1}{\theta+1}\right) e^{-y/(\theta+1)}, & y > 0,\ \theta > -1, \\ 0, & \text{elsewhere.} \end{cases}$$

Suggest a suitable statistic to use as an unbiased estimator for θ. [*Hint:* Consider $\overline{Y}$.]

$$f(y) = \begin{cases} \frac{1}{\theta+1} \cdot e^{-y/\theta+1} & y > 0 \\ 0 & \text{elsewhere} \end{cases}$$

exponential distribution:

$$\lambda = \frac{1}{\theta+1}$$

$$E(Y_i) = \frac{1}{\lambda} = \theta + 1$$

$$var(Y_i) = \left(\frac{1}{\lambda}\right)^2 = (\theta+1)^2$$

$\hat{\theta} = \overline{Y}$:

$$E(\hat{\theta}) = E(\overline{Y})$$

$$= E\left(\frac{Y_1 + Y_2 + \ldots + Y_n}{n}\right)$$

$$= \frac{1}{n} \cdot E(Y_1) + \frac{1}{n} \cdot E(Y_2) + \ldots + \frac{1}{n} \cdot E(Y_n)$$

$$= \frac{1}{n}(\theta+1) + \frac{1}{n}(\theta+1) + \ldots + \frac{1}{n}(\theta+1)$$

$$= n \cdot \frac{1}{n}(\theta+1)$$

$$= \theta + 1$$

$$E(\overline{Y}) = \theta + 1$$

$$E(\overline{Y}) - 1 = \theta$$

$$E(\overline{Y}) - E(1) = \theta$$

$$E(\overline{Y} - 1) = \theta$$

$$\boxed{\text{suggested statistic: } \hat{\theta} = \overline{Y} - 1}$$

8.36 If $Y_1, Y_2, \ldots, Y_n$ denote a random sample from an exponential distribution with mean θ, then $E(Y_i) = \theta$ and $V(Y_i) = \theta^2$. Thus, $E(\overline{Y}) = \theta$ and $V(\overline{Y}) = \theta^2/n$, or $\sigma_{\overline{Y}} = \theta/\sqrt{n}$. Suggest an unbiased estimator for θ and provide an estimate for the standard error of your estimator.

since $E(\overline{Y}) = \theta$, $\hat{\theta} = \overline{Y}$ is an unbiased estimator.

standard error for $\hat{\theta} = \overline{Y}$: $\sigma_{\overline{Y}} = \dfrac{\theta}{\sqrt{n}}$

estimate for standard error:

$= \text{estimate}\left(\dfrac{\theta}{\sqrt{n}}\right)$

$= \dfrac{1}{\sqrt{n}} \cdot \text{estimate}(\theta)$

$= \dfrac{\overline{Y}}{\sqrt{n}}$

$\boxed{\text{unbiased estimator } \hat{\theta} = \overline{Y} \qquad \text{estimate for standard error} = \overline{Y}/\sqrt{n}}$

Problem 6 $Y_1, \ldots, Y_n$ is a random sample from a normal distribution with mean μ and variance σ^2.

Consider two estimators for σ^2, the sample variance S^2 and the MLE estimator $\hat{\sigma}^2$,

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(Y_i - \overline{Y})^2, \text{ and}$$

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(Y_i - \overline{Y})^2.$$

1) Find $E(S^2)$, $B(S^2)$, $Var(S^2)$, and $MSE(S^2)$.

2) Find $E(\hat{\sigma}^2)$, $B(\hat{\sigma}^2)$, $Var(\hat{\sigma}^2)$, and $MSE(\hat{\sigma}^2)$.

3) Conclude which estimator is unbiased and which estimator has smaller MSE.

$$E\left[\sum_{i=1}^{n}(Y_i-\bar{Y})^2\right] = E\left[\sum_{i=1}^{n}(Y_i^2 - 2\bar{Y}\cdot Y_i + \bar{Y}^2)\right]$$

$$= E\left(\sum_{i=1}^{n} Y_i^2\right) - 2\bar{Y}\cdot E\left(\sum_{i=1}^{n} Y_i\right) + n\cdot E(\bar{Y}^2)$$

$$= \sum_{i=1}^{n} E(Y_i^2) - 2n\cdot\bar{Y}^2 + n\cdot E(\bar{Y}^2)$$

$var(X) = E(X^2) - [E(X)]^2$

$\therefore E(X^2) = var(X) + [E(X)]^2$

$$= \sum_{i=1}^{n}\left[var(Y_i) + [E(Y_i)]^2\right] - n\cdot\left[var(\bar{Y}) + [E(\bar{Y})]^2\right]$$

$$= \sum_{i=1}^{n}(\sigma^2 + \mu^2) - n\cdot\left(\frac{\sigma^2}{n} + \mu^2\right)$$

$$= n\cdot\sigma^2 + n\cdot\mu^2 - \frac{n}{n}\cdot\sigma^2 - n\cdot\mu^2$$

$$= (n-1)\cdot\sigma^2$$

(1) $$E(S^2) = E\left[\frac{1}{n-1}\sum_{i=1}^{n}(Y_i-\bar{Y})^2\right]$$

$$= \frac{1}{n-1}\cdot E\left[\sum_{i=1}^{n}(Y_i-\bar{Y})^2\right]$$

$$= \frac{1}{n-1}\cdot(n-1)\cdot\sigma^2$$

$= \sigma^2 \longrightarrow S^2$ is unbiased

Theorem 7.3: $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$

$$var(S^2) = Var\left(\frac{\sigma^2}{n-1}\cdot\chi^2_{n-1}\right)$$

$$= \left(\frac{\sigma^2}{n-1}\right)^2\cdot var(\chi^2_{n-1})$$

$var(\chi^2_k) = 2k$

$$= \left(\frac{\sigma^2}{n-1}\right)^2\cdot 2(n-1)$$

$$= \frac{2\cdot\sigma^4}{n-1}$$

$$MSE(S^2) = var(S^2) + bias(S^2)$$ since S^2 is an unbiased estimator, $bias(S^2) = 0$.

$$= var(S^2)$$

$$= \frac{2 \cdot \sigma^4}{n-1}$$

$$\boxed{\begin{aligned} E(S^2) &= \sigma^2 \\ var(S^2) &= \frac{2\cdot\sigma^4}{n-1} \\ bias(S^2) &= 0 \\ MSE(S^2) &= \frac{2\cdot\sigma^4}{n-1} \end{aligned}}$$

(2) $$E(\hat{\sigma}^2) = E\left[\frac{1}{n}\sum_{i=1}^{n}(Y_i - \bar{Y})^2\right]$$

$$= \frac{1}{n} \cdot E\left[\sum_{i=1}^{n}(Y_i - \bar{Y})^2\right]$$

$$= \frac{1}{n} \cdot (n-1) \cdot \sigma^2 \neq \sigma^2 \longrightarrow \hat{\sigma}^2 \text{ is biased}$$

$$var(\hat{\sigma}^2) = ? \longrightarrow$$

We know: $\begin{cases} \hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(Y_i - \bar{Y})^2 \\ \frac{\sum_{i=1}^{n}(Y_i-\bar{Y})^2}{\sigma^2} \sim \chi^2_{n-1} \end{cases}$

therefore: $$\frac{n\cdot\hat{\sigma}^2}{\sigma^2} = \frac{\sum_{i=1}^{n}(Y_i-\bar{Y})^2}{\sigma^2} \sim \chi^2_{n-1}$$

$$= var\left(\frac{\sigma^2}{n} \cdot \chi^2_{n-1}\right) \longleftarrow$$

$$= \left(\frac{\sigma^2}{n}\right)^2 \cdot var(\chi^2_{n-1})$$

$$= \left(\frac{\sigma^2}{n}\right)^2 \cdot 2(n-1) \longrightarrow var(\chi^2_k) = 2k$$

$$= \frac{2\cdot\sigma^4\cdot(n-1)}{n^2}$$

$$bias(\hat{\sigma}^2) = E(\hat{\sigma}^2) - \sigma^2$$

$$= \frac{1}{n} \cdot (n-1) \cdot \sigma^2 - \sigma^2$$

$$= \frac{n}{n} \cdot \sigma^2 - \frac{1}{n} \cdot \sigma^2 - \sigma^2$$

$$= -\frac{1}{n} \cdot \sigma^2$$

$$\text{MSE}(\hat{\sigma}^2) = \text{var}(\hat{\sigma}^2) + \text{bias}(\hat{\sigma}^2)^2$$

$$= \frac{2 \cdot \sigma^4 \cdot (n-1)}{n^2} + \frac{1}{n^2} \cdot \sigma^4$$

$$= \frac{2 \cdot n - 1}{n^2} \cdot \sigma^4$$

$$\boxed{\begin{aligned} E(\hat{\sigma}^2) &= \frac{1}{n} \cdot (n-1) \cdot \sigma^2 \\ \text{var}(\hat{\sigma}^2) &= \frac{2 \cdot (n-1) \cdot \sigma^4}{n^2} \\ \text{bias}(\hat{\sigma}^2) &= -\frac{1}{n} \cdot \sigma^2 \\ \text{MSE}(\hat{\sigma}^2) &= \frac{2 \cdot n - 1}{n^2} \cdot \sigma^4 \end{aligned}}$$

(3)

$$\text{bias}(S^2) = 0$$

$$\text{bias}(\hat{\sigma}^2) = -\frac{1}{n} \cdot \sigma^2$$

$$\text{MSE}(S^2) = \frac{2 \cdot \sigma^4}{n-1}$$

$$\text{MSE}(\hat{\sigma}^2) = \frac{2 \cdot n - 1}{n^2} \cdot \sigma^4$$

consistently smaller

n	$\frac{2n-1}{n^2}$	$\frac{2}{n-1}$
1	1	undefined
5	0.36	0.667
10	0.19	0.222
50	0.0396	0.0408
100	0.0199	0.202

S^2 is unbiased, whereas $\hat{\sigma}^2$ is biased.

$\text{MSE}(\hat{\sigma}^2)$ is smaller than $\text{MSE}(S^2)$.