2.

Problem 2.22

With noise $E_{out}(g^{(D)}) = \mathbb{E}_{x,y}\left[\left(g^{(D)}(\vec{x}) - y(\vec{x})\right)^2\right]$

where $y(\vec{x}) = f(\vec{x}) + \varepsilon$ (ε zero mean w/var σ^2)

Show bias-variance decomp becomes

$$\mathbb{E}_D[E_{out}(g^{(D)})] = \sigma^2 + \text{bias} + \text{var}.$$

- Redo book derivation with

$$\mathbb{E}_D[\mathbb{E}_{out}(g^{(D)})] = \mathbb{E}_D\left[\mathbb{E}_{xy}\left[\left(g^{(D)}(\vec{x}) - (f(\vec{x}) + \varepsilon)\right)^2\right]\right]$$

$$= \mathbb{E}_D\left[\mathbb{E}_{xy}\left[g^{(D)}(\vec{x})^2 - 2g^{(D)}(\vec{x})(f(\vec{x}) + \varepsilon) + (f(\vec{x}) + \varepsilon)^2\right]\right]$$

- switch $\mathbb{E}_D$ and $\mathbb{E}_{xy}$, distribute $\mathbb{E}_D$

$$= \mathbb{E}_{xy}\left[\mathbb{E}_D[g^{(D)}(\vec{x})^2] - 2\mathbb{E}_D[g^{(D)}(\vec{x})](f(\vec{x}) + \varepsilon) + (f(\vec{x}) + \varepsilon)^2\right]$$

- add in a $-\bar{g}(\vec{x})^2 + \bar{g}(\vec{x})^2$

$$= \mathbb{E}_{xy}\left[\mathbb{E}_D[g^{(D)}(\vec{x})^2] - \bar{g}(\vec{x})^2 + \bar{g}(\vec{x})^2 - 2\bar{g}(\vec{x})(f(\vec{x}) + \varepsilon) + (f(\vec{x}) + \varepsilon)^2\right]$$

$$= \mathbb{E}_{xy}\left[\mathbb{E}_D[g^{(D)}(\vec{x})^2 - \bar{g}(\vec{x})^2] + \underline{\bar{g}(\vec{x})^2} - \underline{2\bar{g}(\vec{x})f(\vec{x})} - 2\bar{g}(\vec{x})\varepsilon + \underline{f(\vec{x})^2} + 2f(\vec{x})\varepsilon + \varepsilon^2\right]$$

$$= \mathbb{E}_{xy}\left[\text{variance} + \bar{g}(\vec{x})^2 - 2\bar{g}(\vec{x})f(\vec{x}) + f(\vec{x})^2 - 2\bar{g}(\vec{x})\varepsilon + 2f(\vec{x})\varepsilon + \varepsilon^2\right]$$

$$= \mathbb{E}_{xy}\left[\text{variance} + (\bar{g}(\vec{x}) - f(\vec{x}))^2 + \varepsilon^2 + (2f(\vec{x}) - \bar{g}(\vec{x}))\varepsilon\right]$$

$$= \text{Variance} + \text{bias} + \mathbb{E}_{xy}[\varepsilon^2] - 2\mathbb{E}_{xy}\left[(\bar{g}(\vec{x}) - f(\vec{x}))\varepsilon\right]$$

this is 0 b/c $\mathbb{E}[\varepsilon] = 0$

$$= \boxed{\text{variance} + \text{bias} + \sigma^2}$$

3. 2.24 $x \in [-1, 1]$, $f(x) = x^2$

$D = \{(x_1, x_1^2), (x_2, x_2^2)\}$

$h \in \{h(x) = ax+b\}$

(a) Give an expression for $\bar{g}(x)$. ok

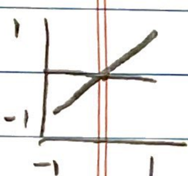
$a = \frac{x_2^2 - x_1^2}{x_2 - x_1}$ $\quad b = \frac{x_2 x_1^2 - x_1 x_2^2}{x_2 - x_1}$

Find $\mathbb{E}_D[ax+b]$

$$\bar{g}(x) = \mathbb{E}_D\left[\frac{x_2^2 - x_1^2}{x_2 - x_1}x + \frac{x_2 x_1^2 - x_1 x_2^2}{x_2 - x_1}\right]$$

$$= x\frac{1}{2}\frac{1}{2}\int_{-1}^{1}\int_{-1}^{1} \frac{x_2^2 - x_1^2}{x_2 - x_1} dx_1 dx_2 + \frac{1}{2}\frac{1}{2}\int_{-1}^{1}\int_{-1}^{1} \frac{x_1 x_2^2 - x_2 x_1^2}{x_2 - x_1} dx_1 dx_2$$

$$= \frac{x}{4}\int_{-1}^{1}\int_{-1}^{1} \frac{\cancel{(x_2 - x_1)}(x_2 + x_1)}{\cancel{x_2 - x_1}} dx_1 dx_2 + \frac{1}{4}\int_{-1}^{1}\int_{-1}^{1} \frac{-x_1 x_2 \cancel{(x_2 - x_1)}}{\cancel{x_2 - x_1}} dx_1 dx_2$$



$$= \frac{x}{4}\iint_{-1}^{1} x_2 + x_1 \, dx_1 dx_2 + \frac{1}{4}\int_{-1}^{1}\int_{-1}^{1} x_1 x_2 \, dx_1 dx_2$$

$$= \frac{x}{4} 0 + 0 + \frac{1}{4} 0 0 = \boxed{0}$$

(b) Numerically determine $\bar{g}(x)$, $\mathbb{E}[E_{out}]$, bias & Var

- $\bar{g}(x)$: generate N datasets $((x_1, x_1^2), (x_2, x_2^2))$
  - Calculate $a = \frac{1}{N}\sum_{n=1}^{N} (x_1 + x_2)$
    $b = \frac{1}{N}\sum_{n=1}^{N} -x_1 x_2$
- bias, Var, $\mathbb{E}[E_{out}]$:
  - sample N datasets, calculate $\overset{\text{many}}{g^D(x)}$ and $\overset{\text{one}}{\bar{g}(x)}$
  - resample above M times
  - Variance: $\mathbb{E}_x[\mathbb{E}_D[(g^D(x) - \bar{g}(x))^2]]$

    $= \boxed{\frac{1}{M}\frac{1}{N}\sum^{M,N}(g_N^D(x) - \bar{g}_M(x))^2}$

• Bias $= \mathbb{E}_x[\mathbb{E}_D[(\bar{g}(x) - f(x))^2]]$

$\boxed{\text{Numerically: } \frac{1}{MN}\sum_1^M \sum_1^N (\bar{g}_m(x) - f_{m,n}(x))^2}$

• $\mathbb{E}[E_{out}]: \mathbb{E}_x[\mathbb{E}_D[(g^D(x) - f(x))^2]$

$\boxed{\text{Numerically: } \frac{1}{MN}\sum_1^M \sum_1^N (g^D_{m,n}(x) - f_{m,n}(x))^2}$

(c) Variance $= 0.336$

Bias $= 0.207$

$\mathbb{E}[E_{out}] = 0.544$

Var + bias $= 0.543$

pretty close (figure on next page)

(d) Analytically compute $\mathbb{E}[E_{out}]$, bias, var.

$$\text{bias} = \mathbb{E}_x[\mathbb{E}_D[(\bar{g}(x) - f(x))^2]]$$
$$= \mathbb{E}_x[(0 - f(x))^2]$$
$$= \mathbb{E}_x[f(x)^2] = \mathbb{E}_x[x^4] = \frac{1}{2}\int_{-1}^{1} x^4 dx = \frac{1}{2}\frac{x^5}{5}\Big|_{-1}^{1}$$

$$\text{bias} = \frac{1}{2}\frac{2}{5} = \boxed{1/5}$$

$$\text{Variance} = \mathbb{E}_x[\mathbb{E}_D[(g^D(x) - \underbrace{\bar{g}(x)}_{\to 0})^2]]$$
$$= \mathbb{E}_x[\mathbb{E}_D[(ax+b)^2]] = \mathbb{E}_x[\mathbb{E}_D[((x_1+x_2)x - x_1x_2)^2]]$$
$$= \mathbb{E}_x[\mathbb{E}_D[(x_1+x_2)^2x^2 - 2x_1x_2(x_1+x_2)x + x_1^2x_2^2]]$$
$$= \mathbb{E}_x[x^2\mathbb{E}_D[(x_1+x_2)^2] - 2x\mathbb{E}_D[x_1x_2(x_1+x_2)] + \mathbb{E}_D[x_1^2x_2^2]]$$
$$= \mathbb{E}_x[x^2\mathbb{E}_D[x_1^2+x_2^2+2x_1x_2] - 2x\mathbb{E}_D[x_1^2x_2 + x_2^2x_1] + \mathbb{E}_D[x_1^2x_2^2]]$$

$$\mathbb{E}_D[x_1] = 0, \quad \mathbb{E}_D[x_1^2] = \frac{1}{2}\int_{-1}^{1} x^2 dx = \frac{1}{2}\frac{x^3}{3}\Big|_{-1}^{1} = \frac{1}{2}\frac{2}{3} = \frac{1}{3}$$

$$= \mathbb{E}_x[x^2(\tfrac{1}{3}+\tfrac{1}{3}) - 2x(\tfrac{1}{3}(0) + \tfrac{1}{3}(0)) + (\tfrac{1}{3}\tfrac{1}{3})]$$
$$= \frac{1}{3}\left(\frac{2}{3}\right) - 0 + \frac{1}{9} = 3/9 = \boxed{1/3} = \text{Variance}$$

(d) cont.

$$\mathbb{E}[E_{out}] = \mathbb{E}_x[\mathbb{E}_D[(g(x)-f(x))^2]]$$ (forget about $\mathbb{E}_D$ for a sec)

$$= \mathbb{E}_x[(ax+b-x^2)^2] = \mathbb{E}_x[x^4 - 2x^2(ax+b) + (ax+b)^2]$$

$$= \mathbb{E}_x[x^4] - 2a\,\mathbb{E}_x[x^3] - 2b\,\mathbb{E}_x[x^2] + a^2\mathbb{E}[x^2] + 2ab\,\mathbb{E}[x] + b^2$$

$$= \frac{1}{5} - 0 - 2b\left(\frac{1}{3}\right) + a^2\left(\frac{1}{3}\right) + b^2$$

$$= \frac{1}{5} + \frac{1}{3}(a^2 - 2b) + b^2$$

$$= \mathbb{E}_D\left[\frac{1}{5} + \frac{1}{3}(a^2-2b) + b^2\right] \qquad a = x_1 + x_2, \quad b = -x_1 x_2$$

$$= \frac{1}{5} + \frac{1}{3}\mathbb{E}_D[(x_1+x_2)^2 + 2x_1x_2] + \mathbb{E}_D[(x_1x_2)^2]$$

$$= \frac{1}{5} + \frac{1}{3}\mathbb{E}_D[x_1^2 + 2x_1x_2 + x_2^2 + 2x_1x_2] + \mathbb{E}_D[x_1^2x_2^2]$$

$$= \frac{1}{5} + \frac{1}{3}\left(\frac{1}{3} + 0 + \frac{1}{3} + 0\right) + \left(\frac{1}{3}\right)\frac{1}{3}$$

$$= \frac{1}{5} + \frac{2}{9} + \frac{1}{9} = \boxed{8/15} = \mathbb{E}[E_{out}]$$ (also equals bias + var)

4. 3.4 Adaptive Linear Neuron (adaline)

(a) $e_n(\vec{w}) = (\max(0, 1 - y_n \vec{w}^T\vec{x}))^2$

- show $e_n(\vec{w})$ is continious and differentiable.
- Write down $\nabla e_n(\vec{w})$

- 0 is continious and differentiable $(1-y_n\vec{w}^T\vec{x} < 0)$
- $(1-y_n\vec{w}^T\vec{x})^2$ is continious and differentiable $(1-y_n\vec{w}^T\vec{x} > 0)$
- if $1-y_n\vec{w}^T\vec{x} = 0$, then $y_n = \vec{w}^T\vec{x}$ and there is no error!
  - $\rightarrow e_n(\vec{w}) = 0$
  - $\rightarrow$ so this function is continious between both its pieces

- $\nabla e_n(\vec{w}) = 2(1-y_n\vec{w}^T\vec{x})(-y_n\vec{x})$
- This is 0 at $1-y_n\vec{w}^T\vec{x}$.
  - $\rightarrow$ so this function is differentiable between both its pieces.

(b) Show $e_n(\vec{w})$ is an upper bound for $[\![\text{sign}(\vec{w}^T\vec{x}_n) \neq y_n]\!]$

- Two cases:
  1. $\text{sign}(\vec{w}^T\vec{x}_n) = y_n$
     - here $y_n\vec{w}^T\vec{x} \geq 0$, $[\![\text{sign}(\vec{w}^T\vec{x}_n) \neq y_n]\!] = 0$,
     - $e_n(\vec{w}) \geq 0$ if $y_n\vec{w}^T\vec{x} < 1$
     - $e_n(\vec{w}) = 0$ if $y_n\vec{w}^T\vec{x} \geq 1$
     - Thus: $[\![\text{sign}(\vec{w}^T\vec{x}_n) \neq y_n]\!] \leq e_n(\vec{w})$ (in this case)
  2. $\text{sign}(\vec{w}^T\vec{x}_n) \neq y_n$
     - here $y_n\vec{w}^T\vec{x} \leq 0$, $[\![\text{sign}(\vec{w}^T\vec{x}_n) \neq y_n]\!] = 1$
     - $e_n(\vec{w}) \geq 1$
     - Thus $[\![\text{sign}(\vec{w}^T\vec{x}_n) \neq y_n]\!] \leq e_n(\vec{w})$ (in both cases now)

Thus $\boxed{E_{in}(\vec{w}) = \frac{1}{N}\sum_{n=1}^{N}[\![\text{sign}(\vec{w}^T\vec{x}_n) \neq y_n]\!] \leq \frac{1}{N}\sum_{n=1}^{N} e_n(\vec{w})}$

(c) Argue Adaline algorithm performs stochastic gradient descent on $\frac{1}{N}\sum_{n=1}^{N} e_n(\vec{w})$.

$$s(t) = \vec{w}^T(t)\,\vec{x}(t)$$

$$\vec{w}(t+1) \leftarrow \vec{w}(t) - \eta \nabla e_n(\vec{w})$$ gradient descent

$$\vec{w}(t+1) \leftarrow \vec{w}(t) - \eta\left(2(1-y_n\vec{w}^T\vec{x})(-y_n\vec{x})\right)$$

$$\leftarrow \vec{w}(t) - 2\eta\,(1-y_n s(t))(-y_n\vec{x})$$

$$\leftarrow \vec{w}(t) + 2\eta\,(y_n - \cancel{y_n^2}\,s(t))\,\vec{x}$$

$$\leftarrow \vec{w}(t) + 2\eta\,(y_n - s(t))\,\vec{x}$$

$$\boxed{\vec{w}(t+1) \leftarrow \vec{w}(t) + 2\eta\,(y(t) - s(t))\,\vec{x}(t)}$$

This is Adaline algorithm (subsistute $\eta = 2\eta$).

5. 3.19 Is learning feasible?

(a) $\Phi(\vec{x}) = \begin{cases} (0, \ldots 0, 1, 0, \ldots) & \text{if } \vec{x} = \vec{x}_n \\ (0, 0 \ldots 0) & \text{otherwise} \end{cases}$

This maps points of dimension 2 to dimension N (where N is total number of points). This dramatically increases the generazization error as N increases.

$$E_{out}(g) \leq E_{in}(g) + O\left(\sqrt{d_{vc} \frac{\ln N}{N}}\right)$$

here d_{vc} will grow with N, so the generalization error increases with N ☹

(b) $\phi_n(\vec{x}) = \exp\left(-\frac{\|\vec{x} - \vec{x}_n\|^2}{2\gamma^2}\right)$

This seems ok. It seems to just normaliz $\vec{x}_n$ to $\vec{x}$ an a standard $2\gamma^2$. Looks like the polar form of something. Since the dimensionality stays the same after the transform, the VC dimension should stay the same and generalization should decrease with N.

(c) $\phi_{i,j}(\vec{x}) = \exp\left(\frac{-\|\vec{x} - (i,j)\|^2}{2\gamma^2}\right)$ $\quad i \in \{0, \frac{1}{100}, \ldots\}$, $j \in \{0, \frac{1}{100}, \ldots\}$

This converts every datapoint of dimension 2 to dimension 101×101. Dratimically increasing the dimensionality of the dataset. This gives us the same problem as in (a).