WSABI Summer Research Lab **Summer 2025**

Lab 11: Priors and Fake Data

Instructor: Jonathan Pipping *Authors: RB, JP*

11.1 Posterior Credible Intervals in Free Throws

11.1.1 The Model

Recall the Beta-Binomial model we derived in class:

$$\text{Likelihood: } W \sim \text{Binomial}(W + L, p)$$
$$\text{Prior: } p \sim \text{Beta}(\alpha, \beta)$$

where W is the number of successful free throws, L is the number of failed free throws, and p is the true probability of success on a single free throw.

11.1.2 Posterior Distribution

From Bayes' Rule, we have that the posterior distribution of $p \mid W, L$ is proportional to the product of the likelihood and the prior:

$$\mathbb{P}(p \mid W, L) \propto p^{W+\alpha-1}(1-p)^{L+\beta-1}$$

This is the kernel of a Beta distribution, so we have that

$$p \mid W, L \sim \text{Beta}(\alpha + W, \beta + L)$$

This reflects our beliefs about p after setting a prior and observing the data (W, L).

11.1.3 Posterior Credible Intervals

Since we now have a **full distribution** encoding our beliefs (and uncertainty) about p given the data (W, L), we can create a **Bayesian posterior credible interval** to summarize our uncertainty in p. We do this by finding values A and B such that

$$\mathbb{P}(A \leq p \mid W, L \leq B) = \mathbb{P}(A \leq \text{Beta}(\alpha + W, \beta + L) \leq B) = 1 - \alpha$$

where α is the **confidence level** of the interval. In practice, this means extracting the $\alpha/2$ and $1 - \alpha/2$ quantiles of $\mathbb{P}(p \mid W, L)$, the posterior distribution of p given the data (W, L). *Note that this confidence interval is a function which depends on our prior hyperparameters α and β.*

We would interpret this interval in the following way:

> *"We are 95% confident that the true probability of success on a single free throw is between A and B."*

This Bayesian interpretation is notably different from the frequentist confidence interval, which would be interpreted as

> *"If we were to repeat this experiment many times, 95% of the time the true probability of success on a single free throw would be between A and B."*

Between these two, the Bayesian credible interval is more intuitive. This is because it directly expresses our updated belief about the parameter p after observing the data rather than as some long-run frequency of intervals capturing the true value **under repeated sampling**, the frequentist interpretation.

11.1.4 Your Task

Using the free throw data from Labs 8 and 9, create credible intervals for various combinations of α and β and compare them to the Wald, Agresti, and bootstrapped confidence intervals. You will do this as follows:

1. Define a grid of α and β values.
2. For each combination of α and β, generate a 95% posterior credible interval for p.
3. Compare the widths of these intervals to the widths of the Wald, Agresti, and bootstrapped confidence intervals. Visualize this comparison.
4. In your simulated data, compare the coverage of these intervals to the coverage of the Wald, Agresti, and bootstrapped confidence intervals. Visualize this comparison.