

## Lab 10: Kelly Betting

*Instructor: Jonathan Pipping**Authors: RB, JP*

## 10.1 A Smart -EV Bet

### 10.1.1 Kelly's Algorithm

In 1956, John L. Kelly extended his analysis to horses. Consider making a series of bets on  $N$  horse races, each with  $m$  horses running in the race. Following Kelly's notation, we define the following quantities:

- Let  $i \in \{1, \dots, m\}$  represent the horse.
- Let  $p_i$  represent the probability that horse  $i$  wins the race.
- Let  $\alpha_i$  represent the decimal odds for horse  $i$ .
- Let  $\mathbf{f}$  represent the payroll fractions  $(f_1, \dots, f_m)$  for each horse.

In his paper, he used Lagrange multipliers and Karush-Kuhn-Tucker conditions to maximize the **expected log bankroll** after  $N$  bets, yielding the Kelly allocations  $\mathbf{f}^*$  via the following algorithm:

1. Permute indices so that  $p_i \alpha_i \geq p_{i+1} \alpha_{i+1}$
2. Set  $b$  equal to the minimum positive value of

$$\frac{1 - p_t}{1 - \sigma_t}$$

where  $p_t = \sum_{i=1}^t p_i$  and  $\sigma_t = \sum_{i=1}^t \frac{1}{\alpha_i}$ . This represents the proportion of the bankroll we keep in reserve.

3. Set  $f_i = \max(0, p_s - \frac{b}{\alpha_s})$ . If the odds are fair,  $\sum_{i=1}^m f_i = 1 - b$ .

You will use this algorithm in a simulated horse race to determine optimal Kelly allocations.

### 10.1.2 Your Task

1. Consider a horse race with  $m = 3$  horses. Find an example of true horse win probabilities  $(p_1, p_2, p_3)$  and decimal odds  $(\alpha_1, \alpha_2, \alpha_3)$  such that it is Kelly-optimal to bet on a -EV horse.  
Note that according to the above algorithm, the Kelly bet allocations  $(f_1, f_2, f_3)$  must have a  $f_i > 0$  for some horse  $i$ , and that betting on a horse is -EV if  $p_i \alpha_i - 1 < 0$ .
2. Then,  $M = 100$  times, simulate  $N = 1000$  bets on this horse race using two different betting strategies:
  - Strategy 1: Following the Kelly allocations.
  - Strategy 2: Following Kelly for +EV horses, but setting  $f_i = 0$  for any -EV horses.
3. For each strategy, plot the average bankroll after  $N$  bets across the  $M$  simulations. Which yields more wealth on average? How much more?

### 10.1.3 Discussion

**Why is it possible that making a -EV bet is smart?** Since Kelly staking works by growing wealth *multiplicatively* (or linearly on the log scale), it can be highly-sensitive to big draw-downs: meaning that a single wipe-out (or massive loss) hurts massively and can't be undone by later wins.

In a mutually-exclusive race, allocating a small fraction to a short-priced favorite (even if the edge is negative) hedges the inevitable variance in win-loss outcomes created by betting long shots that carry most of the positive edge. This smoother return path increases the **geometric mean growth rate**, which is what ultimately compounds to grow wealth.

Put differently, spreading out some of your budget onto the "least bad" remaining outcomes reduces the chance of catastrophic loss *faster* than it erodes expected profit. This means that every dollar wagered either **earns a positive edge** or **buys enough variance reduction** to raise the overall long-run growth.

## References

[JLK] Kelly, J. L., *A New Interpretation of Information Rate*, Bell System Technical Journal, 1956.