

## Lab 11: Priors and Fake Data

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## 11.1 Posterior Credible Intervals in Free Throws

### 11.1.1 The Model

Recall the Beta-Binomial model we derived in class:

$$\text{Likelihood: } W \sim \text{Binomial}(W + L, p)$$

$$\text{Prior: } p \sim \text{Beta}(\alpha, \beta)$$

where  $W$  is the number of successful free throws,  $L$  is the number of failed free throws, and  $p$  is the true probability of success on a single free throw.

### 11.1.2 Posterior Distribution

From Bayes' Rule, we have that the posterior distribution of  $p \mid W, L$  is proportional to the product of the likelihood and the prior:

$$\mathbb{P}(p \mid W, L) \propto p^{W+\alpha-1} (1-p)^{L+\beta-1}$$

This is the kernel of a Beta distribution, so we have that

$$p \mid W, L \sim \text{Beta}(\alpha + W, \beta + L)$$

This reflects our beliefs about  $p$  after setting a prior and observing the data  $(W, L)$ .

### 11.1.3 Posterior Credible Intervals

Since we now have a **full distribution** encoding our beliefs (and uncertainty) about  $p$  given the data  $(W, L)$ , we can create a **Bayesian posterior credible interval** to summarize our uncertainty in  $p$ . We do this by finding values  $A$  and  $B$  such that

$$\mathbb{P}(A \leq p \mid W, L \leq B) = \mathbb{P}(A \leq \text{Beta}(\alpha + W, \beta + L) \leq B) = 1 - \alpha$$

where  $\alpha$  is the **confidence level** of the interval. In practice, this means extracting the  $\alpha/2$  and  $1 - \alpha/2$  quantiles of  $\mathbb{P}(p \mid W, L)$ , the posterior distribution of  $p$  given the data  $(W, L)$ . *Note that this confidence interval is a function which depends on our prior hyperparameters  $\alpha$  and  $\beta$ .*

We would interpret this interval in the following way:

*"We are 95% confident that the true probability of success on a single free throw is between  $A$  and  $B$ ."*

This Bayesian interpretation is notably different from the frequentist confidence interval, which would be interpreted as

*"If we were to repeat this experiment many times, 95% of the time the true probability of success on a single free throw would be between  $A$  and  $B$ ."*

Between these two, the Bayesian credible interval is more intuitive. This is because it directly expresses our updated belief about the parameter  $p$  after observing the data rather than as some long-run frequency of intervals capturing the true value **under repeated sampling**, the frequentist interpretation.

#### 11.1.4 Your Task

Using the free throw data from Labs 8 and 9, create credible intervals for various combinations of  $\alpha$  and  $\beta$  and compare them to the Wald, Agresti, and bootstrapped confidence intervals. You will do this as follows:

1. Define a grid of  $\alpha$  and  $\beta$  values.
2. For each combination of  $\alpha$  and  $\beta$ , generate a 95% posterior credible interval for  $p$ .
3. Compare the widths and coverages of these intervals to the widths and coverages of the Wald, Agresti, and bootstrapped confidence intervals. Visualize these comparisons.