WSABI Summer Research Lab **Summer 2025**

Lab 11: Priors and Fake Data

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11.1 Posterior Credible Intervals in Free Throws

11.1.1 The Model

Recall the Beta-Binomial model we derived in class:

$$\text{Likelihood: } W \sim \text{Binomial}(W + L, p)$$
$$\text{Prior: } p \sim \text{Beta}(\alpha, \beta)$$

where W is the number of successful free throws, L is the number of failed free throws, and p is the true probability of success on a single free throw.

11.1.2 Posterior Distribution

From Bayes' Rule, we have that the posterior distribution of $p \mid W, L$ is proportional to the product of the likelihood and the prior:

$$\mathbb{P}(p \mid W, L) \propto p^{W+\alpha-1}(1-p)^{L+\beta-1}$$

This is the kernel of a Beta distribution, so we have that

$$p \mid W, L \sim \text{Beta}(\alpha + W, \beta + L)$$

This reflects our beliefs about p after setting a prior and observing the data (W, L).

11.1.3 Posterior Credible Intervals

Since we now have a **full distribution** encoding our beliefs (and uncertainty) about p given the data (W, L), we can create a **Bayesian posterior credible interval** to summarize our uncertainty in p. We do this by finding values A and B such that

$$\mathbb{P}(A \leq p \mid W, L \leq B) = \mathbb{P}(A \leq \text{Beta}(\alpha + W, \beta + L) \leq B) = 1 - \alpha$$

where α is the **confidence level** of the interval. In practice, this means extracting the $\alpha/2$ and $1 - \alpha/2$ quantiles of $\mathbb{P}(p \mid W, L)$, the posterior distribution of p given the data (W, L). *Note that this confidence interval is a function which depends on our prior hyperparameters α and β.*

We would interpret this interval in the following way:

> *"We are 95% confident that the true probability of success on a single free throw is between A and B."*

This Bayesian interpretation is notably different from the frequentist confidence interval, which would be interpreted as

> *"If we were to repeat this experiment many times, 95% of the time the true probability of success on a single free throw would be between A and B."*

Between these two, the Bayesian credible interval is more intuitive. This is because it directly expresses our updated belief about the parameter p after observing the data rather than as some long-run frequency of intervals capturing the true value **under repeated sampling**, the frequentist interpretation.

11.1.4 Your Task

Using the free throw data from Lab 9, create credible intervals for various combinations of α and β and compare them to the Wald, Agresti, and bootstrapped confidence intervals. You will do this as follows:

1. Define a grid of α and β values.
2. For each combination of α and β, generate a 95% posterior credible interval for p.
3. Compare the widths and coverage of these intervals to the widths and coverage of the Wald, Agresti, and bootstrapped confidence intervals. Visualize these comparisons.