WSABI Summer Research Lab **Summer 2024**

Lecture 19: Kelly Betting

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19.1 Card Game Example

19.1.1 Instructions

1. Split into pairs and grab a deck of cards. Let player A be the dealer and player B be the bettor, and each starts with \$100.
2. Player A keeps the cards face down and flips them over one at a time until 30 cards are revealed.
3. Before each flip, Player B guesses whether the next card will be red or black, and bets as much of their bankroll as they want that their guess is correct.
4. After each flip, record the amount of money Player B has earned.
5. After 30 flips, change roles and repeat the game. Whoever makes the most money wins!

19.1.2 Strategy

Think about what an optimal strategy for player B would be. There are two components to the strategy:

1. Which color to bet on.
2. How much to bet on each flip.

With the knowledge that there are 26 red cards and 26 black cards, the strategy for guessing the color of the next card is simple: bet on red if there have been more black cards revealed so far, and bet on black if there have been more red cards revealed so far.

However, the strategy for how much to bet is a bit more complicated. We will find an optimal strategy mathematically.

19.2 Setting Up the Problem

Suppose we want to bet on NBA games over the course of an entire season and beyond with an initial bankroll $\boldsymbol{B}$ of \$100. Let i denote each game, and define the following variables:

- p_i: the true probability that team A_i beats team B_i.
- α_{A_i}: the Vegas (decimal) odds for team A_i to beat team B_i. This is the number of dollars returned for every dollar bet on team A_i, or \$1 + profit from \$1 bet.

- α_{B_i}: the Vegas (decimal) odds for team B_i to beat team A_i. This is the number of dollars returned for every dollar bet on team B_i, or \$1 + profit from \$1 bet.

How should we bet? A frequently thought-of strategy is to maximize the expected value of our bets, which we'll calculate below.

19.2.1 Expected Value

The expected value of a \$1 bet on team A in game i is:

$$\begin{aligned}EV_{A_i} &= \mathbb{E}[\text{profit}_{A_i}] \\ &= \mathbb{P}(A_i \text{ wins}) \cdot \text{profit if } A_i \text{ wins} + \mathbb{P}(A_i \text{ loses}) \cdot \text{profit if } A_i \text{ loses} \\ &= p_i(\alpha_{A_i} - 1) + (1-p_i)(-1) \\ &= p_i\alpha_{A_i} - 1\end{aligned}$$

Similarly, the expected value of a \$1 bet on team B in game i is:

$$\begin{aligned}EV_{B_i} &= \mathbb{E}[\text{profit}_{B_i}] \\ &= \mathbb{P}(B_i \text{ wins}) \cdot \text{profit if } B_i \text{ wins} + \mathbb{P}(B_i \text{ loses}) \cdot \text{profit if } B_i \text{ loses} \\ &= (1-p_i)(\alpha_{B_i} - 1) + p_i(-1) \\ &= (1-p_i)\alpha_{B_i} - 1\end{aligned}$$

Let's say $EV_{A_i} < 0$. How much should we bet on team A_i? Most people would say you shouldn't take that bet. Similarly, let's say $EV_{A_i} > 0$. Most people would say to make a bet! Still, even if $EV_{A_i} > 0$, if you bet your entire bankroll on A_i and lose, you're out of money! How should we account for this?

19.2.2 Flat Staking

One way to account for the risk of losing your entire bankroll is to bet a fixed portion of your bankroll on each game. For example, over N bets, split up your money evenly between them and bet $\frac{\boldsymbol{B}}{N}$ on team A_i each time? Since our EV is positive, we should make money over time, but can we make even more? **What are we missing?**

The answer? We're not accounting for the **sequential** nature of the bets, or that we're betting on games one after another. For example, if we make money on the first bet, we can use that profit to bet more on the second bet! This **compounding** effect allows us to capitalize on our wins and make more money over time. How do we achieve this?

19.2.3 Fractional Staking

Instead of betting a fixed amount on each game, we instead bet a **fraction** of our bankroll on each game. We call this fraction $f \in (0, 1)$. Now in game 1, we bet $f \cdot \boldsymbol{B}$ on team A_1. Our profit options in this case are

$$P_{A_1} = \begin{cases} (\alpha_{A_1} - 1) \cdot f \cdot \boldsymbol{B} & \text{w.p. } p_1 \\ -f \cdot \boldsymbol{B} & \text{w.p. } 1 - p_1 \end{cases}$$

Compactly, we can write our profit as

$$P_{A_1} = f_1 \cdot \boldsymbol{B}\,(\alpha_{A_1} X_1 - 1)$$

where X_1 is a Bernoulli(p_1) random variable representing whether team A_1 wins.

Is the profit of each individual bet really what we want to maximize though? What do we actually want to have at the end? **A high bankroll!** Let's set up our problem mathematically.

19.2.4 Maximizing Our Bankroll

Let our bankroll after game i be $\boldsymbol{B}_i$. We express our bankroll after game 1 as

$$\begin{aligned} \boldsymbol{B}_1 &= \boldsymbol{B} + P_{A_1} \\ &= \boldsymbol{B} + f_1 \cdot \boldsymbol{B}\left(\alpha_{A_1} X_1 - 1\right) \\ &= \boldsymbol{B}\left[1 + f_1\left(\alpha_{A_1} X_1 - 1\right)\right] \end{aligned}$$

Then for our second bet, we bet $f_2 \cdot \boldsymbol{B}_1$ on team A_2, and our profit can be expressed as

$$P_{A_2} = \boldsymbol{B}\left[1 + f_1\left(\alpha_{A_1} X_1 - 1\right)\right] \cdot f_2\left(\alpha_{A_2} X_2 - 1\right)$$

by the same logic. Our bankroll after game 2 is then

$$\boldsymbol{B}_2 = \boldsymbol{B}\left[1 + f_1\left(\alpha_{A_1} X_1 - 1\right)\right] \cdot \left[1 + f_2\left(\alpha_{A_2} X_2 - 1\right)\right]$$

If we repeat this process for N games, we get an expression for our bankroll of

$$\boldsymbol{B}_N = \boldsymbol{B} \prod_{i=1}^{N} \left[1 + f_i\left(\alpha_{A_i} X_i - 1\right)\right]$$

where $\boldsymbol{B}$ is our initial bankroll, α_{A_i} is the Vegas odds for team A to beat team B in game i, X_i is a Bernoulli(p_i) random variable representing whether team A wins in game i, and f_i is the fraction of our bankroll we bet on team A in game i.

Our goal is to maximize this expression, but note that our Bankroll $\boldsymbol{B}_N$ is a random variable which depends on the $\boldsymbol{X} = (X_1, X_2, \ldots, X_N)$. What we should do is maximize the **expected value** of our bankroll. How do we express this mathematically?

$$\arg\max_{\boldsymbol{f}} \mathbb{E}[\boldsymbol{B}_N] = \arg\max_{\boldsymbol{f}} \mathbb{E}\left[\boldsymbol{B} \prod_{i=1}^{N} \left[1 + f_i\left(\alpha_{A_i} X_i - 1\right)\right]\right]$$

This is a difficult problem to solve as-is due to the product of random variables. However, the logarithm function is **monotonic**, which means that the $\boldsymbol{f}$ that maximizes $\mathbb{E}[\boldsymbol{B}_N]$ is the same as the $\boldsymbol{f}$ that maximizes $\mathbb{E}[\log(\boldsymbol{B}_N)]$. So our problem is to solve the following optimization problem:

$$\arg\max_{\boldsymbol{f}} \mathbb{E}[\log(\boldsymbol{B}_N)] = \arg\max_{\boldsymbol{f}} \mathbb{E}\left[\log\left(\boldsymbol{B} \prod_{i=1}^{N} \left[1 + f_i\left(\alpha_{A_i} X_i - 1\right)\right]\right)\right]$$

19.3 The Kelly Criterion

In 1956, J.L. Kelly Jr. applied the Shannon-McMillan-Breiman theorem to gambling games, proving that the allocation function $\boldsymbol{f}$ that maximizes $\mathbb{E}[\log(\boldsymbol{B}_N)]$ makes more money as $N \to \infty$ than any other allocation function. We will now derive this result.

19.3.1 Deriving the Kelly Criterion

We begin with the same expression we derived in the previous section:

$$\begin{aligned}\arg\max_{\boldsymbol{f}} \mathbb{E}[\log(\boldsymbol{B}_N)] &= \arg\max_{\boldsymbol{f}} \mathbb{E}\left[\log\left(\boldsymbol{B}\prod_{i=1}^{N}\left[1+f_i\left(\alpha_{A_i}X_i-1\right)\right]\right)\right] \\ &= \arg\max_{\boldsymbol{f}} \mathbb{E}\left[\log(\boldsymbol{B})+\sum_{i=1}^{N}\log\left(1+f_i\left(\alpha_{A_i}X_i-1\right)\right)\right]\end{aligned}$$

Since $\log(\boldsymbol{B})$ is a constant, it does not affect the optimization. Then,

$$\arg\max_{\boldsymbol{f}} \mathbb{E}[\log(\boldsymbol{B}_N)] = \arg\max_{\boldsymbol{f}} \mathbb{E}\left[\sum_{i=1}^{N}\log\left(1+f_i\left(\alpha_{A_i}X_i-1\right)\right)\right]$$

By the linearity of expectation, we have that

$$\arg\max_{\boldsymbol{f}} \mathbb{E}[\log(\boldsymbol{B}_N)] = \arg\max_{\boldsymbol{f}} \sum_{i=1}^{N}\mathbb{E}[\log\left(1+f_i\left(\alpha_{A_i}X_i-1\right)\right)]$$

Since X_i is a Bernoulli(p_i) random variable, we have that

$$\arg\max_{\boldsymbol{f}} \mathbb{E}[\log(\boldsymbol{B}_N)] = \arg\max_{\boldsymbol{f}} \sum_{i=1}^{N}\log\left(1+f_i\left(\alpha_{A_i}-1\right)\right)p_i+\log\left(1-f_i\right)\left(1-p_i\right)$$

Then, since each term i depends only on one f_i, we can find the optimal f_i for each term independently, and it will take the same form for each term. We can then rewrite the expression as

$$\arg\max_{f_i} \log(1+f_i(\alpha_{A_i}-1))p_i+\log(1-f_i)(1-p_i)$$

and this is solvable with calculus. We take the derivative with respect to f_i and set it to 0 to find the optimal f_i:

$$\begin{aligned}\frac{\partial}{\partial f_i}\left(\log(1+f_i(\alpha_{A_i}-1))p_i+\log(1-f_i)(1-p_i)\right) &= 0 \\ \frac{\alpha_{A_i}-1}{1+f_i(\alpha_{A_i}-1)}p_i-\frac{1}{1-f_i}(1-p_i) &= 0\end{aligned}$$

Rearranging, we see that

$$\frac{1-p_i}{1-f_i}=\frac{p_i(\alpha_{A_i}-1)}{1+f_i(\alpha_{A_i}-1)}$$

And solving for f_i, we get

$$f_i^*=\frac{p_i\alpha_{A_i}-1}{\alpha_{A_i}-1}$$

This is known as the **Kelly fraction**. Note that $f_i^*<0$ corresponds to making a negative bet, which is not possible. So our final expression for the optimal f_i is

$$f_i^*=\max\left(0,\frac{p_i\alpha_{A_i}-1}{\alpha_{A_i}-1}\right)$$

And our set of functions $\boldsymbol{f}^*$ is

$$\boldsymbol{f}^* = (f_1^*, f_2^*, \ldots, f_N^*)$$

Let's consider some examples where we might use the Kelly Criterion.

19.3.2 The Kelly Criterion in Practice

Example 19.1 (Guaranteed Wins)**.** Suppose the outcome of a game is guaranteed to be a win. Then $p_i = 1$, and $f_i^* = 1$. This means we should bet our entire bankroll on this game, which makes sense!

Example 19.2 (Standard Odds)**.** Suppose we are making a bet at the traditional price of -110. Then the decimal odds of A_i winning are

$$\alpha_{A_i} = 1 + \frac{100}{110} = \frac{210}{110} \approx 1.909$$

Then for f_i^* to be positive (meaning we should bet something on A_i), we need

$$\frac{p_i \alpha_{A_i} - 1}{\alpha_{A_i} - 1} > 0$$

which is equivalent to

$$p_i > \frac{1}{\alpha_{A_i}} \approx 0.524$$

Note that this is the same break-even percentage you calculated in Lab 8! What does this imply about the **fair odds** for a game?

Example 19.3 (Fair Odds)**.** The fair odds of a game are the odds that would make the expected value of a bet exactly 0. We can calculate this by setting the Kelly fraction to 0:

$$\begin{aligned} f_i^* = \frac{p_i \alpha_{A_i} - 1}{\alpha_{A_i} - 1} &= 0 \\ \implies p_i \alpha_{A_i} - 1 &= 0 \\ \implies p_i \alpha_{A_i} &= 1 \\ \implies \alpha_{A_i} &= \frac{1}{p_i} \end{aligned}$$

This means that the fair decimal odds are inversely proportional to the probability of the event occurring.

Example 19.4 (Betting Your Edge)**.** Suppose the odds of a game are $\alpha_{A_i} = \frac{1}{p_i} + \delta$, where $\delta > 0$ represents your "edge". Then the optimal stake is

$$\begin{aligned} f_i^* &= \frac{p_i \alpha_{A_i} - 1}{\alpha_{A_i} - 1} \\ &= \frac{p_i \left(\frac{1}{p_i} + \delta\right) - 1}{\left(\frac{1}{p_i} + \delta\right) - 1} \\ &= \frac{(1 + p_i \delta) - 1}{\frac{1}{p_i} + \delta - 1} \\ &= \frac{p_i \delta}{\frac{1 - p_i}{p_i} + \delta} \end{aligned}$$

This is an increasing function of δ, so as your edge δ increases, so does the optimal stake f_i^*. This makes sense! If you have a higher edge, you should bet more of your bankroll on that game.

We continue with some notes about applying the Kelly Criterion to real-world betting.

19.3.3 Applying Kelly Under Uncertainty

Note that in practice, the true win probability p_i of a horse, player, or team is not an observable or known quantity (like a deck of cards is). This means we must **estimate it** from the data! Define this estimate as $\widehat{p}_i$. How does Kelly betting change under this condition?

Ideally, our estimator $\widehat{p}_i$ is **unbiased**, meaning that

$$\mathbb{E}[\widehat{p}_i] = p_i$$

but still subject to some uncertainty, which we call

$$\mathrm{Var}(\widehat{p}_i) = \tau^2$$

The greater this uncertainty, the **less** of our bankroll we should bet.

19.3.4 Fractional Kelly

Under a fixed fractional Kelly system, we bet some fixed fraction $K \in (0, 1)$ of the original Kelly betting fraction f_i^*. This means that our new betting fraction is

$$\begin{aligned} f_i &= K f_i^* \\ &= K \max\left(0, \frac{\widehat{p}_i \alpha_{A_i} - 1}{\alpha_{A_i} - 1}\right) \end{aligned}$$

We can modify this slightly to account for the uncertainty in our estimate. Instead of having K be a constant value, let this new K' be some function of τ, denoted

$$K' = K(\tau)$$

Then as τ varies, the fraction of the bankroll we bet should also change. This should follow the following properties:

$$\begin{aligned} \lim_{\tau \to 0} K(\tau) &= 1 \\ \lim_{\tau \to \infty} K(\tau) &= 0 \end{aligned}$$

meaning that if our uncertainty τ in our win probability estimate $\widehat{p}_i$ is close to 0, we should use the original Kelly betting fraction f_i^*, whereas if we have high uncertainty in our estimate, we should bet nothing.

Note: A K-fractional Kelly with an uncertain $\widehat{p}_i$ is equivalent to a full Kelly with a shrinkage estimator for $\widehat{p}$, where we shrink more the larger τ^2 is.

References

[JLK] Kelly, J. L., *A New Interpretation of Information Rate*, Bell System Technical Journal, 1956.