WSABI Summer Research Lab **Summer 2025**

Lab 10: Kelly Betting

Instructor: Jonathan Pipping *Authors: RB, JP*

10.1 A Smart -EV Bet

10.1.1 Kelly's Algorithm

In 1956, John L. Kelly extended his analysis to horses. Consider making a series of bets on N horse races, each with m horses running in the race. Following Kelly's notation, we define the following quantities:

- Let $i \in \{1, \ldots, m\}$ represent each horse.
- Let p_i represent the probability that horse i wins the race.
- Let α_i represent the decimal odds for horse i.
- Let $\boldsymbol{f}$ represent the payroll fractions $(f_1, \ldots, f_m)$ for each horse.

In his paper, he used Lagrange multipliers and Karush-Kuhn-Tucker conditions to maximize the **expected log bankroll** after N bets, yielding the Kelly allocations $\boldsymbol{f}^*$ via the following algorithm:

1. Permute indices so that $p_i\alpha_i \geq p_{i+1}\alpha_{i+1}$
2. Set b equal to the minimum positive value of
$$\frac{1-p_t}{1-\sigma_t}$$
where $p_t = \sum_{i=1}^{t} p_i$ and $\sigma_t = \sum_{i=1}^{t} \frac{1}{\alpha_i}$. This represents the proportion of the bankroll we keep in reserve.
3. Set $f_i = \max(0, p_s - \frac{b}{\alpha_s})$. If the odds are fair, $\sum_{i=1}^{m} f_i = 1 - b$.

You will use this algorithm in a simulated horse race to determine optimal Kelly allocations.

10.1.2 Your Task

1. Consider a horse race with $m = 3$ horses. Find an example of true horse win probabilities (p_1, p_2, p_3) and decimal odds $(\alpha_1, \alpha_2, \alpha_3)$ such that it is Kelly-optimal to bet on a -EV horse.

   Note that according to the above algorithm, the Kelly bet allocations (f_1, f_2, f_3) must have a $f_i > 0$ for some horse i, and that betting on a horse is -EV if $p_i\alpha_i - 1 < 0$.
2. Then, $M = 100$ times, simulate $N = 1000$ bets on this horse race using two different betting strategies:
   - Strategy 1: Following the Kelly allocations.
   - Strategy 2: Following Kelly for +EV horses, but setting $f_i = 0$ for any -EV horses.
3. For each strategy, plot the average bankroll after N bets across the M simulations. Which yields more wealth on average? How much more?

10.1.3 Discussion

Why is it possible that making a -EV bet is smart? Since Kelly staking works by growing wealth *multiplicatively* (or linearly on the log scale), it can be highly-sensitive to big draw-downs: meaning that a single wipe-out (or massive loss) hurts massively and can't be undone by later wins.

In a mutually-exclusive race, allocating a small fraction to a short-priced favorite (even if the edge is negative) hedges the inevitable variance in win-loss outcomes created by betting long shots that carry most of the positive edge. This smoother return path increases the **geometric mean growth rate**, which is what ultimately compounds to grow wealth.

Put differently, spreading out some of your budget onto the "least bad" remaining outcomes reduces the chance of catastrophic loss *faster* than it erodes expected profit. This means that every dollar wagered either **earns a positive edge** or **buys enough variance reduction** to raise the overall long-run growth.

References

[JLK] Kelly, J. L., *A New Interpretation of Information Rate*, Bell System Technical Journal, 1956.