

# Conditional Laws for the Maximal Values of Anchored Random Walk Functionals

Jonathan Pipping Jiaoyang Huang Abraham J. Wyner

University of Pennsylvania, Department of Statistics & Data Science

## Problem Setup

**Setup:** Let  $(X_k)_{k=0}^N$  be a random walk with  $X_0 = 0$  and i.i.d. increments  $\Delta X_k$  of mean  $\mu$  and variance  $\sigma^2$ , adapted to  $(\mathcal{F}_k)$ . For a position  $d$  at time  $k$ , define the functional

$$f(k, X_k) = \mathbb{P}(X_N > 0 \mid X_k = d)$$

Then the maximum of  $f$  over the time interval  $[0, N]$  is

$$M = \max_{0 \leq k \leq N} f(k, X_k).$$

**Question:** What is the law of  $M$  conditional on the event  $X_N < 0$ ? A closed-form solution is difficult to obtain, so we consider the limiting case.

**Brownian Limit:** By Donsker's invariance principle, as  $N \rightarrow \infty$ ,

$$\frac{X_{\lfloor Nt \rfloor}}{\sigma\sqrt{N}} \xrightarrow{d} B_t + \mu^* t, \quad \mu^* = \frac{\sqrt{N}\mu}{\sigma},$$

In the limit,

$$f(t, B_t) = \mathbb{P}(B_1 \geq -\mu^* \mid \mathcal{F}_t) = \Phi\left(\frac{B_t + \mu^*(1-t)}{\sqrt{1-t}}\right),$$

This represents a bounded martingale with  $f(0, B_0) = \Phi(\mu^*)$  whose terminal value is in  $\{0, 1\}$ . The analogous maximum over the time interval  $[0, 1]$  is

$$M = \sup_{0 \leq t \leq 1} f(t, B_t),$$

and we study the law of  $M$  conditional on the event  $B_1 < -\mu^*$ .

## Symmetric Random Walks

**Functional:** Since  $\mu = 0$ , the functional simplifies to

$$f(t, B_t) = \mathbb{P}(B_1 \geq 0 \mid \mathcal{F}_t) = \Phi\left(\frac{B_t}{\sqrt{1-t}}\right),$$

**Unconditional Distribution:** Optional stopping at the first hitting time of level  $x$  gives the unconditional distribution of  $M$ :

$$F_M(x) = \begin{cases} 0, & x < \frac{1}{2} \\ 1 - \frac{1}{2x}, & \frac{1}{2} \leq x < 1 \\ 1, & x = 1 \end{cases}$$

**Conditional Distribution:** The conditional distribution of  $M$  given  $B_1 < 0$  is

$$F_{M|B_1<0}(x) = 2 - \frac{1}{x}, \quad x \in [\frac{1}{2}, 1]$$

**Density Function:** Its corresponding density function is

$$f_{M|B_1<0}(x) = \frac{1}{x^2}, \quad x \in (\frac{1}{2}, 1)$$

## Asymmetric Random Walks

**Functional:** In the presence of drift ( $\mu^* \neq 0$ ), the functional

$$f(t, B_t) = \mathbb{P}(B_1 \geq -\mu^* \mid \mathcal{F}_t) = \Phi\left(\frac{B_t + \mu^*(1-t)}{\sqrt{1-t}}\right),$$

is a bounded martingale with  $f(0, B_0) = \Phi(\mu^*)$ .

**Unconditional Distribution:** Optional stopping at the first hitting time of level  $x$  gives the distribution of  $M$ :

$$F_M(x) = \begin{cases} 0, & x < \Phi(\mu^*) \\ 1 - \frac{\Phi(\mu^*)}{x}, & \Phi(\mu^*) \leq x < 1 \\ 1, & x = 1 \end{cases}$$

**Conditional Distribution:** Conditional on  $B_1 < -\mu^*$ ,

$$F_{M|B_1<-\mu^*}(x) = 1 - \frac{\Phi(\mu^*)}{\Phi(-\mu^*)} \cdot \frac{1-x}{x}, \quad x \in [\Phi(\mu^*), 1]$$

**Density Function:**

$$f_{M|B_1<-\mu^*}(x) = \frac{\Phi(\mu^*)}{\Phi(-\mu^*)} \cdot \frac{1}{x^2}, \quad x \in (\Phi(\mu^*), 1).$$

## Complementary Processes

**Definition:** Consider the complementary martingales

$$p_t = \mathbb{P}(B_1 \geq -\mu^* \mid \mathcal{F}_t), \quad q_t = 1 - p_t = \mathbb{P}(B_1 < -\mu^* \mid \mathcal{F}_t).$$

**Terminal Behavior:** At time 1, exactly one of  $\{p_t, q_t\}$  converges to 1, while the other converges to 0 (a.s.). Our object of interest is defined as

$$M = \sup_{0 \leq t < 1} (p_t \mathbf{1}_{\{p_t=0\}} + q_t \mathbf{1}_{\{q_t=0\}}),$$

i.e. the running supremum of whichever process ultimately vanishes.

**Branch Distributions:** For a process that starts at  $s \in (0, 1)$  and is conditioned to terminate at 0, denote by  $F_s(x)$  the distribution function of its maximum:

$$F_s(x) = \mathbb{P}(\sup_{0 \leq t < 1} p_t \leq x \mid p_0 = s, p_1 = 0)$$

**Distribution of the Eventual Vanishing Process:** Let  $p_0 := \Phi(\mu^*)$ . Then the law of the eventual vanishing process is the mixture

$$F_{\text{vanish}}(x) = \Phi(\mu^*) F_{1-\Phi(\mu^*)}(x) + (1 - \Phi(\mu^*)) F_{\Phi(\mu^*)}(x)$$

with support  $x \in [\min\{\Phi(\mu^*), 1 - \Phi(\mu^*)\}, 1]$ .

## Monte Carlo Simulation

**Validation:** We validate our theoretical results with Monte Carlo simulations. The symmetric case is shown in Figure 1.

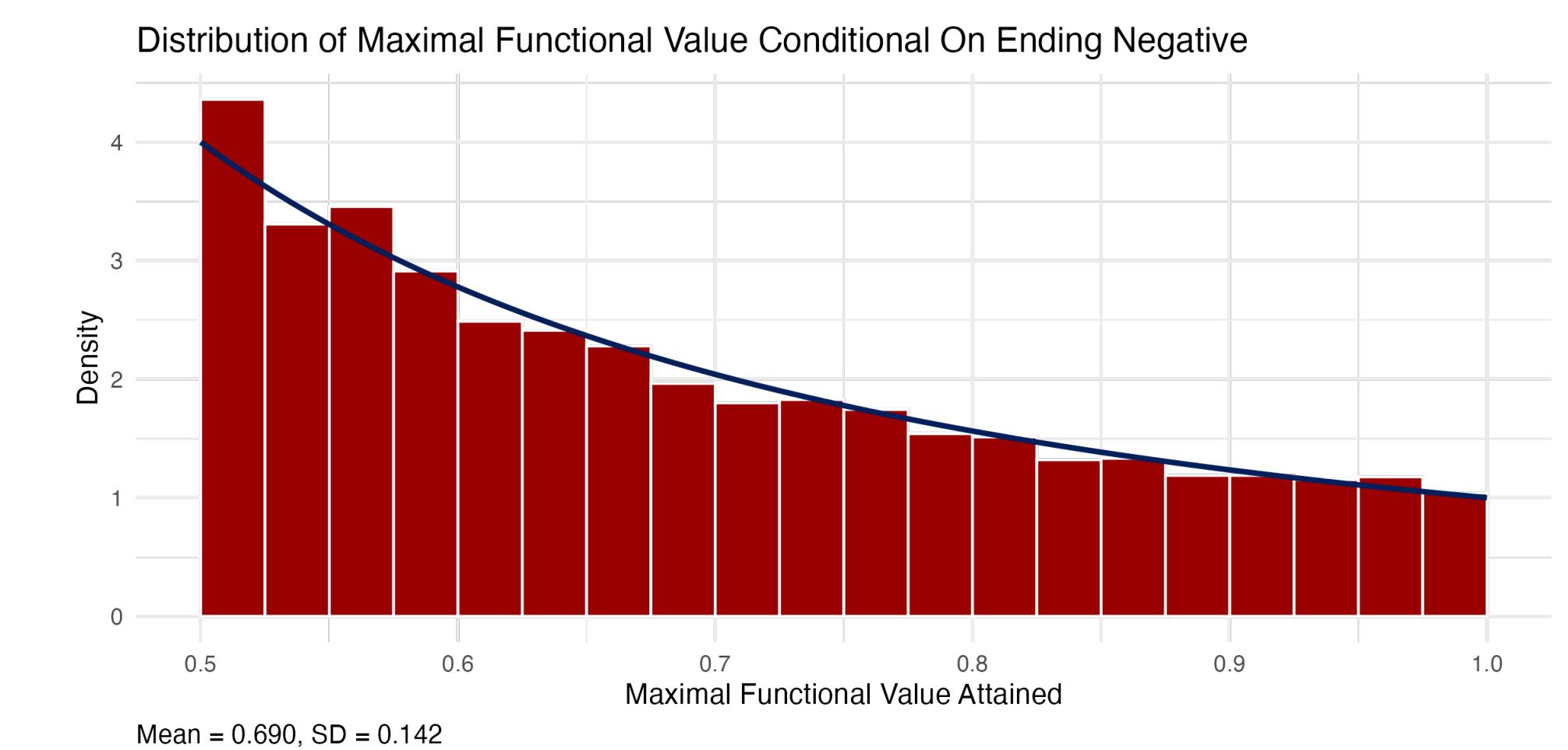


Figure 1. Symmetric case: simulated distribution (red) vs. theoretical density (blue).

## Application to In-Game Win Probabilities

**NFL Win Probabilities:** Win probability trajectories from closely-matched games approximately follow our theoretical symmetric distribution.

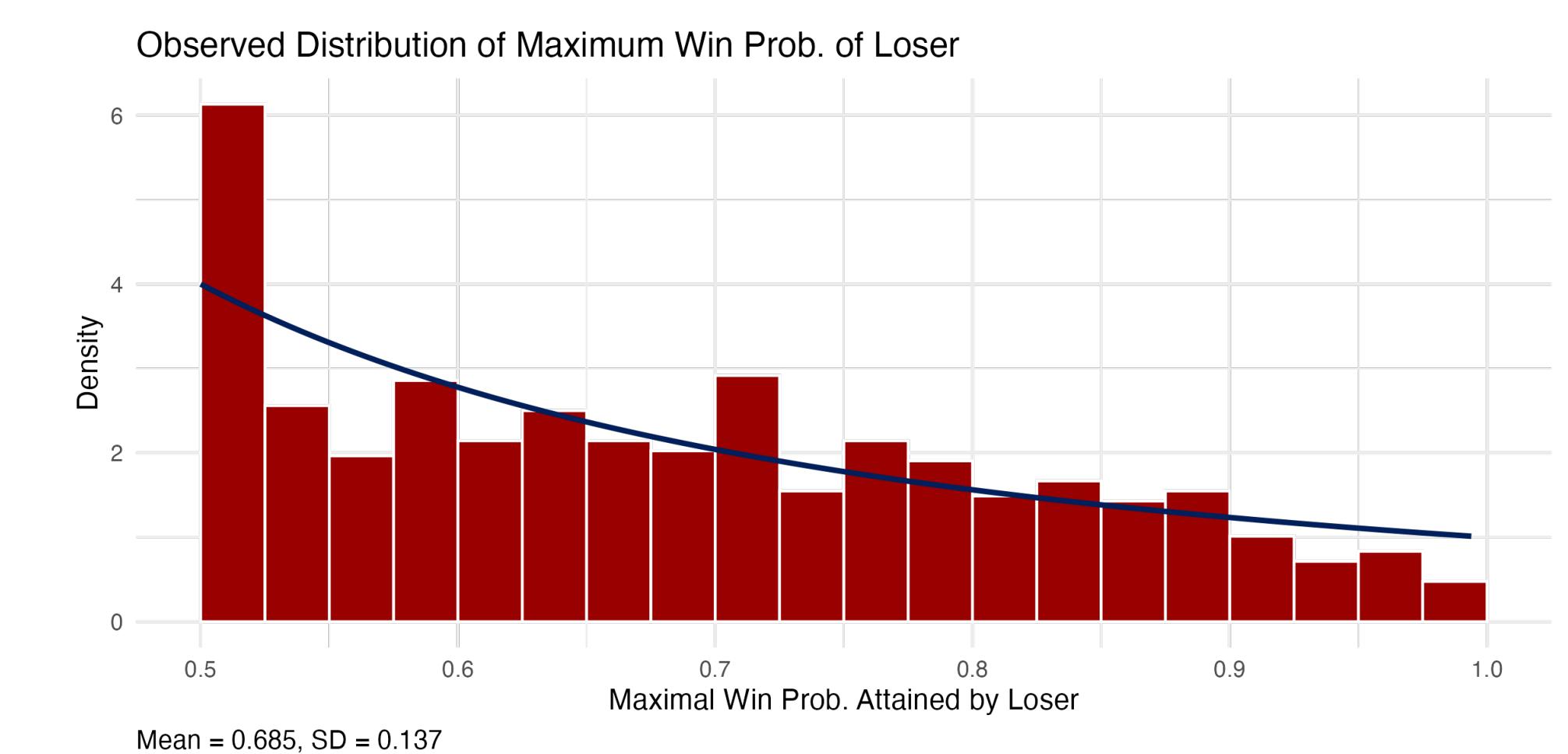


Figure 2. Distribution of maximal win probabilities of eventual NFL losers (2002-2024).

## References

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