The Blown Lead Paradox: Conditional Laws for the Running Maximum of Binary Doob Martingales

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Abstract

Live win-probability forecasts are now ubiquitous in sports broadcasts, and retrospective commentary often cites the largest win probability attained by a team that ultimately loses as evidence of a "collapse." Interpreting such extrema requires a reference distribution under correct specification. Modeling the forecast sequence as the Doob martingale of conditional win probabilities for a binary terminal outcome, we derive sharp distributional laws for its path maximum, including the conditional law given an eventual loss. In discrete time, we quantify explicit correction terms (last-step crossings and overshoots); under continuous-path regularity these corrections disappear, yielding exact identities. We further obtain closed-form distributions for two extensions: the maximal win probability attained by the eventual loser in a two-player game and the minimal win probability attained by the eventual winner in an n-player game. The resulting formulas furnish practical benchmarks for diagnosing sequential forecast calibration.

1 Introduction

Modern decision systems routinely display real-time probability forecasts for binary terminal events. A particularly visible example is *win probability* in sports broadcasts, which reports at each timestep the estimated chance that a designated team ultimately wins. In Super Bowl LVII, for instance, ESPN's win probability model assigned the Philadelphia Eagles a peak win probability of 78.4% early in the third quarter (ESPN, 2023). The Eagles nevertheless lost the game on a last-second field goal. Post hoc commentary often treats such episodes as extraordinary "collapses," implicitly suggesting that a team losing from a 78.4% position should be rare.

This interpretation is misleading, however. Even under perfect calibration, the distribution of the peak win probability *conditional on ultimately losing* differs from the naive fixed-time intuition suggested by a single displayed probability. Accordingly, the relevant object is not the forecast at a fixed time, but the path maximum attained by the eventual loser. Without a reference distribution for this conditional maximum under correct probabilistic specification, the evidentiary value of a quoted peak win probability is unclear.

We study this phenomenon in its canonical probabilistic form. Let $Y \in \{0, 1\}$ be the terminal outcome of interest (e.g., team A wins), revealed at a terminal time N, and let $(\mathcal{F}_k)_{k=0}^{N}$ be the natural filtration generated by the observable state process up to time k. Define the (ideal) forecast process

$$p_k \equiv \mathbb{P}(Y = 1 \mid \mathcal{F}_k), \qquad k = 0, 1, \ldots, N.$$

Then (p_k) is a bounded Doob martingale with $p_0 \in (0, 1)$ and $p_N = Y$. The primary path functional we analyze is the path maximum

$$M_N \equiv \max_{0 \le k \le N} p_k,$$

with continuous-path analogue $M = \sup_{0 \le t \le 1} p_t$. We study the terminal running maximum M_N (the path maximum over the horizon). We are particularly interested in behavior on the event $\{Y = 0\}$ ("eventual loss"). In two-player games, one often reports the eventual loser's maximal win probability M_λ, while in n-player settings one may analogously consider the eventual winner's minimal probability M_ω; both are natural extremes of coupled Doob martingales induced by conditional winning probabilities.

Related Work. Classical martingale theory controls the upper tail of M_N (and its continuous-path analogue M) through maximal inequalities (e.g., Doob's inequality) and Ville-type bounds for nonnegative (super)martingales, implying that under correct martingale specification large values are not, by themselves, exceptional (Doob, 1953; Ville, 1939). What is typically missing in applications is a *distributional* benchmark, an explicit law for M_N and related extremes, especially conditional on the terminal outcome. Under continuous-path regularity, our unconditional law for $\sup_t p_t$ can be viewed as a special case of Doob's maximal identity for continuous nonnegative local martingales; see, e.g., Nikeghbali and Yor (2006) and textbook treatments such as Revuz and Yor (1999). Our main focus is on the conditional law given the terminal outcome and on the multi-player competitive extensions. Our perspective also aligns with the martingale viewpoint in sequential analysis (Wald, 1947) and with modern "betting martingale" or e-value formulations that provide time-uniform guarantees via nonnegative supermartingales (Howard et al., 2020).

Contributions. This paper supplies a distributional benchmark for path extrema of bounded Doob martingales with $\{0, 1\}$-valued terminal outcomes. The main technical contribution is a set of sharp closed-form laws—exact under continuous-path regularity and otherwise accompanied by explicit last-step/overshoot corrections—that sharpen classical Doob/Ville maximal bounds which control only the upper tail. Concretely:

- We derive the unconditional law of M_N and the conditional law of M_N given $Y = 0$ in discrete time, identifying precisely when equalities hold and quantifying the discrete-time correction terms from last-step crossings and overshoots.
- We extend these results to the two-player eventual-loser functional M_λ and to the n-player eventual-winner functional M_ω, yielding closed-form distribution functions that depend only on the initial probabilities.
- We show that under continuous-path limits, terminal-step crossings and overshoots vanish and the discrete-time bounds become exact distributional identities.

Organization. Section 2 develops the discrete-time theory via optional stopping. Section 3 treats the continuous-path case and derives exact distributional identities. Section 4 illustrates how the resulting reference distributions can be used to interpret and test "collapse" claims in applied settings. Section 5 concludes with limitations and extensions.

2 Discrete-Time Case

We first analyze the problem in discrete time. Our starting point is a binary terminal event and its associated conditional probability (win-probability) martingale, whose maximum records how close the process comes to the event over the course of the game or experiment. In this section, we work with an abstract filtered probability space and derive general distributional laws for this maximum, both unconditionally and conditionally on the event not occurring.

2.1 Problem Setup

Let $(\Omega, \mathcal{F}, (\mathcal{F}_k)_{k=0}^N, \mathbb{P})$ be a filtered probability space, and let $Y \in \{0,1\}$ be $\mathcal{F}_N$-measurable. Define the associated Doob martingale of conditional event probabilities

$$p_k \equiv \mathbb{E}[Y \mid \mathcal{F}_k], \quad 0 \le k \le N.$$

Then $p_k \in [0,1]$, with

$$p_0 = \mathbb{P}(Y=1) \in (0,1),$$

and $p_N = Y \in \{0,1\}$. We define the path maximum of the martingale and, for each fixed $x \in [0,1]$, the first time the process reaches or exceeds level x:

$$M_N \equiv \max_{0 \le k \le N} p_k, \qquad \tau_x \equiv \inf\{k \ge 0 : p_k \ge x\},$$

with the convention $\inf \emptyset = \infty$ and with optional stopping applied at $\tau_x \wedge N$. Our goal in this section is to characterize the distribution of M_N and its conditional law given $Y = 0$ under this minimal martingale structure.

2.2 Unconditional Distribution

Theorem 1. The cumulative distribution function of M_N satisfies

$$F_{M_N}(x) \ge 1 - \frac{p_0}{x}, \quad x \in [p_0, 1),$$

with equality when $\mathbb{P}(\tau_x = N) = 0$ and $p_{\tau_x} = x$ a.s. on $\{\tau_x < N\}$ (e.g. under continuous-path limits). For $x < p_0$, we have $F_{M_N}(x) = 0$. In particular, M_N has an atom at $x = 1$ of mass $\mathbb{P}(M_N = 1) = p_0$. For finite N, the remainder of the law is supported on the discrete set of attainable values of M_N in $[p_0, 1)$.

Proof. See Appendix A for the full proof. □

2.3 Conditional Distribution

Theorem 2. The cumulative distribution function of M_N conditional on the terminal event $Y = 0$ satisfies

$$F_{M_N|Y=0}(x) \ge 1 - \left(\frac{p_0}{1-p_0}\right) \cdot \left(\frac{1-x}{x}\right), \quad x \in [p_0, 1),$$

with equality when $\mathbb{P}(\tau_x = N) = 0$ and $p_{\tau_x} = x$ a.s. on $\{\tau_x < N\}$ (e.g. under continuous-path limits). For $x < p_0$, we have $F_{M_N|Y=0}(x) = 0$. For finite N, the law is supported on the discrete set of attainable values of M_N in $[p_0, 1)$.

Proof. See Appendix A for the full proof. □

3 Continuous-Path Case

In the discrete-time setting, we studied the martingale

$$p_k = \mathbb{E}[Y \mid \mathcal{F}_k],$$

for a binary terminal outcome Y, and derived sharp inequalities for the law of its maximum $M_N = \max_{0 \le k \le N} p_k$. Equality in discrete time requires no overshoot at first passage and no terminal-step crossings (i.e., $\mathbb{P}(p_{\tau_x} > x,\ \tau_x < N) = 0$ and $\mathbb{P}(\tau_x = N) = 0$); under continuous paths these obstructions vanish, yielding exact identities.

3.1 Problem Setup

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,1]}, \mathbb{P})$ be a filtered probability space, and let $Y \in \{0,1\}$ be $\mathcal{F}_1$-measurable. Define the Doob martingale of conditional event probabilities

$$p_t \equiv \mathbb{E}[Y \mid \mathcal{F}_t], \quad 0 \le t \le 1.$$

Assume (p_t) has continuous sample paths. Then $p_t \in [0,1]$, with $p_0 = \mathbb{P}(Y = 1) \in (0,1)$ and $p_1 = Y$ almost surely. Define the running supremum and, for each fixed $x \in [0,1]$, the first passage time:

$$M = \sup_{0\le t\le 1} p_t, \qquad \tau_x = \inf\{t \in [0,1] : p_t \ge x\},$$

with the convention $\inf \emptyset = \infty$. Continuity implies $\{\tau_x < 1\} = \{M \ge x\}$ and $\mathbb{P}(\tau_x \ge 1, Y = 1) = 0$ for $x < 1$.

3.2 Unconditional Distribution

Theorem 3. The cumulative distribution function of M satisfies

$$F_M(x) = \begin{cases} 0 & \text{for } x < p_0, \\ 1 - \frac{p_0}{x} & \text{for } x \in [p_0, 1), \\ 1 & \text{for } x = 1. \end{cases}$$

In particular, M has an atom at $x = 1$ of mass $\mathbb{P}(M = 1) = p_0$.

Proof. See Appendix B for the full proof. □

3.3 Conditional Distribution

Theorem 4. The cumulative distribution function of M conditional on the terminal event $Y = 0$ satisfies

$$F_{M|Y=0}(x) = \begin{cases} 0 & \text{for } x < p_0, \\ 1 - \left(\frac{p_0}{1-p_0}\right) \cdot \left(\frac{1-x}{x}\right) & \text{for } x \in [p_0, 1), \\ 1 & \text{for } x = 1. \end{cases}$$

Proof. See Appendix B for the full proof. □

4 Examples and Applications

The results above are purely martingale-theoretic and make no reference to the domain that generates the win-probability paths. In particular, the conditional laws and competitive extensions provide explicit distributional benchmarks for extremes, which quantify how often large swings (e.g. an eventual loser ever reaching 90% win probability) occur even under a perfectly calibrated model. We illustrate these baseline expectations in head-to-head win-probability models and in model-based financial trading strategies.

4.1 Maximum Win Probability of the Eventual Loser in Two-Player Games

In sports and other competitive settings, it is common to see statements such as "Team A blew a 90% chance to win." Formally, this refers to the maximum win probability attained at any time by the team that ultimately loses. Our theory provides the natural reference distribution for this quantity under a correctly calibrated win-probability model.

Consider a two-team game where

$$Y = \mathbf{1}\{X_1 \geq 0\} \in \{0, 1\}$$

indicates that team A wins. The associated win-probability martingale for team A is

$$p_t = \mathbb{E}[Y \mid \mathcal{F}_t] = \mathbb{P}(Y = 1 \mid \mathcal{F}_t), \qquad 0 \leq t \leq 1,$$

so that p_t is the model-implied probability at time t that team A ultimately wins. The complementary martingale

$$q_t = 1 - p_t = \mathbb{E}[1 - Y \mid \mathcal{F}_t]$$

is the win probability for team B.

At the terminal time $t = 1$, exactly one team wins, so

$$(p_1, q_1) \in \{(1, 0), (0, 1)\}.$$

On the event $\{Y = 0\}$ (team A loses), the loser's win-probability path is (p_t), and

$$M_\lambda = \sup_{0 \leq t \leq 1} p_t \quad \text{on } \{Y = 0\},$$

while on $\{Y = 1\}$ (team B loses) the loser's path is (q_t), and

$$M_\lambda = \sup_{0 \leq t \leq 1} q_t \quad \text{on } \{Y = 1\}.$$

The maximum win probability attained by the eventual loser is then given by the mixture definition

$$M_\lambda \equiv \sup_{0 \leq t \leq 1} \Big(p_t \mathbf{1}\{Y = 0\} + q_t \mathbf{1}\{Y = 1\} \Big), \tag{1}$$

which selects the appropriate process to track based on the outcome of the game: on $\{Y = 0\}$ we track (p_t), while on $\{Y = 1\}$ we track (q_t). We are interested in the distribution of M_λ, which we obtain from the law of total probability:

$$\begin{aligned} F_{M_\lambda}(x) = \mathbb{P}(M_\lambda \leq x) &= \mathbb{P}(M_\lambda \leq x \mid Y = 0)\mathbb{P}(Y = 0) + \mathbb{P}(M_\lambda \leq x \mid Y = 1)\mathbb{P}(Y = 1), \\ &= \mathbb{P}(M_\lambda \leq x \mid Y = 0)(1 - p_0) + \mathbb{P}(M_\lambda \leq x \mid Y = 1)p_0, \end{aligned} \tag{2}$$

where $p_0 = \mathbb{P}(Y = 1)$ is the prior probability that team A wins. The results in Sections 2.3 and 3.3 give the conditional distributions immediately. In the continuous-path case of Theorem 4, when team A loses we have

$$\begin{aligned} F_{M_\lambda \mid Y=0}(x) = \mathbb{P}(M_\lambda \leq x \mid Y = 0) &= \mathbb{P}\left(\sup_{0 \leq t \leq 1} p_t \leq x \;\middle|\; Y = 0 \right) \\ &= \begin{cases} 0 & \text{for } x < p_0, \\ 1 - \left(\frac{p_0}{1 - p_0}\right)\left(\frac{1 - x}{x}\right) & \text{for } x \in [p_0, 1), \\ 1 & \text{for } x = 1. \end{cases} \end{aligned} \tag{3}$$

where (p_t) is the win-probability martingale for team A with prior probability $p_0 = \mathbb{P}(Y = 1)$. Since (q_t) is the win-probability martingale for the event $\{Y = 0\}$ (team B wins) with prior probability $q_0 = \mathbb{P}(Y = 0) = 1 - p_0$, applying Theorem 4 with p_0 replaced by $q_0 = 1 - p_0$ gives

$$\begin{aligned} F_{M_\lambda \mid Y=1}(x) = \mathbb{P}(M_\lambda \leq x \mid Y = 1) &= \mathbb{P}\left(\sup_{0 \leq t \leq 1} q_t \leq x \;\middle|\; Y = 1 \right) \\ &= \begin{cases} 0 & \text{for } x < 1 - p_0, \\ 1 - \left(\frac{1 - p_0}{p_0}\right)\left(\frac{1 - x}{x}\right) & \text{for } x \in [1 - p_0, 1), \\ 1 & \text{for } x = 1. \end{cases} \end{aligned} \tag{4}$$

Combining (3) and (4) yields the piecewise CDF (derivation in Appendix C). Without loss of generality, assume $p_0 \geq \frac{1}{2}$ (otherwise swap team labels), so $1 - p_0 \leq p_0$:

$$F_{M_\lambda}(x) = \mathbb{P}(M_\lambda \leq x) = \begin{cases} 0 & \text{for } 0 \leq x < 1 - p_0, \\ 1 - \frac{1-p_0}{x} & \text{for } 1 - p_0 \leq x < p_0, \\ 2 - \frac{1}{x} & \text{for } p_0 \leq x < 1, \\ 1 & \text{for } x = 1. \end{cases}$$

4.1.1 Symmetric Case

For a symmetric matchup with $p_0 = 1/2$, substituting into the general piecewise formula gives $1 - p_0 = p_0 = 1/2$, so the intermediate region disappears. The distribution simplifies to

$$F_{M_\lambda}(x) = \mathbb{P}(M_\lambda \leq x) = \begin{cases} 0 & \text{for } 0 \leq x < \frac{1}{2}, \\ 2 - \frac{1}{x} & \text{for } \frac{1}{2} \leq x < 1, \\ 1 & \text{for } x = 1. \end{cases}$$

For example, in the symmetric case, $\mathbb{P}(M_\lambda \geq 2/3) = 1/2$. Additional numerical examples are provided in the Supplementary Material, Section S1.1 (Pipping-Gamón and Wyner, 2026).

4.1.2 Asymmetric Case

When $p_0 \neq 1/2$, the key difference from the symmetric case is the intermediate region $[1 - p_0, p_0)$ that creates a kink in the distribution. Figure 1 displays the theoretical cumulative distribution function $F_{M_\lambda}(x)$ for several values of p_0. As p_0 increases from 0.5 to 1, the curves rise more quickly in the intermediate region and rejoin the symmetric ($p_0 = 0.5$) curve at $x = p_0$, where all curves converge to the common form $F_{M_\lambda}(x) = 2 - 1/x$ for $x \geq p_0$.

4.2 Minimum Win Probability of the Eventual Winner in n-Player Games

A natural extension of the two-player framework is to consider games between $n \geq 3$ players or teams with one winner. In such settings, the analogous quantity is the *minimum* win probability attained by the eventual *winner*.

Consider an n-player game where, for each player $i \in \{1, \ldots, n\}$, the win-probability martingale is

$$p_t^{(i)} = \mathbb{E}[Y^{(i)} \mid \mathcal{F}_t], \qquad 0 \leq t \leq 1,$$

where $Y^{(i)} = \mathbf{1}\{\text{player } i \text{ wins}\} \in \{0, 1\}$. Since exactly one player wins, we have $\sum_{i=1}^{n} p_t^{(i)} = 1$ for all t, and at the terminal time $t = 1$, exactly one player i^* satisfies $p_1^{(i^*)} = 1$ while $p_1^{(j)} = 0$ for all $j \neq i^*$.

On the event $\{Y^{(i)} = 1\}$ (player i wins), the winner's win-probability path is $(p_t^{(i)})$, and

$$M_\omega = \inf_{0 \leq t \leq 1} p_t^{(i)} \quad \text{on } \{Y^{(i)} = 1\}.$$

So the minimum win probability of the eventual winner is given by the mixture definition

$$M_\omega \equiv \inf_{0 \leq t \leq 1} \sum_{i=1}^{n} p_t^{(i)} \mathbf{1}\{Y^{(i)} = 1\}, \tag{5}$$

Figure 1: Theoretical cumulative distribution function $F_{M_\lambda}(x)$ of the maximum win probability of the eventual loser for different pre-game win probabilities p_0. The symmetric case ($p_0 = 0.5$) shows a smooth curve, while asymmetric cases exhibit a kink at $x = p_0$ where both conditional distributions begin to contribute.

which selects the appropriate process to track based on the outcome. We are interested in the distribution of M_ω, which we obtain from the law of total probability:

$$\begin{aligned} F_{M_\omega}(x) = \mathbb{P}(M_\omega \leq x) &= \sum_{i=1}^{n} \mathbb{P}(M_\omega \leq x \mid Y^{(i)} = 1)\mathbb{P}(Y^{(i)} = 1), \\ &= \sum_{i=1}^{n} \mathbb{P}(M_\omega \leq x \mid Y^{(i)} = 1) p_0^{(i)}, \end{aligned} \tag{6}$$

where $p_0^{(i)} = \mathbb{P}(Y^{(i)} = 1)$ is the prior probability that player i wins.

Since $\inf_t p_t^{(i)} \leq x$ if and only if $\sup_t(1 - p_t^{(i)}) \geq 1 - x$, the conditional distributions follow from Theorem 4 applied to the complementary martingale $(1 - p_t^{(i)}) = \mathbb{P}(Y^{(i)} = 0 \mid \mathcal{F}_t)$. On $\{Y^{(i)} = 1\}$ we have $(1 - p_1^{(i)}) = 0$, so this is exactly the "loser" case with prior probability $(1 - p_0^{(i)})$. Applying Theorem 4 gives

$$\begin{aligned} \mathbb{P}(M_\omega \leq x \mid Y^{(i)} = 1) &= \mathbb{P}\left(\inf_{0 \leq t \leq 1} p_t^{(i)} \leq x \,\middle|\, Y^{(i)} = 1 \right) \\ &= \mathbb{P}\left(\sup_{0 \leq t \leq 1} (1 - p_t^{(i)}) \geq 1 - x \,\middle|\, Y^{(i)} = 1 \right) \\ &= \begin{cases} 0 & \text{for } x < 0, \\ \left(\frac{1 - p_0^{(i)}}{p_0^{(i)}} \right) \left(\frac{x}{1 - x} \right) & \text{for } x \in [0, p_0^{(i)}), \\ 1 & \text{for } x \geq p_0^{(i)}. \end{cases} \end{aligned} \tag{7}$$

The last case follows because the minimum cannot exceed the starting value $p_0^{(i)}$.

Substituting (7) into (6) yields the distribution:

$$F_{M_\omega}(x) = \mathbb{P}(M_\omega \le x) = \begin{cases} 0 & \text{for } x < 0, \\ \sum_{i: x \ge p_0^{(i)}} p_0^{(i)} + \left(\frac{x}{1-x}\right) \sum_{i: x < p_0^{(i)}} (1 - p_0^{(i)}) & \text{for } x \in [0, \max_i p_0^{(i)}), \\ 1 & \text{for } x \ge \max_i p_0^{(i)}. \end{cases} \tag{8}$$

Note that $M_\omega \le x$ is certain when $x \ge \max_i p_0^{(i)}$, since the minimum cannot exceed any player's initial probability. Derivations appear in Appendix D.

4.2.1 Symmetric Case

For the symmetric case where all players have equal prior probabilities $p_0^{(i)} = 1/n$ for all i, the distribution simplifies. Since $M_\omega \le 1/n$ almost surely, the CDF is nontrivial only on $x \in [0, 1/n)$, where $\mathbf{1}\{x < p_0^{(i)}\} = 1$ for all i.

$$\begin{aligned} F_{M_\omega}(x) &= \sum_{i=1}^{n} (1 - 1/n) \left(\frac{x}{1-x}\right) \\ &= \left(\sum_{i=1}^{n} \frac{n-1}{n}\right) \left(\frac{x}{1-x}\right) \\ &= \frac{(n-1)x}{1-x}, \quad x \in [0, 1/n). \end{aligned}$$

Therefore, the cumulative distribution function is

$$F_{M_\omega}(x) = \mathbb{P}(M_\omega \le x) = \begin{cases} 0 & \text{for } x < 0, \\ \frac{(n-1)x}{1-x} & \text{for } x \in [0, 1/n), \\ 1 & \text{for } x \ge 1/n. \end{cases}$$

Even in a perfectly calibrated model for a symmetric three-player game, the minimum win probability can be quite small. Figure 2 displays the cumulative distribution function $F_{M_\omega}(x)$ for several values of n. As n increases, the distribution shifts leftward, reflecting that eventual winners in larger games are more likely to have had very low win probabilities at some point (with minimums no lower than $1/n$).

For example, in the symmetric three-player case, $\mathbb{P}(M_\omega \le 0.2) = 1/2$. Additional numerical benchmarks are provided in the Supplementary Material, Section S1.2 (Pipping-Gamón and Wyner, 2026).

4.3 Financial Trading Strategies

The same principles apply to model-based trading strategies. Let $(\mathcal{F}_t)_{t \in [0,1]}$ be the filtration generated by observable market information and model inputs. Define the terminal indicator

$$Y = \mathbf{1}\{\text{trade finishes with nonnegative PnL}\} \in \{0, 1\}$$

and the model-implied win-probability process

$$p_t = \mathbb{E}[\, Y \mid \mathcal{F}_t \,], \quad 0 \le t \le 1.$$

Here p_t is the model's current estimate of the probability that the position eventually ends profitably, given all information up to time t. As in the win-probability setting, we idealize the model-implied probabilities as the true conditional probabilities $\mathbb{P}(Y = 1 \mid \mathcal{F}_t)$; in applications, they serve as approximations to this martingale.

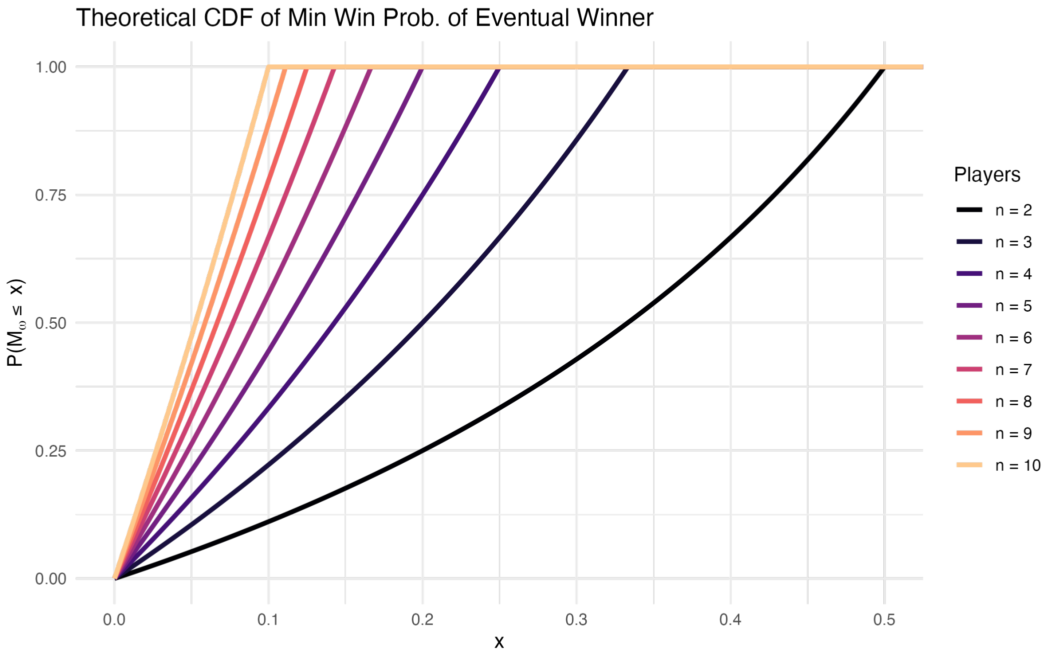

Figure 2: Theoretical cumulative distribution function $F_{M_\omega}(x)$ of the minimum win probability of the eventual winner for symmetric n-player games. As n increases, winners are more likely to have had very low win probabilities at some point during the game.

In retrospective performance analysis it is natural to focus on *losing* trades ($Y = 0$) and ask whether the model was systematically overconfident. A convenient scalar diagnostic is the maximal model-implied win probability attained along each losing trade:

$$M_{\text{loss}} = \sup_{0 \le t \le 1} p_t, \qquad \text{evaluated conditional on } Y = 0.$$

If the model is well-calibrated and (p_t) is approximately continuous, Theorem 4 yields a sharp target distribution for M_{loss} conditional on $Y = 0$:

$$F_{M_{\text{loss}}|Y=0}(x) = \mathbb{P}(M_{\text{loss}} \le x \mid Y = 0) = \begin{cases} 0 & \text{for } x < p_0, \\ 1 - \left(\frac{p_0}{1-p_0}\right)\left(\frac{1-x}{x}\right) & \text{for } x \in [p_0, 1), \\ 1 & \text{for } x = 1, \end{cases} \tag{9}$$

where $p_0 = \mathbb{P}(Y = 1)$ is the model-implied probability, at trade inception, that the trade finishes profitably.

Figure 3 displays the theoretical cumulative distribution function $F_{M_{\text{loss}}|Y=0}(x)$ for several values of p_0.

For example, when $p_0 = 1/2$, $\mathbb{P}(M_{\text{loss}} \ge 2/3 \mid Y = 0) = 1/2$. Additional benchmark thresholds appear in the Supplementary Material, Section S1.3 (Pipping-Gamón and Wyner, 2026).

4.4 Interpretation and Implications

The distributions derived in Sections 4.1, 4.2, and 4.3 provide model-agnostic baselines that translate narrative claims about exceptional outcomes into precise distributional statements. In applications, these baselines serve two roles: *expectation calibration* (how often large swings should occur under correct specification) and *diagnostic validation* (whether empirical CDFs deviate from the theoretical targets in ways that indicate over- or underconfidence or violations of the martingale assumptions). Formal

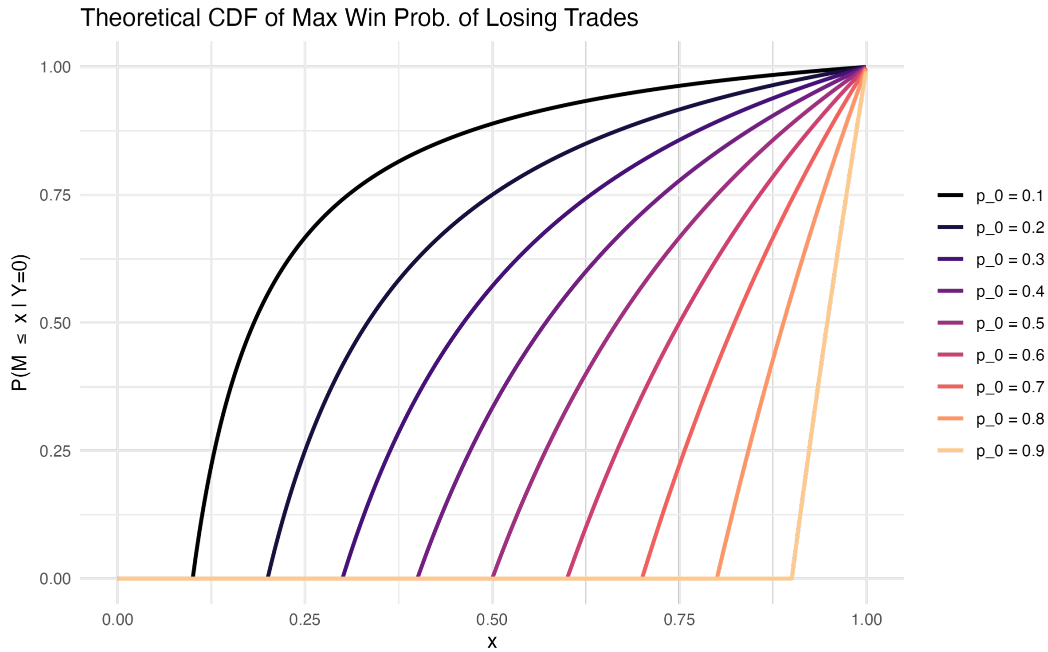

Figure 3: Theoretical cumulative distribution function $F_{M_{\text{loss}}|Y=0}(x)$ of the maximum win probability of losing trades for different inception probabilities p_0. As p_0 increases, the distribution shifts rightward, reflecting that trades starting with higher success probability are more likely to reach higher peak probabilities before losing.

testing procedures, practical guidance, and empirical validation on NFL and NBA data appear in the Supplementary Material (Sections S2 and S3) (Pipping-Gamón and Wyner, 2026).

5 Discussion

This work provides sharp distributional laws for the maximum of bounded Doob martingales with binary terminal outcomes. We derive the unconditional law of M, its conditional law given $Y = 0$, and competitive extensions for two-player and n-player settings. In discrete time the results are sharp with explicit last-step/overshoot corrections; under continuous-path regularity these corrections vanish, yielding exact closed-form expressions that serve as accurate approximations for discrete-time processes with many updates.

Returning to the motivating broadcast narrative, our results formalize why statements such as "the Eagles reached a win probability of 78.4% and still lost" are not self-evidently anomalous under correct calibration. The relevant comparison is not to a fixed-time probability such as $1 - 0.784$, but to the conditional distribution of the path maximum attained along trajectories that end in a loss. This provides a principled baseline against which such claims can be assessed: a peak win probability is informative only insofar as it is extreme relative to this conditional reference law, and systematic departures from the benchmark indicate model misspecification or non-martingale behavior, rather than providing standalone evidence of an exceptional collapse.

5.1 Limitations and Assumptions

Our results rely on two fundamental assumptions. First, the discrete-time bounds require that (p_k) is an exact Doob martingale, $p_k = \mathbb{E}[Y \mid \mathcal{F}_k]$. In practice, win-probability models are estimates of this idealized process, and systematic deviations from the martingale property (e.g., due to model misspecification or non-stationarity) will manifest as discrepancies between empirical and theoretical distributions.

Second, the exact identities in the continuous-path case assume path regularity, such as continuity of (p_t) (or, equivalently for our purposes, the absence of terminal-step crossings and overshoots). With jumps or coarse update schedules, last-step crossings and overshoots may be non-negligible; the discrete-time bounds in Section 2 remain valid but may be loose when such effects are frequent. We also assume binary terminal outcomes $p_N \in \{0, 1\}$; extensions to ties or continuous outcomes would require different techniques.

5.2 Extensions and Future Work

Several natural extensions suggest themselves. One could relax the binary terminal condition to allow $p_N \in [0, 1]$ (e.g., games with ties or trades with continuous PnL outcomes). The distribution of other path statistics, such as the time at which the maximum is attained or the joint distribution of the maximum and terminal value, could provide additional diagnostic power. Non-martingale processes (e.g., supermartingales or processes with predictable drift) would require different techniques but might capture systematic biases in model updates.

We provide empirical validation on NFL and NBA data in the Supplementary Material (Pipping-Gamón and Wyner, 2026); additional large-scale datasets (e.g., financial trading records or A/B test outcomes) would further strengthen the practical utility of these distributions and reveal domain-specific patterns in model calibration.

In summary, these distributional laws provide a principled foundation for interpreting maximum win-probability statistics, converting subjective narratives into objective, testable claims about model calibration and competitive dynamics.

Appendix

A Proofs for Discrete-Time Results

A.1 Unconditional Distribution

Theorem 5 (Discrete-Time Unconditional)**.** The cumulative distribution function of M_N satisfies

$$F_{M_N}(x) \geq 1 - \frac{p_0}{x}, \quad x \in [p_0, 1),$$

with equality when $\mathbb{P}(\tau_x = N) = 0$ and $p_{\tau_x} = x$ a.s. on $\{\tau_x < N\}$ (e.g. under continuous-path limits). For $x < p_0$, we have $F_{M_N}(x) = 0$. In particular, M_N has an atom at $x = 1$ of mass $\mathbb{P}(M_N = 1) = p_0$. For finite N, the remainder of the law is supported on the discrete set of attainable values of M_N in $[p_0, 1)$.

Proof. If $x \leq p_0$, then $\tau_x = 0$ a.s., so $M_N \geq x$ a.s. and $F_{M_N}(x) = 0$. Now consider $x > p_0$. By the optional stopping theorem (Doob, 1953) applied to the bounded martingale (p_k) at $\tau_x \wedge N$,

$$\mathbb{E}[p_{\tau_x \wedge N}] = p_0.$$

Since $p_{\tau_x \wedge N} = p_{\tau_x} \mathbf{1}\{\tau_x \leq N\} + p_N \mathbf{1}\{\tau_x > N\}$ and $\tau_x > N$ implies $Y = 0$ (hence $p_N = 0$) for $x > 0$, we obtain

$$p_0 = \mathbb{E}[p_{\tau_x} \mathbf{1}\{\tau_x < N\}] + \mathbb{P}(\tau_x = N).$$

On $\{\tau_x < N\}$ we have $p_{\tau_x} \geq x$, so

$$p_0 = x\mathbb{P}(\tau_x < N) + \mathbb{P}(\tau_x = N) + \mathbb{E}[(p_{\tau_x} - x)\mathbf{1}\{\tau_x < N\}].$$

Since $\mathbb{P}(M_N \geq x) = \mathbb{P}(\tau_x < N) + \mathbb{P}(\tau_x = N)$, we can rewrite this as

$$p_0 = x\mathbb{P}(M_N \geq x) + (1-x)\mathbb{P}(\tau_x = N) + \mathbb{E}[(p_{\tau_x} - x)\mathbf{1}\{\tau_x < N\}].$$

For $x < 1$, both correction terms are nonnegative, so

$$\mathbb{P}(M_N \geq x) \leq \frac{p_0}{x}, \quad x \in [p_0, 1),$$

with equality when $\mathbb{P}(\tau_x = N) = 0$ and $p_{\tau_x} = x$ a.s. on $\{\tau_x < N\}$. When $x = 1$, we have $M_N = 1$ if and only if $Y = 1$, so

$$\mathbb{P}(M_N \geq 1) = \mathbb{P}(M_N = 1) = p_0. \qquad \square$$

A.2 Conditional Distribution

Theorem 6 (Discrete-Time Conditional)**.** The cumulative distribution function of M_N conditional on the terminal event $Y = 0$ satisfies

$$F_{M_N|Y=0}(x) \geq 1 - \left(\frac{p_0}{1-p_0}\right) \cdot \left(\frac{1-x}{x}\right), \quad x \in [p_0, 1),$$

with equality when $\mathbb{P}(\tau_x = N) = 0$ and $p_{\tau_x} = x$ a.s. on $\{\tau_x < N\}$ (e.g. under continuous-path limits). For $x < p_0$, we have $F_{M_N|Y=0}(x) = 0$. For finite N, the law is supported on the discrete set of attainable values of M_N in $[p_0, 1)$.

Proof. Decompose the event $\{M_N \geq x\}$ according to the terminal outcome:

$$\mathbb{P}(M_N \geq x) = \mathbb{P}(M_N \geq x, Y = 0) + \mathbb{P}(M_N \geq x, Y = 1).$$

On $\{Y = 1\}$ we have $p_N = 1 \geq x$ a.s., so $\{M_N \geq x\} \cap \{Y = 1\} = \{Y = 1\}$ and hence

$$\begin{aligned}
\mathbb{P}(M_N \geq x) &= \mathbb{P}(Y = 1) + \mathbb{P}(M_N \geq x, Y = 0) \\
&= \mathbb{P}(Y = 1) + \mathbb{P}(Y = 0)\mathbb{P}(M_N \geq x \mid Y = 0) \\
&= p_0 + (1 - p_0)\mathbb{P}(M_N \geq x \mid Y = 0).
\end{aligned}$$

Plugging in the bound from Theorem 5 yields

$$\frac{p_0}{x} \geq p_0 + (1 - p_0)\mathbb{P}(M_N \geq x \mid Y = 0),$$

and solving for $\mathbb{P}(M_N \geq x \mid Y = 0)$ gives

$$\mathbb{P}(M_N \geq x \mid Y = 0) \leq \left(\frac{p_0}{1-p_0}\right) \cdot \left(\frac{1-x}{x}\right), \quad x \in [p_0, 1),$$

with equality when $\mathbb{P}(\tau_x = N) = 0$ and $p_{\tau_x} = x$ a.s. on $\{\tau_x < N\}$. $\square$

B Proofs for Continuous-Path Results

B.1 Unconditional Distribution

Theorem 7 (Continuous-Path Unconditional)**.** The cumulative distribution function of M satisfies

$$F_M(x) = \begin{cases} 0 & \text{for } x < p_0, \\ 1 - \frac{p_0}{x} & \text{for } x \in [p_0, 1), \\ 1 & \text{for } x = 1. \end{cases}$$

In particular, M has an atom at $x = 1$ of mass $\mathbb{P}(M = 1) = p_0$.

Proof. For $x < p_0$, we have $F_M(x) = 0$ since p_t starts at p_0, so $M \geq p_0 > x$ always.

For $x \in [p_0, 1)$, by the optional stopping theorem (Doob, 1953) applied to the bounded martingale (p_t) at $\tau_x \wedge 1$,

$$\mathbb{E}[p_{\tau_x \wedge 1}] = p_0.$$

Decomposing this expectation yields

$$p_0 = \mathbb{E}[p_{\tau_x} \mathbf{1}\{\tau_x < 1\}] + \mathbb{E}[p_1 \mathbf{1}\{\tau_x \geq 1\}].$$

On $\{\tau_x < 1\}$ we have $p_{\tau_x} = x$ by continuity, and on $\{\tau_x \geq 1\}$ we have $p_1 = Y$, so

$$p_0 = x\mathbb{P}(\tau_x < 1) + \mathbb{P}(Y = 1, \tau_x \geq 1).$$

For $x < 1$, continuity implies $\mathbb{P}(Y = 1, \tau_x \geq 1) = 0$, hence

$$p_0 = x\mathbb{P}(\tau_x < 1).$$

Since $\mathbb{P}(M \geq x) = \mathbb{P}(\tau_x < 1)$, we obtain

$$\mathbb{P}(M \geq x) = \frac{p_0}{x}, \quad x \in [p_0, 1).$$

Therefore, $F_M(x) = \mathbb{P}(M \leq x) = 1 - \mathbb{P}(M \geq x) = 1 - p_0/x$ for $x \in [p_0, 1)$.

When $x = 1$, we have $M = 1$ if and only if $Y = 1$, so

$$\mathbb{P}(M \geq 1) = \mathbb{P}(M = 1) = p_0,$$

and therefore $F_M(1) = \mathbb{P}(M \leq 1) = 1$. □

B.2 Conditional Distribution

Theorem 8 (Continuous-Path Conditional)**.** The cumulative distribution function of M conditional on the terminal event $Y = 0$ satisfies

$$F_{M|Y=0}(x) = \begin{cases} 0 & \text{for } x < p_0, \\ 1 - \left(\frac{p_0}{1-p_0}\right) \cdot \left(\frac{1-x}{x}\right) & \text{for } x \in [p_0, 1), \\ 1 & \text{for } x = 1. \end{cases}$$

Proof. For $x < p_0$, we have $F_{M|Y=0}(x) = 0$ since p_t starts at p_0, so $M \geq p_0 > x$ always.

For $x \in [p_0, 1)$, as in the discrete-time case, decompose the event $\{M \geq x\}$:

$$\mathbb{P}(M \geq x) = \mathbb{P}(M \geq x, Y = 0) + \mathbb{P}(M \geq x, Y = 1).$$

On $\{Y = 1\}$ we have $p_1 = 1 \geq x$ a.s., so $\{M \geq x\} \cap \{Y = 1\} = \{Y = 1\}$, and hence

$$\begin{aligned} \mathbb{P}(M \geq x) &= \mathbb{P}(Y = 1) + \mathbb{P}(M \geq x, Y = 0) \\ &= \mathbb{P}(Y = 1) + \mathbb{P}(Y = 0)\mathbb{P}(M \geq x \mid Y = 0) \\ &= p_0 + (1 - p_0)\mathbb{P}(M \geq x \mid Y = 0). \end{aligned}$$

Using Theorem 7 (which gives $\mathbb{P}(M \geq x) = p_0/x$ for $x \in [p_0, 1)$), we obtain

$$\frac{p_0}{x} = p_0 + (1 - p_0)\mathbb{P}(M \geq x \mid Y = 0),$$

and solving for $\mathbb{P}(M \geq x \mid Y=0)$ yields

$$\mathbb{P}(M \geq x \mid Y=0) = \left(\frac{p_0}{1-p_0}\right) \cdot \left(\frac{1-x}{x}\right), \quad x \in [p_0, 1).$$

Therefore, $F_{M|Y=0}(x) = \mathbb{P}(M \leq x \mid Y=0) = 1 - \mathbb{P}(M \geq x \mid Y=0) = 1 - (p_0/(1-p_0)) \cdot ((1-x)/x)$ for $x \in [p_0, 1)$.

When $x = 1$, note that if $p_t = 1$ for some $t \leq 1$, then $\mathbb{P}(Y = 1 \mid \mathcal{F}_t) = 1$ and hence $Y = 1$ almost surely on that event. Therefore on $\{Y = 0\}$ the process never attains 1, so $M < 1$ a.s., and $F_{M|Y=0}(1) = \mathbb{P}(M \leq 1 \mid Y = 0) = 1$. □

C Distribution for M_λ (Two Players)

Recall the two-player setting with win-probability martingales (p_t) and (q_t), where $q_t = 1 - p_t$ and $p_0 = \mathbb{P}(Y = 1)$; define p such that $p_0 \geq 1/2$ without loss of generality. The maximum win probability attained by the eventual loser is

$$M_\lambda \equiv \sup_{0 \leq t \leq 1} \Big(p_t \mathbf{1}\{Y=0\} + q_t \mathbf{1}\{Y=1\}\Big). \tag{10}$$

By the law of total probability,

$$F_{M_\lambda}(x) = \mathbb{P}(M_\lambda \leq x) = \mathbb{P}(M_\lambda \leq x \mid Y=0)(1-p_0) + \mathbb{P}(M_\lambda \leq x \mid Y=1)p_0. \tag{11}$$

From the conditional distributions of the previous section, when team A loses we have

$$F_{M_\lambda|Y=0}(x) = \mathbb{P}(M_\lambda \leq x \mid Y=0) = \begin{cases} 0 & \text{for } x < p_0, \\ 1 - \left(\frac{p_0}{1-p_0}\right)\left(\frac{1-x}{x}\right) & \text{for } x \in [p_0, 1), \\ 1 & \text{for } x = 1. \end{cases} \tag{12}$$

Applying the same formula with p_0 replaced by $1 - p_0$ gives

$$F_{M_\lambda|Y=1}(x) = \mathbb{P}(M_\lambda \leq x \mid Y=1) = \begin{cases} 0 & \text{for } x < 1-p_0, \\ 1 - \left(\frac{1-p_0}{p_0}\right)\left(\frac{1-x}{x}\right) & \text{for } x \in [1-p_0, 1), \\ 1 & \text{for } x = 1. \end{cases} \tag{13}$$

Assume $p_0 \geq 1/2$ (team A is favored), so $p_0 \geq 1 - p_0$. Substituting (12) and (13) into (11) gives three regions:

1. For $0 \leq x < 1 - p_0$, neither (12) nor (13) contributes, so $F_{M_\lambda}(x) = 0$.
2. For $1 - p_0 \leq x < p_0$, only (13) contributes:

$$\begin{aligned} F_{M_\lambda}(x) &= (1-p_0) \cdot 0 + p_0 \cdot \left[1 - \left(\frac{1-p_0}{p_0}\right)\left(\frac{1-x}{x}\right)\right] \\ &= p_0 - (1-p_0)\left(\frac{1-x}{x}\right) = 1 - \frac{1-p_0}{x}. \end{aligned}$$

3. For $x \geq p_0$, both (12) and (13) contribute:

$$\begin{aligned}
F_{M_\lambda}(x) &= (1-p_0)\left[1-\left(\frac{p_0}{1-p_0}\right)\left(\frac{1-x}{x}\right)\right] + p_0\left[1-\left(\frac{1-p_0}{p_0}\right)\left(\frac{1-x}{x}\right)\right] \\
&= \left[(1-p_0) - p_0\left(\frac{1-x}{x}\right)\right] + \left[p_0 - (1-p_0)\left(\frac{1-x}{x}\right)\right] \\
&= 1 - \frac{1-x}{x} = \frac{2x-1}{x} = 2 - \frac{1}{x}.
\end{aligned}$$

Summarizing, the piecewise cumulative distribution function is:

$$F_{M_\lambda}(x) = \mathbb{P}(M_\lambda \leq x) = \begin{cases} 0 & \text{for } 0 \leq x < 1-p_0, \\ 1 - \frac{1-p_0}{x} & \text{for } 1-p_0 \leq x < p_0, \\ 2 - \frac{1}{x} & \text{for } p_0 \leq x < 1, \\ 1 & \text{for } x = 1. \end{cases}$$

D Distribution for M_ω (n Players)

Consider an n-player game with win-probability martingales $(p_t^{(i)})_{0 \leq t \leq 1}$ for $i \in \{1, \dots, n\}$, where $Y^{(i)} = \mathbf{1}\{\text{player } i \text{ wins}\}$ and $p_0^{(i)} = \mathbb{P}(Y^{(i)} = 1)$. Define the eventual-winner minimum

$$M_\omega \equiv \inf_{0 \leq t \leq 1} \sum_{i=1}^{n} p_t^{(i)} \mathbf{1}\{Y^{(i)} = 1\}. \tag{14}$$

On the event $\{Y^{(i)} = 1\}$, the winner's path is $(p_t^{(i)})$, so

$$M_\omega = \inf_{0 \leq t \leq 1} p_t^{(i)} \quad \text{on } \{Y^{(i)} = 1\}.$$

Since $\inf_t p_t^{(i)} \leq x$ if and only if $\sup_t (1 - p_t^{(i)}) \geq 1 - x$, the conditional distribution follows from Theorem 8 applied to the complementary martingale $(1 - p_t^{(i)}) = \mathbb{P}(Y^{(i)} = 0 \mid \mathcal{F}_t)$ with initial value $1 - p_0^{(i)}$:

$$\begin{aligned}
\mathbb{P}(M_\omega \leq x \mid Y^{(i)} = 1) &= \mathbb{P}\left(\sup_{0 \leq t \leq 1} (1 - p_t^{(i)}) \geq 1 - x \,\middle|\, Y^{(i)} = 1\right) \\
&= \begin{cases} 0 & \text{for } x < 0, \\ \left(\frac{1-p_0^{(i)}}{p_0^{(i)}}\right)\left(\frac{x}{1-x}\right) & \text{for } x \in [0, p_0^{(i)}), \\ 1 & \text{for } x \geq p_0^{(i)}. \end{cases}
\end{aligned} \tag{15}$$

Finally, by the law of total probability,

$$\begin{aligned}
F_{M_\omega}(x) = \mathbb{P}(M_\omega \leq x) &= \sum_{i=1}^{n} \mathbb{P}(M_\omega \leq x \mid Y^{(i)} = 1)\mathbb{P}(Y^{(i)} = 1) \\
&= \sum_{i=1}^{n} \mathbb{P}(M_\omega \leq x \mid Y^{(i)} = 1) p_0^{(i)}.
\end{aligned} \tag{16}$$

Substituting (15) into (16) yields the piecewise distribution reported in the main text:

$$F_{M_\omega}(x) = \mathbb{P}(M_\omega \leq x) = \begin{cases} 0 & \text{for } x < 0, \\ \sum_{i: x \geq p_0^{(i)}} p_0^{(i)} + \left(\frac{x}{1-x}\right) \sum_{i: x < p_0^{(i)}} (1 - p_0^{(i)}) & \text{for } x \in [0, \max_i p_0^{(i)}), \\ 1 & \text{for } x \geq \max_i p_0^{(i)}. \end{cases} \tag{17}$$

Supplementary Material

Supplementary Material. Supplement to "The Blown Lead Paradox: Conditional Laws for the Running Maximum of Binary Doob Martingales". It contains additional proofs, simulation details, and empirical diagnostics referenced in the main text.

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