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The three-body problem in the spherical geometry

by

Jesus David Prada Gonzalez

Advisor: PhD Alonso Botero

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Abstract

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by Jesus David Prada Gonzalez

In this document an insightful formalism for the study of the planar three-body problem by Botero and Leyvraz is reproduced with the objective to be mapped to its spherical counterpart. We deduced that a straightforward analogue is not trivial and studied the advantages and flaws of the Cartesian, spherical and stereographic coordinates in terms of feasibility of the aforesaid mapping. We discovered that the most suitable set of coordinates, with which a very similar analogue of Botero's formalism can be deduced, is the stereographic projection, due to its nature and the obtained results. However, the spherical analogue of the studied planar three-body problem was demonstrated to be not obvious nor simple. We leave the proper study of this map for future work.

Full document in english can be found on:
https://github.com/jdprada1760/Thesis/blob/master/Document/Thesis.pdf

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Resumen

Facultad de ciencias
Departamento de Física

por Jesus David Prada Gonzalez

En este documento se reproduce un formalismo para estudiar el problema de tres cuerpos en el plano, desarrollado por Botero y Leyvraz [1], con el objetivo de traducirlo al caso esférico. Deducimos que un análogo directo no es trivial al intentar esta traducción en coordenadas cartesianas, esféricas y estereográficas. En cada caso expusimos las ventajas y las falencias de cada sistema de coordenadas en téminos de viabilidad de esta traducción. Deducimos que si buscamos un mapeo de dicho formalismo, que no cambie mucho su esencia, el mejor sistema de coordenadas para estudiar el problema es la proyección estereográfica, dada su naturaleza y los resultados obtenidos. De todas maneras, como el problema demostró no ser obvio ni simple, dejamos el estudio de la apropiada traducción del formalismo como trabajo futuro.

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Contents

Chapter 1

Introduction

The N-body problem is a highly-known and studied issue in Physics. It consists on researching the trajectories that N point masses would follow when interacting with external and internal forces with certain defined characteristics, given all the information of the initial conditions.

At first, the principal interest in the study of this problem was the exact prediction of the path of celestial bodies. However, as the problem was known more, it was understood that its study is of great importance not only for astrophysics but for the theoretical comprehension of classical mechanics. Great minds of physics and mathematics have worked in the restricted problem of three bodies, as Poincaré [2] and Jacobi [3]. It was this way that Poincaré, in an attempt of solving the three body problem, discovered that it is not integrable in general, and formulated the bases of what is known nowadays as chaos theory [2].

Independently of the formalism chosen to define the system, the N-body problem is reduced to the integration of the equations of motion for the N particles. As this problem has been known to be non-integrable for the $N \geq 3$ cases, with the exception of few occurrences that involve forces with strange features as explicit dependence of the position an velocities [4], the study of realistic 3 body problems is a very interesting question in physics.

Regarding quantum mechanics, there are many puzzling aspects about its fundamentals, as the role that symmetries play in the quantization of states. In this case, the analysis of N-body problems may be useful to get some intuition about the meaning and cause of those perplexing facts, given the strong correlation between classical and quantum mechanics via the Hamiltonian formalism.

1.1 Motivation

One of the particular realistic cases of the three body problem that is known to be integrable, is the system of three particles on the plane with mass m and charge e under the influence of a large constant magnetic field perpendicular to the surface and forces whose potentials are invariant under rotations and translations in the plane [1].

This problem is integrable by virtue of the action of the big magnetic potential, which decouples the movement of the particle into two degrees of freedom known as guiding centres and linear momenta. Further analysis of the movement of the particles can be performed in terms of a canonical transformation that encodes the shape of the generated triangle into a Bloch sphere.

The analysis carried out in [1] can be extended to the quantum formalism via the canonical quantization rules [6], where the role of the $SU(2)$ symmetry becomes clear with the implementation of a Schwinger angular momentum associated with the Bloch sphere variables. The classical decoupling of the guiding centres degrees of freedom is associated with the fact that, in the quantum formalism, electrons are confined to the ground state of energy due to the Landau level gap modulated by the magnetic field, an effect that is at the basis of the Quantum Fractional Hall Effect (QFHE). The clear relation between both quantum and classical formalisms, together with the fact that the spherical QFHE shows similar behaviour than the planar case on the large magnetic field regimen, tempts us to intend a generalization of the procedure for the spherical geometry.

To do so, as a first approach to the problem, we are going reproduce the calculations and analysis from [1] for the classical and quantum system in great detail. Then, we are going to study some important aspects of the classical one-body problem on the sphere under the influence of a magnetic monopole to obtain some intuition about the analogies of the movement of the particle in this case. Then, an analogue formalism to study the motion of the three-body problem in the sphere is going to be intended, giving some insights as why the translation of the analysis from the plane to the sphere is not trivial.

Chapter 2

The three body problem in the plane

In this chapter a classic approach of a somehow general case of the three body problem in 2 dimensions is going to be presented. This problem was developed by Botero and Leyvraz in [1]. It will give some necessary intuition to develop the analogous problem in the spherical geometry. To begin with, the problem is going to be described in great detail; then its integrability is going to be proven; and finally, a brief description of the movement of the particles is going to be presented.

2.1 The definition of the problem

The three-body problem presented here is that of three particles of electrical charge e and mass m confined to a plane, under the influence of a strong magnetic field perpendicular to it and forces whose potentials satisfy translational and rotational symmetries in the plane. To make the problem more interesting, we consider a harmonic potential with frequency ω.

Given this information, the Hamiltonian associated with this system has the form:

$$H = \sum_{i=1}^{3} \frac{1}{2m} \left\| \vec{p_i} - e\vec{A}\left(\vec{r_i}\right) \right\|^2 + V\left(\vec{r_1}, \vec{r_2}, \vec{r_3}\right) + \frac{m\omega}{2} \sum_{i=1}^{3} \left\| \vec{r_i} \right\|^2 , \tag{2.1}$$

where $\vec{r_i} = x_i\hat{\imath} + y_i\hat{\jmath}$, $\vec{p_i} = p_{x_i}\hat{\imath} + p_{y_i}\hat{\jmath}$ and $\vec{A}(\vec{r})$ is the magnetic vector potential on its symmetric gauge, which satisfies $\nabla \times \vec{A} = B\hat{k}$.

Besides, the potential $V(\vec{r_1}, \vec{r_2}, \vec{r_3})$ satisfies the symmetries:

$$V(R\vec{r_1} + \vec{a}, R\vec{r_2} + \vec{a}, R\vec{r_3} + \vec{a}) = V(\vec{r_1}, \vec{r_2}, \vec{r_3}), \tag{2.2}$$

for any rotation R about $\hat{k}$ and any translation $\vec{a}$ in the plane.

Here it is important to note that a magnetic field that is parallel to the plane will have no effect on the dynamics of the system. In fact, one can choose a convenient gauge in which the vector potential has the only non-null component along the z axis. This way, given the restriction $\dot{z} = 0$, there will be no change in the Lagrangian function.

2.2 The canonical transformation of the guiding centres

For the proof of integrability for this system, and for further analysis of the trajectories of the particles, let us perform the well-known transformation of the guiding centres.

This transformation is defined by the following two equations:

$$\vec{\pi_i} = \vec{p_i} - e\vec{A}(\vec{q_i}), \tag{2.3}$$

$$\vec{R_i} = \vec{r_i} - \frac{\hat{k} \times \vec{\pi_i}}{eB}. \tag{2.4}$$

The equation (2.3) passes from the canonical momentum $\vec{p_i}$ to the linear momentum $\vec{\pi_i}$, which is much more intuitive and understandable; while the equation (2.4) transforms the general position $\vec{r_i}$ to the position of the instantaneous guiding center $\vec{R_i}$.

In a system without the interaction potentials, the electrically charged particles are known to perform the circular motion of the cyclotron with radii that depend on the initial linear momenta. In this case, the guiding centres would be constant in time as would be the linear momenta. However, with the introduction of an interacting potential, the momentum of each particle may vary making the guiding centre change too, which is why the instantaneous interpretation of the guiding centres is necessary.

Now, let us calculate the Poisson brackets for this new set of coordinates for a specific particle:

$$
\begin{aligned}
\{\pi_1, \pi_2\} &= \frac{\partial \pi_1}{\partial r_\alpha}\frac{\partial \pi_2}{\partial p_\alpha} - \frac{\partial \pi_2}{\partial r_\alpha}\frac{\partial \pi_1}{\partial p_\alpha} \\
&= -e\delta_{\alpha 2}\frac{\partial A_1}{\partial r_\alpha} + e\delta_{\alpha 1}\frac{\partial A_2}{\partial r_\alpha} \\
&= -e\frac{\partial A_1}{\partial y} + e\frac{\partial A_2}{\partial x} \\
&= e(\nabla \times \vec{A})_3 = eB,
\end{aligned}
$$

$$
\begin{aligned}
\{R_1, R_2\} &= \{r_1, r_2\} + \left\{r_1, -\frac{\pi_1}{eB}\right\} + \left\{\frac{\pi_2}{eB}, r_2\right\} + \left\{\frac{\pi_2}{eB}, -\frac{\pi_1}{eB}\right\} \\
&= \frac{1}{eB}\left(\cancelto{-1}{\{p_1, r_1\}} - \cancelto{0}{e\{A_1, r_1\}} + \cancelto{-1}{\{p_2, r_2\}} \cancelto{0}{e\{A_2, r_2\}}\right) + \frac{eB}{(eB)^2} \\
&= \frac{-2}{eB} + \frac{1}{eB} = -(eB)^{-1},
\end{aligned}
$$

$$
\begin{aligned}
\{R_1, \pi_2\} &= \{r_1, \pi_2\} + \cancelto{0}{\left\{\frac{\pi_2}{eB}, \pi_2\right\}} \\
&= \cancelto{0}{\{r_1, p_2\}} - e\cancelto{0}{\{r_1, A_2\}} \\
&= \{R_2, \pi_1\} = 0.
\end{aligned}
$$

This Poisson brackets can be generalised to the transformation for the three particles. Taking $i, j = \{1, 2, 3\}$ and $\alpha, \beta = \{1, 2\}$:

$$\{\pi_{i,\alpha}, \pi_{j,\beta}\} = (eB)\,\delta_{ij}\epsilon_{\alpha\beta}, \tag{2.5}$$

$$\{R_{i,\alpha}, R_{j,\beta}\} = -\,(eB)^{-1}\,\delta_{ij}\epsilon_{\alpha\beta}, \tag{2.6}$$

$$\{R_{i,\alpha}, \pi_{j,\beta}\} = 0. \tag{2.7}$$

Equations (2.5)-(2.6) allow us to identify the proposed transformation as canonical. However, this is not the usual canonical transformation where the position coordinates and the momentum coordinates are canonical conjugates. In this special case, one component of the momentum is canonical conjugate with the other momentum coordinate, and similarly for the guiding center position components.

Now, with a very large magnetic field, if the potential of the interaction forces does not vary abruptly in space, we can use the approximation $\vec{R}_i \approx \vec{q}_i$ to average the potentials over the guiding centres; that is, we can replace $\vec{q}_i$ for $\vec{R}_i$ in $V(\vec{q}_1, \vec{q}_2, \vec{q}_3)$.

We can support the last approximation as follows: In the cyclotron problem, the radius of the circular motion described is proportional to the linear momentum and inversely proportional to the magnetic field. Then, in the presence of a big magnetic field B, the radius of the cyclotron would shrink to a very small size. Regarding the case we are working with, the radii of the instantaneous cyclotron motions would be proportional to $\left\|(\hat{k} \times \vec{\pi}_i)(eB)^{-1}\right\| = \left\|\vec{\pi}_i(eB)^{-1}\right\|$ and its frequency to $\sqrt{B}$. As the potential V does not vary abruptly in the radii scale, the averaging of this motion over the guiding centres means that this potential does not sense that circular motion. Moreover, given the big frequency of the cyclotrons and the scale of variance of the potential, the scale of time of the local circular motions is far smaller than that of the motion of the guiding centres. Therefore, we can ignore the instantaneous quality of the circular motion, and take it as constant in a scale of time small enough for the motion of the guiding centres. In this sense we say that the coordinates for the guiding centres decouple from those of the linear momenta of the particles.

Before replacing the new set of coordinates in the Hamiltonian, it is necessary to do a scale transformation to obtain the proper Poisson brackets for the formal definition of canonical transformation; that is:

$$
\begin{aligned}
\vec{\pi}_i &\to (eB)^{-1/2}\vec{\pi}_i \\
\vec{R}_i &\to \sqrt{eB}\vec{R}_i.
\end{aligned}
$$

With this consideration, the Hamiltonian of the system in the new set of rescaled coordinates is given by:

$$H = \sum_{i=1}^{3} \frac{eB}{2m} \|\vec{\pi}_i\|^2 + V\left((eB)^{-1/2}\vec{R}_1, (eB)^{-1/2}\vec{R}_2, (eB)^{-1/2}\vec{R}_3\right) + \frac{m\omega}{2eB}\sum_{i=1}^{3}\left\|\vec{R}_i\right\|^2 . \quad (2.8)$$

This Hamiltonian, given equation (2.7), can be decomposed in a Hamiltonian that describes the movement of the guiding centres, and other that describes de movement of the linear momenta. In one hand, the Hamiltonian for the linear momenta is easily identified with the harmonic oscillator, whereas the one that characterises the movement of the guiding centres needs a deeper analysis.

2.3 Integrability of the system

As the Hamiltonian describing the trajectories of the linear momenta of the particles is that of an harmonic oscillator, this part of the problem is integrable and its solutions are widely known. The guiding centre Hamiltonian, in turn, needs to be analysed more deeply. For this purpose, let us take the following convention:

$$H_{gc} = \frac{\omega}{2}\sum_{i=1}^{3}\left\|\bar{x}^2\right\| + \left\|\bar{y}^2\right\| + V\left(\bar{x}, \bar{y}\right), \quad (2.9)$$

where $\bar{x} = (x_1, x_2, x_3)$ and $\bar{y} = (y_1, y_2, y_3)$, being x_i, y_i the rescaled coordinates of the guiding centres of the particles. For simplicity, the potential V and the frequency ω have been rescaled to take into account the scale transform of the coordinates and maintain the original form of the Hamiltonian:

$$\frac{eB\omega}{m} \to \omega$$
$$V\left(\frac{\bar{x}}{\sqrt{eB}}, \frac{\bar{y}}{\sqrt{eB}}\right) \to V\left(\bar{x}, \bar{y}\right).$$

Clearly, the scaled potential V still has the symmetries expressed in the equation (2.2). Furthermore, with this scale transformation the Poisson brackets take the form:

$$\{y_i, x_j\} = \delta_{ij}. \quad (2.10)$$

Now that the guiding center Hamiltonian has been expressed in terms of the proper canonical set of coordinates, the fastest way to prove the integrability of the system is via the Liouville-Arnol'd theorem [7, Sect. 49]. For this theorem, it is only necessary to find 2 more independent integrals in involution (besides the Hamiltonian).

To get these 2 integrals, let us exploit the symmetries of the guiding centres Hamiltonian. We then define the generators of translations and rotation in the plane, which are symmetries of the potential:

$$\begin{aligned} T_x &= \sum_{i=1}^{3} x_i \\ T_y &= \sum_{i=1}^{3} y_i \end{aligned} , \tag{2.11}$$

$$R_z = \frac{1}{2} \sum_{i=1}^{3} \left(x_i^2 + y_i^2 \right) . \tag{2.12}$$

It is easily verifiable that these are indeed the symmetry generators. To see that, take the first order infinitesimal transformations of translations and rotations on the plane:

$$x_i \to x_i + \epsilon,$$

$$y_i \to y_i + \epsilon,$$

$$(x_i, y_i) \to (x_i + \epsilon y_i, y_i - \epsilon x_i) .$$

Now note that for the infinitesimal translations, the potential of the transformed coordinates is related to the potential of the normal coordinates by a directional derivative, which can be identified with the Poisson bracket of the potential V and each generator:

$$\begin{aligned} 0 = V(\bar{x} + \epsilon, \bar{y}) - V(\bar{x}, \bar{y}) = \epsilon \sum_{i=1}^{3} \frac{\partial V(\bar{x}, \bar{y})}{\partial x_i} = \epsilon \{V, T_x\} = 0 \\ 0 = V(\bar{x}, \bar{y} + \epsilon) - V(\bar{x}, \bar{y}) = \epsilon \sum_{i=1}^{3} \frac{\partial V(\bar{x}, \bar{y})}{\partial y_i} = \epsilon \{T_y, V\} = 0. \end{aligned}$$

For the infinitesimal rotation, the relation is analogous:

$$0 = V\left(\bar{x} + \epsilon\bar{y}, \bar{y} - \epsilon\bar{y}\right) - V\left(\bar{x}, \bar{y}\right) = \frac{\partial V\left(\bar{x}, \bar{y}\right)}{\partial x_i}\left(\epsilon y_i\right) - \frac{\partial V\left(\bar{x}, \bar{y}\right)}{\partial y_i}\left(\epsilon x_i\right) = \epsilon\left\{V, R_z\right\}.$$

Therefore, we conclude that the generators of translations and rotations in the plane commute with the potential V due to its symmetries. Besides, the generator of rotations is exactly equal to the harmonic-like part of the guiding center Hamiltonian which validates that R_z is other integral in involution. The generators of translations are not integrals in involution for they do not commute with the harmonic potential. However, we can calculate a quantity in terms of these generators, which already commute with the potential V, to make it commute with the remaining part of H_{gc}:

$$L \coloneqq \frac{1}{6}\left(T_x^2 + T_y^2\right). \tag{2.13}$$

This new quantity L clearly commutes with the potential V because the Poisson bracket is a linear differential operator in one component and it obeys the Leibniz rule. Moreover, it also commutes with the rotation generator $J \coloneqq R_z$:

$$\begin{aligned}
\left\{T_x^2 + T_y^2, J\right\} &= \sum_{i,j,k}\left\{x_i x_j + y_i y_j, x_k^2 + y_k^2\right\} \\
&= \sum_{i,j,k}\left\{x_i x_j, y_k^2\right\} + \left\{y_i y_j, x_k^2\right\} = \sum_{i,j,k} y_k\left\{x_i x_j, y_k\right\} + x_k\left\{y_i y_j, x_k\right\} \\
&= \sum_{i,j,k} y_k x_i \delta_{jk} + y_k x_j \delta_{ik} - x_k y_i \delta_{jk} - x_k y_j \delta_{ik} \\
&= 2\sum_{i,j} x_i y_j - x_i y_j = 0.
\end{aligned}$$

As we found L as the last integral in involution, we conclude, by the Liouville-Arnol'd theorem, that the subsystem of guiding centres is integrable by quadratures.

2.4 The spinorial representation and the Bloch sphere mapping

Taking the motion integrals obtained in the previous section for the guiding center Hamiltonian (2.9) the motion of the particles can be broken down in that of the center of mass, and the relative trajectories of the particles. To see that, note that as $T_x^2 + T_y^2$ in (2.13) is an integral in involution with the Hamiltonian, it is a conserved quantity. The conservation of this term is clearly interpreted as a circular motion of the center of mass of the system $\left(\frac{T_x}{3}, \frac{T_y}{3}\right)$.

The latter analysis can be clarified in terms of the decoupling of the guiding center trajectories from the general motion of the particles. In that sense, one can deduce that the guiding center movement is only affected by the interaction potential V and the external harmonic potential. This can be confirmed by the lack of magnetic terms in the guiding centre Hamiltonian in equation (2.8).

Taking this into account, as the potential V is a central potential that produces only internal forces that do not affect the movement of the centre of mass, one may assure that its motion is determined by the only term of external interaction in the Hamiltonian, the harmonic potential.

With this, the equations of motion for the center of mass of the system will be given by $\dot{T}_i = \{T_i, \omega J\}$. Now, from equations above about the commutation of L and J, it is simple to deduce:

$$
\begin{aligned}
\left\{T_x^2, \omega J\right\} &= 2T_x \left\{T_x, J\right\} = 2\omega \sum_{i,j} x_i y_j = -\left\{T_y^2, \omega J\right\} \\
&= 2\omega T_x T_y \\
\left\{T_x, \omega J\right\} &= \omega T_y \\
\left\{T_y, \omega J\right\} &= -\omega T_x,
\end{aligned}
$$

which tells us that the movement of (T_x, T_y) mass is besides circular, uniform. The problem is then reduced to the analysis of the coordinates relative to the center of mass.

To carry on with this procedure, let us take the spinor of relative position defined as follows:

$$\Psi = \frac{1}{2\sqrt{3}} \begin{pmatrix} \sqrt{3}\,(z_2 - z_1) \\ z_2 + z_1 - 2z_2 \end{pmatrix}, \tag{2.14}$$

with $z_i = x_i + iy_i$. If we define the center of mass as $T = \frac{1}{3}(T_x + iT_y) = \frac{1}{3}(z_1 + z_2 + z_3)$, we can retrieve the general position of each particle:

$$\begin{aligned} z_1 &= T + \frac{1}{\sqrt{3}}\Psi_2 - \frac{1}{3}\Psi_1 \\ z_2 &= T + \frac{1}{\sqrt{3}}\Psi_2 + \frac{1}{3}\Psi_1 \\ z_3 &= T - \frac{2}{\sqrt{3}}\Psi_2. \end{aligned} \tag{2.15}$$

The spinor components were chosen to satisfy the canonical commutation relations:

$$\begin{aligned} \{\Psi_\alpha, \Psi_\alpha\} &= 0 \\ \{\Psi_1, \Psi_2\} = \{\Psi_1^*, \Psi_2^*\}^* &= \left\{\frac{1}{2}(z_2 - z_1), \frac{1}{2\sqrt{3}}(z_2 + z_1 - 2z_3)\right\} \\ &= i\left\{\frac{1}{2}(x_2 - x_1), \frac{1}{2\sqrt{3}}(y_2 + y_1 - 2y_3)\right\} \\ &+ i\left\{\frac{1}{2}(y_2 - y_1), \frac{1}{2\sqrt{3}}(x_2 + x_1 - 2x_3)\right\} = 0, \end{aligned}$$

$$\begin{aligned} \{\Psi_1, \Psi_2^*\} &= \{\Psi_1^*, \Psi_2\}^* = -\frac{i}{2\sqrt{3}}\{x_2 - x_1, y_2 + y_1 - 2y_3\} = 0 \\ \{\Psi_1, \Psi_1^*\} &= \frac{-2i}{4}\{x_2 - x_1, y_2 - y_1\} = -i \\ \{\Psi_2, \Psi_2^*\} &= \frac{-i}{6}\{x_1 + x_2 - 2x_3, y_1 + y_2 - 2y_3\} = -i, \end{aligned}$$

which can be summarized:

$$\{\Psi_\alpha, \Psi_\beta\} = \left\{\Psi_\alpha^*, \Psi_\beta^*\right\} = 0 \tag{2.16}$$

$$\{\Psi_\alpha^*, \Psi_\beta\} = i\delta_{\alpha,\beta}. \tag{2.17}$$

This spinorial representation of the relative coordinates of the particles with respect to the center of mass is analogous to the transformation performed to analyse the two body problem, only adapted to take into account one more particle and one less dimension. This spinor, besides representing a canonical transformation of the guiding centres, has norm equal to the spin quantity $S = J - L$ which is also an integral of motion:

$$\begin{aligned}
\Psi^\dagger\Psi &= \|\Psi_1\|^2 + \|\Psi_2\|^2 \\
&= \frac{1}{4}\left(\|z_1\|^2 + \|z_2\|^2 - 2\,\mathrm{Re}\left(z_1 z_2^*\right)\right) + \frac{1}{12}\left(\|z_1\|^2 + \|z_2\|^2 + \|z_3\|^2 + \mathrm{Re}\left(2z_1 z_2^* - 4z_1 z_3^* - 4z_2 z_3^*\right)\right) \\
&= \frac{1}{3}\sum_{i=1}^{3} x_i^2 + y_i^2 - \frac{1}{3}\sum_{i>j} x_i x_j + y_i y_j \\
&= \frac{1}{2}\sum_{i=1}^{3} x_i^2 + y_i^2 - \frac{1}{6}\left(T_x^2 + T_y^2\right) \\
&= J - L = S.
\end{aligned}$$

This quantity S can be easily interpreted as proportional to the moment of inertia of the triangle rotating about its center of mass, which gives more intuition to the interpretation of S as spin momentum:

$$\begin{aligned}
I = m\sum_{i=1}^{3}\|\vec{r_i} - \vec{r}_{cm}\|^2 &= m\sum_{i=1}^{3}\|\vec{r_i}\|^2 + 3\,\|\vec{r}_{cm}\| - 2\vec{r}_{cm}\cdot\sum_{i=1}^{3}\vec{r_i} = m\left(2J - 3\,\|\vec{r}_{cm}\|\right) \\
&= m\left(2J - \frac{1}{3}\left(T_x^2 + T_y^2\right)\right) = m\left(2J - 2L\right) = 2mS.
\end{aligned}$$

On the other hand, it is observed that any phase multiplication to the spinor Ψ is equivalent to a rotation about the center of mass or about the origin, as they have the same effect on relative vectors $\vec{r_i} - \vec{r_j}$:

$$\begin{aligned}
e^{i\phi}z &= \left(\cos\phi + i\sin\phi\right)\left(x + iy\right) = \left(x\cos\phi - y\sin\phi\right) + i\left(y\cos\phi + x\sin\phi\right) \\
\Psi &= \frac{1}{2\sqrt{3}}\begin{pmatrix}\sqrt{3}\left(z_2 - z_1\right) \\ z_2 + z_1 - 2z_3\end{pmatrix} = \frac{1}{2\sqrt{3}}\begin{pmatrix}\sqrt{3}\left(z_2 - z_1\right) \\ z_2 - z_3 + z_1 - z_3\end{pmatrix} \\
e^{i\phi}\Psi &= \frac{1}{2\sqrt{3}}\begin{pmatrix}\sqrt{3}e^{i\phi}\left(z_2 - z_1\right) \\ e^{i\phi}\left(z_2 - z_3\right) + e^{i\phi}\left(z_1 - z_3\right)\end{pmatrix}.
\end{aligned}$$

Moreover, scale transformations will be clearly given by a scalar multiplication. Consequently, a general transformation performed to the triangle relative coordinates can be represented by a scalar and a phase, in other words, a complex number. Now, taking this into account, we can take the equivalence class of shapes associated with a spinor Ψ as $[\Psi] = \{z\Psi | z \in \mathbb{C}/\{0\}\}$. Here the equivalence class $[\Psi]$ represents a shape of the triangle formed by the particles, and together with the norm of the spinor, it codifies the actual information needed of the triangle for the Hamiltonian of guiding centres (2.8).

In fact, if we take the space of normalised representatives $\Psi/\sqrt{S}$ we can note that the Hilbert spaces associated to the shapes of the triangles and to the wave vectors of a Qubit are equivalent, and can be mapped to a Bloch sphere. To comprehend better this mapping, let us define the vector consisting of the expected value of the Pauli matrices $\vec{\sigma}$ in terms of the equivalence class representatives:

$$\vec{\zeta} = \frac{1}{S}\Psi^{\dagger}\vec{\sigma}\Psi. \tag{2.18}$$

The interpretation of each component of this vector becomes clear with some heavy algebra. Here we present the result in terms of the position vectors $\vec{r}_i$ of each particle:

$$\begin{aligned}
\zeta_1 &= \frac{1}{2\sqrt{3}S}\left(\|\vec{r}_2 - \vec{r}_3\|^2 - \|\vec{r}_1 - \vec{r}_3\|^2\right) \\
\zeta_2 &= \frac{1}{\sqrt{3}S}\left(\vec{r}_1 \times \vec{r}_2 + \vec{r}_2 \times \vec{r}_3 + \vec{r}_3 \times \vec{r}_1\right) \cdot \hat{z} = \frac{2A}{\sqrt{3}S} \\
\zeta_3 &= \frac{1}{6S}\left(2\|\vec{r}_2 - \vec{r}_1\|^2 - \|\vec{r}_3 - \vec{r}_1\|^2 - \|\vec{r}_3 - \vec{r}_2\|^2\right).
\end{aligned}$$

From this, the second component of the vector can be easily interpreted as proportional to the signed area A of the triangle whose sign encodes the chirality of the system.

The first and third components, in turn, share a similar composition with the components of the vector Ψ, mapping the complex quantities z_i that encode the vertices coordinates, to the squared norm of the opposite side of the triangle. In other words, we can obtain expressions for the squared lengths of the sides of the triangle as we did in equations (2.15):

$$S = \frac{1}{3}\sum_{i=1}^{3} x_i^2 + y_i^2 - \frac{1}{3}\sum_{i>j} x_i x_j + y_i y_j = \frac{1}{6}\left(\|\vec{r}_1 - \vec{r}_2\|^2 + \|\vec{r}_2 - \vec{r}_3\|^2 + \|\vec{r}_3 - \vec{r}_1\|^2\right),$$

$$
\begin{aligned}
\rho_1 &:= \|\vec{r}_2 - \vec{r}_3\|^2 = 2S\left(1 + \frac{\sqrt{3}}{2}\zeta_1 - \frac{1}{2}\zeta_3\right) \\
\rho_2 &:= \|\vec{r}_3 - \vec{r}_1\|^2 = 2S\left(1 - \frac{\sqrt{3}}{2}\zeta_1 - \frac{1}{2}\zeta_3\right) \\
\rho_3 &:= \|\vec{r}_2 - \vec{r}_1\|^2 = 2S\left(1 + \zeta_3\right).
\end{aligned}
\tag{2.19}
$$

The information carried by the three components of the vector $\vec{\zeta}$ and the scalar S presented before, can then be compressed elegantly in this fashion:

$$
\begin{aligned}
\rho_k &= 2S\left(1 + \vec{m}_k \cdot \vec{\zeta}\right) \qquad (1 \le k \le 3) \\
\vec{m}_k &= \left(\sin\frac{2\pi k}{3}, 0, \cos\frac{2\pi k}{3}\right) \\
A &= \frac{\sqrt{3}S}{2}\zeta_2.
\end{aligned}
\tag{2.20}
$$

Now, to know better the distribution of shapes in the Bloch sphere, let us take the coordinate ζ_2 as the vertical axis which defines the poles and the rest as the ones defining the remaining perpendicular plane. In this terms, the pole vectors represent triangles of maximal area, while the equatorial line represents triangles of minimal null area. The triangles from the north hemisphere differ from their specular image with respect to the equatorial plane in the south hemisphere by only a sign in the area, in other words, they have the same shape but different chirality.

Moreover, isosceles triangles require the vector $\vec{\zeta}$ to be perpendicular to $\vec{m}_i - \vec{m}_j$ for some $i \neq j$. Given that $\{\vec{m}_i\}$ define an equilateral triangle on the equatorial plane, the previous requirement is equivalent to $\vec{\zeta}$ being parallel to some $\vec{m}_i$. Besides, given that the vectors $\vec{m}_i$ lie in the equatorial plane, it can be deduced that any isosceles triangle can be found on the lines on the sphere connecting the poles with the vectors $\vec{m}_i$. From this, it is clear that the polar triangles are equilateral.

Taking the analysis a little bit further, it can be seen from equation (2.20), that directions $\vec{\zeta} = \vec{m}_k$ on the equatorial plane of the Bloch sphere describe triangles where none of its sides ρ_k are null. Hence, those directions will determine isosceles triangles with null area where two of its sides sum up to the other. On the other hand, if we look at directions $\vec{\zeta} = -\vec{m}_k := \vec{n}_k$, we find that one of the sides will be null. We then find on those directions, isosceles triangles with one of the sides equal to zero, which accounts for its null area.

One may be curious about the direction of equilateral triangles with null area; that is, all the points in the same position. The answer is that we cannot find them in the Bloch sphere because, as we stated earlier, it is a representation for normalised vectors $\Psi/\sqrt{S}$ and this triangle would require $S = 0$. However, if we take the normalisation factor S as the radius of the sphere, and consider the family of Bloch spheres for different radii filling $\mathbb{R}^3/\{0\}$, we surely would find these triangles in any direction on the limit $S \to 0$.

2.5 Analysis of the motion

Taking into account the decoupling of the center of mass movement with respect to the relative coordinates, which are encoded in the canonical spinor Ψ, the addition of terms that depend only on the center of mass components to the Hamiltonian will not affect the equations of motion of the relative coordinates. We can then add a term $-\omega L$ from the Hamiltonian in equation (2.9) with no effect on Ψ:

$$\begin{aligned} H_\Psi &= H_{gc} - \omega L = V(\Psi, \Psi^*) + \omega(J - L) = V(\Psi, \Psi^*) + \omega S \\ &= V(\Psi_\alpha, \Psi^*_\alpha) + \omega \Psi_\alpha \Psi^*_\alpha. \end{aligned}$$

With this new Hamiltonian for relative coordinates, one may obtain the temporal evolution of the spinor Ψ, taking the Poisson bracket with it, the time evolution generator. However, to express the Poisson bracket in terms of the new spinorial coordinates, we can reformulate the canonical relations in (2.17) to resemble more the classical relations of x and p_x:

$$\{\Psi_1, i\Psi_1^*\} = \{\Psi_2, i\Psi_2^*\} = 1.$$

With this clarification, we can take the usual Poisson bracket expressions with the canonical variables $(\Psi_\alpha, i\Psi^*_\alpha)$:

$$\{f,g\} = \frac{1}{i}\sum_{\alpha=1}^{2}\left(\frac{\partial f}{\partial \Psi_\alpha}\frac{\partial g}{\partial \Psi_\alpha^*} - \frac{\partial f}{\partial \Psi_\alpha^*}\frac{\partial g}{\partial \Psi_\alpha}\right).$$

And the temporal evolution of Ψ_α takes the simple form:

$$i\dot{\Psi}_\alpha = i\left\{\Psi_\alpha, H_\Psi\right\} = \frac{\partial V}{\partial \Psi_\alpha^*} + \omega\Psi_\alpha.$$

We can go further in this analysis if we take into account the Bloch sphere mapping. We then can pass from the spinorial representation given by (Ψ, Ψ^*) to the shape space given by $(\vec{\zeta}, S)$. In this fashion, the time evolution of the spinor component Ψ_α will be given by:

$$\begin{aligned}
i\dot{\Psi}_\alpha &= \frac{\partial V}{\partial \Psi_\alpha^*} + \omega\Psi_\alpha \\
&= \frac{\partial V}{\partial S}\frac{\partial S}{\partial \Psi_\alpha^*} + \frac{\partial V}{\partial \zeta_j}\frac{\partial \zeta_j}{\partial \Psi_\alpha^*} + \omega\Psi_\alpha \\
&= \omega\Psi_\alpha + \frac{\partial V}{\partial S}\frac{\partial}{\partial \Psi_\alpha^*}\left(\Psi_\beta^*\Psi_\beta\right) + \frac{\partial V}{\partial \zeta_j}\frac{\partial}{\partial \Psi_\alpha^*}\left(\frac{1}{S}\Psi_\delta^*\sigma_{\delta\beta}^j\Psi_\beta\right) \\
&= \omega\Psi_\alpha + \frac{\partial V}{\partial S}\Psi_\alpha + \frac{\partial V}{\partial \zeta_j}\left(\frac{-1}{S^2}\Psi_\delta^*\sigma_{\delta\beta}^j\Psi_\beta\frac{\partial S}{\partial \Psi_\alpha^*} + \frac{1}{S}\sigma_{\alpha\beta}^j\Psi_\beta\right) \\
&= \left(\omega\Psi + \frac{\partial V}{\partial S}\Psi + \frac{1}{S}\frac{\partial V}{\partial \zeta_j}\left(-\zeta_j\Psi + \sigma^j\Psi\right)\right)_\alpha .
\end{aligned}$$

This result can be compressed taking into account all components of Ψ:

$$i\dot{\Psi} = \left(\omega + \frac{\partial V}{\partial S} + \frac{1}{S}\frac{\partial V}{\partial \zeta_j}\left(-\zeta_j\mathbb{I} + \sigma^j\right)\right)\Psi. \tag{2.21}$$

Now, as we know that S is a constant of movement, the time evolution of the components ζ_j of the vector of shapes will be given by:

$$\dot{\zeta}_j = \frac{1}{S}\left(\dot{\Psi}^\dagger \sigma^j \Psi + \Psi^\dagger \sigma^j \dot{\Psi}\right) = \frac{1}{S}\left(\dot{\Psi}^\dagger {\sigma^j}^\dagger \Psi + \Psi^\dagger \sigma^j \dot{\Psi}\right)$$

$$= \frac{2}{S}\Re\left\{\Psi^\dagger \sigma^j \dot{\Psi}\right\}.$$

From equation (2.21) this expression can be simplified:

$$\frac{1}{S}\Psi^\dagger \sigma^k \dot{\Psi} = -i\left(\omega + \frac{\partial V}{\partial S} - \frac{1}{S}\frac{\partial V}{\partial \zeta_j}\zeta_j\right)\frac{1}{S}\Psi^\dagger \sigma^k \dot{\Psi} - i\frac{1}{S}\Psi^\dagger \sigma^k \left(\frac{1}{S}\frac{\partial V}{\partial \zeta_j}\sigma^j\right)\Psi$$

$$\frac{2}{S}\Re\left\{\Psi^\dagger \sigma^k \dot{\Psi}\right\} = 2\Re\left\{-i\frac{1}{S}\frac{\partial V}{\partial \zeta_j}\frac{1}{S}\Psi^\dagger \sigma^k \sigma^j \Psi\right\}$$

$$\dot{\zeta}_k = -i\frac{1}{S}\frac{\partial V}{\partial \zeta_j}\frac{1}{S}\Psi^\dagger \sigma^k \sigma^j \Psi + i\frac{1}{S}\frac{\partial V}{\partial \zeta_j}\frac{1}{S}\Psi^\dagger \sigma^j \sigma^k \Psi$$

$$= -i\left(\frac{1}{S}\frac{\partial V}{\partial \zeta_j}\right)\frac{1}{S}\Psi^\dagger\left[\sigma^k, \sigma^j\right]\Psi,$$

where $[a, b] = ab - ba$ is the commutator between a and b.

It is widely known that the Pauli matrices $\vec{\sigma}$ satisfy the commutation relations $[\sigma^i, \sigma^j] = 2i\epsilon_{ijk}\sigma_k$, that are in fact the angular momentum relations. Taking this into account, we finally obtain an expression for the time evolution of the shape vector $\vec{\zeta}$:

$$\dot{\zeta}_k = 2\epsilon_{kjl}\left(\frac{1}{S}\frac{\partial V}{\partial \zeta_j}\right)\frac{1}{S}\Psi^\dagger \sigma_l \Psi = 2\epsilon_{kjl}\left(\frac{1}{S}\frac{\partial V}{\partial \zeta_j}\right)\zeta_l$$

$$\dot{\vec{\zeta}} = \frac{2}{S}\left(\nabla_{\vec{\zeta}} V\right) \times \vec{\zeta}. \tag{2.22}$$

Having the time evolution for the vector $\vec{\zeta}$ and knowing the potential V, the shape of the triangle can be easily retrieved from the relations in (2.19). Regarding the orientation of the triangle, the analysis requires more work because when passing from the spinor representation (Ψ, Ψ^*) to the shape space $(\vec{\zeta}, S)$ we loose the information of the phase of the spinor. This phase, as we deduced before, encodes the rotation (orientation) of the triangle and is not important when analysing shapes.

To retrieve the orientation information of the triangle, we would like to obtain a definition of phase for infinitesimally separated spinors that would correspond to an infinitesimal evolution in time of the trajectories. To achieve this, let us take an infinitesimal rotation in the spinorial representation and compare with the infinitesimal separation of the spinor:

$$
\begin{aligned}
e^{-id\chi}\Psi &\approx \Psi - id\chi\Psi = \Psi + d\Psi \\
d\chi\Psi &= id\Psi \\
d\chi\Psi^{\dagger}\Psi &= i\Psi^{\dagger}d\Psi \\
d\chi &= i\frac{\Psi^{\dagger}d\Psi}{\Psi^{\dagger}\Psi}.
\end{aligned}
$$

Now, if we identify Ψ with the state of the triangle at a time t and $\Psi + d\Psi$ with a time $t + dt$, we can obtain an expression for the dynamical angular velocity of the triangle:

$$
\begin{aligned}
\omega_r^{(dyn)} &:= \frac{d\chi}{dt} = i\frac{\Psi^{\dagger}\dot{\Psi}}{\Psi^{\dagger}\Psi} \\
\omega_r^{(dyn)} &= \left(\omega + \frac{\partial V}{\partial S} - \cancel{\frac{1}{S}\frac{\partial V}{\partial \zeta_j}\zeta_j}\right) + \cancel{\left(\frac{1}{S}\frac{\partial V}{\partial \zeta_j}\right)\frac{\Psi^{\dagger}\sigma^{j}\Psi}{\Psi^{\dagger}\Psi}} \\
\omega_r^{(dyn)} &= \omega + \frac{\partial V}{\partial S}.
\end{aligned}
\tag{2.23}
$$

From the equation (2.23) for the dynamical angular velocity of the triangle, one may define a phase between two finite-time separated spinors in the trajectory by simple integration. However, if the initial and final shapes are not the same, the interpretation of this phase as an angle of rotation of the triangle is not possible.

If we then define the phase $\Delta\chi = \int_0^{T_s} \omega_r^{(dyn)} dt$ over a period T_s of the shape motion, as the initial and final shapes must be equal over this period, one would expect that $\Delta\chi$ may be interpreted as the angle of rotation of the shape. This is wrong as long as a geometrical phase or Hannay angle [8] must be taken into account due to the change of shape of the triangle in the period. To see this more clearly, let us parametrise the spinor Ψ:

$$\Psi = \sqrt{S} e^{-i\gamma} \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{-i\phi}\sin\frac{\theta}{2} \end{pmatrix}.$$

This parametrisation is chosen so that the angles (θ, ϕ) correspond to spherical angular coordinates. The vector $\vec{\zeta}$ under this parametrisation is then given by:

$$\vec{\zeta} = \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix}.$$

Then, if we apply the definition of $\omega_r^{(dyn)}$ we obtain:

$$\begin{aligned} \omega_r^{(dyn)} &= i\frac{1}{S}\Psi^\dagger\dot{\Psi} \\ &= \left(e^{i\gamma}\begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \end{pmatrix}\right) \cdot e^{-i\gamma}\left(\dot{\gamma}\begin{pmatrix} \cos\frac{\theta}{2} \\ e^{-i\phi}\sin\frac{\theta}{2} \end{pmatrix} + \begin{pmatrix} 0 \\ \dot{\phi}e^{-i\phi}\sin\frac{\theta}{2} \end{pmatrix} + \frac{\dot{\theta}}{2}\begin{pmatrix} -\sin\frac{\theta}{2} \\ e^{-i\phi}\cos\frac{\theta}{2} \end{pmatrix}\right) \\ &= \dot{\gamma} + \dot{\phi}\sin^2\frac{\theta}{2}, \end{aligned}$$

being γ the rotation phase of the spinor.

Over a period T_s one can truthfully identify $\Delta\gamma$ with the angle of rotation of that shape, obtaining then the overall angular velocity $\omega_r = \Delta\gamma/T_s$. With this clarification, the expression for ω_r in terms of the dynamical and geometrical angular velocities is easily deduced:

$$\Delta\gamma = \Delta\chi - \int_0^{T_s} \dot{\phi} \sin^2 \frac{\theta}{2} dt$$

$$\omega_r = \frac{\Delta\gamma}{T_s} = \frac{1}{T_s} \int_0^{T_s} \omega_r^{(dyn)} dt - \frac{1}{T_s} \oint \sin^2 \frac{\theta}{2} d\phi$$

$$\omega_r = \left\langle \omega_r^{(dyn)} \right\rangle + \omega_r^{(geo)},$$

with $\omega_r^{(geo)} = -\frac{1}{2T_s} \oint 2\sin^2 \frac{\theta}{2} d\phi = -\frac{1}{2T_s} \Omega(E,S)$ being proportional to the solid angle $\Omega(E,S)$ on the Bloch sphere enclosed by the level curve of the shape trajectory corresponding to (S,E):

$$\begin{aligned} \Omega(E,S) &= \int_0^{2\pi} \int_0^{\theta(\phi)} \sin\theta d\theta d\phi = \oint (1-\cos\theta) d\phi \\ &= \oint 2\sin^2 \frac{\theta}{2} d\phi. \end{aligned}$$

The system may be analysed in terms of angle-action variables and with this method the frequencies deduced here can be formalised as canonical variables with unambiguous meaning. However, the purpose of this chapter is to present the 2-dimensional formalism developed by Botero et. al to describe the three body problem in presence of a strong magnetic field as a preamble and motivation to look for a similarly elegant formalism to describe the three-body problem on the sphere.

Chapter 3

Quantum analysis of the three body problem in the plane

In this chapter we continue the analysis implemented in [1], explaining in detail the quantum treatment of the three body problem in the plane. To achieve this goal, the quantum problem will be defined at the same time as the quantum analogous of the classical reduction of problem is briefly discussed. The Schwinger oscillator is going to be explained to get some insights about the quantum analogue analysis, and finally it is going to be applied to the problem.

3.1 The quantum reduction of the problem

The quantum treatment of the three-body problem on the plane is not very different from the classical approach, given that we thoroughly worked the classical system out in the Hamiltonian formalism. In fact, all the transformations and reductions from the classical approach work; however, we have to be careful when treating topics referring equations of movement. The only thing we have to do is to apply the principles of canonical quantization [6]. To make the problem of passing to the quantum formalism less complicated, let us carry the analysis in fundamental units $\hbar = 1$.

Including the last considerations, the quantum Hamiltonian is going to be the same as the one expressed in (2.1), and as canonical transformations still work in quantum mechanics, a similar reduction of the problem can be performed.

First, given the big magnetic field decoupling, the problem can be reduced to the analysis of the guiding centres motion given by the Hamiltonian (2.8). The decoupled system of linear momenta is equally identified as a quantum harmonic oscillator, which is also very well known amongst the physical sciences community.

Therefore, we are left with the reduced Hamiltonian for the guiding centres (2.8) with the same definition of the canonical coordinates. The integrability analysis carried out in Chapter 2 where there were found two integrals of motion in involution with the Hamiltonian is not meaningful on the quantum formalism. Actually, speaking of integrability or chaos on quantum systems is somewhat complicated since the Schrödinger equation is lineal. The analysis of the consequences of integrability of classical systems on their quantum counterparts is a topic that goes beyond the scope of this monograph.

3.2 The spinorial transformation

There are some remarkable aspects referring the canonical variables $(\bar{x}, \bar{y})$ associated with (2.8). With these one can form creation-annihilation operators given by:

$$
\begin{aligned}
a_j &= \frac{1}{\sqrt{2}}(x_j + iy_j) \\
a_j^\dagger &= \frac{1}{\sqrt{2}}(x_j - iy_j) \\
\left[a_j, a_k^\dagger\right] &= \delta_{kj}.
\end{aligned}
$$

We then note that the variables called z_i in the classical formalism will correspond to annihilation operators in the quantum formalism.

Following the reduction of the problem analogously to the classical approach, we can proceed to decouple the center of mass coordinates form the relative coordinates. The spinor Ψ is known to follow relations of creation-annihilation operators (taking into account the canonical quantization). However, the center of mass was not formalised as a canonical variable in the classical approach. To take this into account, we choose the constants to make it also an annihilation operator. If we encode the center of the triangle in b, the decoupling transformation can be expressed in the following way:

$$\begin{pmatrix} b \\ \Psi_1 \\ \Psi_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ 1_3 \end{pmatrix},$$

where the new variables satisfy the creation-annihilation relations:

$$\begin{aligned} \left[\Psi_\alpha, \Psi_\beta^\dagger\right] &= \delta_{\alpha\beta} \\ \left[b, b^\dagger\right] &= 1 \\ \left[b, \Psi_\alpha^\dagger\right] &= [b, \Psi_\alpha] = 0. \end{aligned}$$

From this transformation it is worth noticing:

$$\begin{aligned} b^\dagger b &= \frac{1}{3}\left(\sum_i a_i^\dagger\right)\left(\sum_j a_j\right) = \frac{1}{6}\left(\sum_i x_i - iy_i\right)\left(\sum_j x_j + iy_j\right) \\ &= \frac{1}{6}\sum_i\sum_j (x_i - iy_i)(x_j - iy_j) = \frac{1}{6}\sum_i\sum_j x_i x_j + y_i y_j + i\cancelto{i\delta_{ij}}{[x_i, y_j]} \\ &= L - \frac{1}{2}, \end{aligned}$$

$$\begin{aligned} \Psi^\dagger\Psi &= \Psi_1^\dagger\Psi_1 + \Psi_2^\dagger\Psi_2 \\ &= \frac{1}{2}(a_2^\dagger - a_1^\dagger)(a_2 - a_1) + \frac{1}{6}(a_2^\dagger + a_1^\dagger - 2a_3^\dagger)(a_2 + a_1 - 2a_3) \\ &= \frac{1}{2}(x_2 - x_1 - i(y_2 - y_1))(x_2 - x_1 + i(y_2 - y_1)) \\ &+ \frac{1}{6}(x_2 + x_1 - 2x_3 - i(y_2 + y_1 - 2y_3))(x_2 + x_1 - 2x_3 + i(y_2 + y_1 - 2y_3)) \\ &= \frac{1}{2}(x_2 - x_1)^2 + (y_2 - y_1)^2 + i\cancelto{2i}{[x_2 - x_1, y_2 - y_1]} \\ &+ \frac{1}{6}(x_2 + x_1 - 2x_3)^2 + (y_2 + y_1 - 2y_3)^2 + i\cancelto{3i}{[x_2 + x_1 - 2x_3, y_2 + y_1 - 2y_3]} \\ &= S - 1. \end{aligned}$$

So far we have reduced the problem to the study of the center of mass and relative coordinates spinor. In these terms, the Hamiltonian for these variables will be given by:

$$
\begin{aligned}
H_{gc} &= V\left(\Psi, \Psi^{\dagger}\right) + \omega(S + L) \\
&= V\left(\Psi, \Psi^{\dagger}\right) + \omega(\Psi^{\dagger}\Psi + b^{\dagger}b + \frac{3}{2}) \\
&= V\left(\Psi, \Psi^{\dagger}\right) + \omega(\Psi_1^{\dagger}\Psi_1 + \Psi_2^{\dagger}\Psi_2 + 1) + \omega(b^{\dagger}b + \frac{1}{2}).
\end{aligned}
$$

Ignoring the effects of the center of mass b, what we have here is the Hamiltonian of two oscillators (Ψ_1, Ψ_2) coupled by the potential V which does not affect the center of the triangle. What we will see next is that in fact, the two oscillators from Ψ induce a Schwinger angular momentum [9].

In quantum mechanics the symmetries of a system play a substantial role in the solution of the associated problem. Actually, the role of symmetries in quantum mechanics is more important than in classical mechanics for it is more sensitive to the internal structure of the symmetry groups [10]. In quantum mechanics they determine the algebra of operators, and consequently, the eigenvectors and eigenvalues that generate the states of the Hilbert space. In this case, we know that the potential V follows the symmetries associated with the special Euclidean group in 2D ($SE(2)$), that, given the simplification of the problem and the canonical spinor variables, is transformed into $SU(2)$.

The role of $SU(2)$ can be understood as the operators Ψ_α implement the mentioned Schwinger oscillator [9] associated with this symmetry.

3.3 The Schwinger oscillator and the angular momentum representation

To understand better the role of the symmetry of Ψ in the problem, let us explain the theory associated with the Schwinger oscillators. In a nutshell, the Schwinger formalism demonstrates that the annihilation-creation operator algebra associated to two decoupled quantum oscillators induce a $SU(2)$ algebra of angular momentum.

To understand this, let us first consider two decoupled harmonic oscillators with the same frequency ω. The Hamiltonian of this system in terms of the annihilation-creation operators (a_1, a_2) of each oscillator, will be given by:

$$H_{SO} = \omega \left(a_1^\dagger a_1 + a_2^\dagger a_2 + 1 \right).$$

As the oscillators are decoupled, we can define the excitation number hermitian operators $n_i = a_i^\dagger a_i$. Some important commutation relations between these operators are then given by:

$$\begin{aligned}
\left[a_i, a_j^\dagger\right] &= \delta_{ij} \\
[a_i, a_j] &= 0 \\
[n_i, a_j] &= \left[n_i, a_j^\dagger\right] = [n_i, n_j] = 0.
\end{aligned}$$

Now we are going to show that this algebra of annihilation operators induces an angular momentum algebra with symmetry $SU(2)$. To do that, let us define an angular momentum in terms of the $\{a_i\}$ and see its commutation relations:

$$\begin{aligned}
J_i &= \frac{1}{2}\sigma^i_{\alpha\beta} a_\alpha^\dagger a_\beta \\
J_1 &= \frac{1}{2}\left(a_2^\dagger a_1 + a_1^\dagger a_2 \right) \\
J_2 &= \frac{i}{2}\left(a_2^\dagger a_1 - a_1^\dagger a_2 \right) \\
J_3 &= \frac{1}{2}\left(n_1 - n_2 \right),
\end{aligned} \tag{3.1}$$

$$\begin{aligned}
[J_i, J_j] &= \frac{1}{4}\left[\sigma^i_{\alpha\beta} a_\alpha^\dagger a_\beta, \sigma^j_{\gamma\delta} a_\gamma^\dagger a_\delta\right] = \frac{1}{4}\sigma^i_{\alpha\beta}\sigma^j_{\gamma\delta}\left[a_\alpha^\dagger a_\beta, a_\gamma^\dagger a_\delta\right] \\
&= \frac{1}{4}\sigma^i_{\alpha\beta}\sigma^j_{\gamma\delta}\left(a_\alpha^\dagger [a_\beta, a_\gamma^\dagger] a_\delta + a_\gamma^\dagger [a_\alpha^\dagger, a_\delta] a_\beta \right) \\
&= \frac{1}{4}\left(\sigma^i_{\alpha\beta}\sigma^j_{\beta\delta} a_\alpha^\dagger a_\delta - \sigma^i_{\alpha\beta}\sigma^j_{\gamma\alpha} a_\gamma^\dagger a_\beta \right) = \frac{1}{4} a_\alpha^\dagger a_\beta \left[\sigma^i, \sigma_j\right]_{\alpha\beta} \\
&= i\epsilon_{ijk}\frac{1}{4}\sigma^k_{\alpha\beta} a_\alpha^\dagger a_\beta,
\end{aligned}$$

$$[J_i, J_j] = i\epsilon_{ijk} J_k.$$

Then, as well as with any other angular momentum, one can define the operators $J^\pm = J_1 \pm J_2$ and obtain, based on these, the rules for the quantised numbers (j, m) that define the eigenvalues of (J^2, J_3). Then, let us calculate J^2 in terms of the hermitian excitation number operators (n_1, n_2):

$$
\begin{aligned}
J^2 &= J_1 J_1 + J_2 J_2 + J_3 J_3 \\
&= \frac{1}{4}((n_1 - n_2)^2 + (-\cancel{a_2^\dagger a_1 a_2^\dagger a_1} - \cancel{a_1^\dagger a_2 a_1^\dagger a_2} + n_1 a_2 a_2^\dagger + n_2 a_1 a_1^\dagger) \\
&+ (\cancel{a_2^\dagger a_1 a_2^\dagger a_1} + \cancel{a_1^\dagger a_2 a_1^\dagger a_2} + n_1 a_2 a_2^\dagger + n_2 a_1 a_1^\dagger)) \\
&= \frac{1}{4}\left((n_1 - n_2)^2 + 2n_1(n_2+1) + 2n_2(n_1+1)\right) = \frac{1}{4}\left((n_1+n_2)(n_1+n_2+4)\right) \\
&= \frac{n_1+n_2}{2}\left(\frac{n_1+n_2}{2} + 1\right).
\end{aligned}
$$

Now, comparing this result with the eigenvalues $j(j+1)$ of J^2 it is clear that $j = \frac{n_1+n_2}{2}$. Furthermore, the eigenvalues m of J_3 are given by $m = \frac{n_1-n_2}{2}$. As a consequence, given that $n_1 + n_2 = 1, 2, ...$, we conclude that the quantum number j is going to be some semi-integer $j = \frac{1}{2}, 1, \frac{3}{2}, ...$. These results lead us to conclude that the symmetry group associated with this angular momentum algebra is SU(2).

To sum up, given the proceedings in this section, the algebra of creation-annihilation operators that characterise a set of 2 decoupled harmonic oscillators, induces an angular momentum algebra with symmetry $SU(2)$.

3.4 Quantum diagonalization of the Hamiltonian

Now that we introduced some important aspects about Schwinger oscillators and its relation to $SU(2)$ symmetries, we are able to apply this theory to the analysis of our reduced problem. The only thing we have to do is to take the operators Ψ_α as the annihilation-creation operators of the decoupled harmonic oscillators. Given this, we obtain an angular momentum in accordance to equation (3.1) given by:

$$
\begin{aligned}
F_i &= \frac{1}{2}\Psi_\alpha^\dagger \sigma_{\alpha,\beta}^i \Psi_\beta \\
\vec{F} &= \frac{1}{2}\Psi^\dagger \vec{\sigma} \Psi.
\end{aligned}
$$

Certainly, as seen before, this angular momentum satisfies $[F_i, F_j] = i\epsilon_{ijk}F_k$. Moreover, this angular momentum has the same definition as the Bloch sphere shape vector $\vec{\zeta}$ with the exception of the constants, namely $\frac{2}{S}$. We can indeed relate its components to the shape of the triangle and its area in a similar way as we did in (2.20):

$$\rho_i = 2S + 4\vec{m}_i \cdot \vec{F}$$
$$A = \sqrt{3}F_2.$$

On the other hand, the operator F^2, in analogy with J^2 studied before, will be given by:

$$\begin{aligned}
F^2 &= \frac{n_1 + n_2}{2}\left(\frac{n_1 + n_2}{2} + 1\right) = \frac{\Psi_1^\dagger \Psi_1 + \Psi_2^\dagger \Psi_2}{2}\left(\frac{\Psi_1^\dagger \Psi_1 + \Psi_2^\dagger \Psi_2}{2} + 1\right) \\
&= \frac{S-1}{2}\left(\frac{S-1}{2} + 1\right) \\
&= \frac{1}{4}(S^2 - 1).
\end{aligned}$$

The $SU(2)$ algebra relation becomes clear. First, because it is already related to the symmetry of the shape vector $\vec{\zeta}$, and then because we deduced it in the Schwinger oscillator implementation.

The quantum number associated with this angular momentum $\vec{F}$ are clearly given by $(s = S, m)$ where m is an eigenvalue of some component of the shape vector $\vec{F}$. This, together with the quantum number $l = L$, characterises an eigenstate of our system H_{gc}. The eigen-energies associated with an eigenstate (l, m, s) are then given by:

$$E_{l,m,s} = V_{s,m} + \omega(s + l).$$

With this, we consider the quantum problem fully analysed, at least for the purpose of this document.

Chapter 4

The problem of a charged particle in the magnetic field of a monopole

With the objective to obtain some intuition about the N-body problem restricted to a spherical geometry, we study in this chapter the symmetries and trajectories of a charged particle under the influence of the magnetic field of a monopole. To achieve this, we present the deduction, via the Lagrangian formalism, of the so called Poincaré cone [11] that characterises the trajectory of a particle in this situation. We then extrapolate the important symmetries used in the Lagrangian formalism to the Hamiltonian formalism, to retrieve some important aspects of the classical counterparts of the known Haldane formalism [12].

We deduce in this chapter that the Poincaré cone confinement, together with the restriction to the spherical surface, results in the particles describing uniform circular motion on the sphere, in analogy with the case of the plane. This suggests that a similar analysis in terms of guiding centres can be performed in the spherical problem.

4.1 Definition of the Lagrangian

Let L be the Lagrangian of a charged particle of charge $-e$ and mass m, under the influence of a magnetic monopole of magnitude g. Then L takes the form:

$$L\left(\vec{r},\dot{\vec{r}}\right) = \frac{m}{2}\left\|\dot{\vec{r}}\right\|^2 - e\vec{A}_{\hat{u}}(\vec{r}) \cdot \dot{\vec{r}}, \tag{4.1}$$

where $\vec{A}(\vec{r})$ is the vector potential of the magnetic monopole with singularity along the direction defined by the unit vector $\hat{u}$. This singularity arises from our inability to reproduce the magnetic field of the monopole with the electromagnetic theory, and it has no physical meaning [12]. In addition, the different vector potentials identified by different unit vectors $\hat{u}$ are related by gauge transformations which leave the trajectories invariant. This family of vector potentials is given by [13]:

$$\vec{A}_{\hat{u}}(\vec{r}) = \frac{g}{r}\frac{\hat{u}\times\hat{r}}{1+\hat{u}\cdot\hat{r}}. \tag{4.2}$$

4.2 The symmetries and its conserved quantities

The first thing to note in the previously defined Lagrangian is its time independence, which yields to the conservation of the Jacobi integral:

$$\frac{\partial L}{\partial \dot{\vec{r}}}\cdot\dot{\vec{r}} - L = m\left\|\dot{\vec{r}}\right\|^2 = m\left\|\dot{\vec{r_0}}\right\|^2,$$

with $\dot{\vec{r_0}}$ the initial velocity of the particle. Here, the Jacobi integral clearly represents the kinetic energy of the particle, which is conserved because constant magnetic fields do no work.

Now, due to the simplicity of the problem, one can expect some other symmetries. The next symmetry presented here is not associated with the Lagrangian, but rather with the action invariance due to the associated transformation involving the time variable. If we define the action as in the equation (4.3), it can be seen that it may be invariant under a proper scale transform of position and time. It is not difficult to find this transform, and it is presented in equation (4.4).

$$S = \int_{t_1}^{t_2} L\left(\vec{r},\dot{\vec{r}}\right)dt = \int_{t_1}^{t_2}\left(\frac{m}{2}\left\|\dot{\vec{r}}\right\|^2 - e\vec{A}_{\hat{u}}(\vec{r})\cdot\dot{\vec{r}}\right)dt, \tag{4.3}$$

$$\begin{aligned} \vec{r}' &= e^s\vec{r} \\ t' &= e^{2s}t. \end{aligned} \tag{4.4}$$

As a result of this symmetry, by the general form of Noether's theorem, there must be a conserved quantity implied. To calculate it we prefer the method stated in [14, 2.19

Noether's Thm]; however, to apply this method, the Lagrangian must be parametrised to include the time t as a generalised coordinate.

To achieve this, note that:

$$\dot{\vec{r}} = \frac{d\vec{r}}{dt} = \frac{\frac{d\vec{r}}{d\tau}}{\frac{dt}{d\tau}} := \frac{\mathring{\vec{r}}}{\mathring{t}}.$$

Then the action can be written in terms of the new parameter τ:

$$S = \int_{t_1}^{t_2} L\left(\vec{r}, \dot{\vec{r}}\right) dt = \int_{\tau_1}^{\tau_2} \mathring{t} L\left(\vec{r}, \mathring{\vec{r}}\mathring{t}^{-1}\right) d\tau = \int_{\tau_1}^{\tau_2} L'\left(\vec{r}, \mathring{\vec{r}}, \mathring{t}\right) d\tau,$$

which, by analogy with equation (4.3), gives the new parametrised Lagrangian L':

$$L'\left(\vec{r}, \mathring{\vec{r}}, \mathring{t}\right) = \mathring{t} L\left(\vec{r}, \mathring{\vec{r}}\mathring{t}^{-1}\right) = \frac{m}{2\mathring{t}} \left\|\mathring{\vec{r}}\right\|^2 - e\vec{A}_{\hat{u}}(\vec{r}) \cdot \mathring{\vec{r}}.$$

As we converted the symmetry of the action S in a symmetry of the Lagrangian L', the invariance given by equation (4.4) becomes clear. Now, using Noether's theorem [14], we obtain the conserved quantity according to the transformation (4.4):

$$G = \frac{\partial L'}{\partial \mathring{q}_i} \frac{\partial q_i'}{\partial s}\bigg|_{s=0} = -m \left\| \frac{\mathring{\vec{r}}}{\mathring{t}} \right\|^2 t + \left(\frac{m\mathring{\vec{r}}}{\mathring{t}} - e\vec{A}_{\hat{u}} \right) \cdot \vec{r}$$

$$G = m\dot{\vec{r}_0} \cdot \vec{r_0} = -m \left\|\dot{\vec{r}}\right\|^2 t + m\dot{\vec{r}} \cdot \vec{r}.$$

Working the previous conserved quantity one can obtain an equation for the magnitude of the position in function of time:

$$2\dot{\vec{r}} \cdot \vec{r} = \frac{d\left\|r\right\|^2}{dt} = 2\left\|\dot{\vec{r_0}}\right\|^2 t + 2\dot{\vec{r_0}} \cdot \vec{r_0}$$

$$r^2 = r_0^2 + 2\dot{\vec{r_0}} \cdot \vec{r_0} t + \left\|\dot{\vec{r_0}}\right\|^2 t^2,$$

$$r^2 = \left\| \vec{r_0} + \dot{\vec{r_0}} t \right\|^2 . \tag{4.5}$$

From equation (4.5) it is important to note that the only way the radius of the particle stays constant is that the initial velocity of the particle is zero, otherwise, the time term will always contribute to the change of that radius. Furthermore, as constant magnetic fields only affect perpendicular velocities, one expect equation (4.5) to be fulfilled not only in norm, but in direction when the initial velocity $\dot{\vec{r_0}}$ is parallel to the initial radius $\vec{r_0}$. We then retrieve the formula for uniform motion with no acceleration. To verify this suspicion, let us analyse other quantities associated with the symmetries of the problem.

Taking our attention to the usual symmetries on the original Lagrangian L in equation (4.1) , we note that once chosen a unit vector $\hat{u}$ for the vector potential $\vec{A}_{\hat{u}}$, L is invariant under rotations around said unit vector $\hat{u}$. To obtain the conserved quantity, take following infinitesimal transformation:

$$\vec{r'} = \vec{r} + s\left(\hat{u} \times \vec{r}\right). \tag{4.6}$$

As this transformation does not include the time t, we can perform the calculation of the conserved quantity over the original Lagrangian:

$$\begin{aligned}
G_2 &= \left.\frac{\partial L}{\partial \dot{\vec{r}}} \cdot \frac{\partial \vec{r}}{\partial s}\right|_{s=0} \\
G_2 &= \left(m\dot{\vec{r}} - e\vec{A}_{\hat{u}}\right) \cdot \left(\hat{u} \times \vec{r}\right) \\
G_2 &= \left(m\vec{r} \times \dot{\vec{r}}\right) \cdot \hat{u} - eg\frac{\left\|\hat{u} \times \hat{r}\right\|^2}{\left(1 + \hat{r} \cdot \hat{u}\right)} \\
G_2 &= \left(m\vec{r} \times \dot{\vec{r}}\right) \cdot \hat{u} - eg\left(1 - \hat{r} \cdot \hat{u}\right) \\
J_{\hat{u}} &:= \left(m\vec{r} \times \dot{\vec{r}} + eg\hat{r}\right) \cdot \hat{u} = const.
\end{aligned}$$

Now, the last argument is valid for any unit vector $\hat{u}$ chosen, and this yields the conservation of the known Poincaré vector in equation (4.7).

$$\vec{J} = m\vec{r} \times \dot{\vec{r}} + eg\hat{r}. \tag{4.7}$$

This last symmetry is very meaningful because it restricts the trajectory of the particle to a cone centred in the origin with central vector $\hat{J}$. To verify this, it is only necessary to see that the radial component of the Poincaré vector is constant for all points in the trajectory, which means that the angle between $\vec{J}$ and $\vec{r}(t)$ is a constant, in other words, the path of the particle is restricted to a cone:

$$\vec{J} \cdot \hat{r} = eg.$$

Furthermore, we can deduce from the conservation of the Poincaré vector that the angular momentum $\mathbb{L} = m\vec{r} \times \dot{\vec{r}}$ of the particle is constant in magnitude and that it determines the aperture of the cone of restriction:

$$\begin{aligned} \left\|\vec{J}\right\|^2 &= \|\mathbb{L}\|^2 + (gc)^2 \\ \|\mathbb{L}\| &= const \\ \cos\theta &= \frac{\vec{J} \cdot \hat{r}}{\left\|\vec{J}\right\|} = \sqrt{\frac{(ge)^2}{\|\mathbb{L}\|^2 + (ge)^2}}. \end{aligned} \tag{4.8}$$

From this equations it can be seen that the angle of aperture of the Poincaré cone is zero when the angular momentum $\mathbb{L}$ cancel, which means that the particle performs rectilinear motion, as deduced before from the other symmetries of the problem.

It is important to note here that if we restrict the trajectories of the particle to be in a sphere, we cannot carry a scale transform, hence we would not obtain the radius trajectory described in equation (4.5). However, a rotation is consistent with the norm conservation of the restriction to the sphere, and consequently, we can obtain the Poincaré's vector invariance. The cone confinement together with the restriction of the trajectories to a constant radius would result in the particle describing a circular trajectory on the sphere.

Moreover, in the sphere restriction, the position vector and the velocity vector must be perpendicular, therefore, as the norm of the position is the constant radius of the sphere, the conservation of the angular momentum would result in the conservation of the linear velocity, meaning that the circular motion of the particle is in fact uniform, in analogy with the planar problem.

Another important aspect that can be deduced from the set of equations (4.8), is that if we choose a magnetic monopole with big charge g compared to the angular momentum (in proper units), the angle of the Poincaré cone would tend to zero. In the case of the particle restricted to the sphere, this would mean that the radius of the circular motion would also tend to zero, as happens with the already studied case of the particle in the plane. This gives us some clues as where to look for the analogue guiding center formalism in the case of the magnetic monopole.

4.3 Important quantities in the Hamiltonian formalism

From the analysis carried out before in the Lagrangian formalism, we can obtain some important quantities that are useful to describe the movement of particles in the presence of a magnetic monopole, some of which are associated with certain symmetries of the problem. In the Hamiltonian formalism, we would like to study specially the angular momentum $\mathbb{L} = m\vec{r} \times \dot{\vec{r}}$ and the Poincaré vector $\vec{J} = \left(m\vec{r} \times \dot{\vec{r}}\right) + eg\hat{r}$. To do that, let us first calculate the Hamiltonian for the particle in the magnetic field of a monopole, this time restricting the radius r of the sphere to a constant:

$$H\left(\vec{r}, \vec{p}\right) = \frac{1}{2m} \left\| \vec{p} + e\vec{A}_{\hat{u}}(\vec{r}) \right\|^2 \Bigg|_{S^2}. \tag{4.9}$$

As the Hamiltonian is the Legendre transform of the Lagrangian, we obtain the generalised momentum $\vec{p}$ in terms of the velocity of the particle. Moreover, we can see that the Hamiltonian is just the kinetic energy of the particle:

$$\vec{p} = \frac{\partial L}{\partial \dot{\vec{r}}} = m\dot{\vec{r}} - e\vec{A}_{\hat{u}} := \vec{\pi} - e\vec{A}_{\hat{u}}$$

$$H = \frac{1}{2m} \left\| \vec{\pi} \right\|^2 \Bigg|_{S^2}.$$

From here we can propose the quantity $\mathbb{L} = \vec{r} \times \vec{\pi}$ as a canonical angular momentum. To verify this, it is useful to calculate first some Poisson brackets related to the linear momentum $\vec{\pi} = \vec{p} + e\vec{A}$, as it was done in the first chapter:

$$
\begin{aligned}
\{\pi_i, \pi_j\} &= \frac{\partial \pi_i}{\partial r_l}\frac{\partial \pi_j}{\partial p_l} - \frac{\partial \pi_j}{\partial r_l}\frac{\partial \pi_i}{\partial p_l} \\
&= e\delta_{lj}\frac{\partial A_i}{\partial r_l} - e\delta_{li}\frac{\partial A_j}{\partial r_l} = e\left(\delta_{lj}\delta_{mi} - \delta_{li}\delta_{mj}\right)\frac{\partial A_m}{\partial r_l} \\
&= -e\epsilon_{ijk}\epsilon_{lmk}\frac{\partial A_m}{\partial r_l} \\
&= -e\epsilon_{ijk}(\nabla \times \vec{A})_k = e\epsilon_{ijk}B_k = -\frac{eg}{r^3}\epsilon_{ijk}r_k,
\end{aligned}
$$

$$
\{\pi_i, r_j\} = \frac{\partial \pi_i}{\partial r_l}\cancelto{0}{\frac{\partial r_j}{\partial p_l}} - \frac{\partial r_j}{\partial r_l}\frac{\partial \pi_i}{\partial p_l} = -\delta_{ij}.
$$

Then the algebra becomes easier for $\mathbb{L}$

$$
\begin{aligned}
\{L_i, L_j\} &= \{\epsilon_{iab}r_a\pi_b, \epsilon_{jcd}r_c\pi_d\} = \epsilon_{iab}\epsilon_{jcd}\{r_a\pi_b, r_c\pi_d\} \\
&= \epsilon_{iab}\epsilon_{jcd}\left(r_ar_c\{\pi_b,\pi_d\} + \pi_br_c\{r_a,\pi_d\} + r_a\pi_d\{\pi_b,r_c\} + \pi_b\pi_d\cancelto{0}{\{r_a,r_c\}}\right) \\
&= \epsilon_{iab}\epsilon_{jcd}\left(-\frac{eg}{r^3}\epsilon_{bdk}r_ar_cr_k + \pi_br_c\delta_{ad} - \pi_dr_a\delta_{bc}\right) \\
&= -\frac{eg}{r^3}\epsilon_{iab}\left(\delta_{jk}\delta_{cb} - \delta_{jb}\delta_{ck}\right)r_ar_cr_k + \left(\delta_{bj}\delta_{ic} - \cancel{\delta_{bc}\delta_{ij}}\right)\pi_br_c - \left(\delta_{aj}\delta_{ib} - \cancel{\delta_{ia}\delta_{jb}}\right)\pi_dr_a \\
&= -\frac{eg}{r^3}\left(r_j\cancelto{0}{\epsilon_{iab}r_ar_b} - \epsilon_{aji}r_ar_br_b\right) + \left(\delta_{ia}\delta_{jb} - \delta_{ib}\delta_{ja}\right)r_a\pi_b \\
&= \epsilon_{ijk}\left(\epsilon_{abk}r_a\pi_b - eg\hat{r}_k\right) = \epsilon_{ijk}\left(L_k - eg\hat{r}_k\right).
\end{aligned}
$$

Now, we can observe that $\mathbb{L}$ does not follow the canonical relations for angular momentum, which is not surprising because the momentum $\vec{\pi}$ used in the definition of $\mathbb{L}$ is not the canonical momentum $\vec{p}$. We can fix this by taking a slight variation in $\mathbb{L}$:

$$
\mathbb{J} \coloneqq \mathbb{L} + eg\hat{r}. \tag{4.10}
$$

The Poisson bracket relations for the quantity $\mathbb{J}$ are given by:

$$
\begin{aligned}
\{J_i, J_j\} &= \left\{L_i + \frac{eg}{r} r_i, L_j + \frac{eg}{r} r_j\right\} \\
&= \{L_i, L_j\} + \frac{eg}{r} \left(\{L_i, r_j\} + \{r_i, L_j\}\right) + \left(\frac{eg}{r}\right)^2 \cancelto{0}{\{r_i, r_j\}} \\
&= \epsilon_{ijk} \left(L_k - eg\hat{r}_k\right) + \frac{eg}{r} \left(-\delta_{lj} \frac{\partial \epsilon_{iab} r_a \left(p_b + eA_b\right)}{\partial p_l} + \delta_{il} \frac{\partial \epsilon_{jcd} r_c \left(p_d + eA_d\right)}{\partial p_l}\right) \\
&= \epsilon_{ijk} \left(L_k - eg\hat{r}_k\right) + \frac{eg}{r} \left(-\epsilon_{jia} r_a + \epsilon_{ijc} r_c\right) \\
&= \epsilon_{ijk} \left(L_k + eg\hat{r}_k\right) = \epsilon_{ijk} J_k.
\end{aligned}
$$

Then we deduce that $\mathbb{J}$ satisfies the canonical relations for angular momentum. It is not difficult to see that this is indeed the Poincaré vector in equation (4.7) and that, as seen before, it determines the center of the circular motion performed by the particle:

$$
\mathbb{J} = \vec{r} \times \vec{\pi} + eg\hat{r} = m\vec{r} \times \dot{\vec{r}} + eg\hat{r} = \vec{J}.
$$

Moreover, we can calculate the Poisson brackets of this angular momentum with other important vectors of the system, namely $\vec{r}$ and $\vec{L}$:

$$
\begin{aligned}
\{J_i, r_j\} &= \{L_i + eg\hat{r}_i, r_j\} = \epsilon_{iab} r_a \{\pi_b, r_j\} \\
&= \epsilon_{ijk} r_k
\end{aligned}
$$

$$
\begin{aligned}
\{J_i, L_j\} &= \left\{J_i, J_j - \frac{eg}{r} r_i\right\} = \epsilon_{ijk} J_k - \frac{eg}{r} \{L_i, r_j\} = \epsilon_{ijk}(J_k - eg\hat{r}_k) \\
&= \epsilon_{ijk} L_k.
\end{aligned}
$$

With this, we conclude that the angular momentum $\mathbb{J}$ is also the generator of rotations over the sphere.

Here it is important to clarify that this angular momentum is gauge-invariant and arises, as seen before in the Lagrangian formalism, from the rotational symmetry of the system that leaves invariant the equations of movement. This is indeed the difference between this angular momentum $\mathbb{J}$ and the canonical angular momentum $\vec{r} \times \vec{p}$.

Chapter 5

The three body problem in the sphere

Given the introduction about the one-body problem under the influence of a monopolar magnetic field, we are now able to generalize the three-body problem restricted to the sphere. As we did in chapter 2 and 3, we would like to look for a similar analysis for the three-body problem on the sphere on the large magnetic field regimen. As we will discuss later in this chapter, an analogous analysis is not trivial, however, we explain some important aspects of the problem and give some general guidelines that may, trough future work, carry to the so wanted analogous analysis.

This chapter is organized in the following way: First, the problem is going to be formally defined. Then, a brief discussion of the advantages and flaws of spherical and Cartesian coordinate systems, in the development of the analysis of the problem, is going to be presented. Finally, the stereographic projection, as an alternative coordinate system to analyse similarities with the case of the plane, is going to be introduced.

5.1 Definition of the problem

In this problem, we practically have three Hamiltonians (4.9) of free particles in a monopolar magnetic field, coupled by a potential V with the three-dimensional rotational symmetries. To make things easier, we choose the same gauge for the vector potentials associated with each particle. In these terms, the three-body Hamiltonian will have the form:

$$\begin{aligned} H &= \frac{1}{2m}\left(\|\vec{\pi}_1\|^2 + \|\vec{\pi}_2\|^2 + \|\vec{\pi}_3\|^2\right) + V\left(\vec{r}_1, \vec{r}_2, \vec{r}_3\right)\Big|_{S^2} \\ &= \frac{1}{2m}\left(\left\|\vec{p}_1 + e\vec{A}_{\hat{u}}(\vec{r}_1)\right\|^2 + \left\|\vec{p}_2 + e\vec{A}_{\hat{u}}(\vec{r}_2)\right\|^2 + \left\|\vec{p}_3 + e\vec{A}_{\hat{u}}(\vec{r}_3)\right\|^2\right) + V\left(\vec{r}_1, \vec{r}_2, \vec{r}_3\right)\Big|_{S^2}, \end{aligned} \tag{5.1}$$

where $\vec{\pi}_i = \vec{p}_i + e\vec{A}_{\hat{u}}(\vec{r}_i)$ is the linear momentum of each particle, $\vec{A}_{\hat{u}}(\vec{r})$ is defined in equation (4.2), and the potential V satisfies:

$$V\left(R\vec{r}_1, R\vec{r}_2, R\vec{r}_3\right) = V\left(\vec{r}_1, \vec{r}_2, \vec{r}_3\right),$$

for any three-dimensional rotation R.

5.2 The analogous canonical transformation of the guiding centres

Since the canonical coordinates of each particle are independent, we can define the Poincaré vector for each particle in the same way as in equation (4.10) without loosing its angular momentum Poisson relations.

$$\begin{aligned} \vec{J}_i &= \vec{r}_i \times \vec{\pi}_i + eg\hat{r}_i \\ &= \vec{r}_i \times \left(\vec{p}_i + e\vec{A}_{\hat{u}}(\vec{r}_i)\right) + eg\hat{r}_i \\ \{J_i, J_j\} &= \epsilon_{ijk} J_k. \end{aligned}$$

Now, as seen before, $\vec{J}_i$ is the generator of rotations, in this case, of the coordinate system of each particle. To obtain the generator of rotations for the general system, we simply add up the three Poincaré vectors producing $\mathbb{J}$, which will also follow the angular momentum relations.

It was previously discussed that the Poincaré vector points in the direction of the guiding center of a particle. We therefore are tempted to take this vector as the analogue of the canonical guiding center transformation performed in the plane:

$$\vec{R}_i = \frac{\vec{J}_i}{\frac{eg}{r}} = \vec{r}_i + \frac{\hat{r}_i \times \vec{\pi}_i}{\frac{eg}{r^2}}$$
$$= \vec{r}_i + \frac{\hat{r}_i \times \vec{\pi}_i}{eB}. \tag{5.2}$$

Indeed, the previous transformation (5.2) has the same form of that of the plane on equation (2.4). The problem is completely analogous: in stead of $\hat{k}$ indicating the direction of the constant magnetic field on the plane, we obtain $\hat{r}$ that is actually the direction of the magnetic field of the monopole.

This can get things complicated as $\vec{r}$ is not constant in time, but points to the position of the particle on the sphere at any time. This situation is reflected in the fact that this transformation is not canonical, but it follows the relations of angular momentum with the exemption of constants, namely $\frac{eg}{r}$.

Anyway, given taking equation (5.2), in the big magnetic field regimen, we can also make the assumption that the guiding center is approximately equal to the position of the particle, simplifying the problem in a similar way to the plane case. The next step on this problem would be to find the analogue to the linear momentum that characterises the radius of the guiding center and the position in the circle. As the guiding center variables follow the relations of angular momentum, it would be ideal to find this analogue quantity to follow the same relations.

However, the principal idea of doing the guiding center transformation is to obtain some independent canonical quantities with null Poisson brackets, with the objective of separating the former Hamiltonian into two decoupled problems under a big magnetic field B. Although it would be ideal to find the complementary quantity of the spherical guiding center to satisfy the angular momentum relations, for the purpose of the decoupling of the problem we only require this complementary quantity to satisfy the null Poisson brackets with its guiding center counterpart. Besides, in analogy with the problem on the plane, we require the kinetic part of the Hamiltonian to be a function of these new complementary quantities or at least to be separable into some function of the guiding centres and other of the remaining coordinates.

One can try to find a vector which follows these conditions. However, a little bit of thinking tells us that it can not be found at least in the Cartesian coordinate system.

The reason is that, as it was demonstrated before, the guiding center transformation produces three generators of rotations for the particular coordinate systems. In these terms, no matter what quantities we define, if they are vectors we automatically deduce that the Poisson bracket between this quantity and its respective guiding center coordinate is not null but represents the infinitesimal rotation of this vector.

We can propose a transformation that encodes the counterpart of the guiding center coordinates in some scalar quantities so that the Poisson bracket with the guiding centres becomes null, and we can then perform the separation of the problem into two different decoupled Hamiltonians. The finding of these quantities is not easy and anyway would have not helped much in the development of a sphere formalism similar to that of the plane. The problem resides in the characterisation of the guiding center coordinates as the rotation generator in the Cartesian coordinate system, which impedes a proper separation of variables.

To see this clearly, let us assume that we found some proper non-vectorial quantities $\{\vec{\Lambda}_i\}$ that commute with the quantities $\{\vec{R}_i\}$, which generate the kinetic part of the Hamiltonian. We then can separate this Hamiltonian into the guiding centres Hamiltonian and other for $\{\vec{\Lambda}_i\}$. The remaining Hamiltonian would be simply the potential V averaged over the guiding centres, and properly rescaled:

$$H_{gc} = V(\vec{J}_1, \vec{J}_2, \vec{J}_3)$$
$$\{J_{\alpha,i}, J_{\beta,j}\} = \delta_{\alpha\beta}\epsilon_{ijk}J_{\alpha,k},$$

where Greek indices label the particle, and Latin indices the coordinate.

The next step here would be to find some transformation that decouples the movement of the center of mass of guiding centres from the relative coordinates. However, the relative coordinates are vectorial quantities and would definitely not commute with the center of mass coordinates, nor between them. Moreover, the subtraction of angular momenta is not necessarily another angular momentum.

We could try the same argument as before of finding some scalar quantities to decouple the center of mas from the relative coordinates, nonetheless, we would loose the coordinate-like transformation and would not be able to easily relate the quantities with the characteristics of the triangle.

The issue with Cartesian coordinates, besides the principal quantities being the generators of rotations, is that there is no simple way to introduce the restriction to the sphere. We will always be working with more coordinates than we need. Taking this into account, an obvious alternative to the solution of this problem is the use of spherical coordinates, where we would get rid of the radial coordinate and momentum, obtaining a problem properly described by two angular coordinates.

5.3 The problem in spherical coordinates

We redefine the problem in spherical coordinates to verify if it produces a suitable set of quantities to carry out a similar analysis than that of the plane. To do this, let us calculate the Lagrangian of a single particle in the magnetic field of a monopole restricted to the spherical surface, but first, let us define the spherical coordinate transformation and some important relations:

$$
\begin{aligned}
x &= r\sin\theta\cos\phi & \hat{r} &= \frac{\partial\vec{r}}{\partial r}\Big/\left\|\frac{\partial\vec{r}}{\partial r}\right\| \\
y &= r\sin\theta\sin\phi & \hat{\theta} &= \frac{\partial\vec{r}}{\partial\theta}\Big/\left\|\frac{\partial\vec{r}}{\partial\theta}\right\| \\
z &= r\cos\theta & \hat{\phi} &= \frac{\partial\vec{r}}{\partial\phi}\Big/\left\|\frac{\partial\vec{r}}{\partial\phi}\right\| \\
\hat{r}\times\hat{\theta} &= \hat{\phi}.
\end{aligned}
$$

Here, θ is the polar angle and ϕ the azimuthal angle.

We begin with the definition of the linear velocity of a single particle in terms of the angular coordinates (θ, ϕ) and their unit vectors $(\hat{\theta}, \hat{\phi})$, which is not difficult to deduce with some algebra:

$$\dot{\vec{r}} = r\dot{\theta}\hat{\theta} + r\sin\theta\dot{\phi}\hat{\phi}. \tag{5.3}$$

Now, transforming the vector potential with the convenient gauge along the z axis, we obtain:

$$\vec{A}_{\hat{k}} = \frac{eg}{r} \tan \frac{\theta}{2} \hat{\phi}.$$

As these unit vectors are orthogonal, we obtain the following Lagrangian:

$$L = \frac{mr^2}{2} \left(\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2 \right) + \frac{eg}{r} \tan \frac{\theta}{2} \dot{\phi}.$$

With this Lagrangian, the canonical conjugate momenta are determined:

$$\begin{aligned} p_\theta &= mr^2 \dot{\theta} \\ p_\phi &= mr^2 \sin^2 \theta \dot{\phi} + \frac{eg}{r} \tan \frac{\theta}{2}. \end{aligned}$$

Then the Hamiltonian takes the form:

$$H = \frac{1}{2mr^2} \left(p_\theta^2 + \left(\frac{p_\phi - \frac{eg}{r} \tan \frac{\theta}{2}}{\sin \theta} \right)^2 \right). \tag{5.4}$$

With the linear velocity previously defined, we can then define a linear momentum vector from this Hamiltonian:

$$\begin{aligned} \vec{\pi} &= \frac{p_\theta}{r} \hat{\theta} + \frac{p_\phi - \frac{eg}{r} \tan \frac{\theta}{2}}{r \sin \theta} \hat{\phi} \\ \pi_\theta &= \frac{p_\theta}{r} \\ \pi_\phi &= \frac{p_\phi - \frac{eg}{r} \tan \frac{\theta}{2}}{r \sin \theta}. \end{aligned}$$

The purpose of this more or less long introduction to the problem in spherical coordinates was to obtain the linear momenta formula. With this expression, one can calculate the Poisson bracket between the two components (π_θ, π_ϕ) and realize that, given the dependence of π_ϕ on both canonical momenta and angular coordinates, this Poisson bracket will have a rather complicated expression that is not worth writing down. This tells us that the counterpart for the guiding center transformation on spherical coordinates will not work, at least, for the linear momentum.

One may justify that the correct analogous of the linear momenta in the plane are the conjugate angular momenta on the sphere in stead of the linear velocities. Although this argument is right and makes sense, we find that even with this consideration we obtain a set of variables that is not much different from before and possesses the same problem:

$$
\begin{aligned}
L_\theta &= p_\theta \\
L_\phi &= p_\phi - \frac{eg}{r}\tan\frac{\theta}{2}.
\end{aligned}
$$

When we calculate the Poisson brackets between these two quantities, we obtain the following result:

$$
\begin{aligned}
\{L_\theta, L\phi\} &= \cancelto{0}{\{p_\theta, p_\phi\}} - \left\{p_\theta, \frac{eg}{r}\tan\frac{\theta}{2}\right\} \\
&= \frac{eg}{r}\frac{\partial \tan\frac{\theta}{2}}{\partial\theta} \\
&= \frac{eg}{r}\frac{1}{1+\cos\theta}.
\end{aligned}
$$

While this expression is very close to something simple, as happens in the case of the plane, we can not characterise this transformation as canonical. The reason is that this Poisson bracket can get indefinitely big when $\theta \to \pi$. If we approximate this Poisson bracket to $\frac{eg}{r}$, which is valid when $\theta \to 0$, we would obtain the so wanted canonical transformation for the momentum. However, with this approximation, we are taking a region on the north pole of the sphere as if it was a plane, which is a drastic reduction of the problem.

On the other hand, regarding the transformation to the guiding center angular momentum, one can simply transform the momentum $\vec{J}$ to spherical coordinates. Certainly, the same rotational symmetries analysed on chapter 4 have to produce the same conserved quantity.

$$
\begin{aligned}
\vec{J} &= \vec{r} \times \vec{\pi} + eg\hat{r} \\
&= r\pi_\theta \cancelto{\hat{\phi}}{(\hat{r} \times \hat{\theta})} + r\pi_\phi \cancelto{-\hat{\theta}}{(\hat{r} \times \hat{\phi})} + eg\hat{r} \\
&= -r\pi_\phi \hat{\theta} + r\pi_\theta \hat{\phi} + eg\hat{r}.
\end{aligned}
$$

We then discover that the guiding center angular momentum has angular components that are the same than the linear momentum, besides a constant radial component. The Poisson bracket relations do not change at all in comparison to that of the linear momentum components. Besides not finding a nice expression for the algebra followed by the guiding center and the linear momenta, we find in the spherical coordinates a bigger problem: The linear momentum and the guiding center cannot be properly decoupled because they are practically the same.

These reasons leaded us to abandon the study of the problem in spherical coordinates.

5.4 The stereographic projection

Even though a simple analogue formalism for the sphere cannot be deduced from spherical and Cartesian coordinates, we can try one more time with an uncommon system of coordinates for spheres: the stereographic projection.

To understand the stereographic projection, let us choose one point on the surface of the sphere, that we will call the projection point. Picture one plane that is tangent to the diametrical opposite of the projection point, which will be addressed as target plane. To map any point of the sphere to the plane, draw a straight line from the projection point to the mapping point and prolong it until it intersects the target plane. It is clear that the projection point is not well defined under this transformation.

With some algebra, one can obtain the formulas for the transformation when the projection point lies at $z = -r$:

$$
\begin{aligned}
X &= \frac{x}{1+z/r} \\
Y &= \frac{y}{1+z/r}.
\end{aligned}
$$

The inverse transform is obtained easily:

$$
\begin{aligned}
x &= \frac{X}{\frac{1}{2r^2}(r^2+X^2+Y^2)} \\
y &= \frac{Y}{\frac{1}{2r^2}(r^2+X^2+Y^2)} \\
z &= \frac{\frac{1}{2r}(r^2-X^2-Z^2)}{\frac{1}{2r^2}(r^2+X^2+Y^2)}.
\end{aligned}
$$

Some aspects are remarkable about this transformation, which can be verified with a simple analysis of the equations. The first and most important of it is that this projection does not preserve areas. This is easily recognizable due to the factor $1 + z/r$ which gets arbitrarily small when $z \to -r$ producing a divergence on the differential forms (dX, dY). This non-conservation of the area is actually expected since the area of the plane is infinite, while the one of the sphere is finite.

The other important aspect of this transformation is that it conserves the azimuthal angle ϕ between the coordinates (x, y) or (X, Y), which is verifiable through the transformation equations. In fact, this is valid for all angles formed by three points on the sphere or the plane. These aspects characterise the transformation as conformal.

Now, applying the transformation directly in the Hamiltonian formalism is not simple because we do not know well how to apply it to the canonical momenta, which also keeps information from the magnetic vector potential. Therefore, we choose to apply it first to the Lagrangian of one particle, to generalize from there to the Hamiltonian formalism.

In stereographic coordinates, the Lagrangian of one particle under the presence of a magnetic monopole, is given by:

$$L = \frac{m}{2}\left(\frac{\dot{X}^2+\dot{Y}^2}{\left(\frac{1}{2r^2}\left(r^2+X^2+Y^2\right)\right)}^2\right)+\frac{eg}{r^2}\frac{X\dot{Y}-Y\dot{X}}{\frac{1}{2r^2}\left(r^2+X^2+Y^2\right)}$$
$$= \frac{m}{2}\left(\left(\frac{\dot{X}}{K}\right)^2+\left(\frac{\dot{Y}}{K}\right)^2\right)+eB\left(X\left(\frac{\dot{Y}}{K}\right)-Y\left(\frac{\dot{X}}{K}\right)\right), \tag{5.5}$$

where we defined $K = \frac{1}{2r^2}\left(r^2+X^2+Y^2\right)$. Conveniently, we chose the magnetic vector potential in a gauge with singular axis matching the stereographic projection point.

We note then that if we interpret $(\dot{X}, \dot{Y})$ as velocities on the plane, some sort of scaled velocities will be given by $(\frac{\dot{X}}{K}, \frac{\dot{Y}}{K})$. In these terms, the magnetic field, in terms of the new coordinates, will be analogue to the one on the planar case:

$$\vec{V} = \frac{1}{K}\left(\dot{X}\hat{X}+\dot{Y}\hat{Y}\right) = \frac{1}{K}\begin{pmatrix}\dot{X}\\ \dot{Y}\\ 0\end{pmatrix}$$
$$\vec{B} = B\begin{pmatrix}-Y\\ X\\ 0\end{pmatrix} = \vec{B}\times\vec{R}.$$

The scale term K accounts for the non-conservation of the area by the stereographic transformation. Although this problem is now very similar to that of the plane, the scale factor is complicated to treat. However, we can note that the scale factor depends on the distance from the origin of the stereographic plane, and is always accompanying the time variable. We presume that this can be simplified with a coordinate-time transformation in analogy with the time-radius scale symmetry studied in the previous chapter. Further analysis of this transformation is not trivial and would require more time to be studied.

We conclude this study with some remarks on the Hamiltonian formalism for the stereographic projection:

$$
\begin{aligned}
P_X &= \frac{m}{K^2}\dot{X} + \frac{eB}{A}Y \\
P_Y &= \frac{m}{K^2}\dot{Y} - \frac{eB}{A}X \\
H &= \frac{m}{2}\left(\left(\frac{\dot{X}}{K}\right)^2 + \left(\frac{\dot{Y}}{K}\right)^2\right) \\
&= \frac{1}{2m}\left((KP_X - eBY)^2 + (KP_Y + eBX)^2\right).
\end{aligned}
$$

Once again we find the scale factor in our way. We can ignore it and try to find the linear momentum on the stereographic plane in analogy with the usual case of the plane. We find that, with some heavy algebra, we obtain a short expression:

$$
\begin{aligned}
\pi_X &= KP_X - eBY \\
\pi_Y &= KP_Y + eBX \, \{\pi_X, \pi_Y\} \qquad &= -K\left(2B + \frac{1}{r^2}\left(P_Y X - P_X Y\right)\right).
\end{aligned}
$$

However, we do not confide in the stereographic canonical momenta because they can contain information about the magnetic field. We would like to study this expression in terms of the time derivatives of the coordinates, that is, replacing the canonical momenta for their Lagrangian expressions:

$$
\{\pi_X, \pi_Y\} = -B + \frac{m}{K}\left(Y\dot{X} - X\dot{Y}\right).
$$

We find then that in a regime where $B \gg m\left(Y\frac{\dot{X}}{K} - X\frac{\dot{Y}}{K}\right)$ which is a comparison between the strength of the magnetic field and the scaled angular momentum of the particle in the stereographic plane. We then find that this is a canonical set of conjugate coordinates. Nevertheless, this regime is translated to only one hemisphere or a small region near a pole of the sphere, as in the divergent hemisphere, this angular momentum is scaled and becomes arbitrarily big. This approximation is not trivial and requires deep analysis of the effects of the scale factor on the angular and linear velocities.

However, we can note that the scale factor affects specially the pole where we choose the projection point. We can take a gauge and a projection point to coincide with the initial position of the particle, which is constant. In these terms, the scale factor K will temporarily equal to 1, and we would obtain, on stereographic coordinates, a problem that is at least locally equivalent to that of the plane on chapter 2.

We can use this same trick for each particle on the three body problem, using a different gauge and a different projection points associated with each initial position. The sphere problem is also locally analogous to the one of the plane in stereographic coordinates. However, this approximation can produce huge errors when particles displace much from their initial position.

At first we started the study of the problem on stereographic coordinates due to its use in the Haldane formalism [12, 15, 5]. On some of these works [15, 5], it is explicitly expressed that the three-dimensional rotational symmetries $SO(3)$ map to the euclidean symmetries on the plane $SE(2)$ when the radius of the sphere is big enough:

$$
\begin{aligned}
L_z &\to rP_z \\
L_x &\to -rP_y \\
L_y &\to P_x.
\end{aligned}
$$

However, we consider this mapping a rough approximation of the sphere which is not useful for the purpose of this chapter.

Chapter 6

Conclusions

In this thesis we reproduced a very insightful approach developed by Botero and Leyvraz [1], to analyse the three-body problem in the plane. We showed that in the classical case, this problem is integrable due to the decoupling of the guiding centres and the linear momenta of the particles, in the regimen of big magnetic fields. Moreover, the canonical transformation of relative coordinates can be manipulated to codify the shape of the triangle in a Bloch sphere. With this analysis, the study of the orientation and shape of the triangle can be studied naturally in terms of the dynamical and geometrical angular velocity of the system, which are in fact related to the angular variables in the angle-action formalism.

In the quantum counterpart of the problem, a similar analysis was shown, where the symmetry of the system has a more noticeable role. The mentioned relative coordinate transformation implements a Schwinger quantum angular momentum $SU(2)$ connected to the Bloch sphere mapping of the triangle shapes. Moreover, the reduction of the degrees of freedom of the problem due to the big magnetic field can be compared to the quantum Hall effect on the plane, where the electron states are restricted to the ground state in this regimen.

With the study carried by Botero and Leyvraz [1], the correlation of the classic and quantum formalisms becomes more noticeable through the mapping of the quantum and classic symmetries $SU(2)$ of the relative triangular coordinates, and the connection between the classic decoupling of the guiding centres degrees of freedom and the quantum confinement of the particles to the ground state on the big magnetic field regimen. This, together with the fact that Haldane's formalism [12] of the quantum base states for Hall effect on the sphere is very similar to that of the planar quantum Hall effect, leaded us to believe that a formalism similar to Botero's and Leyvraz's on the regime of big magnetic fields could be easily developed for the sphere.

Moreover, as we analysed the one-body problem on the sphere as an introduction to the three-body formalism, we discovered that in terms of trajectories, there were many similarities that indicated the finding of the wanted analogue analysis was actually possible.

When we tried to develop the mentioned formalism on the sphere, we found that although there are some analogues for some quantities and some coincidences on the trajectories, the structure of the algebra of the guiding center transformation on the sphere is very different from that of the plane, impeding further simplification of the problem trough the crucial relative coordinate transformation.

We studied the three-body problem on the sphere in Cartesian, spherical and stereographic coordinates, all of which have their advantages and flaws. We found that if we want a very similar analysis from that of the plane, the stereographic projection, due to its nature and its results, is the most suitable candidate. Otherwise, the mapping of the plane formalism to the sphere is not trivial, nor even similar. This is supported by the fact that on the Haldane formalism, some spinor representation, introduced to describe the ground states, is deeply related to the stereographic projection.

Given all these hints, we believe that a proper development of a spherical formalism similar to the planar from Botero et al. [1], can be performed through the use of stereographic coordinates. However, as this mapping demonstrated to be not trivial, we leave this study for future work.

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