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Three body problem in the spherical geometry

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Contents

Chapter 1

Introduction

The N body problem is a highly known and studied issue in Physics. It consists in researching the trajectories that N point masses would follow when interacting with external and internal forces with certain defined characteristics, given all the information of the initial conditions.

At first, the principal interest in the study of this problem was the exact prediction of the path of celestial bodies. However, as the problem was known more, it was understood that its study is of great importance not only for astrophysics but for the theoretical comprehension of classical mechanics. Great minds of physics and mathematics have worked in the restricted problem of three bodies, as Poincaré [1] and Jacobi [2]. It was this way that Poincaré, in an attempt of solving the three body problem, discovered that it is not integrable in general, and formulated the bases of what is known nowadays as chaos theory [1].

Independently of the formalism chosen to define the system, the N body problem is reduced to the integration of the equations of motion for the N particles. As this problem has been known to be non-integrable for the $N \geq 3$ cases, with the exception of few occurrences that involve forces with strange features as explicit dependence of the position an velocities [3], the study of realistic 3 body problems is a very interesting question in physics.

1.1 Motivation

One of the particular realistic cases of the three body problem that is known to be integrable, is the system of three particles on the plane with mass m and charge e under the influence of a huge constant magnetic field perpendicular to the surface and forces whose potentials are invariant under rotations and translations in the plane [4].

This problem is integrable by virtue of the action of the big magnetic potential, which decouples the movement of the particle into two degrees of freedom known as the guiding centres and the linear momenta. The analysis carried out in [4] can be extended to the quantum formalism and it explains the quantum Hall effect (QHE) fact that particles at low temperatures are confined to the base state to energy due to Landau level gap modulated by the magnetic field.

The study of this problem may be applied to classical and quantum Hall effect models. These models are principally centred in the analysis of the bulk properties, and are not interested in the effect of border effects. Consequently, these models are usually defined over geometries with no borders as the infinite plane, the torus and the sphere [5]. Having this into account, the principal objective of this work is to try to expand the analysis of the three body problem on the plane carried out in [4] to the spherical geometry.

To do so, as a first approach to the problem, we are going reproduce the calculations and analysis from [4] of the classical system in great detail. Then, we are going to study some important aspects of the classical one body problem on the sphere under the influence of a magnetic monopole to obtain some intuition about the important quantities and the analogies of the movement of the particle in this case and the one on the plane. After that, we are going to proceed with the analysis of the quantum three body problem in the plane. Then some intuition is going to be presented about the quantum one body problem in the sphere, known as Haldane's formalism [6].

Chapter 2

The three body problem in the plane

In this chapter a classic approach of a somehow general case of the three body problem in 2 dimensions is going to be presented. This problem was developed by Alonso Botero et. al in [4]. It will give some necessary intuition to develop the analogous problem in the spherical geometry. To begin with, the problem is going to be described in great detail; then its integrability is going to be proven; and finally, a brief description of the movement of the particles is going to be presented.

2.1 The definition of the problem

The three body problem presented here is that of three particles of electrical charge e and mass m confined to a plane, under the influence of a strong magnetic field perpendicular to it and forces whose potentials satisfy translational and rotational symmetries in the plane. To make the problem more interesting, we consider a harmonic potential with frequency ω

Given this information, the Hamiltonian associated with this system has the form:

$$H = \sum_{i=1}^{3} \frac{1}{2m} \left\| \vec{p_i} - e\vec{A}\left(\vec{r_i}\right) \right\|^2 + V\left(\vec{r_1}, \vec{r_2}, \vec{r_3}\right) + \frac{m\omega}{2} \sum_{i=1}^{3} \left\| \vec{r_i} \right\|^2 \tag{2.1}$$

Where $\vec{r_i} = x_i\hat{\imath} + y_i\hat{\jmath}$, $\vec{p_i} = p_{x_i}\hat{\imath} + p_{y_i}\hat{\jmath}$ and $\vec{A}(\vec{r})$ is the magnetic vector potential on its symmetric gauge, which satisfies $\nabla \times \vec{A} = B\hat{k}$.

Besides, the potential $V(\vec{r_1}, \vec{r_2}, \vec{r_3})$ satisfies the symmetries:

$$V(R\vec{r_1} + \vec{a}, R\vec{r_2} + \vec{a}, R\vec{r_3} + \vec{a}) = V(\vec{r_1}, \vec{r_2}, \vec{r_3}) \tag{2.2}$$

For any rotation R about $\hat{k}$ and any translation $\vec{a}$ in the plane.

Here it is important to note that a magnetic field that is perpendicular to the plane will have no effect on the dynamics of the system. In fact, one can choose a convenient gauge in which the vector potential has the only non-null component along the z axis. This way, given the restriction $\dot{z} = 0$, there will be no change in the Lagrangian function.

2.2 The canonical transformation of the guiding centres

For the proof of integrability for this system, and for further analysis of the trajectories of the particles, let us perform the well known transformation of the guiding centres.

This transformation is defined by the following two equations:

$$\vec{\pi_i} = \vec{p_i} - e\vec{A}(\vec{q_i}) \tag{2.3}$$

$$\vec{R_i} = \vec{r_i} - \frac{\hat{k} \times \vec{\pi_i}}{eB} \tag{2.4}$$

The equation (2.3) passes from the canonical momentum $\vec{p_i}$ to the linear momentum $\vec{\pi_i}$, which is much more intuitive and understandable; while the equation (2.4) transforms the general position $\vec{r_i}$ to the position of the instantaneous guiding centre $\vec{R_i}$.

In a system without the interaction potentials, the electrically charged particles are known to perform the circular motion of the cyclotron with radii that depends on the initial linear momenta. In this case, the guiding centres would be constant in time as would be the linear momenta. However, with the introduction of an interacting potential, the momentum of each particle may vary making the guiding centre change too, which is why the instantaneous interpretation of the guiding centres is necessary.

Now, let us calculate the Poisson brackets for this new set of coordinates for a specific particle.

$$
\begin{aligned}
\{\pi_1, \pi_2\} &= \frac{\partial \pi_1}{\partial r_\alpha}\frac{\partial \pi_2}{\partial p_\alpha} - \frac{\partial \pi_2}{\partial r_\alpha}\frac{\partial \pi_1}{\partial p_\alpha} \\
&= -e\delta_{\alpha 2}\frac{\partial A_1}{\partial r_\alpha} + e\delta_{\alpha 1}\frac{\partial A_2}{\partial r_\alpha} \\
&= -e\frac{\partial A_1}{\partial y} + e\frac{\partial A_2}{\partial x} \\
&= e(\nabla \times \vec{A})_3 = eB
\end{aligned}
$$

$$
\begin{aligned}
\{R_1, R_2\} &= \{r_1, r_2\} + \left\{r_1, -\frac{\pi_1}{eB}\right\} + \left\{\frac{\pi_2}{eB}, r_2\right\} + \left\{\frac{\pi_2}{eB}, -\frac{\pi_1}{eB}\right\} \\
&= \frac{1}{eB}\left(\cancelto{-1}{\{p_1, r_1\}} - \cancelto{0}{e\{A_1, r_1\}} + \cancelto{-1}{\{p_2, r_2\}} \cancelto{0}{e\{A_2, r_2\}}\right) + \frac{eB}{(eB)^2} \\
&= \frac{-2}{eB} + \frac{1}{eB} = -(eB)^{-1}
\end{aligned}
$$

$$
\begin{aligned}
\{R_1, \pi_2\} &= \{r_1, \pi_2\} + \cancelto{0}{\left\{\frac{\pi_2}{eB}, \pi_2\right\}} \\
&= \cancelto{0}{\{r_1, p_2\}} - e\cancelto{0}{\{r_1, A_2\}} \\
&= \{R_2, \pi_1\} = 0
\end{aligned}
$$

This Poisson brackets can be generalised to the transformation for the three particles. Taking $i, j = \{1, 2, 3\}$ and $\alpha, \beta = \{1, 2\}$:

$$\{\pi_{i,\alpha}, \pi_{j,\beta}\} = (eB)\,\delta_{ij}\epsilon_{\alpha\beta} \tag{2.5}$$

$$\{R_{i,\alpha}, R_{j,\beta}\} = -\,(eB)^{-1}\,\delta_{ij}\epsilon_{\alpha\beta} \tag{2.6}$$

$$\{R_{i,\alpha}, \pi_{j,\beta}\} = 0 \tag{2.7}$$

Equations (2.5)-(2.6) allow us to identify the proposed transformation as canonical. However, this is not the usual canonical transformation where the position coordinates and the momentum coordinates are canonical conjugates. In this special case, one component of the momentum is canonical conjugate with the other momentum coordinate, and similarly for the guiding center position components.

Now, with a huge magnetic field, if the potential of the interaction forces does not vary abruptly in space, we can use the approximation $\vec{R}_i \approx \vec{q}_i$ to average the potentials over the guiding centres, that is, we can replace $\vec{q}_i$ for $\vec{R}_i$ in $V(\vec{q_1}, \vec{q_2}, \vec{q_3})$.

We can support the last approximation as follows: In the cyclotron problem, the radius of the circular motion described is proportional to the linear momentum and inversely proportional to the magnetic field. Then, in the presence of a big magnetic field B, the radius of the cyclotron would shrink to a very small size. Regarding the case we are working with, the radii of the instantaneous cyclotron motion would be proportional to $\left\|(\hat{k} \times \vec{\pi_i})(eB)^{-1}\right\| = \left\|\vec{\pi_i}(eB)^{-1}\right\|$ and its frequency to $\sqrt{B}$. As the potential V does not vary abruptly in the radii scale, the averaging of this motion over the guiding centres means that this potential does not sense that circular motion. Moreover, given the big frequency of the cyclotrons and the scale of variance of the potential, the scale of time of the local circular motions is far smaller than that of the motion of the guiding centres. Therefore, we can ignore the instantaneous quality of the circular motion, and take it as constant in a scale of time small enough for the motion of the guiding centres. In this sense we say that the coordinates for the guiding centres decouple from that of the linear momenta of the particles.

Before replacing the new set of coordinates in the Hamiltonian, it is necessary to do a scale transformation to obtain the proper Poisson brackets for the formal definition of canonical transformation, that is:

$$\begin{aligned}
\vec{\pi}_i &\to (eB)^{-1/2}\vec{\pi}_i \\
\vec{R}_i &\to \sqrt{eB}\vec{R}_i
\end{aligned}$$

With this consideration, the Hamiltonian of the system in the new set of rescaled coordinates is given by:

$$H = \sum_{i=1}^{3} \frac{eB}{2m} \left\| \vec{\pi}_i \right\|^2 + V\left((eB)^{-1/2}\vec{R}_1, (eB)^{-1/2}\vec{R}_2, (eB)^{-1/2}\vec{R}_3 \right) + \frac{m\omega}{2eB} \sum_{i=1}^{3} \left\| \vec{R}_i \right\|^2 \quad (2.8)$$

This Hamiltonian, given equation (2.7) can be decomposed in a Hamiltonian that describes the movement of the guiding centres, and other that describes de movement of the linear momenta. In one hand, the Hamiltonian for the linear momenta is easily identified with the harmonic oscillator, whereas the one that characterises the movement of the guiding centres needs a deeper analysis.

2.3 Integrability of the system

As the Hamiltonian describing the trajectories of the linear momenta of the particles is that of an harmonic oscillator, this part of the problem is integrable and its solutions are widely known. The guiding centre Hamiltonian, in turn, needs to be analysed more deeply. For this purpose, let us take the following convention:

$$H_{gc} = \frac{\omega}{2} \sum_{i=1}^{3} \left\| \bar{x}^2 \right\| + \left\| \bar{y}^2 \right\| + V\left(\bar{x}, \bar{y} \right) \quad (2.9)$$

Where $\bar{x} = (x_1, x_2, x_3)$ and $\bar{y} = (y_1, y_2, y_3)$, being x_i, y_i the rescaled coordinates of the guiding centres of the particles. For simplicity, the potential V and the frequency ω have been rescaled to take into account the scale transform of the coordinates and maintain the original form of the Hamiltonian:

$$\frac{eB\omega}{m} \to \omega$$
$$V\left(\frac{\bar{x}}{\sqrt{eB}}, \frac{\bar{y}}{\sqrt{eB}} \right) \to V\left(\bar{x}, \bar{y} \right)$$

Clearly, the scaled potential V still has the symmetries expressed in the equation (2.2). Furthermore, with this scale transformation the Poisson brackets take the form:

$$\{y_i, x_j\} = \delta_{ij} \quad (2.10)$$

Now that the guiding center Hamiltonian has been expressed in terms of the proper canonical set of coordinates, the fastest way to prove the integrability of the system is via the Liouville-Arnol'd theorem [7, Sect. 49]. For this theorem, it is only necessary to find 2 more independent integrals in involution (besides the Hamiltonian).

To get this 2 integrals, let us exploit the symmetries of the guiding centres Hamiltonian. We then define the generators of translations and rotation in the plane, which are symmetries of the potential:

$$\begin{aligned} T_x &= \sum_{i=1}^{3} x_i \\ T_y &= \sum_{i=1}^{3} y_i \end{aligned} \tag{2.11}$$

$$R_z = \frac{1}{2}\sum_{i=1}^{3}\left(x_i^2 + y_i^2\right) \tag{2.12}$$

It is easily verifiable that these are indeed the symmetries generators. To see that, take the first order infinitesimal transformations of translations and rotations on the plane:

$$x_i \to x_i + \epsilon$$

$$y_i \to y_i + \epsilon$$

$$(x_i, y_i) \to (x_i + \epsilon y_i, y_i - \epsilon x_i)$$

Now note that for the infinitesimal translations, the potential of the transformed coordinates is related to the potential of the normal coordinates by a directional derivative, which can be identified with the Poisson bracket of the potential V and each generator:

$$\begin{aligned} 0 &= V\left(\bar{x}+\epsilon, \bar{y}\right) - V\left(\bar{x}, \bar{y}\right) = \epsilon \sum_{i=1}^{3} \frac{\partial V\left(\bar{x}, \bar{y}\right)}{\partial x_i} = \epsilon \left\{V, T_x\right\} = 0 \\ 0 &= V\left(\bar{x}, \bar{y}+\epsilon\right) - V\left(\bar{x}, \bar{y}\right) = \epsilon \sum_{i=1}^{3} \frac{\partial V\left(\bar{x}, \bar{y}\right)}{\partial y_i} = \epsilon \left\{T_y, V\right\} = 0 \end{aligned}$$

For the infinitesimal rotation, the relation is analogous:

$$0 = V\left(\bar{x} + \epsilon\bar{y}, \bar{y} - \epsilon\bar{y}\right) - V\left(\bar{x}, \bar{y}\right) = \frac{\partial V\left(\bar{x}, \bar{y}\right)}{\partial x_i}\left(\epsilon y_i\right) - \frac{\partial V\left(\bar{x}, \bar{y}\right)}{\partial y_i}\left(\epsilon x_i\right) = \epsilon\left\{V, R_z\right\}$$

Therefore, we conclude that the generators of translations and rotations in the plane commute with the potential V due to its symmetries. Besides, the generator of rotations is exactly equal to the harmonic-like part of the guiding center Hamiltonian which validates that R_z is other integral in involution. The generators of translations are not integrals in involution for they do not commute with the harmonic potential. However, we can calculate a quantity in terms of these generators, which already commute with the potential V, to make it commute with the remaining part of H_{gc}:

$$L \coloneqq \frac{1}{6}\left(T_x^2 + T_y^2\right) \tag{2.13}$$

This new quantity L clearly commutes with the potential V because the Poisson bracket is a linear differential operator in one component and it obeys the Leibniz rule. Moreover, it also commutes with the rotation generator $J \coloneqq R_z$:

$$\begin{aligned}
\left\{T_x^2 + T_y^2, J\right\} &= \sum_{i,j,k}\left\{x_i x_j + y_i y_j, x_k^2 + y_k^2\right\} \\
&= \sum_{i,j,k}\left\{x_i x_j, y_k^2\right\} + \left\{y_i y_j, x_k^2\right\} = \sum_{i,j,k} y_k\left\{x_i x_j, y_k\right\} + x_k\left\{y_i y_j, x_k\right\} \\
&= \sum_{i,j,k} y_k x_i \delta_{jk} + y_k x_j \delta_{ik} - x_k y_i \delta_{jk} - x_k y_j \delta_{ik} \\
&= 2\sum_{i,j} x_i y_j - x_i y_j = 0
\end{aligned}$$

As we found L as the last integral in involution, we conclude, by the Liouville-Arnol'd theorem, that the subsystem of guiding centres is integrable by quadratures.

2.4 The spinorial representation and the Bloch sphere mapping

Taking the motion integrals obtained in the previous section for the guiding center Hamiltonian (2.9) the motion of the particles can be broken down in that of the center of mass, and the relative trajectories of the particles. To see that, note that as $T_x^2 + T_y^2$ in (2.13) is an integral in involution with the Hamiltonian, it is a conserved quantity. The conservation of this term is clearly interpreted as a circular motion of the center of mass of the system $\left(\frac{T_x}{3}, \frac{T_y}{3}\right)$.

The latter analysis can be clarified in terms of the decoupling of the guiding center trajectories from the general motion of the particles. In that sense one can deduce that the guiding centre movement is only affected by the interaction potential V and the external harmonic potential. This can be confirmed by the lack of magnetic terms in the guiding centre Hamiltonian in equation (2.8).

Taking this into account, as the potential V is a central potential that produces only internal forces that do not affect the movement of the centre of mass, one can assure that its motion is determined by the only term of external interaction in the Hamiltonian, the harmonic potential.

With this, the equations of motion for the center of mass of the system will be given by $\dot{T}_i = \{T_i, \omega J\}$. Now, from equations above about the commutation of L and J, it is simple to deduce:

$$
\begin{aligned}
\left\{T_x^2, \omega J\right\} &= 2T_x\left\{T_x, J\right\} = 2\omega\sum_{i,j} x_i y_j = -\left\{T_y^2, \omega J\right\} \\
&= 2\omega T_x T_y \\
\left\{T_x, \omega J\right\} &= \omega T_y \\
\left\{T_y, \omega J\right\} &= -\omega T_x
\end{aligned}
$$

Which tells us that the movement of (T_x, T_y) mass is besides circular, uniform. The problem is then reduced to the analysis of the coordinates relative to the center of mass.

To carry on with this procedure, let us take the spinor of relative position defined as follows:

$$\Psi = \frac{1}{2\sqrt{3}} \begin{pmatrix} \sqrt{3}\,(z_2 - z_1) \\ z_2 + z_1 - 2z_2 \end{pmatrix} \tag{2.14}$$

With $z_i = x_i + iy_i$. If we call define the center of mass as $T = \frac{1}{3}(T_x + iT_y) = \frac{1}{3}(z_1 + z_2 + z_3)$ we can retrieve the general position of each particle:

$$\begin{aligned} z_1 &= T + \frac{1}{\sqrt{3}}\Psi_2 - \frac{1}{3}\Psi_1 \\ z_2 &= T + \frac{1}{\sqrt{3}}\Psi_2 + \frac{1}{3}\Psi_1 \\ z_3 &= T - \frac{2}{\sqrt{3}}\Psi_2 \end{aligned} \tag{2.15}$$

The spinor components were chosen to satisfy the canonical commutation relations:

$$\begin{aligned} \{\Psi_\alpha, \Psi_\alpha\} &= 0 \\ \{\Psi_1, \Psi_2\} &= \{\Psi_1^*, \Psi_2^*\}^* = \left\{\frac{1}{2}(z_2 - z_1), \frac{1}{2\sqrt{3}}(z_2 + z_1 - 2z_3)\right\} \\ &= i\left\{\frac{1}{2}(x_2 - x_1), \frac{1}{2\sqrt{3}}(y_2 + y_1 - 2y_3)\right\} \\ &+ i\left\{\frac{1}{2}(y_2 - y_1), \frac{1}{2\sqrt{3}}(x_2 + x_1 - 2x_3)\right\} = 0 \end{aligned}$$

$$\begin{aligned} \{\Psi_1, \Psi_2^*\} &= \{\Psi_1^*, \Psi_2\}^* = -\frac{i}{2\sqrt{3}}\{x_2 - x_1, y_2 + y_1 - 2y_3\} = 0 \\ \{\Psi_1, \Psi_1^*\} &= \frac{-2i}{4}\{x_2 - x_1, y_2 - y_1\} = -i \\ \{\Psi_2, \Psi_2^*\} &= \frac{-i}{6}\{x_1 + x_2 - 2x_3, y_1 + y_2 - 2y_3\} = -i \end{aligned}$$

Which can be resumed:

$$\{\Psi_\alpha, \Psi_\beta\} = \left\{\Psi_\alpha^*, \Psi_\beta^*\right\} = 0 \tag{2.16}$$

$$\{\Psi_\alpha^*, \Psi_\beta\} = i\delta_{\alpha,\beta} \tag{2.17}$$

This spinorial representation of the relative coordinates of the particles with respect to the center of mass is analogous to the transformation performed to analyse the two body problem, only adapted to take into account one more particle and one less dimension. This spinor, besides representing a canonical transformation of the guiding centres, has norm equal to the spin quantity $S = J - L$ which is also an integral of motion:

$$
\begin{aligned}
\Psi^\dagger\Psi &= \|\Psi_1\|^2 + \|\Psi_2\|^2 \\
&= \frac{1}{4}\left(\|z_1\|^2 + \|z_2\|^2 - 2\operatorname{Re}\left(z_1 z_2^*\right)\right) + \frac{1}{12}\left(\|z_1\|^2 + \|z_2\|^2 + \|z_3\|^2 + \operatorname{Re}\left(2z_1 z_2^* - 4z_1 z_3^* - 4z_2 z_3^*\right)\right) \\
&= \frac{1}{3}\sum_{i=1}^{3} x_i^2 + y_i^2 - \frac{1}{3}\sum_{i>j} x_i x_j + y_i y_j \\
&= \frac{1}{2}\sum_{i=1}^{3} x_i^2 + y_i^2 - \frac{1}{6}\left(T_x^2 + T_y^2\right) \\
&= J - L = S
\end{aligned}
$$

This quantity S can be easily interpreted as proportional to the moment of inertia of the triangle rotating about its center of mass, which gives more intuition to the interpretation of S as spin momentum:

$$
\begin{aligned}
I = m\sum_{i=1}^{3}\|\vec{r_i} - \vec{r}_{cm}\|^2 &= m\sum_{i=1}^{3}\|\vec{r_i}\|^2 + 3\|\vec{r}_{cm}\| - 2\vec{r}_{cm}\cdot\sum_{i=1}^{3}\vec{r_i} = m\left(2J - 3\|\vec{r}_{cm}\|\right) \\
= m\left(2J - \frac{1}{3}\left(T_x^2 + T_y^2\right)\right) &= m\left(2J - 2L\right) = 2mS
\end{aligned}
$$

On the other hand, it is observed that any phase multiplication to the spinor Ψ is equivalent to any rotation about the center of mass or about the origin, as they have the same effect on relative vectors $\vec{r_i} - \vec{r_j}$:

$$
\begin{aligned}
e^{i\phi} z &= \left(\cos\phi + i\sin\phi\right)\left(x + iy\right) = \left(x\cos\phi - y\sin\phi\right) + i\left(y\cos\phi + x\sin\phi\right) \\
\Psi &= \frac{1}{2\sqrt{3}}\begin{pmatrix}\sqrt{3}\left(z_2 - z_1\right) \\ z_2 + z_1 - 2z_3\end{pmatrix} = \frac{1}{2\sqrt{3}}\begin{pmatrix}\sqrt{3}\left(z_2 - z_1\right) \\ z_2 - z_3 + z_1 - z_3\end{pmatrix} \\
e^{i\phi}\Psi &= \frac{1}{2\sqrt{3}}\begin{pmatrix}\sqrt{3}e^{i\phi}\left(z_2 - z_1\right) \\ e^{i\phi}\left(z_2 - z_3\right) + e^{i\phi}\left(z_1 - z_3\right)\end{pmatrix}
\end{aligned}
$$

Moreover, scale transformations will be clearly given by a scalar multiplication. Consequently, a general transformation performed to the triangle relative coordinates can be represented by a scalar and a phase, or which is the same, a complex number. Now, taking this into account, we can take the equivalence class of shapes associated with a spinor Ψ as $[\Psi] = \{z\Psi | z \in \mathbb{C}/\{0\}\}$. Here the equivalence class $[\Psi]$ represents a shape of the triangle formed by the particles, and together with the norm of the spinor, it codifies the actual information needed of the triangle for the Hamiltonian of guiding centres (2.8).

In fact, if we take the space of normalised representatives $\Psi/\sqrt{S}$ we can note that the Hilbert spaces associated to the shapes of the triangles and to the wave vectors of a Qubit are equivalent, and can be mapped to a Bloch sphere. To comprehend better this mapping, let us define the vector consisting of the expected value of the Pauli matrices $\vec{\sigma}$ in terms of the equivalence class representatives:

$$\vec{\zeta} = \frac{1}{S}\Psi^\dagger \vec{\sigma} \Psi \tag{2.18}$$

The interpretation of each component of this vector becomes clear with some heavy algebra. Here we present the result in terms of the position vectors $\vec{r}_i$ of each particle:

$$\begin{aligned}
\zeta_1 &= \frac{1}{2\sqrt{3}S}\left(\|\vec{r}_2 - \vec{r}_3\|^2 - \|\vec{r}_1 - \vec{r}_3\|^2\right) \\
\zeta_2 &= \frac{1}{\sqrt{3}S}\left(\vec{r}_1 \times \vec{r}_2 + \vec{r}_2 \times \vec{r}_3 + \vec{r}_3 \times \vec{r}_1\right) \cdot \hat{z} = \frac{2A}{\sqrt{3}S} \\
\zeta_3 &= \frac{1}{6S}\left(2\|\vec{r}_2 - \vec{r}_1\|^2 - \|\vec{r}_3 - \vec{r}_1\|^2 - \|\vec{r}_3 - \vec{r}_2\|^2\right)
\end{aligned}$$

From this, the second component of the vector can be easily interpreted as proportional to the signed area A of the triangle which sign encodes the chirality of the system.

The first and third components, in turn, share a similar composition with the components of the vector Ψ, mapping the complex quantities z_i that encode the vertices coordinates, to the squared norm of the opposite side of the triangle. In other words, we can obtain expressions for the squared lengths of the sides of the triangle as we did in equations (2.15):

$$S = \frac{1}{3}\sum_{i=1}^{3} x_i^2 + y_i^2 - \frac{1}{3}\sum_{i>j} x_i x_j + y_i y_j = \frac{1}{6}\left(\|\vec{r}_1 - \vec{r}_2\|^2 + \|\vec{r}_2 - \vec{r}_3\|^2 + \|\vec{r}_3 - \vec{r}_1\|^2\right)$$

$$\begin{aligned}
\rho_1 &:= \|\vec{r}_2 - \vec{r}_3\|^2 = 2S\left(1 + \frac{\sqrt{3}}{2}\zeta_1 - \frac{1}{2}\zeta_3\right) \\
\rho_2 &:= \|\vec{r}_3 - \vec{r}_1\|^2 = 2S\left(1 - \frac{\sqrt{3}}{2}\zeta_1 - \frac{1}{2}\zeta_3\right) \\
\rho_3 &:= \|\vec{r}_2 - \vec{r}_1\|^2 = 2S\left(1 + \zeta_3\right)
\end{aligned} \tag{2.19}$$

The information carried by the three components of the vector $\vec{\zeta}$ and the scalar S presented before, can then be compressed elegantly in this fashion:

$$\begin{aligned}
\rho_k &= 2S\left(1 + \vec{m}_k \cdot \vec{\zeta}\right) \\
\vec{m}_k &= \left(\sin\frac{2\pi k}{3}, 0, \cos\frac{2\pi k}{3}\right), k \in \{1,2,3\} \\
A &= \frac{\sqrt{3}S}{2}\zeta_2
\end{aligned} \tag{2.20}$$

Now, to know better the distribution of shapes in the Bloch sphere, let us take the coordinate ζ_2 as the vertical axis which defines the poles and the rest as the ones defining the remaining perpendicular plane. In this terms, the pole vectors represent triangles of maximal area and the equatorial line, triangles of minimal null area. The triangles from the north hemisphere differ from their specular image with respect to the equatorial plane in the south hemisphere by only a sign in the area, which means that they have the same shape but different chirality.

Moreover, isosceles triangles require the vector $\vec{\zeta}$ to be perpendicular to $\vec{m}_i - \vec{m}_j$ for some $i \neq j$. It can be easily demonstrated that $\vec{m}_k \perp \vec{m}_i - \vec{m}_j$ for all $i \neq j \neq k$ and as a consequence, any isosceles triangle must follow that $\vec{\zeta} \parallel \vec{m}_i$ for any i. Besides, given that the vectors $\vec{m}$ live in the equatorial plane, it can be deduced that any isosceles triangle can be found on the lines on the sphere connecting the poles with the vectors $\vec{m}$. From this, it is clear that the polar triangles are equilateral.

Taking the analysis a little bit further, it can be seen from the form that takes the sides of the vertices ρ_k in equation (2.20), that in the equatorial plane if we look the directions $\vec{\zeta} = \vec{m}_k$, no side of the triangle will be null. Hence, those directions will determine isosceles triangles with null area, that is, triangles whose two sides sum up to the other. On the other hand, if we look directions $\vec{\zeta} = -\vec{m}_k := \vec{n}_k$, we find that one of the sides will be null. We find then on those directions, isosceles triangles with one of the sides equal to zero, which accounts for its null area.

One can be curious for the direction of equilateral triangles with null area, that is, all the points in the same position. The answer is that we cannot find them in the Bloch sphere because, as we stated earlier, it is a representation for normalised vectors $\Psi/\sqrt{S}$ and this triangle would require $S = 0$. However, if we take the normalisation factor S as the radius of the sphere, and consider the family of Bloch spheres for different radii filling $\mathbb{R}^3/\left\{0\right\}$, we surely would find these triangles in any direction on the limit $S \to 0$.

2.5 Analysis of the motion

Taking into account the decoupling of the center of mass movement with respect to the relative coordinates which are encoded in the canonical spinor Ψ, the addition of terms that depend only on the center of mass components to the Hamiltonian will not affect the equations of motion of the relative coordinates. We then can add a term $-\omega L$ from the Hamiltonian in equation (2.9) with no effect on Ψ:

$$
\begin{aligned}
H_\Psi &= H_{gc} - \omega L = V\left(\Psi, \Psi^*\right) + \omega(J - L) = V\left(\Psi, \Psi^*\right) + \omega S \\
&= V\left(\Psi_\alpha, \Psi_\alpha^*\right) + \omega \Psi_\alpha \Psi_\alpha^*
\end{aligned}
$$

With this new Hamiltonian for relative coordinates, one can obtain the temporal evolution of the spinor Ψ, taking the Poisson bracket with it, the time evolution generator. However, to express the Poisson bracket in terms of the new spinorial coordinates, we can reformulate the canonical relations in (2.17) to resemble more the classical relations of x and p_x:

$$
\{\Psi_1, i\Psi_1^*\} = \{\Psi_2, i\Psi_2^*\} = 1
$$

With this clarification, we can take the usual Poisson bracket expressions with the canonical variables $(\Psi_\alpha, i\Psi^*_\alpha)$:

$$
\begin{aligned}
\{f,g\} &= \sum_{\alpha=1}^{2}\left(\frac{\partial f}{\partial \Psi_\alpha}\frac{\partial g}{\partial i\Psi^*_\alpha} - \frac{\partial f}{\partial i\Psi^*_\alpha}\frac{\partial g}{\partial \Psi_\alpha}\right)\\
i\{f,g\} &= \sum_{\alpha=1}^{2}\left(\frac{\partial f}{\partial \Psi_\alpha}\frac{\partial g}{\partial \Psi^*_\alpha} - \frac{\partial f}{\partial \Psi^*_\alpha}\frac{\partial g}{\partial \Psi_\alpha}\right)
\end{aligned}
$$

And the temporal evolution of Ψ_α takes the simple form:

$$
i\dot{\Psi}_\alpha = i\{\Psi_\alpha, H_\Psi\} = \frac{\partial V}{\partial \Psi^*_\alpha} + \omega\Psi_\alpha
$$

We can go further in this analysis if we take into account the Bloch sphere mapping. We then can pass from the spinorial representation given by (Ψ, Ψ^*) to the shape space given by $(\vec{\zeta}, S)$. In this fashion, the time evolution of the spinor component Ψ_α will be given by:

$$
\begin{aligned}
i\dot{\Psi}_\alpha &= \frac{\partial V}{\partial \Psi^*_\alpha} + \omega\Psi_\alpha\\
&= \frac{\partial V}{\partial S}\frac{\partial S}{\partial \Psi^*_\alpha} + \frac{\partial V}{\partial \zeta_j}\frac{\partial \zeta_j}{\partial \Psi^*_\alpha} + \omega\Psi_\alpha\\
&= \omega\Psi_\alpha + \frac{\partial V}{\partial S}\frac{\partial}{\partial \Psi^*_\alpha}\left(\Psi^*_\beta\Psi_\beta\right) + \frac{\partial V}{\partial \zeta_j}\frac{\partial}{\partial \Psi^*_\alpha}\left(\frac{1}{S}\Psi^*_\delta\sigma^j_{\delta\beta}\Psi_\beta\right)\\
&= \omega\Psi_\alpha + \frac{\partial V}{\partial S}\Psi_\alpha + \frac{\partial V}{\partial \zeta_j}\left(\frac{-1}{S^2}\Psi^*_\delta\sigma^j_{\delta\beta}\Psi_\beta\frac{\partial S}{\partial \Psi^*_\alpha} + \frac{1}{S}\sigma^j_{\alpha\beta}\Psi_\beta\right)\\
&= \left(\omega\Psi + \frac{\partial V}{\partial S}\Psi + \frac{1}{S}\frac{\partial V}{\partial \zeta_j}\left(-\zeta_j\Psi + \sigma^j\Psi\right)\right)_\alpha
\end{aligned}
$$

This result can be compressed taking into account all components of Ψ:

$$i\dot{\Psi} = \left(\omega + \frac{\partial V}{\partial S} + \frac{1}{S}\frac{\partial V}{\partial \zeta_j}\left(-\zeta_j \mathbb{I} + \sigma^j\right)\right)\Psi \tag{2.21}$$

Now, as we know that S is a constant of movement, the time evolution of the components ζ_j of the vector of shapes will be given by:

$$\dot{\zeta}_j = \frac{1}{S}\left(\dot{\Psi}^\dagger \sigma^j \Psi + \Psi^\dagger \sigma^j \dot{\Psi}\right) = \frac{1}{S}\left(\dot{\Psi}^\dagger {\sigma^j}^\dagger \Psi + \Psi^\dagger \sigma^j \dot{\Psi}\right)$$

$$= \frac{2}{S}\Re\left\{\Psi^\dagger \sigma^j \dot{\Psi}\right\}$$

From equation (2.21) this expression can be simplified:

$$\frac{1}{S}\Psi^\dagger \sigma^k \dot{\Psi} = -i\left(\omega + \frac{\partial V}{\partial S} - \frac{1}{S}\frac{\partial V}{\partial \zeta_j}\zeta_j\right)\frac{1}{S}\Psi^\dagger \sigma^k \dot{\Psi} - i\frac{1}{S}\Psi^\dagger \sigma^k \left(\frac{1}{S}\frac{\partial V}{\partial \zeta_j}\sigma^j\right)\Psi$$

$$\frac{2}{S}\Re\left\{\Psi^\dagger \sigma^k \dot{\Psi}\right\} = 2\Re\left\{-i\frac{1}{S}\frac{\partial V}{\partial \zeta_j}\frac{1}{S}\Psi^\dagger \sigma^k \sigma^j \Psi\right\}$$

$$\dot{\zeta}_k = -i\frac{1}{S}\frac{\partial V}{\partial \zeta_j}\frac{1}{S}\Psi^\dagger \sigma^k \sigma^j \Psi + i\frac{1}{S}\frac{\partial V}{\partial \zeta_j}\frac{1}{S}\Psi^\dagger \sigma^j \sigma^k \Psi$$

$$= -i\left(\frac{1}{S}\frac{\partial V}{\partial \zeta_j}\right)\frac{1}{S}\Psi^\dagger\left[\sigma^k, \sigma^j\right]\Psi$$

Where $[a,b] = ab - ba$ is the commutator between a and b. It is widely known that the Pauli matrices $\vec{\sigma}$ satisfy the commutation relations $[\sigma^i, \sigma^j] = 2i\epsilon_{ijk}\sigma_k$, that are in fact the angular momentum relations. Taking this into account, we finally obtain an expression for the time evolution of the shape vector $\vec{\zeta}$:

$$\dot{\zeta}_k = 2\epsilon_{kjl}\left(\frac{1}{S}\frac{\partial V}{\partial \zeta_j}\right)\frac{1}{S}\Psi^\dagger \sigma_l \Psi = 2\epsilon_{kjl}\left(\frac{1}{S}\frac{\partial V}{\partial \zeta_j}\right)\zeta_l$$

$$\dot{\vec{\zeta}} = \frac{2}{S}\left(\nabla_{\vec{\zeta}} V\right)\times\vec{\zeta} \tag{2.22}$$

Having the time evolution for the vector $\vec{\zeta}$ and knowing the potential V, the shape of the triangle can be easily retrieved from the relations in (2.19). Regarding the orientation of the triangle, the analysis requires more work because when passing from the spinor representation (Ψ, Ψ^*) to the shape space $(\vec{\zeta}, S)$ we loose the information of the phase of the spinor. This phase, as we deduced before, encodes the rotation (orientation) of the triangle and is not important when analysing shapes.

To retrieve the orientation information of the triangle, we would like to obtain a definition of phase for infinitesimally separated spinors that would correspond to infinitesimal evolution in time of the trajectories. To achieve this, let us take an infinitesimal rotation in the spinorial representation and compare with the infinitesimal separation of the spinor:

$$\begin{aligned}
e^{-id\chi}\Psi &\approx \Psi - id\chi\Psi = \Psi + d\Psi \\
d\chi\Psi &= id\Psi \\
d\chi\Psi^\dagger\Psi &= i\Psi^\dagger d\Psi \\
d\chi &= i\frac{\Psi^\dagger d\Psi}{\Psi^\dagger\Psi}
\end{aligned}$$

Now, if we identify Ψ with the state of the triangle at a time t and $\Psi + d\Psi$ with a time $t + dt$, we can obtain an expression for the dynamical angular velocity of the triangle:

$$\begin{aligned}
\omega_r^{(dyn)} &:= \frac{d\chi}{dt} = i\frac{\Psi^\dagger\dot{\Psi}}{\Psi^\dagger\Psi} \\
\omega_r^{(dyn)} &= \left(\omega + \frac{\partial V}{\partial S} - \cancel{\frac{1}{S}\frac{\partial V}{\partial \zeta_j}\zeta_j}\right) + \cancel{\left(\frac{1}{S}\frac{\partial V}{\partial \zeta_j}\right)\frac{\Psi^\dagger\sigma^j\Psi}{\Psi^\dagger\Psi}} \\
\omega_r^{(dyn)} &= \omega + \frac{\partial V}{\partial S}
\end{aligned} \tag{2.23}$$

From the equation (2.23) for the dynamical angular velocity of the triangle, one may define a phase between two finite-time separated spinors in the trajectory by simple integration. However, if the initial and final shapes are not the same, the interpretation of this phase as an angle of rotation of the triangle is not possible.

If we then define the phase $\Delta\chi = \int_0^{T_s} \omega_r^{(dyn)} dt$ over a period T_s of the shape motion, as the initial and final shapes must be equal over this period, one would expect that $\Delta\chi$ may be interpreted as the angle of rotation of the shape. This is wrong as long as a geometrical phase or Hannay angle [**?**] must be taken into account due to the change of shape of the triangle in the period. To see this more clearly, let us parametrise the spinor Ψ:obtained

$$\Psi = \sqrt{S} e^{-i\gamma} \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{-i\phi}\sin\frac{\theta}{2} \end{pmatrix}$$

This parametrisation is chosen so that the angles (θ, ϕ) correspond to spherical angular coordinates. The vector $\vec{\zeta}$ under this parametrisation is then given by:

$$\vec{\zeta} = \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix}$$

Then if we apply the definition of $\omega_r^{(dyn)}$ we obtain:

$$\begin{aligned} \omega_r^{(dyn)} &= i\frac{1}{S}\Psi^\dagger\dot{\Psi} \\ &= \left(e^{i\gamma} \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \end{pmatrix} \right) \cdot e^{-i\gamma} \left(\dot{\gamma} \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{-i\phi}\sin\frac{\theta}{2} \end{pmatrix} + \begin{pmatrix} 0 \\ \dot{\phi}e^{-i\phi}\sin\frac{\theta}{2} \end{pmatrix} + \frac{\dot{\theta}}{2} \begin{pmatrix} -\sin\frac{\theta}{2} \\ e^{-i\phi}\cos\frac{\theta}{2} \end{pmatrix} \right) \\ &= \dot{\gamma} + \dot{\phi}\sin^2\frac{\theta}{2} \end{aligned}$$

Being γ the rotation phase of the spinor, over a period T_s one can truthfully identify $\Delta\gamma$ with the angle of rotation of that shape, obtaining then the overall angular velocity $\omega_r = \Delta\gamma/T_s$. With this clarification, the expression for ω_r in terms of the dynamical and geometrical angular velocities is easily deduced:

$$\Delta\gamma = \Delta\chi - \int_0^{T_s} \dot{\phi} \sin^2 \frac{\theta}{2} dt$$

$$\omega_r = \frac{\Delta\gamma}{T_s} = \frac{1}{T_s} \int_0^{T_s} \omega_r^{(dyn)} dt - \frac{1}{T_s} \oint \sin^2 \frac{\theta}{2} d\phi$$

$$\omega_r = \left\langle \omega_r^{(dyn)} \right\rangle + \omega_r^{(geo)}$$

With $\omega_r^{(geo)} = -\frac{1}{2T_s} \oint 2 \sin^2 \frac{\theta}{2} d\phi = -\frac{1}{2T_s} \Omega(E, S)$ being proportional to the solid angle $\Omega(E, S)$ on the Bloch sphere enclosed by the level curve of the shape trajectory corresponding to (S, E):

$$\begin{aligned} \Omega(E, S) &= \int_0^{2\pi} \int_0^{\theta(\phi)} \sin\theta d\theta d\phi = \oint (1 - \cos\theta) d\phi \\ &= \oint 2 \sin^2 \frac{\theta}{2} d\phi \end{aligned}$$

The system may be analysed in terms of angle-action variables and with this method the frequencies deduced here can be formalised as canonical variables with unambiguous meaning. However, the purpose of this chapter is to present the 2-dimensional formalism developed by Botero et. al to describe the three body problem in presence of a strong magnetic field as a preamble and motivation to look for a similarly elegant formalism to describe the three-body problem on the sphere.

Chapter 3

Quantum analysis of the three body problem in the plane

In this chapter we continue the analysis implemented in [4], explaining in detail the quantum treatment of the three body problem in the plane. To achieve this goal, the quantum problem will be defined at the same time as the quantum analogous of the classical reduction of problem is going to be briefly discussed. The Schwinger oscillator is going to be explained to get some insights about the quantum analogue analysis, and finally it is going to be applied to the problem.

3.1 The quantum reduction of the problem

The quantum treatment of the three body problem on the plane is not very different from the classical approach, given that we thoroughly worked the classical system out in the Hamiltonian formalism. In fact, all the transformations and reductions from the classical approach work, however, we have to be careful when treating topics referring equations of movement. The only thing we have to do is to apply the principles of canonical quantization [?]. To make the problem of parsing to the quantum formalism less complicated, let us carry the analysis in fundamental units $\hbar = 1$.

Including the last considerations, the quantum Hamiltonian is going to be the same as the one expressed in (2.1), and as canonical transformations still work in quantum mechanics, a similar reduction of the problem can be performed.

First, given the big magnetic field decoupling, the problem can be reduced to the analysis of the guiding centres motion given by the Hamiltonian (2.8). The decoupled system of linear momenta is equally identified as a quantum harmonic oscillator, which is also very well known amongst the physical sciences community.

Therefore, we are left with the reduced Hamiltonian for the guiding centres (2.8) with the same definition of the canonical coordinates. The integrability analysis carried out in Chapter 2 where there were found two integrals of motion in involution with the Hamiltonian, may be interpreted as the finding of a complete set of commutative observables (CSCO) [**?**]. This means that the Hamiltonian of the guiding centres can be simultaneously diagonalised with the operators L and J that can be interpreted as orbital and total angular momentum respectively.

3.2 The spinorial transformation

There are some remarkable aspects referring the canonical variables $(\bar{x}, \bar{y})$ associated with (2.8). With these one can form creation-annihilation operators given by:
then

$$\begin{aligned} a_j &= \frac{1}{\sqrt{2}}(x_j + iy_j) \\ a_j^\dagger &= \frac{1}{\sqrt{2}}(x_j - iy_j) \\ \left[a_j, a_k^\dagger\right] &= \delta_{kj} \end{aligned}$$

We then note that the variables called z_i in the classical formalism will correspond to annihilation operators in the quantum formalism.

Following the reduction of the problem analogously to the classical approach, we can proceed to decouple the center of mass coordinates form the relative coordinates. The spinor Ψ is known to follow relations of creation-annihilation operators (taking into account the canonical quantisation), however, the center of mass was not formalised as a canonical variable in the classical approach. To take this into account, we choose the constants to make it also an annihilation operator. If we encode the center of the triangle in b, the decoupling transformation can be expressed in the following way:

$$\begin{pmatrix} b \\ \Psi_1 \\ \Psi_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{pmatrix}$$

Where the new variables satisfy the creation-annihilation operators relation:

$$\begin{aligned} \left[\Psi_\alpha, \Psi_\beta^\dagger\right] &= \delta_{\alpha\beta} \\ \left[b, b^\dagger\right] &= 1 \\ \left[b, \Psi_\alpha^\dagger\right] &= [b, \Psi_\alpha] = 0 \end{aligned}$$

From this transformation it is worth noticing:

$$\begin{aligned} b^\dagger b &= \frac{1}{3}\left(\sum_i a_i^\dagger\right)\left(\sum_j a_j\right) = \frac{1}{6}\left(\sum_i x_i - iy_i\right)\left(\sum_j x_j + iy_j\right) \\ &= \frac{1}{6}\sum_i\sum_j (x_i - iy_i)(x_j - iy_j) = \frac{1}{6}\sum_i\sum_j x_i x_j + y_i y_j + i\cancelto{i\delta_{ij}}{[xi, y_j]} \\ &= L - \frac{1}{2} \end{aligned}$$

$$\begin{aligned} \Psi^\dagger\Psi &= \Psi_1^\dagger\Psi_1 + \Psi_2^\dagger\Psi_2 \\ &= \frac{1}{2}(a_2^\dagger - a_1^\dagger)(a_2 - a_1) + \frac{1}{6}(a_2^\dagger + a_1^\dagger - 2a_3^\dagger)(a_2 + a_1 - 2a_3) \\ &= \frac{1}{2}(x_2 - x_1 - i(y_2 - y_1))(x_2 - x_1 + i(y_2 - y_1)) \\ &+ \frac{1}{6}(x_2 + x_1 - 2x_3 - i(y_2 + y_1 - 2y_3))(x_2 + x_1 - 2x_3 + i(y_2 + y_1 - 2y_3)) \\ &= \frac{1}{2}(x_2 - x_1)^2 + (y_2 - y_1)^2 + i\cancelto{2i}{[x_2 - x_1, y_2 - y_1]} \\ &+ \frac{1}{6}(x_2 + x_1 - 2x_3)^2 + (y_2 + y_1 - 2y_3)^2 + i\cancelto{3i}{[x_2 + x_1 - 2x_3, y_2 + y_1 - 2y_3]} \\ &= S - 1 \end{aligned}$$

So far we have reduced the problem to the study of the center of mass and relative coordinates spinor. In this terms, the Hamiltonian for these variables will be given by:

$$
\begin{aligned}
H_{gc} &= V\left(\Psi, \Psi^\dagger\right) + \omega(S + L) \\
&= V\left(\Psi, \Psi^\dagger\right) + \omega(\Psi^\dagger \Psi + b^\dagger b + \frac{3}{2}) \\
&= V\left(\Psi, \Psi^\dagger\right) + \omega(\Psi_1^\dagger \Psi_1 + \Psi_2^\dagger \Psi_2 + 1) + \omega(b^\dagger b + \frac{1}{2})
\end{aligned}
$$

Ignoring the effects of the center of mass b, what we have here is the Hamiltonian of two oscillators (Ψ_1, Ψ_2) coupled by the potential V which does not affect the center of the triangle. What we will see next is that in fact, the two oscillators from Ψ induce a Schwinger angular momentum [**?**].

In quantum mechanics the symmetries of a system play a substantial role in the solution of the associated problem. In fact, the role of the symmetries in quantum mechanics is more important than in classical mechanics [**?**]. In quantum mechanics they determine the algebra of operators, and consequently, the eigenvectors and eigenvalues that generate the states of the Hilbert space. In this case, we know that the potential V follows the symmetries associated with the special Euclidean group in 2D ($SE(2)$), that, given the simplification of the problem and the canonical spinor variables, is transformed into $SU(2)$.

The role of this symmetry can be understood as the operators Ψ_α implement the mentioned Schwinger oscillator [**?**] associated with this symmetry.

3.3 The Schwinger oscillator and the angular momentum representation

To understand better the role of the symmetry of Ψ in the problem, let us explain the theory associated with the Schwinger oscillators. In a nutshell, the Schwinger formalism demonstrates that the annihilation-creation operator algebra associated to two decoupled quantum oscillators induce a $SU(2)$ algebra of angular momentum.

To understand this, let us first consider two decoupled harmonic oscillators with the same frequency ω. The Hamiltonian of this system in terms of the annihilation-creation operators (a_1, a_2) of each oscillator, will be given by:

$$H_{SO} = \omega\left(a_1^\dagger a_1 + a_2^\dagger a_2 + 1\right)$$

As the oscillators are decoupled, we can define the excitation number hermitian operators $n_i = a_i^\dagger a_i$. Some important commutation relations between these operators are then given by:

$$\begin{aligned}
\left[a_i, a_j^\dagger\right] &= \delta_{ij} \\
[a_i, a_j] &= 0 \\
[n_i, a_j] &= \left[n_i, a_j^\dagger\right] = [n_i, n_j] = 0
\end{aligned}$$

Now we are going to show that this algebra of annihilation operators induces an angular momentum algebra with symmetry $SU(2)$. To do that, let us define an angular momentum in terms of the $\{a_i\}$ and see its commutation relations:

$$\begin{aligned}
J_i &= \frac{1}{2}\sigma^i_{\alpha\beta} a^\dagger_\alpha a_\beta \\
J_1 &= \frac{1}{2}\left(a_2^\dagger a_1 + a_1^\dagger a_2\right) \\
J_2 &= \frac{i}{2}\left(a_2^\dagger a_1 - a_1^\dagger a_2\right) \\
J_3 &= \frac{1}{2}\left(n_1 - n_2\right)
\end{aligned} \tag{3.1}$$

$$\begin{aligned}
[J_i, J_j] &= \frac{1}{4}\left[\sigma^i_{\alpha\beta} a^\dagger_\alpha a_\beta, \sigma^j_{\gamma\delta} a^\dagger_\gamma a_\delta\right] = \frac{1}{4}\sigma^i_{\alpha\beta}\sigma^j_{\gamma\delta}\left[a^\dagger_\alpha a_\beta, a^\dagger_\gamma a_\delta\right] \\
&= \frac{1}{4}\sigma^i_{\alpha\beta}\sigma^j_{\gamma\delta}\left(a^\dagger_\alpha [a_\beta, a^\dagger_\gamma] a_\delta + a^\dagger_\gamma [a^\dagger_\alpha, a_\delta] a_\beta\right) \\
&= \frac{1}{4}\left(\sigma^i_{\alpha\beta}\sigma^j_{\beta\delta} a^\dagger_\alpha a_\delta - \sigma^i_{\alpha\beta}\sigma^j_{\gamma\alpha} a^\dagger_\gamma a_\beta\right) = \frac{1}{4} a^\dagger_\alpha a_\beta \left[\sigma^i, \sigma_j\right]_{\alpha\beta} \\
&= i\epsilon_{ijk}\frac{1}{4}\sigma^k_{\alpha\beta} a^\dagger_\alpha a_\beta
\end{aligned}$$

$$[J_i, J_j] = i\epsilon_{ijk} J_k$$

Then as well as with any other angular momentum, one can get the operators $J^{\pm} = J_1 \pm J_2$ and get, based on these, the rules for the quantised numbers (j, m) that define the eigenvalues of (J^2, J_3). Then, let us calculate J^2 in terms of the hermitian excitation number operators (n_1, n_2):

$$\begin{aligned}
J^2 &= J_1J_1 + J_2J_2 + J_3J_3 \\
&= \frac{1}{4}((n_1 - n_2)^2 + (\cancel{-a_2^\dagger a_1 a_2^\dagger a_1} - \cancel{a_1^\dagger a_2 a_1^\dagger a_2} + n_1 a_2 a_2^\dagger + n_2 a_1 a_1^\dagger) \\
&+ (\cancel{a_2^\dagger a_1 a_2^\dagger a_1} + \cancel{a_1^\dagger a_2 a_1^\dagger a_2} + n_1 a_2 a_2^\dagger + n_2 a_1 a_1^\dagger)) \\
&= \frac{1}{4}\left((n_1 - n_2)^2 + 2n_1(n_2+1) + 2n_2(n_1+1)\right) = \frac{1}{4}\left((n_1+n_2)(n_1+n_2+4)\right) \\
&= \frac{n_1+n_2}{2}\left(\frac{n_1+n_2}{2} + 1\right)
\end{aligned}$$

Now, comparing this result with the eigenvalues $j(j+1)$ of J^2 it is clear that $j = \frac{n_1+n_2}{2}$. Furthermore, the eigenvalues m of J_3 are given by $j = \frac{n_1-n_2}{2}$. As a consequence, given that $n_1 + n_2 = 1, 2, ...$, we conclude that the quantum number j is going to be some semi-integer $j = \frac{1}{2}, 1, \frac{3}{2}, ...$ These results lead us to conclude that the symmetry group associated with this angular momentum algebra is SU(2).

To sum up, given the proceedings in this section, the algebra of creation-annihilation operators that characterise a set of 2 decoupled harmonic oscillators induces an angular momentum algebra with symmetry $SU(2)$.

3.4 Quantum diagonalization of the Hamiltonian

Now that we introduced some important aspects about Schwinger oscillators and its relation to $SU(2)$ symmetries, we are now able to apply this theory to the analysis of our reduced problem. The only thing we have to do is take the operators Ψ_α as the annihilation-creation operators of the decoupled harmonic oscillator. Given this, we obtain an angular momentum in accordance to equation (3.1) given by:

$$\begin{aligned}
F_i &= \frac{1}{2}\Psi_\alpha^\dagger \sigma^i_{\alpha,\beta}\Psi_\beta \\
\vec{F} &= \frac{1}{2}\Psi^\dagger \vec{\sigma}\Psi
\end{aligned}$$

In fact, as seen before, this angular momentum satisfies $[F_i, F_j] = i\epsilon_{ijk}F_k$. Moreover, this angular momentum has the same definition as the Bloch sphere shape vector $\vec{\zeta}$ with the exception of the constants, namely $\frac{2}{S}$. We can indeed relate its components to the shape of the triangle and its area in a similar way as we did in (2.20):

$$\begin{aligned} \rho_i &= 2S + 4\vec{m}_i \cdot \vec{F} \\ A &= \sqrt{3}F_2 \end{aligned}$$

On the other hand, the operator F^2, in analogy with J^2 studied before, will be given by:

$$\begin{aligned} F^2 &= \frac{n_1 + n_2}{2}\left(\frac{n_1 + n_2}{2} + 1\right) = \frac{\Psi_1^\dagger \Psi_1 + Psi_2^\dagger \Psi_2}{2}\left(\frac{\Psi_1^\dagger \Psi_1 + \Psi_2^\dagger \Psi_2}{2} + 1\right) \\ &= \frac{S-1}{2}\left(\frac{S-1}{2} + 1\right) \\ &= \frac{1}{4}(S^2 - 1) \end{aligned}$$

The $SU(2)$ algebra relation then becomes clear. First, because it is already related to the symmetry of the shape vector $\vec{\zeta}$ and then because we deduced it in the Schwinger oscillator implementation.

The quantum number associated with this angular momentum $\vec{F}$ are clearly given by $(s = S, m)$ where m is some eigenvalue of some component of the shape vector $\vec{F}$. This, together with the quantum number $l = L$ completely characterises an eigenstate of our system H_{gc}. The eigen-energies associated with an eigenstate (l, m, s) are then given by:

$$E_{l,m,s} = V_{s,m} + \omega(s + l)$$

With this, we consider the quantum problem fully analysed, at least for the purpose of this document.

Chapter 4

The problem of a charged particle in the magnetic field of a monopole

With the objective to obtain some intuition about the N-body problem restricted to a spherical geometry, we study in this chapter the symmetries and trajectories of a charged particle under the influence of the magnetic field of a monopole. To achieve this we present the deduction, via the Lagrangian formalism, of the so called Poincaré cone [8] that characterises the trajectory of a particle in this situation. We then extrapolate the important symmetries used in the Lagrangian formalism to the Hamiltonian formalism to retrieve some important aspects of the classical counterparts of the known Haldane formalism [6].

4.1 Definition of the Lagrangian

Let L be the Lagrangian of a charged particle of charge $-e$ and mass m under the influence of a magnetic monopole of magnitude g. Then L takes the form:

$$L\left(\vec{r},\dot{\vec{r}}\right)=\frac{m}{2}\left\|\dot{\vec{r}}\right\|^{2}-e\vec{A}_{\hat{u}}(\vec{r})\cdot\dot{\vec{r}} \tag{4.1}$$

Where $\vec{A}(\vec{r})$ is the vector potential of the magnetic monopole with singularity along the direction defined by the unit vector $\hat{u}$. This singularity arises from our inability to reproduce the magnetic field of the monopole with the electromagnetic theory and it has no physical meaning [6]. In addition, the different vector potentials identified by different unit vectors $\hat{u}$ are related by a gauge transformation which leaves the trajectories invariant. This family of vector potentials are given by [9]:

$$\vec{A}_{\hat{u}}(\vec{r}) = \frac{g}{r}\frac{\hat{u}\times\hat{r}}{1+\hat{u}\cdot\hat{r}} \tag{4.2}$$

4.2 The symmetries and its conserved quantities

The first thing to note in the previously defined Lagrangian is its time independence, which yields to the conservation of the Jacobi integral:

$$\frac{\partial L}{\partial\dot{\vec{r}}}\cdot\dot{\vec{r}} - L = m\left\|\dot{\vec{r}}\right\|^2 = m\left\|\dot{\vec{r_0}}\right\|^2$$

With $\dot{\vec{r_0}}$ the initial velocity of the particle. Here the Jacobi integral clearly represents the kinetic energy of the particle, which is constant in time because the magnetic field does no work.

Now, due to the simplicity of the problem, one can expect some other symmetries. The next symmetry presented here is not associated with the Lagrangian, but rather with the action invariance due to the transformation involving the time parameter. If we define the action as in the equation (4.3), it can be seen that it may be invariant under a proper scale transform of position and time. It is not difficult to find this transform, and it is presented in equation (4.4).

$$S = \int_{t_1}^{t_2} L\left(\vec{r},\dot{\vec{r}}\right)dt = \int_{t_1}^{t_2}\left(\frac{m}{2}\left\|\dot{\vec{r}}\right\|^2 - e\vec{A}_{\hat{u}}(\vec{r})\cdot\dot{\vec{r}}\right)dt \tag{4.3}$$

$$\begin{aligned}\vec{r}' &= e^s\vec{r}\\ t' &= e^{2s}t\end{aligned} \tag{4.4}$$

As a result of this symmetry, by the general form of Noether's theorem, there must be a conserved quantity implied. To calculate it we prefer the method stated in [10, 2.19 Noether's Thm]; however, to apply this method, the Lagrangian must be parametrised to include the time t as a generalised coordinate.

To achieve this, note that:

$$\dot{\vec{r}} = \frac{d\vec{r}}{dt} = \frac{\frac{d\vec{r}}{d\tau}}{\frac{dt}{d\tau}} := \frac{\mathring{\vec{r}}}{\mathring{t}}$$

Then the action can be written in terms of the new parameter τ:

$$S = \int_{t_1}^{t_2} L\left(\vec{r}, \dot{\vec{r}}\right) dt = \int_{\tau_1}^{\tau_2} \mathring{t} L\left(\vec{r}, \mathring{\vec{r}}\mathring{t}^{-1}\right) d\tau = \int_{\tau_1}^{\tau_2} L'\left(\vec{r}, \mathring{\vec{r}}, \mathring{t}\right) d\tau$$

Which by analogy with equation (4.3), gives the new parametrised Lagrangian L':

$$L'\left(\vec{r}, \mathring{\vec{r}}, \mathring{t}\right) = \mathring{t} L\left(\vec{r}, \mathring{\vec{r}}\mathring{t}^{-1}\right) = \frac{m}{2\mathring{t}} \left\|\mathring{\vec{r}}\right\|^2 - e\vec{A}_{\hat{u}}(\vec{r}) \cdot \mathring{\vec{r}}$$

As we converted the symmetry of the action S in a symmetry of the Lagrangian L', the invariance given by equation (4.4) becomes clear. Now, using Noether's theorem [10], we obtain the conserved quantity in accordance with the transformation (4.4):

$$G = \left.\frac{\partial L'}{\partial \mathring{q}_i} \frac{\partial q'_i}{\partial s}\right|_{s=0} = -m \left\|\frac{\mathring{\vec{r}}}{\mathring{t}}\right\|^2 t + \left(\frac{m\mathring{\vec{r}}}{\mathring{t}} - e\vec{A}_{\hat{u}}\right) \cdot \vec{r}$$

$$G = m\dot{\vec{r}_0} \cdot \vec{r_0} = -m \left\|\dot{\vec{r}}\right\|^2 t + m\dot{\vec{r}} \cdot \vec{r}$$

Working the previous conserved quantity one can obtain an equation for the magnitude of the position in function of time:

$$2\dot{\vec{r}} \cdot \vec{r} = \frac{d\left\|r\right\|^2}{dt} = 2\left\|\dot{\vec{r}_0}\right\|^2 t + 2\dot{\vec{r}_0} \cdot \vec{r_0}$$

$$r^2 = r_0^2 + 2\dot{\vec{r}_0} \cdot \vec{r_0} t + \left\|\dot{\vec{r}_0}\right\|^2 t^2$$

$$r^2 = \left\| \vec{r_0} + \dot{\vec{r_0}} t \right\|^2 \tag{4.5}$$

From equation (4.5) it is important to note that the only way the radius of the particle stays constant is that the initial velocity of the particle is zero, otherwise, the time term will always contribute to the change of that radius. Furthermore, as constant magnetic fields only affect perpendicular velocities, one expect equation (4.5) to be fulfilled not only in norm, but in direction when the initial velocity $\dot{\vec{r_0}}$ is parallel to the initial radius r_0. We then retrieve the formula for uniform motion with no acceleration. To verify this suspicion, let us analyse other quantities associated with the symmetries of the problem.

Taking our attention to the usual symmetries on the original Lagrangian L in equation (4.1) , we note that once chosen a unit vector $\hat{u}$ for the vector potential $\vec{A}_{\hat{u}}$, it is quite clear that L is invariant under rotations around said unit vector $\hat{u}$. To obtain the conserved quantity, take following infinitesimal transformation:

$$\vec{r}' = \vec{r} + s\left(\hat{u} \times \vec{r}\right) \tag{4.6}$$

As this transformation does not include the time t, we can perform the calculation of the conserved quantity over the original Lagrangian:

$$\begin{aligned}
G_2 &= \frac{\partial L}{\partial \dot{\vec{r}}} \cdot \frac{\partial \vec{r}}{\partial s}\bigg|_{s=0} \\
G_2 &= \left(m\dot{\vec{r}} - e\vec{A}_{\hat{u}}\right) \cdot \left(\hat{u} \times \vec{r}\right) \\
G_2 &= \left(m\vec{r} \times \dot{\vec{r}}\right) \cdot \hat{u} - eg\frac{\left\|\hat{u} \times \hat{r}\right\|^2}{\left(1 + \hat{r} \cdot \hat{u}\right)} \\
G_2 &= \left(m\vec{r} \times \dot{\vec{r}}\right) \cdot \hat{u} - eg\left(1 - \hat{r} \cdot \hat{u}\right) \\
J_{\hat{u}} &:= \left(m\vec{r} \times \dot{\vec{r}} + eg\hat{r}\right) \cdot \hat{u} = const
\end{aligned}$$

Now, the last argument is valid for any unit vector $\hat{u}$ chosen and this yields the conservation of the known as Poincaré vector in equation (4.7).

$$\vec{J} = \left(m\vec{r} \times \dot{\vec{r}}\right) + eg\hat{r} \tag{4.7}$$

This last symmetry is very meaningful because it restricts the trajectory of the particle to a cone centred in the origin with central vector $\hat{J}$. To verify this, it is only necessary to see that the radial component of the Poincaré vector is constant for all points in the trajectory, which means that the angle between $\vec{J}$ and $\vec{r}(t)$ is a constant and that the path of the particle is restricted to a cone:

$$\vec{J} \cdot \hat{r} = eg$$

Furthermore, we can deduce from the conservation of the Poincaré vector that the angular momentum $\mathbb{L} = m\vec{r} \times \dot{\vec{r}}$ of the particle is constant in magnitude and that it determines the aperture of the cone of restriction.:

$$\begin{aligned}
\left\|\vec{J}\right\|^2 &= \|\mathbb{L}\|^2 + (gc)^2 \\
\|\mathbb{L}\| &= const \\
\cos\theta &= \frac{\vec{J} \cdot \hat{r}}{\left\|\vec{J}\right\|} = \sqrt{\frac{(ge)^2}{\|\mathbb{L}\|^2 + (ge)^2}}
\end{aligned} \tag{4.8}$$

From this equations it can be seen that the angle of aperture of the Poincaré cone is zero when the angular momentum $\mathbb{L}$ cancel, which means that the particle performs rectilinear motion, as deduced before from the other symmetries of the problem.

It is important to note here that if we restrict the trajectories of the particle to be in a sphere, we cannot carry a scale transform, hence we would not obtain the radius trajectory described in equation (4.5). However, a rotation is consistent with the norm conservation of the restriction to the sphere, and consequently we can obtain the Poincaré's vector invariance. The cone confinement together with the restriction of the trajectories to a constant radius would result in the particle describing a circular trajectory on the sphere.

Moreover in the sphere restriction the position vector and the velocity vector must be perpendicular, therefore, as the norm of the position is the constant radius of the sphere, the conservation of the angular momentum would result in the conservation of the linear velocity, meaning that the circular motion of the particle is in fact uniform, in analogy with the problem in the sphere.

Another important aspect that can be deduced from the set of equations (4.8) is that if we choose a magnetic monopole with big charge g compared with the angular momentum (in proper units), the angle of the Poincaré cone would tend to zero, which in the case of the particle restricted to the sphere, would mean that the radius of the circular motion would also tend to zero, as happens with the already studied case of the particle in the plane. This gives us some clues as where to look for the analogue guiding center formalism in the case of the magnetic monopole.

4.3 Important quantities in the Hamiltonian formalism

From the analysis carried out before, we can see some important quantities that are useful to describe the movement of particles in the presence of a magnetic monopole, some of which are associated with certain symmetries of the problem. Here we would like to study specially the angular momentum $\mathbb{L} = m\vec{r} \times \dot{\vec{r}}$ and the Poincaré vector $\vec{J} = \left(m\vec{r} \times \dot{\vec{r}}\right) + eg\hat{r}$. To do that, let us first calculate the Hamiltonian for the particle in the magnetic field of a monopole, this time restricting the radius r of the sphere to a constant:

$$H\left(\vec{r}, \vec{p}\right) = \frac{1}{2m} \left\|\vec{p} + e\vec{A}_{\hat{u}}(\vec{r})\right\|^2 \Big|_{S^2} \tag{4.9}$$

As the Hamiltonian is the Legendre transform of the Lagrangian, we obtain the generalised momentum $\vec{p}$ in terms of the velocity of the particle. Moreover, we can see that the Hamiltonian is just the kinetic energy of the particle:

$$\vec{p} = \frac{\partial L}{\partial \dot{\vec{r}}} = m\dot{\vec{r}} - e\vec{A}_{\hat{u}} := \vec{\pi} - e\vec{A}_{\hat{u}}$$

$$H = \frac{1}{2m} \left\|\vec{\pi}\right\|^2 \Big|_{S^2}$$

From here we can propose the angular momentum of the particle as $\mathbb{L} = \vec{r} \times \vec{\pi}$ and we can calculate its Poisson bracket. To do so, it is useful to calculate first some Poisson brackets related to the linear momentum $\vec{\pi} = \vec{p} + e\vec{A}$, as it was done in the first chapter:

$$
\begin{aligned}
\{\pi_i, \pi_j\} &= \frac{\partial \pi_i}{\partial r_l}\frac{\partial \pi_j}{\partial p_l} - \frac{\partial \pi_j}{\partial r_l}\frac{\partial \pi_i}{\partial p_l} \\
&= e\delta_{lj}\frac{\partial A_i}{\partial r_l} - e\delta_{li}\frac{\partial A_j}{\partial r_l} = e\left(\delta_{lj}\delta_{mi} - \delta_{li}\delta_{mj}\right)\frac{\partial A_m}{\partial r_l} \\
&= -e\epsilon_{ijk}\epsilon_{lmk}\frac{\partial A_m}{\partial r_l} \\
&= -e\epsilon_{ijk}(\nabla \times \vec{A})_k = e\epsilon_{ijk}B_k = -\frac{eg}{r^3}\epsilon_{ijk}r_k
\end{aligned}
$$

$$
\{\pi_i, r_j\} = \frac{\partial \pi_i}{\partial r_l}\cancelto{0}{\frac{\partial r_j}{\partial p_l}} - \frac{\partial r_j}{\partial r_l}\frac{\partial \pi_i}{\partial p_l} = -\delta_{ij}
$$

Then the algebra becomes a little bit easier for $\mathbb{L}$

$$
\begin{aligned}
\{L_i, L_j\} &= \{\epsilon_{iab}r_a\pi_b, \epsilon_{jcd}r_c\pi_d\} = \epsilon_{iab}\epsilon_{jcd}\{r_a\pi_b, r_c\pi_d\} \\
&= \epsilon_{iab}\epsilon_{jcd}\left(r_ar_c\{\pi_b, \pi_d\} + \pi_br_c\{r_a, \pi_d\} + r_a\pi_d\{\pi_b, r_c\} + \pi_b\pi_d\cancelto{0}{\{r_a, r_c\}}\right) \\
&= \epsilon_{iab}\epsilon_{jcd}\left(-\frac{eg}{r^3}\epsilon_{bdk}r_ar_cr_k + \pi_br_c\delta_{ad} - \pi_dr_a\delta_{bc}\right) \\
&= -\frac{eg}{r^3}\epsilon_{iab}\left(\delta_{jk}\delta_{cb} - \delta_{jb}\delta_{ck}\right)r_ar_cr_k + \left(\delta_{bj}\delta_{ic} - \cancel{\delta_{bc}\delta_{ij}}\right)\pi_br_c - \left(\delta_{aj}\delta_{ib} - \cancel{\delta_{ia}\delta_{jb}}\right)\pi_dr_a \\
&= -\frac{eg}{r^3}\left(r_j\cancelto{0}{\epsilon_{iab}r_ar_b} - \epsilon_{aji}r_ar_br_b\right) + \left(\delta_{ia}\delta_{jb} - \delta_{ib}\delta_{ja}\right)r_a\pi_b \\
&= \epsilon_{ijk}\left(\epsilon_{abk}r_a\pi_b - eg\hat{r}_k\right) = \epsilon_{ijk}\left(L_k - eg\hat{r}_k\right)
\end{aligned}
$$

Now, we can observe that $\mathbb{L}$ does not follow the canonical relations for angular momentum, which is not surprising because the momentum $\vec{\pi}$ used in the definition of $\mathbb{L}$ is not the canonical momentum p. We can fix this by taking a slight variation in $\mathbb{L}$:

$$
\mathbb{J} := \mathbb{L} + eg\hat{r} \tag{4.10}
$$

The Poisson bracket relation for this angular momentum is given by:

$$
\begin{aligned}
\{J_i, J_j\} &= \left\{L_i + \frac{eg}{r} r_i, L_j + \frac{eg}{r} r_j\right\} \\
&= \{L_i, L_j\} + \frac{eg}{r}\left(\{L_i, r_j\} + \{r_i, L_j\}\right) + \left(\frac{eg}{r}\right)^2 \cancelto{0}{\{r_i, r_j\}} \\
&= \epsilon_{ijk}\left(L_k - eg\hat{r}_k\right) + \frac{eg}{r}\left(-\delta_{lj}\frac{\partial \epsilon_{iab} r_a \left(p_b + eA_b\right)}{\partial p_l} + \delta_{il}\frac{\partial \epsilon_{jcd} r_c \left(p_d + eA_d\right)}{\partial p_l}\right) \\
&= \epsilon_{ijk}\left(L_k - eg\hat{r}_k\right) + \frac{eg}{r}\left(-\epsilon_{jia} r_a + \epsilon_{ijc} r_c\right) \\
&= \epsilon_{ijk}\left(L_k + eg\hat{r}_k\right) = \epsilon_{ijk} J_k
\end{aligned}
$$

Then we deduce that $\mathbb{J}$ satisfies the canonical relations for angular momentum. It is not difficult to see that this is in fact the Poincaré vector in equation (4.7) and that, as seen before, it determines the center of the circular motion performed by the particle:

$$
\mathbb{J} = \vec{r} \times \vec{\pi} + eg\hat{r} = m\vec{r} \times \dot{\vec{r}} + eg\hat{r} = \vec{J}
$$

Moreover, we can calculate the Poisson brackets of this angular momentum with other important vectors of the system, namely $\vec{r}$ and $\vec{L}$:

$$
\begin{aligned}
\{J_i, r_j\} &= \{L_i + eg\hat{r}_i, r_j\} = \epsilon_{iab} r_a \{\pi_b, r_j\} \\
&= \epsilon_{ijk} r_k
\end{aligned}
$$

$$
\begin{aligned}
\{J_i, L_j\} &= \left\{J_i, J_j - \frac{eg}{r} r_i\right\} = \epsilon_{ijk} J_k - \frac{eg}{r}\{L_i, r_j\} = \epsilon_{ijk}(J_k - eg\hat{r}_k) \\
&= \epsilon_{ijk} L_k
\end{aligned}
$$

With this, we conclude that the angular momentum $\mathbb{J}$ is also the generator of rotations over the sphere.

Here it is important to clarify that this angular momentum is gauge-invariant and arises, as seen before in the Lagrangian formalism, from the rotational symmetry of the system that leaves invariant the equations of movement. This is indeed the difference between this angular momentum and the canonical momentum $\vec{r} \times \vec{p}$.

Chapter 5

The three body problem in the sphere

Given the introduction about the one-body problem under the influence of a monopolar magnetic field, we are now able to generalize the three-body problem restricted to the sphere. As we did in chapter 2 and 3, we would like to search for a similar analysis for the three-body problem on the sphere on huge magnetic field regimen. As we will discuss later in this chapter, an analogous analysis is not trivial, however, we explain some important aspects of the problem and give some general guidelines that may, trough future work, carry to the so wanted analogous analysis.

This chapter is organized the following way: First, the problem is going to be formally defined. Then, a brief discussion of the integrability in spherical and Cartesian coordinates and some disadvantages of these coordinate systems are going to be presented. Finally, the stereographic projection, as an alternative coordinate system to analyse similarities is going to be introduced.

5.1 Definition of the problem

In this problem, we practically have three Hamiltonians (4.9) of free particles in a monopolar magnetic field, coupled by a potential V with the three-dimensional rotational symmetries. To make things easier, we choose the same gauge for the vector potentials associated with each particle. In this terms, the three-body Hamiltonian will have the form:

$$H = \frac{1}{2m}\left(\|\vec{\pi}_1\|^2 + \|\vec{\pi}_2\|^2 + \|\vec{\pi}_3\|^2\right) + V\left(\vec{r}_1, \vec{r}_2, \vec{r}_3\right)\Big|_{S^2}$$
$$= \frac{1}{2m}\left(\left\|\vec{p}_1 + e\vec{A}_{\hat{u}}(\vec{r}_1)\right\|^2 + \left\|\vec{p}_2 + e\vec{A}_{\hat{u}}(\vec{r}_2)\right\|^2 + \left\|\vec{p}_3 + e\vec{A}_{\hat{u}}(\vec{r}_3)\right\|^2\right) + V\left(\vec{r}_1, \vec{r}_2, \vec{r}_3\right)\Big|_{S^2} \quad (5.1)$$

Where $\vec{\pi}_i = \vec{p}_i + e\vec{A}_{\hat{u}}(\vec{r}_i)$ is the linear momentum of each particle, $\vec{A}_{\hat{u}}(\vec{r})$ is defined in equation (4.2), and the potential V satisfies:

$$V\left(R\vec{r}_1, R\vec{r}_2, R\vec{r}_3\right) = V\left(\vec{r}_1, \vec{r}_2, \vec{r}_3\right)$$

For any three-dimensional rotation.

5.2 The analogous canonical transformation of the guiding centres in Cartesian coordinates

Since the canonical coordinates of each particle are independent, we can define the Poincaré vector for each particle in the same way as in equation (4.10) without loosing its angular momentum Poisson relations.

$$\begin{aligned}\vec{J}_i &= \vec{r}_i \times \vec{\pi}_i + eg\hat{r}_i \\ &= \vec{r}_i \times \left(\vec{p}_i + e\vec{A}_{\hat{u}}(\vec{r}_i)\right) + eg\hat{r}_i \\ \{J_i, J_j\} &= \epsilon_{ijk} J_k\end{aligned}$$

Now, as seen before, $\vec{J}_i$ is the generator of rotations, in this case, of the coordinate system of each particle. To obtain the generator of rotations for the general system, we simply add up the three Poincaré vectors producing $\mathbb{J}$, which will also follow the angular momentum relations.

It was previously discussed that the Poincaré vector points in the direction of the guiding center of a particle. We therefore are tempted to take this vector as the analogue of the canonical guiding center transformation performed in the plane:

$$\vec{R}_i = \frac{\vec{J}_i}{\frac{eg}{r}} = \vec{r}_i + \frac{\hat{r}_i \times \vec{\pi}_i}{\frac{eg}{r^2}}$$
$$= \vec{r}_i + \frac{\hat{r}_i \times \vec{\pi}_i}{eB} \tag{5.2}$$

In fact, the previous transformation (5.2) has the same form of that of the plane on equation (2.4). The problem is completely analogous: in stead of $\hat{k}$ indicating the direction of the constant magnetic field on the plane, we obtain $\hat{r}$ that is actually the direction of the magnetic field of the monopole.

This can get things complicated as $\vec{r}$ is not constant in time, but points to the position of the particle on the sphere at any time. This situation is reflected in the fact that this transformation is not canonical, but it follows the relations of angular momentum with the exemption of constants, namely $\frac{eg}{r}$.

Anyway, given the relation on equation (5.2) in the big magnetic field regime we can also make the assumption that the guiding center is approximately equal to the position of the particle, simplifying the problem in a similar way. The next step on this problem would be to find the analogue to the linear momentum that characterises the radius of the guiding center and the position in the circle. As the guiding center variables follow the relations of angular momentum, it would be ideal to find this analogue quantity to follow the same relations.

However, the principal idea of doing the guiding center transformation is to obtain some separated canonical quantities with null Poisson brackets with the objective of separating the former Hamiltonian into two decoupled problems under a big magnetic field B. Then, while it is ideal to find the complementary quantity of the spherical guiding center to satisfy the angular momentum relations, we would be okay with it not necessarily being an angular momentum but unavoidably satisfying the null Poisson brackets with its guiding center counterpart. Besides, in analogy with the problem on the plane, we require the kinetic part of the Hamiltonian to be a function of these new complementary quantities or at least to be separable into some function of the guiding centres and other of the remaining coordinates.

One can try to find a vector which follows this condition. However, a little bit of thinking tells us that it can not be found at least in the Cartesian coordinate system. The reason is that, as it was demonstrated before, the guiding center transformation produces three generators of rotations of the particular coordinate system. In these terms, no matter what quantities we define, if they are vectors we automatically deduce that the Poisson bracket between this quantity and its respective guiding center coordinate is not null, but represents the infinitesimal rotation of this vector.

We can propose a transformation that encodes the counterpart of the guiding center coordinates in some scalar quantities so that the Poisson bracket with the guiding centres become null, and we can then perform the separation of the problem into two different decoupled Hamiltonians. The finding of these quantities is not easy and anyway would have not helped much in the development of a sphere formalism similar to that of the plane. The problem resides in the characterisation of the guiding center coordinates as the rotation generator in the Cartesian coordinate system, which impedes a proper separation of variables.

To see this clearly, let us assume that we found some proper non-vectorial quantities $\{\Lambda_i\}$ that commute with the quantities $\{\vec{R}_i\}$,and which generate the kinetic part of the Hamiltonian. We then can separate this Hamiltonian into the guiding centres Hamiltonian and other for $\{\Lambda_i\}$. The remaining Hamiltonian would be simply the potential V averaged over the guiding centres and properly rescaled:

$$\begin{aligned} H_{gc} &= V(\vec{J}_1, \vec{J}_2, \vec{J}_3) \\ \{J_{\alpha,i}, J_{\beta,j}\} &= \delta_{\alpha\beta}\epsilon_{ijk}J_{\alpha,k} \end{aligned}$$

Where Greek indices label the particle and Latin indices the coordinate. The next step here would be to find some transformation that decouples the movement of the center of mass of guiding centres from the relative coordinates. However, the relative coordinates are vectorial quantities and would definitely not commute with the center of mass coordinates, nor between them. Moreover, the subtraction of angular momentum is not necessarily another angular momentum.

We would try the same argument as before of finding some scalar quantities to decouple the center of mas from the relative coordinates. Nonetheless, we would loose the coordinate-like transformation and would not be able to easily relate the quantities with the characteristics of the triangle.

The issue with Cartesian coordinates, besides the principal quantities being the generators of rotations, is that there is no simple way to introduce the restriction of the coordinates to the sphere. We will always be working with more coordinates than we need. Taking this into account, an obvious alternative to the solution of this problem is the use of spherical coordinates, where we would get rid of the radial coordinate and momentum, obtaining a problem properly described by two coordinates.

5.3 The problem in spherical coordinates

We redefine then the problem in spherical coordinates to verify if this is a suitable set of quantities to carry out a similar analysis than that of the plane. To do this let us start with the Lagrangian of a single particle in the magnetic field of a monopole restricted to the spherical surface. But first let us define the spherical coordinate transformation and some important relations [**?**]:

$$
\begin{aligned}
x &= r\sin\theta\cos\phi & \hat{r} &= \frac{\partial\vec{r}}{\partial r}\Big/\left\|\frac{\partial\vec{r}}{\partial r}\right\| \\
y &= r\sin\theta\sin\phi & \hat{\theta} &= \frac{\partial\vec{r}}{\partial\theta}\Big/\left\|\frac{\partial\vec{r}}{\partial\theta}\right\| \\
z &= r\cos\theta & \hat{\phi} &= \frac{\partial\vec{r}}{\partial\phi}\Big/\left\|\frac{\partial\vec{r}}{\partial\phi}\right\| \\
\hat{r}\times\hat{\theta} &= \hat{\phi}
\end{aligned}
$$

We begin with the definition of the linear velocity of a single particle in terms of the angular coordinates (θ, ϕ) and their unit vectors $(\hat{\theta}, \hat{\phi})$, which is not difficult to deduce with some algebra:

$$\dot{\vec{r}} = r\dot{\theta}\hat{\theta} + r\sin\theta\dot{\phi}\hat{\phi} \tag{5.3}$$

Now, transforming the vector potential with the convenient gauge along the z axis, we obtain:

$$\vec{A}_{\hat{k}} = \frac{eg}{r}\tan\frac{\theta}{2}\hat{\phi}$$

As these unit vectors are orthogonal we obtain the following Lagrangian:

$$L = \frac{mr^2}{2}\left(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right) + \frac{eg}{r}\tan\frac{\theta}{2}\dot{\phi}$$

With this Lagrangian, the canonical conjugate momenta are determined:

$$p_\theta = mr^2\dot{\theta} p_\phi = mr^2\sin^2\theta\dot{\phi} + \frac{eg}{r}\tan\frac{\theta}{2}$$

Then the Hamiltonian is determined:

$$H = \frac{1}{2mr^2}\left(p_\theta^2 + \left(\frac{p_\phi - \frac{eg}{r}\tan\frac{\theta}{2}}{\sin\theta}\right)^2\right) \tag{5.4}$$

We can then define a linear momentum vector in analogy with this Hamiltonian and with the linear velocity previously defined:

$$\begin{aligned}
\vec{\pi} &= \frac{p_\theta}{r}\hat{\theta} + \frac{p_\phi - \frac{eg}{r}\tan\frac{\theta}{2}}{r\sin\theta}\hat{\phi} \\
\pi_\theta &= \frac{p_\theta}{r} \\
\pi_\phi &= \frac{p_\phi - \frac{eg}{r}\tan\frac{\theta}{2}}{r\sin\theta}
\end{aligned}$$

The purpose of this more or less long introduction to the problem in spherical coordinates was to obtain the linear momenta formula. With this formula one can calculate the Poisson bracket between the two components (π_θ, π_ϕ) and realize that, given the dependence of π_ϕ on both momenta and coordinates, this Poisson bracket will have a rather complicated expression that is not worth writing down. This tells us that the counterpart for the guiding center transformation on spherical coordinates wont function at least for the linear momentum.

One may argument that the correct analogous of the linear momentum in the plane are the conjugate angular momenta on the sphere in stead of the same linear momentum. While this argument is right and makes a lot of sense, we find that even with this consideration we obtain a set of variables that are not much different from before and posses the same problem:

$$\begin{aligned}
\vec{L} &= p_\theta \hat{\theta} + \left(p_\phi - \frac{eg}{r} \tan \frac{\theta}{2} \right) \hat{\phi} \\
L_\theta &= p_\theta \\
L_\phi &= p_\phi - \frac{eg}{r} \tan \frac{\theta}{2}
\end{aligned}$$

When we calculate the Poisson brackets between these two quantities, one obtain the following result:

$$\begin{aligned}
\{L_\theta, L\phi\} &= \cancelto{0}{\{p_\theta, p_\phi\}} - \left\{ p_\theta, \frac{eg}{r} \tan \frac{\theta}{2} \right\} \\
&= \frac{eg}{r} \frac{\partial \tan \frac{\theta}{2}}{\partial \theta} \\
&= \frac{eg}{r} \frac{1}{1 + \cos \theta}
\end{aligned}$$

While this expression is very close to something simple as happened in the case of the plane, we can not characterise this transformation as canonical. The reason is that this Poisson bracket can get as big as possible when $\theta \to \pi$. If we approximate this Poisson bracket to $\frac{eg}{r}$, which is valid when $\theta \to 0$, we would obtain the so wanted canonical transformation for the momentum. However, with this approximation, we are taking a region on the north pole of the sphere as if it was a plane, and the problem on the sphere its curvature.

On the other hand, regarding the transformation to the guiding center angular momentum, one can simply transform the momentum $\vec{J}$ to spherical coordinates. In fact, the same rotational symmetries analysed on chapter 4 have to produce the same conserved quantity.

$$
\begin{aligned}
\vec{J} &= \vec{r} \times \vec{\pi} + eg\hat{r} \\
&= r\pi_\theta \cancelto{\hat{\phi}}{(\hat{r} \times \hat{\theta})} + r\pi_\phi \cancelto{-\hat{\theta}}{(\hat{r} \times \hat{\phi})} + eg\hat{r} \\
&= -r\pi_\phi \hat{\theta} + r\pi_\theta \hat{\phi} + eg\hat{r}
\end{aligned}
$$

We then discover that the guiding center angular momentum has angular components that are the same than the linear momentum, besides a constant radial component. The Poisson bracket relations do not change a little bit in comparison to that of the linear momentum components. Besides not finding a nice expression for the algebra followed by the guiding center and the linear momenta, we find in the spherical coordinates a bigger problem: The linear momentum and the guiding center cannot be properly decoupled because they are practically the same.

These reasons leaded us to abandon the study of the problem in spherical coordinates.

5.4 Integrability of the system

5.5 Analysis of the movement of the guiding centres

Chapter 6

Conclusions

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