

UNIVERSITY NAME (IN BLOCK CAPITALS)

# Three body problem in the spherical geometry

by

Jesus David Prada Gonzalez

A thesis submitted in partial fulfillment for the  
degree of Doctor of Philosophy

in the

Faculty Name

Department or School Name

February 2016

# Declaration of Authorship

I, AUTHOR NAME, declare that this thesis titled, 'THESIS TITLE' and the work presented in it are my own. I confirm that:

- This work was done wholly or mainly while in candidature for a research degree at this University.
- Where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated.
- Where I have consulted the published work of others, this is always clearly attributed.
- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work.
- I have acknowledged all main sources of help.
- Where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself.

Signed:

---

Date:

---

*“Write a funny quote here.”*

If the quote is taken from someone, their name goes here

UNIVERSITY NAME (IN BLOCK CAPITALS)

# *Abstract*

Faculty Name

Department or School Name

Doctor of Philosophy

by Jesus David Prada Gonzalez

The Thesis Abstract is written here (and usually kept to just this page). The page is kept centered vertically so can expand into the blank space above the title too...

# *Acknowledgements*

The acknowledgements and the people to thank go here, don't forget to include your project advisor...

# Contents

<b>Declaration of Authorship</b>	<b>i</b>
<b>Abstract</b>	<b>iii</b>
<b>Acknowledgements</b>	<b>iv</b>
<b>List of Figures</b>	<b>vi</b>
<b>List of Tables</b>	<b>vii</b>
<b>Abbreviations</b>	<b>viii</b>
<b>Physical Constants</b>	<b>ix</b>
<b>Symbols</b>	<b>x</b>
<b>1 Introduction</b>	<b>1</b>
1.1 The three body problem in the plane . . . . .	1
1.1.1 The canonical transformation of the guiding centres . . . . .	2
1.2 Another Section . . . . .	4
<b>A An Appendix</b>	<b>5</b>

# List of Figures

# List of Tables



# Abbreviations

**LAH** List Abbreviations **Here**

# Physical Constants

Speed of Light  $c = 2.997\,924\,58 \times 10^8 \text{ ms}^{-\text{s}}$  (exact)

# Symbols

$a$	distance	m
$P$	power	W ( $\text{Js}^{-1}$ )
$\omega$	angular frequency	$\text{rads}^{-1}$

*For/Dedicated to/To my...*

# Chapter 1

## Introduction

In this chapter a classic approach of a somehow general case of the three body problem in 2 dimensions is going to be presented. This will give some necessary intuition to develop the analogous problem in the spherical geometry. To begin with, the problem is going to be described in great detail; then its integrability is going to be proven; and finally, a formalism to describe the movement of the particles is going to be presented.

### 1.1 The three body problem in the plane

The three body problem presented here is that of three particles of electrical charge  $e$  and mass  $m$  confined to a plane, under the influence of a strong magnetic field perpendicular to it, and forces whose potentials satisfy translational and rotational symmetries in the plane.

Given this information, the Hamiltonian associated with this system has the form:

$$H = \sum_{i=1}^3 \frac{1}{2m_i} \left\| \vec{p}_i - e\vec{A}(\vec{q}_i) \right\|^2 + V(\vec{q}_1, \vec{q}_2, \vec{q}_3) + \frac{\omega_c^2}{2} \sum_{i=1}^3 \|\vec{q}_i\|^2 \quad (1.1)$$

Where  $\vec{q}_i = x_i\hat{i} + y_i\hat{j}$ ,  $\vec{p}_i = p_{x_i}\hat{i} + p_{y_i}\hat{j}$  and  $\vec{A}(\vec{q})$  is the magnetic vector potential, which satisfies  $\nabla \times \vec{A} = B\hat{k}$ .

Besides, the potential  $V(\vec{q}_1, \vec{q}_2, \vec{q}_3)$  satisfies the symmetries:

$$V(R\vec{q}_1 + \vec{a}, R\vec{q}_2 + \vec{a}, R\vec{q}_3 + \vec{a}) = V(\vec{q}_1, \vec{q}_2, \vec{q}_3) \quad (1.2)$$

For any rotation  $R$  and translation  $\vec{a}$  in the plane.

### 1.1.1 The canonical transformation of the guiding centres

For the proof of integrability for this system, and for further analysis of the trajectories of the particles, let us perform the well known transformation of the guiding centres.

This transformation is defined by the following two equations:

$$\vec{\pi}_i = \vec{p}_i - e\vec{A}(\vec{q}_i) \quad (1.3)$$

$$\vec{R}_i = \vec{q}_i - \frac{\hat{k} \times \vec{\pi}_i}{eB} \quad (1.4)$$

The equation 1.3 passes from the canonical momentum  $\vec{p}_i$  to the linear momentum  $\vec{\pi}_i$ , which is much more intuitive and understandable; while the equation 1.4 transforms the general position  $\vec{q}_i$  to the position of the instantaneous guiding centre  $\vec{R}_i$ .

In a system without the interaction potentials, the electrically charged particles are known to perform the circular motion of the cyclotron with radii that depends on the initial linear momenta. In this case, the guiding centres would be constant in time as would be the linear momenta. However, with the introduction of an interacting potential, the momentum of each particle may vary making the guiding centre change too, which is why the instantaneous interpretation of the guiding centres is necessary.

Now, let us calculate the Poisson brackets for this new set of coordinates in a specific particle.

$$\begin{aligned}
 \{\pi_1, \pi_2\} &= \frac{\partial \pi_1}{\partial q_\alpha} \frac{\partial \pi_2}{\partial p_\alpha} - \frac{\partial \pi_2}{\partial q_\alpha} \frac{\partial \pi_1}{\partial p_\alpha} \\
 &= -e \delta_{\alpha 2} \frac{\partial A_1}{\partial q_\alpha} + e \delta_{\alpha 1} \frac{\partial A_2}{\partial q_\alpha} \\
 &= -e \frac{\partial A_1}{\partial q_2} + e \frac{\partial A_2}{\partial q_1} \\
 &= e(\nabla \times \vec{A})_3 = eB
 \end{aligned}$$

$$\begin{aligned}
 \{R_1, R_2\} &= \{q_1, q_2\} + \left\{q_1, -\frac{\pi_1}{eB}\right\} + \left\{\frac{\pi_2}{eB}, q_2\right\} + \left\{\frac{\pi_2}{eB}, -\frac{\pi_1}{eB}\right\} \\
 &= \frac{1}{eB} \left( \cancel{\{p_1, q_1\}}^{-1} - e \cancel{\{A_1, q_1\}}^0 + \cancel{\{p_2, q_2\}}^{-1} - e \cancel{\{A_2, q_2\}}^0 \right) + \frac{eB}{(eB)^2} \\
 &= \frac{-2}{eB} + \frac{1}{eB} = -(eB)^{-1}
 \end{aligned}$$

$$\begin{aligned}
 \{R_1, \pi_2\} &= \{q_1, \pi_2\} + \left\{\frac{\pi_2}{eB}, \pi_2\right\} \\
 &= \cancel{\{q_1, p_2\}}^0 - e \cancel{\{q_1, A_2\}}^0 \\
 &= \{R_2, \pi_1\} = 0
 \end{aligned}$$

This Poisson brackets can be generalised to the transformation for the three particles. Taking  $i, j = \{1, 2, 3\}$  and  $\alpha, \beta = \{1, 2\}$ :

$$\{\pi_{i,\alpha}, \pi_{j,\beta}\} = (eB) \delta_{ij} \epsilon_{\alpha\beta} \quad (1.5)$$

$$\{R_{i,\alpha}, R_{j,\beta}\} = -(eB)^{-1} \delta_{ij} \epsilon_{\alpha\beta} \quad (1.6)$$

$$\{R_{i,\alpha}, \pi_{j,\beta}\} = 0 \quad (1.7)$$

Equations ?? allow us to identify the proposed transformation as canonical. However, this is not the usual canonical transformation where the position coordinates and the momentum coordinates are canonical conjugates. In this special case, one component of the momentum is canonical conjugate with the other momentum coordinate, and equally for the position coordinates.

Now, with a huge magnetic field, if the potential of the interaction forces does not vary abruptly in space, we can use the approximation  $\vec{R}_i \approx \vec{q}_i$  to average the potentials over the guiding centres, that is, we can replace  $\vec{q}_i$  for  $\vec{R}_i$  in  $V(\vec{q}_1, \vec{q}_2, \vec{q}_3)$ .

We can interpret the last approximation as follows: In the cyclotron problem, the radius of the circular motion described is proportional to the linear momentum and inversely proportional to the magnetic field. Then, in the presence of a big  $B$ , the radius of the cyclotron would shrink to a very small size. In the case we are working, the radius of the instantaneous cyclotron motion would be proportional to  $\left\|(\hat{k} \times \vec{\pi}_i)(eB)^{-1}\right\|$ , then, in a presence of a huge magnetic field, the instantaneous radii of the particles would be small enough that the potential would not sense the instantaneous circular motion and would sense the particle as if it were always in the position of its guiding centre.

## 1.2 Another Section

Phasellus nisi quam, volutpat non ullamcorper eget, congue fringilla leo. Cras et erat et nibh placerat commodo id ornare est. Nulla facilisi. Aenean pulvinar scelerisque eros eget interdum. Nunc pulvinar magna ut felis varius in hendrerit dolor accumsan. Nunc pellentesque magna quis magna bibendum non laoreet erat tincidunt. Nulla facilisi.

Duis eget massa sem, gravida interdum ipsum. Nulla nunc nisl, hendrerit sit amet commodo vel, varius id tellus. Lorem ipsum dolor sit amet, consectetur adipiscing elit. Nunc ac dolor est. Suspendisse ultrices tincidunt metus eget accumsan. Nullam facilisis, justo vitae convallis sollicitudin, eros augue malesuada metus, nec sagittis diam nibh ut sapien. Duis blandit lectus vitae lorem aliquam nec euismod nisi volutpat. Vestibulum ornare dictum tortor, at faucibus justo tempor non. Nulla facilisi. Cras non massa nunc, eget euismod purus. Nunc metus ipsum, euismod a consectetur vel, hendrerit nec nunc.



## Appendix A

# An Appendix

Lorem ipsum dolor sit amet, consectetur adipiscing elit. Vivamus at pulvinar nisi. Phasellus hendrerit, diam placerat interdum iaculis, mauris justo cursus risus, in viverra purus eros at ligula. Ut metus justo, consequat a tristique posuere, laoreet nec nibh. Etiam et scelerisque mauris. Phasellus vel massa magna. Ut non neque id tortor pharetra bibendum vitae sit amet nisi. Duis nec quam quam, sed euismod justo. Pellentesque eu tellus vitae ante tempus malesuada. Nunc accumsan, quam in congue consequat, lectus lectus dapibus erat, id aliquet urna neque at massa. Nulla facilisi. Morbi ullamcorper eleifend posuere. Donec libero leo, faucibus nec bibendum at, mattis et urna. Proin consectetur, nunc ut imperdiet lobortis, magna neque tincidunt lectus, id iaculis nisi justo id nibh. Pellentesque vel sem in erat vulputate faucibus molestie ut lorem.

Quisque tristique urna in lorem laoreet at laoreet quam congue. Donec dolor turpis, blandit non imperdiet aliquet, blandit et felis. In lorem nisi, pretium sit amet vestibulum sed, tempus et sem. Proin non ante turpis. Nulla imperdiet fringilla convallis. Vivamus vel bibendum nisl. Pellentesque justo lectus, molestie vel luctus sed, lobortis in libero. Nulla facilisi. Aliquam erat volutpat. Suspendisse vitae nunc nunc. Sed aliquet est suscipit sapien rhoncus non adipiscing nibh consequat. Aliquam metus urna, faucibus eu vulputate non, luctus eu justo.

Donec urna leo, vulputate vitae porta eu, vehicula blandit libero. Phasellus eget massa et leo condimentum mollis. Nullam molestie, justo at pellentesque vulputate, sapien velit ornare diam, nec gravida lacus augue non diam. Integer mattis lacus id libero ultrices sit amet mollis neque molestie. Integer ut leo eget mi volutpat congue. Vivamus sodales, turpis id venenatis placerat, tellus purus adipiscing magna, eu aliquam nibh dolor id nibh. Pellentesque habitant morbi tristique senectus et netus et malesuada fames ac turpis egestas. Sed cursus convallis quam nec vehicula. Sed vulputate neque eget odio fringilla ac sodales urna feugiat.

Phasellus nisi quam, volutpat non ullamcorper eget, congue fringilla leo. Cras et erat et nibh placerat commodo id ornare est. Nulla facilisi. Aenean pulvinar scelerisque eros eget interdum. Nunc pulvinar magna ut felis varius in hendrerit dolor accumsan. Nunc pellentesque magna quis magna bibendum non laoreet erat tincidunt. Nulla facilisi.

Duis eget massa sem, gravida interdum ipsum. Nulla nunc nisl, hendrerit sit amet commodo vel, varius id tellus. Lorem ipsum dolor sit amet, consectetur adipiscing elit. Nunc ac dolor est. Suspendisse ultrices tincidunt metus eget accumsan. Nullam facilisis, justo vitae convallis sollicitudin, eros augue malesuada metus, nec sagittis diam nibh ut sapien. Duis blandit lectus vitae lorem aliquam nec euismod nisi volutpat. Vestibulum ornare dictum tortor, at faucibus justo tempor non. Nulla facilisi. Cras non massa nunc, eget euismod purus. Nunc metus ipsum, euismod a consectetur vel, hendrerit nec nunc.