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Three body problem in the spherical geometry

by

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degree of Doctor of Philosophy

in the
Faculty Name
Department or School Name

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Abstract

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Acknowledgements

The acknowledgements and the people to thank go here, don't forget to include your project advisor...

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Abbreviations

LAH	**L**ist **A**bbreviations **H**ere

Physical Constants

Speed of Light $c = 2.997\ 924\ 58 \times 10^{8}\ \text{ms}^{-\text{S}}$ (exact)

Symbols

a	distance	m
P	power	W (Js^{-1})
ω	angular frequency	$rads^{-1}$

For/Dedicated to/To my…

Chapter 1

Introduction

In this chapter a classic approach of a somehow general case of the three body problem in 2 dimensions is going to be presented. This will give some necessary intuition to develop the analogous problem in the spherical geometry. To begin with, the problem is going to be described in great detail; then its integrability is going to be proven; and finally, a formalism to describe the movement of the particles is going to be presented.

1.1 The three body problem in the plane

The three body problem presented here is that of three particles of electrical charge e and mass m confined to a plane, under the influence of a strong magnetic field perpendicular to it, and forces whose potentials satisfy translational and rotational symmetries in the plane.

Given this information, the Hamitonian associated with this system has the form:

$$H = \sum_{i=1}^{3} \frac{1}{2m} \left\| \vec{p_i} - e\vec{A}\left(\vec{q_i}\right) \right\|^2 + V\left(\vec{q_1}, \vec{q_2}, \vec{q_3}\right) + \frac{\omega_c^2}{2m} \sum_{i=1}^{3} \|\vec{q_i}\|^2 \tag{1.1}$$

Where $\vec{q_i} = x_i\hat{\imath} + y_i\hat{\jmath}$, $\vec{p_i} = p_{x_i}\hat{\imath} + p_{y_i}\hat{\jmath}$ and $\vec{A}\left(\vec{q}\right)$ is the magnetic vector potential, which satisfies $\nabla \times \vec{A} = B\hat{k}$.

Besides, the potential $V\left(\vec{q_1}, \vec{q_2}, \vec{q_3}\right)$ satisfies the symmetries:

$$V\left(R\vec{q_1}+\vec{a},R\vec{q_2}+\vec{a},R\vec{q_3}+\vec{a}\right)=V\left(\vec{q_1},\vec{q_2},\vec{q_3}\right) \tag{1.2}$$

For any rotation R and translation $\vec{a}$ in the plane.

1.1.1 The canonical transformation of the guiding centres

For the proof of integrability for this system, and for further analysis of the trajectories of the particles, let us perform the well known transformation of the guiding centres.

This transformation is defined by the following two equations:

$$\vec{\pi_i}=\vec{p_i}-e\vec{A}\left(\vec{q_i}\right) \tag{1.3}$$

$$\vec{R_i}=\vec{q_i}-\frac{\hat{k}\times\vec{\pi_i}}{eB} \tag{1.4}$$

The equation 1.3 passes from the canonical momentum $\vec{p_i}$ to the linear momentum $\vec{\pi_i}$, which is much more intuitive and understandable; while the equation 1.4 transforms the general position $\vec{q_i}$ to the position of the instantaneous guiding centre $\vec{R_i}$.

In a system without the interaction potentials, the electrically charged particles are known to perform the circular motion of the cyclotron with radii that depends on the initial linear momenta. In this case, the guiding centres would be constant in time as would be the linear momenta. However, with the introduction of an interacting potential, the momentum of each particle may vary making the guiding centre change too, which is why the instantaneous interpretation of the guiding centres is necessary.

Now, let us calculate the Poisson brackets for this new set of coordinates in a specific particle.

$$
\begin{aligned}
\{\pi_1, \pi_2\} &= \frac{\partial \pi_1}{\partial q_\alpha}\frac{\partial \pi_2}{\partial p_\alpha} - \frac{\partial \pi_2}{\partial q_\alpha}\frac{\partial \pi_1}{\partial p_\alpha} \\
&= -e\delta_{\alpha 2}\frac{\partial A_1}{\partial q_\alpha} + e\delta_{\alpha 1}\frac{\partial A_2}{\partial q_\alpha} \\
&= -e\frac{\partial A_1}{\partial q_2} + e\frac{\partial A_2}{\partial q_1} \\
&= e(\nabla \times \vec{A})_3 = eB
\end{aligned}
$$

$$
\begin{aligned}
\{R_1, R_2\} &= \{q_1, q_2\} + \left\{q_1, -\frac{\pi_1}{eB}\right\} + \left\{\frac{\pi_2}{eB}, q_2\right\} + \left\{\frac{\pi_2}{eB}, -\frac{\pi_1}{eB}\right\} \\
&= \frac{1}{eB}\left(\cancelto{-1}{\{p_1, q_1\}} - e\cancelto{0}{\{A_1, q_1\}} + \cancelto{-1}{\{p_2, q_2\}} \cancelto{0}{e\{A_2, q_2\}}\right) + \frac{eB}{(eB)^2} \\
&= \frac{-2}{eB} + \frac{1}{eB} = -(eB)^{-1}
\end{aligned}
$$

$$
\begin{aligned}
\{R_1, \pi_2\} &= \{q_1, \pi_2\} + \cancelto{0}{\left\{\frac{\pi_2}{eB}, \pi_2\right\}} \\
&= \cancelto{0}{\{q_1, p_2\}} - e\cancelto{0}{\{q_1, A_2\}} \\
&= \{R_2, \pi_1\} = 0
\end{aligned}
$$

This Poisson brackets can be generalised to the transformation for the three particles. Taking $i, j = \{1, 2, 3\}$ and $\alpha, \beta = \{1, 2\}$:

$$\{\pi_{i,\alpha}, \pi_{j,\beta}\} = (eB)\,\delta_{ij}\epsilon_{\alpha\beta} \tag{1.5}$$

$$\{R_{i,\alpha}, R_{j,\beta}\} = -(eB)^{-1}\,\delta_{ij}\epsilon_{\alpha\beta} \tag{1.6}$$

$$\{R_{i,\alpha}, \pi_{j,\beta}\} = 0 \tag{1.7}$$

Equations **??** allow us to identify the proposed transformation as canonical. However, this is not the usual canonical transformation where the position coordinates and the momentum coordinates are canonical conjugates. In this special case, one component of the momentum is canonical conjugate with the other momentum coordinate, and equally for the position coordinates.

Now, with a huge magnetic field, if the potential of the interaction forces does not vary abruptly in space, we can use the approximation $\vec{R}_i \approx \vec{q}_i$ to average the potentials over the guiding centres, that is, we can replace $\vec{q}_i$ for $\vec{R}_i$ in $V(\vec{q}_1, \vec{q}_2, \vec{q}_3)$.

We can interpret the last approximation as follows: In the cyclotron problem, the radius of the circular motion described is proportional to the linear momentum and inversely proportional to the magnetic field. Then, in the presence of a big B, the radius of the cyclotron would shrink to a very small size. In the case we are working, the radia of the instantaneous cyclotron motion would be proportional to $\left\|(\hat{k} \times \vec{\pi}_i)(eB)^{-1}\right\|$, then, in a presence of a huge magnetic field, the instantaneous radii of the particles would be small enough that the potential would not sense the instantaneous circular motion and would sense the particle as if it where always in the position of its guiding centre.

Before replacing the new set of coordinates in the Hamiltonian, it is necessary to do a scale transformation to obtain the proper Poisson brackets for the formal definition of canonical transformation, that is:

$$
\begin{aligned}
\vec{\pi}_i &\to (eB)^{-1/2} \vec{\pi}_i \\
\vec{R}_i &\to \sqrt{eB} \vec{R}_i
\end{aligned}
$$

With this consideration, the Hamiltonian of the system in the new set of rescaled coordinates is given by:

$$
H = \sum_{i=1}^{3} \frac{eB}{2m} \|\vec{\pi}_i\|^2 + V\left((eB)^{-1/2}\vec{R}_1, (eB)^{-1/2}\vec{R}_2, (eB)^{-1/2}\vec{R}_3\right) + \frac{\omega_c^2}{2meB} \sum_{i=1}^{3} \left\|\vec{R}_i\right\|^2 \quad (1.8)
$$

This Hamiltonian, given equation 1.7 can be decomposed in a Hamiltonian that describes the movement of the guiding centres, and other that describes de movement of the linear momenta. In one hand, the Hamiltonian for the linear momenta is easily identified with the harmonic oscillator, whereas the one that characterises the movement of the guiding centres needs a deeper analysis.

1.2 Integrability of the system

As the Hamiltonian describing the trajectories of the linear momenta of the particles is that of an harmonic oscillator, this part of the problem is integrable and its solutions are widely known. The guiding centre Hamiltonian, in turn, needs to be analysed more deeply. For this purpose, take the following convention:

$$H_{gc} = \frac{\omega_c^{*2}}{2m} \sum_{i=1}^{3} \left\| \bar{x}^2 \right\| + \left\| \bar{y}^2 \right\| + V^* \left(\bar{x}, \bar{y} \right) \tag{1.9}$$

Where $\bar{x} = (x_1, x_2, x_3)$ and $\bar{y} = (y_1, y_2, y_3)$, being x_i, y_i the rescaled coordinates of the guiding centres of the particles. Besides, the potential V and the constant ω_c have been rescaled to take into account this rescale and maintain the original form of the Hamiltonian:

$$\omega^* = \frac{\omega}{\sqrt{eB}}$$
$$V^* \left(\bar{x}, \bar{y} \right) = V \left(eB\bar{x}, eB\bar{y} \right)$$

Clearly, the new potential V^* still has the symmetries expressed in the equation 1.2. Furthermore, in the new order for the scaled guiding centres coordinates the Poisson brackets take the form:

$$\{y_i, x_j\} = \delta_{ij} \tag{1.10}$$

Appendix A

An Appendix

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