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Three body problem in the spherical geometry

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Contents

Chapter 1

Introduction

The N body problem is a highly known and studied issue in Physics. It consists in researching the trajectories that N point masses would follow when interacting with external and internal forces with certain defined characteristics, given all the information of the initial conditions.

At first, the principal interest in the study of this problem was the exact prediction of the path of celestial bodies. However, as the problem was known more, it was understood that its study is of great importance not only for astrophysics but for the theoretical comprehension of classical mechanics. Great minds of physics and mathematics have worked in the restricted problem of three bodies, as Poincaré [1] and Jacobi [2]. It was this way that Poincaré, in an attempt of solving the three body problem, discovered that it is not integrable in general, and formulated the bases of what is known nowadays as chaos theory [1].

Independently of the formalism chosen to define the system, the N body problem is reduced to the integration of the equations of motion for the N particles. As this problem has been known to be non-integrable for the $N \geq 3$ cases, with the exception of few occurrences that involve forces with strange features as explicit dependence of the position an velocities [3], the study of realistic 3 body problems is a very interesting question in physics.

1.1 Motivation

One of the particular realistic cases of the three body problem that is known to be integrable, is the system of three particles on the plane with mass m and charge e under the influence of a huge constant magnetic field perpendicular to the surface and forces whose potentials are invariant under rotations and translations in the plane [4].

This problem is integrable by virtue of the action of the big magnetic potential, which decouples the movement of the particle into two degrees of freedom known as the guiding centres and the linear momenta. The analysis carried out in [4] can be extended to the quantum formalism and it explains the quantum Hall effect (QHE) fact that particles at low temperatures are confined to the base state to energy due to Landau level gap modulated by the magnetic field.

The study of this problem may be applied to classical and quantum Hall effect models. These models are principally centred in the analysis of the bulk properties, and are not interested in the effect of border effects. Consequently, these models are usually defined over geometries with no borders as the infinite plane, the torus and the sphere [5]. Having this into account, the principal objective of this work is to try to expand the analysis of the three body problem on the plane carried out in [4] to the spherical geometry.

To do so, as a first approach to the problem, we are going reproduce the calculations and analysis from [4] of the classical system in great detail. Then, we are going to study some important aspects of the classical one body problem on the sphere under the influence of a magnetic monopole to obtain some intuition about the important quantities and the analogies of the movement of the particle in this case and the one on the plane. After that, we are going to proceed with the analysis of the quantum three body problem in the plane. Then some intuition is going to be presented about the quantum one body problem in the sphere, known as Haldane's formalism [6].

Chapter 2

The three body problem in the plane

In this chapter a classic approach of a somehow general case of the three body problem in 2 dimensions is going to be presented. This problem was developed by Alonso Botero et. al in [4]. It will give some necessary intuition to develop the analogous problem in the spherical geometry. To begin with, the problem is going to be described in great detail; then its integrability is going to be proven; and finally, a brief description of the movement of the particles is going to be presented.

2.1 The definition of the problem

The three body problem presented here is that of three particles of electrical charge e and mass m confined to a plane, under the influence of a strong magnetic field perpendicular to it and forces whose potentials satisfy translational and rotational symmetries in the plane.

Given this information, the Hamiltonian associated with this system has the form:

$$H = \sum_{i=1}^{3} \frac{1}{2m} \left\| \vec{p_i} - e\vec{A}\left(\vec{r_i}\right) \right\|^2 + V\left(\vec{r_1}, \vec{r_2}, \vec{r_3}\right) + \frac{m\omega_c^2}{2} \sum_{i=1}^{3} \|\vec{r_i}\|^2 \tag{2.1}$$

Where $\vec{r_i} = x_i\hat{\imath} + y_i\hat{\jmath}$, $\vec{p_i} = p_{x_i}\hat{\imath} + p_{y_i}\hat{\jmath}$ and $\vec{A}\,(\vec{r})$ is the magnetic vector potential, which satisfies $\nabla \times \vec{A} = B\hat{k}$.

Besides, the potential $V\,(\vec{r_1}, \vec{r_2}, \vec{r_3})$ satisfies the symmetries:

$$V\,(R\vec{r_1} + \vec{a}, R\vec{r_2} + \vec{a}, R\vec{r_3} + \vec{a}) = V\,(\vec{r_1}, \vec{r_2}, \vec{r_3}) \tag{2.2}$$

For any rotation R about $\hat{k}$ and any translation $\vec{a}$ in the plane.

2.2 The canonical transformation of the guiding centres

For the proof of integrability for this system, and for further analysis of the trajectories of the particles, let us perform the well known transformation of the guiding centres.

This transformation is defined by the following two equations:

$$\vec{\pi_i} = \vec{p_i} - e\vec{A}\,(\vec{q_i}) \tag{2.3}$$

$$\vec{R_i} = \vec{r_i} - \frac{\hat{k} \times \vec{\pi_i}}{eB} \tag{2.4}$$

The equation (2.3) passes from the canonical momentum $\vec{p_i}$ to the linear momentum $\vec{\pi_i}$, which is much more intuitive and understandable; while the equation (2.4) transforms the general position $\vec{r_i}$ to the position of the instantaneous guiding centre $\vec{R_i}$.

In a system without the interaction potentials, the electrically charged particles are known to perform the circular motion of the cyclotron with radii that depends on the initial linear momenta. In this case, the guiding centres would be constant in time as would be the linear momenta. However, with the introduction of an interacting potential, the momentum of each particle may vary making the guiding centre change too, which is why the instantaneous interpretation of the guiding centres is necessary.

Now, let us calculate the Poisson brackets for this new set of coordinates for a specific particle.

$$
\begin{aligned}
\{\pi_1, \pi_2\} &= \frac{\partial \pi_1}{\partial r_\alpha}\frac{\partial \pi_2}{\partial p_\alpha} - \frac{\partial \pi_2}{\partial r_\alpha}\frac{\partial \pi_1}{\partial p_\alpha} \\
&= -e\delta_{\alpha 2}\frac{\partial A_1}{\partial r_\alpha} + e\delta_{\alpha 1}\frac{\partial A_2}{\partial r_\alpha} \\
&= -e\frac{\partial A_1}{\partial y} + e\frac{\partial A_2}{\partial x} \\
&= e(\nabla \times \vec{A})_3 = eB
\end{aligned}
$$

$$
\begin{aligned}
\{R_1, R_2\} &= \{r_1, r_2\} + \left\{r_1, -\frac{\pi_1}{eB}\right\} + \left\{\frac{\pi_2}{eB}, r_2\right\} + \left\{\frac{\pi_2}{eB}, -\frac{\pi_1}{eB}\right\} \\
&= \frac{1}{eB}\left(\cancelto{-1}{\{p_1, r_1\}} - \cancelto{0}{e\{A_1, r_1\}} + \cancelto{-1}{\{p_2, r_2\}} \cancelto{0}{e\{A_2, r_2\}}\right) + \frac{eB}{(eB)^2} \\
&= \frac{-2}{eB} + \frac{1}{eB} = -(eB)^{-1}
\end{aligned}
$$

$$
\begin{aligned}
\{R_1, \pi_2\} &= \{r_1, \pi_2\} + \cancelto{0}{\left\{\frac{\pi_2}{eB}, \pi_2\right\}} \\
&= \cancelto{0}{\{r_1, p_2\}} - e\cancelto{0}{\{r_1, A_2\}} \\
&= \{R_2, \pi_1\} = 0
\end{aligned}
$$

This Poisson brackets can be generalised to the transformation for the three particles. Taking $i, j = \{1, 2, 3\}$ and $\alpha, \beta = \{1, 2\}$:

$$
\{\pi_{i,\alpha}, \pi_{j,\beta}\} = (eB)\,\delta_{ij}\epsilon_{\alpha\beta} \tag{2.5}
$$

$$
\{R_{i,\alpha}, R_{j,\beta}\} = -\,(eB)^{-1}\,\delta_{ij}\epsilon_{\alpha\beta} \tag{2.6}
$$

$$
\{R_{i,\alpha}, \pi_{j,\beta}\} = 0 \tag{2.7}
$$

Equations (2.5)-(2.6) allow us to identify the proposed transformation as canonical. However, this is not the usual canonical transformation where the position coordinates and the momentum coordinates are canonical conjugates. In this special case, one component of the momentum is canonical conjugate with the other momentum coordinate, and similarly for the guiding center position components.

Now, with a huge magnetic field, if the potential of the interaction forces does not vary abruptly in space, we can use the approximation $\vec{R}_i \approx \vec{q}_i$ to average the potentials over the guiding centres, that is, we can replace $\vec{q}_i$ for $\vec{R}_i$ in $V(\vec{q}_1, \vec{q}_2, \vec{q}_3)$.

We can support the last approximation as follows: In the cyclotron problem, the radius of the circular motion described is proportional to the linear momentum and inversely proportional to the magnetic field. Then, in the presence of a big magnetic field B, the radius of the cyclotron would shrink to a very small size. Regarding the case we are working with, the radii of the instantaneous cyclotron motion would be proportional to $\left\|(\hat{k} \times \vec{\pi}_i)(eB)^{-1}\right\| = \left\|\vec{\pi}_i(eB)^{-1}\right\|$ and its frequency to $\sqrt{B}$. As the potential V does not vary abruptly in the radii scale, the averaging of this motion over the guiding centres means that this potential does not sense that circular motion. Moreover, given the big frequency of the cyclotrons and the scale of variance of the potential, the scale of time of the local circular motions is far smaller than that of the motion of the guiding centres. Therefore, we can ignore the instantaneous quality of the circular motion, and take it as constant in a scale of time small enough for the motion of the guiding centres. In this sense we say that the coordinates for the guiding centres decouple from that of the linear momenta of the particles.

Before replacing the new set of coordinates in the Hamiltonian, it is necessary to do a scale transformation to obtain the proper Poisson brackets for the formal definition of canonical transformation, that is:

$$\begin{aligned} \vec{\pi}_i &\to (eB)^{-1/2}\vec{\pi}_i \\ \vec{R}_i &\to \sqrt{eB}\vec{R}_i \end{aligned}$$

With this consideration, the Hamiltonian of the system in the new set of rescaled coordinates is given by:

$$H = \sum_{i=1}^{3} \frac{eB}{2m} \left\|\vec{\pi}_i\right\|^2 + V\left((eB)^{-1/2}\vec{R}_1, (eB)^{-1/2}\vec{R}_2, (eB)^{-1/2}\vec{R}_3\right) + \frac{m\omega_c^2}{2eB} \sum_{i=1}^{3} \left\|\vec{R}_i\right\|^2 \quad (2.8)$$

This Hamiltonian, given equation (2.7) can be decomposed in a Hamiltonian that describes the movement of the guiding centres, and other that describes de movement of the linear momenta. In one hand, the Hamiltonian for the linear momenta is easily identified with the harmonic oscillator, whereas the one that characterises the movement of the guiding centres needs a deeper analysis.

2.3 Integrability of the system

As the Hamiltonian describing the trajectories of the linear momenta of the particles is that of an harmonic oscillator, this part of the problem is integrable and its solutions are widely known. The guiding centre Hamiltonian, in turn, needs to be analysed more deeply. For this purpose, let us take the following convention:

$$H_{gc} = \frac{1}{2}\sum_{i=1}^{3} \left\|\bar{x}^2\right\| + \left\|\bar{y}^2\right\| + V^*\left(\bar{x}, \bar{y}\right) \tag{2.9}$$

Where $\bar{x} = (x_1, x_2, x_3)$ and $\bar{y} = (y_1, y_2, y_3)$, being x_i, y_i the rescaled coordinates of the guiding centres of the particles. For simplicity, the potential V and has been rescaled to take into account the scale transform of the coordinates and maintain the original form of the Hamiltonian:

$$\omega_c^2 = \frac{eB}{m}$$
$$V^*\left(\bar{x}, \bar{y}\right) = V\left(\frac{\bar{x}}{\sqrt{eB}}, \frac{\bar{y}}{\sqrt{eB}}\right)$$

Clearly, the new potential V^* still has the symmetries expressed in the equation (2.2). Furthermore, with this scale transformation the Poisson brackets take the form:

$$\{y_i, x_j\} = \delta_{ij} \tag{2.10}$$

Now that the guiding center Hamiltonian has been expressed in terms of the proper canonical set of coordinates, the fastest way to prove the integrability of the system is via the Liouville-Arnol'd theorem [9, Sect. 49]. For this theorem, it is only necessary to find 2 more independent integrals in involution (besides the Hamiltonian).

To get this 2 integrals, let us exploit the symmetries of the guiding centres Hamiltonian. We then define the generators of translations and rotation in the plane, which are symmetries of the potential:

$$\begin{aligned} T_x &= \sum_{i=1}^{3} x_i \\ T_y &= \sum_{i=1}^{3} y_i \end{aligned} \tag{2.11}$$

$$R_z = \frac{1}{2}\sum_{i=1}^{3}\left(x_i^2 + y_i^2\right) \tag{2.12}$$

It is easily verifiable that these are indeed the symmetries generators. To see that, take the first order infinitesimal transformations of translations and rotations on the plane:

$$x_i \to x_i + \epsilon$$

$$y_i \to y_i + \epsilon$$

$$(x_i, y_i) \to (x_i + \epsilon y_i, y_i - \epsilon x_i)$$

Now note that for the infinitesimal translations, the potential of the transformed coordinates is related to the potential of the normal coordinates by a directional derivative, which can be identified with the Poisson bracket of the potential V^* and each generator:

$$\begin{aligned} 0 = V^*\left(\bar{x} + \epsilon, \bar{y}\right) - V^*\left(\bar{x}, \bar{y}\right) = \epsilon \sum_{i=1}^{3} \frac{\partial V^*\left(\bar{x}, \bar{y}\right)}{\partial x_i} = \epsilon \left\{V^*, T_x\right\} = 0 \\ 0 = V^*\left(\bar{x}, \bar{y} + \epsilon\right) - V^*\left(\bar{x}, \bar{y}\right) = \epsilon \sum_{i=1}^{3} \frac{\partial V^*\left(\bar{x}, \bar{y}\right)}{\partial y_i} = \epsilon \left\{T_y, V^*\right\} = 0 \end{aligned}$$

For the infinitesimal rotation, the relation is analogous:

$$0 = V^*\left(\bar{x} + \epsilon\bar{y}, \bar{y} - \epsilon\bar{y}\right) - V^*\left(\bar{x}, \bar{y}\right) = \frac{\partial V^*\left(\bar{x}, \bar{y}\right)}{\partial x_i}\left(\epsilon y_i\right) - \frac{\partial V^*\left(\bar{x}, \bar{y}\right)}{\partial y_i}\left(\epsilon x_i\right) = \epsilon \left\{V^*, R_z\right\}$$

Therefore, we conclude that the generators of translations and rotations in the plane commute with the potential V^* due to its symmetries. Besides, the generator of rotations is exactly equal to the harmonic-like part of the guiding center Hamiltonian which validates that R_z is other integral in involution. The generators of translations are not integrals in involution for they do not commute with the harmonic potential. However, we can calculate a quantity in terms of these generators, which already commute with the potential V^*, to make it commute with the remaining part of H_{gc}:

$$L \coloneqq \frac{1}{6}\left(T_x^2 + T_y^2\right) \tag{2.13}$$

This new quantity L clearly commutes with the potential V^* because the Poisson bracket is a linear differential operator in one component and it obeys the Leibniz rule. Moreover, it also commutes with the rotation generator $J \coloneqq R_z$:

$$\begin{aligned}
\left\{T_x^2 + T_y^2, J\right\} &= \sum_{i,j,k} \left\{x_i x_j + y_i y_j, x_k^2 + y_k^2\right\} \\
&= \sum_{i,j,k} \left\{x_i x_j, y_k^2\right\} + \left\{y_i y_j, x_k^2\right\} = \sum_{i,j,k} y_k \left\{x_i x_j, y_k\right\} + x_k \left\{y_i y_j, x_k\right\} \\
&= \sum_{i,j,k} y_k x_i \delta_{jk} + y_k x_j \delta_{ik} - x_k y_i \delta_{jk} - x_k y_j \delta_{ik} \\
&= \sum_{i,j} y_j x_i + y_i x_j - x_j y_i - x_i y_j = 0
\end{aligned}$$

As we found L as the last integral in involution, we conclude, by the Liouville-Arnol'd theorem, that the subsystem of guiding centres is integrable by quadratures.

2.4 Brief analysis of the motion

Taking the motion integrals obtained in the previous section for the guiding center Hamiltonian (2.9) the motion of the particles can be broken down in that of the center of mass, and the relative trajectories of the particles. To see that, note that as $T_x^2 + T_y^2$ in (2.13) is an integral in involution with the Hamiltonian, it is a conserved quantity. The conservation of this term is clearly interpreted as a circular motion of the center of mass of the system $\left(\frac{T_x}{3}, \frac{T_y}{3}\right)$.

The latter analysis can be clarified in terms of the decoupling of the guiding center trajectories from the general motion of the particles. In that sense one can deduce that the guiding centre movement is only affected by the interaction potential V and the external harmonic potential. This can be confirmed by the lack of magnetic terms in the guiding centre Hamiltonian in equation (2.8).

Taking this into account, as the potential V is a central potential that produces only internal forces that do not affect the movement of the centre of mass, one can assure that its motion is determined by the only term of external interaction in the Hamiltonian, the harmonic potential.

Now, as the center of mass variables are clearly canonical conjugates, we deduce that $\left(\frac{T_x}{3}, \frac{T_y}{3}\right)$ performs the motion of a harmonic oscillator. In other words, the motion is besides circular, uniform. The problem is then reduced to the analysis of the motion of the coordinates relative to the centre of mass.

To carry on with this procedure, let us take the spinor of relative position defined as follows:

$$\Psi = \frac{1}{2\sqrt{3}} \begin{pmatrix} \sqrt{3}\,(z_2 - z_1) \\ z_2 + z_1 - 2z_2 \end{pmatrix} \tag{2.14}$$

With $z_i = x_i + iy_i$. If we call define the center of mass as $T = \frac{1}{3}\left(T_x + iT_y\right) = \frac{1}{3}\left(z_1 + z_2 + z_3\right)$ we can retrieve the general position of each particle:

$$\begin{aligned} z_1 &= T + \frac{1}{\sqrt{3}}\Psi_2 - \frac{1}{3}\Psi_1 \\ z_2 &= T + \frac{1}{\sqrt{3}}\Psi_2 + \frac{1}{3}\Psi_1 \\ z_3 &= T - \frac{2}{\sqrt{3}}\Psi_2 \end{aligned} \tag{2.15}$$

The spinor components were chosen to satisfy the canonical commutation relations:

$$\{\Psi_\alpha, \Psi_\alpha\} = 0$$

$$
\begin{aligned}
\{\Psi_1, \Psi_2\} = \{\Psi_1^*, \Psi_2^*\}^* = &\left\{ \frac{1}{2}\left(z_2 - z_1\right), \frac{1}{2\sqrt{3}}\left(z_2 + z_1 - 2z_3\right) \right\} \\
&= i \left\{ \frac{1}{2}\left(x_2 - x_1\right), \frac{1}{2\sqrt{3}}\left(y_2 + y_1 - 2y_3\right) \right\} \\
&+ i \left\{ \frac{1}{2}\left(y_2 - y_1\right), \frac{1}{2\sqrt{3}}\left(x_2 + x_1 - 2x_3\right) \right\} = 0
\end{aligned}
$$

$$
\begin{aligned}
\{\Psi_1, \Psi_2^*\} &= \{\Psi_1^*, \Psi_2\}^* = -\frac{i}{2\sqrt{3}}\left\{x_2 - x_1, y_2 + y_1 - 2y_3\right\} = 0 \\
\{\Psi_1, \Psi_1^*\} &= \frac{-2i}{4}\left\{x_2 - x_1, y_2 - y_1\right\} = -i \\
\{\Psi_2, \Psi_2^*\} &= \frac{-i}{6}\left\{x_1 + x_2 - 2x_3, y_1 + y_2 - 2y_3\right\} = -i
\end{aligned}
$$

Which can be resumed:

$$
\left\{\Psi_\alpha, \Psi_\beta\right\} = \left\{\Psi_\alpha^*, \Psi_\beta^*\right\} = 0 \tag{2.16}
$$

$$
\left\{\Psi_\alpha^*, \Psi_\beta\right\} = i\delta_{\alpha,\beta} \tag{2.17}
$$

With this canonical transformation, given the decoupling of the center of mass coordinates from the relative coordinates, we are left with a very simple Hamiltonian for the relative positions of the guiding centers:

$$
H_{rc}\left(\Psi, \Psi^*\right) = V^*\left(\Psi, \Psi^*\right) \tag{2.18}
$$

From here, the solution of the problem can be done by simple integration of the equations of motion, which given the spinorial canonical transformation, can be expressed in the following compact form:

$$
\frac{d\Psi}{dt} = i\frac{\partial H_{rc}}{\partial \Psi *} = i\frac{\partial V^*}{\partial \Psi^*} \tag{2.19}
$$

Chapter 3

The problem of a charged particle in the magnetic field of a monopole

With the objective to obtain some intuition about the N-body problem restricted to a spherical geometry, we study in this chapter the symmetries and trajectories of a charged particle under the influence of the magnetic field of a monopole. To achieve this we present the deduction, via the Lagrangian formalism, of the so called Poincaré cone [7] that characterises the trajectory of a particle in this situation. We then extrapolate the important symmetries used in the Lagrangian formalism to the Hamiltonian formalism to retrieve some important aspects of the classical counterparts of the known Haldane formalism [6].

3.1 Definition of the Lagrangian

Let L be the Lagrangian of a charged particle of charge $-e$ and mass m under the influence of a magnetic monopole of magnitude g. Then L takes the form:

$$L\left(\vec{x},\dot{\vec{x}}\right) = \frac{m}{2}\left\|\dot{\vec{x}}\right\|^2 - e\vec{A}_{\hat{u}}(\vec{x})\cdot\dot{\vec{x}} \tag{3.1}$$

Where $\vec{A}(\vec{x})$ is the vector potential of the magnetic monopole with singularity along the direction defined by the unit vector $\hat{u}$. It is know that there is no vector potential that is finite in $\mathbb{R}^3$ which reproduces the magnetic field of a monopole; in addition, the different vector potentials identified by different unit vectors $\hat{u}$ are related by a gauge transformation which leaves the trajectories invariant. This family of vector potentials are given by [8]:

$$\vec{A}_{\hat{u}}(\vec{x}) = \frac{g}{r}\frac{\hat{u}\times\hat{r}}{1+\hat{u}\cdot\hat{r}} \tag{3.2}$$

3.2 The symmetries and its conserved quantities

The first thing to note in the previously defined Lagrangian is its time independence, which yields to the conservation of the Jacobi integral:

$$\frac{\partial L}{\partial \dot{\vec{x}}}\cdot\dot{\vec{x}} - L = m\left\|\dot{\vec{x}}\right\|^2 = m\left\|\dot{\vec{x_0}}\right\|^2$$

With $\dot{\vec{x_0}}$ the initial velocity of the particle. Here the Jacobi integral clearly represents the kinetic energy of the particle, which is constant in time because the magnetic field does no work.

Now, due to the simplicity of the problem, one can suspect of many other symmetries. The next symmetry presented here is not associated with Lagrangian, but rather with the action invariance. If we define the action as in the equation (3.3), it can be seen that it may be invariant under a proper scale transform of position and time. It is not difficult to find this transform, and it is presented in equation (3.4).

$$S = \int_{t_1}^{t_2} L\left(\vec{x},\dot{\vec{x}}\right)dt = \int_{t_1}^{t_2}\left(\frac{m}{2}\left\|\dot{\vec{x}}\right\|^2 - e\vec{A}_{\hat{u}}(\vec{x})\cdot\dot{\vec{x}}\right)dt \tag{3.3}$$

$$\begin{aligned} \vec{x}' &= e^{s}\vec{x} \\ t' &= e^{2s}t \end{aligned} \tag{3.4}$$

As a result of this symmetry, by Noether's theorem there must be a conserved quantity implied. To calculate it we prefer the method stated in [9, Thm on Integrable Systems]; however, to apply this method, the Lagrangian must be parametrised to include the time t as a generalised coordinate. To achieve this, note that:

$$\dot{\vec{x}} = \frac{d\vec{x}}{dt} = \frac{\frac{d\vec{x}}{d\tau}}{\frac{dt}{d\tau}} := \frac{\mathring{\vec{x}}}{\mathring{t}}$$

Then the action can be written in terms of the new parameter τ:

$$S = \int_{t_1}^{t_2} L\left(\vec{x}, \dot{\vec{x}}\right) dt = \int_{\tau_1}^{\tau_2} \mathring{t} L\left(\vec{x}, \mathring{\vec{x}}\mathring{t}^{-1}\right) d\tau = \int_{\tau_1}^{\tau_2} L'\left(\vec{x}, \mathring{\vec{x}}, \mathring{t}\right) d\tau$$

Which by analogy with equation (3.3), gives the new parametrised Lagrangian L':

$$L'\left(\vec{x}, \mathring{\vec{x}}, \mathring{t}\right) = \mathring{t} L\left(\vec{x}, \mathring{\vec{x}}\mathring{t}^{-1}\right) = \frac{m}{2\mathring{t}} \left\|\mathring{\vec{x}}\right\|^2 - e\vec{A}_{\hat{u}}(\vec{x}) \cdot \mathring{\vec{x}}$$

As we converted the symmetry of the action S in a symmetry of the Lagrangian L', the invariance given by equation (3.4) becomes clear. Now, using [9, Thm on Integrable Systems], we obtain the conserved quantity in accordance with the transformation (3.4):

$$G = \left.\frac{\partial L'}{\partial \mathring{q}_i} \frac{\partial q_i'}{\partial s}\right|_{s=0}$$

$$G = -m \left\|\frac{\mathring{\vec{x}}}{\mathring{t}}\right\|^2 t + \left(\frac{m\mathring{\vec{x}}}{\mathring{t}} - e\vec{A}_{\hat{u}}\right) \cdot \vec{x}$$

$$G = m\dot{\vec{x_0}} \cdot \vec{x_0} = -m \left\|\dot{\vec{x_0}}\right\|^2 t + m\dot{\vec{x}} \cdot \vec{x}$$

Working the previous conserved quantity one can obtain an equation for the magnitude of the position in function of time:

$$2\dot{\vec{x}} \cdot \vec{x} = \frac{d\left\|x\right\|^2}{dt} = 2\left\|\dot{\vec{x_0}}\right\|^2 t + 2\dot{\vec{x_0}} \cdot \vec{x_0}$$

$$x^2 = x_0^2 + 2\dot{\vec{x_0}} \cdot \vec{x_0} t + \left\|\dot{\vec{x_0}}\right\|^2 t^2 \tag{3.5}$$

From equation (3.5) it is important to note that the only way the radius of the particle stays constant is that the initial velocity of the particle is zero, otherwise, the time-quadratic term will always contribute to the increase of that radius. Furthermore, if the angular momentum of the particle $m\dot{\vec{x_0}} \cdot \vec{x_0}$ is zero, we retrieve the trajectory of a particle in rectilinear motion. This means that the radial velocity of the particle is not affected by the magnetic field of the monopole, as expected from the symmetry of the field.

On the other hand, we note that once chosen a unit vector $\hat{u}$ for the vector potential $\vec{A}_{\hat{u}}$, it is quite obvious that the original Lagrangian L in equation (3.1) is invariant under rotations around the unit vector $\hat{u}$, that is, under the following infinitesimal transformation:

$$\vec{x}' = \vec{x} + s\,(\hat{u} \times \vec{x}) \tag{3.6}$$

Then, as this transformation does not include the time t, we can perform the calculation of the conserved quantity over the original Lagrangian:

$$\begin{aligned}
G_2 &= \frac{\partial L}{\partial \dot{\vec{x}}} \cdot \frac{\partial \vec{x}}{\partial s}\bigg|_{s=0} \\
G_2 &= \left(m\dot{\vec{x}} - e\vec{A}_{\hat{u}}\right) \cdot (\hat{u} \times \vec{x}) \\
G_2 &= \left(m\vec{x} \times \dot{\vec{x}}\right) \cdot \hat{u} - eg\frac{\|\hat{u} \times \hat{x}\|^2}{(1 + \hat{x} \cdot \hat{u})} \\
G_2 &= \left(m\vec{x} \times \dot{\vec{x}}\right) \cdot \hat{u} - eg\,(1 - \hat{x} \cdot \hat{u}) \\
J_{\hat{u}} &\coloneqq \left(m\vec{x} \times \dot{\vec{x}} + eg\hat{x}\right) \cdot \hat{u} = const
\end{aligned}$$

Now, the last argument is valid for any unit vector $\hat{u}$ chosen and this yields the conservation of the known as Poincaré vector in equation (3.7).

$$\vec{J} = \left(m\vec{x} \times \dot{\vec{x}}\right) + eg\hat{x} \tag{3.7}$$

This last symmetry is very meaningful because it restricts the trajectory of the particle to a cone centred in the origin with central vector $\hat{J}$. To verify this, it is only necessary to see that the radial component of the Poincaré vector is constant for all points in the trajectory, which means that the angle between $\vec{J}$ and $\vec{x}(t)$ is a constant and that the path of the particle is restricted to a cone:

$$\vec{J} \cdot \hat{x} = eg$$

Furthermore, we can deduce from the conservation of the Poincaré vector that the angular momentum $\mathbb{L} = m\vec{x} \times \dot{\vec{x}}$ of the particle is constant in magnitude and that it determines the aperture of the cone of restriction.:

$$\begin{aligned} \left\|\vec{J}\right\|^2 &= \|\mathbb{L}\|^2 + (gc)^2 \\ \|\mathbb{L}\| &= const \\ \cos\theta &= \frac{\vec{J} \cdot \hat{x}}{\left\|\vec{J}\right\|} = \sqrt{\frac{(ge)^2}{\|\mathbb{L}\|^2 + (ge)^2}} \end{aligned} \tag{3.8}$$

From this equations it can be seen that the angle of aperture of the Poincaré cone is zero when the angular momentum $\mathbb{L}$ cancel, which means that the particle performs rectilinear motion, as deduced before from the other symmetries of the problem.

It is important to note here that if we restrict the trajectories of the particle to be in a sphere, we cannot carry a scale transform, hence we would not obtain the radius trajectory described in equation (3.5). However, a rotation is consistent with the norm conservation of the restriction to the sphere, and consequently we can obtain the Poincaré's vector invariance. The cone confinement together with the restriction of the trajectories to a constant radius would result in the particle describing a circular trajectory on the sphere.

Moreover in the sphere restriction the position vector and the velocity vector must be perpendicular, therefore, as the norm of the position is the constant radius of the sphere, the conservation of the angular momentum would result in the conservation of the linear velocity, meaning that the circular motion of the particle is in fact uniform, in analogy with the problem in the sphere.

Another important aspect that can be deduced from the set of equations (3.8) is that if we choose a magnetic monopole with big charge g compared with the angular momentum (in proper units), the angle of the Poincaré cone would tend to zero, which in the case of the particle restricted to the sphere, would mean that the radius of the circular motion would also tend to zero, as happens with the already studied case of the particle in the plane. This gives us some clues as where to look for the analogue guiding center formalism in the case of the magnetic monopole.

3.3 Important quantities in the Hamiltonian formalism

From the analysis carried out before, we can see some important quantities that are useful to describe the movement of particles in the presence of a magnetic monopole, some of which are associated with certain symmetries of the problem. Here we would like to study specially the angular momentum $\mathbb{L} = m\vec{x} \times \dot{\vec{x}}$ and the Poincaré vector $\vec{J} = \left(m\vec{x} \times \dot{\vec{x}}\right) + eg\hat{x}$. To do that, let us first calculate the Hamiltonian for the particle in the magnetic field of a monopole, this time restricting the problem to a sphere of radius R:

$$H\left(\vec{x}, \vec{p}\right) = \frac{1}{2m} \left\| \vec{p} + e\vec{A}_{\hat{u}}(\vec{x}) \right\|^2 \Big|_{S^2} \tag{3.9}$$

As the Hamiltonian is the Legendre transform of the Lagrangian, we obtain the generalised momentum $\vec{p}$ in terms of the velocity of the particle. Moreover, we can see that the Hamiltonian is just the kinematic energy of the particle:

$$\vec{p} = \frac{\partial L}{\partial \dot{\vec{x}}} = m\dot{\vec{x}} - e\vec{A}_{\hat{u}} := \vec{\pi} - e\vec{A}_{\hat{u}}$$

$$H = \frac{1}{2m} \left\| \vec{\pi} \right\|^2 \Big|_{S^2}$$

From here we can propose the angular momentum of the particle as $\mathbb{L} = \vec{x} \times \vec{\pi}$ and we can calculate its Poisson bracket. To do so, it is useful to calculate first some Poisson brackets related to the linear momentum $\vec{\pi} = \vec{p} + e\vec{A}$, as it was done in the first chapter:

$$\begin{aligned}
\{\pi_i, \pi_j\} &= \frac{\partial \pi_i}{\partial x_l}\frac{\partial \pi_j}{\partial p_l} - \frac{\partial \pi_j}{\partial x_l}\frac{\partial \pi_i}{\partial p_l} \\
&= e\delta_{lj}\frac{\partial A_i}{\partial x_l} - e\delta_{li}\frac{\partial A_j}{\partial x_l} = e\left(\delta_{lj}\delta_{mi} - \delta_{li}\delta_{mj}\right)\frac{\partial A_m}{\partial x_l} \\
&= -e\epsilon_{ijk}\epsilon_{lmk}\frac{\partial A_m}{\partial x_l} \\
&= -e\epsilon_{ijk}(\nabla \times \vec{A})_k = e\epsilon_{ijk}B_k = -\frac{eg}{R^3}\epsilon_{ijk}x_k
\end{aligned}$$

$$\{\pi_i, x_j\} = \frac{\partial \pi_i}{\partial x_l}\cancelto{0}{\frac{\partial x_j}{\partial p_l}} - \frac{\partial x_j}{\partial x_l}\frac{\partial \pi_i}{\partial p_l} = -\delta_{ij}$$

Then the algebra becomes a little bit easier for $\mathbb{L}$

$$\begin{aligned}
\{L_i, L_j\} &= \{\epsilon_{iab}x_a\pi_b, \epsilon_{jcd}x_c\pi_d\} = \epsilon_{iab}\epsilon_{jcd}\{x_a\pi_b, x_c\pi_d\} \\
&= \epsilon_{iab}\epsilon_{jcd}\left(x_ax_c\{\pi_b,\pi_d\} + \pi_bx_c\{x_a,\pi_d\} + x_a\pi_d\{\pi_b,x_c\} + \pi_b\pi_d\cancelto{0}{\{x_a,x_c\}}\right) \\
&= \epsilon_{iab}\epsilon_{jcd}\left(-\frac{eg}{R^3}\epsilon_{bdk}x_ax_cx_k + \pi_bx_c\delta_{ad} - \pi_dx_a\delta_{bc}\right) \\
&= -\frac{eg}{R^3}\epsilon_{iab}\left(\delta_{jk}\delta_{cb} - \delta_{jb}\delta_{ck}\right)x_ax_cx_k + \left(\delta_{bj}\delta_{ic} - \cancel{\delta_{bc}\delta_{ij}}\right)\pi_bx_c - \left(\delta_{aj}\delta_{ib} - \cancel{\delta_{ia}\delta_{jb}}\right)\pi_dx_a \\
&= -\frac{eg}{R^3}\left(x_j\cancelto{0}{\epsilon_{iab}x_ax_b} - \epsilon_{aji}x_ax_bx_b\right) + \left(\delta_{ia}\delta_{jb} - \delta_{ib}\delta_{ja}\right)x_a\pi_b \\
&= \epsilon_{ijk}\left(\epsilon_{abk}x_a\pi_b - eg\hat{x}_k\right) = \epsilon_{ijk}\left(L_k - eg\hat{x}_k\right)
\end{aligned}$$

Now, we can observe that $\mathbb{L}$ does not follow the canonical relations for angular momentum, which is not surprising because the momentum $\vec{\pi}$ used in the definition of $\mathbb{L}$ is not the canonical momentum p. We can fix this by taking a slight variation in $\mathbb{L}$:

$$\mathbb{J} := \mathbb{L} + eg\hat{x} \tag{3.10}$$

The Poisson bracket relation for this angular momentum is given by:

$$\begin{aligned}
\{J_i, J_j\} &= \left\{L_i + \frac{eg}{R}x_i, L_j + \frac{eg}{R}x_j\right\} \\
&= \{L_i, L_j\} + \frac{eg}{R}\left(\{L_i, x_j\} + \{x_i, L_j\}\right) + \left(\frac{eg}{R}\right)^2\cancelto{0}{\{x_i, x_j\}} \\
&= \epsilon_{ijk}\left(L_k - eg\hat{x}_k\right) + \frac{eg}{R}\left(-\delta_{lj}\frac{\partial\epsilon_{iab}x_a\left(p_b + eA_b\right)}{\partial p_l} + \delta_{il}\frac{\partial\epsilon_{jcd}x_c\left(p_d + eA_d\right)}{\partial p_l}\right) \\
&= \epsilon_{ijk}\left(L_k - eg\hat{x}_k\right) + \frac{eg}{R}\left(-\epsilon_{jia}x_a + \epsilon_{ijc}x_c\right) \\
&= \epsilon_{ijk}\left(L_k + eg\hat{x}_k\right) = \epsilon_{ijk}J_k
\end{aligned}$$

Then we deduce that $\mathbb{J}$ is a canonical angular momentum. It is not difficult to see that this is in fact the Poincaré vector in equation (3.7) and that, as seen before, it determines the center of the circular motion performed by the particle:

$$\mathbb{J} = \vec{x} \times \vec{\pi} + eg\hat{x} = m\vec{x} \times \dot{\vec{x}} + eg\hat{x} = \vec{J}$$

Chapter 4

The three body problem in the sphere

4.1 Definition of the problem

4.2 Integrability of the system

4.3 Analysis of the movement of the guiding centres

Chapter 5

Quantum analysis of the three body problem in the plane

In this chapter we continue the analysis implemented in [4], explaining in detail the quantum treatment of the three body problem in the plane. To achieve this goal, the quantum problem will be defined at the same time as the quantum analogous of the classical reduction of problem is going to be briefly discussed; then, the remaining problem is going to be analysed; and finally, some intriguing and rare aspects of this problem are going to be clarified.

MODIFY LAST PART

5.1 The quantum three body problem on the plane

The quantum treatment of the three body problem on the plane is not very different from the classical approach, given that we thoroughly worked the classical system out in the Hamiltonian formalism. The only thing we have to do is to apply the principles of canonical quantization [**?**].

Including the last considerations, the quantum Hamiltonian is going to be the same as the one expressed in (2.1), and as canonical transformations still work in quantum mechanics, a similar reduction of the problem can be performed.

First, given the big magnetic field decoupling, the problem can be reduced to the analysis of the guiding centres motion given by the Hamiltonian (2.8). The decoupled system of linear momenta is equally identified as a quantum harmonic oscillator, which is also very well known amongst the physical sciences community.

Now, we are left with the reduced Hamiltonian for the guiding centres (2.8) with the same definition of the canonical coordinates. The integrability analysis carried out in Chapter 2 where there were found two integrals of motion in involution with the Hamiltonian, may be interpreted as the finding of a complete set of commutative observables (CSCO) [?]. This means that the Hamiltonian of the guiding centres can be simultaneously diagonalised with the operators L and J that can be interpreted as orbital and total angular momentum. However, as the problem can refined to a better extent, this occurrence is currently of no interest for us.

Proceeding with the reduction of the problem, we can decouple here the motion of the center of mass, which describes quantum harmonic motion, from the guiding centre Hamiltonian and then take the same canonical transformation given by the spinor in (2.14):

$$\Psi = \frac{1}{2\sqrt{3}} \begin{pmatrix} \sqrt{3}\,(z_2 - z_1) \\ z_2 + z_1 - 2z_2 \end{pmatrix} \tag{5.1}$$

The Hamiltonian associated with the relative coordinates will be the same as (2.18):

$$H_{rc}\left(\Psi, \Psi^*\right) = V^*\left(\Psi, \Psi^*\right) \tag{5.2}$$

In quantum mechanics the symmetries of a system play a substantial role in the solution of the associated problem. They determine the algebra of operators, and consequently, the eigenvectors and eigenvalues that generate the states of the Hilbert space. In this case, we know that the potential V^* follows the symmetries associated with the special Euclidean group in 2D ($SE(2)$), but given the simplification of the problem, this symmetry slightly vary for $V^*\left(\Psi, \Psi^*\right)$.

From here, the following step in analysing the quantum 2-body problem restricted to the plane is to look for the symmetries in the most simplified Hamiltonian (2.18).

5.2 The spinorial representation and the Bloch sphere mapping

This spinorial representation of the relative coordinates of the particles with respect to the center of mass is analogous to the transformation performed to analyse the two body problem, only adapted to take into account one more particle and one less dimension. This spinor, besides representing a canonical transformation of the guiding centres, has norm equal to the spin quantity $S = J - L$ which is also an integral of motion:

$$\begin{aligned}
\Psi^\dagger\Psi &= \|\Psi_1\|^2 + \|\Psi_2\|^2 \\
&= \frac{1}{4}\left(\|z_1\|^2 + \|z_2\|^2 - 2\,\mathrm{Re}\left(z_1 z_2^*\right)\right) + \frac{1}{12}\left(\|z_1\|^2 + \|z_2\|^2 + \|z_3\|^2 + \mathrm{Re}\left(2z_1 z_2^* - 4z_1 z_3^* - 4z_2 z_3^*\right)\right) \\
&= \frac{1}{3}\sum_{i=1}^{3} x_i^2 + y_i^2 - \frac{1}{3}\sum_{i>j} x_i x_j + y_i y_j \\
&= \frac{1}{2}\sum_{i=1}^{3} x_i^2 + y_i^2 - \frac{1}{6}\left(T_x^2 + T_y^2\right) \\
&= J - L = S
\end{aligned}$$

This quantity S can be easily interpreted as proportional to the moment of inertia of the triangle rotating about its center of mass, which gives more intuition to the interpretation of S as spin momentum:

$$\begin{aligned}
I = m\sum_{i=1}^{3}\|\vec{r}_i - \vec{r}_{cm}\|^2 &= m\sum_{i=1}^{3}\|\vec{r}_i\|^2 + 3\,\|\vec{r}_{cm}\| - 2\vec{r}_{cm}\cdot\sum_{i=1}^{3}\vec{r}_i = m\left(2J - 3\,\|\vec{r}_{cm}\|\right) \\
&= m\left(2J - \frac{1}{3}\left(T_x^2 + T_y^2\right)\right) = m\left(2J - 2L\right) = 2mS
\end{aligned}$$

On the other hand, it is observed that any phase multiplication to the spinor Ψ is equivalent to any rotation about the center of mass or about the origin, as they have the same effect on relative vectors $\vec{r}_i - \vec{r}_j$:

$$
\begin{aligned}
e^{i\phi}z &= (\cos\phi + i\sin\phi)\,(x+iy) = (x\cos\phi - y\sin\phi) + i\,(y\cos\phi + x\sin\phi)\\
\Psi &= \frac{1}{2\sqrt{3}}\begin{pmatrix}\sqrt{3}\,(z_2-z_1)\\ z_2+z_1-2z_3\end{pmatrix} = \frac{1}{2\sqrt{3}}\begin{pmatrix}\sqrt{3}\,(z_2-z_1)\\ z_2-z_3+z_1-z_3\end{pmatrix}\\
e^{i\phi}\Psi &= \frac{1}{2\sqrt{3}}\begin{pmatrix}\sqrt{3}e^{i\phi}\,(z_2-z_1)\\ e^{i\phi}\,(z_2-z_3)+e^{i\phi}\,(z_1-z_3)\end{pmatrix}
\end{aligned}
$$

Moreover, scale transformations will be clearly given by a scalar multiplication. Consequently, a general transformation performed to the triangle relative coordinates can be represented by a scalar and a phase, or which is the same, a complex number. Now, taking this into account, we can take the equivalence class of shapes associated with a spinor Ψ as $[\Psi] = \{z\Psi | z \in \mathbb{C}/\,\{0\}\}$. Here the equivalence class $[\Psi]$ represents a shape of the triangle formed by the particles, and together with the norm of the spinor, it codifies the actual information needed of the triangle for the Hamiltonian of guiding centres (**??**).

In fact, if we take the space of normalised representatives $\Psi/\sqrt{S}$ we can note that the Hilbert spaces associated to the shapes of the triangles and to the wave vectors of a Qubit are equivalent, and can be mapped to a Bloch sphere. To comprehend better this mapping, let us define the vector consisting of the expected value of the Pauli matrices $\vec{\sigma}$ in terms of the equivalence class representatives:

$$
\vec{\zeta} = \frac{1}{S}\Psi^{\dagger}\vec{\sigma}\Psi \tag{5.3}
$$

The interpretation of each component of this vector becomes clear with some heavy algebra. Here we present the result in terms of the position vectors $\vec{r}_i$ of each particle:

$$
\begin{aligned}
\zeta_1 &= \frac{1}{2\sqrt{3}S}\left(\|\vec{r}_2-\vec{r}_3\|^2 - \|\vec{r}_1-\vec{r}_3\|^2\right)\\
\zeta_2 &= \frac{1}{\sqrt{3}S}\left(\vec{r}_1\times\vec{r}_2 + \vec{r}_2\times\vec{r}_3 + \vec{r}_3\times\vec{r}_1\right)\cdot\hat{z} = \frac{2A}{\sqrt{3}S}\\
\zeta_3 &= \frac{1}{6S}\left(2\,\|\vec{r}_2-\vec{r}_1\|^2 - \|\vec{r}_3-\vec{r}_1\|^2 - \|\vec{r}_3-\vec{r}_2\|^2\right)
\end{aligned}
$$

From this, the second component of the vector can be easily interpreted as proportional to the signed area A of the triangle which sign encodes the chirality of the system.

The first and third components, in turn, share a similar composition with the components of the vector Ψ, mapping the complex quantities z_i that encode the vertices coordinates, to the squared norm of the opposite side of the triangle. In other words, we can obtain expressions for the squared lengths of the sides of the triangle as we did in equations (2.15):

$$
\begin{aligned}
S &= \frac{1}{3}\sum_{i=1}^{3} x_i^2 + y_i^2 - \frac{1}{3}\sum_{i>j} x_i x_j + y_i y_j = \frac{1}{6}\left(\|\vec{r}_1 - \vec{r}_2\|^2 + \|\vec{r}_2 - \vec{r}_3\|^2 + \|\vec{r}_3 - \vec{r}_1\|^2\right)\\
\rho_1 &:= \|\vec{r}_2 - \vec{r}_3\|^2 = 2S\left(1 + \frac{\sqrt{3}}{2}\zeta_1 - \frac{1}{2}\zeta_3\right)\\
\rho_2 &:= \|\vec{r}_3 - \vec{r}_1\|^2 = 2S\left(1 - \frac{\sqrt{3}}{2}\zeta_1 - \frac{1}{2}\zeta_3\right)\\
\rho_3 &:= \|\vec{r}_2 - \vec{r}_1\|^2 = 2S\left(1 + \zeta_3\right)
\end{aligned}
$$

The information carried by the three components of the vector $\vec{\zeta}$ and the scalar S presented before, can then be compressed elegantly in this fashion:

$$
\begin{aligned}
\rho_k &= 2S\left(1 + \vec{m}_k \cdot \vec{\zeta}\right)\\
\vec{m}_k &= \left(\sin\frac{2\pi k}{3}, 0, \cos\frac{2\pi k}{3}\right), k \in \{1, 2, 3\}\\
A &= \frac{\sqrt{3}S}{2}\zeta_2
\end{aligned}
\tag{5.4}
$$

Now, to know better the distribution of shapes in the Bloch sphere, let us take the coordinate ζ_2 as the vertical axis which defines the poles and the rest as the ones defining the remaining perpendicular plane. In this terms, the pole vectors represent triangles of maximal area and the equatorial line, triangles of minimal null area. The triangles from the north hemisphere differ from their specular image with respect to the equatorial plane in the south hemisphere by only a sign in the area, which means that they have the same shape but different chirality.

Moreover, isosceles triangles require the vector $\vec{\zeta}$ to be perpendicular to $\vec{m}_i - \vec{m}_j$ for some $i \neq j$. It can be easily demonstrated that $\vec{m}_k \perp \vec{m}_i - \vec{m}_j$ for all $i \neq j \neq k$ and as a consequence, any isosceles triangle must follow that $\vec{\zeta} \parallel \vec{m}_i$ for any i. Besides, given that the vectors $\vec{m}$ live in the equatorial plane, it can be deduced that any isosceles triangle can be found on the lines on the sphere connecting the poles with the vectors $\vec{m}$. From this, it is clear that the polar triangles are equilateral.

Taking the analysis a little bit further, it can be seen from the form that takes the sides of the vertices ρ_k in equation (5.4), that in the equatorial plane if we look the directions $\vec{\zeta} = \vec{m}_k$, no side of the triangle will be null. Hence, those directions will determine isosceles triangles with null area, that is, triangles whose two sides sum up to the other. On the other hand, if we look directions $\vec{\zeta} = -\vec{m}_k := \vec{n}_k$, we find that one of the sides will be null. We find then on those directions, isosceles triangles with one of the sides equal to zero, which accounts for its null area.

One can be curious for the direction of equilateral triangles with null area, that is, all the points in the same position. The answer is that we cannot find them in the Bloch sphere because, as we stated earlier, it is a representation for normalised vectors $\Psi/\sqrt{S}$ and this triangle would require $S = 0$. However, if we take the normalisation factor S as the radius of the sphere, and consider the family of Bloch spheres for different radii filling $\mathbb{R}^3/\left\{0\right\}$, we surely would find these triangles in any direction on the limit $S \to 0$.

5.3 The Schwinger oscillator and the angular momentum representation

Chapter 6

The quantum three body problem on the sphere

Chapter 7

Conclusions

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