Study Guide **JD Russo**
PHYS 311 Exam Date: 09/14/2016
Prof. Amthor

Chapters 1-4

0.1 Vector operations (dot, cross, gradient, magnitude), including familiarity with polar coordinates.

These should be fairly straightforward.
Notecard: Basic vector equations for the above operations.

0.2 Newtons Second Law for the determination of equations of motion

Changes with coordinate system
Notecard: $\vec{F}_{net} = m\vec{a}$
Notecard: $\vec{F}_r = m(\ddot{r} - r\dot{\phi}^2)$
Notecard: $\vec{F}_\phi = m(r\ddot{\phi} + 2\dot{r}\dot{\phi})$

0.3 Inertial and noninertial reference frames

An inertial reference frame is nonaccelerating, which also implies nonrotating.

0.4 Simple Harmonic Oscillator

What about it?

0.5 Solutions of our favorite differential equations

$\ddot{\phi} = -\omega^2\phi$
$\phi = A\sin\omega t + B\cos\omega t$
$= \tilde{A}e^{i\omega(t+\phi_0)}$

0.6 Separation of variables technique

Straightforward

0.7 Drag forces

Linear
$F_{drag} = m\dot{v} = -bv$
$v(t) = Ae^{-kt}$

Quadratic
$F_{drag} = m\dot{v} = cv^2$
Separation of variables on the above leads you to $v = \frac{v_0}{1+cv_0t/m}$

Notecard: $F_{drag} = m\dot{v} = -bv$
Notecard: $v(t) = Ae^{-kt}$
Notecard: $F_{drag} = m\dot{v} = cv^2$
Notecard: $v = \frac{v_0}{1+cv_0t/m}$

0.8 Investigation of the qualities of motion in the limit, whether or not analytical solutions are possible

What?

0.9 Use of approximations, i.e. Taylor expansion for oscillations about an equilibrium position

Oscillation around equilibrium? What is this expansion?
Casting things into a binomial expansion is often helpful! Think of what Ned said about having a main term and a correcting term.
Notecard: Taylor Series - $f(z) = f(a) + f'(a)(z-a) + \frac{1}{2!}f''(a)(z-a)^2$
Notecard: Binomial expansion - $(1+z)^n = 1 + nz + \frac{n(n-1)}{2!}z^2 + ...$

0.10 Linear and Angular Momentum - Definitions, conservations, and rate of change given external forces/torques

Momenta
Linear: $\vec{p} = m\vec{v}$
Angular: $\vec{\ell} = r \times p$
Remember that force is the derivative of momentum, so the derivative of angular momentum has the force acting on the particle in it (product rule works for cross product).

Conservation
Linear: If no external **forces** are acting, $\vec{p}$ stays constant.
Angular: If no external **torques** are present, the total angular momentum $L = \sum r_\alpha \times p_\alpha$ is constant.
These are both evident in the below rates of change.

Rates of Change
Linear: $\dot{p} = F$
Angular: $\dot{\vec{\ell}} = r \times F = \Gamma$ (where Γ = torque)

0.11 Moment of inertia and CoM for both point mass (summation form) and continuous form (integral form) distributions

Center of mass position $R = \frac{1}{M}\sum_{i=1} Nm_ir_i$
$R = \frac{1}{M}\int rdm = \frac{1}{M}\int \rho rdV$ where ρ is the density of the object

Moments of inertia

- Component of angular momentum $L_x = I\omega$ where ω is the angular rotation about the x axis

- Uniform disk rotating about its axis: $I = 1/2MR^2$
- Uniform solid sphere: $I = 2/5MR^2$
- Generally: $I = \sum m_\alpha p_\alpha^2$ where p_α is the distance of the mass m_α from the axis of rotation.

Notecard: $I = 1/2MR^2$
Notecard: $I = 2/5MR^2$
Notecard: $I = \sum m_\alpha p_\alpha^2$

Angular momentum about CoM?

0.12 Work and Kinetic Energy - Work theorem

Work-KE Theorem: $\delta KE \equiv KE_2 - KE_1 = \sum F \cdot dr$
$W = \int F \cdot d\vec{r}$

0.13 Conservative and non-conservative forces (define, identify, use)

Define: A conservative force does the same work for all paths, and has F dependent only on R.
Identify: Check if $\nabla \times F = 0$
Use: For a conservative force, you can pick an easy path to do calculations.

0.14 Derivation of potential function, and relation to the force

Force is the gradient of potential energy.
Review derivation (p. 116)

0.15 Energy of multiple particles given a potential

What?

0.16 Spherical polar coordinates

Check formulas in back of book

Chapter 5

0.17 Harmonic Oscillator solutions in sinusoidal and exponential forms, finding equations of motion based on initial conditions.

Harmonic oscillator solutions
$F_x = -kx$
$U = \frac{1}{2}kx^2$ – This holds true for any sufficiently small displacement from a stable equilibrium. You can prove this by taking a Taylor expansion of a generic U(x), where you'll see it comes out to this. (p. 162)

$\ddot{x} = -\frac{k}{m}x = -\omega^2 x$
$\omega^2 = \frac{k}{m}$

The solutions to this are of the form $x(t) = C_1 e^{i\omega t} + C_2 e^{-i\omega t}$.

Using Euler's Identity (see next section), we can turn this into $B_1 \cos\omega t + B_2 \sin\omega t$, where $B_1 = C_1 + C_2$ and $B_2 = i(C_1 - C_2)$
Finding equations of motion from initial conditions?

0.18 Complex exponentials and Euler relations

Review homework problems on this
Notecard: $e^{\pm i\omega t} = \cos\omega t \pm i\sin\omega t$
Notecard: $\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)$
Notecard: $\sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$

0.19 Damped Oscillators

Adding a Hooke's law force $-kx$ and a resistive force $-b\dot{x}$ yields an equation for a damped 1D oscillator: $m\ddot{x} + b\dot{x} + kx = 0$.
We define constants $\frac{b}{m} = 2\beta$ and $\omega_0 = \sqrt{\frac{k}{m}}$ to get an equation of motion $r^2 + 2\beta r + \omega_0^2 = 0$.
This has solutions $r = -\beta \pm \sqrt{\beta^2 - \omega_0^2}$

Underdamped: A system is underdamped if $\beta < \omega_0$, meaning the amplitude of its oscillations will slowly decrease.
Overdamped: A system is overdamped if $\beta > \omega_0$, meaning it will move out to a maximum displacement once and them asymptotically approach 0 without dipping back below it.
Critically Damped: A system is critically damped if $\beta = \omega_0$. In this case, **it will most quickly settle to its equilibrium state**.
Decay parameter, i.e. how quickly the motion dies out, increases linearly in the underdamped regime, reaches a maximum at critical damping, then decreases exponentially in the undamped regime as the damping constant β goes to infinity.
Notecard: $\frac{b}{m} = 2\beta$, $\omega_0 = \sqrt{\frac{k}{m}}$
Notecard: $r^2 + 2\beta r + \omega_0^2 = 0$.
Notecard: $r_1 = -\beta \pm \sqrt{\beta^2 - \omega_0^2}$

0.20 Driven Damped Harmonic Oscillator

Review bottle in a bucket
By defining the driving force $F(t)$ as a force per unit mass $f(t)$, we can write the equation for a D.D.H.O. as $\ddot{x} + 2\beta\dot{x} + \omega_0^2 = f(t)$
Notecard: $\ddot{x} + 2\beta\dot{x} + \omega_0^2 = f(t)$

0.20.1 General solution of D.E.

The **particular solution** of this system is the solution to $x_p(t) = f$. This is in contrast to the **homogeneous solution** $x_h(t)$, which has the form $x_h(t) = C_1e^{r_1t} + C_2e^{r_2t}$. You can see both exponentials here die out as $t \to \infty$, so this is the transient response.

These solutions obey superposition.

The general solution (p. 184) is $x(t) = A\cos(\omega t - \delta) + C_1 e^{r_1 t} + C_2 e^{r_2 t}$. This has two clear parts, the transient exponential and the steady-state sinusoidal terms. The long-term behavior is dominated by the cosine term.

This only applies to systems where the restoring force and resistive force are linear.

Notecard: $x(t) = A\cos(\omega t - \delta) + C_1 e^{r_1 t} + C_2 e^{r_2 t}$

0.20.2 Resonance

The long-term behavior of the oscillator after the transience dies out is $x(t) = A\cos(\omega t - \delta)$. The amplitude of this is given by $A^2 = \frac{f_0^2}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}$.

From this, we can see if the difference between the driving frequency ω and the natural frequency ω_0 is large, then the amplitude of the oscillations will be very small. However, as the ω approaches ω_0, we approach resonance, where the amplitude grows to a maximum.
Notecard: $A^2 = \frac{f_0^2}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}$

Chapter 12

0.21 Driven damped pendulum

This describes a damped pendulum driven by a sinusoidal force $F(t) = F_0 \cos \omega t$.
$\ddot{\phi} + 2\beta\dot{\phi} + \omega_0^2 \sin\phi = \gamma\omega_0^2 \cos\omega t$
γ is the drive strength, or $\frac{F_0}{mg}$ the ratio of the drive amplitude to the weight of the pendulum.
Do I know what all the components of this equation mean?

Notecard: $\ddot{\phi} + 2\beta\dot{\phi} + \omega_0^2 \sin\phi = \gamma\omega_0^2 \cos\omega t$

0.22 Linear and nonlinear systems

A differential equation is **linear** if and only if it involves the dependent variable and its derivatives only linearly. A harmonic oscillator is linear, since its equation of motion depends only linearly on the oscillator's position, regardless of the form of the driving force. Conversely, a pendulum is nonlinear since it has a $\sin\phi$ dependence on the equation of motion.

Linear systems are actually more the exception than the rule in terms of ubiquity.
Notecard: Pendulum equation of motion: $mL^2\ddot{\phi} = -mgL\sin\phi$

0.23 Chaotic and non-chaotic motion

Chaotic motion is highly sensitive to initial conditions, and requires nonlinearity, and a "somewhat complicated" equation of motion. It is also common in nonlinear systems.

Plotting $\delta\phi$ for a chaotic system will show it exponentially increase. For a linear system, you will see it decay exponentially. This characterizes sensitivity to initial conditions.

Notecard: A linear cannot exhibit chaos, but nonlinearity does not guarantee chaos.
Notecard: Nonlinear equations do not obey superposition.
Notecard: $\delta\phi$ characterizes sensitivity to initial conditions.

0.24 Periodic and non-periodic motion

Straightforward. Be aware of the period doubling cascade.

0.25 Requirements for chaotic motion

- Nonlinearity
- "Complicated equation of motion"
- Sensitivity to initial conditions
- At least 3 dimensions of state space – **this is actually pretty wild**

0.26 Lyapunov Exponent: Definition, use, and implications

The $\delta\phi$ between two identical DDPs with different I.C.s can be characterized as $|\delta\phi(t)| \propto Ke^{\lambda t}$. λ is the Lyapunov exponent.

If the long-term motion is nonchaotic, this exponent is negative. If it is chaotic, then this is positive.

Notecard: $|\delta\phi(t)| \propto Ke^{\lambda t}$. λ is the Lyapunov exponent.
Notecard: $\lambda < 0$: nonchaotic, $\lambda > 0$: chaotic

0.27 Feigenbaum Number

The interval between control parameter thresholds that kick the system into the next period state is given by the geometric progression (Feigenbaum Relation) $(\gamma_{n+1} - \gamma_n) \approx \frac{1}{\delta}(\gamma_n - \gamma_{n-1})$.
Notecard: The inverse of the Feigenbaum number is the scaling factor between bifurcation points.

0.28 Finite limit of a series

?

0.29 Representations of motion

Plots 4 dayz