## The ME model: details

This note gives the missing details in the ME model.

The model is

$$y = \beta x + \epsilon$$

where y is the response,  $\beta$  the effect of the true covariate x with zero mean Gaussian noise  $\epsilon$ . The issue is that x is not observed directly, but only through  $x_{\text{obs}}$ , where

$$x_{\text{obs}} = x + \nu$$

where  $\nu$  is zero mean Gaussian noise. The parameters are:  $\beta$  has prior  $\pi(\beta)$ , x is apriori  $\mathcal{N}(\mu_x I, \tau_x I)$ , and  $\tau_{\text{obs}}$  is the observation precision for x (ie  $\text{Prec}(x_{\text{obs}}|x))^1$ .

Assume that the precision of the observations y,  $\tau_y$ , is known as it does not influence the calculations. Let  $\theta = (\beta, \tau_x, \tau_{\text{obs}}, \mu_x)$ . The full posterior is

$$\pi(x, \theta|y, x_{\text{obs}}) \propto \pi(\theta) \pi(x|\theta) \pi(x_{\text{obs}}|x, \theta) \pi(y|x, \theta)$$

Using that

$$\pi(x|\theta) \ \pi(x_{\text{obs}}|x,\theta) = \pi(x|x_{\text{obs}},\theta) \ \pi(x_{\text{obs}}|\theta)$$

we get

$$\pi(x, \theta|y, x_{\text{obs}}) \propto \pi(\theta) \pi(x|x_{\text{obs}}, \theta) \pi(x_{\text{obs}}|\theta) \pi(y|x, \theta).$$

This means that x only enters in *one term* (apart from the likelihood) hence can be used as an ordinary latent model f(). Its easy to derive that

$$x|x_{\text{obs}}, \theta \sim \mathcal{N}\left(\frac{\tau_x \mu_x I + \tau_{\text{obs}} x_{\text{obs}}}{\tau_x + \tau_{\text{obs}}}, (\tau_x + \tau_{\text{obs}})I\right).$$

and

$$x_{\rm obs}|\theta \sim \mathcal{N}\left(\mu_x I, \frac{1}{1/\tau_x + 1/\tau_{\rm obs}} I\right).$$

Note that  $x_{\text{obs}}|\theta$  does not depend on x, hence conditionally on  $\theta$ , its a constant. But it do need to be included in the model, as its log-density is

$$-\frac{n}{2}\log(2\pi) + \frac{n}{2}\log(\frac{1}{1/\tau_x + 1/\tau_{\text{obs}}}) - \frac{1}{2}\frac{1}{1/\tau_x + 1/\tau_{\text{obs}}}(x_{\text{obs}} - \mu_x I)^T(x_{\text{obs}} - \mu_x I)$$

and do depend on  $\theta$ .

The last tweak, is that we do the change of variable from  $(x,\beta)$  to  $(u,\beta)$ , where  $u=\beta x$ , so that

$$y = u + \epsilon$$
.

This makes the implementation more convenient. Then we get

$$\pi(u, \theta|y, x_{\text{obs}}) \propto \pi(\theta)$$
 (1)

$$\pi(u|x_{\text{obs}}, \theta) \ \pi(x_{\text{obs}}|\theta)$$
 (2)

$$\pi(y|u,\theta). \tag{3}$$

where

$$u|\theta, x_{\text{obs}} \sim \mathcal{N}\left(\beta \frac{\tau_x \mu_x I + \tau_{\text{obs}} x_{\text{obs}}}{\tau_x + \tau_{\text{obs}}}, \frac{\tau_x + \tau_{\text{obs}}}{\beta^2} I\right).$$

<sup>&</sup>lt;sup>1</sup>Note: The second argument in  $\mathcal{N}(,)$  is the precision not the variance.