The BYM-case

$$\eta = \ldots + \underbrace{u}_{\text{space}} + \underbrace{v}_{\text{iid.}}$$

$\text{Prec}(u) = Q$ when $\text{gen.var}(u) = 1$

$\text{Prec}(v) = \tau I$

Rewrite this as

$$\eta = \ldots + \frac{1}{\sqrt{\tau}}\left(\sqrt{1-\phi}\, v + \sqrt{\phi}\, u\right)$$

So that $\phi = 0$ corresponds to an iid model.

Since $\text{Var}\left(\sqrt{1-\phi}\, v + \sqrt{\phi}\, u\right) = 1-\phi + \phi = 1$

then the prior for τ remains unchanged.

So the reference is the distribution for v, i.e. $N(0, I)$, and as ϕ increase, then we mix in dependency all the way to $\phi = 1$. Note that the actual way of parametrisation of ϕ, as $\sqrt{1-\phi}$ and $\sqrt{\phi}$ does not matter. {as long as $\text{Var}(\sqrt{\cdot}\, v + \sqrt{\cdot}\, u) = 1$}.

$$\sqrt{1-\phi}\, v + \sqrt{\phi}\, u \sim N\Big(0,\ \text{Var} = (1-\phi) I + \phi \underbrace{Q^{-}}_{\text{gen. inv proper scaled}}\Big)$$

So we need the KL between

$\text{Var} = (1-\phi) I + \phi Q^{-1}$ — dense matrix, which makes this a bit awkward for large $\dim(Q)$.

and

$\text{Var} = I$

we need for the KL, to compute

$$|(1-\varrho)I + \varrho Q^{-}|$$

or, of course,

$$|[(1-\varrho)I + \varrho Q^{-}]^{-1}|$$

Now using that

$$(I+A^{-1})^{-1} = A(A+I)^{-1} \qquad [149, \text{ in MC}]$$

we get

$$\left((1-\varrho)I + \varrho Q^{-}\right)^{-1} = \left[(1-\varrho)\left\{I + \frac{\varrho}{1-\varrho} Q^{-}\right\}\right]^{-1}$$

$$= \left[(1-\varrho)\left\{I + \left(\frac{1-\varrho}{\varrho} Q\right)^{-1}\right\}\right]^{-1}$$

$$= \frac{1}{1-\varrho}\left(I + \underbrace{\left(\frac{1-\varrho}{\varrho} Q\right)}_{A}^{-1}\right)^{-1}$$

$$= \frac{1}{1-\varrho}\left[\frac{1-\varrho}{\varrho} Q \left(\frac{1-\varrho}{\varrho} Q + I\right)^{-1}\right]$$

$$= \frac{1}{\varrho} Q \left(\frac{1-\varrho}{\varrho} Q + I\right)^{-1}$$

so

$$|(\ )^{-1}| = \frac{1}{\varrho^r} |Q| \Big/ \left|\frac{1-\varrho}{\varrho} Q + I\right|$$

so

$$\boxed{|(1-\varrho)I + \varrho Q^{-1}| = \frac{\left|\frac{1-\varrho}{\varrho} Q + I\right|}{\frac{1}{\varrho^r}|Q|}}$$

Some details

$$\frac{1}{\sqrt{\tau}}\left(\sqrt{1-\varrho}\, v + \sqrt{\varrho}\, u\right)$$

when $u \sim N(0, Q_2)$ [Q_2 scaled] an $v \sim \mathcal{N}(0, I)$.

So let $w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ where $w_2 = u$,

$w_1 | w_2 \sim N\left(\sqrt{\frac{\varrho}{\tau}}\, w_2, \frac{\tau}{1-\varrho} I\right)$, then $w_1 \stackrel{d}{=} \frac{1}{\sqrt{\tau}}\left(\sqrt{1-\varrho}\, v + \sqrt{\varrho}\, u\right)$

and $\pi(w)$ is found from.

$$\cancel{\exp}\; -\frac{1}{2} w_2^T Q_2 w_2 - \frac{1}{2}\left(w_1 - \sqrt{\frac{\varrho}{\tau}} w_2\right)^T \left[\frac{\tau}{1-\varrho} I\right] \left(w_1 - \sqrt{\frac{\varrho}{\tau}} w_2\right)$$

$$= -\frac{1}{2} w_2^T \left[Q_2 + \frac{\varrho}{\tau}\frac{\tau}{1-\varrho} I\right] w_2 - \frac{1}{2} w_1^T \left[\frac{\tau}{1-\varrho} I\right] w_1$$

$$-\frac{1}{2}\left[-2\sqrt{\frac{\varrho}{\tau}}\frac{\tau}{1-\varrho}\, w_2^T w_1\right]$$

$$= -\frac{1}{2}\left[w_1^T\; w_2^T\right]^T \begin{bmatrix} \frac{\tau}{1-\varrho} I & -\frac{\sqrt{\varrho\tau}}{1-\varrho} I \\ & Q_2 + \frac{\varrho}{1-\varrho} I \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

Norm const

$$\left(\frac{1}{\sqrt{2\pi}}\right)^{2\cdot n} \cdot \left(\frac{\tau}{1-\varrho}\right)^{n/2} \cdot |Q|^{1/2}$$

not included.