

# The Hidden Geometry of Particle Collisions

Jesse Thaler



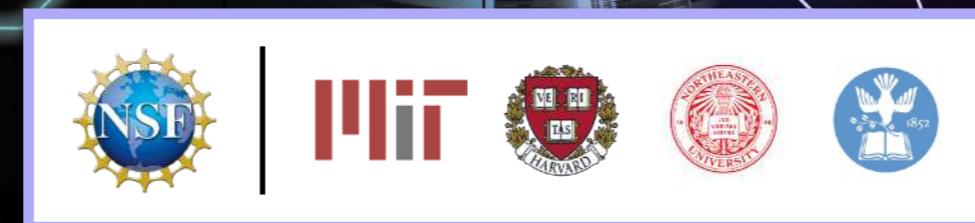
University of Florida / Florida State University Joint HEP Seminar — April 9, 2021

# The NSF AI Institute for Artificial Intelligence and Fundamental Interactions (IAIFI)

“eye-phi”

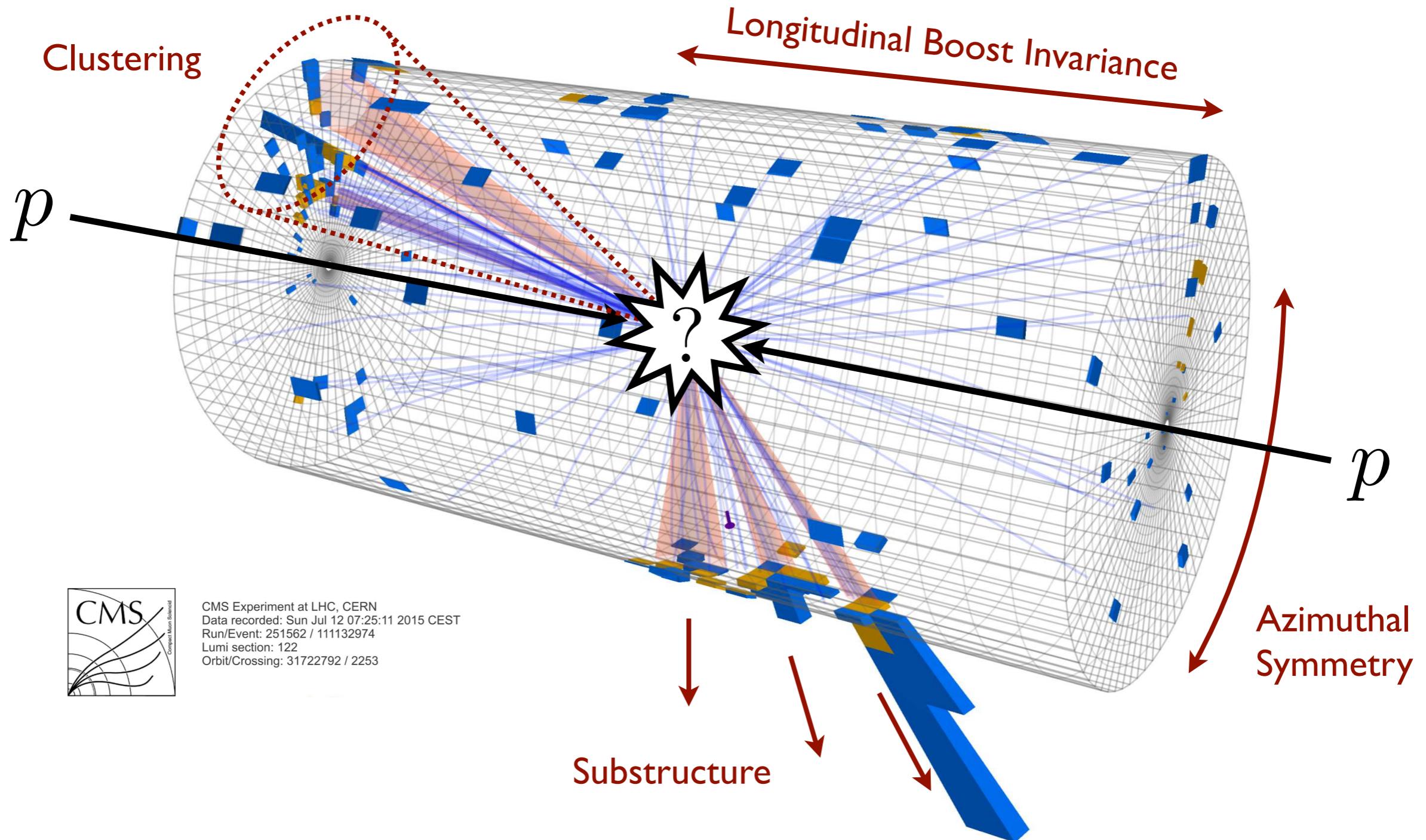


*Advance physics knowledge — from the smallest building blocks of nature  
to the largest structures in the universe — and galvanize AI research innovation*

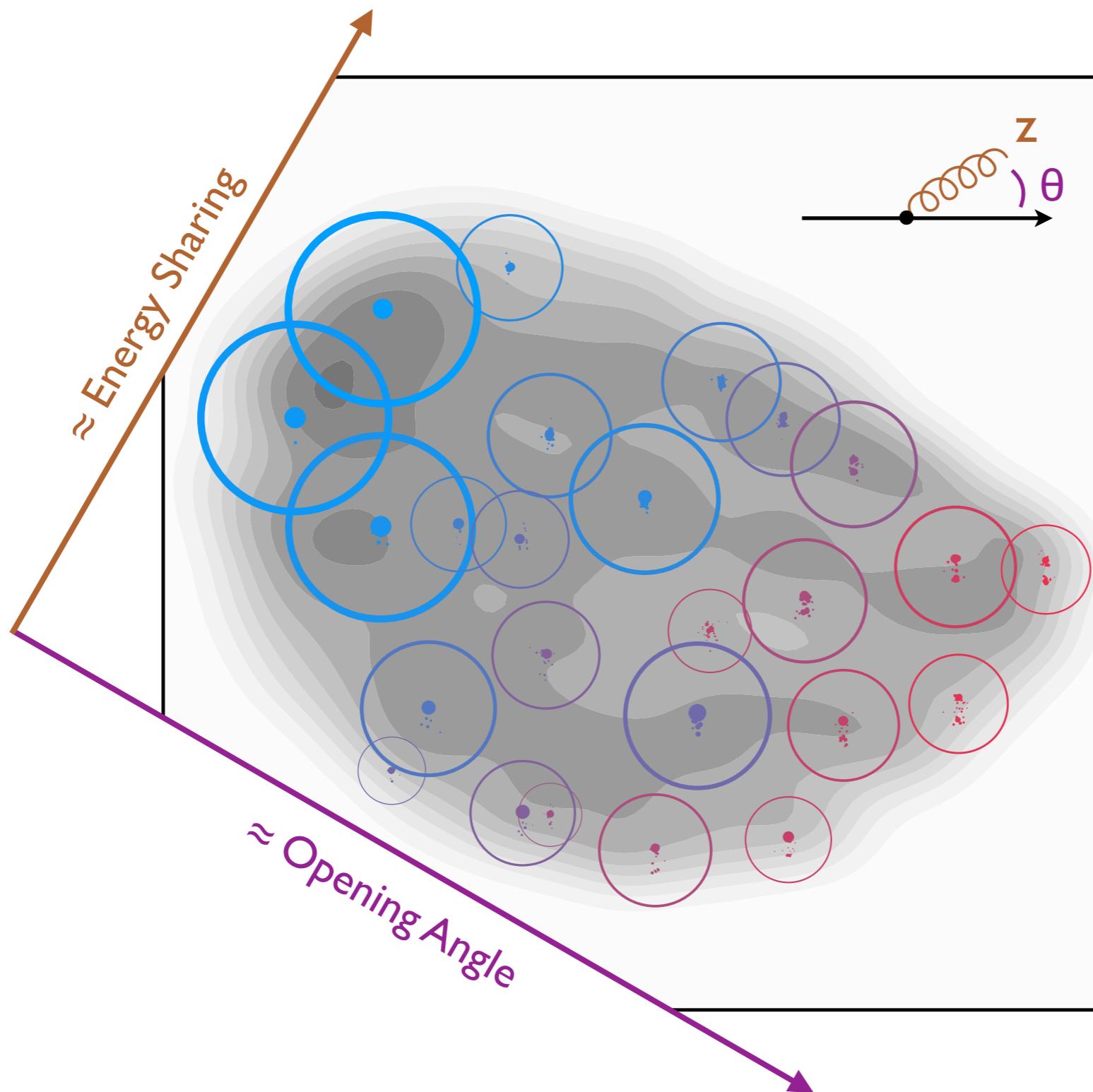


[<http://iaifi.org>, MIT News Announcement]

# The Manifest Geometry of One Collision



# The Emergent Geometry of Many Collisions

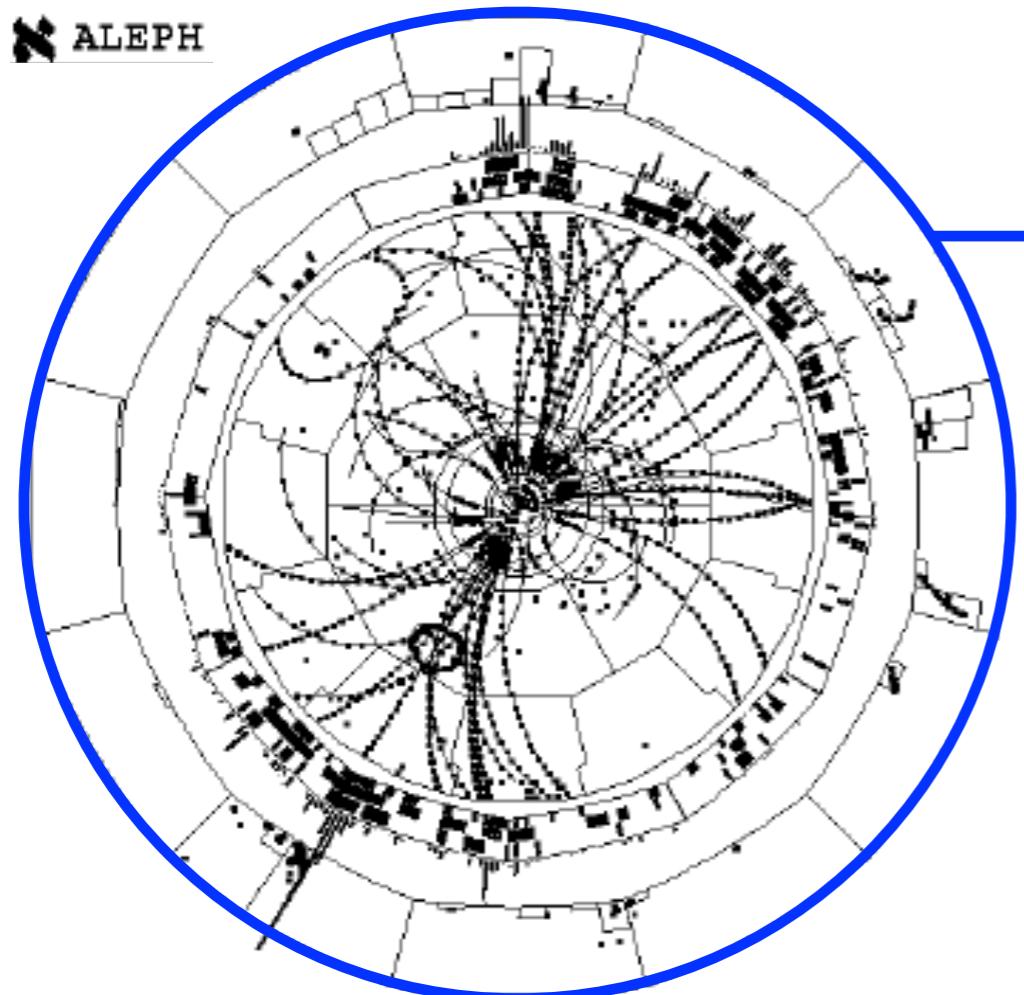


[Komiske, Mastandrea, Metodiev, Naik, JDT, [PRD 2020](#);  
based on Komiske, Metodiev, JDT, [PRL 2019](#); using [EnergyFlow](#) and [CMS Open Data](#)]

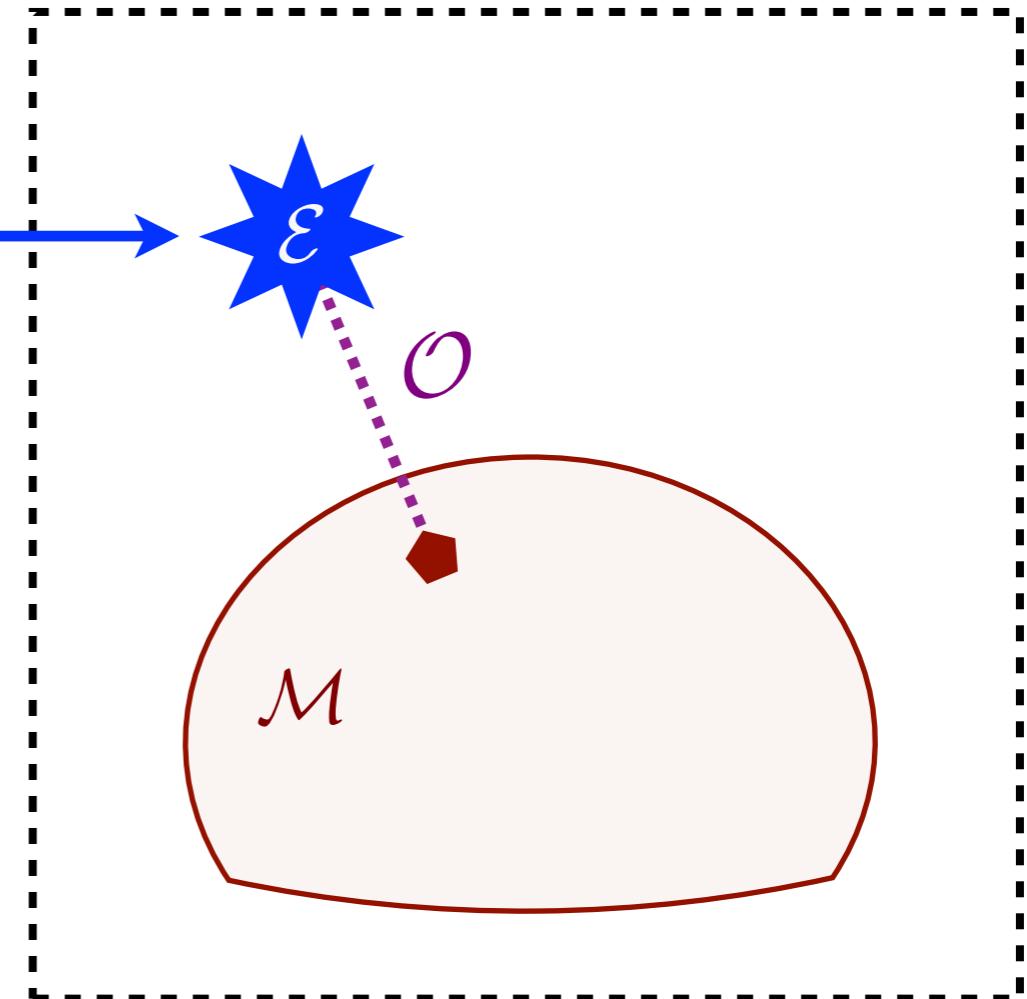


# The Hidden Geometry of Particle Collisions

E.g. Classic QCD Event Shapes



One Electron-Positron Event



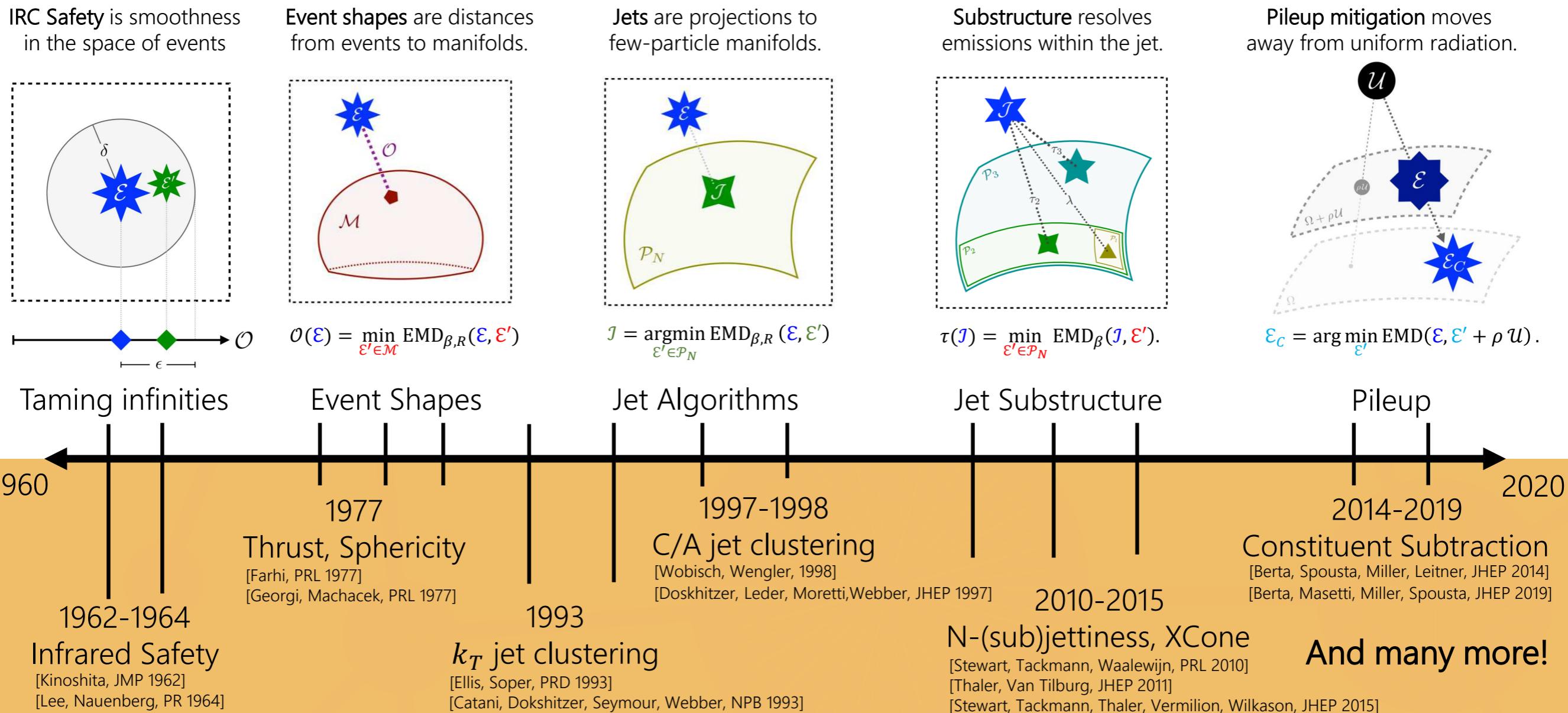
Distance to a Manifold in Event Space

[Komiske, Metodiev, JDT, [JHEP 2020](#)]

[Brandt, Peyrou, Sosnowski, Wroblewski, [PL 1964](#); Farhi, [PRL 1977](#)]

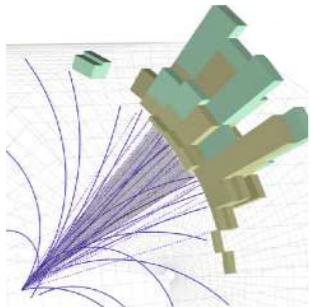


# Six Decades of Collider Physics Translated into a New Geometric Language!

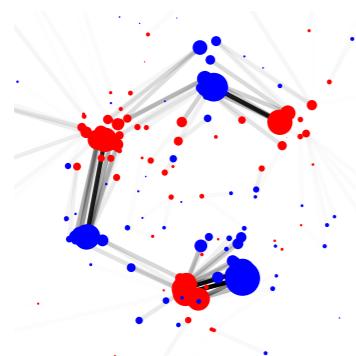


[Timeline from Eric Metodiev]

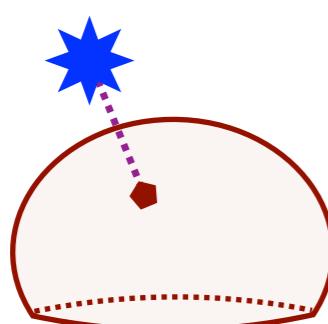
# Outline



What is a Collider Event?



When are Events Similar?



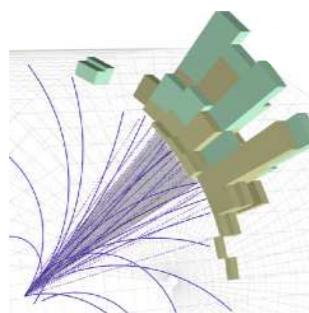
What can be Geometrized?

# Pause

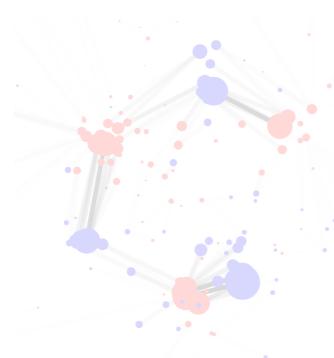
*Interrupt me or drop questions in the chat, and I'll try to answer them as I go*

*Anticipating a coda...*

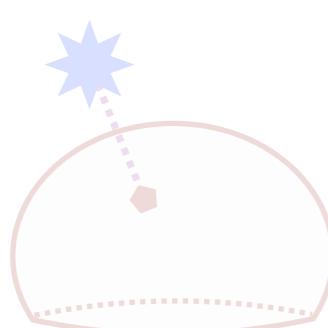




## What is a Collider Event?



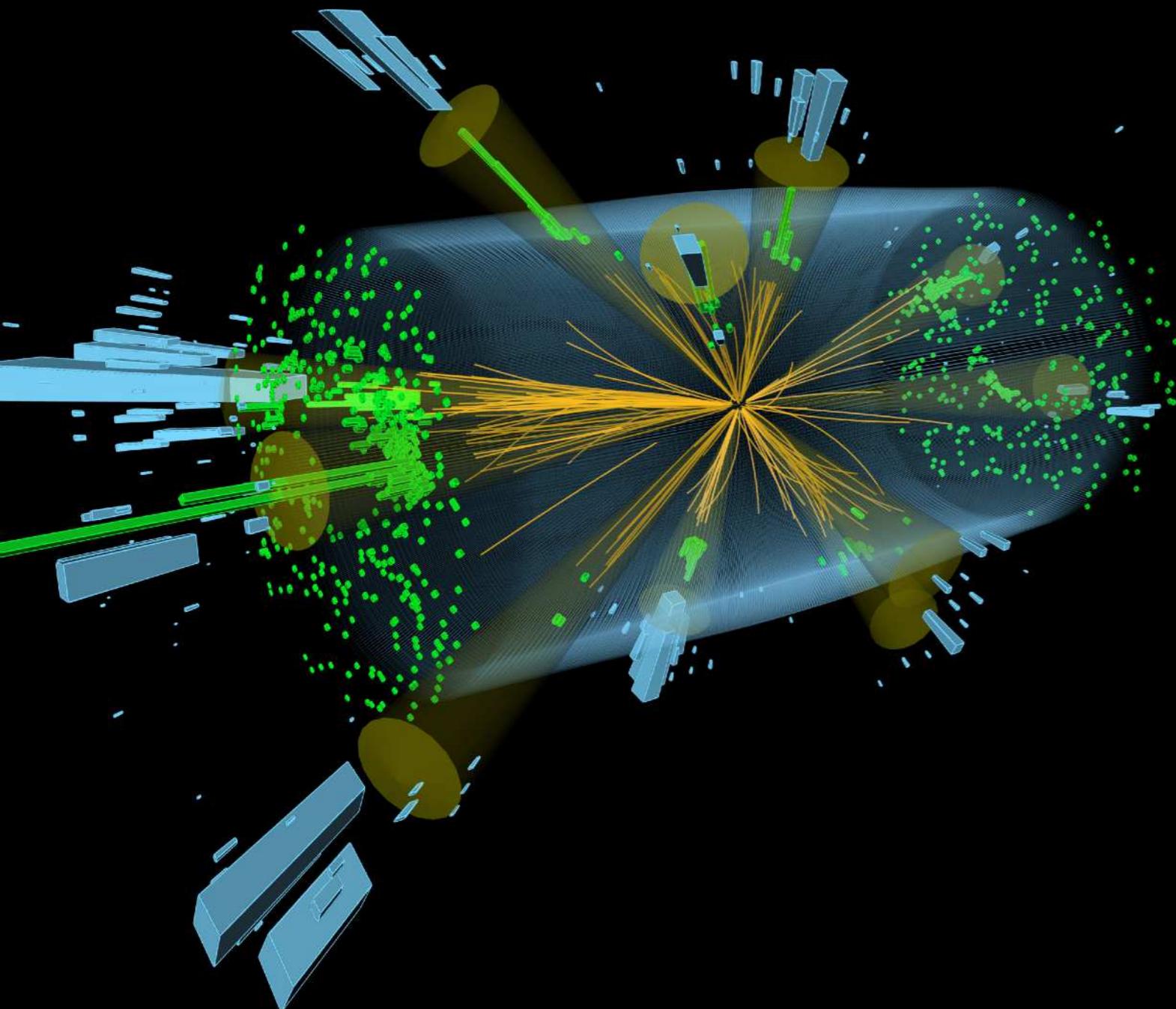
## When are Events Similar?



## What can be Geometrized?

# Collider Event

Collection of points in momentum space



T E H M

 $\gamma$ 

photon

 $e^+$ 

electron

 $\mu^+$ 

muon

 $\pi^+$ 

pion

 $K^+$ 

kaon

 $K_L^0$ 

K-long

 $p/\bar{p}$ 

proton

 $n/\bar{n}$ 

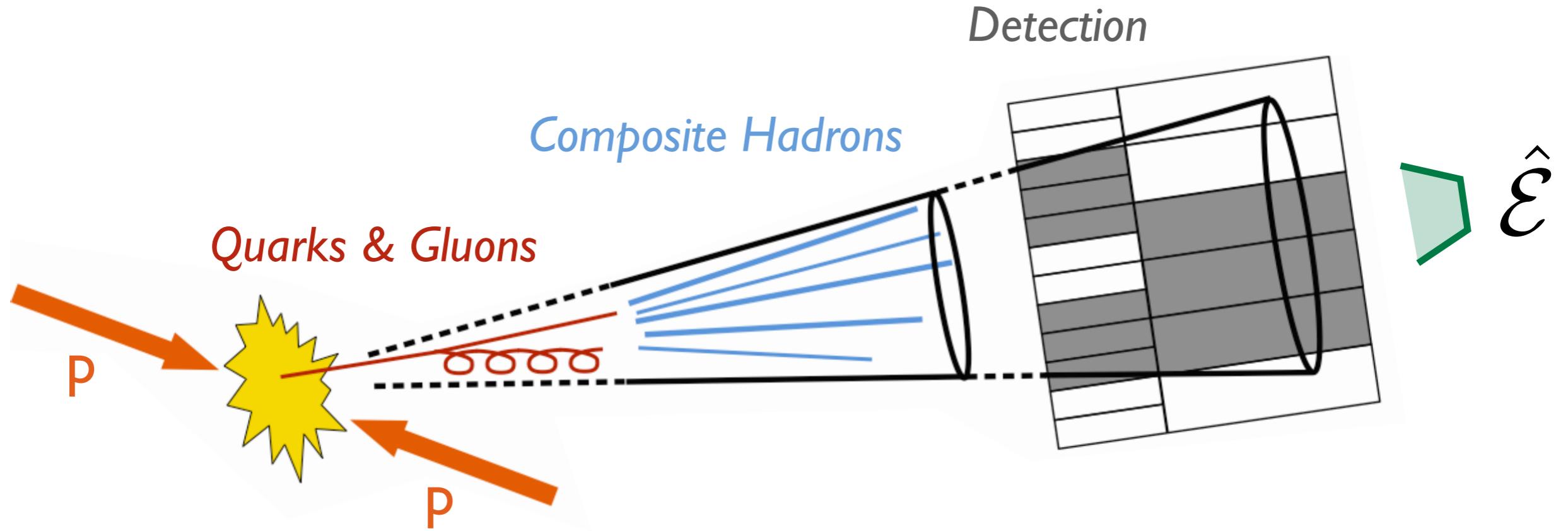
neutron

elementary

composite

# Jet Formation from QCD

Theory



## Energy Flow:

Robust to hadronization and detector effects  
Well-defined for massless gauge theories

$$\hat{\mathcal{E}} \simeq \lim_{t \rightarrow \infty} \hat{n}_i T^{0i}(t, vt\hat{n})$$

[see e.g. Sveshnikov, Tkachov, [PLB 1996](#); Hofman, Maldacena, [JHEP 2008](#); Mateu, Stewart, [JDT, PRD 2013](#); Belitsky, Hohenegger, Korchemsky, Sokatchev, Zhiboedov, [PRL 2014](#); Chen, Moult, Zhang, Zhu, [PRD 2020](#)]

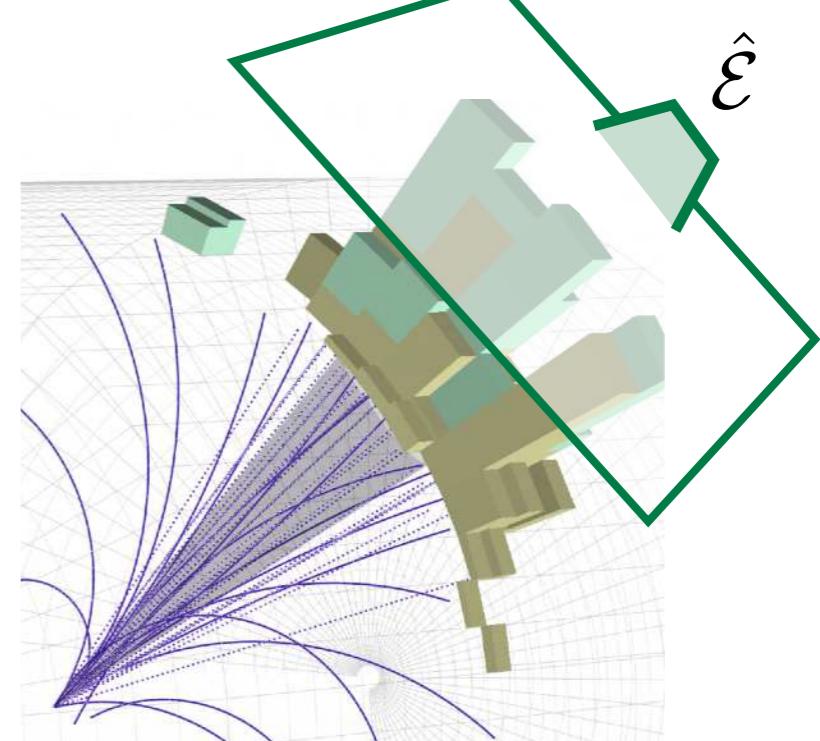
# Jets as Weighted Point Clouds

- Energy-Weighted Directions

$$\vec{p} = \{E, \hat{n}_x, \hat{n}_y, \hat{n}_z\}$$

↑      |  
Energy      Direction

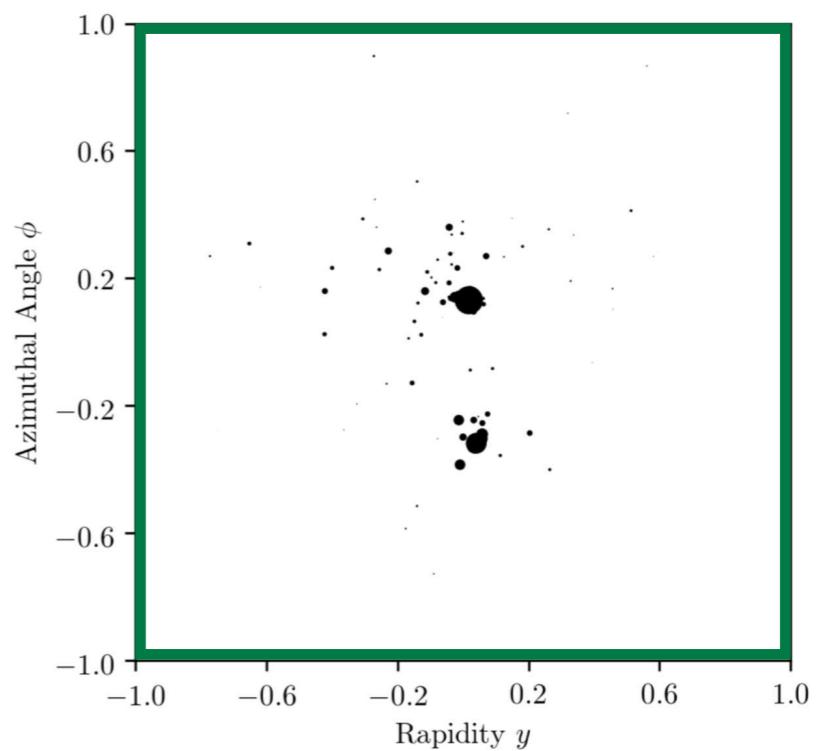
(suppressing “unsafe” charge/flavor information)



- Equivalently: Energy Density

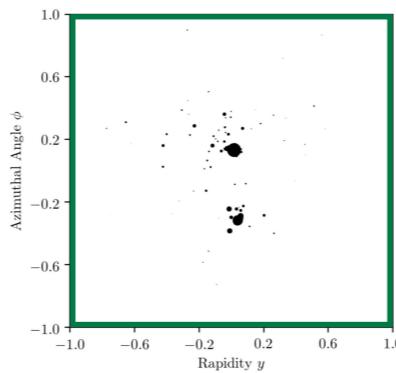
$$\rho(\hat{n}) = \sum_{i \in \mathcal{J}} E_i \delta^{(2)}(\hat{n} - \hat{n}_i)$$

↑      ↑  
Energy      Direction



# Pause

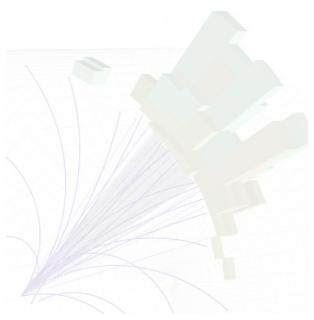
*What is a Collider Event?*



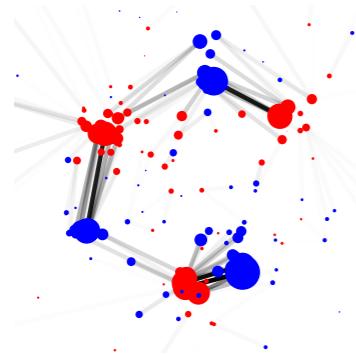
“Calo” Energy Density

$$\mathcal{E}(\hat{n}) = \sum_i E_i \delta(\hat{n} - \hat{n}_i)$$

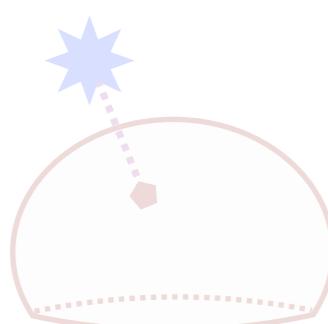
(see backup for relevance to QCD & ML)



## What is a Collider Event?



## When are Events Similar?

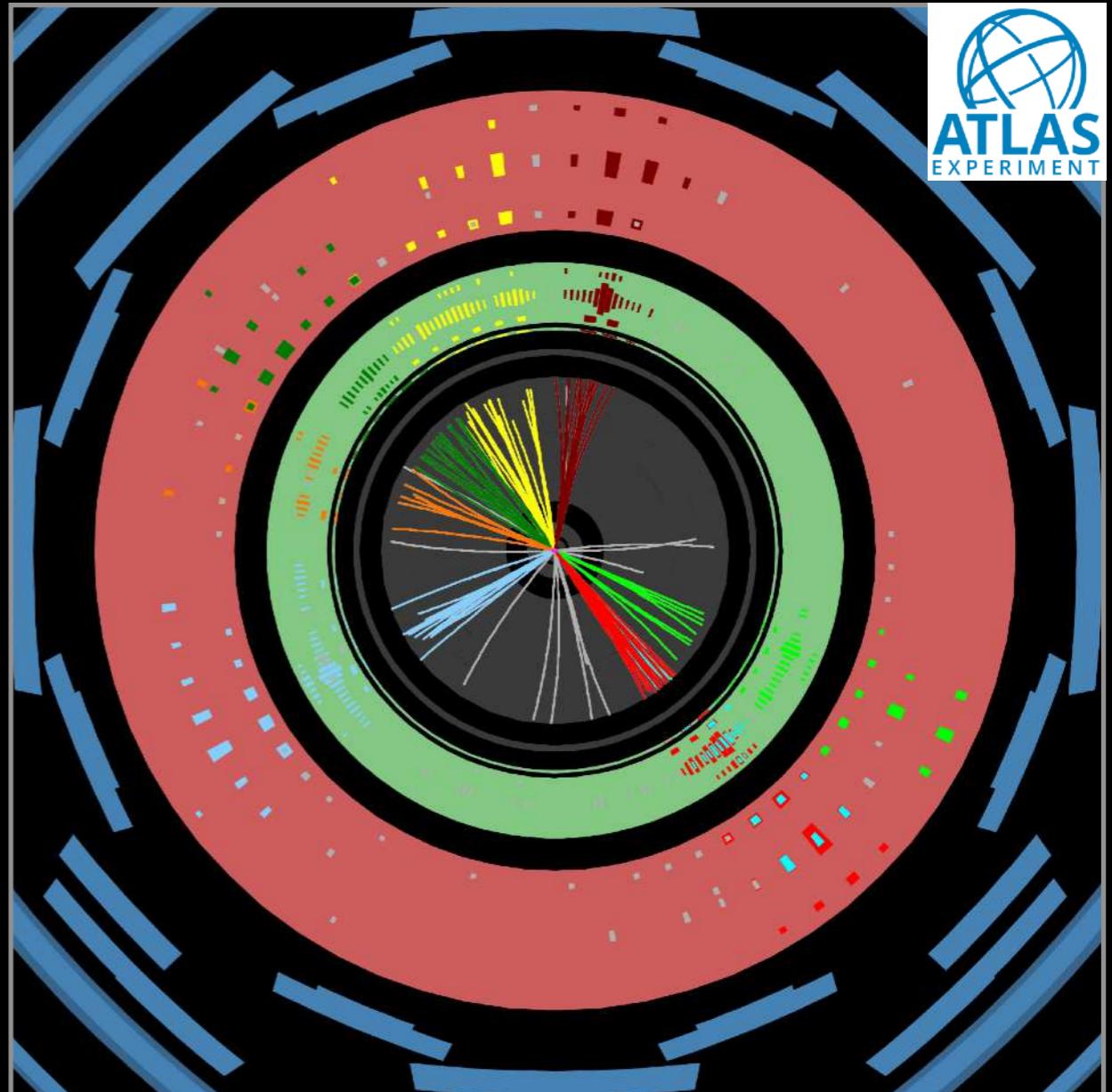
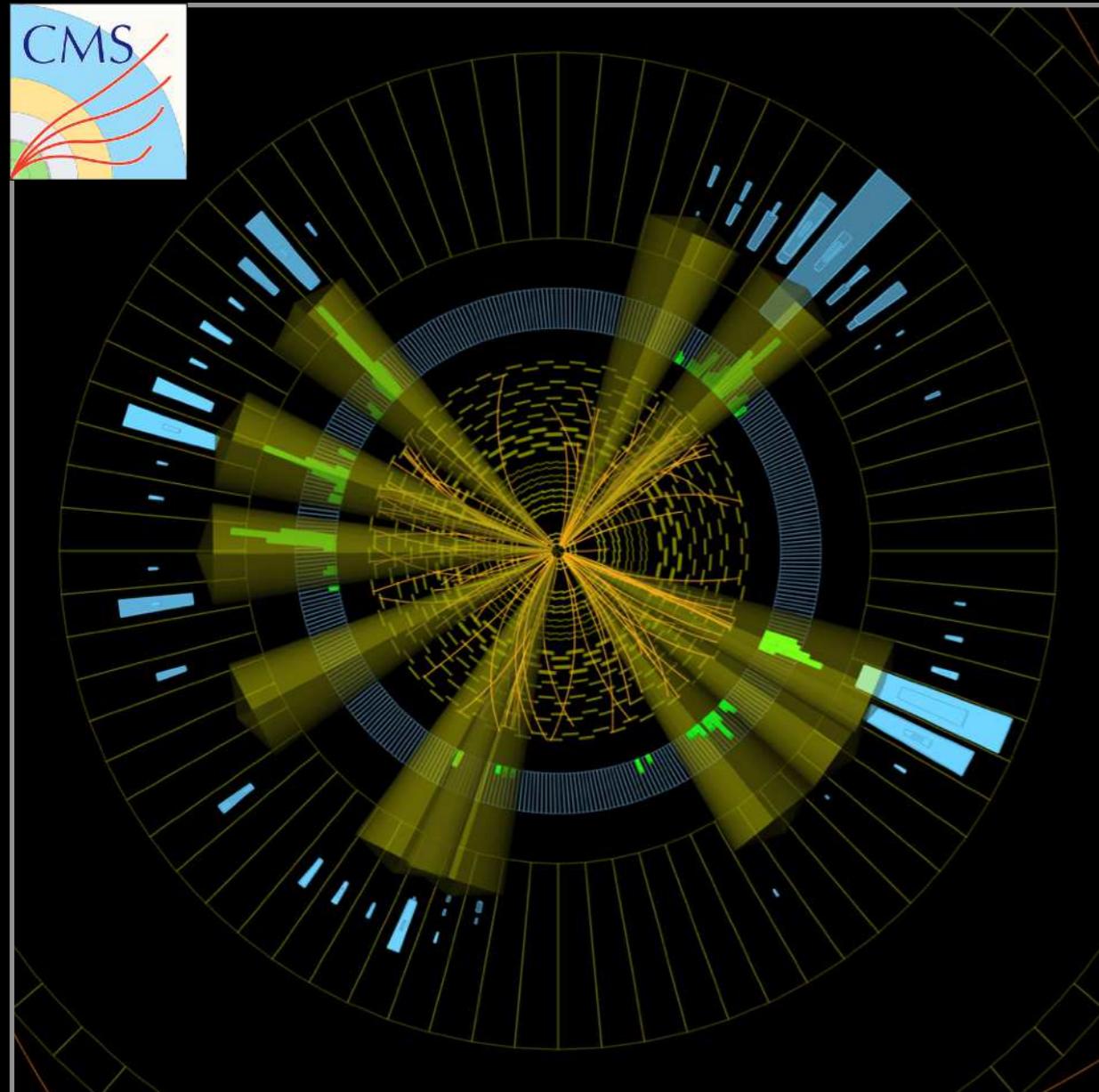


## What can be Geometrized?

# Two Collider Events

Two collections of points in (momentum) space

How “close” are these?



# The Earth Mover's Distance

## Optimal Transport:

[Peleg, Werman, Rom, [IEEE 1989](#);  
Rubner, Tomasi, Guibas, [ICCV 1998](#), [ICCV 2000](#);  
Pele, Werman, [ECCV 2008](#); Pele Taskar, [GSI 2013](#)]

Minimum “work” (stuff  $\times$  distance) to make one distribution look like another distribution



Déblai

Remblai

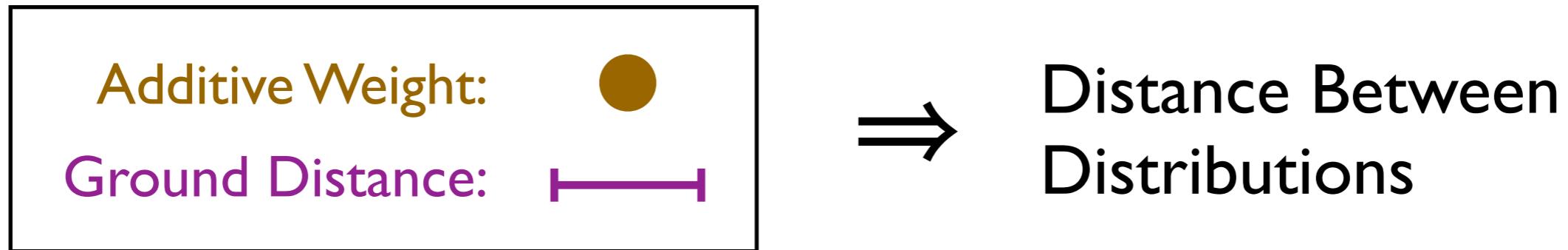
[h/t Niles-Weed, [ML4Jets 2020](#); Monge, 1781; Vaserštejn, 1969; [Wikipedia](#)]

# The Earth Mover's Distance

Optimal Transport:

[Peleg, Werman, Rom, [IEEE 1989](#);  
Rubner, Tomasi, Guibas, [ICCV 1998](#), [ICCV 2000](#);  
Pele, Werman, [ECCV 2008](#); Pele Taskar, [GSI 2013](#)]

Minimum “work” (**stuff** × **distance**) to make  
**one distribution** look like **another distribution**



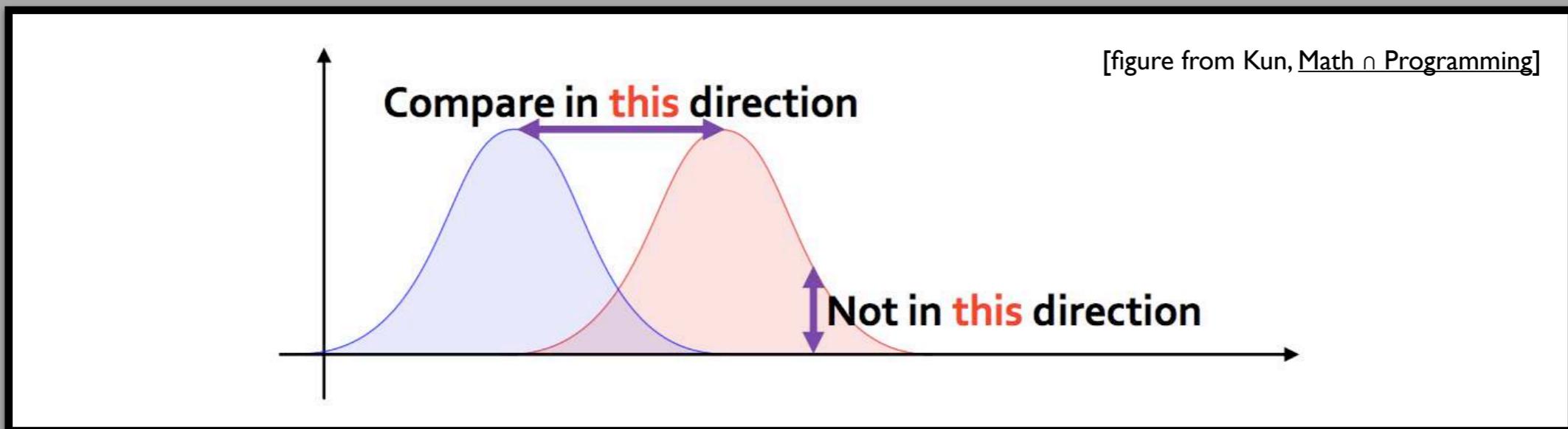
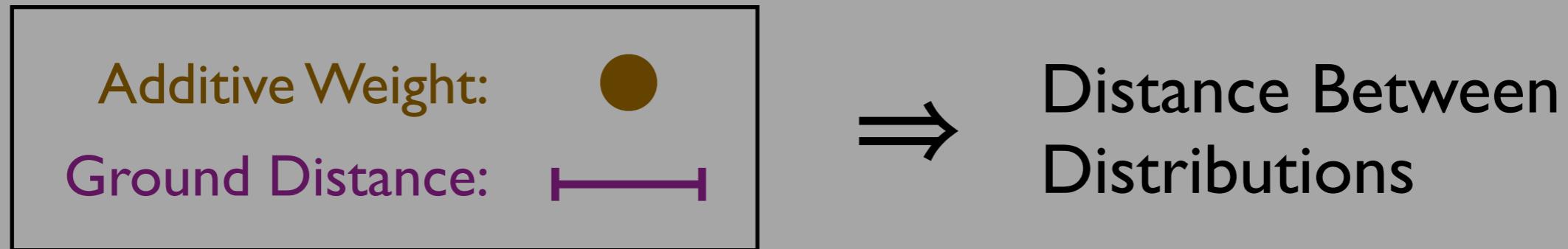
[h/t Niles-Weed, [ML4Jets 2020](#); Monge, 1781; Vaserštejn, 1969; [Wikipedia](#)]

# The Earth Mover's Distance

Optimal Transport:

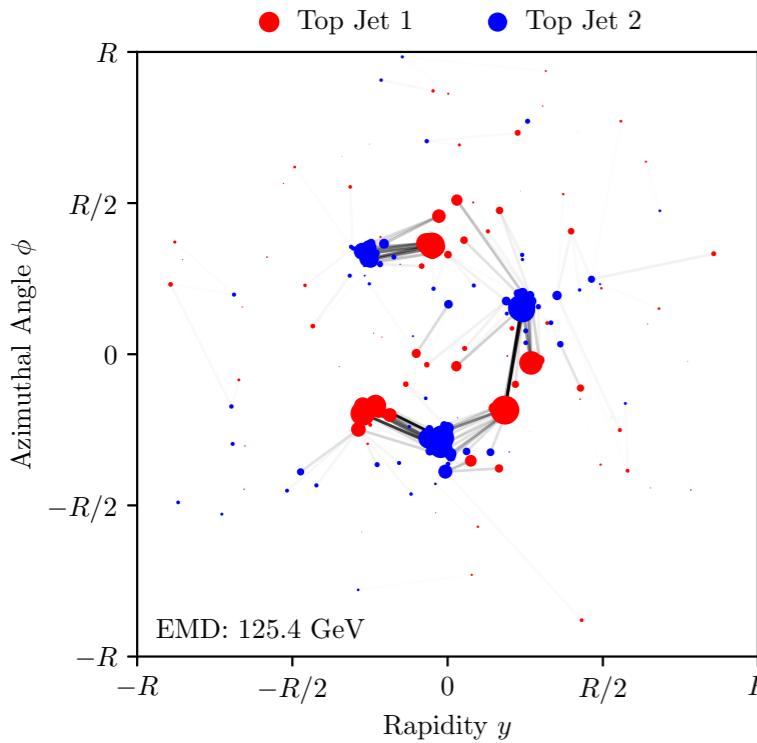
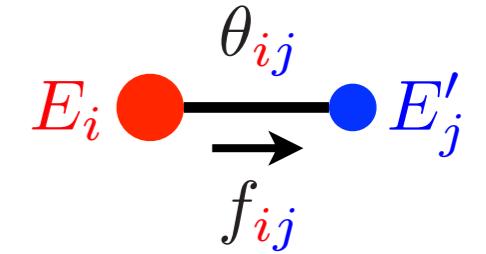
[Peleg, Werman, Rom, [IEEE 1989](#);  
Rubner, Tomasi, Guibas, [ICCV 1998](#), [ICCV 2000](#);  
Pele, Werman, [ECCV 2008](#); Pele Taskar, [GSI 2013](#)]

Minimum “work” (**stuff  $\times$  distance**) to make  
**one distribution** look like **another distribution**



[h/t Niles-Weed, [ML4Jets 2020](#); Monge, 1781; Vaserštejn, 1969; [Wikipedia](#)]

# The Energy Mover's Distance

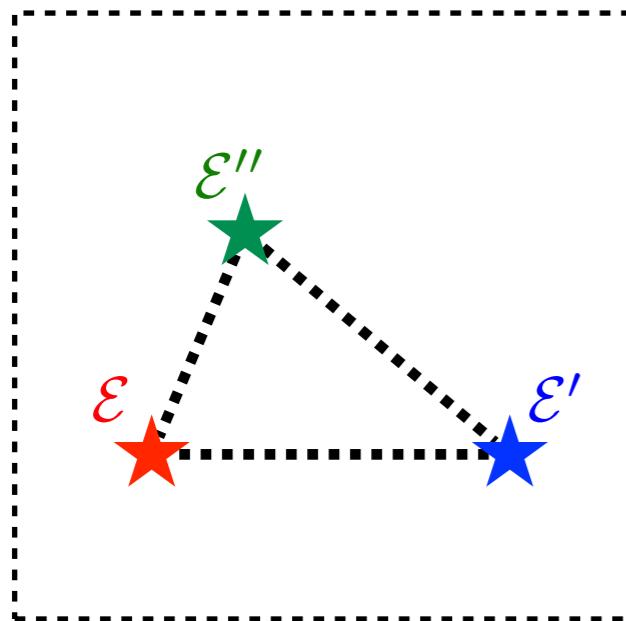


Optimal transport between energy flows...

$$\text{EMD}(\mathcal{E}, \mathcal{E}') = \min_{\{f\}} \sum_i \sum_j f_{ij} \frac{\theta_{ij}}{R} + \left| \sum_i E_i - \sum_j E'_j \right|$$

↑  
in GeV

Cost to move energy      Cost to create energy



...defines a metric on the space of events

$$0 \leq \text{EMD}(\mathcal{E}, \mathcal{E}') \leq \text{EMD}(\mathcal{E}, \mathcal{E}'') + \text{EMD}(\mathcal{E}', \mathcal{E}'')$$

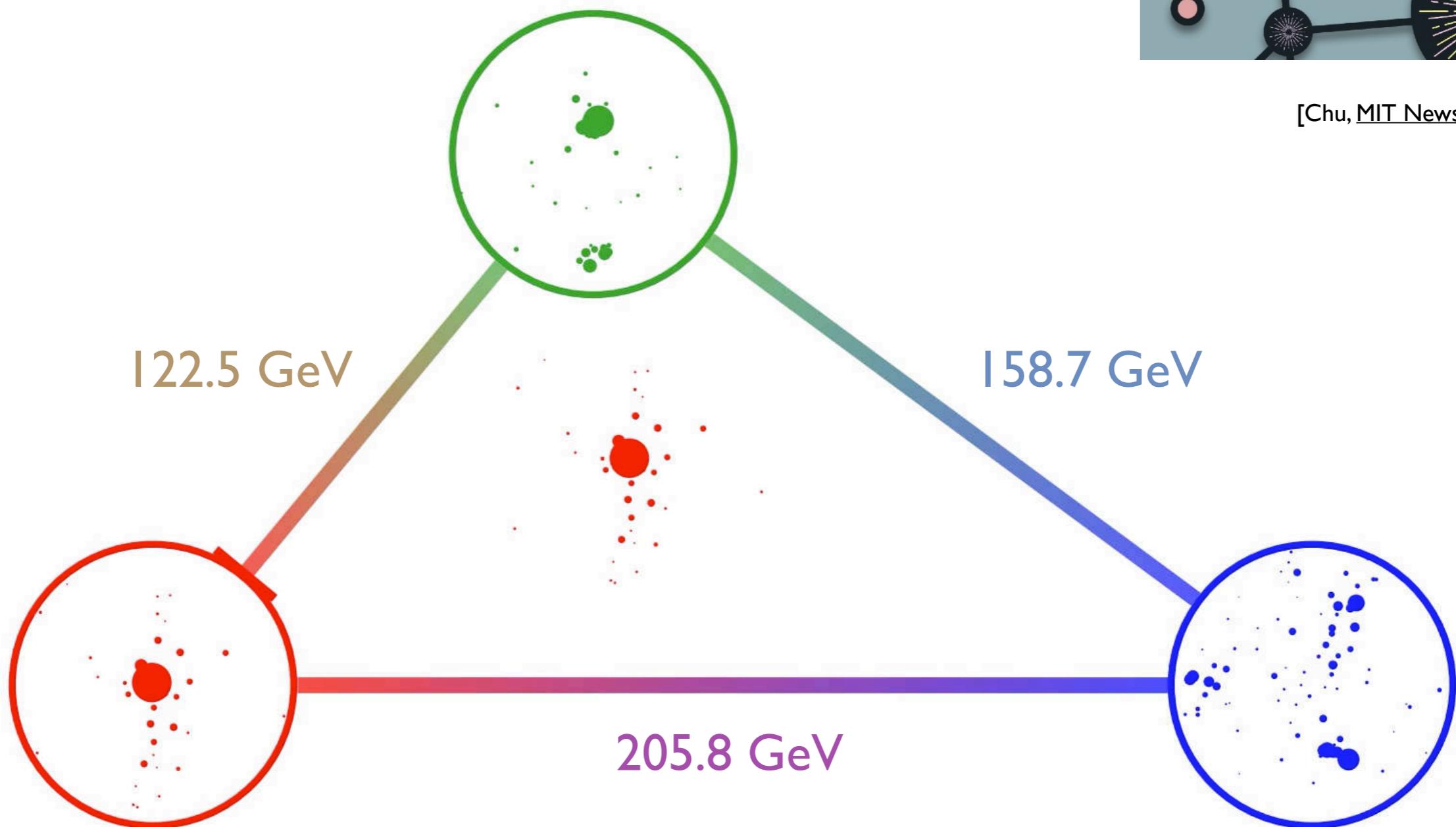
(assuming  $R \geq \theta_{\max}/2$ , i.e.  $R \geq$  jet radius for conical jets)

[Komiske, Metodiev, JDT, PRL 2019;  
see also Pele, Werman, ECCV 2008; Pele, Taskar, GSI 2013;  
[see flavored variant in Crispim Romão, Castro, Milhano, Pedro, Vale, arXiv 2020]  
[see computational speed up in Cai, Cheng, Craig, Craig, arXiv 2020]

# Similarity of Three Energy Flows?



[Chu, MIT News July 2019]



[Komiske, Metodiev, JDT, [PRL 2019](#); code at Komiske, Metodiev, JDT, [energyflow.network](#);  
see alternative graph network approach in Mullin, Pacey, Parker, White, Williams, [arXiv 2019](#)]

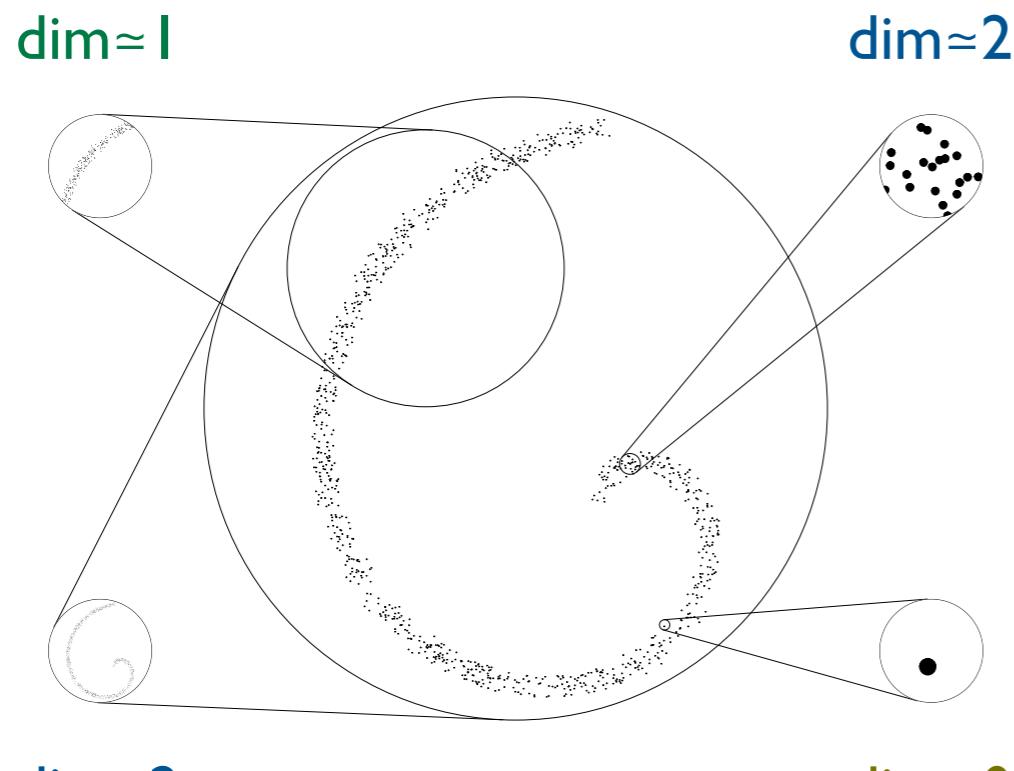
# Dimensionality of Space of Jets



$$N_{\text{neighbors}}(r) \sim r^{\dim}$$

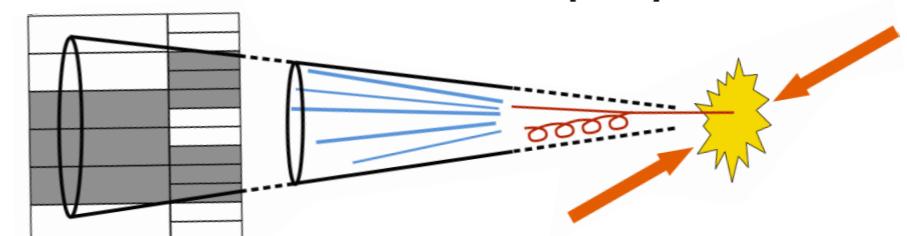
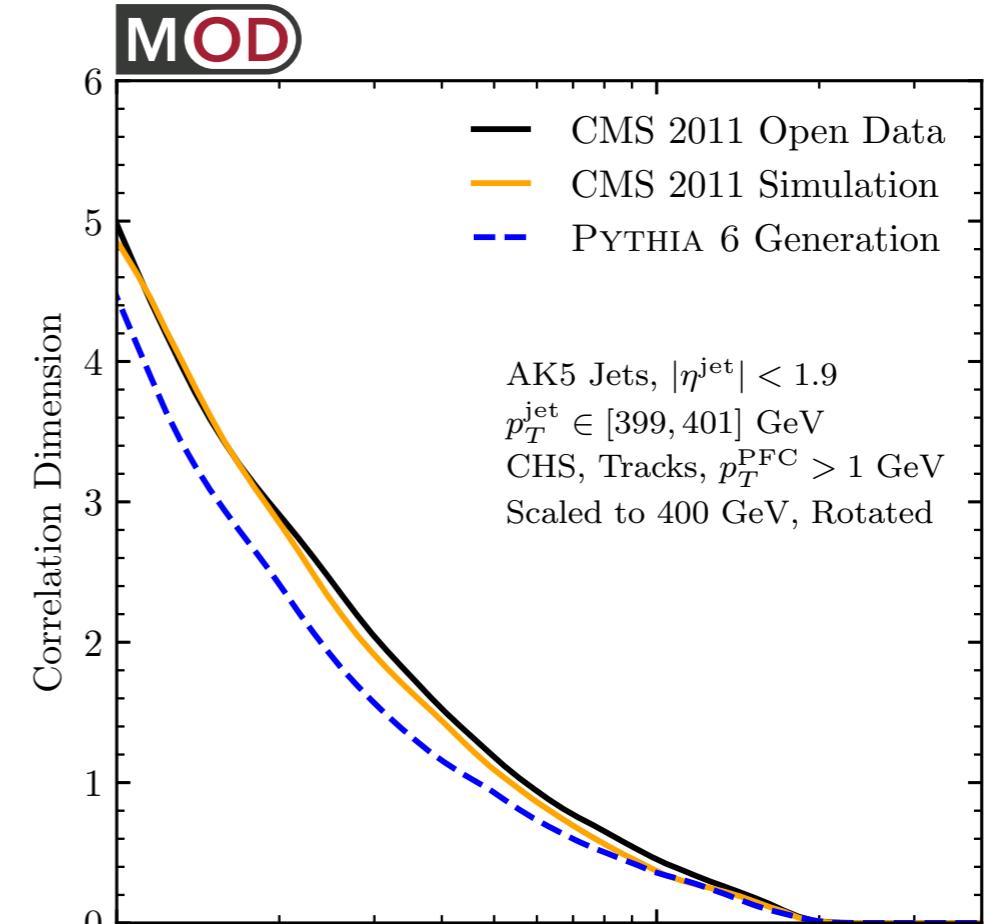
$$\Rightarrow \dim(r) \sim r \frac{\partial}{\partial r} \ln N_{\text{neighbors}}(r)$$

[Grassberger, Procaccia, [PRL 1983](#); Kégl, [NIPS 2002](#)]



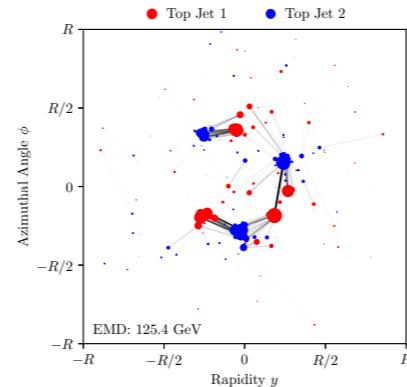
(eventually 0)

[Komiske, Mastandrea, Metodiev, Naik, [JDT, PRD 2020](#);  
using [CMS Open Data](#)]



# Pause

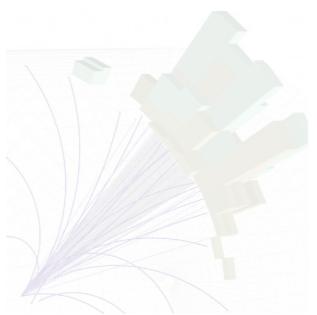
*When are Events Similar?*



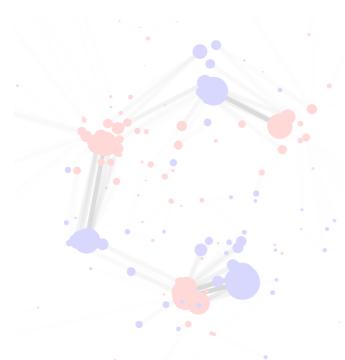
**Small Energy Mover's Distance**

$$\text{EMD}(\mathcal{E}, \mathcal{E}') = \min_{\{f\}} \sum_i \sum_j f_{ij} \frac{\theta_{ij}}{R} + \left| \sum_i E_i - \sum_j E'_j \right|$$

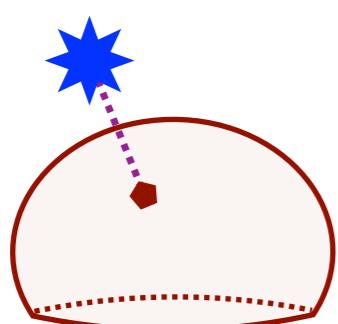
(see backup for more applications)



## What is a Collider Event?



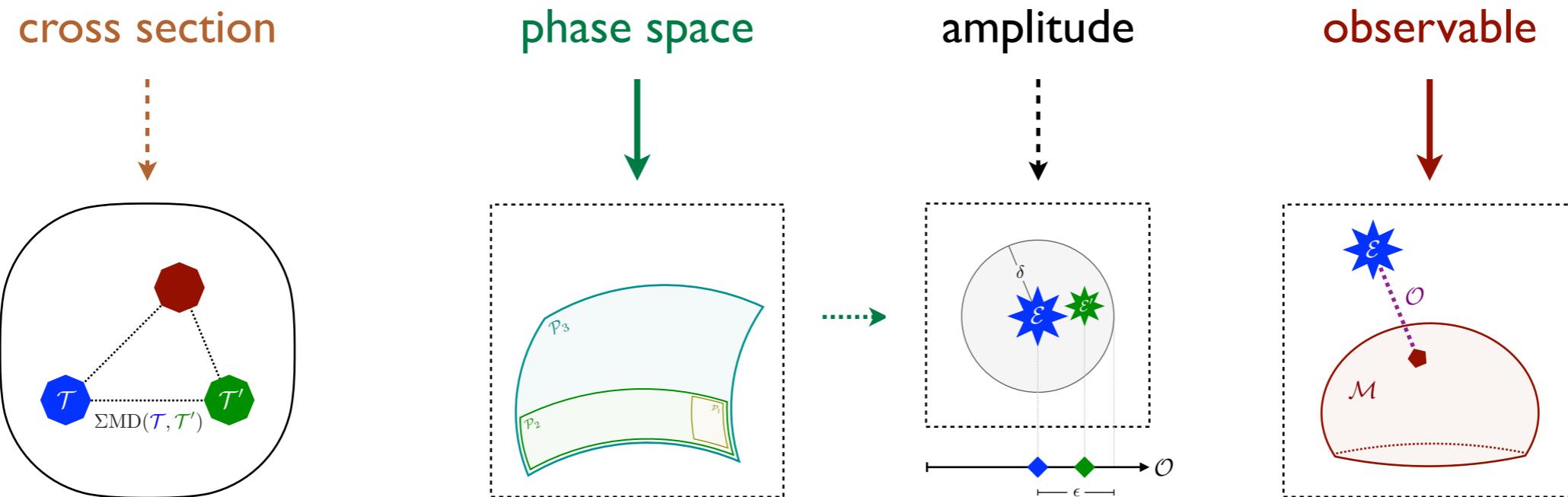
## When are Events Similar?



## What can be Geometrized?

# Master Formula for Collider Physics

$$\sigma_{\text{obs}} \simeq \frac{1}{2E_{\text{CM}}^2} \sum_{n=2}^{\infty} \int d\Phi_n |\mathcal{M}_{AB \rightarrow 12\dots n}|^2 f_{\text{obs}}(\Phi_n)$$

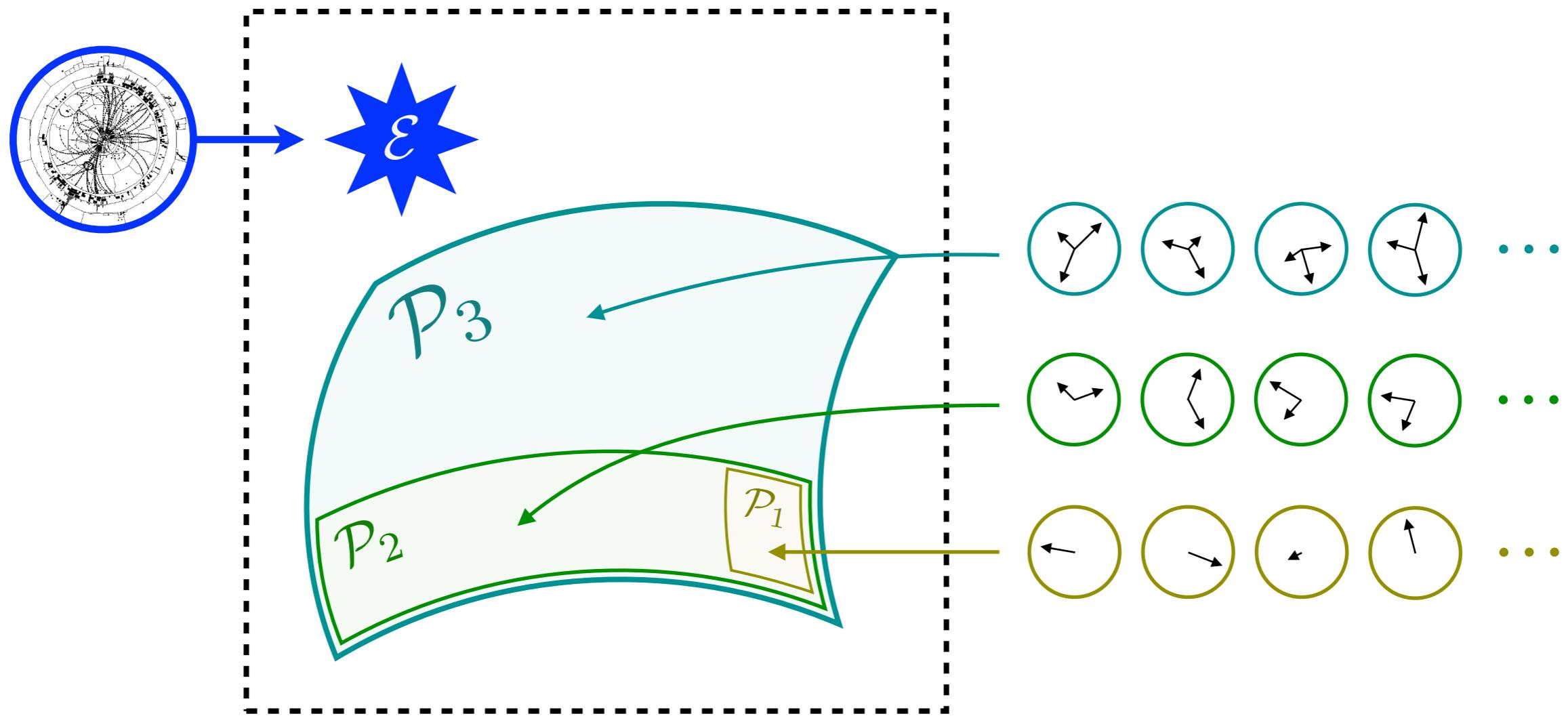


[Komiske, Metodiev, JDT, [JHEP 2020](#)]

# Introducing N-particle Manifolds

$$\sum_{n=2}^{\infty} \int d\Phi_n$$

$\mathcal{P}_N$  = set of all N-particle configurations



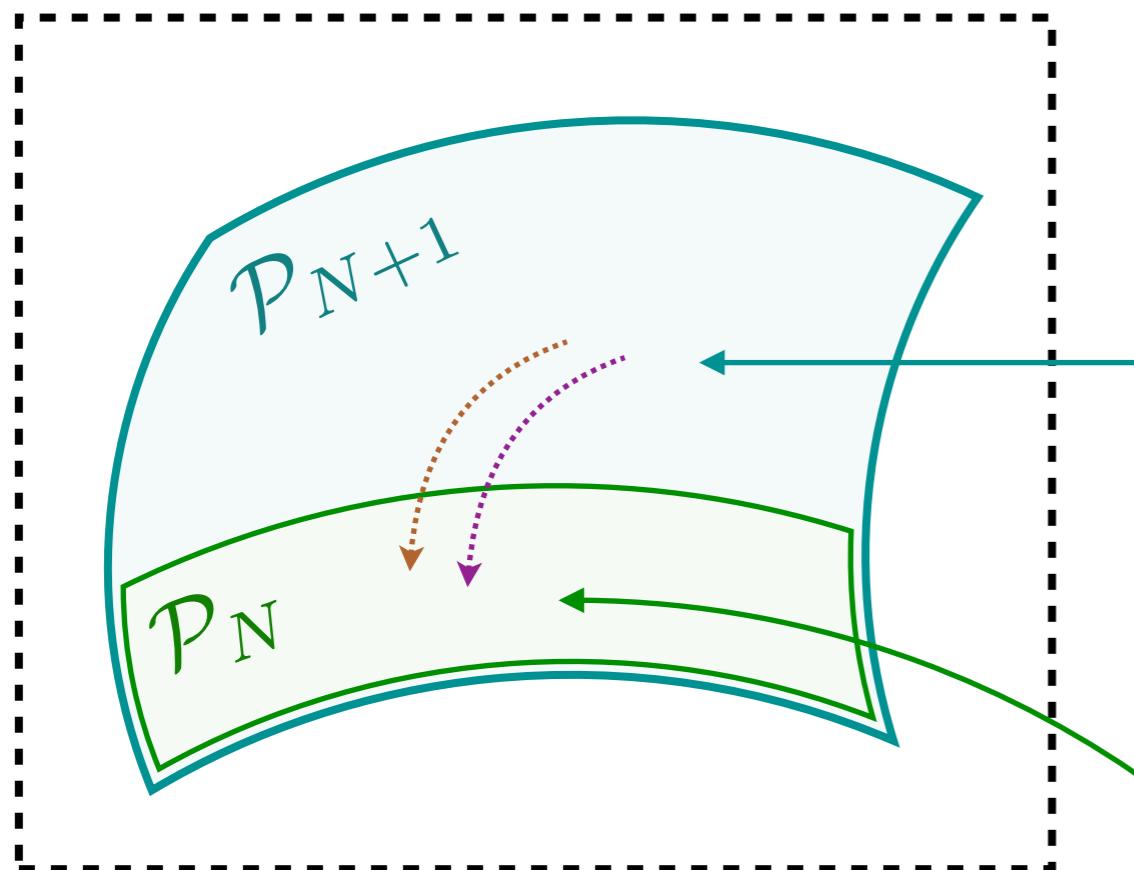
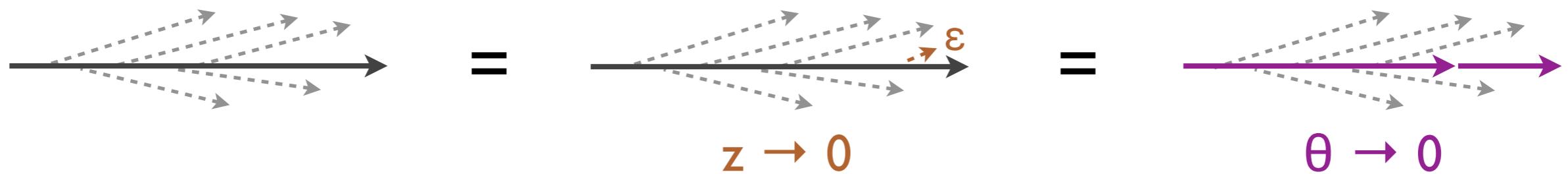
$\mathcal{P}_N \supset \mathcal{P}_{N-1} \supset \dots \supset \mathcal{P}_2 \supset \mathcal{P}_1$  by soft/collinear limits

[see related discussion in Larkoski, Melia, [PRD 2020](#)]

# When are Two Events the Same?

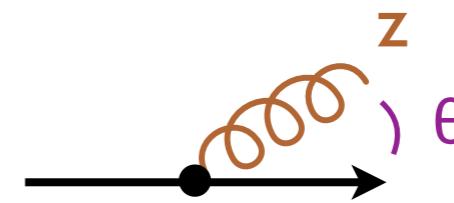
$$\mathcal{E}(\hat{n}) = \sum_i E_i \delta(\hat{n} - \hat{n}_i)$$

*Energy Flow unchanged by infinitesimal soft/collinear emissions*



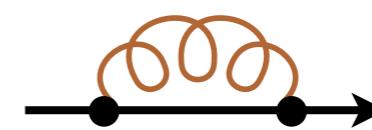
Infrared divergences “live” together!

Real:



$$dP_{i \rightarrow ig} \simeq \frac{2\alpha_s}{\pi} C_i \frac{dz}{z} \frac{d\theta}{\theta}$$

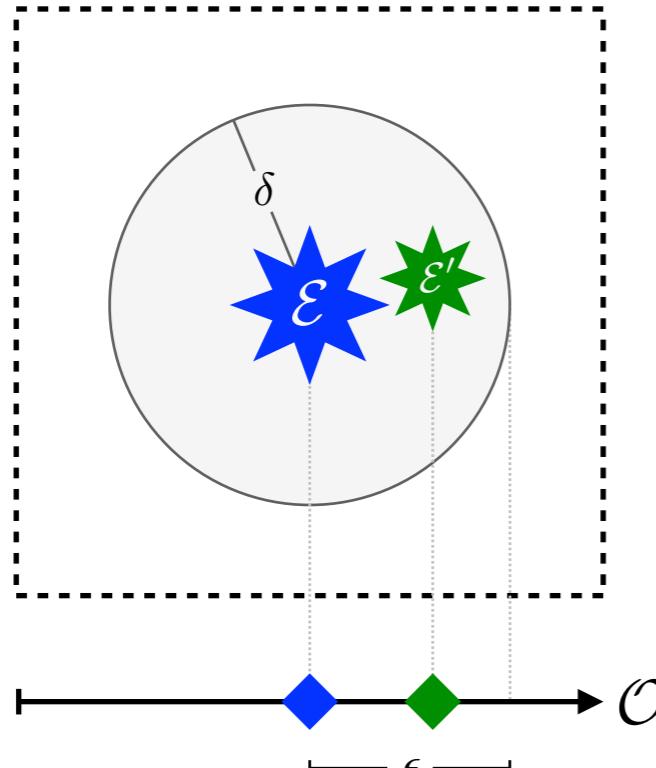
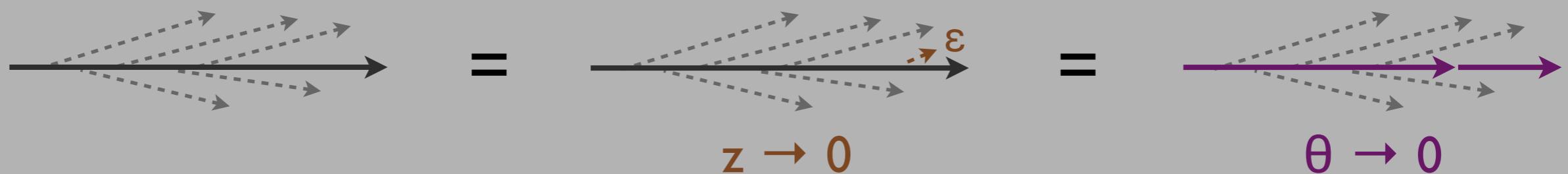
Virtual:



# When are Two Events the Same?

$$\mathcal{E}(\hat{n}) = \sum_i E_i \delta(\hat{n} - \hat{n}_i)$$

*Energy Flow unchanged by infinitesimal soft/collinear emissions*



## Infrared & Collinear Safety

$\approx$  calculable in perturbative quantum field theory

*is\**  $\leftarrow$  (see backup for subtleties)

## Continuity in EMD Space

[Komiske, Metodiev, JDT, [JHEP 2020](#)]

[Sterman, Weinberg, [PRL 1977](#); Sterman, [PRD 1979](#)]

[see also Banfi, Salam, Zanderighi, [JHEP 2005](#); Larkoski, Marzani, JDT, [PRD 2015](#)]

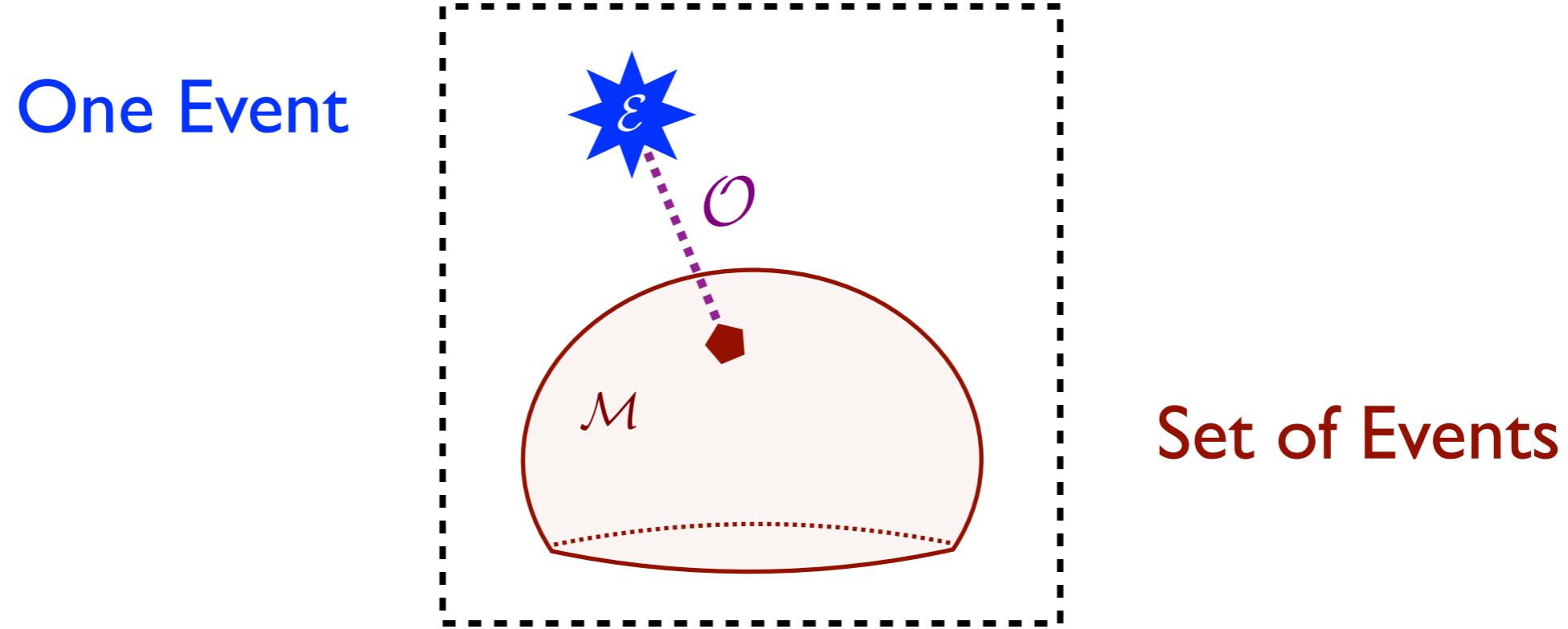
*EMD seems to define the “natural” geometry for massless gauge theories*

Open question: Can you define  $|\mathcal{M}_{AB \rightarrow 12\dots n}|^2$  directly in this space?

What does it mean to “integrate” in this space?

# Manifolds for Observables

$$f_{\text{obs}}(\Phi_n)$$



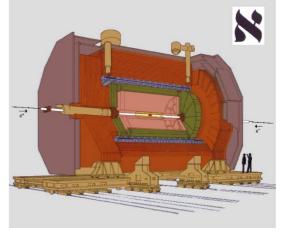
Distance of Closest Approach  $\Rightarrow$  Observable

$$O(\mathcal{E}) = \min_{\mathcal{E}' \in \mathcal{M}} \text{EMD}(\mathcal{E}, \mathcal{E}')$$

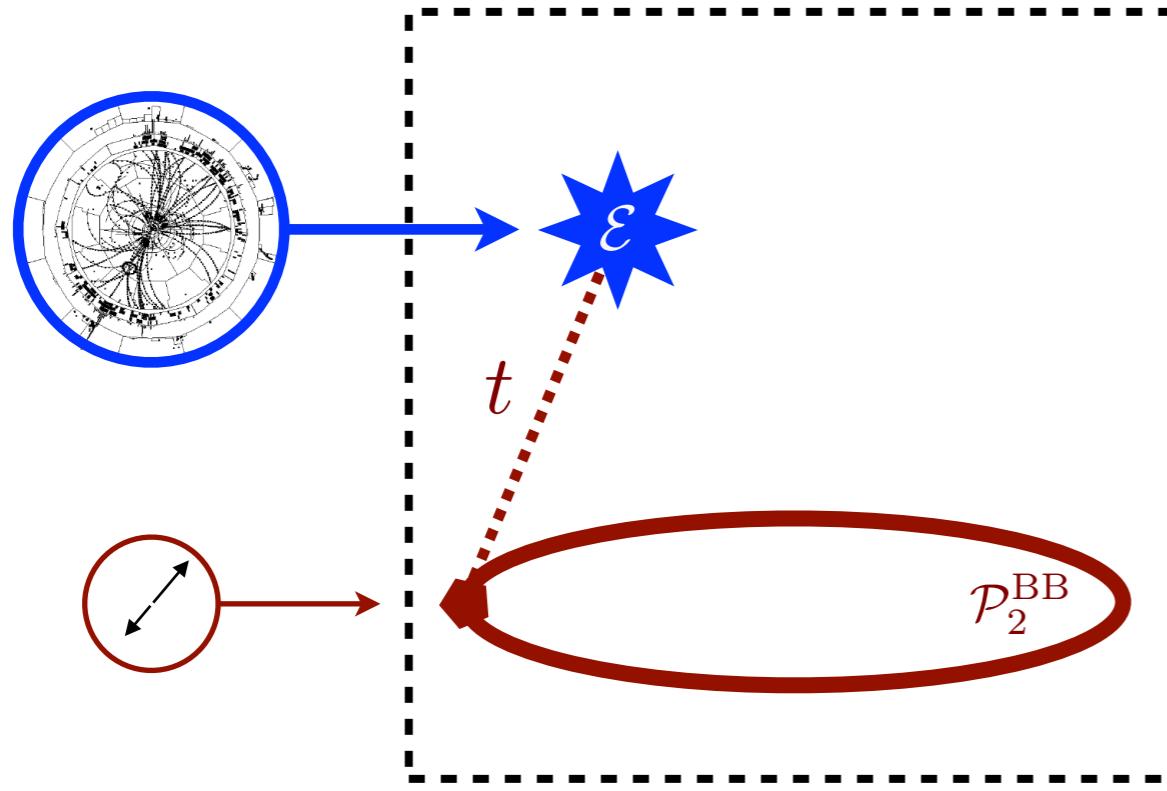
[Komiske, Metodiev, JDT, [JHEP 2020](#)]

# E.g. Thrust

How dijet-like is an event?



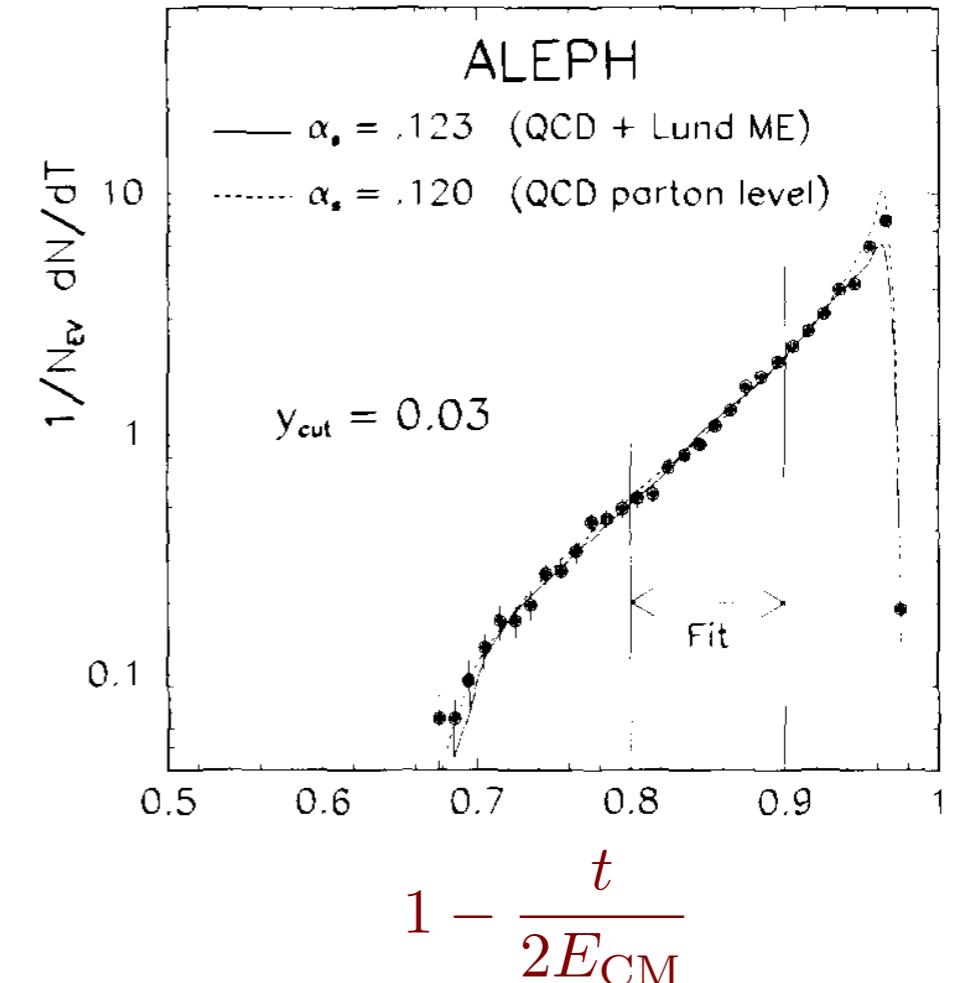
$$t(\mathcal{E}) = \min_{\mathcal{E}' \in \mathcal{P}_2^{\text{BB}}} \text{EMD}_2(\mathcal{E}, \mathcal{E}')$$



All Back-to-Back Two Particle Configurations

$$\mathcal{P}_2^{\text{BB}} = \left\{ \begin{array}{c} \text{red circle with diagonal arrow} \\ \text{red circle with vertical arrow} \\ \text{red circle with horizontal arrow} \\ \text{red circle with double arrows} \end{array} \dots \right\}$$

(using  $\beta=2$  EMD variant)



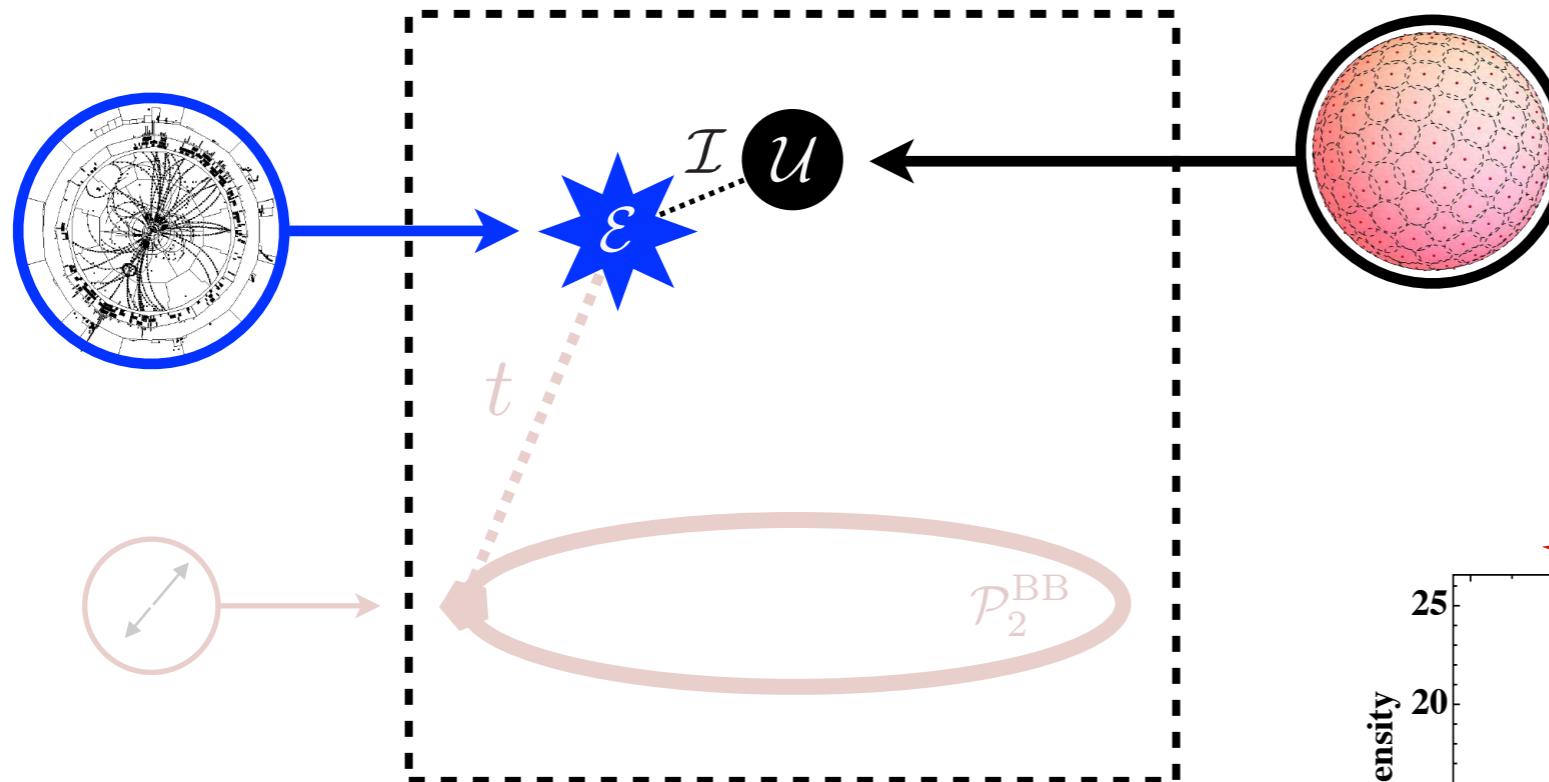
$$1 - \frac{t}{2E_{\text{CM}}}$$

$$\text{cf. } T(\mathcal{E}) = \max_{\hat{n}} \frac{\sum_i |\vec{p}_i \cdot \hat{n}|}{\sum_j |\vec{p}_j|}$$

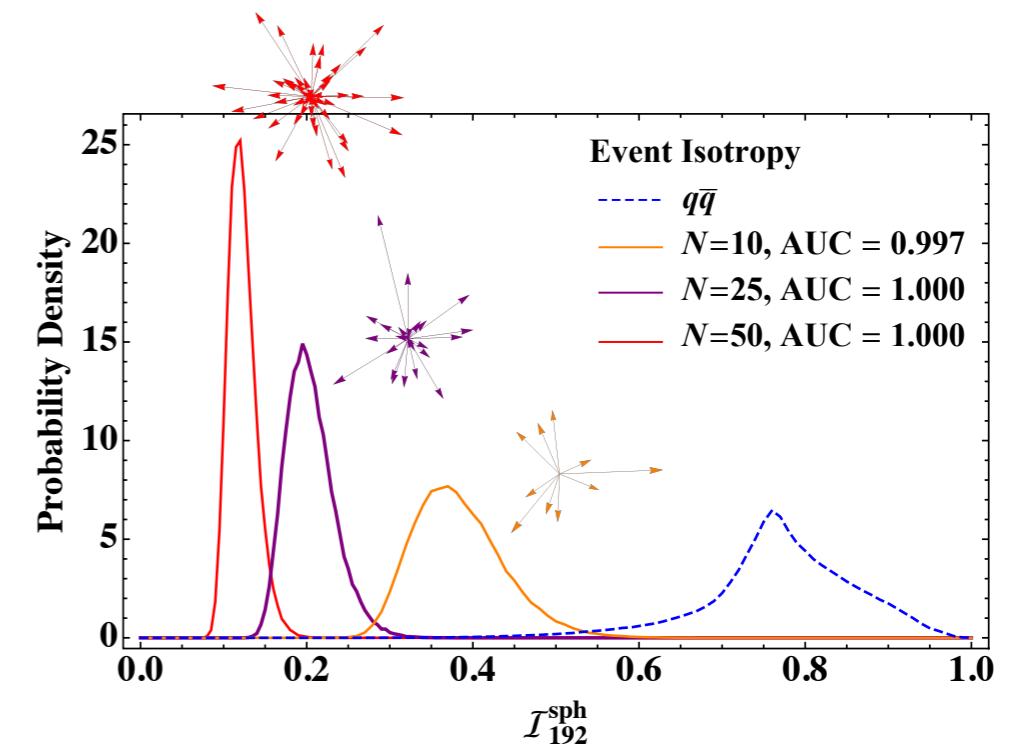
[Komiske, Metodiev, JDT, JHEP 2020]  
 [Brandt, Peyrou, Sosnowski, Wroblewski, PL 1964; Farhi, PRL 1977; ALEPH, PLB 1991]

# New! Event Isotropy

How isotropic is an event?



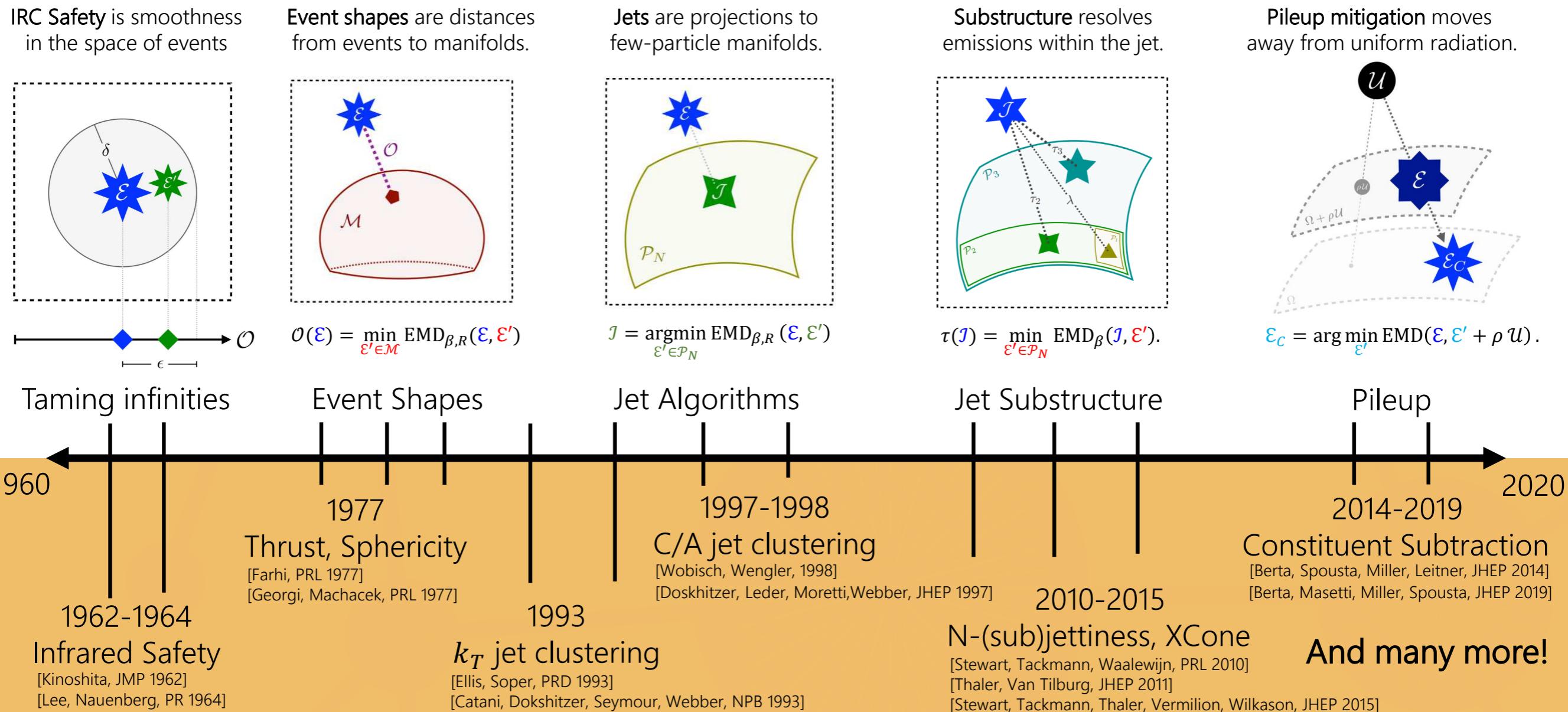
$$\mathcal{I}(\mathcal{E}) = \text{EMD}(\mathcal{E}, \mathcal{U})$$



[Cesarotti, JDT, [JHEP 2020](#);  
see also Cesarotti, Reece, Strassler, [arXiv 2020](#)]



# Six Decades of Collider Physics Translated into a New Geometric Language!

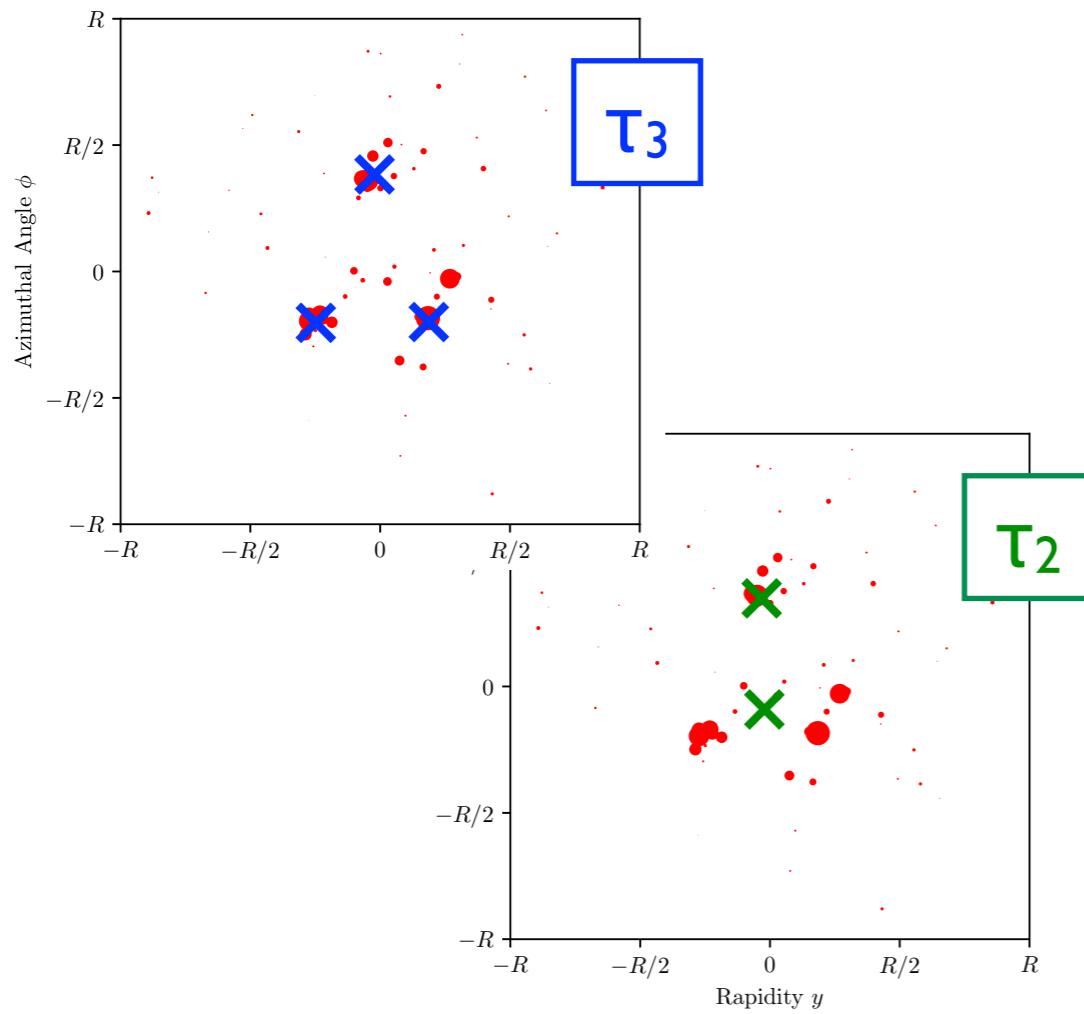


[timeline from Eric Metodiev]

# N-subjettiness

*Ubiquitous jet substructure observable used for almost a decade...*

$$\tau_N(\mathcal{J}) = \min_{N \text{ axes}} \sum_i E_i \min \{\theta_{1,i}, \theta_{2,i}, \dots, \theta_{N,i}\}$$



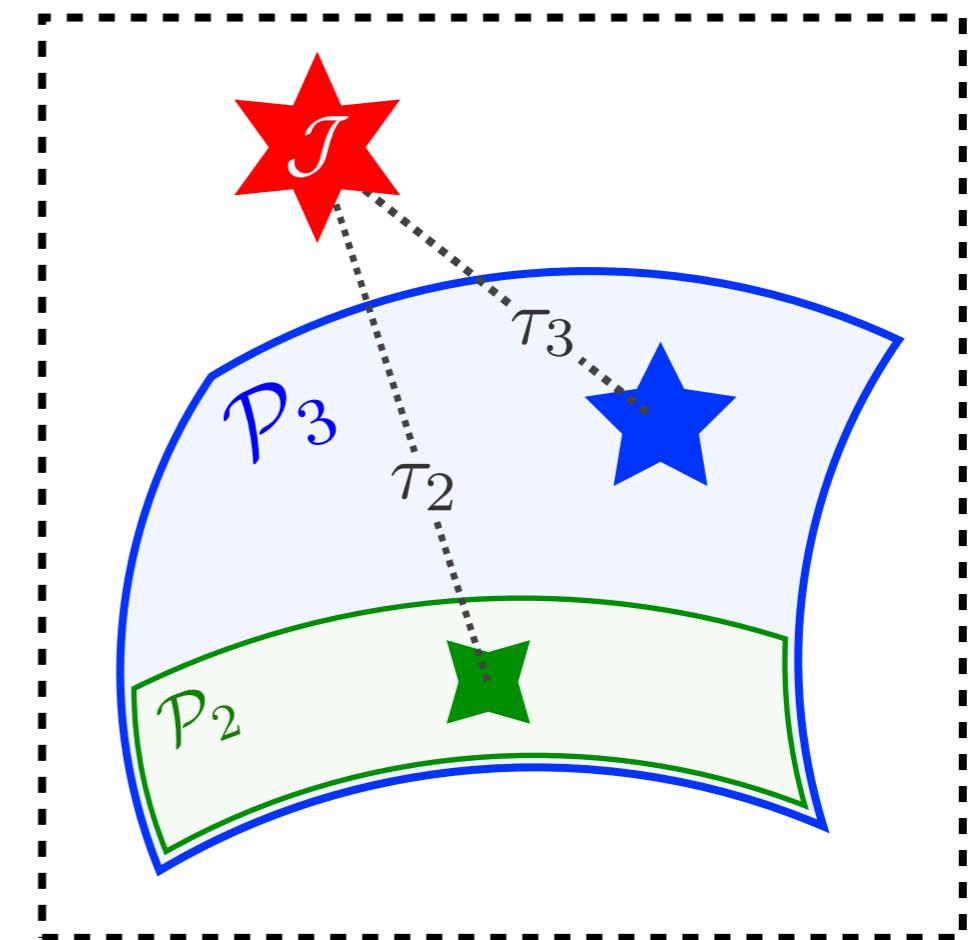
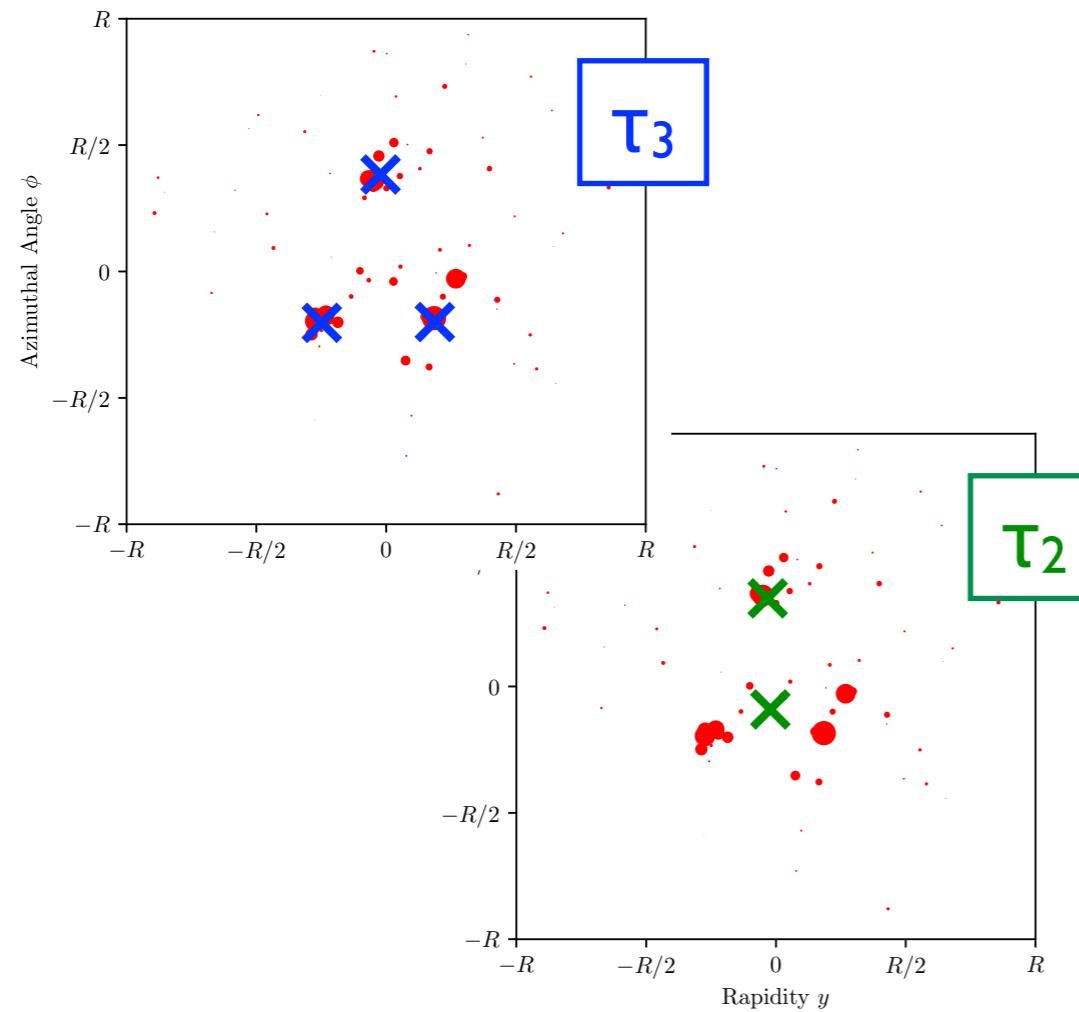
[JDT, Van Tilburg, [JHEP 2011](#), [JHEP 2012](#);  
based on Brandt, Dahmen, [ZPC 1979](#); Stewart, Tackmann, Waalewijn, [PRL 2010](#)]



# N-subjettiness = Point to Manifold EMD

*...is secretly an optimal transport problem*

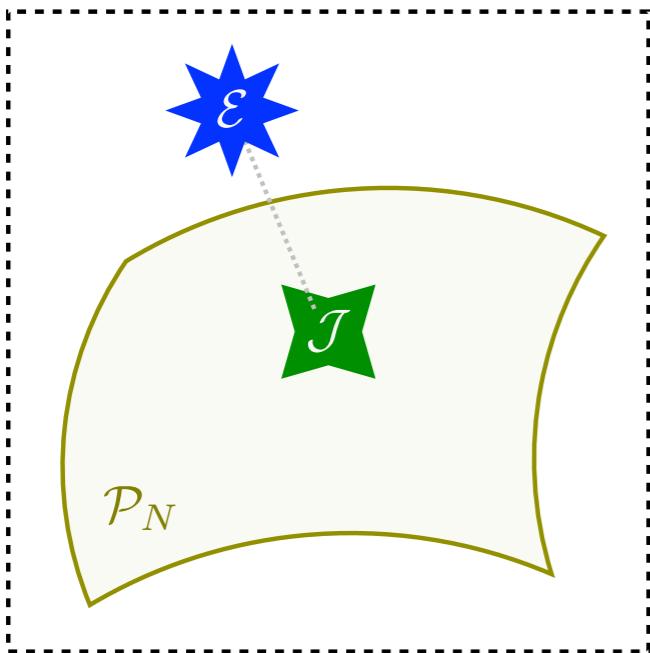
$$\tau_N(\mathcal{J}) = \min_{\mathcal{J}' \in \mathcal{P}_N} \text{EMD}(\mathcal{J}, \mathcal{J}')$$



[JDT, Van Tilburg, [JHEP 2011](#), [JHEP 2012](#);  
rephrased in the language of Komiske, Metodiev, JDT, [PRL 2019](#)]



# More Fun with N-particle Manifolds



## N-jettiness

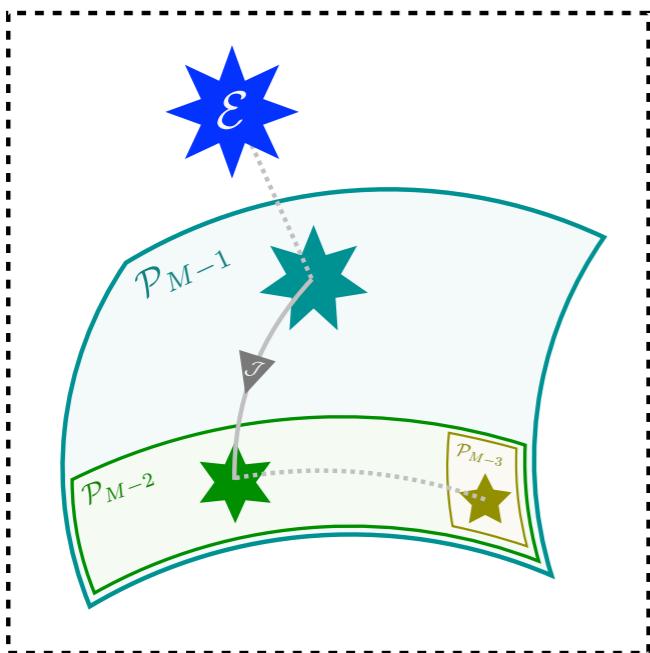
*Distance of closest approach to N-particle manifold*

[Brandt, Dahmen, [ZPC 1979](#); Stewart, Tackmann, Waalewijn, [PRL 2010](#)]

## Exclusive Cone Jet Finding

*Point of closest approach on N-particle manifold*

[Stewart, Tackmann, JDT, Vermilion, Wilkason, [JHEP 2015](#)]



## Sequential Jet Recombination

*Iteratively stepping between various N-particle manifolds*

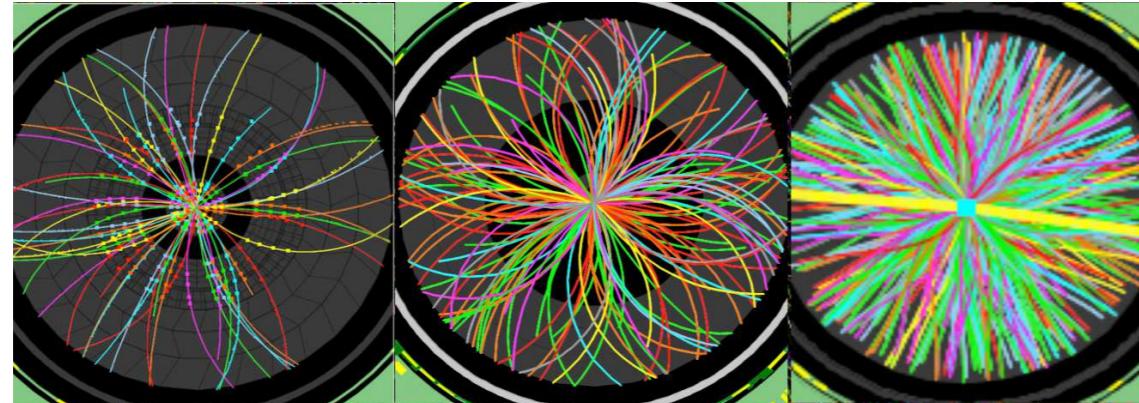
[Catani, Dokshitzer, Seymour, Webber, [NPB 1993](#); Ellis, Soper, [PRD 1993](#)]

[Dokshitzer, Leder, Moretti, Webber, [JHEP 1997](#); Wobisch, Wengler, [arXiv 1999](#)]

[Butterworth, Couchman, Cox, Waugh, [CPC 2003](#); Larkoski, Neill, JDT, [JHEP 2014](#)]

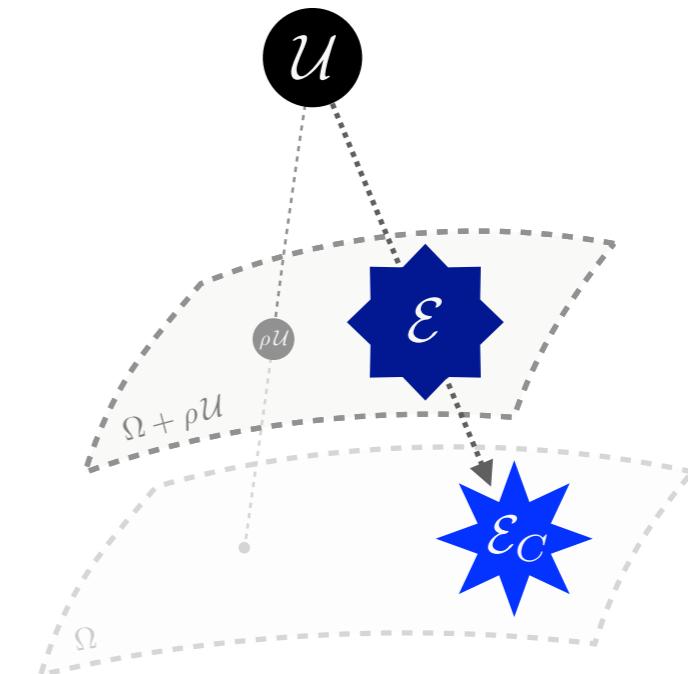
[Komiske, Metodiev, JDT, [JHEP 2020](#)]

# Pileup Mitigation



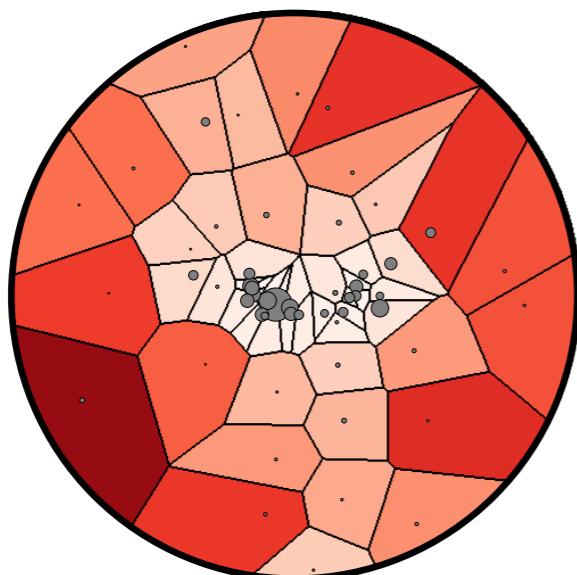
[see review in Soyez, PR 2019]

Uniform event contamination from overlapping proton-proton collisions



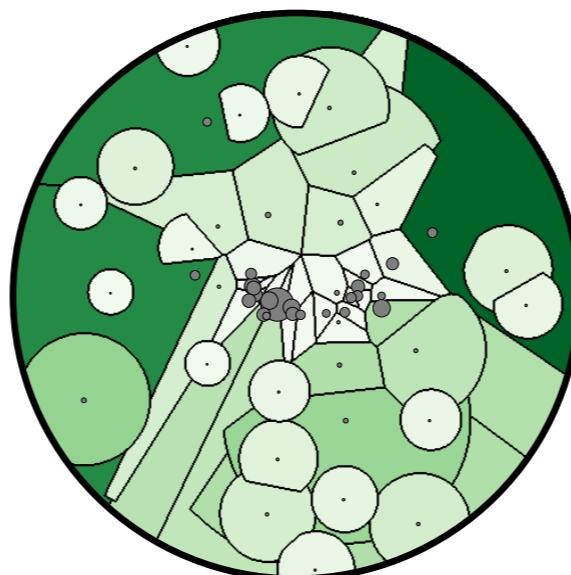
Pileup Mitigation:  
“Move away” from uniform event

Voronoi



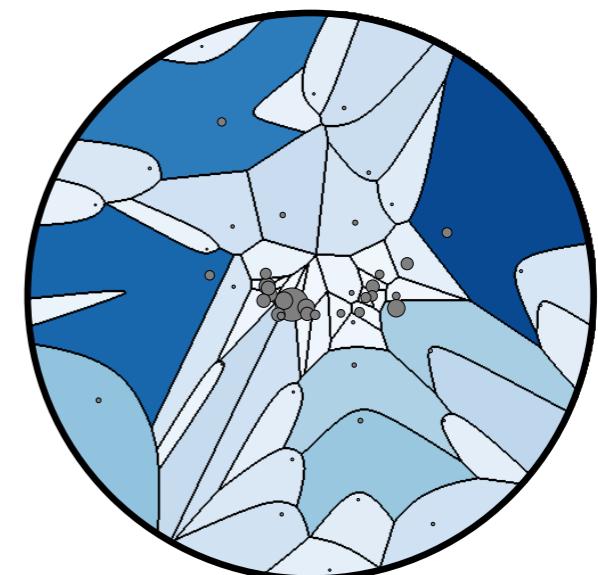
[Cacciari, Salam, Soyez, JHEP 2008]

Constituent Subtraction



[Berta, Spousta, Miller, Leitner, JHEP 2014]

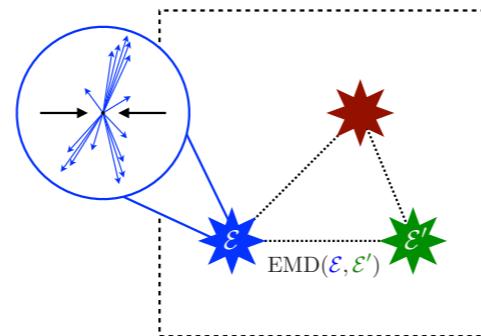
Apollonius



[Komiske, Metodiev, JDT, JHEP 2020]

# Pause

*What can be Geometrized?*



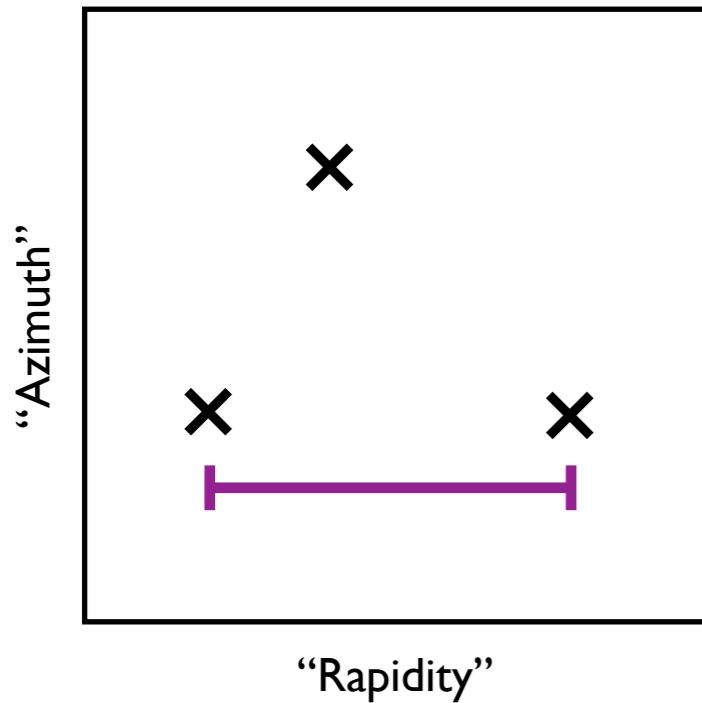
**IRC Safety, Observables,  
Jet Algorithms, Pileup Mitigation**

*And now, the grand finale...*



*How far down does this rabbit hole go?*

# Direction Space



**x** = Direction

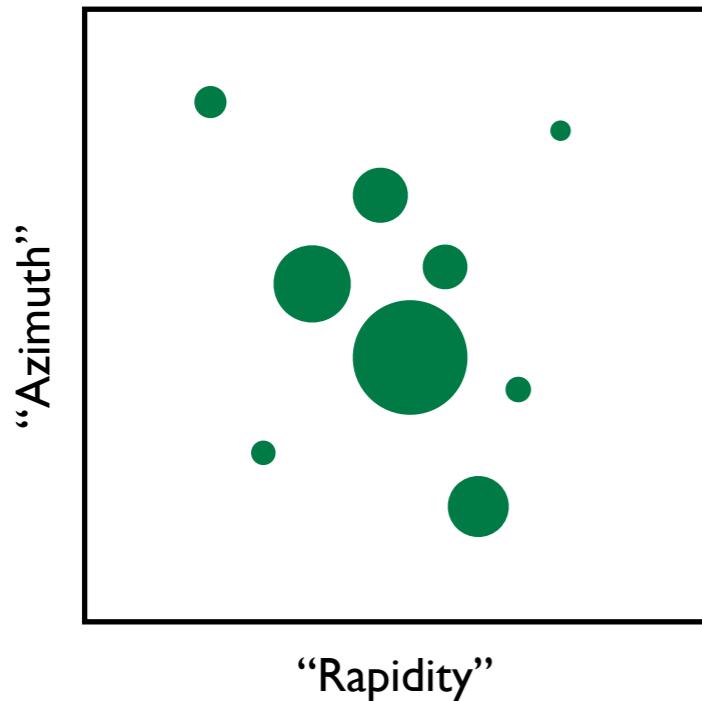
= Angular Distance

$$n_i^\mu = \frac{p_i^\mu}{E_i} = (1, \hat{n})^\mu$$

$$\theta_{ij} = \sqrt{2n_i^\mu n_{j\mu}}$$

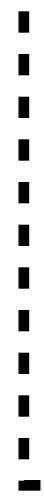
(for massless particles)

# Direction Space Distribution



● = Weighted Direction

— = Angular Distance



★ = Event

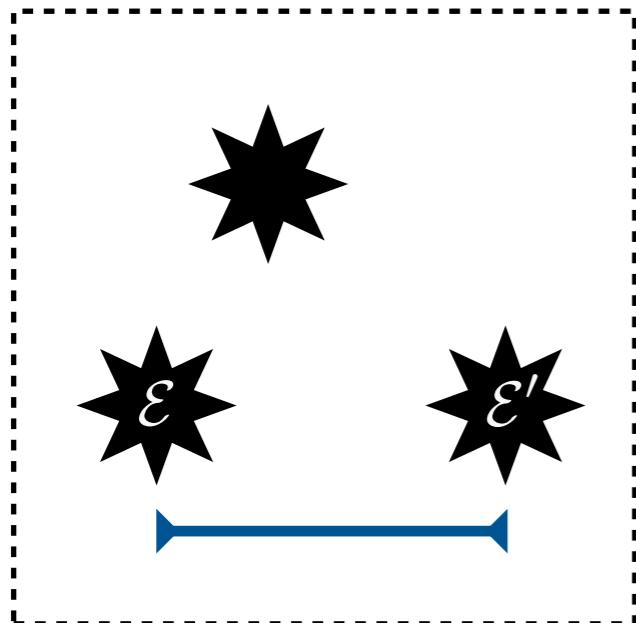
$$n_i^\mu = \frac{p_i^\mu}{E_i} = (1, \hat{n})^\mu$$

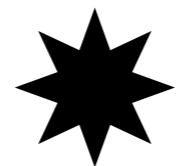
$$w_i = E_i$$

$$\theta_{ij} = \sqrt{2n_i^\mu n_{j\mu}}$$

(for massless particles)

# Event Space



 = Event  
 = EMD  
Energy  
Mover's Distance

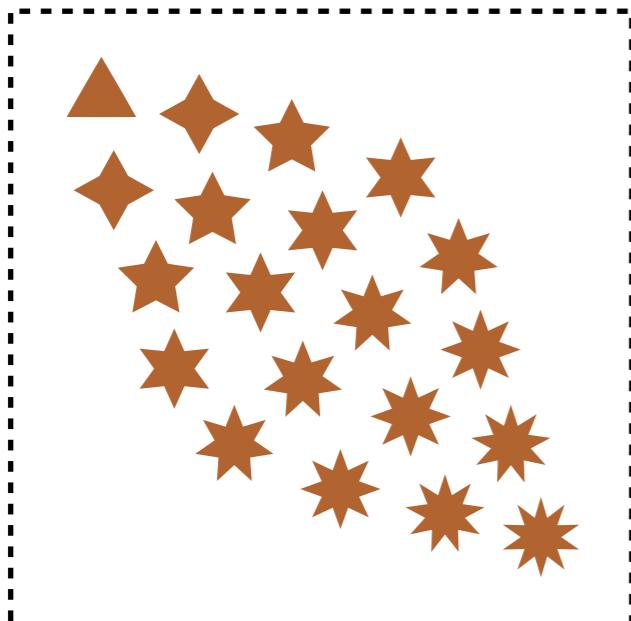
$$\mathcal{E}(\hat{n}) = \sum_i E_i \delta(\hat{n} - \hat{n}_i)$$

$$\text{EMD}(\mathcal{E}, \mathcal{E}') = \min_{\{f\}} \sum_i \sum_j f_{ij} \theta_{ij}$$

(for equal total energy)

[Komiske, Metodiev, JDT, PRL 2019]

# Event Space Distribution



= Weighted Event

$$\mathcal{E}(\hat{n}) = \sum_i E_i \delta(\hat{n} - \hat{n}_i)$$

$$w_a = \sigma_a$$

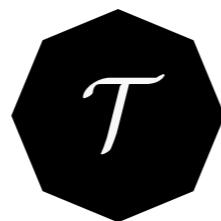
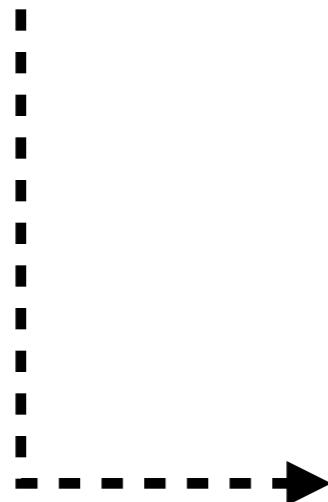


= EMD

Energy  
Mover's Distance

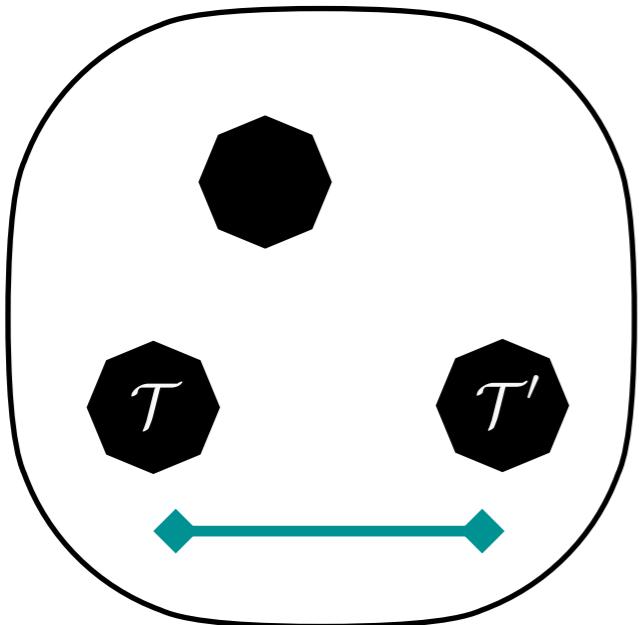
$$\text{EMD}(\mathcal{E}, \mathcal{E}') = \min_{\{f\}} \sum_i \sum_j f_{ij} \theta_{ij}$$

(for equal total energy)



= Theory

# Theory Space



● = Theory

↔ =  $\Sigma\text{MD}$   
Cross-Section  
Mover's Distance

$$\mathcal{T}(\mathcal{E}) = \sum_a \sigma_a \delta(\mathcal{E} - \mathcal{E}_a)$$

$$\Sigma\text{MD}(\mathcal{T}, \mathcal{T}') = \min_{\{\mathcal{F}\}} \sum_a \sum_b \mathcal{F}_{ab} \text{EMD}(\mathcal{E}_a, \mathcal{E}'_b)$$

(for equal total xsec)

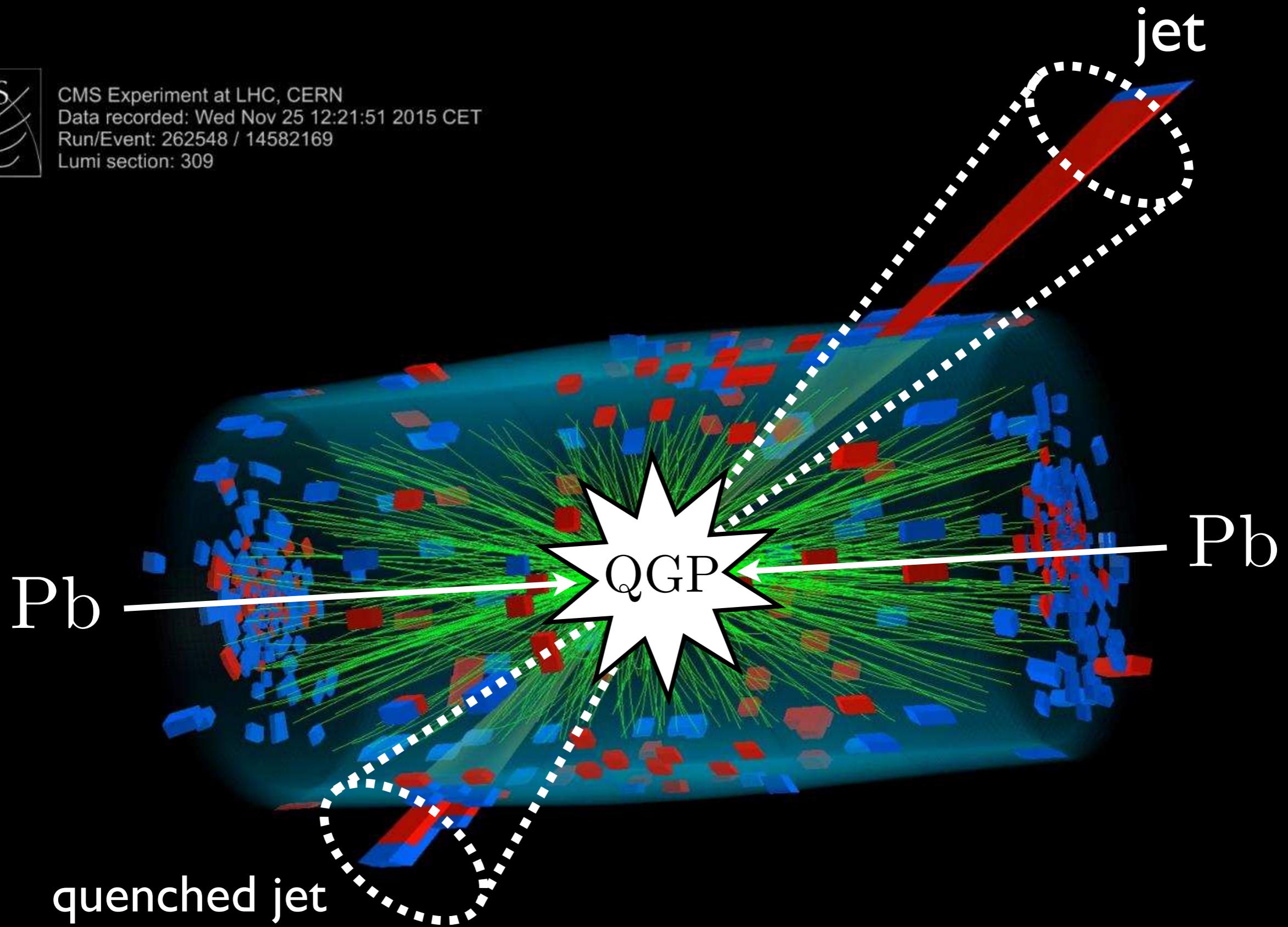
*A distance between theories!*

(e.g. EMD : N-jettiness ::  $\Sigma\text{MD}$  : k-eventiness)

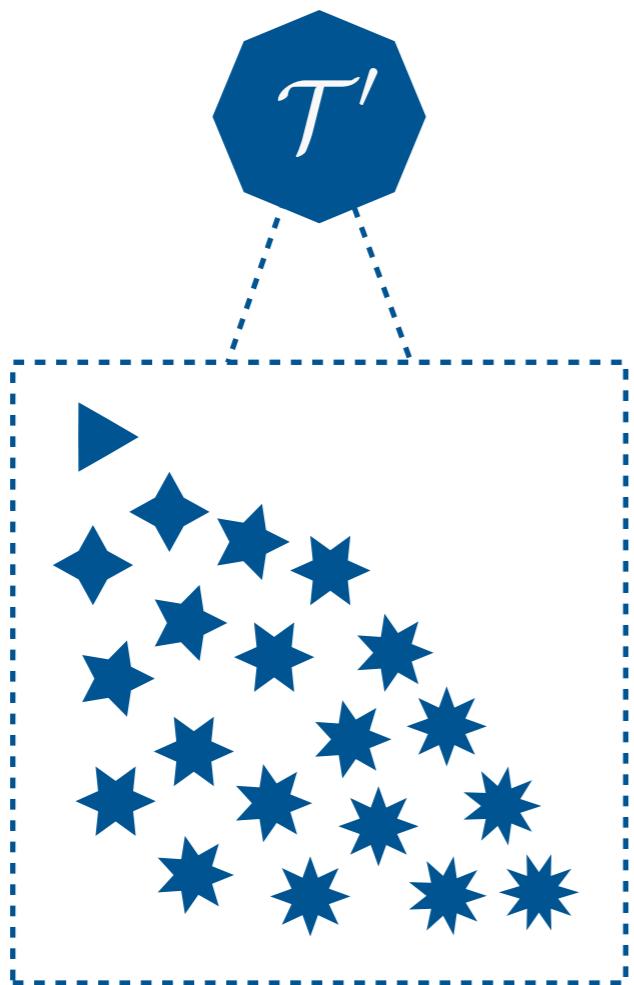
[Komiske, Metodiev, JDT, [JHEP 2020](#)]



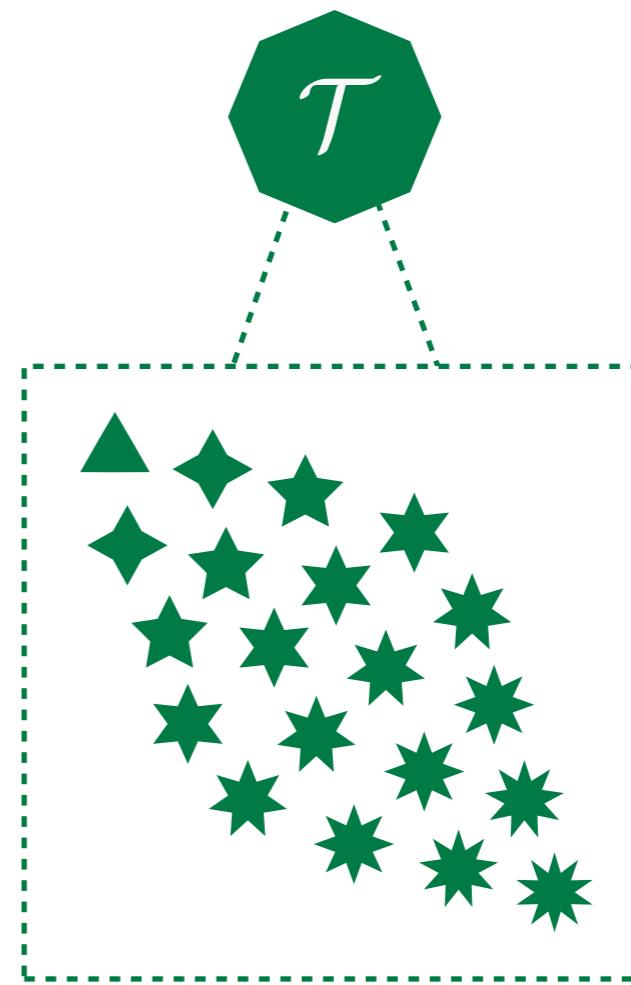
CMS Experiment at LHC, CERN  
Data recorded: Wed Nov 25 12:21:51 2015 CET  
Run/Event: 262548 / 14582169  
Lumi section: 309



## Theory Prime: In-Medium QCD



## Theory: Vacuum QCD

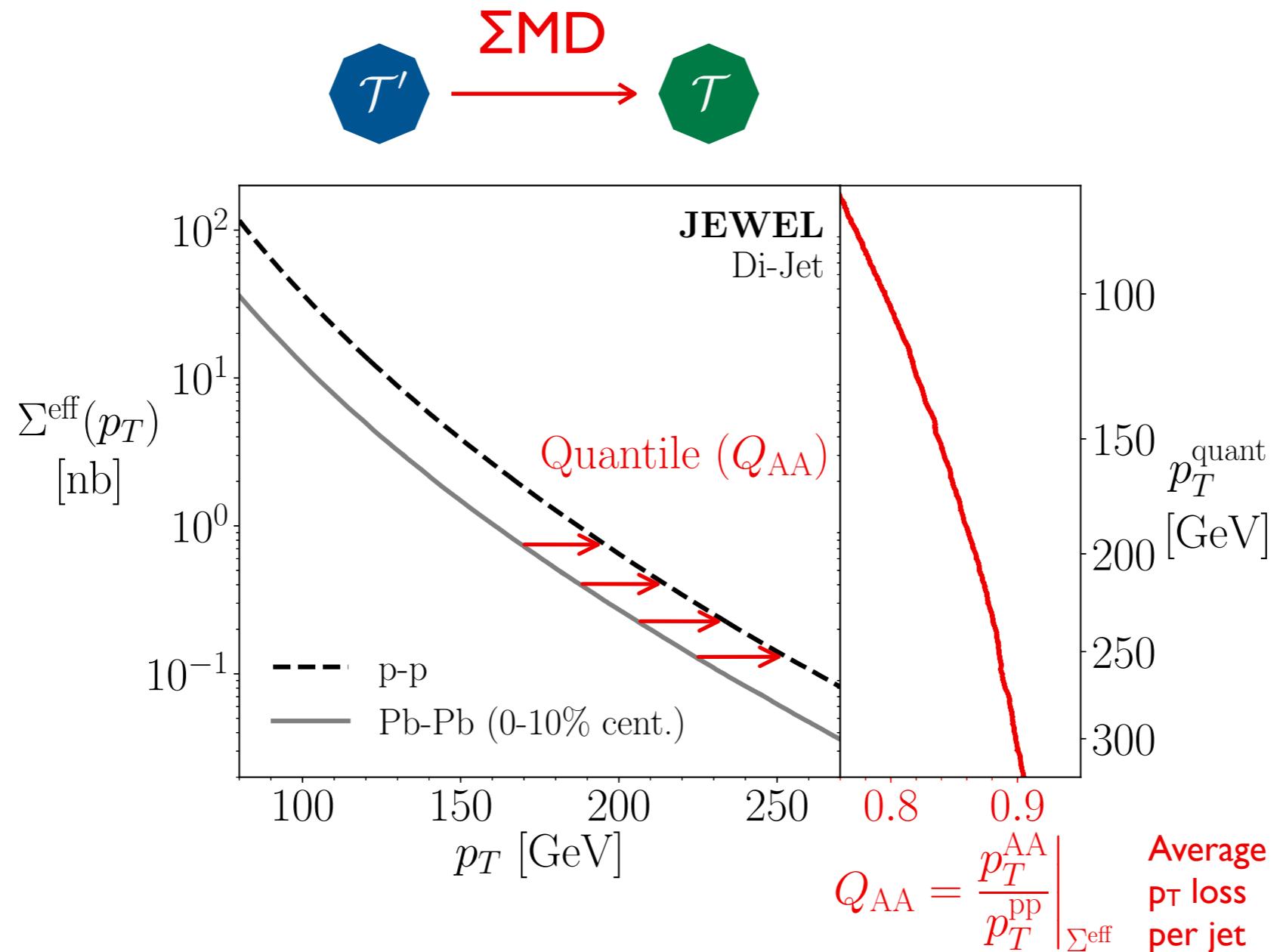


$\Sigma\text{MD}$   
 $\Updownarrow$

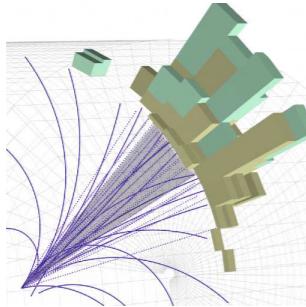
*Optimal transportation plan defines mapping  
between in-medium jets and vacuum jets!*

# Jet Quenching via Quantile Matching

*Equivalent to following a geodesic in theory space (!)*



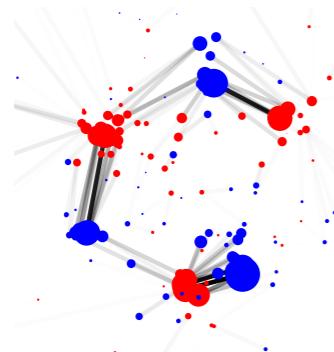
# Summary



## What is a Collider Event?

(ask me about QCD/ML relevance)

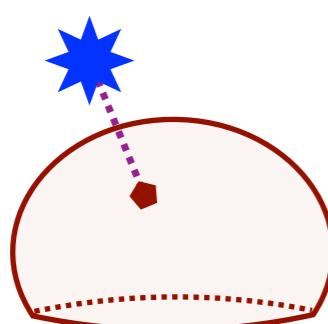
*An unordered set of particles that describes the energy flow away from the collision point*



## When are Events Similar?

(ask me more about EMD)

*When they are close in the geometric space triangulated by the energy mover's distance*



## What can be Geometrized?

(ask me more about safety)

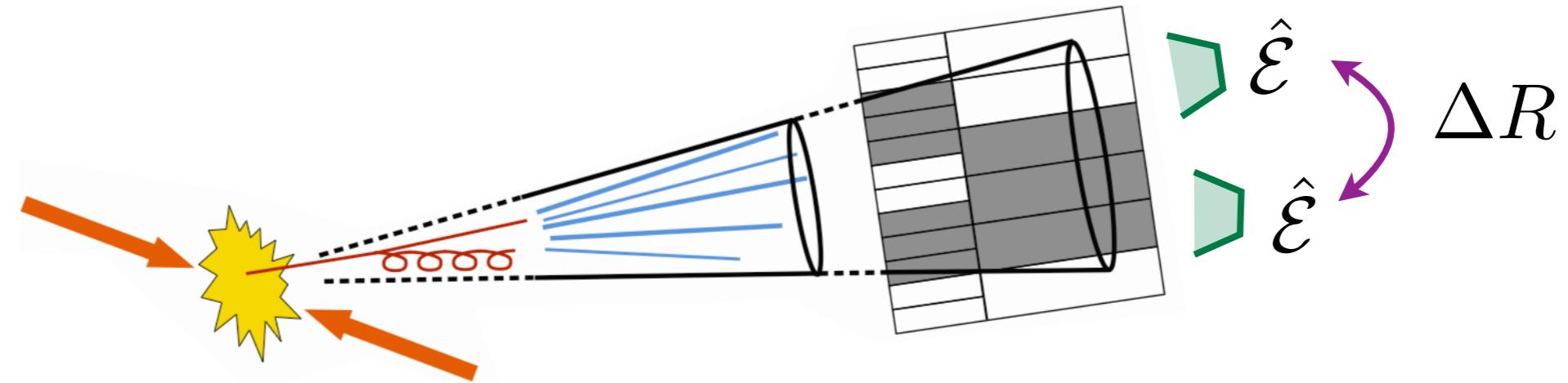
*Many concepts/techniques in quantum field theory and collider physics from the last half century*

# Fin

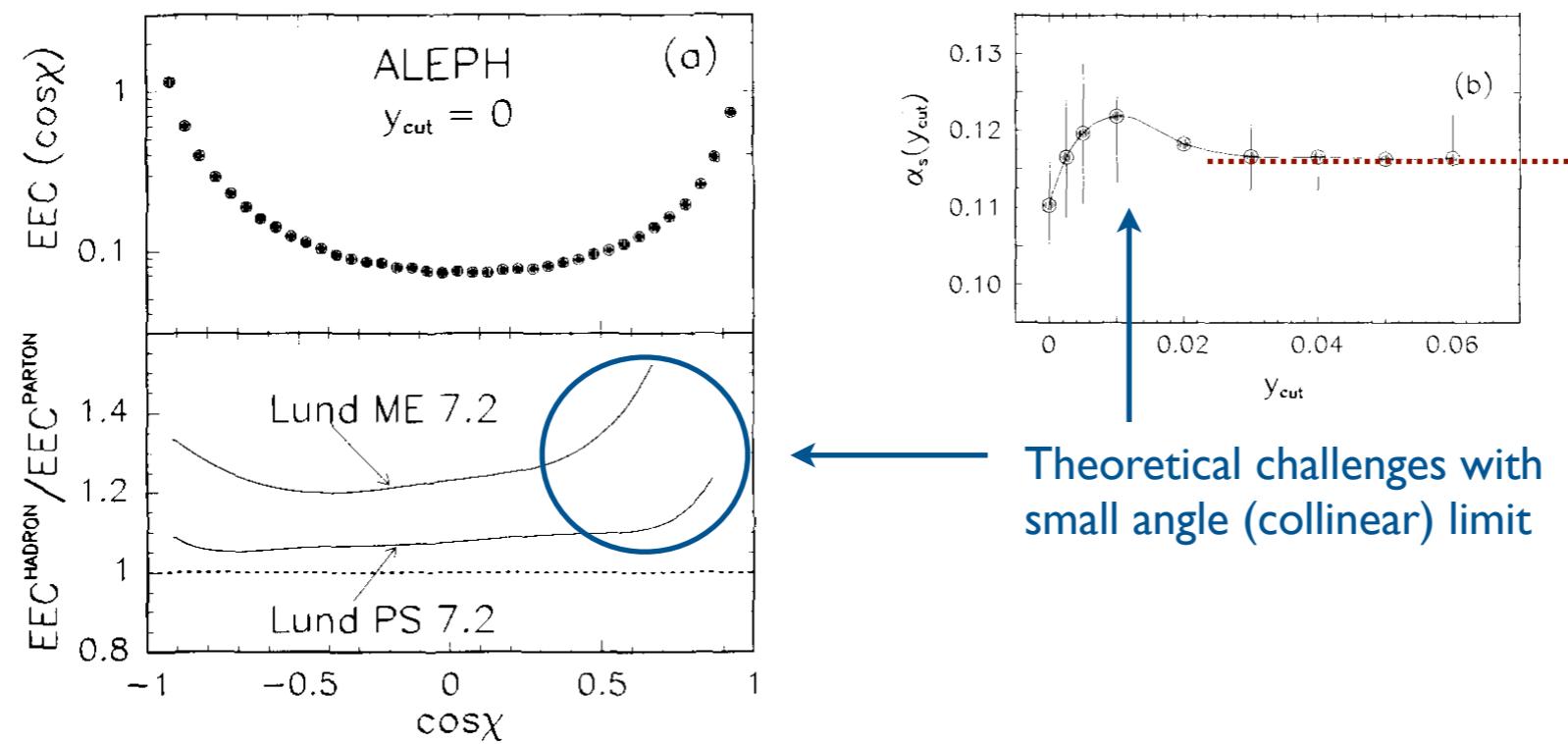
Whew! ↘  
48 of 48

# *Backup Slides*

# Energy-Energy Correlators

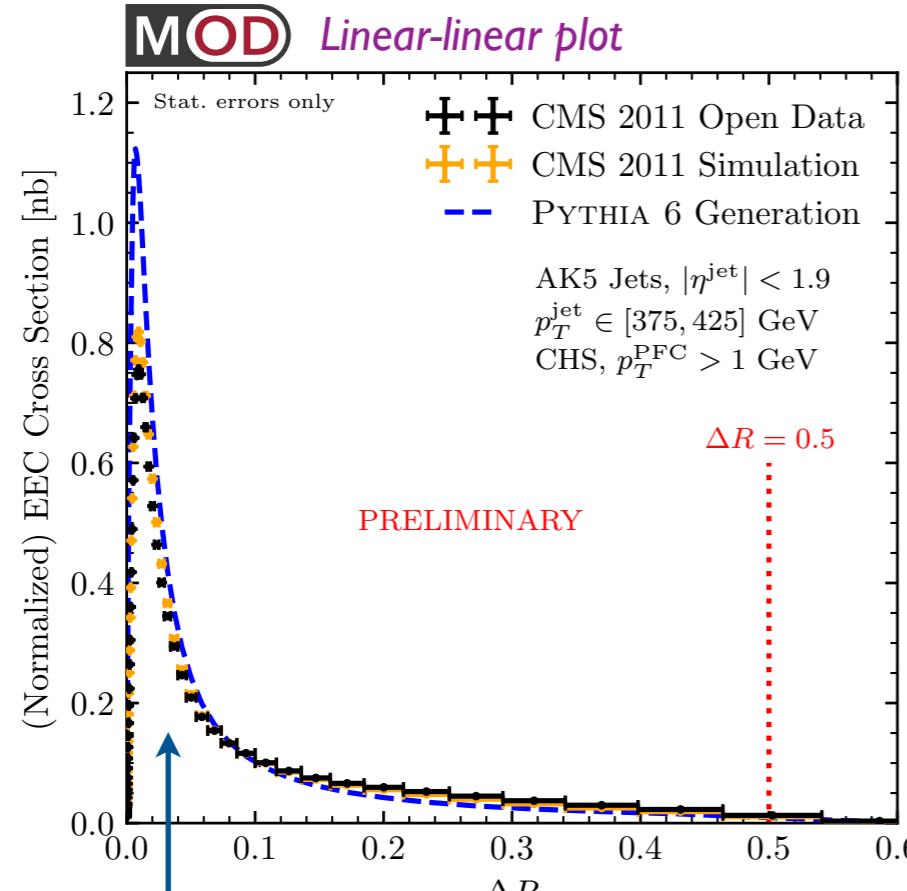


A long history in probing collinear dynamics of QCD



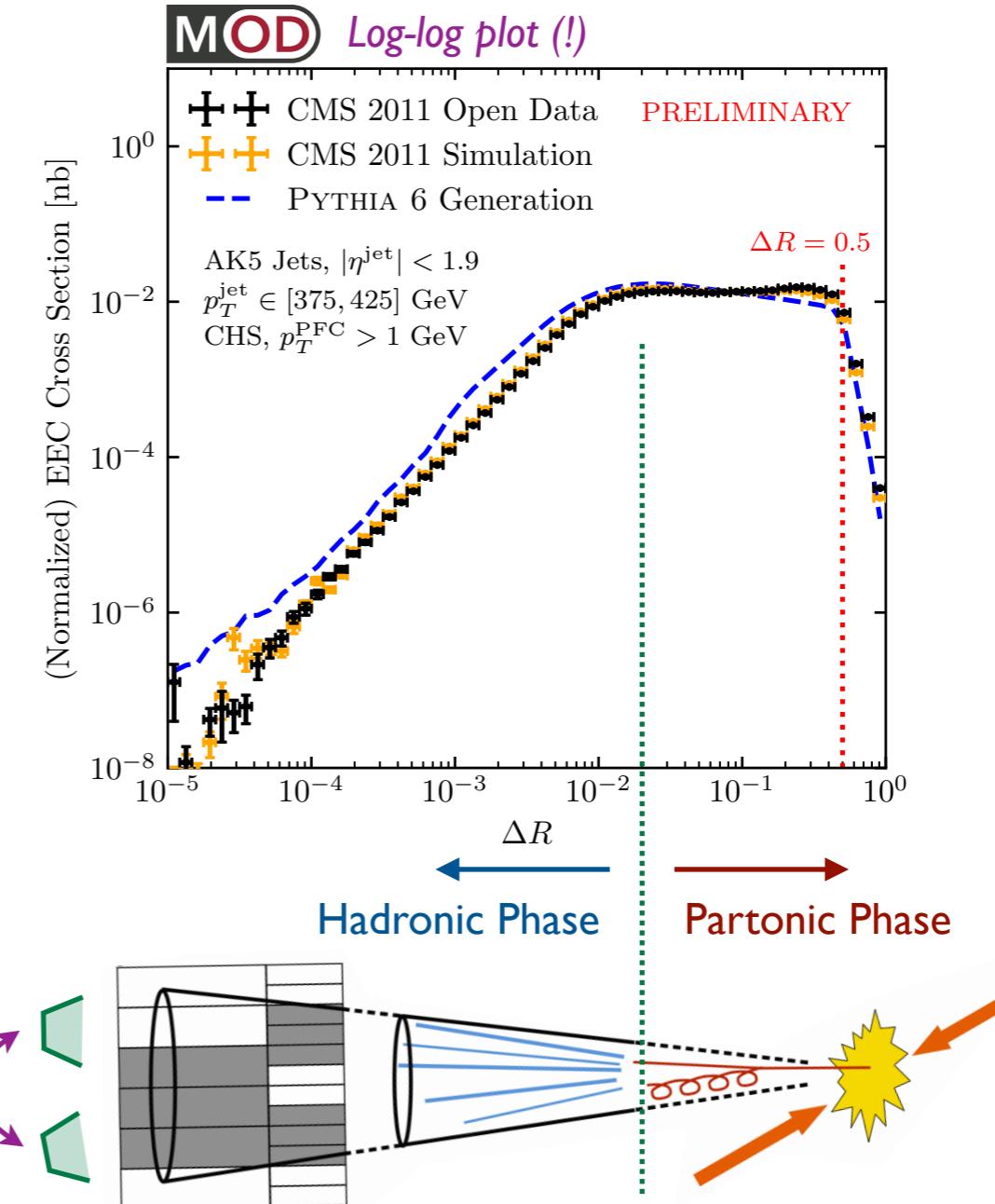
[Basham, Brown, Ellis, Love, [PRL 1978](#); ALEPH, [PLB 1991](#); see Chen, Moult, Zhang, Zhu, [PRD 2020](#)]

# QCD Phase Transition in Jets?



Are we learning something about small angle limit of QCD?

First Jet EEC Plot from the LHC (!)



[Komiske, Moult, JDT, Zhu, in progress; see talks by Moult, [BOOST 2019](#), [BOOST 2020](#)]



# Point Cloud

Collection of points in position space



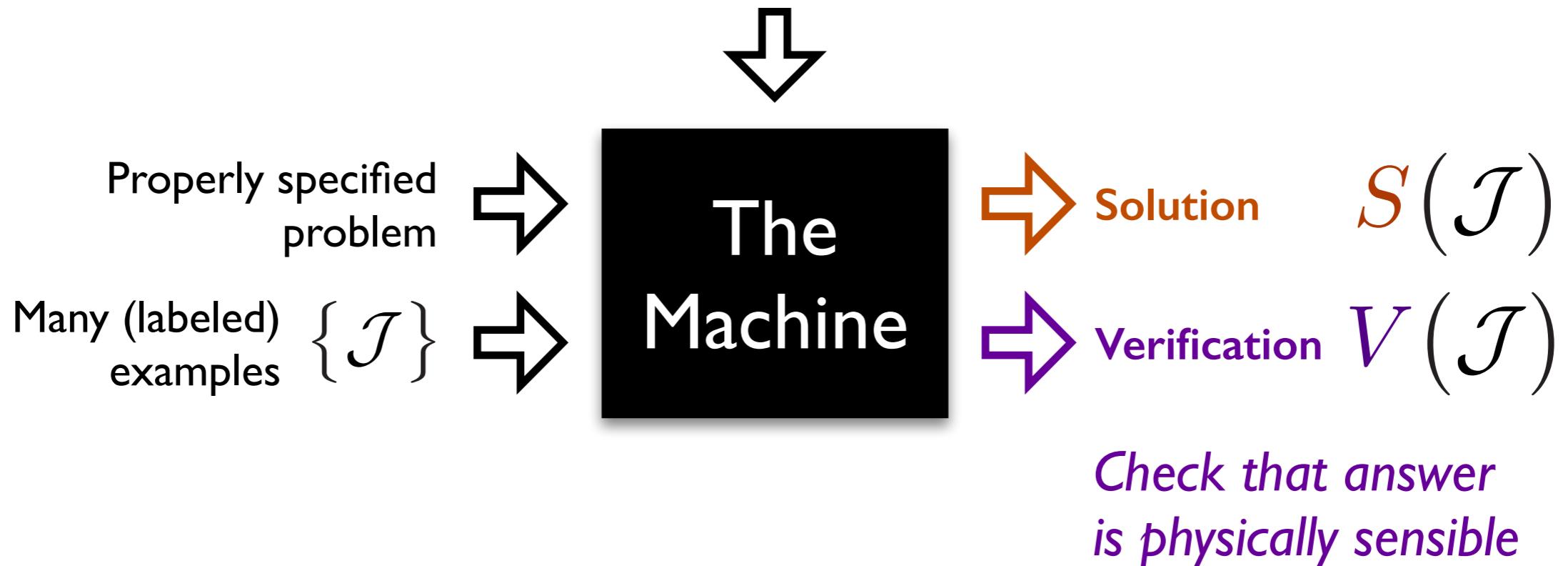
[Popular Science, 2013]

# Aside: Machine Learning for Jets

“ML4Jets”  
NYU, January 2020

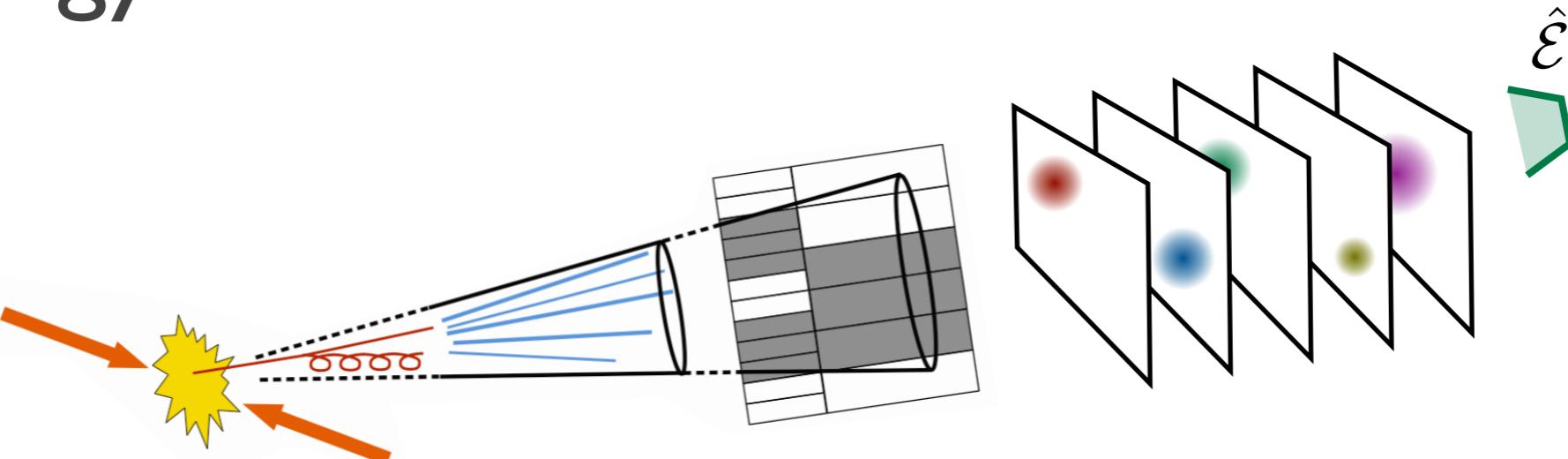
**Symmetry:**  $\mathcal{J} = \{ \vec{p}_1, \vec{p}_2, \vec{p}_3, \dots, \vec{p}_N \}$   
Unordered, Variable Length Set (QM!)

**Safety:**  $\vec{p} = \{ E, \hat{n}_x, \hat{n}_y, \hat{n}_z \}$   
Energy weighting (QFT!)

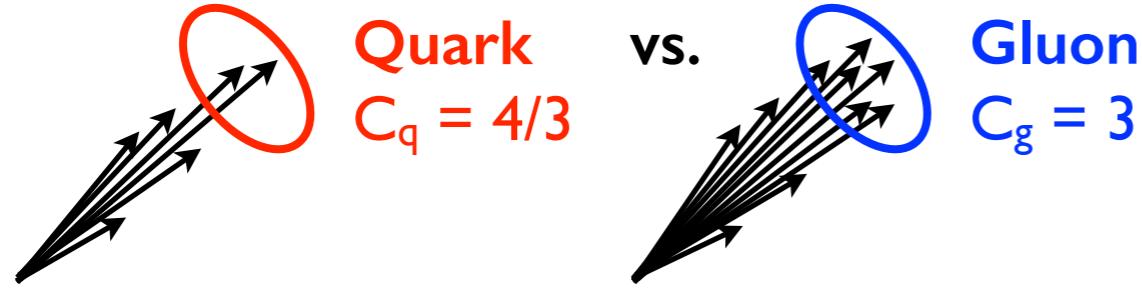


*Check that answer  
is physically sensible*

# E.g. Energy Flow Networks

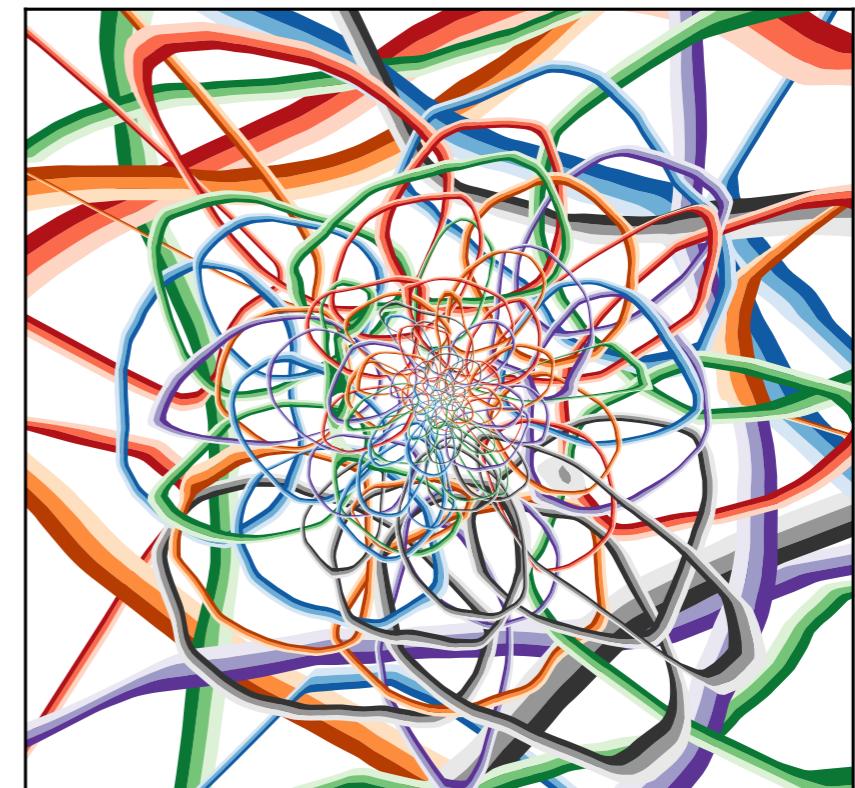
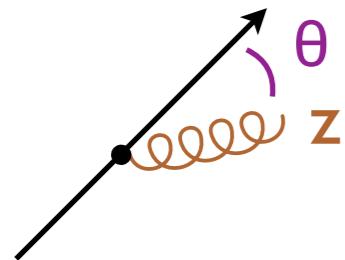


*Learning QCD singularities!*



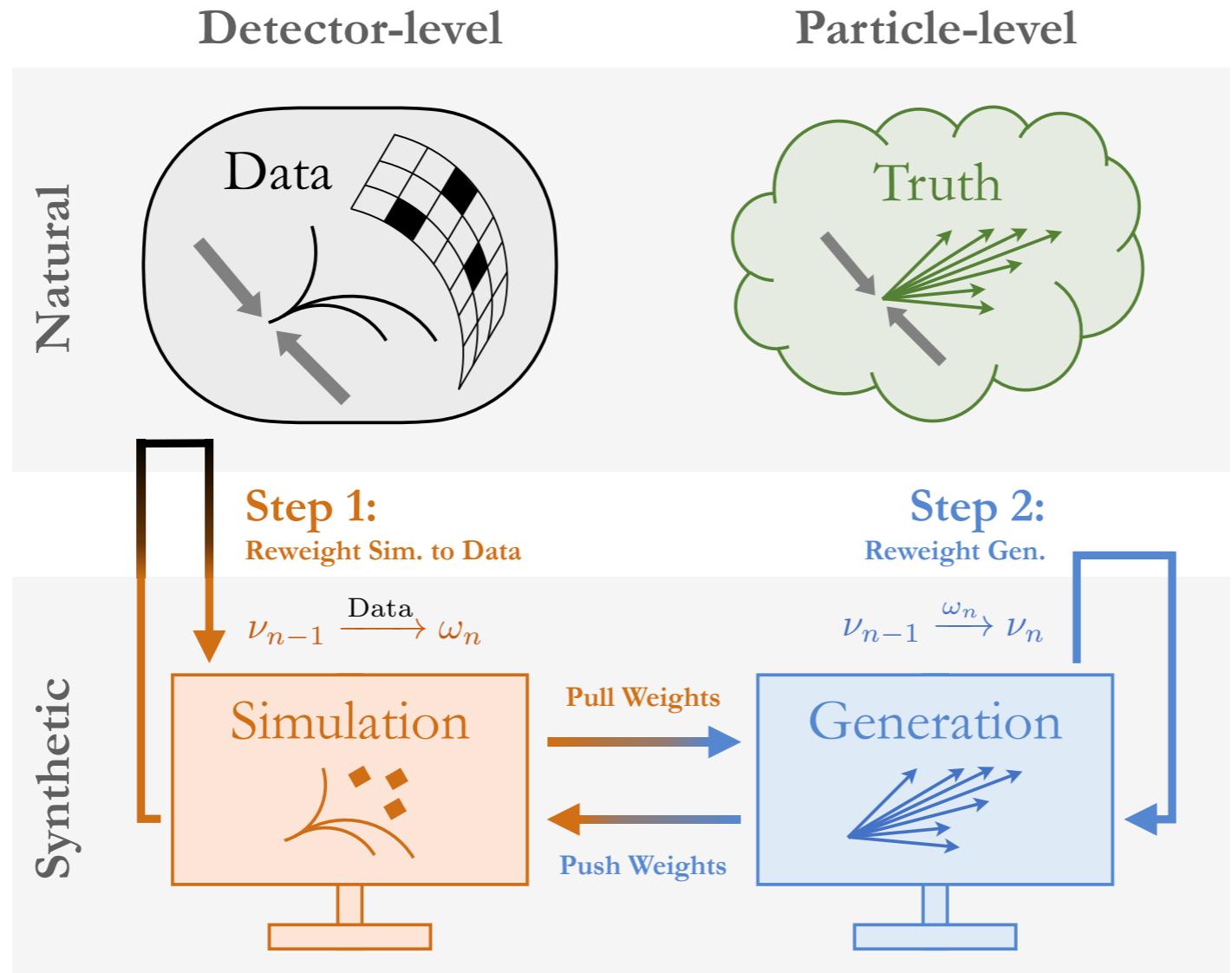
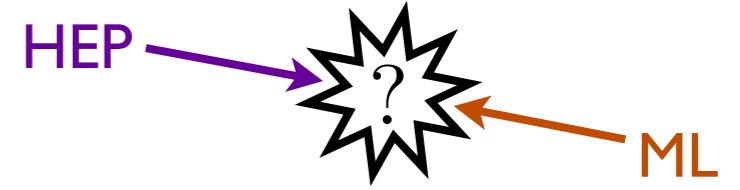
AP splitting probability:

$$dP_{i \rightarrow ig} \simeq \frac{2\alpha_s}{\pi} C_i \frac{d\theta}{\theta} \frac{dz}{z}$$



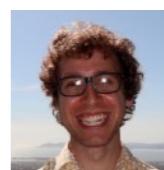
[Komiske, Metodiev, JDT, [JHEP 2019](#); see also Komiske, Metodiev, JDT, [JHEP 2018](#), [PRD 2020](#); special case of Zaheer, Kottur, Ravanbakhsh, Poczos, Salakhutdinov, Smola, [NIPS 2017](#)]

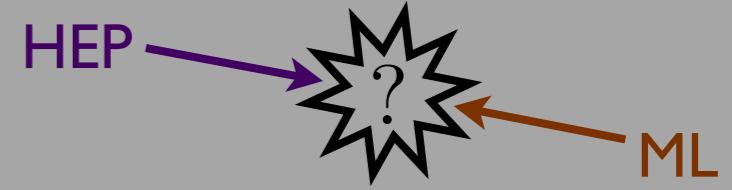
# OmniFold



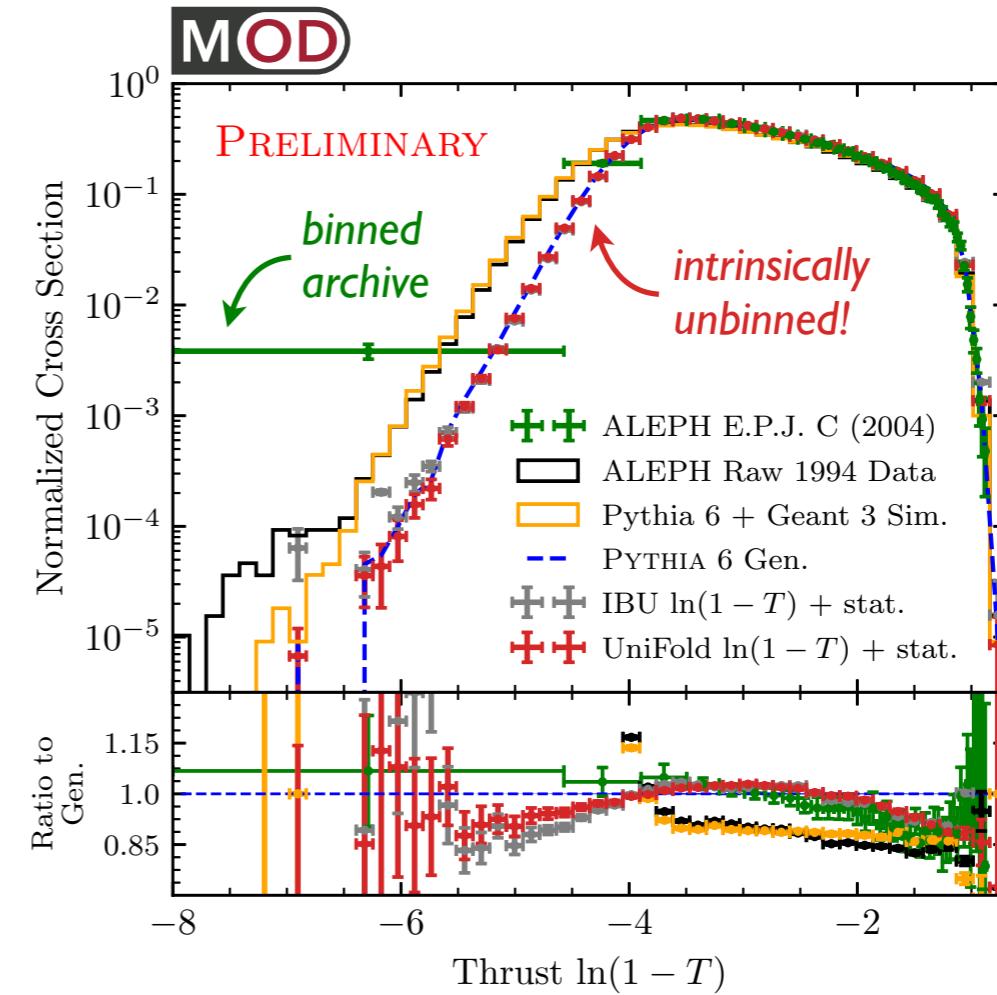
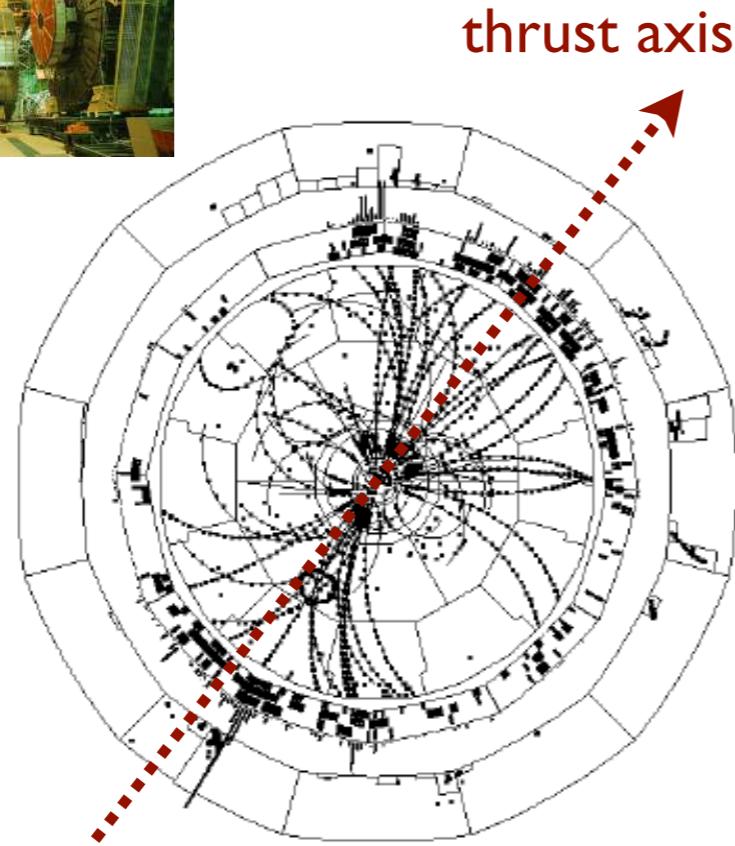
*Multi-dimensional unbinned detector corrections via iterated binary classification*

[Andreassen, Komiske, Metodiev, Nachman, JDT, [PRL 2020](#)]

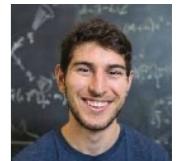




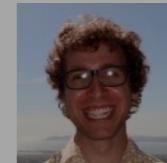
## Back to the Future with ALEPH Archival Data



[talk by Badea, [ICHEP 2020](#); cf. ALEPH, [EPJC 2004](#)  
 [see also Badea, Baty, Chang, Innocenti, Maggi, McGinn, Peters, Sheng, JDT, Lee, [PRL 2019](#)]

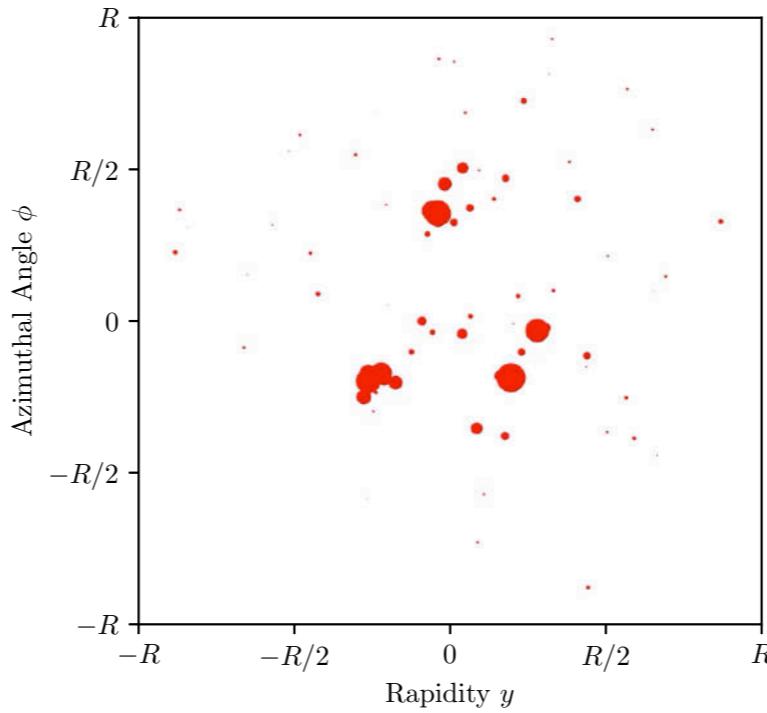


[Andreassen, Komiske, Metodiev, Nachman, JDT, [PRL 2020](#)]

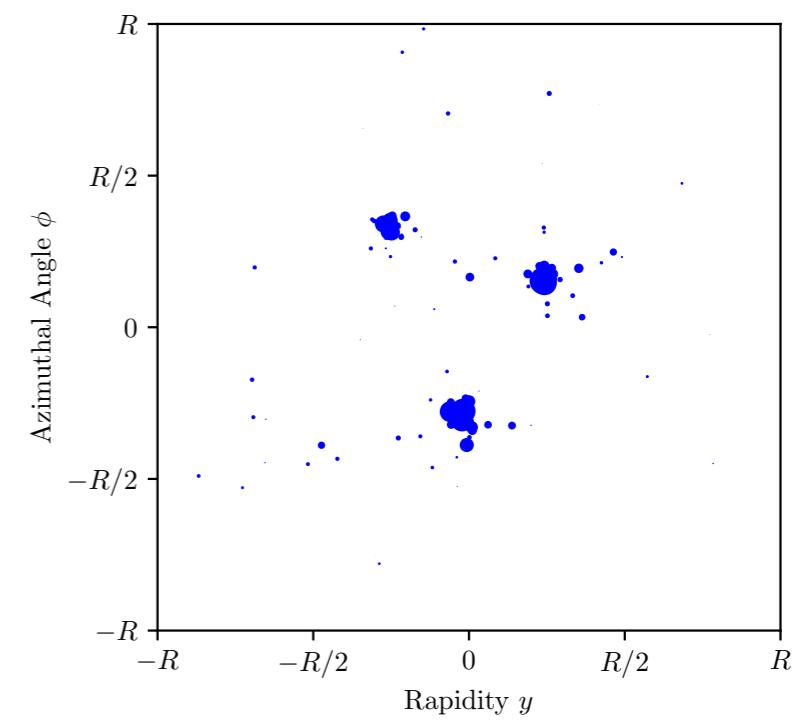
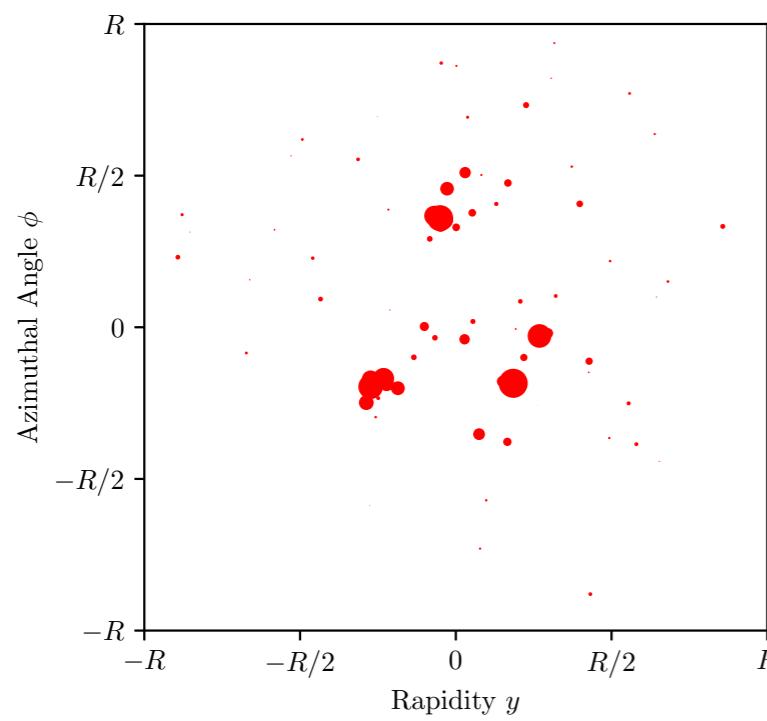


# Similarity of Two Energy Flows?

$$\mathcal{E}(\hat{n}) = \sum_i E_i \delta(\hat{n} - \hat{n}_i)$$



Optimal Transport:  
*Earth Mover's Distance*  
a.k.a.  $l$ -Wasserstein metric



[Komiske, Metodiev, JDT, PRL 2019; code at Komiske, Metodiev, JDT, [energyflow.network](#)]

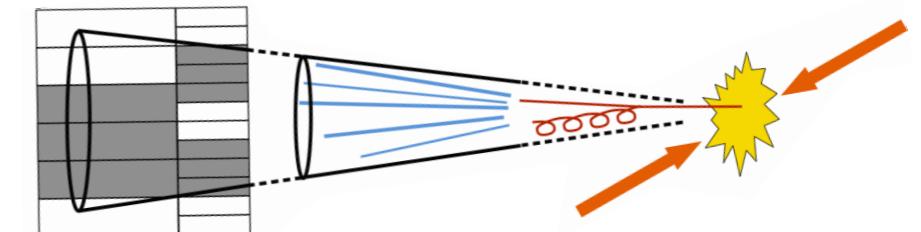
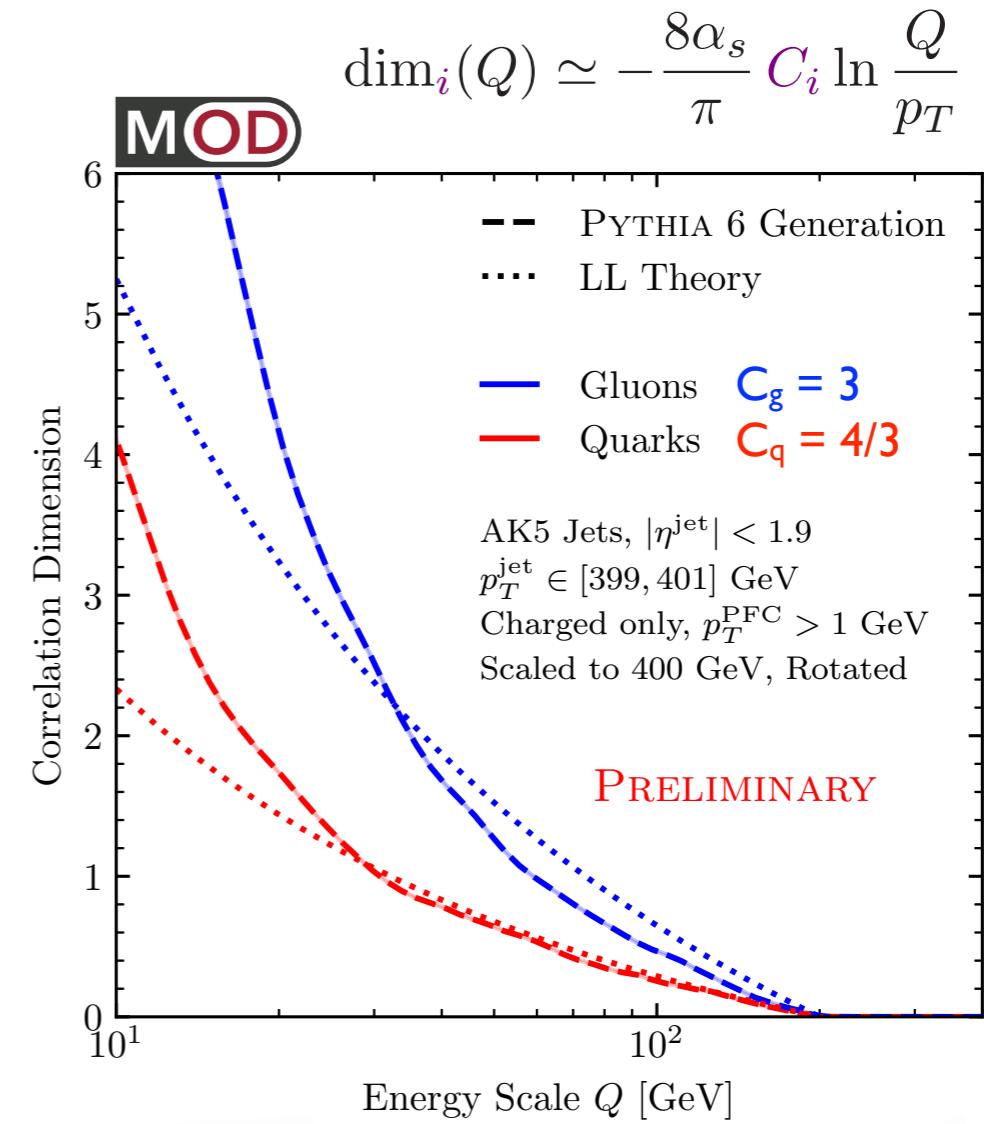
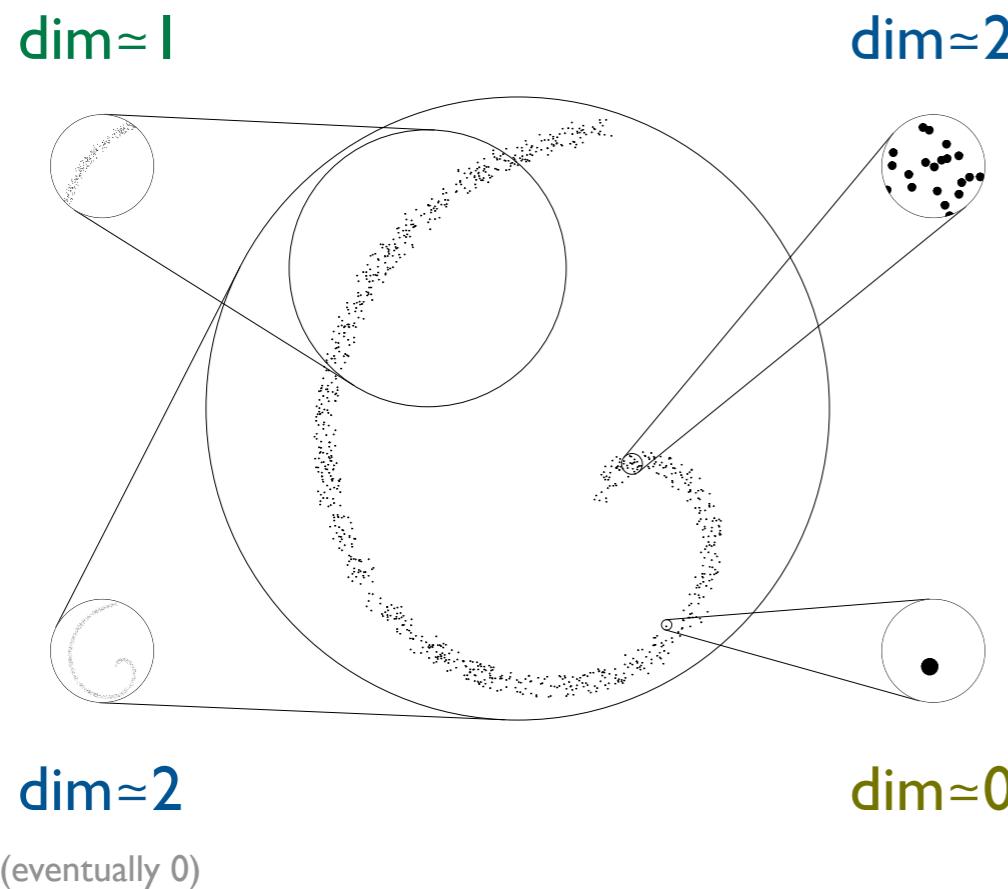
# Dimensionality of Space of Jets



$$N_{\text{neighbors}}(r) \sim r^{\dim}$$

$$\Rightarrow \dim(r) \sim r \frac{\partial}{\partial r} \ln N_{\text{neighbors}}(r)$$

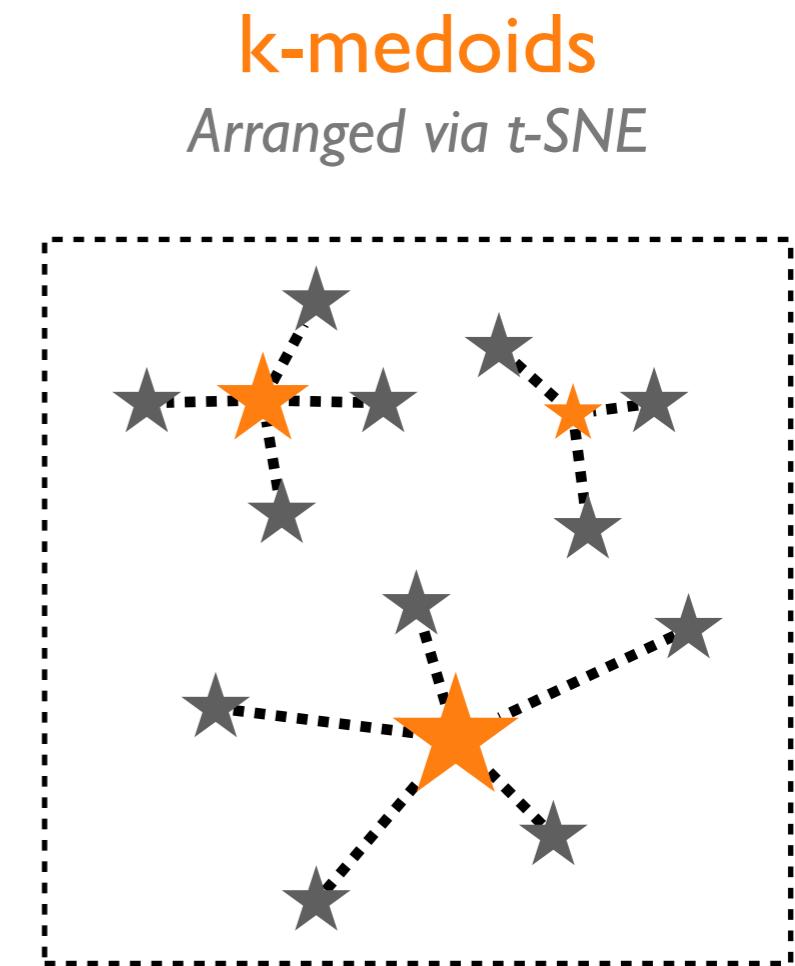
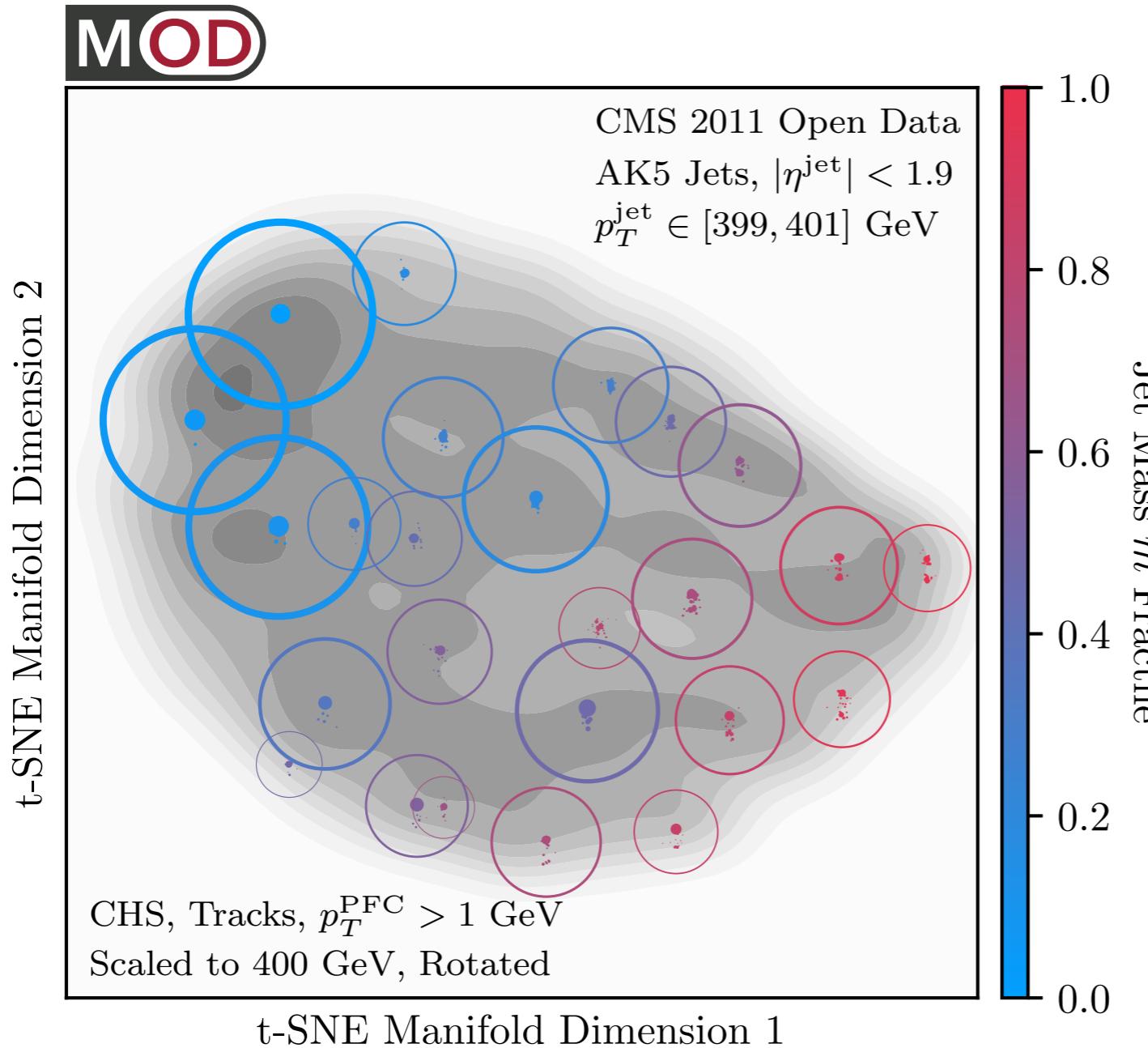
[Grassberger, Procaccia, [PRL 1983](#); Kégl, [NIPS 2002](#)]



# Most Representative Jets



[<http://opendata.cern.ch/>]

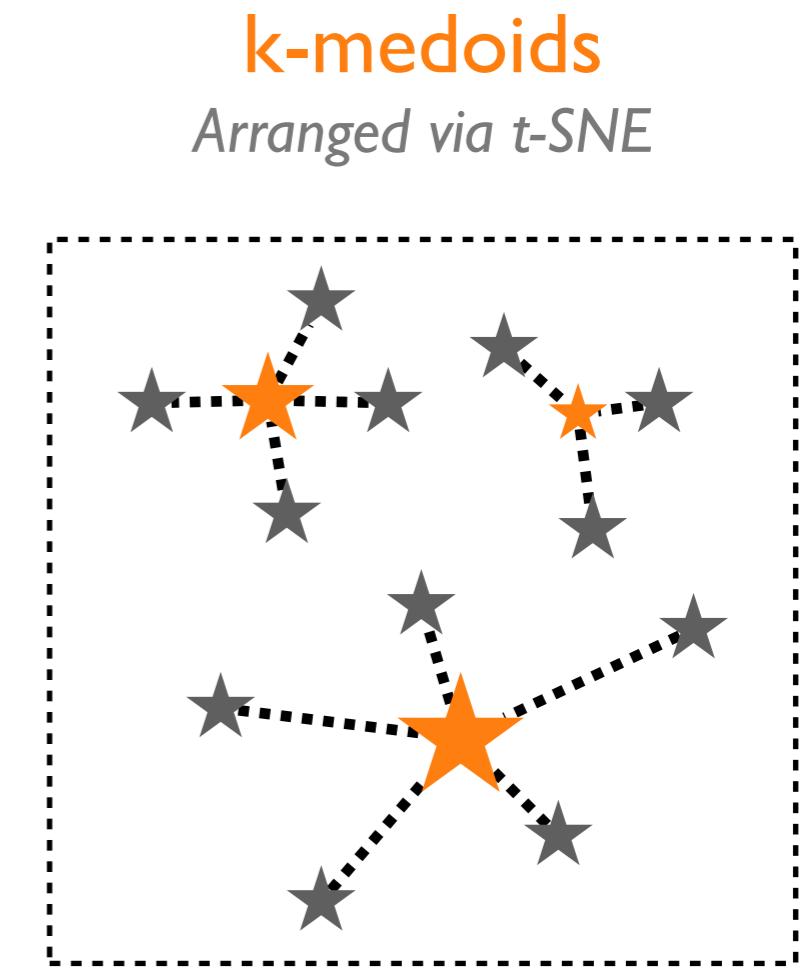
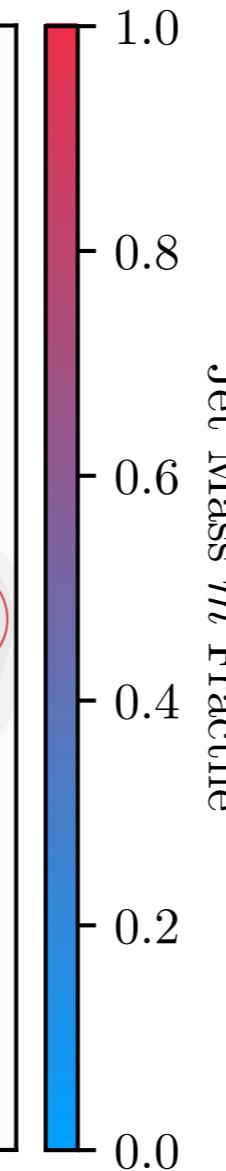
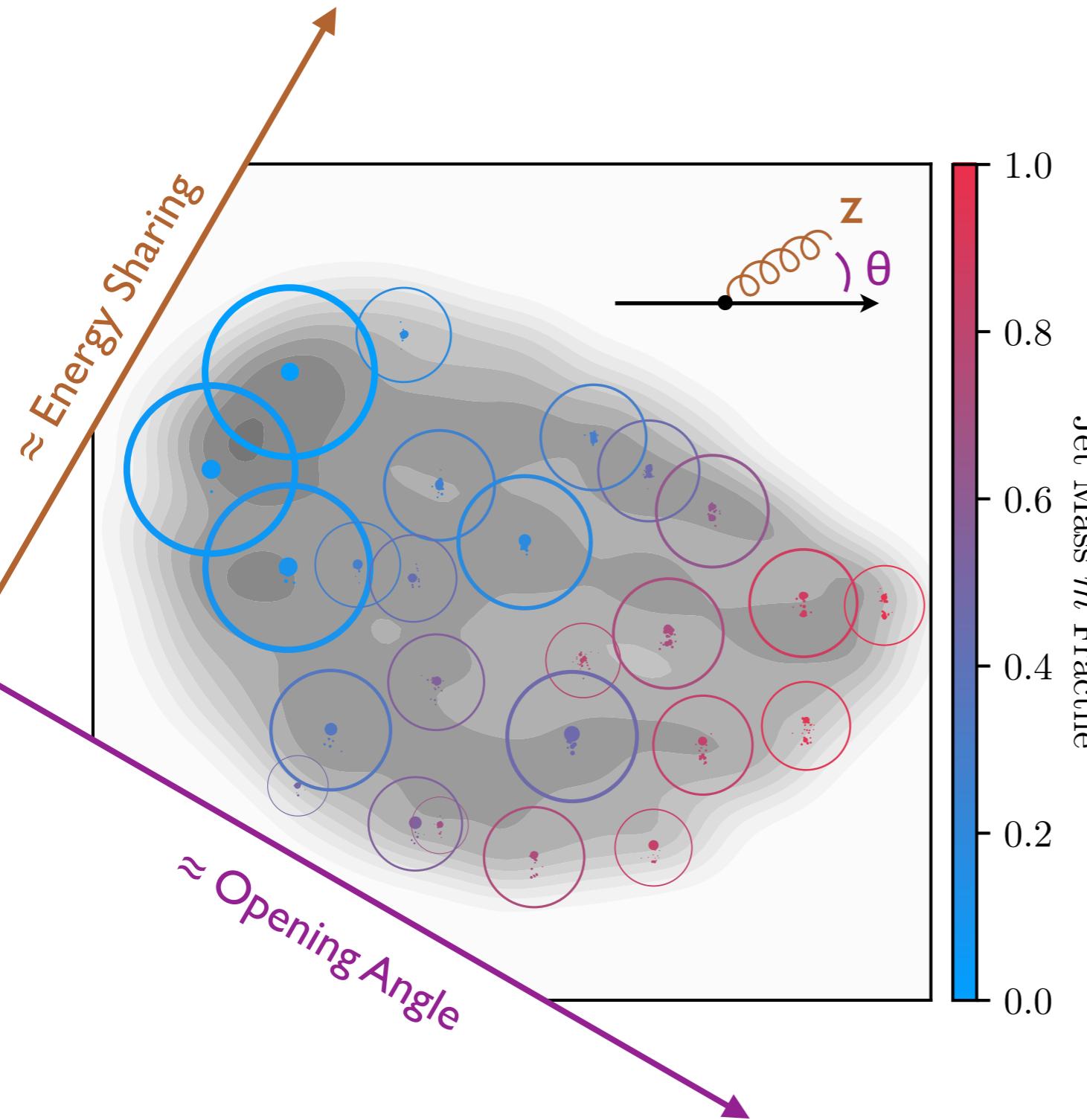


[Komiske, Mastandrea, Metodiev, Naik, JDT, *PRD* 2020; using van der Maaten, Hinton, *JMLR* 2008]

# Most Representative Jets

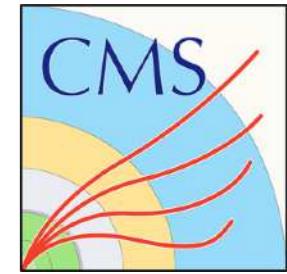


[<http://opendata.cern.ch/>]

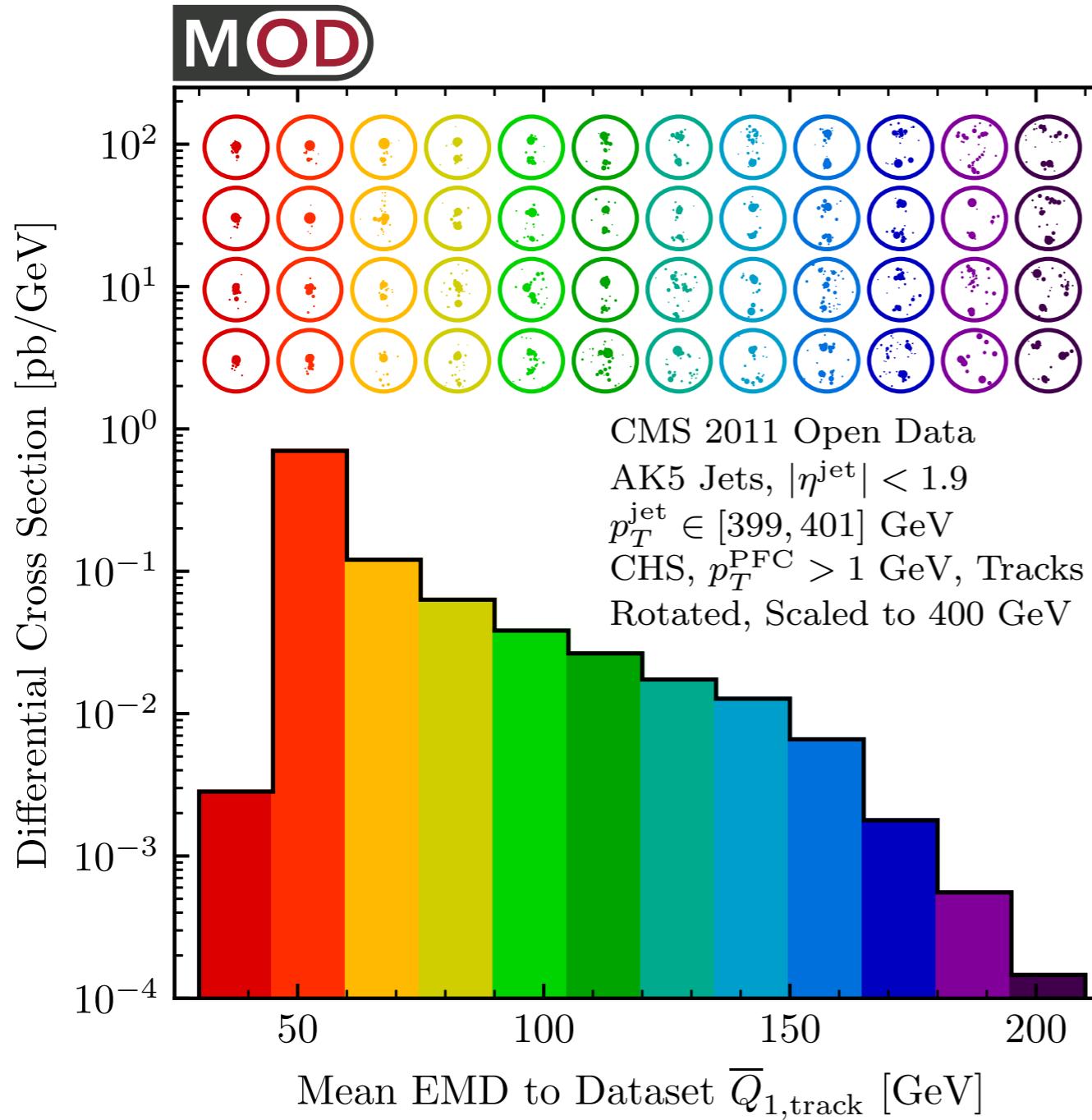


[Komiske, Mastandrea, Metodiev, Naik, JDT, [PRD 2020](#); using van der Maaten, Hinton, [JMLR 2008](#)]

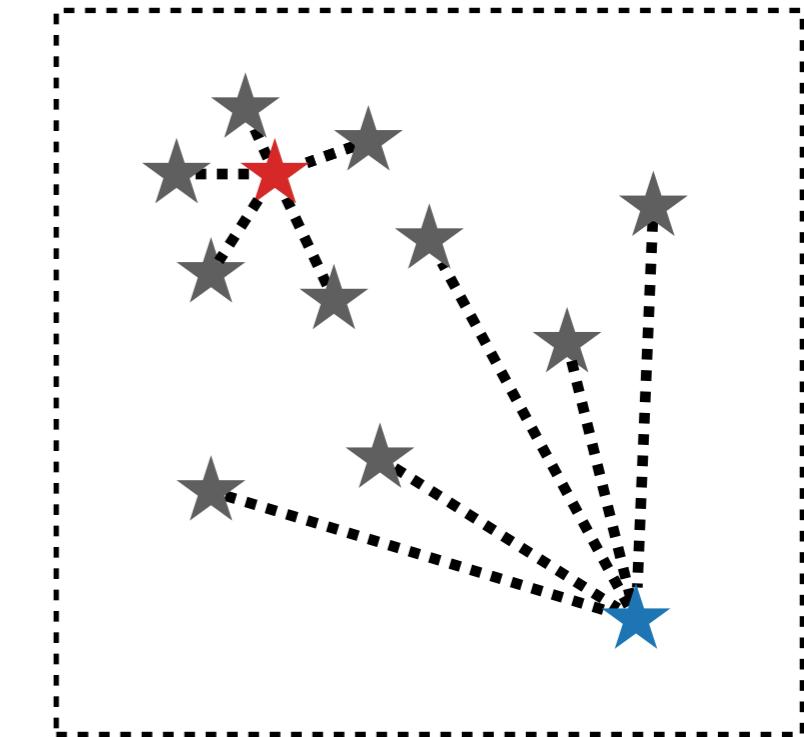
# Least Representative Jets



[<http://opendata.cern.ch/>]

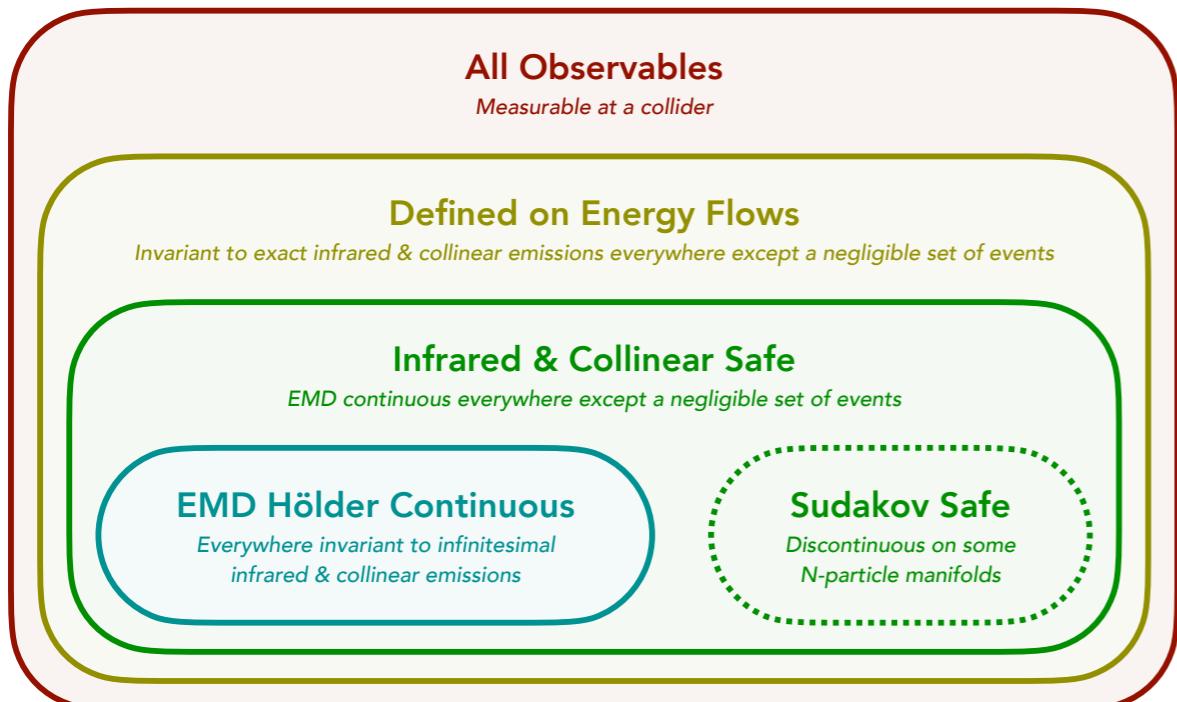


New Physics?  
Or tails of QCD?



[Komiske, Mastandrea, Metodiev, Naik, JDT, [PRD 2020](#)]

# Observable Taxonomy



All Observables	Comments
Multiplicity ( $\sum_i 1$ )	IR unsafe and C unsafe
Momentum Dispersion [65] ( $\sum_i E_i^2$ )	IR safe but C unsafe
Sphericity Tensor [66] ( $\sum_i p_i^\mu p_i^\nu$ )	IR safe but C unsafe
Number of Non-Zero Calorimeter Deposits	C safe but IR unsafe
Defined on Energy Flows	
Pseudo-Multiplicity ( $\min\{N \mid \mathcal{T}_N = 0\}$ )	Robust to exact IR or C emissions
Infrared & Collinear Safe	
Jet Energy ( $\sum_i E_i$ )	Disc. at jet boundary
Heavy Jet Mass [67]	Disc. at hemisphere boundary
Soft-Dropped Jet Mass [38, 68]	Disc. at grooming threshold
Calorimeter Activity [69] ( $N_{95}$ )	Disc. at cell boundary
Sudakov Safe	
Groomed Momentum Fraction [39] ( $z_g$ )	Disc. on 1-particle manifold
Jet Angularity Ratios [37]	Disc. on 1-particle manifold
$N$ -subjettiness Ratios [47, 48] ( $\tau_{N+1}/\tau_N$ )	Disc. on $N$ -particle manifold
$V$ parameter [36] (Eq. (2.11))	Hölder disc. on 3-particle manifold
EMD Hölder Continuous Everywhere	
Thrust [40, 41]	
Spherocity [42]	
Angularities [70]	
$N$ -jettiness [44] ( $\mathcal{T}_N$ )	
$C$ parameter [71–74]	Resummation beneficial at $C = \frac{3}{4}$
Linear Sphericity [72] ( $\sum_i E_i n_i^\mu n_i^\nu$ )	
Energy Correlators [36, 75–77]	
Energy Flow Polynomials [15, 17]	

[Komiske, Metodiev, JDT, [JHEP 2020](#); cf. Sterman, [PRD 1979](#); Banfi, Salam, Zanderighi, [JHEP 2005](#); Larkoski, Marzani, JDT, [PRD 2015](#)]