

The Hidden Geometry of Particle Collisions

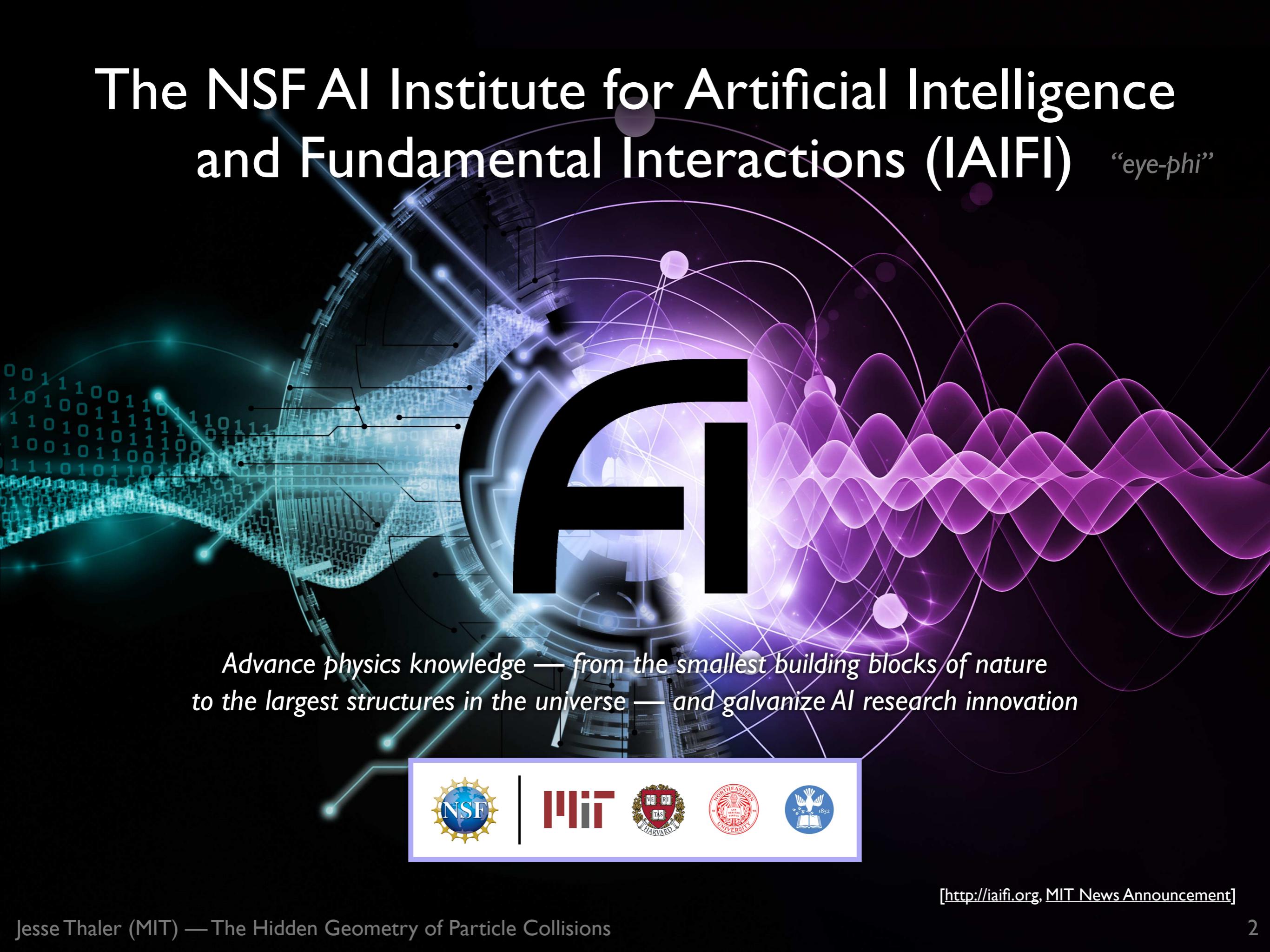
Jesse Thaler



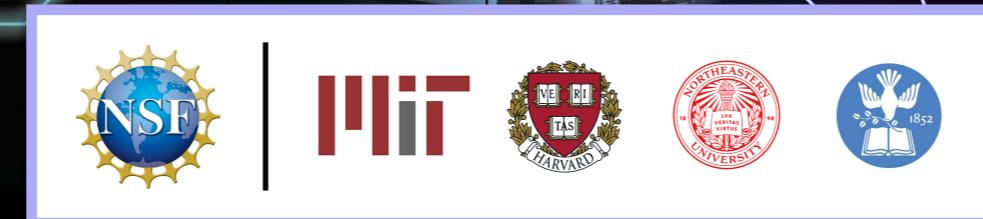
IPPP Seminar, Durham University — May 13, 2021

The NSF AI Institute for Artificial Intelligence and Fundamental Interactions (IAIFI)

“eye-phi”

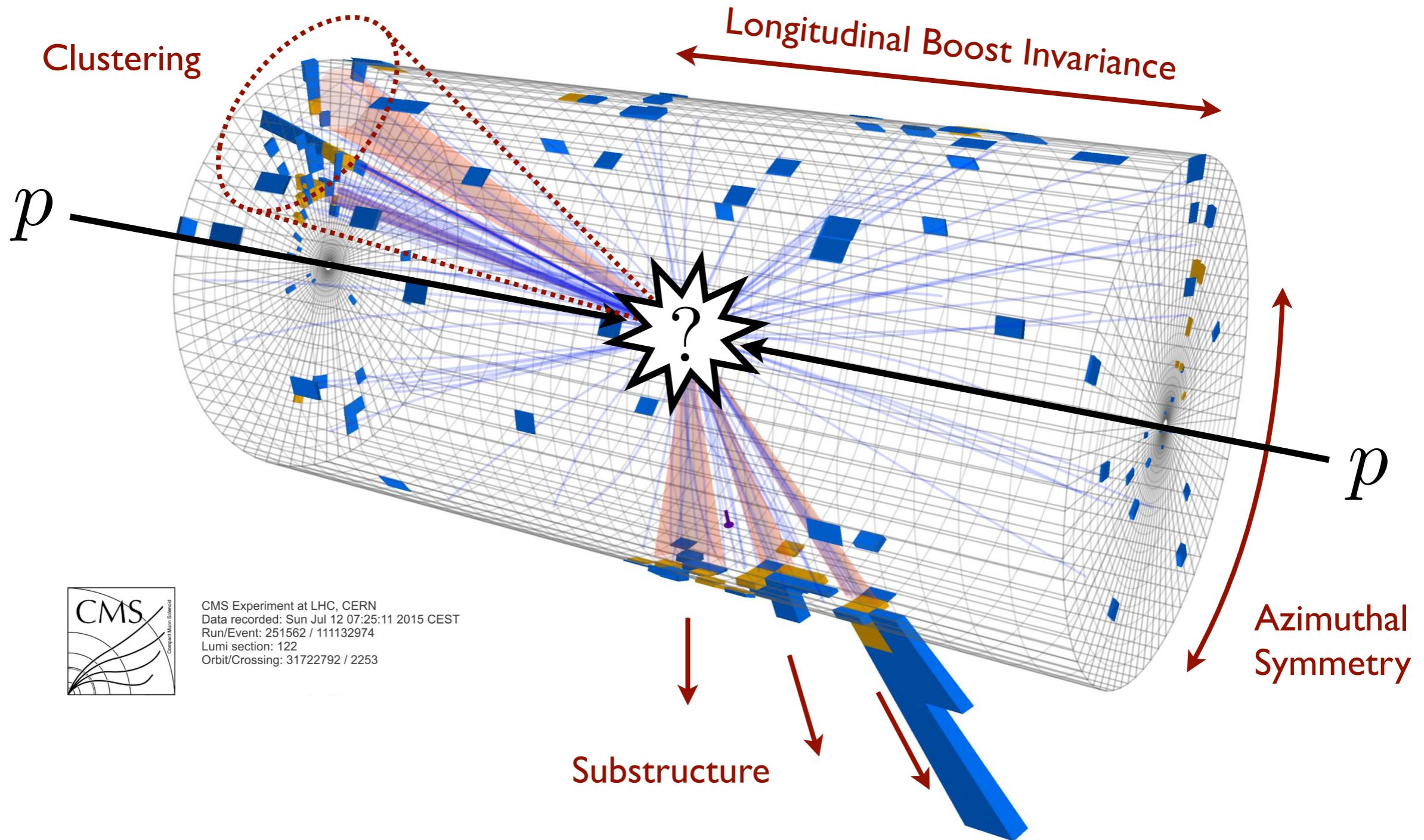


*Advance physics knowledge — from the smallest building blocks of nature
to the largest structures in the universe — and galvanize AI research innovation*

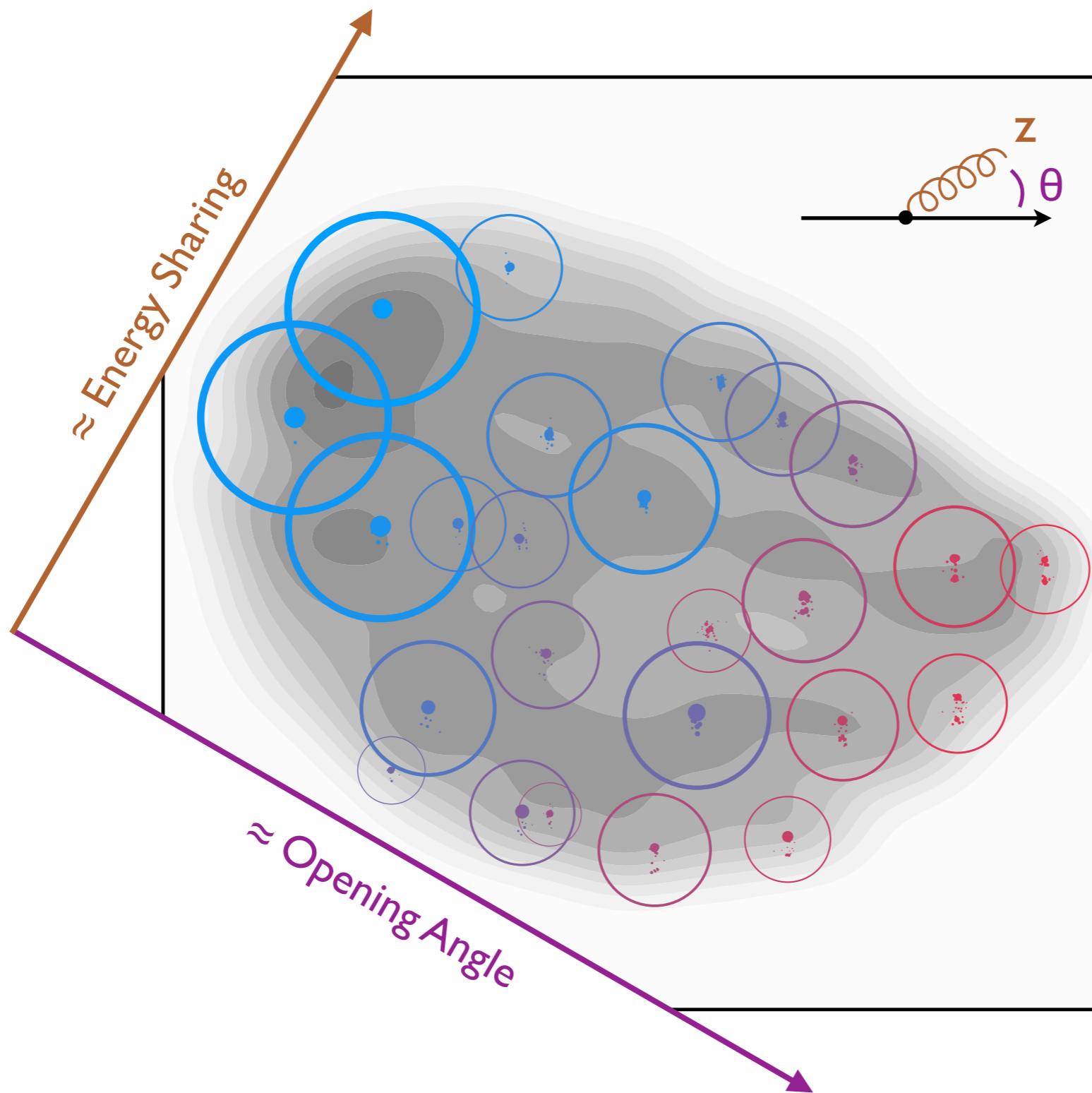


[<http://iaifi.org>, MIT News Announcement]

The Manifest Geometry of One Collision



The Emergent Geometry of Many Collisions

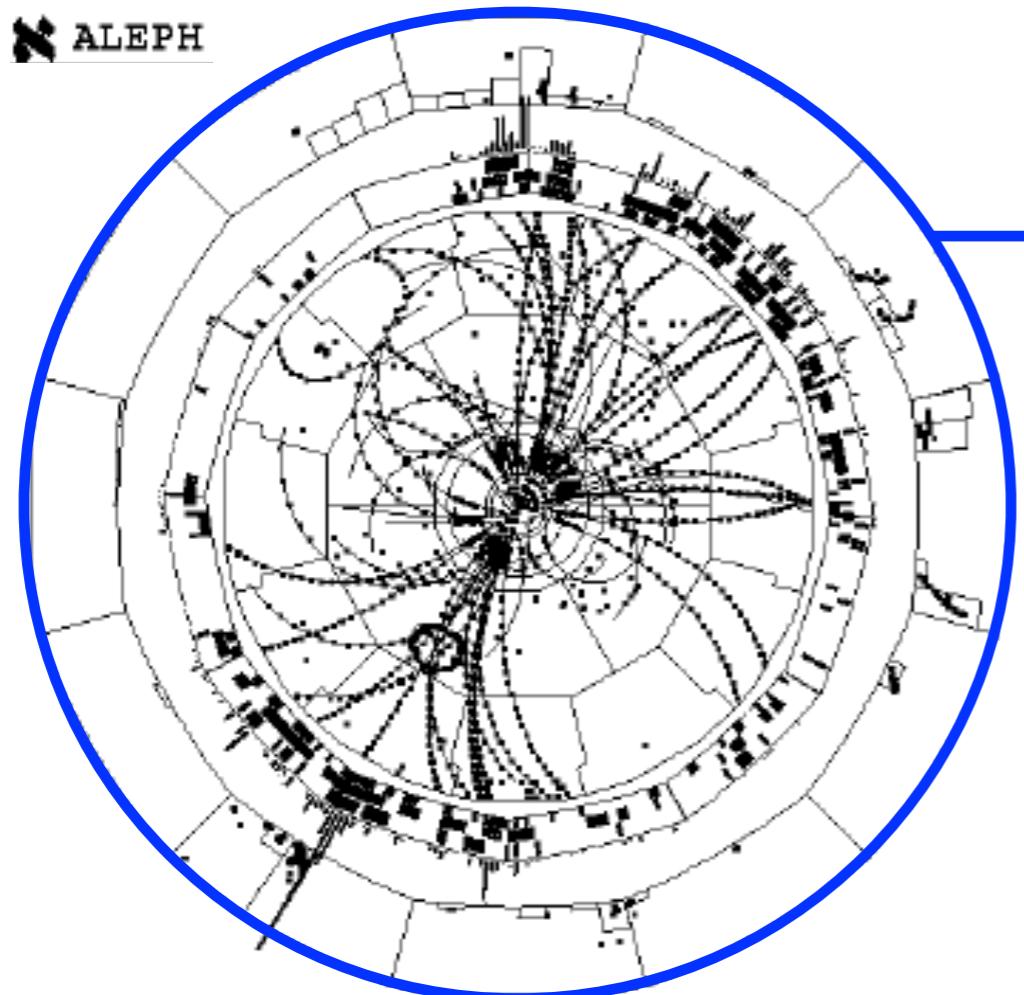


[Komiske, Mastandrea, Metodiev, Naik, JDT, [PRD 2020](#);
based on Komiske, Metodiev, JDT, [PRL 2019](#); using [EnergyFlow](#) and [CMS Open Data](#)]

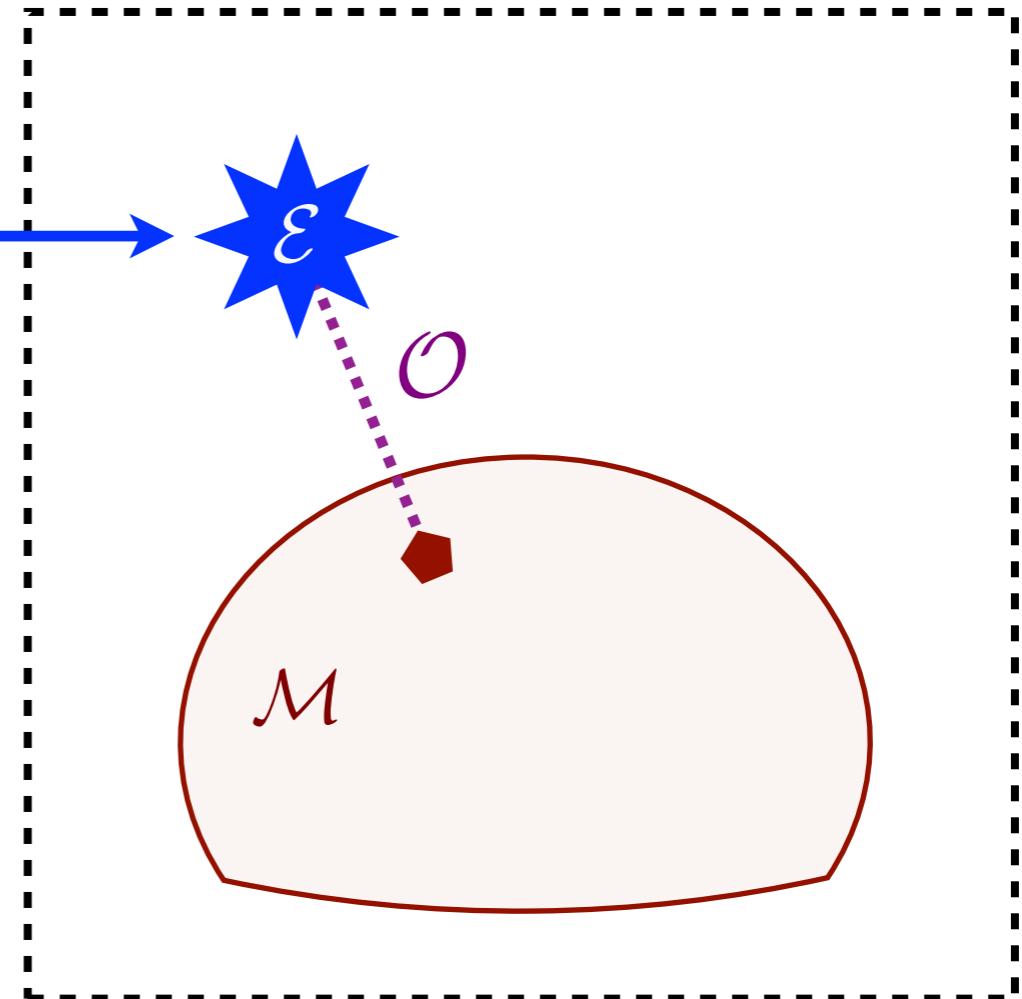


The Hidden Geometry of Particle Collisions

E.g. Classic QCD Event Shapes



One Electron-Positron Event

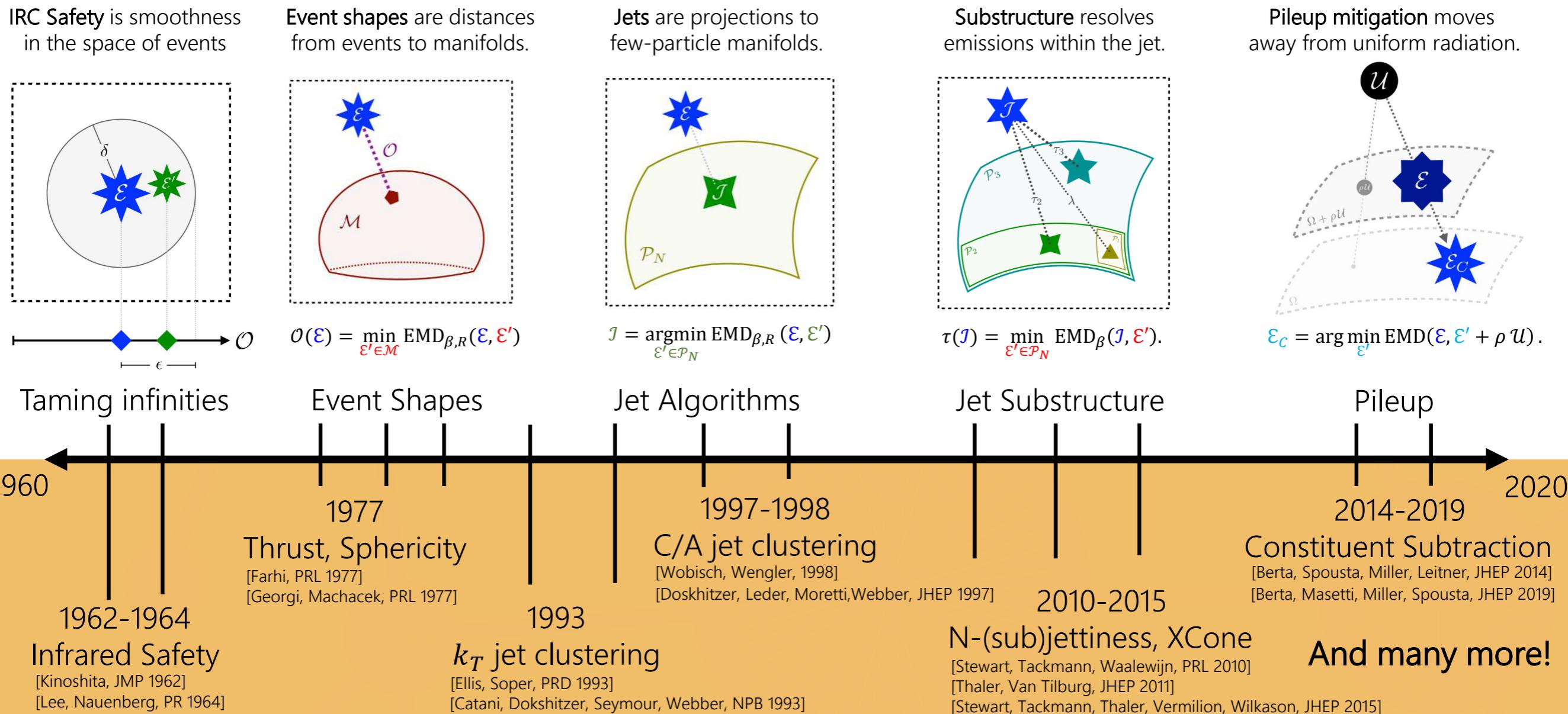


Distance to a Manifold in Event Space

[Komiske, Metodiev, JDT, [JHEP 2020](#)]
[Brandt, Peyrou, Sosnowski, Wroblewski, [PL 1964](#); Farhi, [PRL 1977](#)]

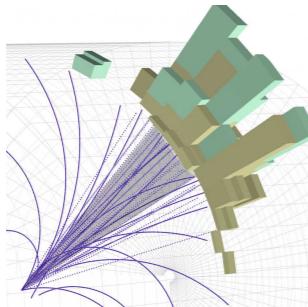


Six Decades of Collider Physics Translated into a New Geometric Language!

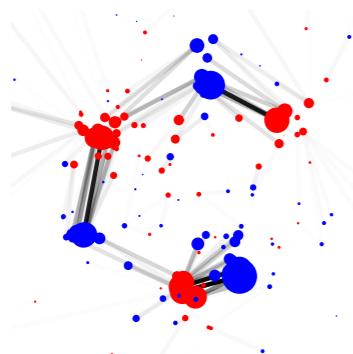


[timeline from Eric Metodiev]

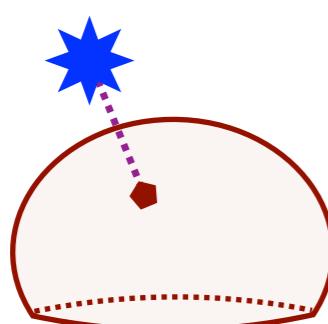
Outline



What is a Collider Event?



When are Events Similar?



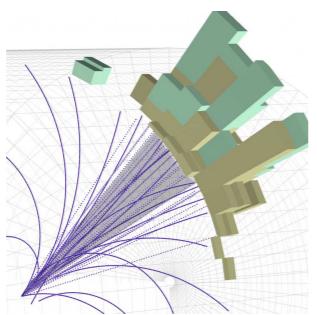
What can be Geometrized?

Pause

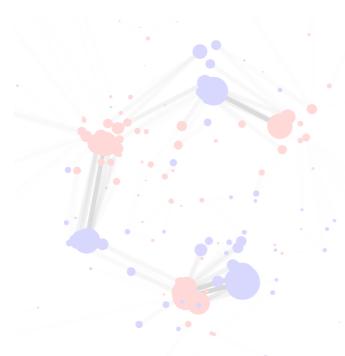
Interrupt me or drop questions in the chat, and I'll try to answer them as I go

Anticipating a coda...

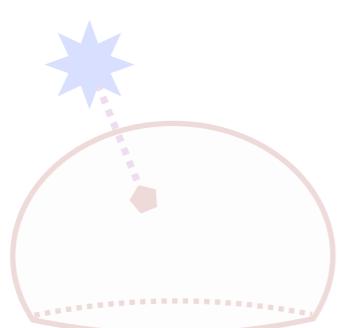




What is a Collider Event?



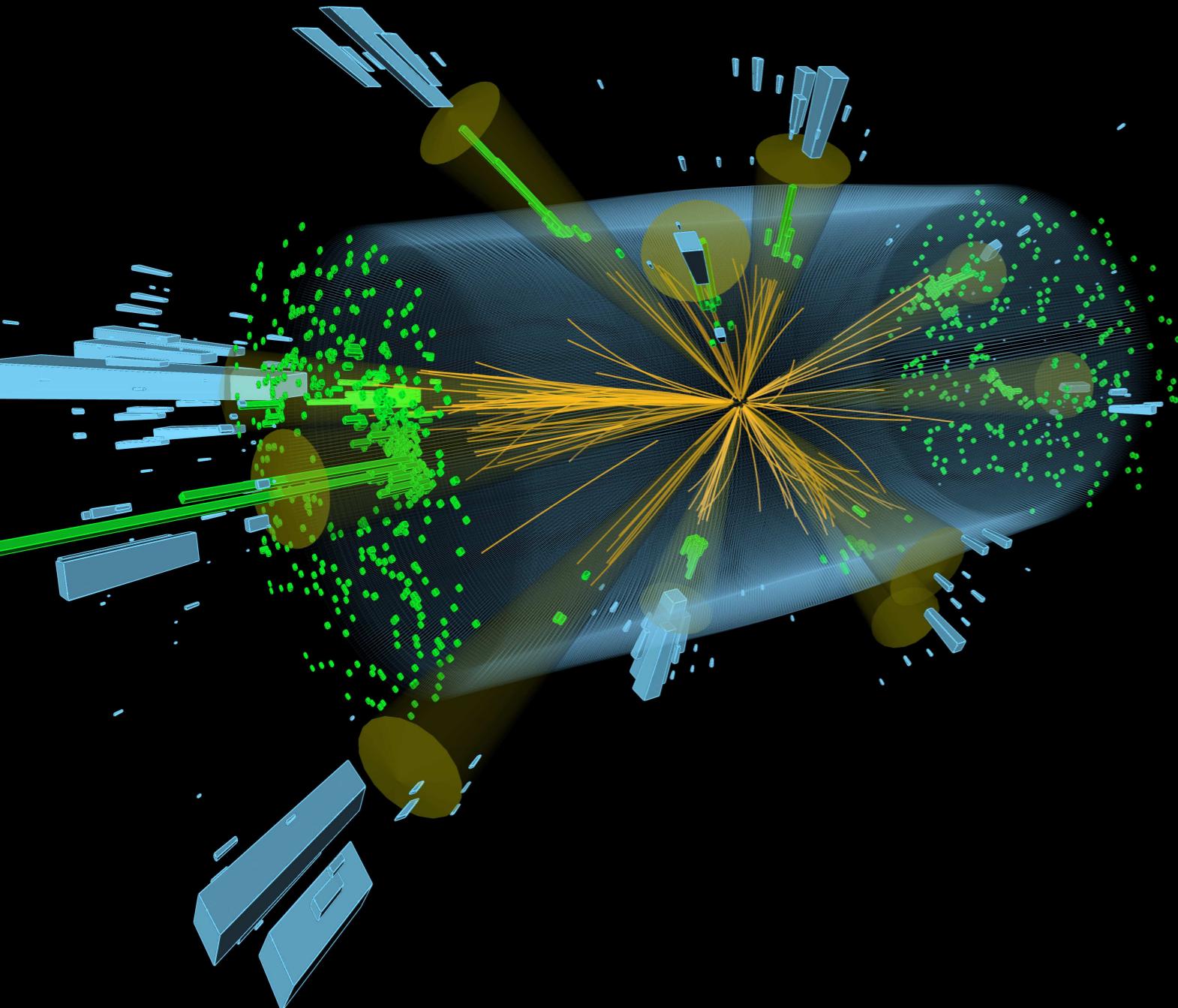
When are Events Similar?



What can be Geometrized?

Collider Event

Collection of points in momentum space



T E H M

 γ

photon

 e^+

electron

 μ^+

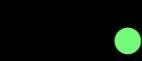
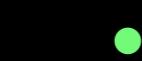
muon

 π^+

pion

 K^+

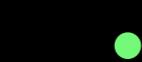
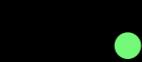
kaon

 K_L^0

K-long

 p/\bar{p}

proton

 n/\bar{n}

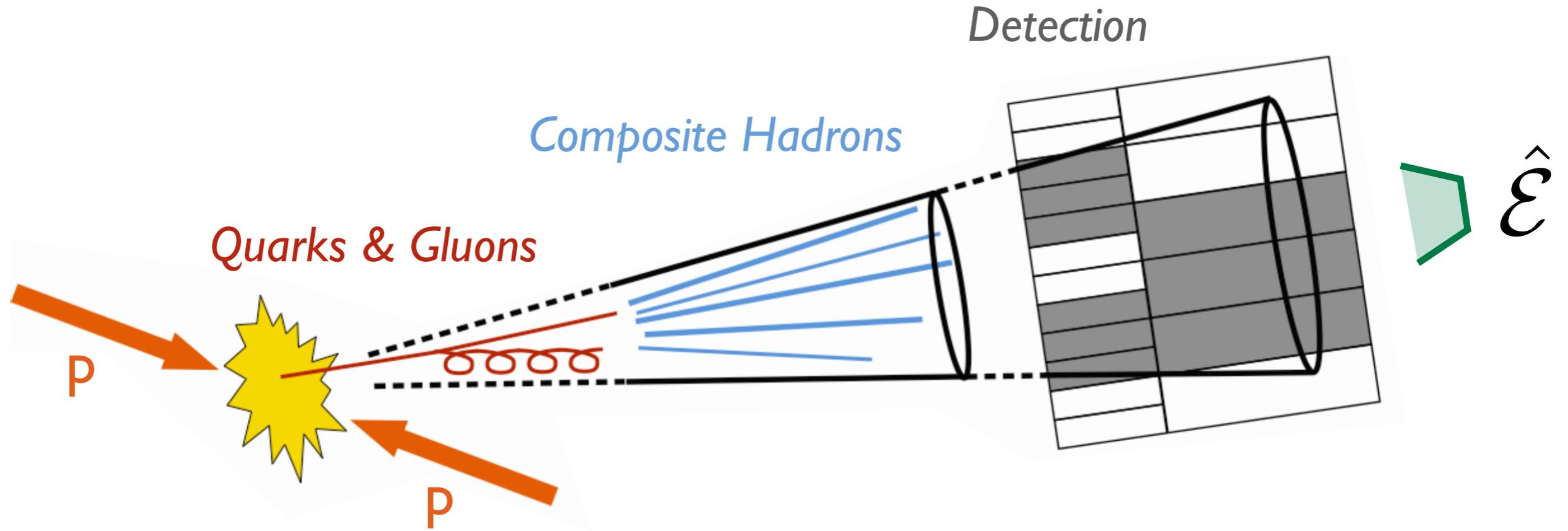
neutron

elementary

composite

Jet Formation from QCD

Theory



Energy Flow:

Robust to hadronization and detector effects
Well-defined for massless gauge theories

$$\hat{\mathcal{E}} \simeq \lim_{t \rightarrow \infty} \hat{n}_i T^{0i}(t, vt\hat{n})$$

[see e.g. Sveshnikov, Tkachov, [PLB 1996](#); Hofman, Maldacena, [JHEP 2008](#); Mateu, Stewart, [JDT, PRD 2013](#); Belitsky, Hohenegger, Korchemsky, Sokatchev, Zhiboedov, [PRL 2014](#); Chen, Moult, Zhang, Zhu, [PRD 2020](#)]

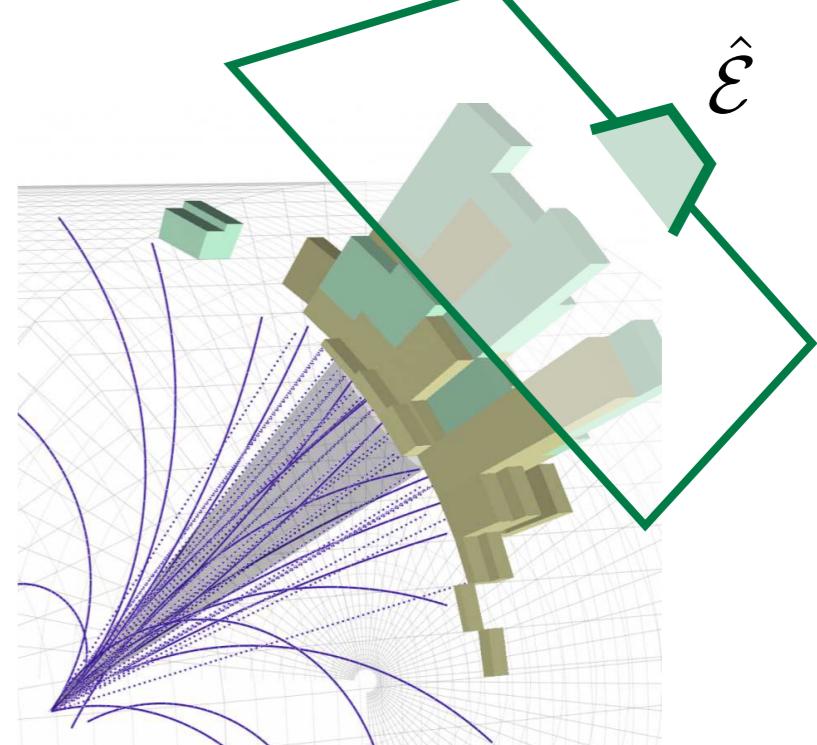
Jets as Weighted Point Clouds

- Energy-Weighted Directions

$$\vec{p} = \{E, \hat{n}_x, \hat{n}_y, \hat{n}_z\}$$

↑ |
Energy Direction

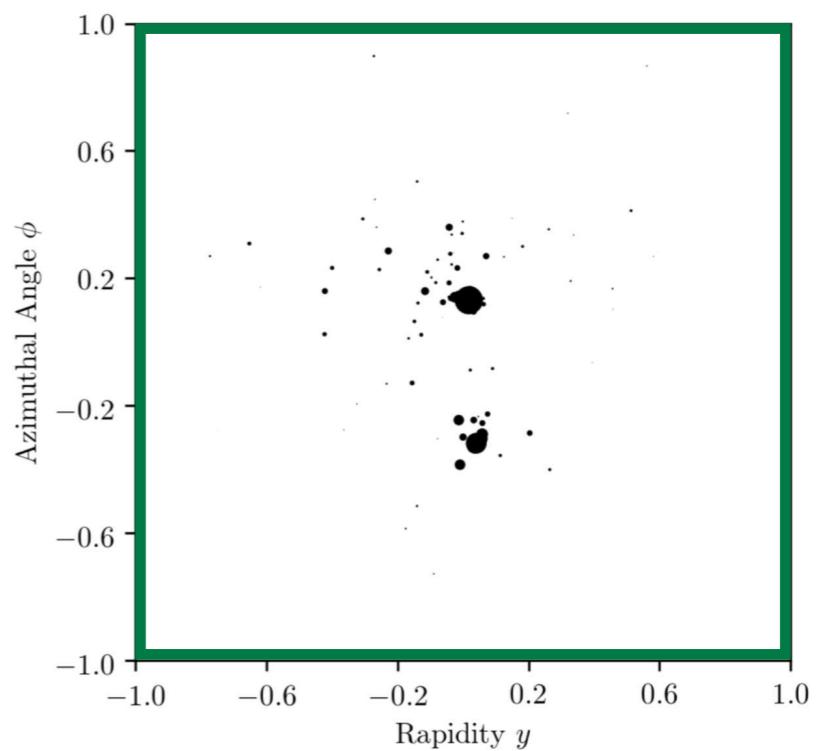
(suppressing “unsafe” charge/flavor information)



- Equivalently: Energy Density

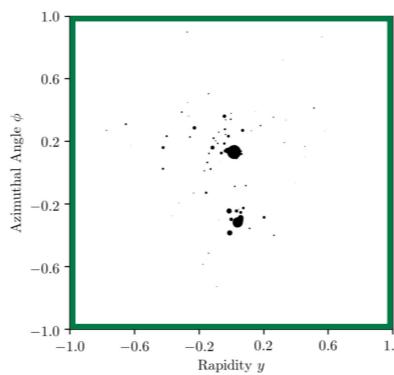
$$\rho(\hat{n}) = \sum_{i \in \mathcal{J}} E_i \delta^{(2)}(\hat{n} - \hat{n}_i)$$

↑ ↑
Energy Direction



Pause

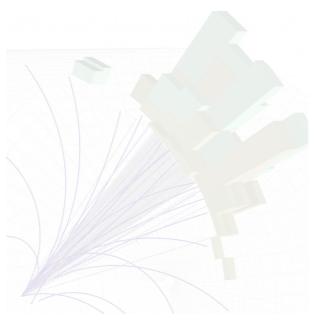
What is a Collider Event?



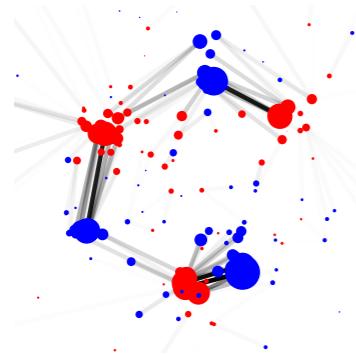
“Calo” Energy Density

$$\mathcal{E}(\hat{n}) = \sum_i E_i \delta(\hat{n} - \hat{n}_i)$$

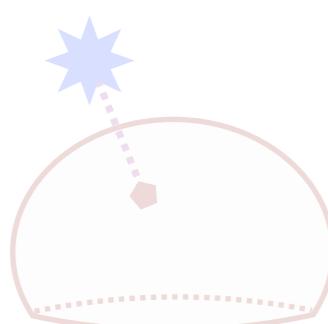
(see backup for relevance to QCD & ML)



What is a Collider Event?



When are Events Similar?

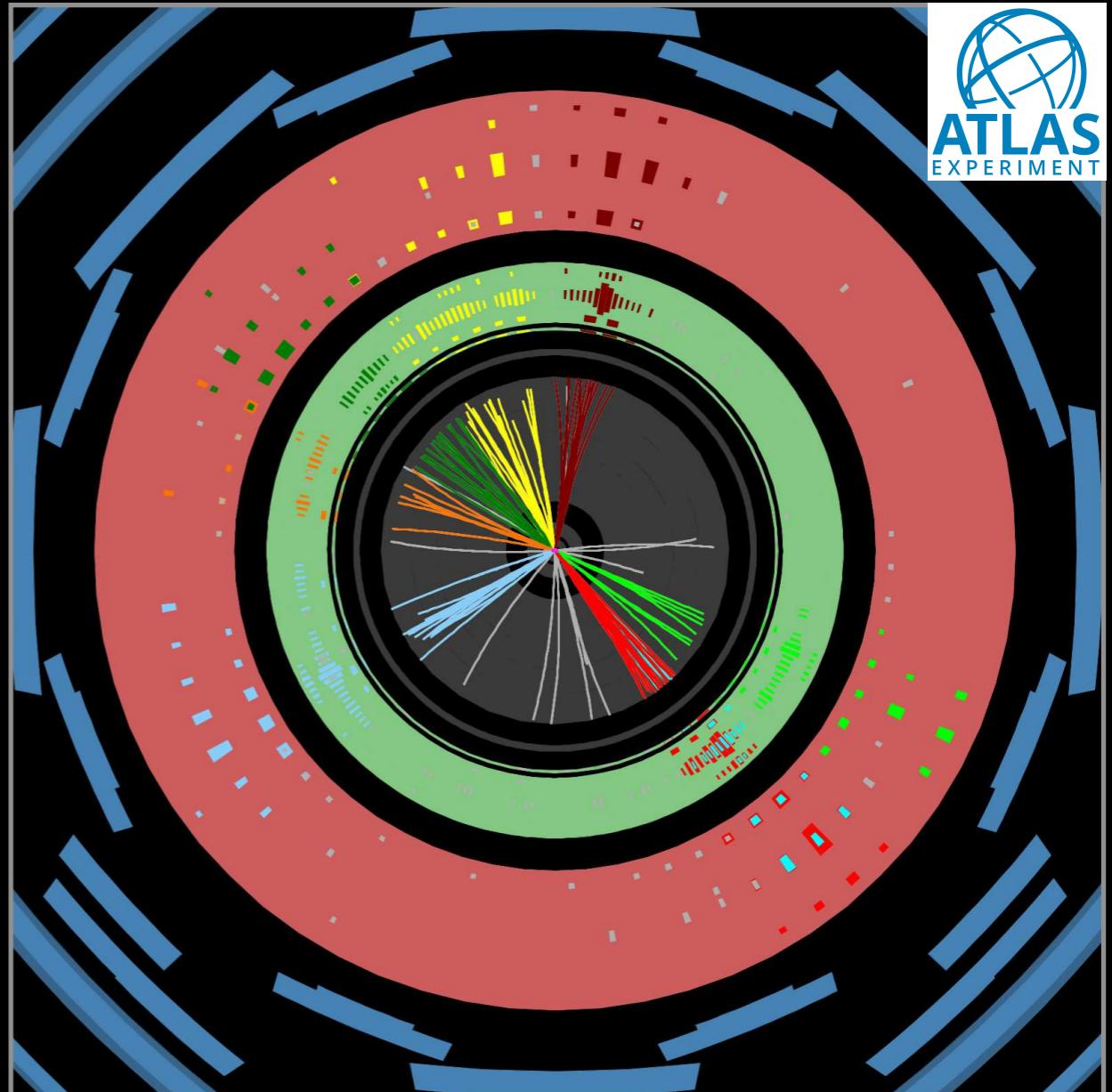
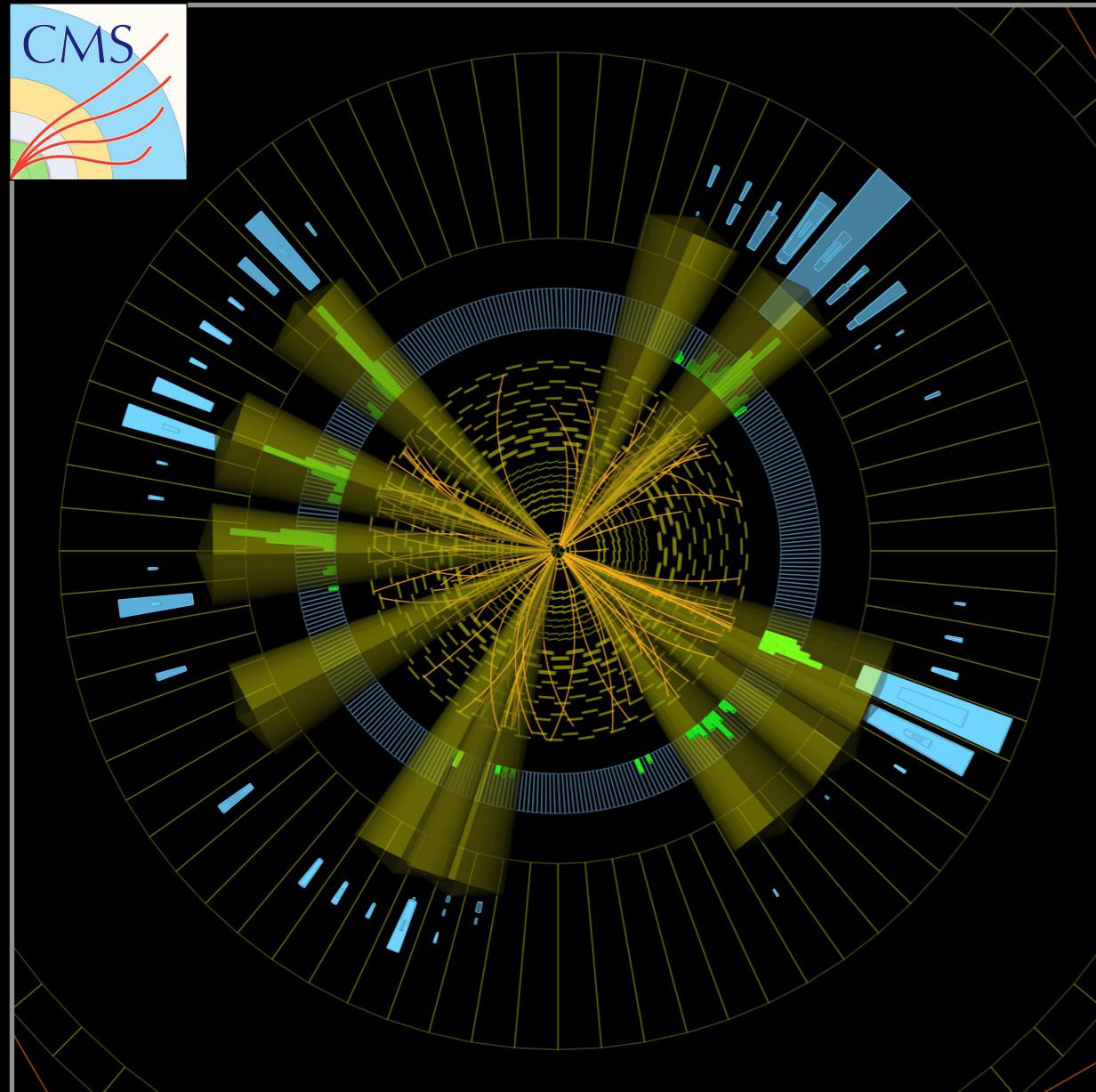


What can be Geometrized?

Two Collider Events

Two collections of points in (momentum) space

How “close” are these?

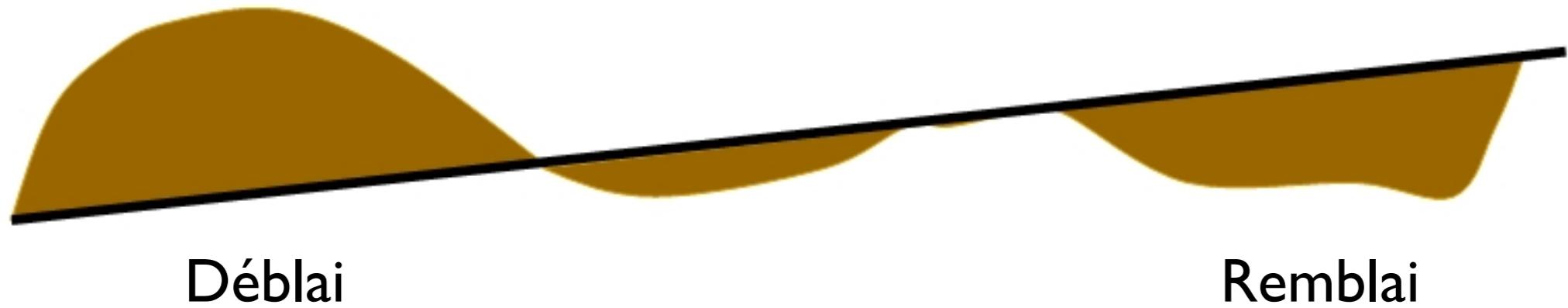


The Earth Mover's Distance

Optimal Transport:

[Peleg, Werman, Rom, [IEEE 1989](#);
Rubner, Tomasi, Guibas, [ICCV 1998](#), [ICCV 2000](#);
Pele, Werman, [ECCV 2008](#); Pele Taskar, [GSI 2013](#)]

Minimum “work” (stuff \times distance) to make one distribution look like another distribution



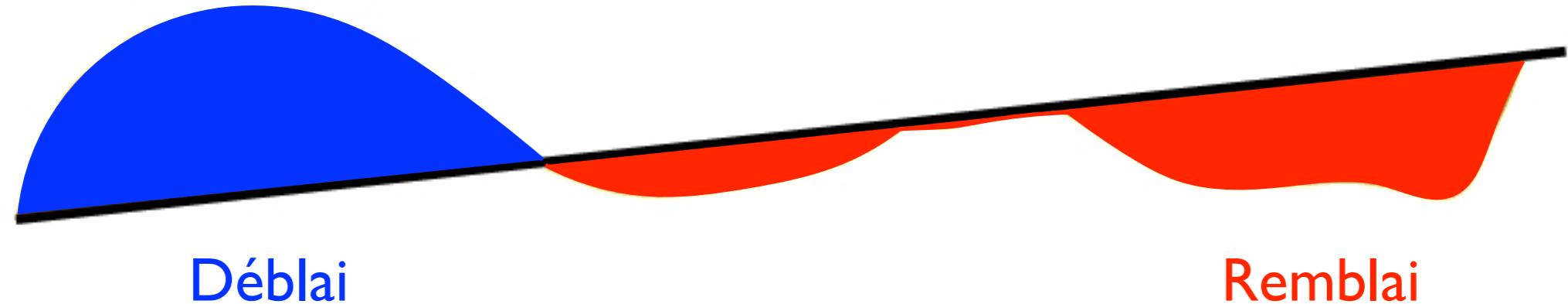
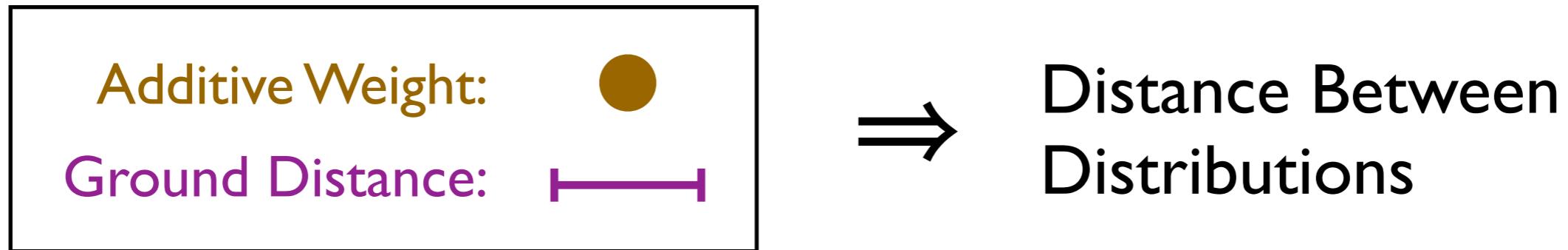
[h/t Niles-Weed, [ML4Jets 2020](#); Monge, 1781; Vaserštejn, 1969; [Wikipedia](#)]

The Earth Mover's Distance

Optimal Transport:

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Pele, Werman, [ECCV 2008](#); Pele Taskar, [GSI 2013](#)]

Minimum “work” (**stuff** × **distance**) to make
one distribution look like **another distribution**



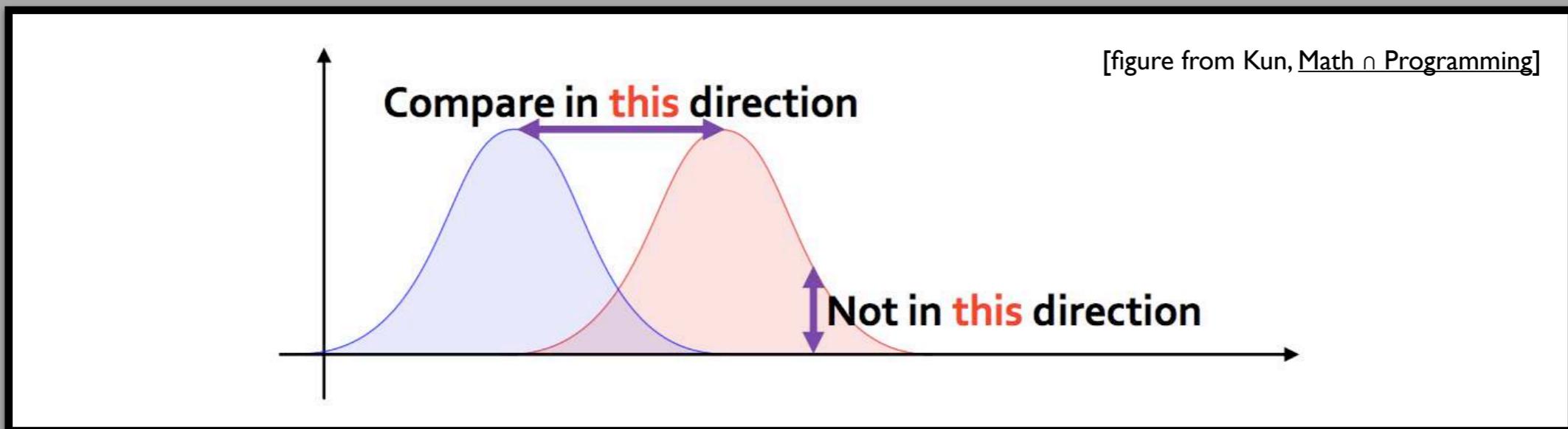
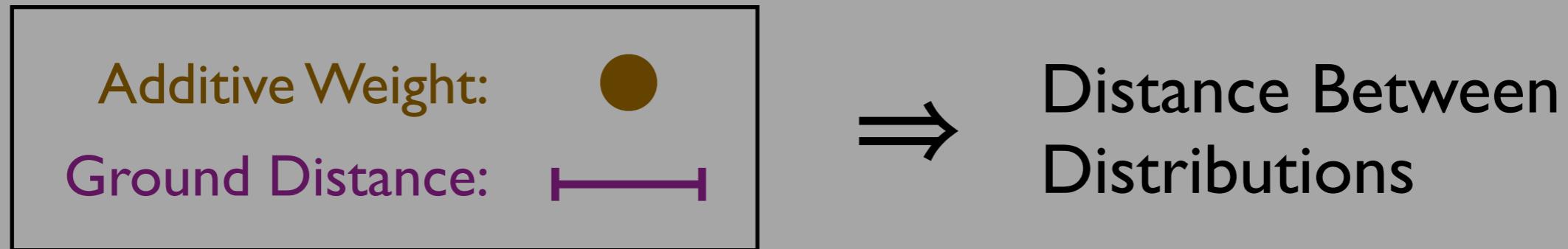
[h/t Niles-Weed, [ML4Jets 2020](#); Monge, 1781; Vaserštejn, 1969; [Wikipedia](#)]

The Earth Mover's Distance

Optimal Transport:

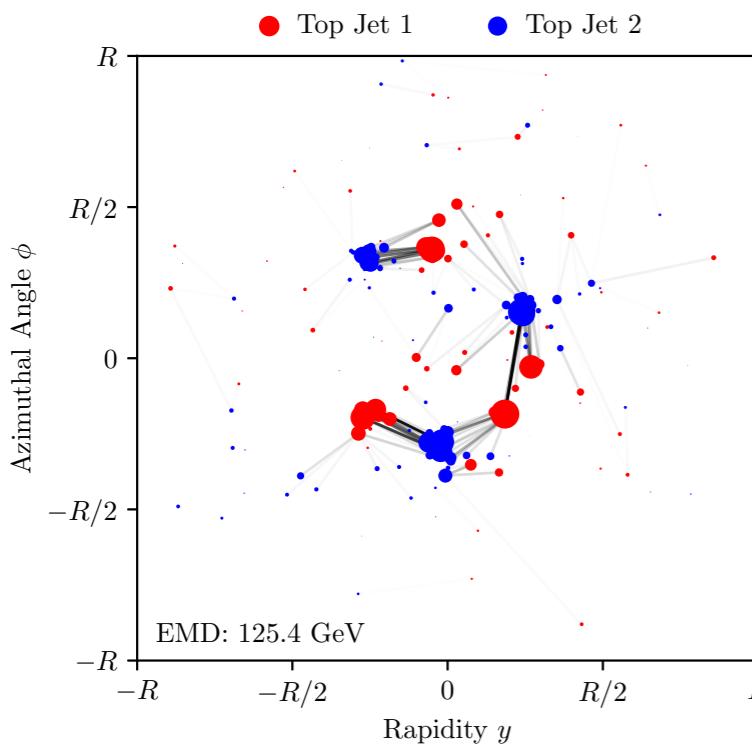
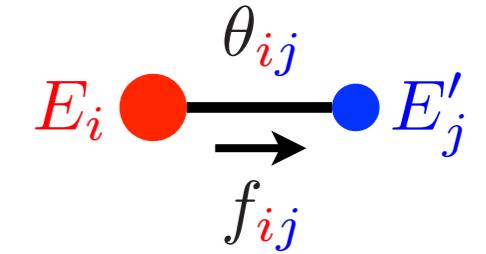
[Peleg, Werman, Rom, [IEEE 1989](#);
Rubner, Tomasi, Guibas, [ICCV 1998](#), [ICJV 2000](#);
Pele, Werman, [ECCV 2008](#); Pele Taskar, [GSI 2013](#)]

Minimum “work” (**stuff \times distance**) to make
one distribution look like **another distribution**



[h/t Niles-Weed, [ML4Jets 2020](#); Monge, 1781; Vaserštejn, 1969; [Wikipedia](#)]

The Energy Mover's Distance

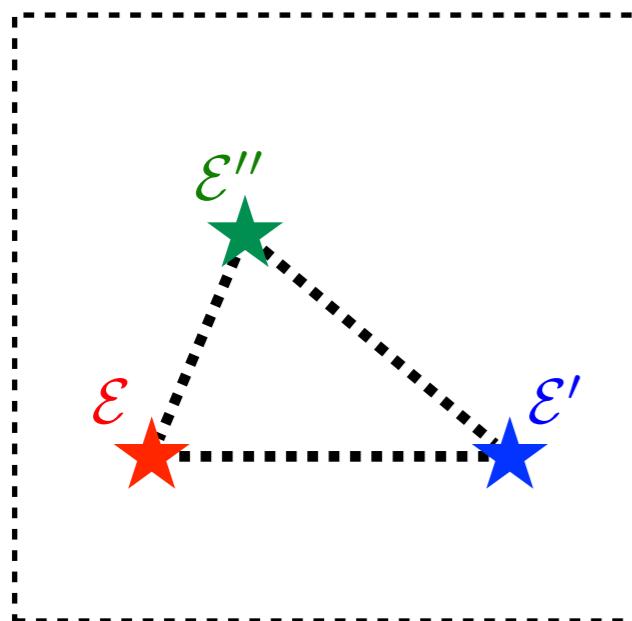


Optimal transport between energy flows...

$$\text{EMD}(\mathcal{E}, \mathcal{E}') = \min_{\{f\}} \sum_i \sum_j f_{ij} \frac{\theta_{ij}}{R} + \left| \sum_i E_i - \sum_j E'_j \right|$$

↑
in GeV

Cost to move energy Cost to create energy



...defines a metric on the space of events

$$0 \leq \text{EMD}(\mathcal{E}, \mathcal{E}') \leq \text{EMD}(\mathcal{E}, \mathcal{E}'') + \text{EMD}(\mathcal{E}', \mathcal{E}'')$$

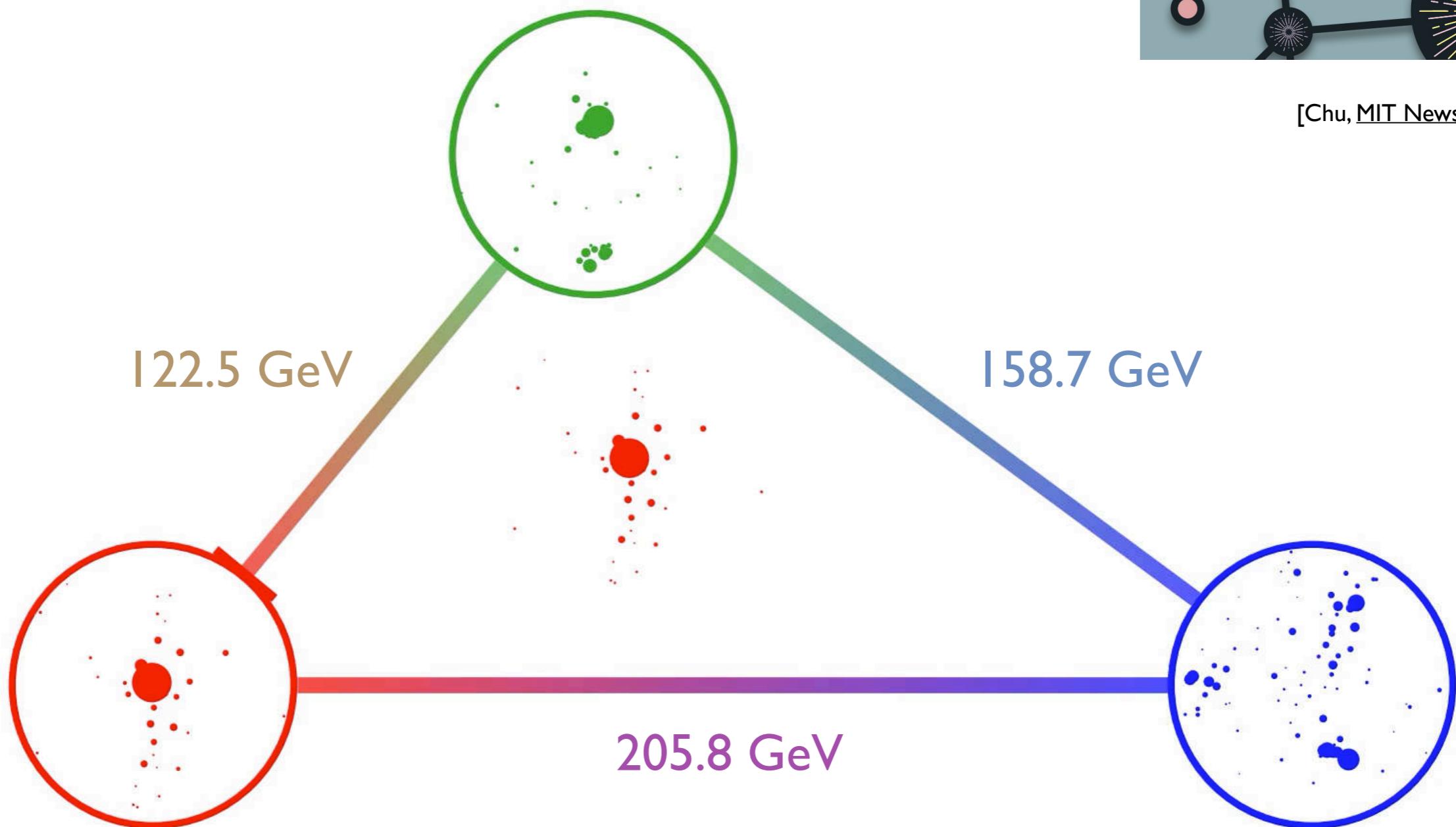
(assuming $R \geq \theta_{\max}/2$, i.e. $R \geq$ jet radius for conical jets)

[Komiske, Metodiev, JDT, [PRL 2019](#);
 see also Pele, Werman, [ECCV 2008](#); Pele, Taskar, [GSI 2013](#);
 [see flavored variant in Crispim Romão, Castro, Milhano, Pedro, Vale, [EPJC 2021](#)]
 [see computational speed up in Cai, Cheng, Craig, [PRD 2020](#)]

Similarity of Three Energy Flows?



[Chu, MIT News July 2019]



[Komiske, Metodiev, JDT, PRL 2019; code at Komiske, Metodiev, JDT, [energyflow.network](#); see alternative graph network approach in Mullin, Pacey, Parker, White, Williams, JHEP 2021]

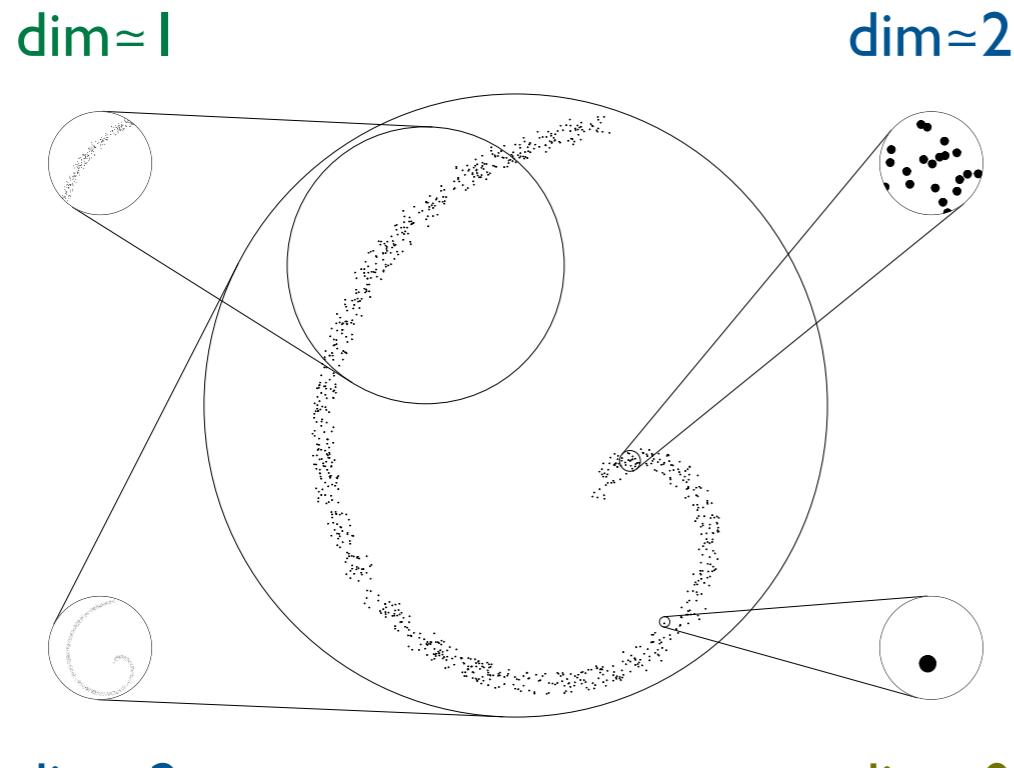
Dimensionality of Space of Jets



$$N_{\text{neighbors}}(r) \sim r^{\dim}$$

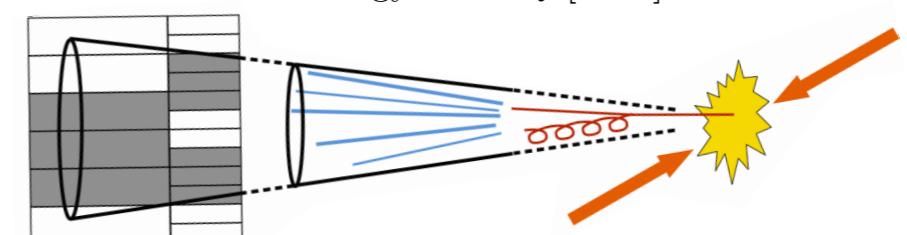
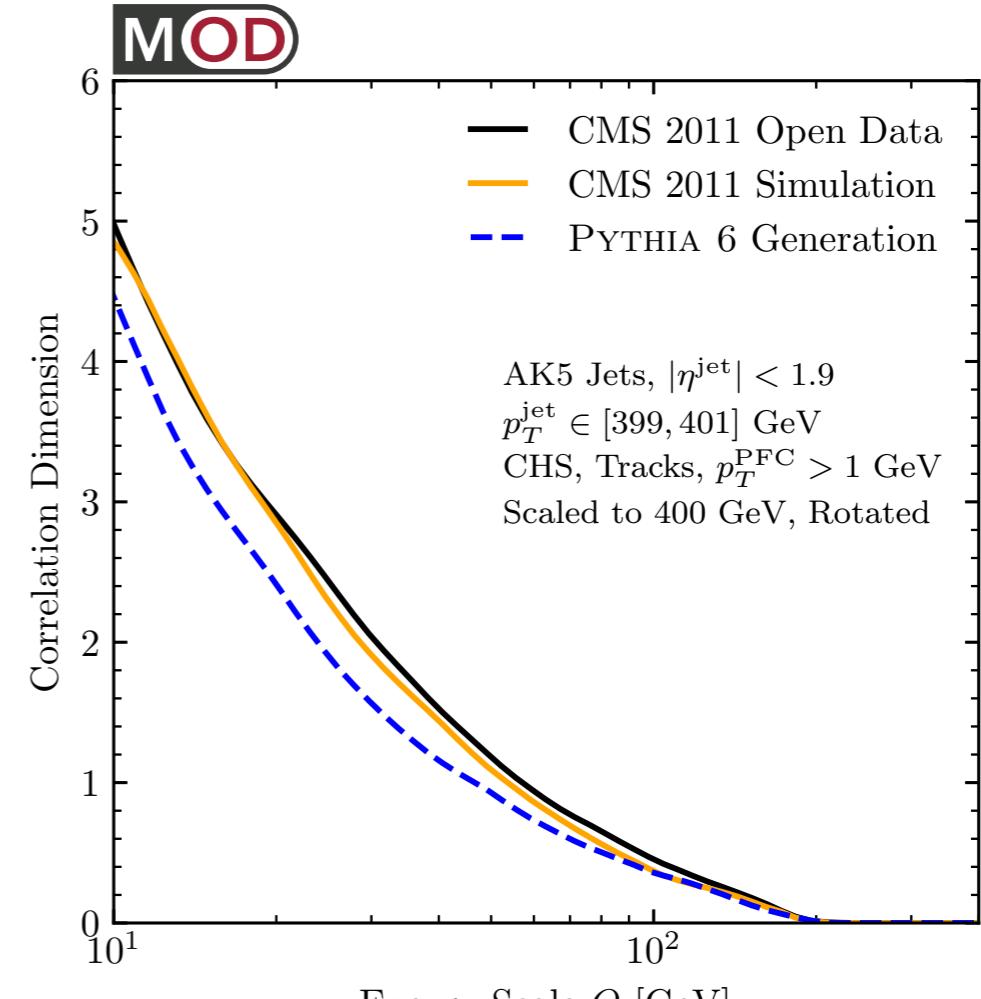
$$\Rightarrow \dim(r) \sim r \frac{\partial}{\partial r} \ln N_{\text{neighbors}}(r)$$

[Grassberger, Procaccia, [PRL 1983](#); Kégl, [NIPS 2002](#)]



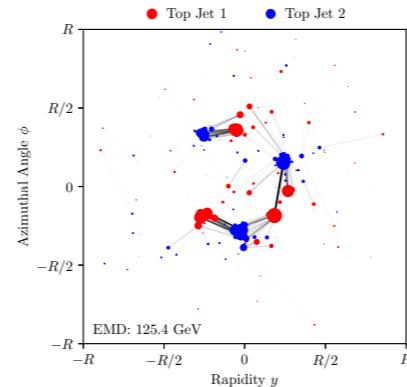
(eventually 0)

[Komiske, Mastandrea, Metodiev, Naik, [JDT, PRD 2020](#);
using [CMS Open Data](#)]



Pause

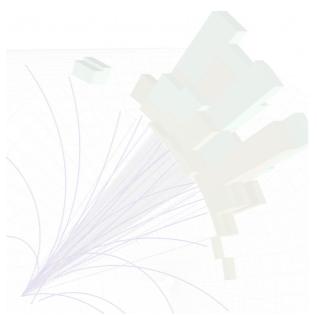
When are Events Similar?



Small Energy Mover's Distance

$$\text{EMD}(\mathcal{E}, \mathcal{E}') = \min_{\{f\}} \sum_i \sum_j f_{ij} \frac{\theta_{ij}}{R} + \left| \sum_i E_i - \sum_j E'_j \right|$$

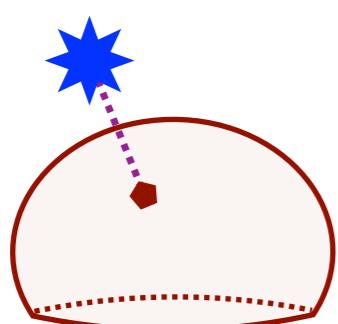
(see backup for more applications)



What is a Collider Event?



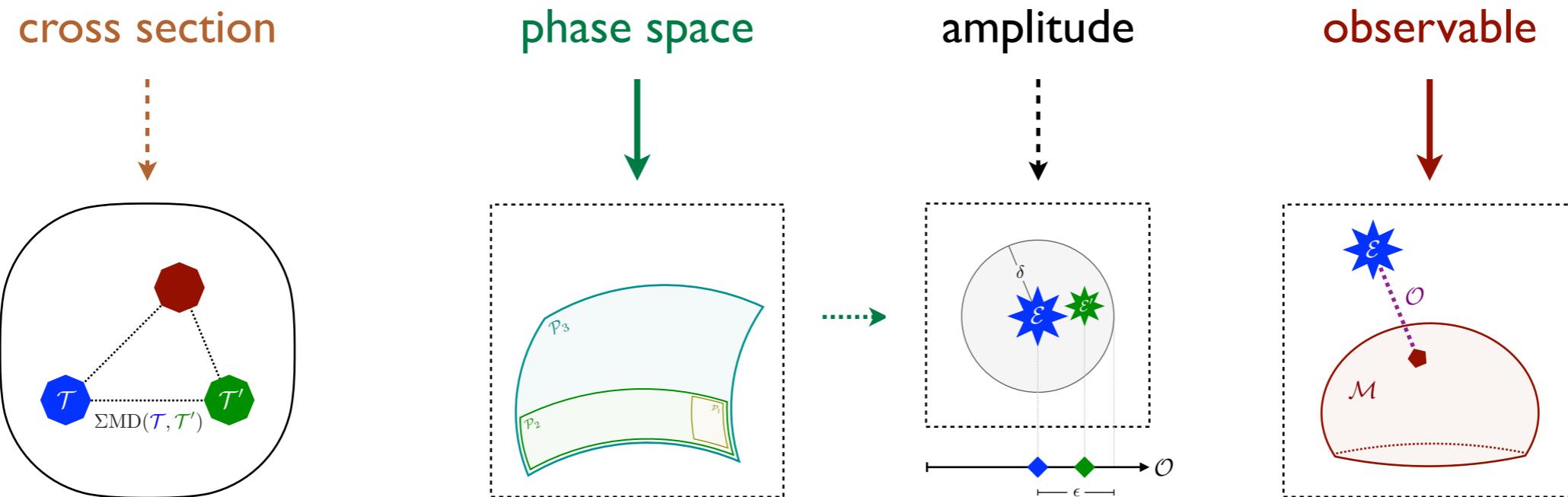
When are Events Similar?



What can be Geometrized?

Master Formula for Collider Physics

$$\sigma_{\text{obs}} \simeq \frac{1}{2E_{\text{CM}}^2} \sum_{n=2}^{\infty} \int d\Phi_n |\mathcal{M}_{AB \rightarrow 12\dots n}|^2 f_{\text{obs}}(\Phi_n)$$

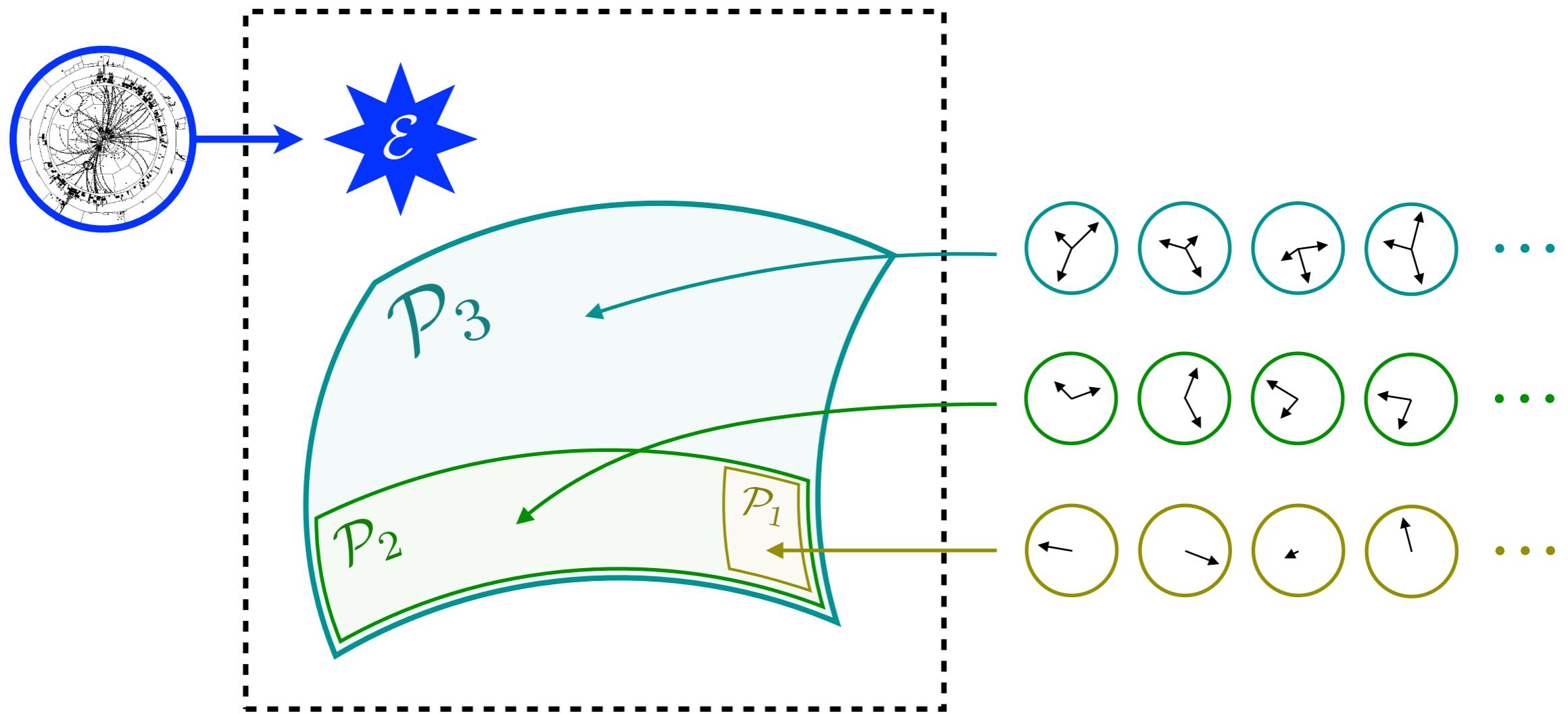


[Komiske, Metodiev, JDT, [JHEP 2020](#)]

Introducing N-particle Manifolds

$$\sum_{n=2}^{\infty} \int d\Phi_n$$

\mathcal{P}_N = set of all N-particle configurations



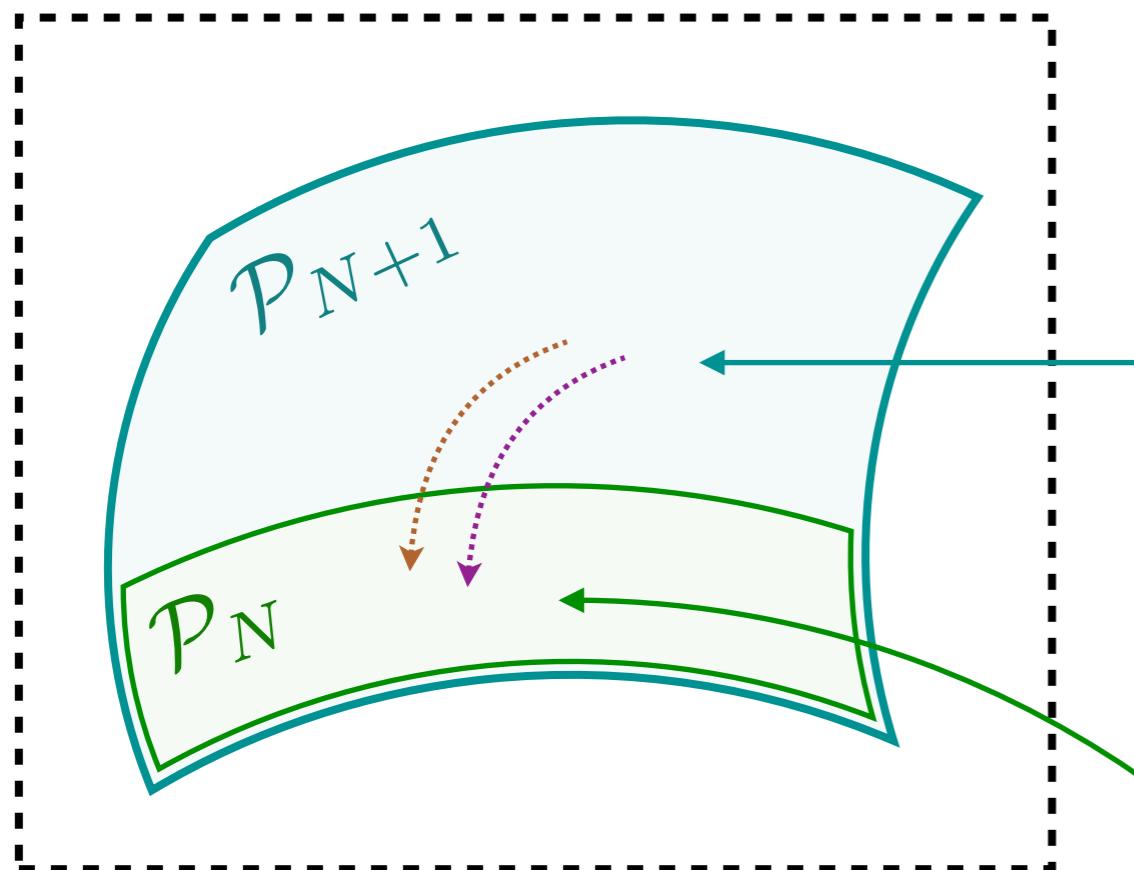
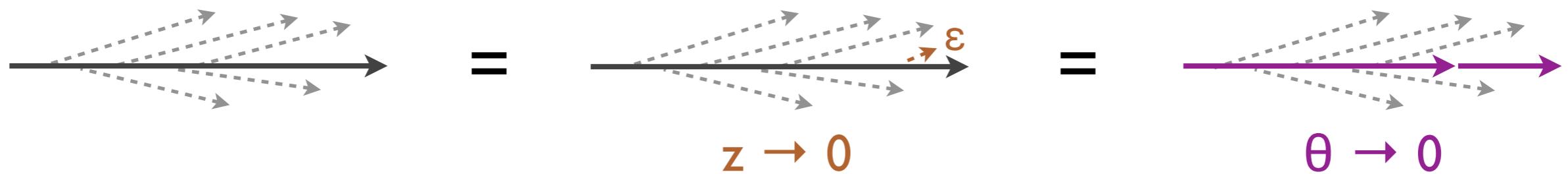
$\mathcal{P}_N \supset \mathcal{P}_{N-1} \supset \dots \supset \mathcal{P}_2 \supset \mathcal{P}_1$ by soft/collinear limits

[see related discussion in Larkoski, Melia, [PRD 2020](#)]

When are Two Events the Same?

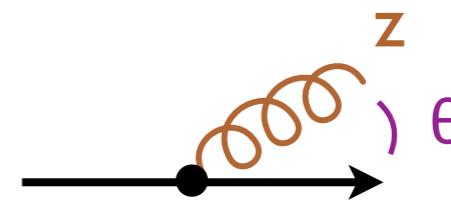
$$\mathcal{E}(\hat{n}) = \sum_i E_i \delta(\hat{n} - \hat{n}_i)$$

Energy Flow unchanged by infinitesimal soft/collinear emissions



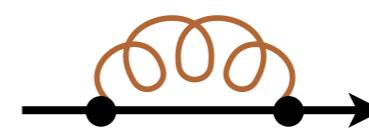
Infrared divergences “live” together!

Real:



$$dP_{i \rightarrow ig} \simeq \frac{2\alpha_s}{\pi} C_i \frac{dz}{z} \frac{d\theta}{\theta}$$

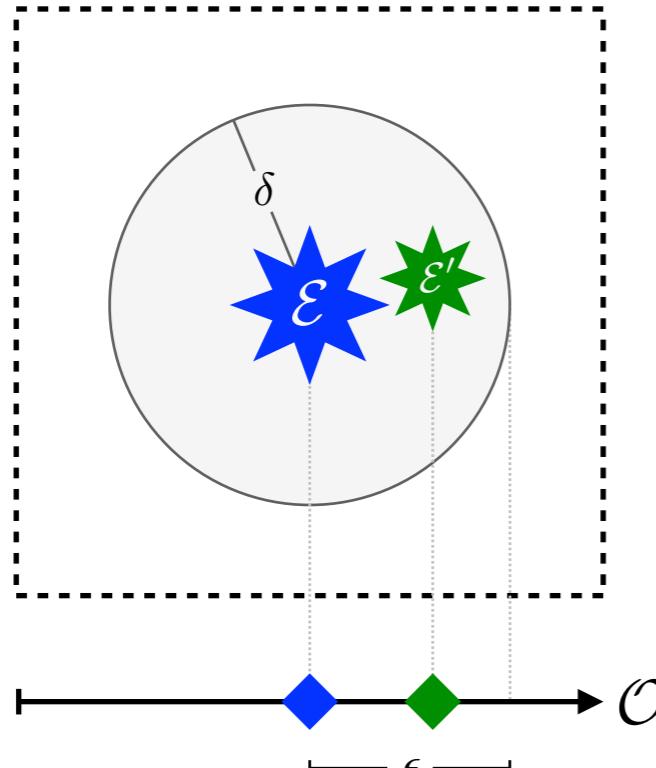
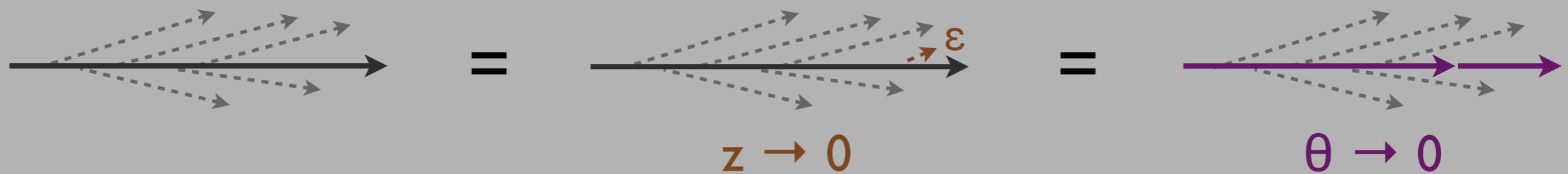
Virtual:



When are Two Events the Same?

$$\mathcal{E}(\hat{n}) = \sum_i E_i \delta(\hat{n} - \hat{n}_i)$$

Energy Flow unchanged by infinitesimal soft/collinear emissions



Infrared & Collinear Safety

\approx calculable in perturbative quantum field theory

*is** \leftarrow (see backup for subtleties)

Continuity in EMD Space

[Komiske, Metodiev, JDT, [JHEP 2020](#)]

[Sterman, Weinberg, [PRL 1977](#); Sterman, [PRD 1979](#)]

[see also Banfi, Salam, Zanderighi, [JHEP 2005](#); Larkoski, Marzani, JDT, [PRD 2015](#)]

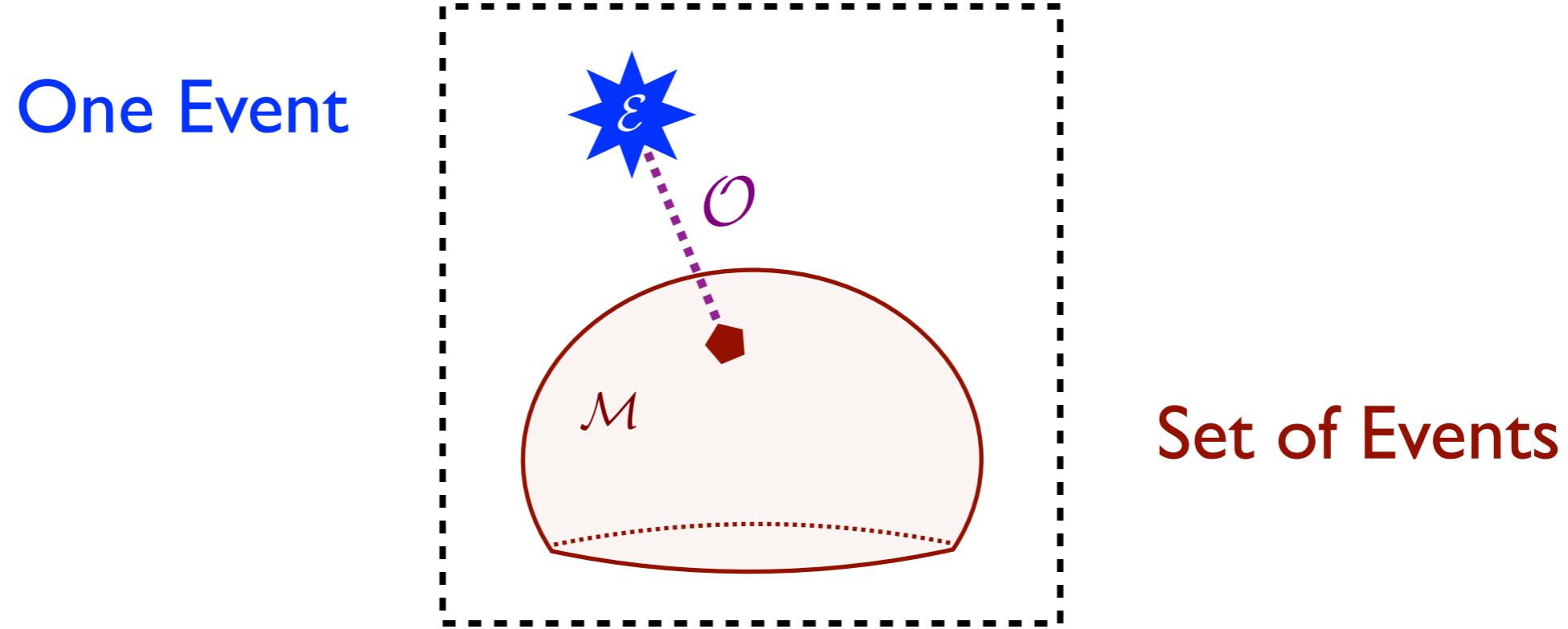
EMD seems to define the “natural” geometry for massless gauge theories

Open question: Can you define $|\mathcal{M}_{AB \rightarrow 12\dots n}|^2$ directly in this space?

What does it mean to “integrate” in this space?

Manifolds for Observables

$$f_{\text{obs}}(\Phi_n)$$



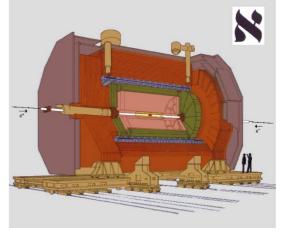
Distance of Closest Approach \Rightarrow Observable

$$O(\mathcal{E}) = \min_{\mathcal{E}' \in \mathcal{M}} \text{EMD}(\mathcal{E}, \mathcal{E}')$$

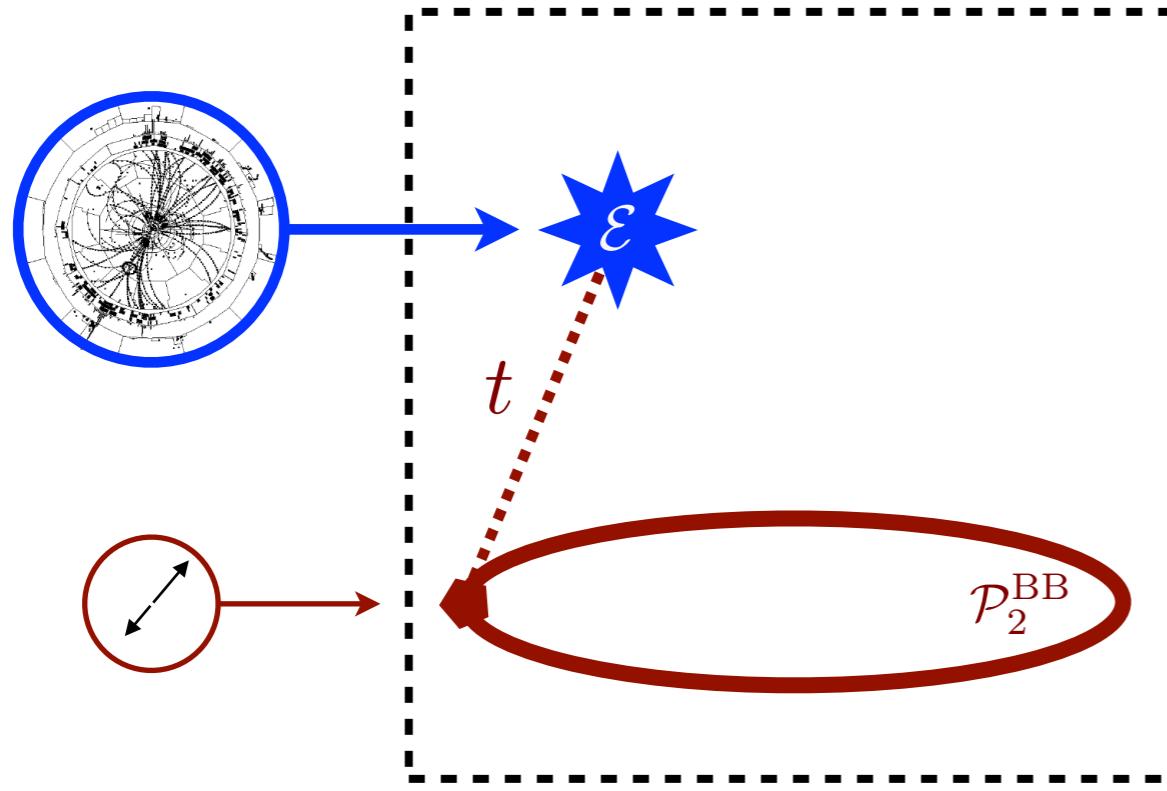
[Komiske, Metodiev, JDT, [JHEP 2020](#)]

E.g. Thrust

How dijet-like is an event?



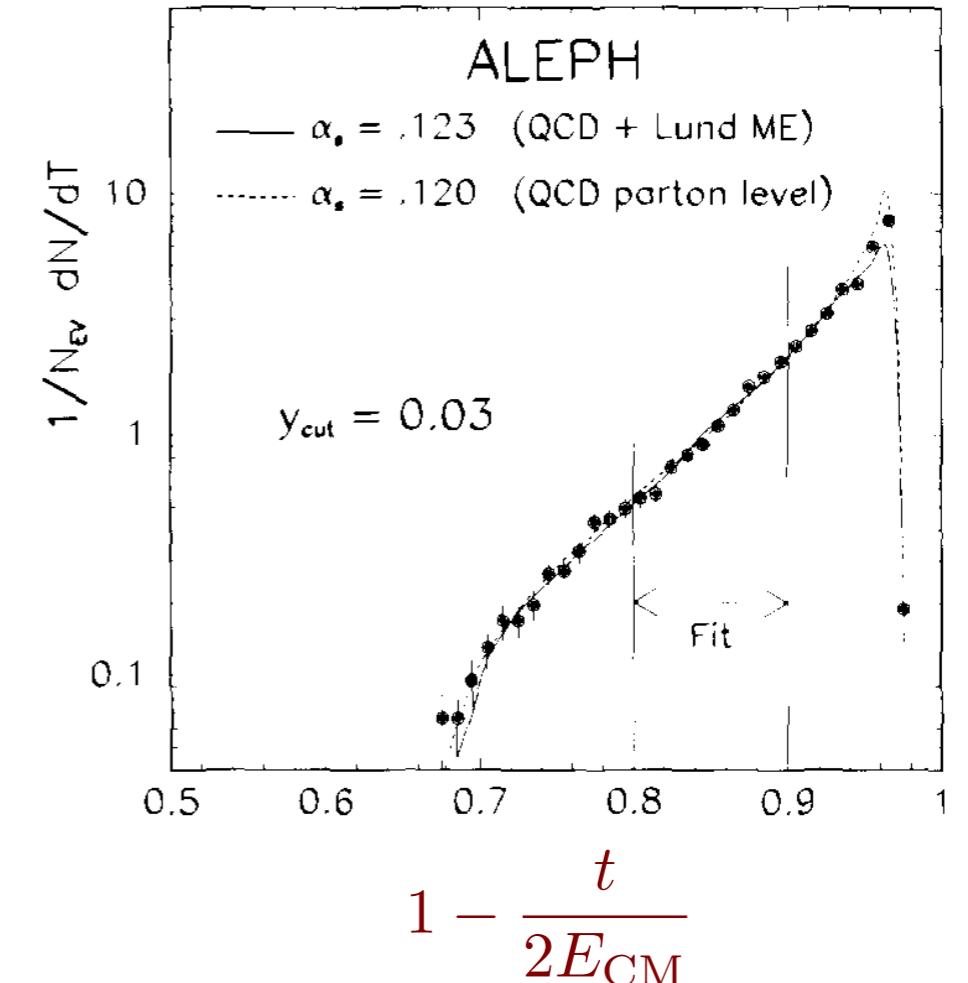
$$t(\mathcal{E}) = \min_{\mathcal{E}' \in \mathcal{P}_2^{\text{BB}}} \text{EMD}_2(\mathcal{E}, \mathcal{E}')$$



All Back-to-Back Two Particle Configurations

$$\mathcal{P}_2^{\text{BB}} = \left\{ \begin{array}{c} \text{red circle with diagonal arrow} \\ \text{red circle with vertical arrow} \\ \text{red circle with horizontal arrow} \\ \text{red circle with double arrows} \end{array} \dots \right\}$$

(using $\beta=2$ EMD variant)



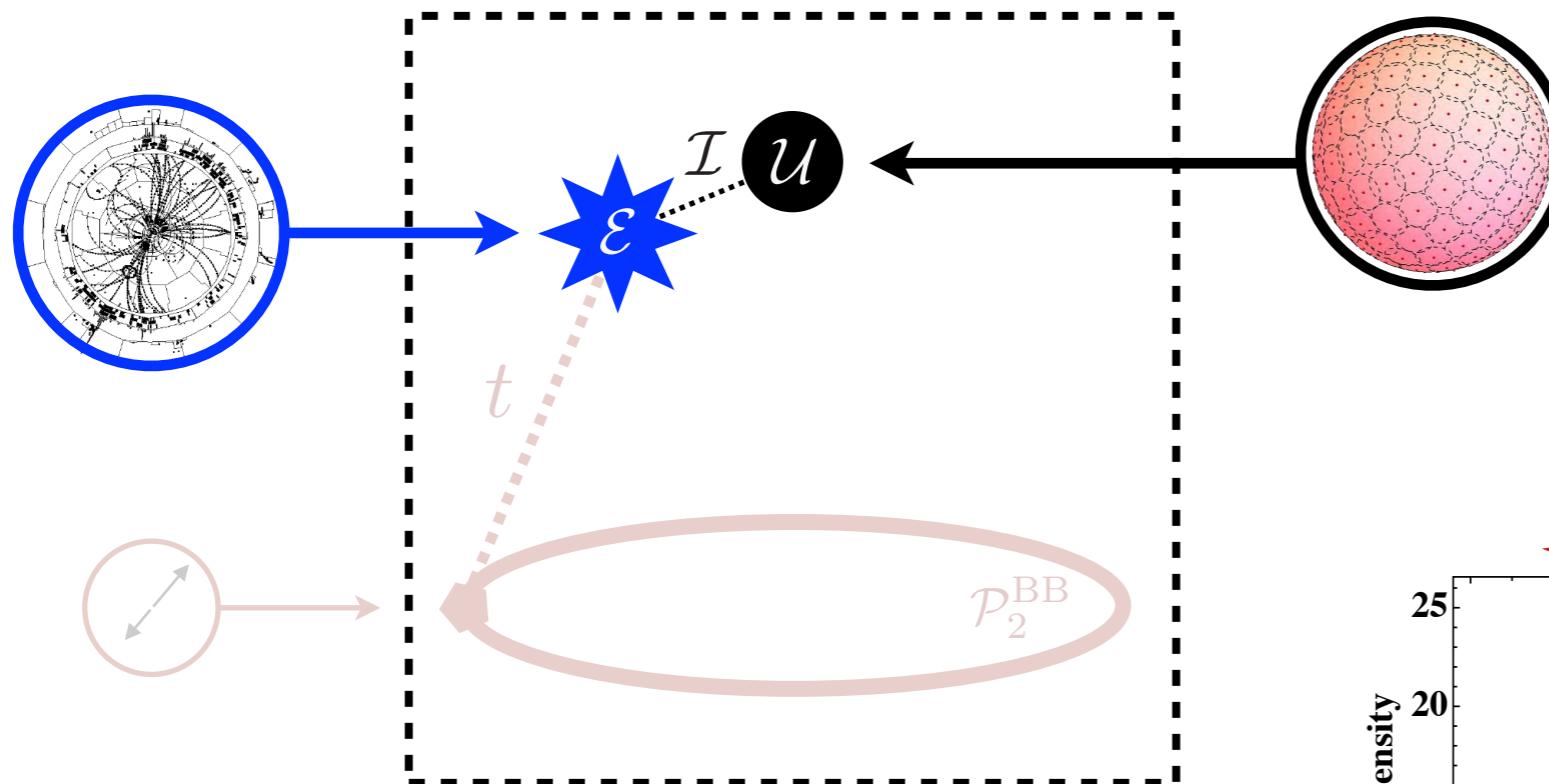
$$1 - \frac{t}{2E_{\text{CM}}}$$

$$\text{cf. } T(\mathcal{E}) = \max_{\hat{n}} \frac{\sum_i |\vec{p}_i \cdot \hat{n}|}{\sum_j |\vec{p}_j|}$$

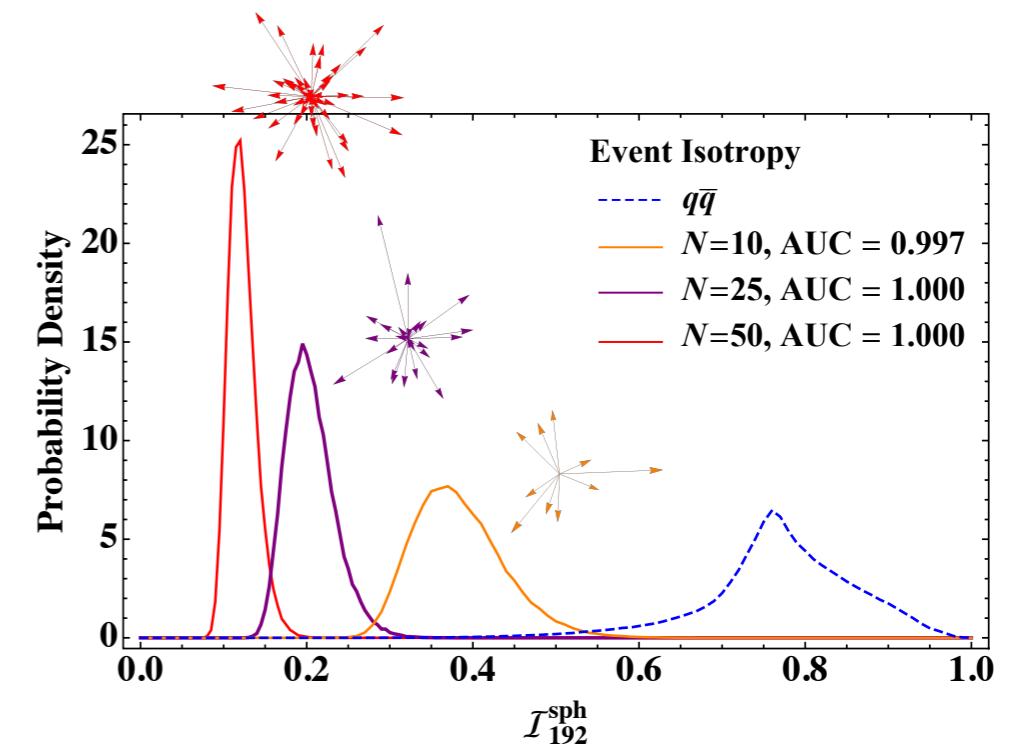
[Komiske, Metodiev, JDT, JHEP 2020]
 [Brandt, Peyrou, Sosnowski, Wroblewski, PL 1964; Farhi, PRL 1977; ALEPH, PLB 1991]

New! Event Isotropy

How isotropic is an event?



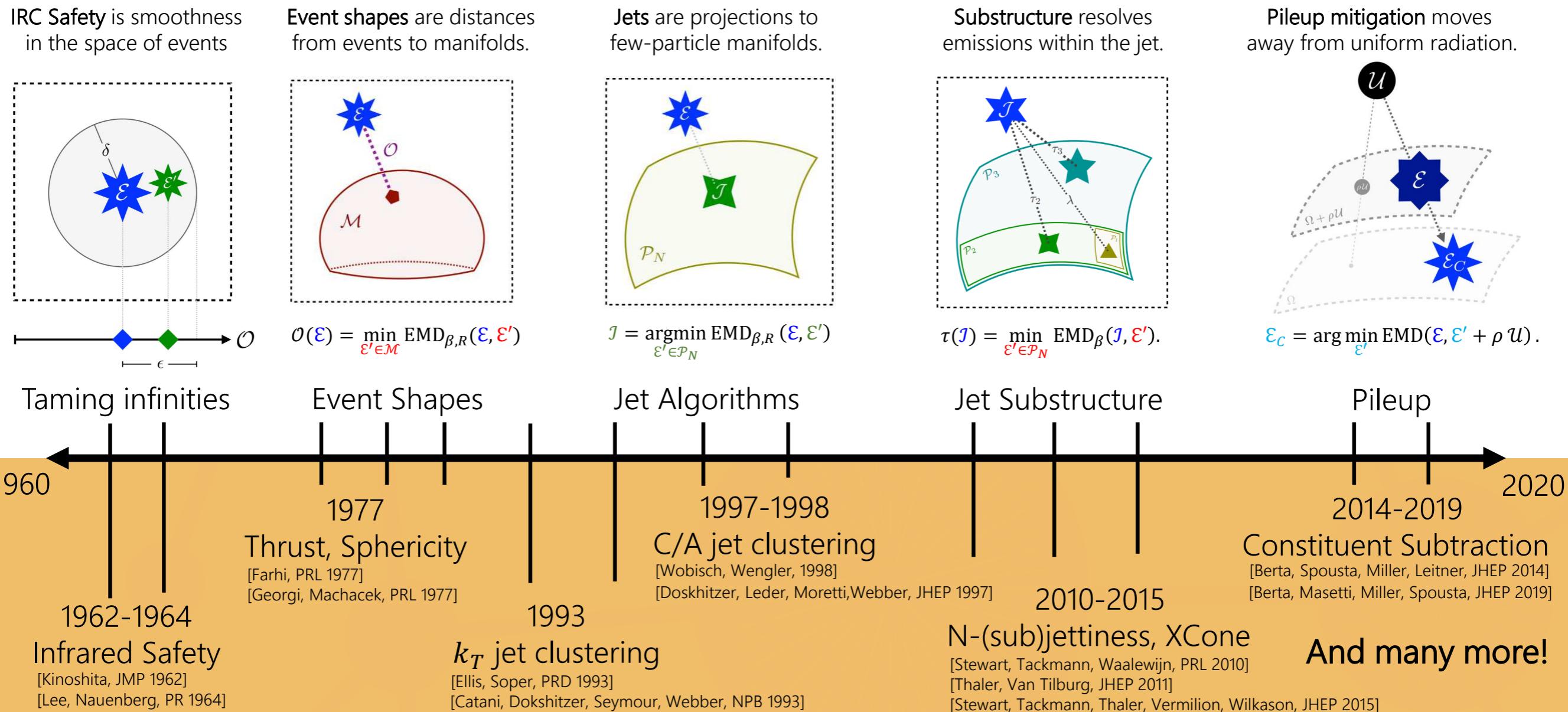
$$\mathcal{I}(\mathcal{E}) = \text{EMD}(\mathcal{E}, \mathcal{U})$$



[Cesarotti, JDT, [JHEP 2020](#);
see also Cesarotti, Reece, Strassler, [arXiv 2020](#)]



Six Decades of Collider Physics Translated into a New Geometric Language!

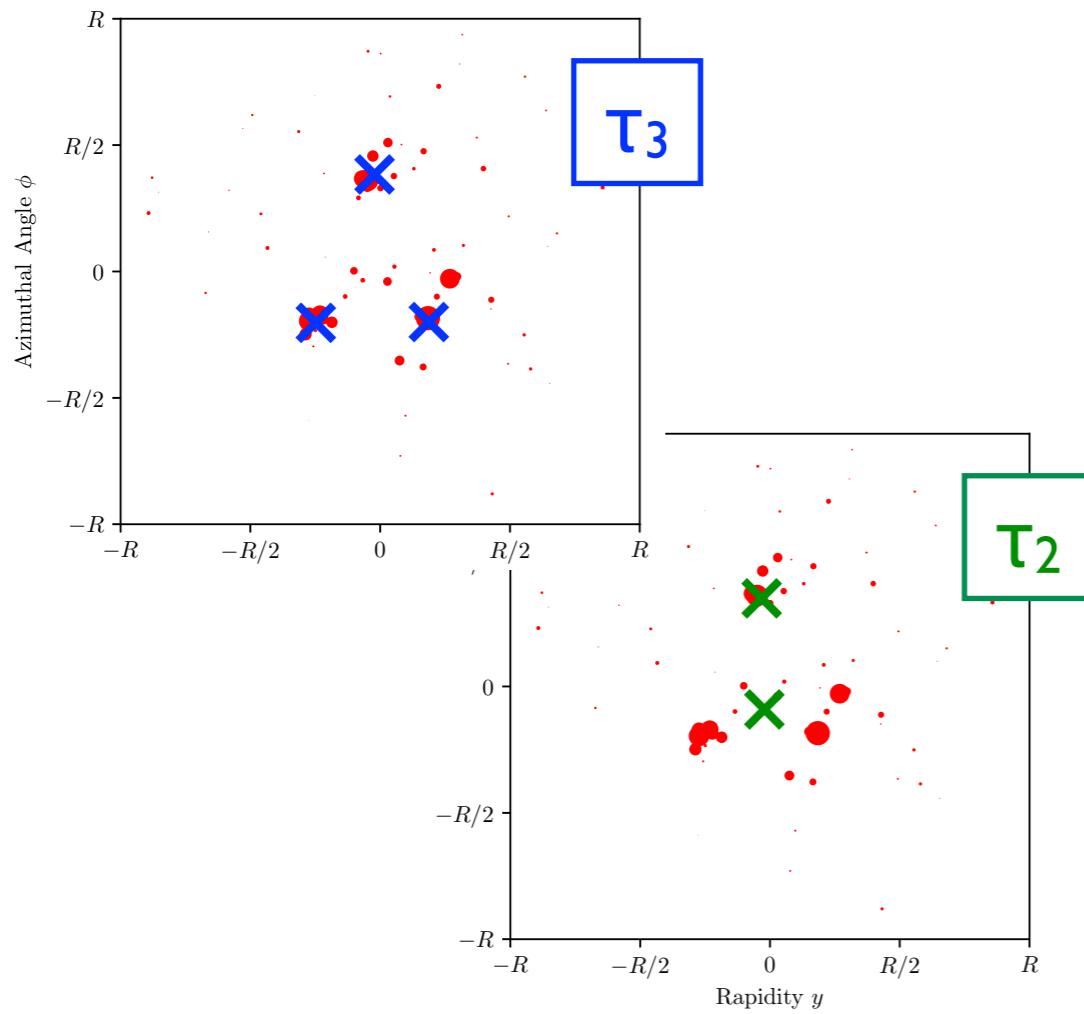


[timeline from Eric Metodiev]

N-subjettiness

Ubiquitous jet substructure observable used for almost a decade...

$$\tau_N(\mathcal{J}) = \min_{N \text{ axes}} \sum_i E_i \min \{\theta_{1,i}, \theta_{2,i}, \dots, \theta_{N,i}\}$$



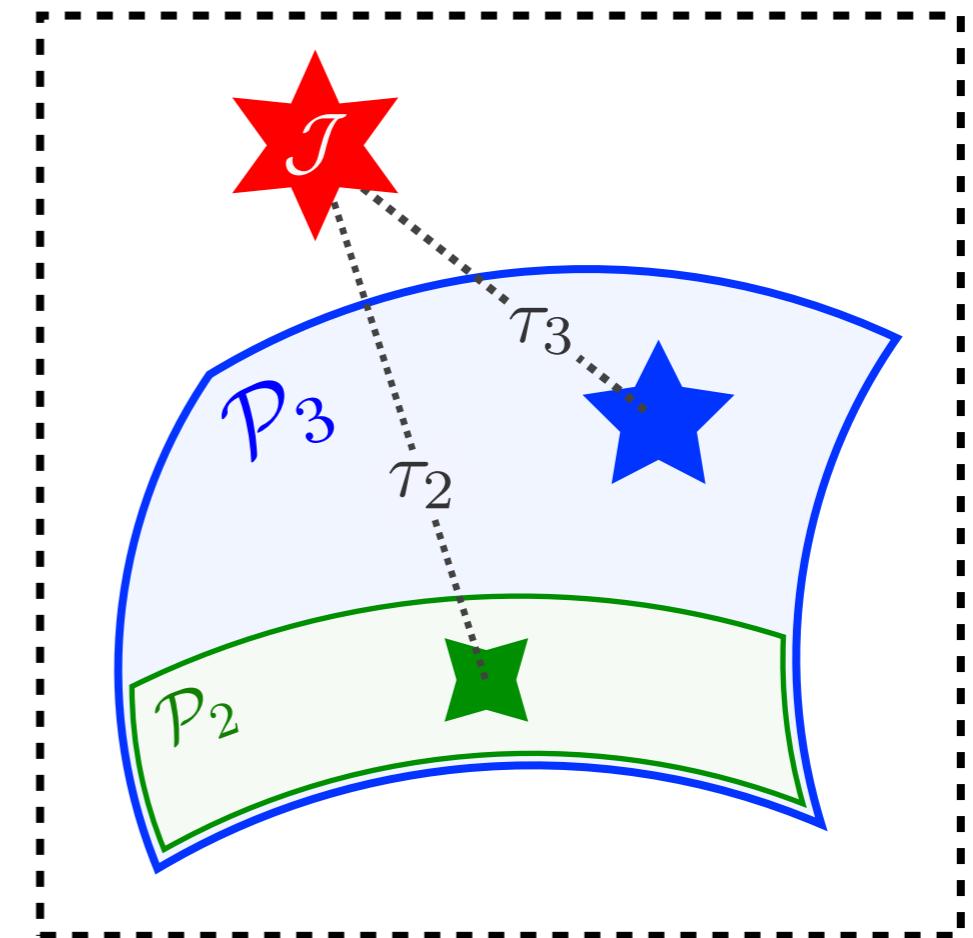
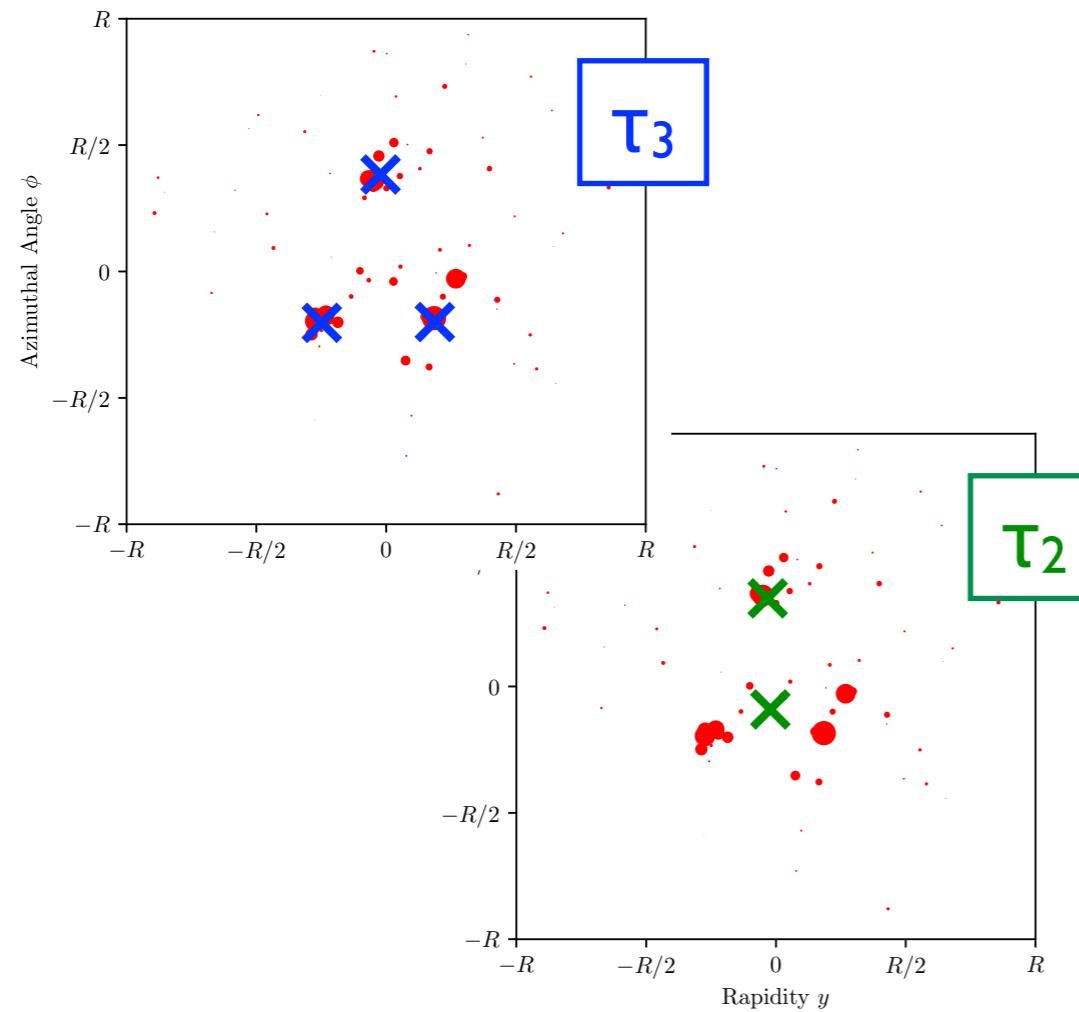
[JDT, Van Tilburg, [JHEP 2011](#), [JHEP 2012](#);
based on Brandt, Dahmen, [ZPC 1979](#); Stewart, Tackmann, Waalewijn, [PRL 2010](#)]



N-subjettiness = Point to Manifold EMD

...is secretly an optimal transport problem

$$\tau_N(\mathcal{J}) = \min_{\mathcal{J}' \in \mathcal{P}_N} \text{EMD}(\mathcal{J}, \mathcal{J}')$$

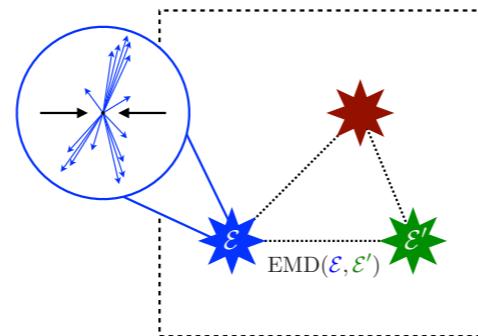


[JDT, Van Tilburg, JHEP 2011, JHEP 2012;
rephrased in the language of Komiske, Metodiev, JDT, PRL 2019]



Pause

What can be Geometrized?



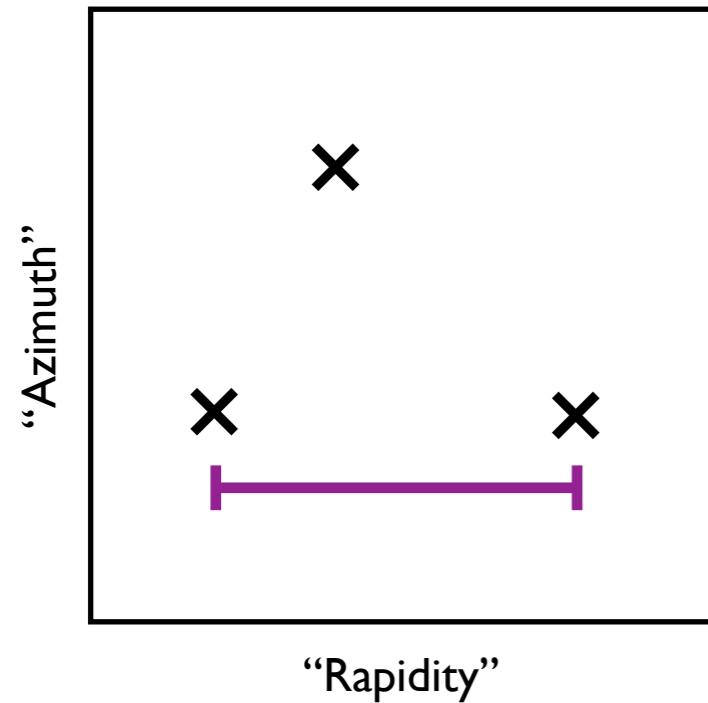
**IRC Safety, Observables,
Jet Algorithms, Pileup Mitigation**

And now, the grand finale...



How far down does this rabbit hole go?

Direction Space



x = Direction

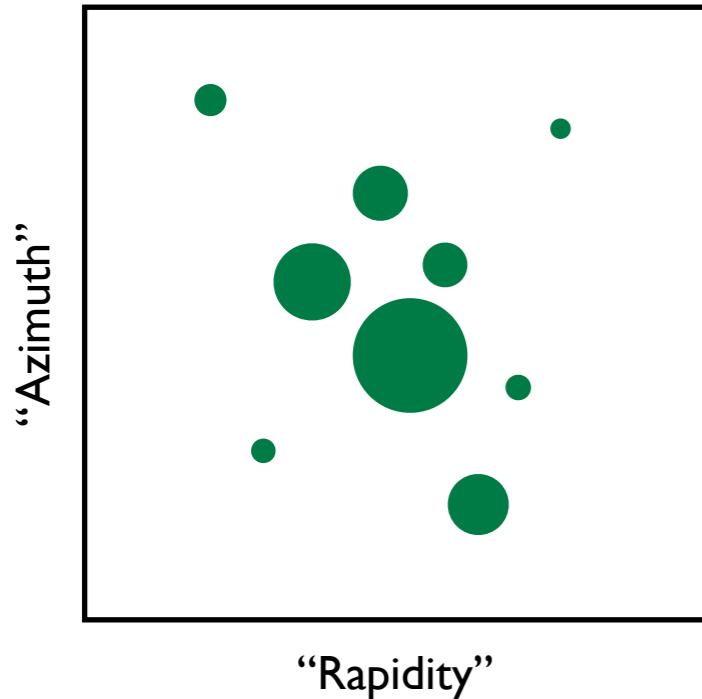
= Angular Distance

$$n_i^\mu = \frac{p_i^\mu}{E_i} = (1, \hat{n})^\mu$$

$$\theta_{ij} = \sqrt{2n_i^\mu n_{j\mu}}$$

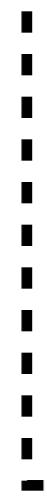
(for massless particles)

Direction Space Distribution



● = Weighted Direction

— = Angular Distance



★ = Event

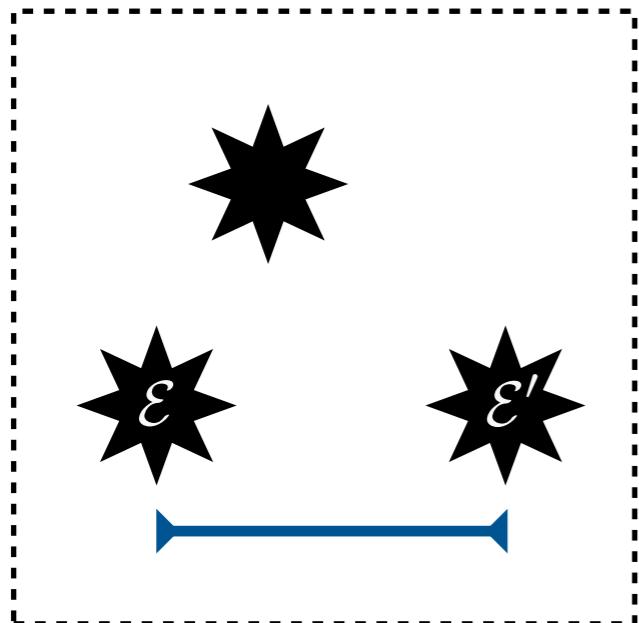
$$n_i^\mu = \frac{p_i^\mu}{E_i} = (1, \hat{n})^\mu$$

$$w_i = E_i$$

$$\theta_{ij} = \sqrt{2n_i^\mu n_{j\mu}}$$

(for massless particles)

Event Space



 = Event
 = EMD
Energy
Mover's Distance

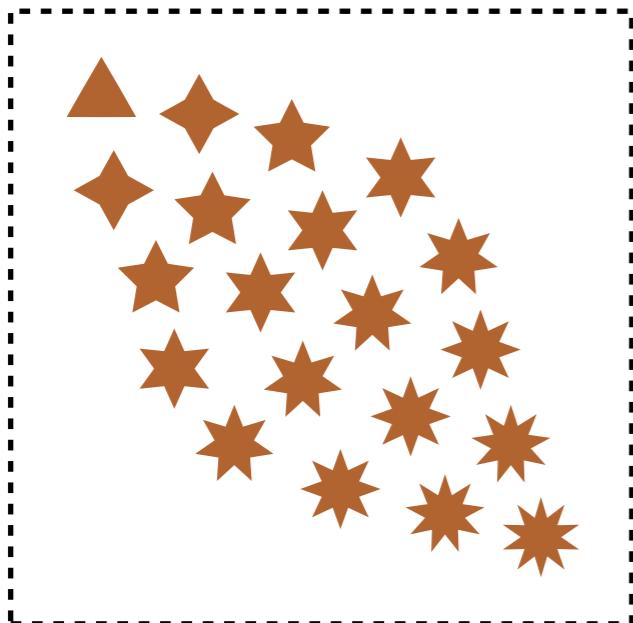
$$\mathcal{E}(\hat{n}) = \sum_i E_i \delta(\hat{n} - \hat{n}_i)$$

$$\text{EMD}(\mathcal{E}, \mathcal{E}') = \min_{\{f\}} \sum_i \sum_j f_{ij} \theta_{ij}$$

(for equal total energy)

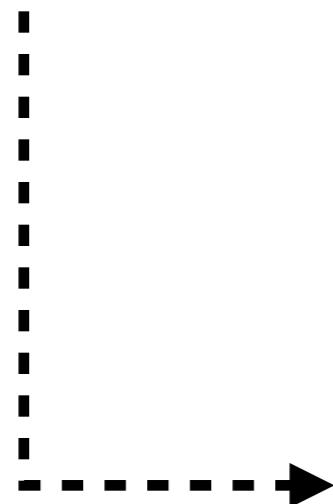
[Komiske, Metodiev, JDT, PRL 2019]

Event Space Distribution



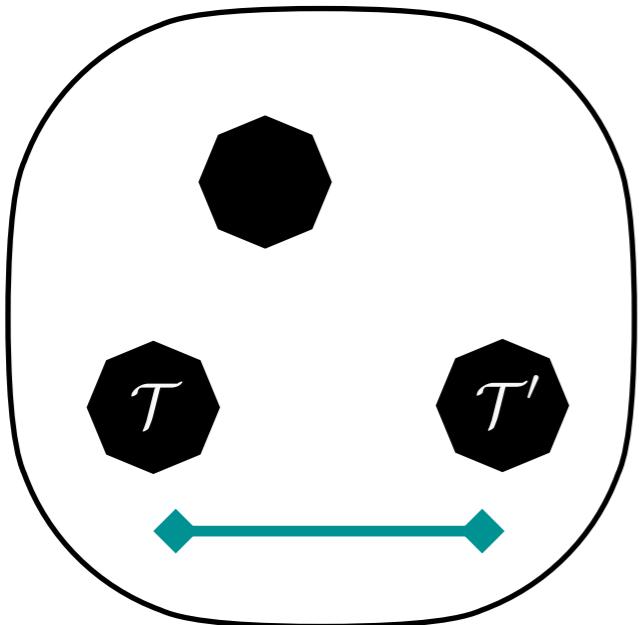
= **Weighted Event** $\mathcal{E}(\hat{n}) = \sum_i E_i \delta(\hat{n} - \hat{n}_i)$
 $w_a = \sigma_a$

= **EMD**
Energy
Mover's Distance $\text{EMD}(\mathcal{E}, \mathcal{E}') = \min_{\{f\}} \sum_i \sum_j f_{ij} \theta_{ij}$
(for equal total energy)



= **Theory**

Theory Space



● = Theory

↔ = ΣMD
Cross-Section
Mover's Distance

$$\mathcal{T}(\mathcal{E}) = \sum_a \sigma_a \delta(\mathcal{E} - \mathcal{E}_a)$$

$$\Sigma\text{MD}(\mathcal{T}, \mathcal{T}') = \min_{\{\mathcal{F}\}} \sum_a \sum_b \mathcal{F}_{ab} \text{EMD}(\mathcal{E}_a, \mathcal{E}'_b)$$

(for equal total xsec)

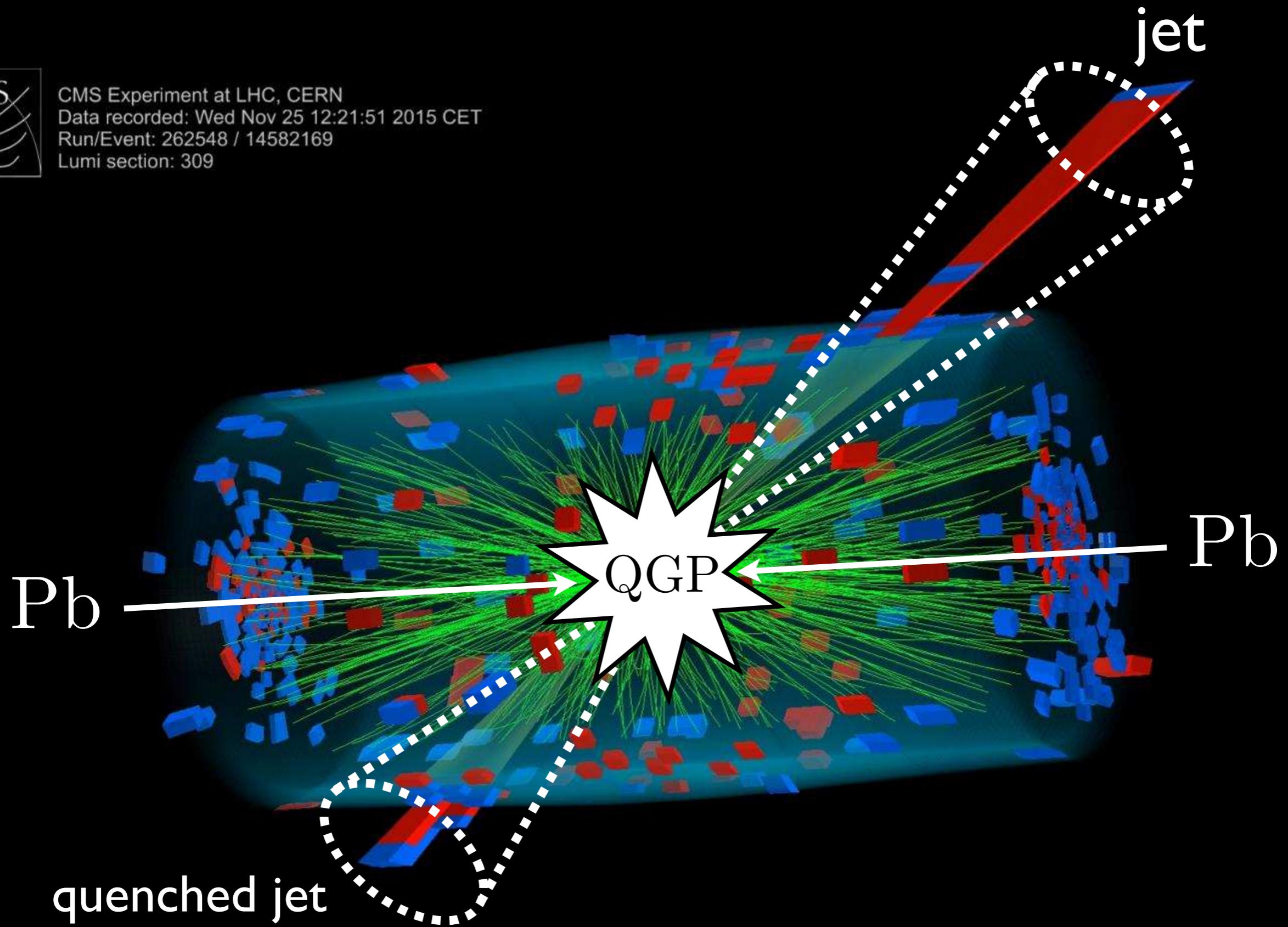
A distance between theories!

(e.g. EMD : N-jettiness :: ΣMD : k-eventiness)

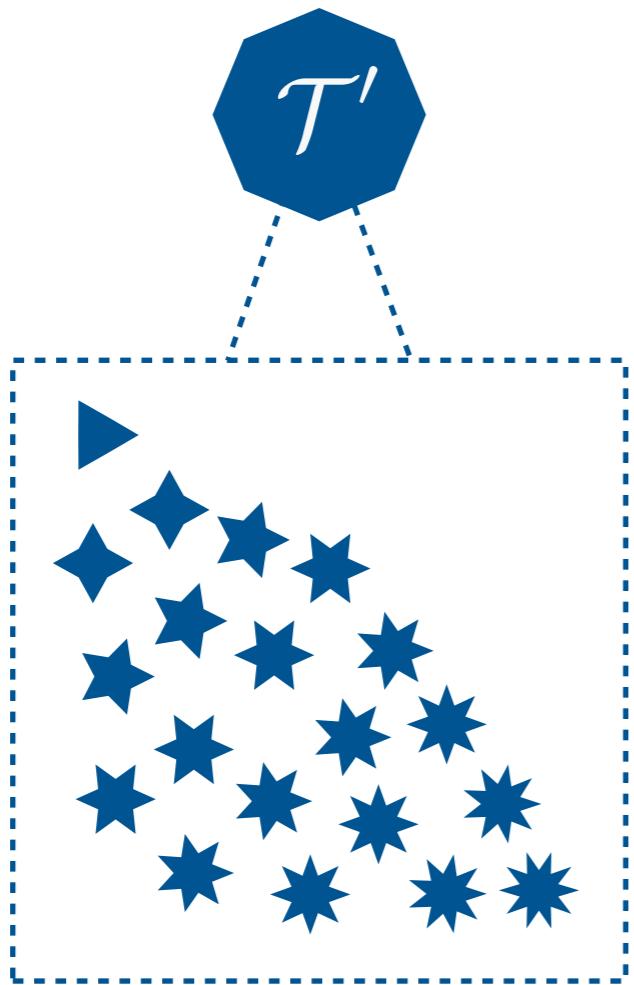
[Komiske, Metodiev, JDT, [JHEP 2020](#)]



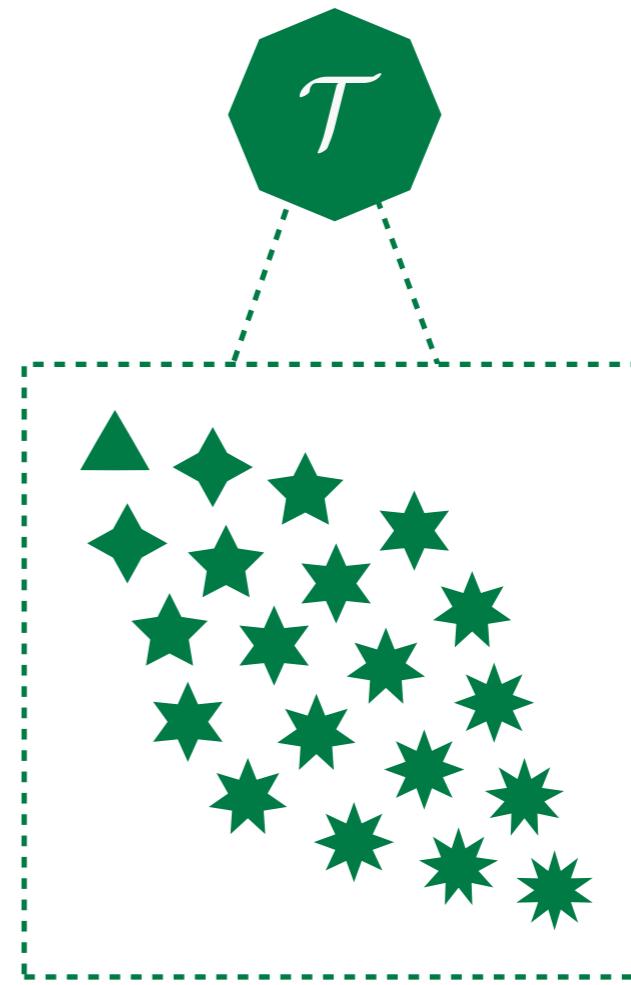
CMS Experiment at LHC, CERN
Data recorded: Wed Nov 25 12:21:51 2015 CET
Run/Event: 262548 / 14582169
Lumi section: 309



Theory Prime: In-Medium QCD



Theory: Vacuum QCD

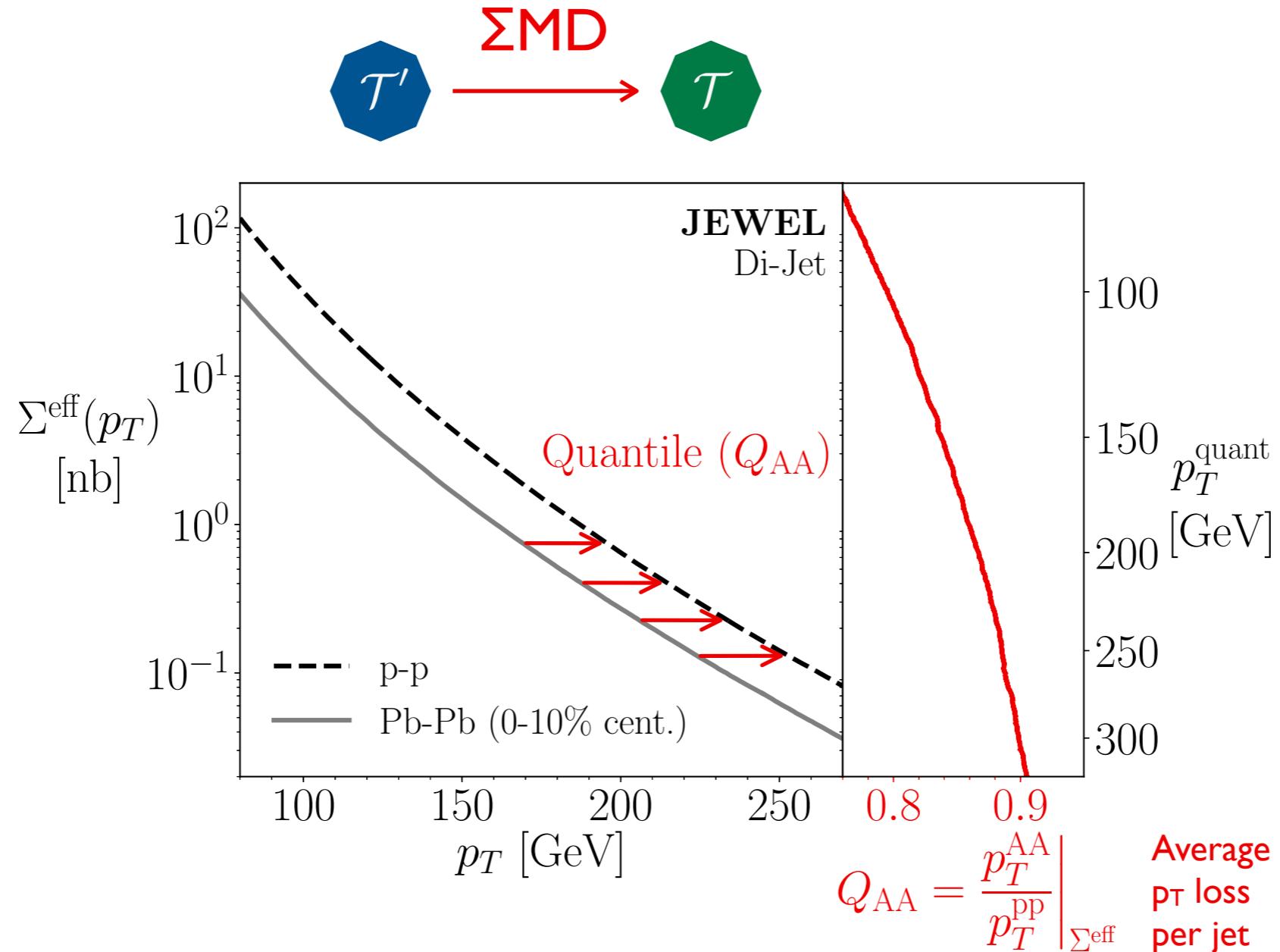


ΣMD
 \Updownarrow

*Optimal transportation plan defines mapping
between in-medium jets and vacuum jets!*

Jet Quenching via Quantile Matching

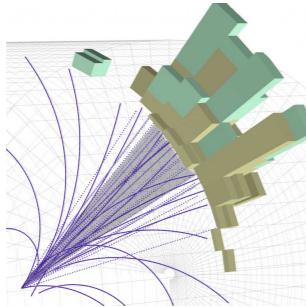
Equivalent to following a geodesic in theory space (!)



[Brewer, Milhano, JDT, PRL 2019]



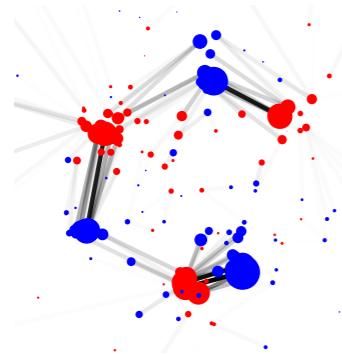
Summary



What is a Collider Event?

(ask me about QCD/ML relevance)

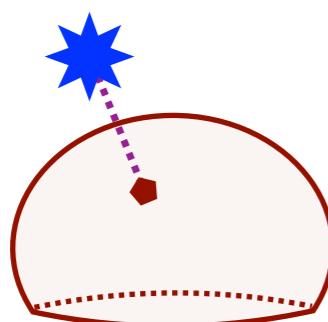
An unordered set of particles that describes the energy flow away from the collision point



When are Events Similar?

(ask me more about EMD)

When they are close in the geometric space triangulated by the energy mover's distance



What can be Geometrized?

(ask me more about safety)

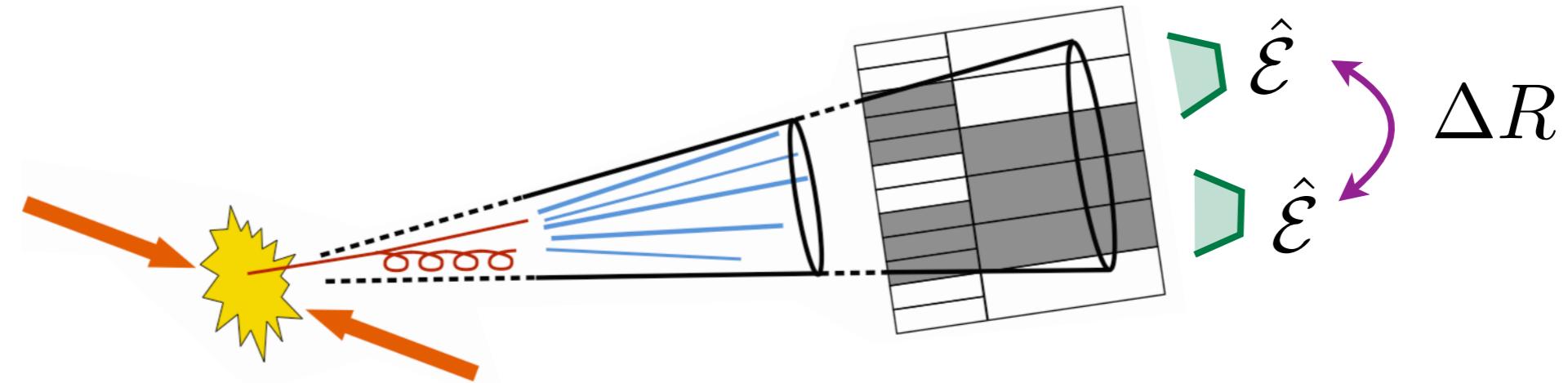
Many concepts/techniques in quantum field theory and collider physics from the last half century

Fin

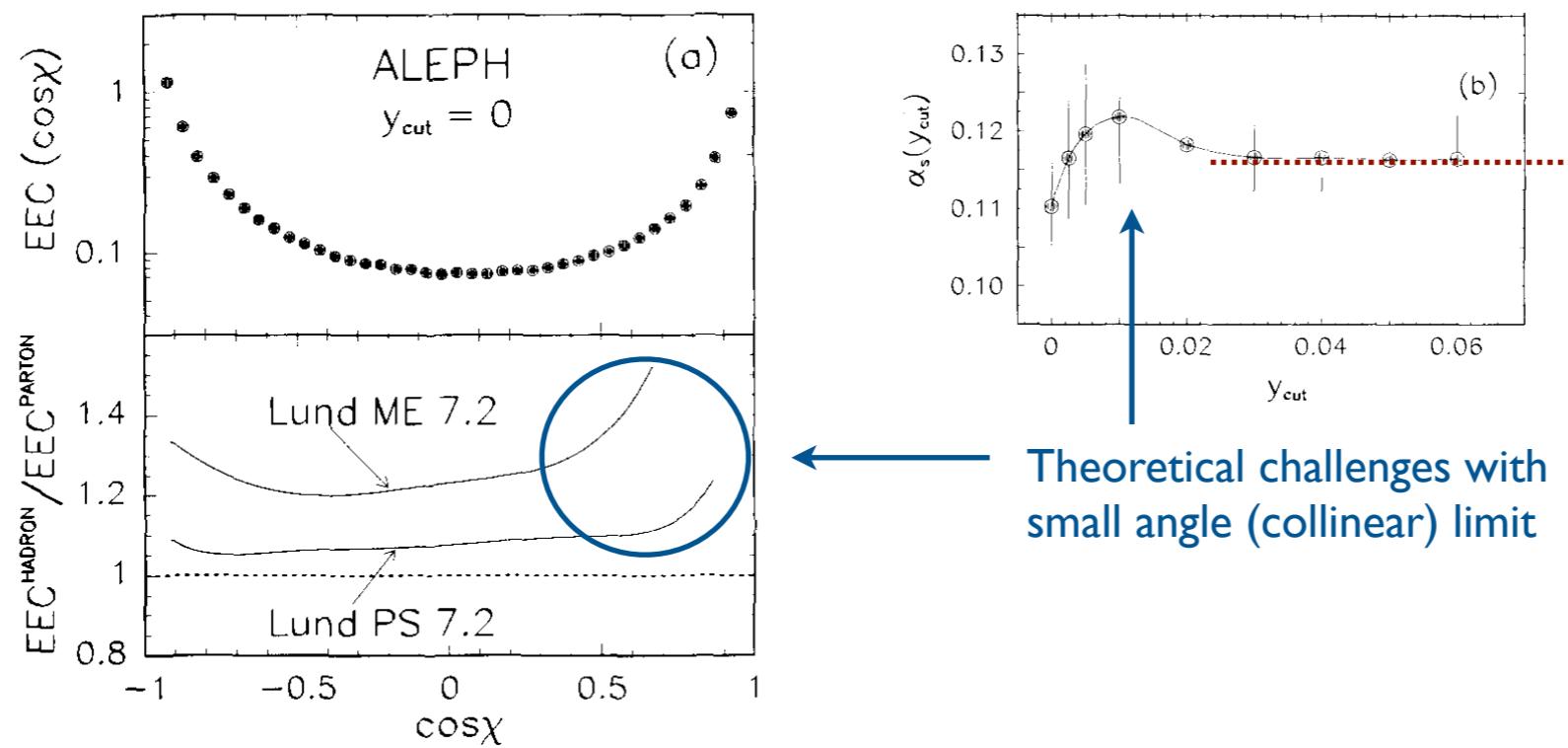
Whew! ↘
46 of 48

Backup Slides

Energy-Energy Correlators

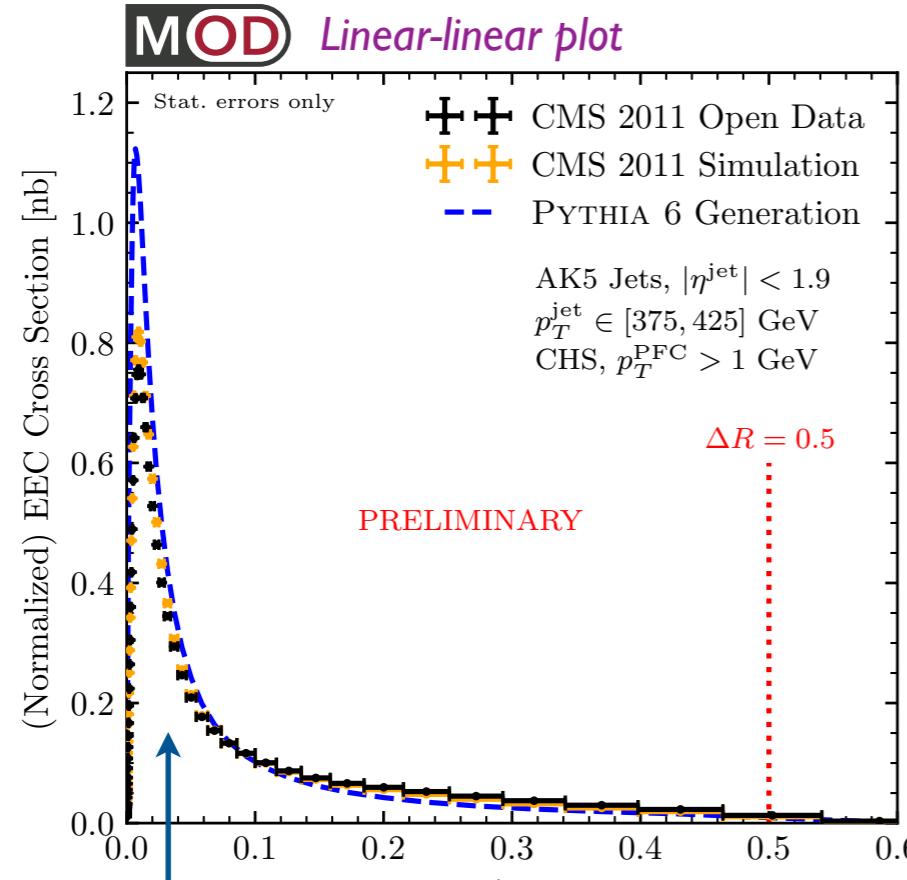


A long history in probing collinear dynamics of QCD



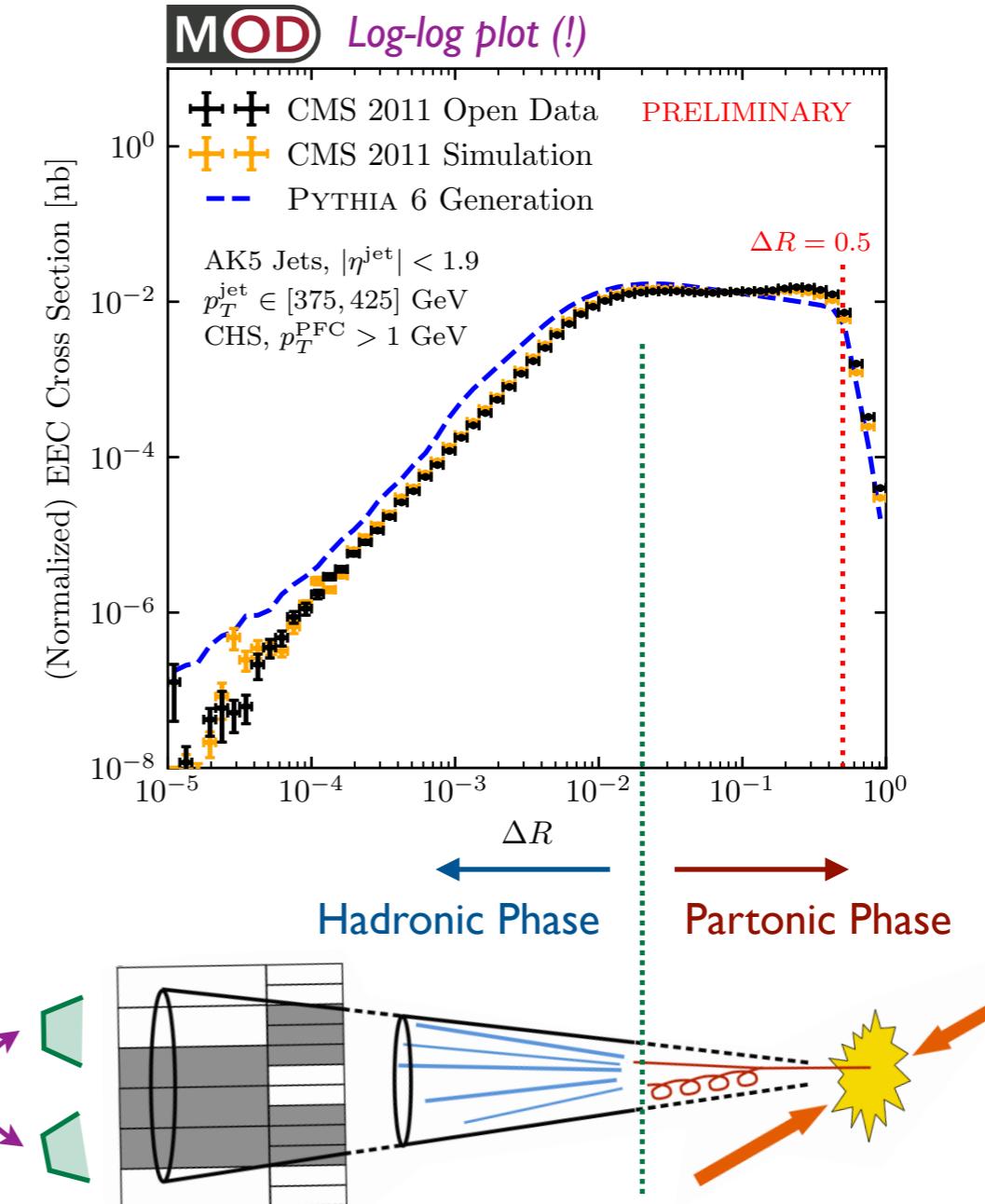
[Basham, Brown, Ellis, Love, [PRL 1978](#); ALEPH, [PLB 1991](#); see Chen, Moult, Zhang, Zhu, [PRD 2020](#)]

QCD Phase Transition in Jets?



Are we learning something about small angle limit of QCD?

First Jet EEC Plot from the LHC (!)



[Komiske, Moult, JDT, Zhu, in progress; see talks by Moult, [BOOST 2019](#), [BOOST 2020](#)]



Point Cloud

Collection of points in position space



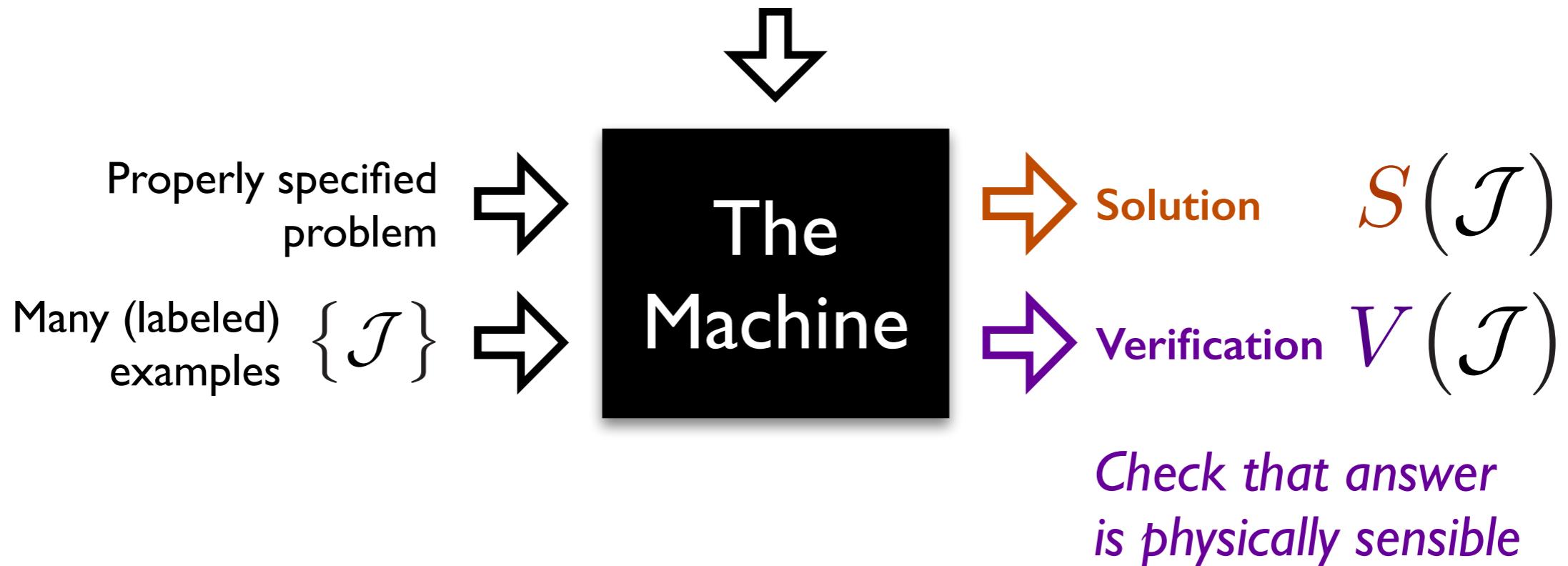
[Popular Science, 2013]

Aside: Machine Learning for Jets

“ML4Jets”
NYU, January 2020

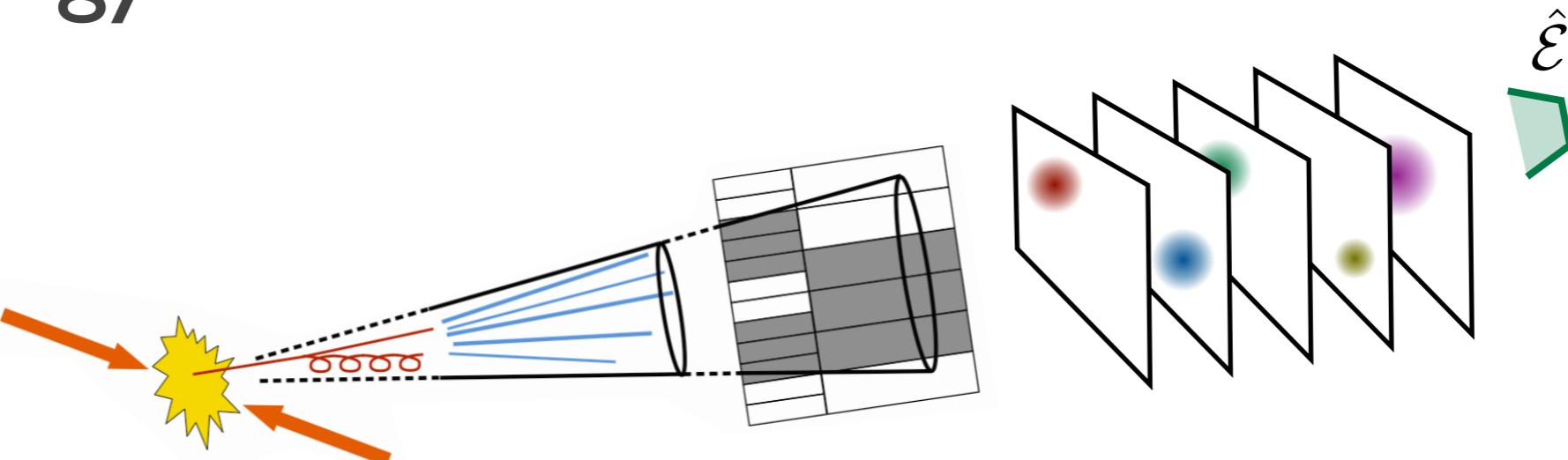
Symmetry: $\mathcal{J} = \{ \vec{p}_1, \vec{p}_2, \vec{p}_3, \dots, \vec{p}_N \}$
Unordered, Variable Length Set (QM!)

Safety: $\vec{p} = \{ E, \hat{n}_x, \hat{n}_y, \hat{n}_z \}$
Energy weighting (QFT!)

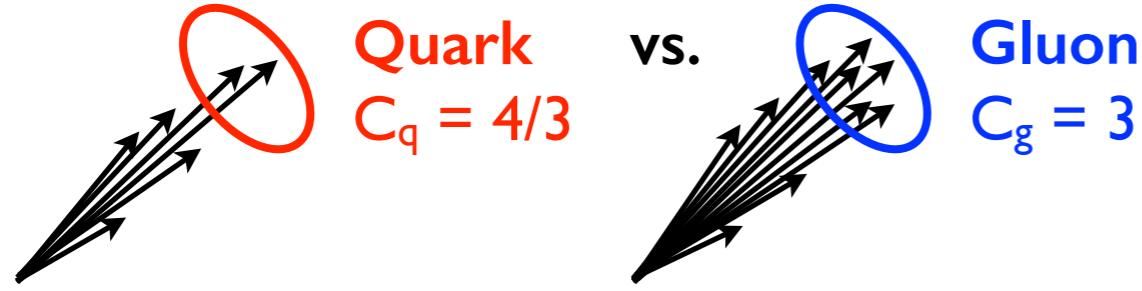


*Check that answer
is physically sensible*

E.g. Energy Flow Networks

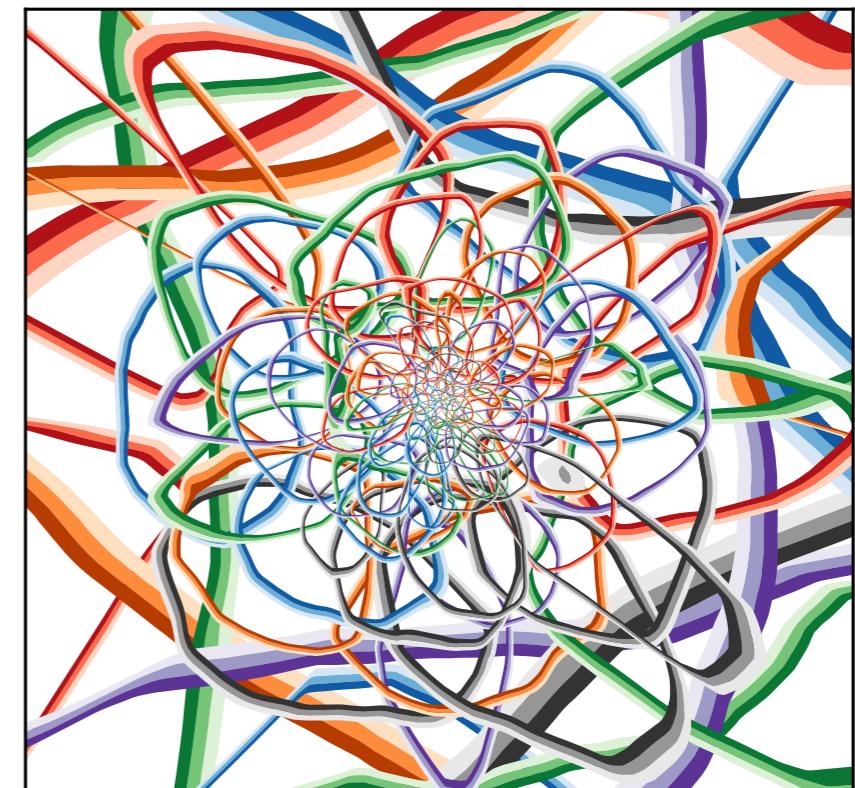
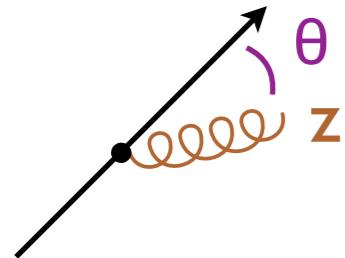


Learning QCD singularities!



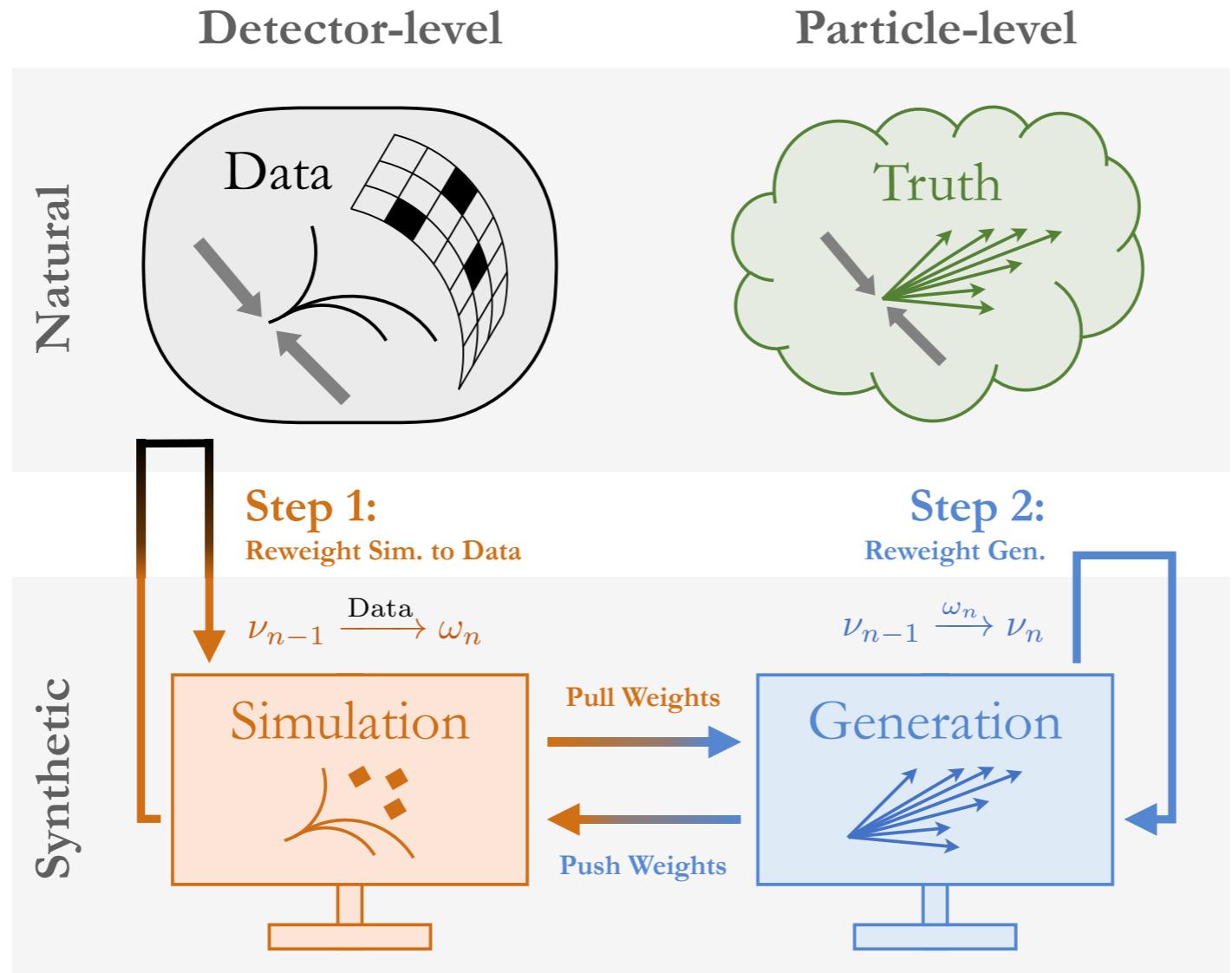
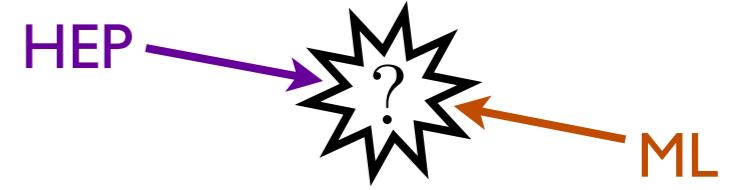
AP splitting probability:

$$dP_{i \rightarrow ig} \simeq \frac{2\alpha_s}{\pi} C_i \frac{d\theta}{\theta} \frac{dz}{z}$$



[Komiske, Metodiev, JDT, [JHEP 2019](#); see also Komiske, Metodiev, JDT, [JHEP 2018](#), [PRD 2020](#); special case of Zaheer, Kottur, Ravanbakhsh, Poczos, Salakhutdinov, Smola, [NIPS 2017](#)]

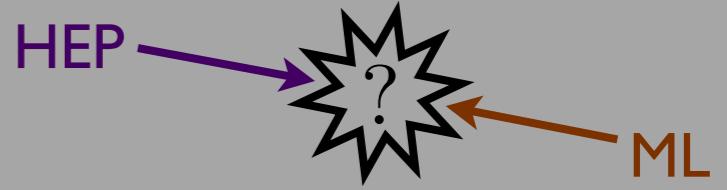
OmniFold



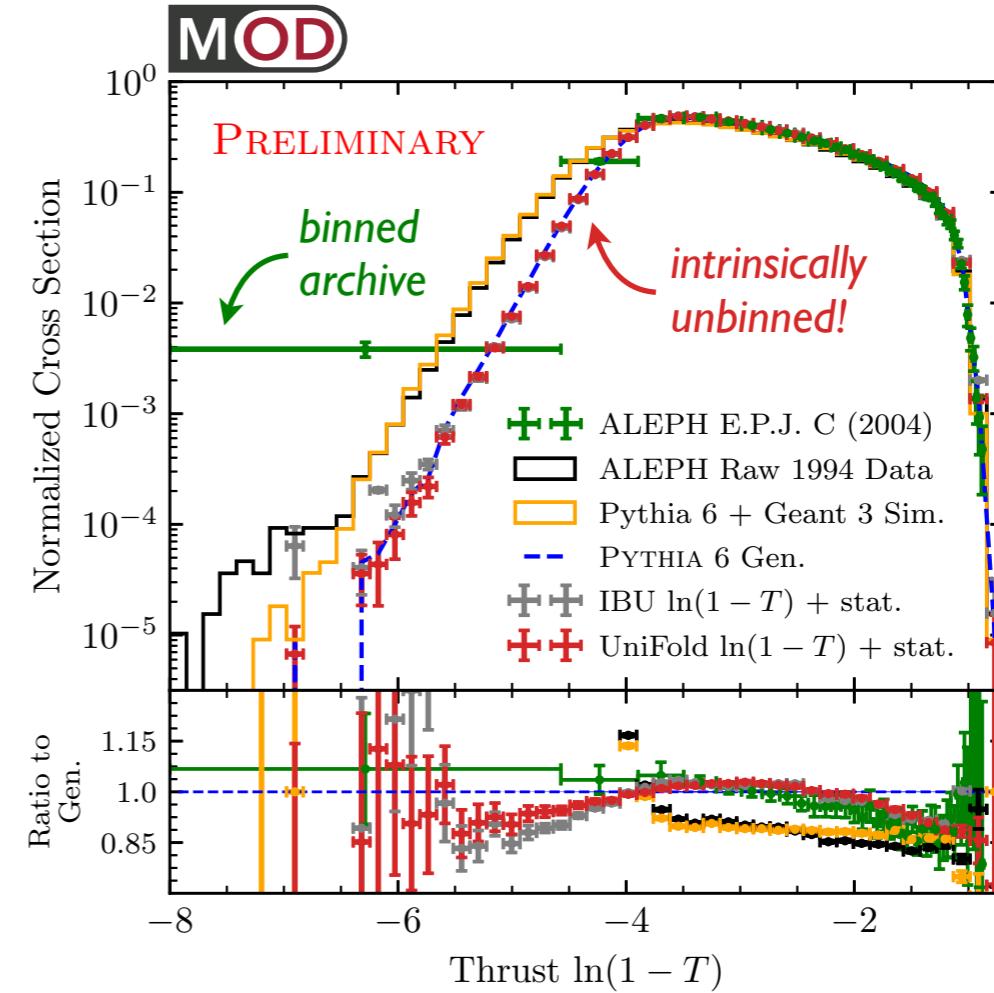
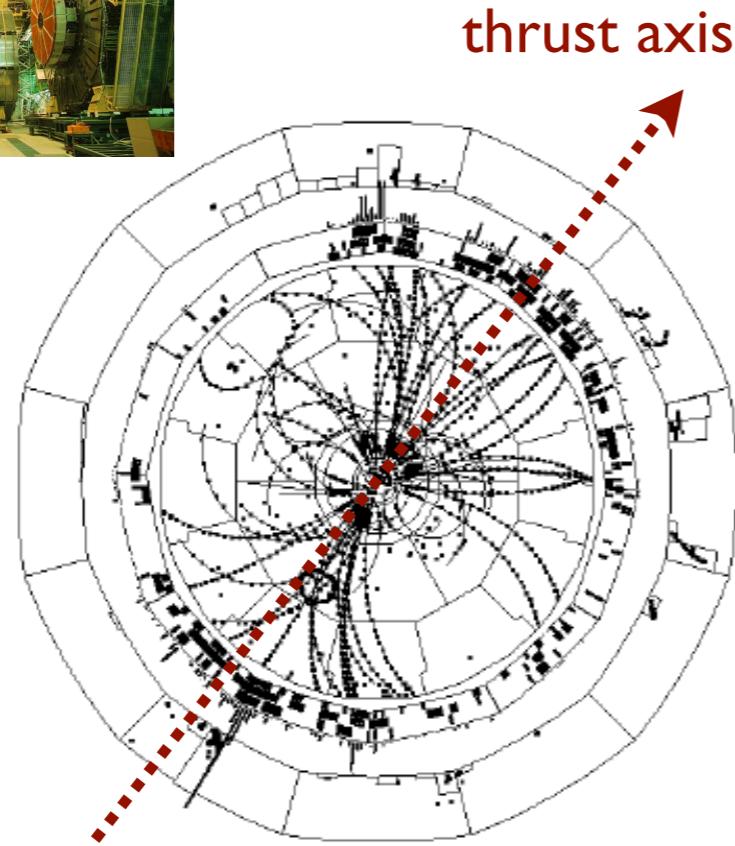
Multi-dimensional unbinned detector corrections via iterated binary classification

[Andreassen, Komiske, Metodiev, Nachman, JDT, [PRL 2020](#)]





Back to the Future with ALEPH Archival Data



[talk by Badea, [ICHEP 2020](#); cf. ALEPH, [EPJC 2004](#)
 [see also Badea, Baty, Chang, Innocenti, Maggi, McGinn, Peters, Sheng, JDT, Lee, [PRL 2019](#)]

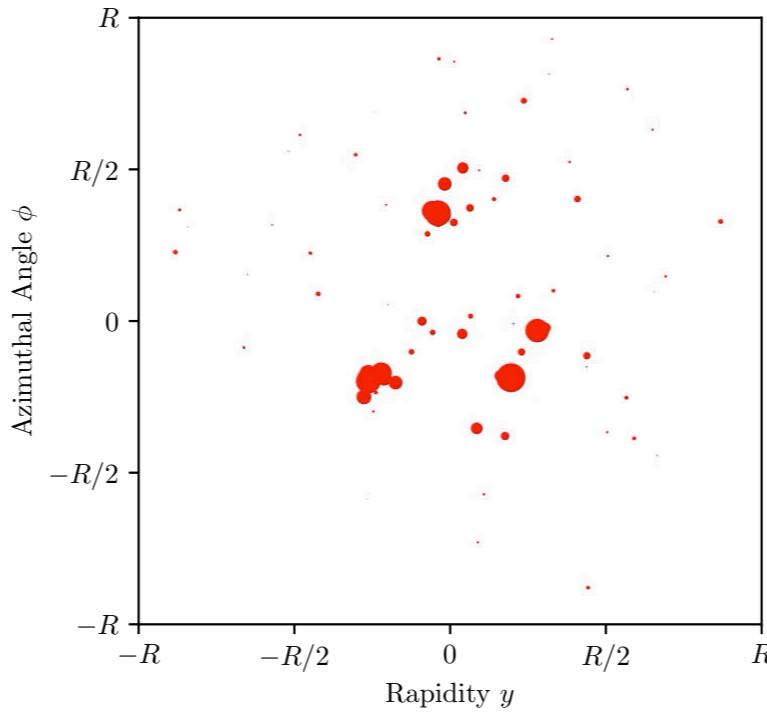


[Andreassen, Komiske, Metodiev, Nachman, JDT, [PRL 2020](#)]

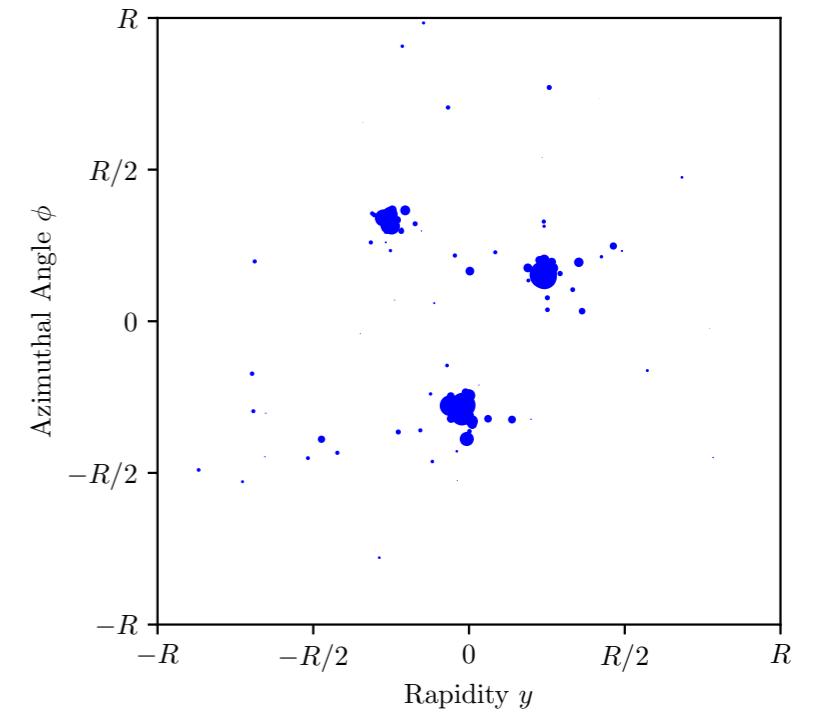
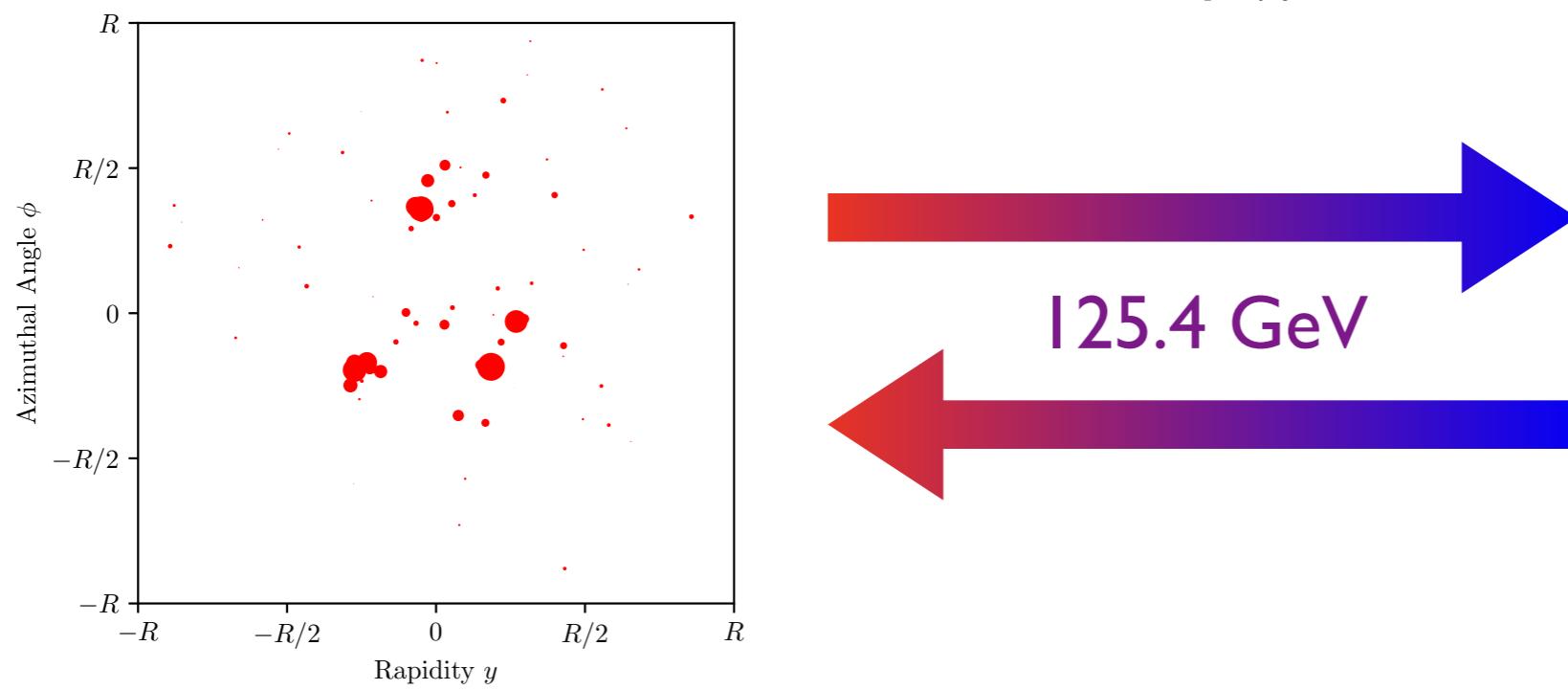


Similarity of Two Energy Flows?

$$\mathcal{E}(\hat{n}) = \sum_i E_i \delta(\hat{n} - \hat{n}_i)$$



Optimal Transport:
Earth Mover's Distance
a.k.a. *1-Wasserstein metric*



[Komiske, Metodiev, JDT, PRL 2019; code at Komiske, Metodiev, JDT, [energyflow.network](#)]

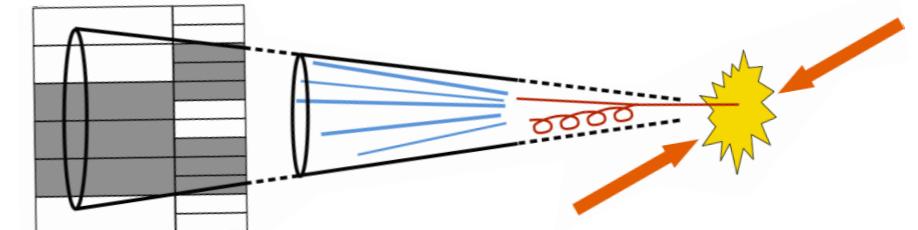
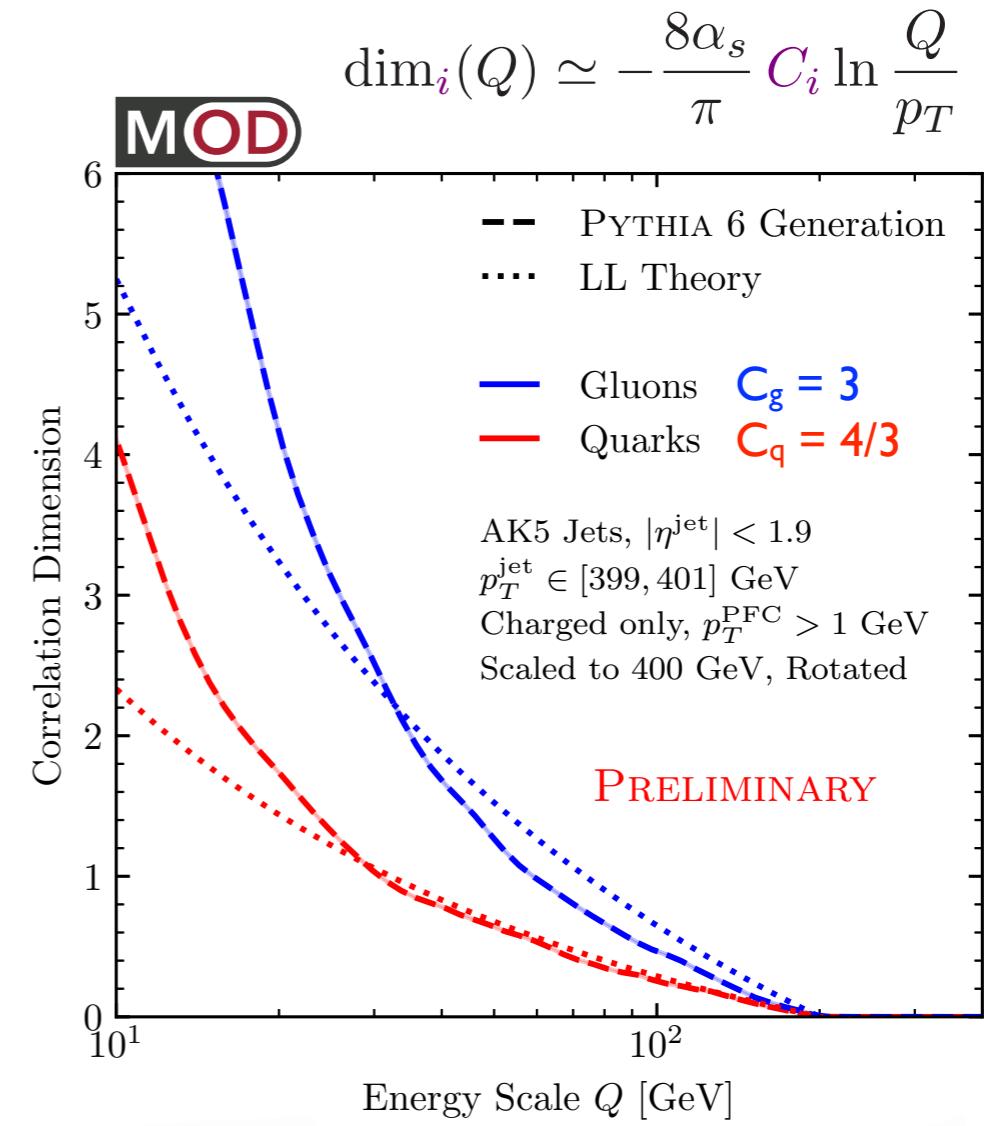
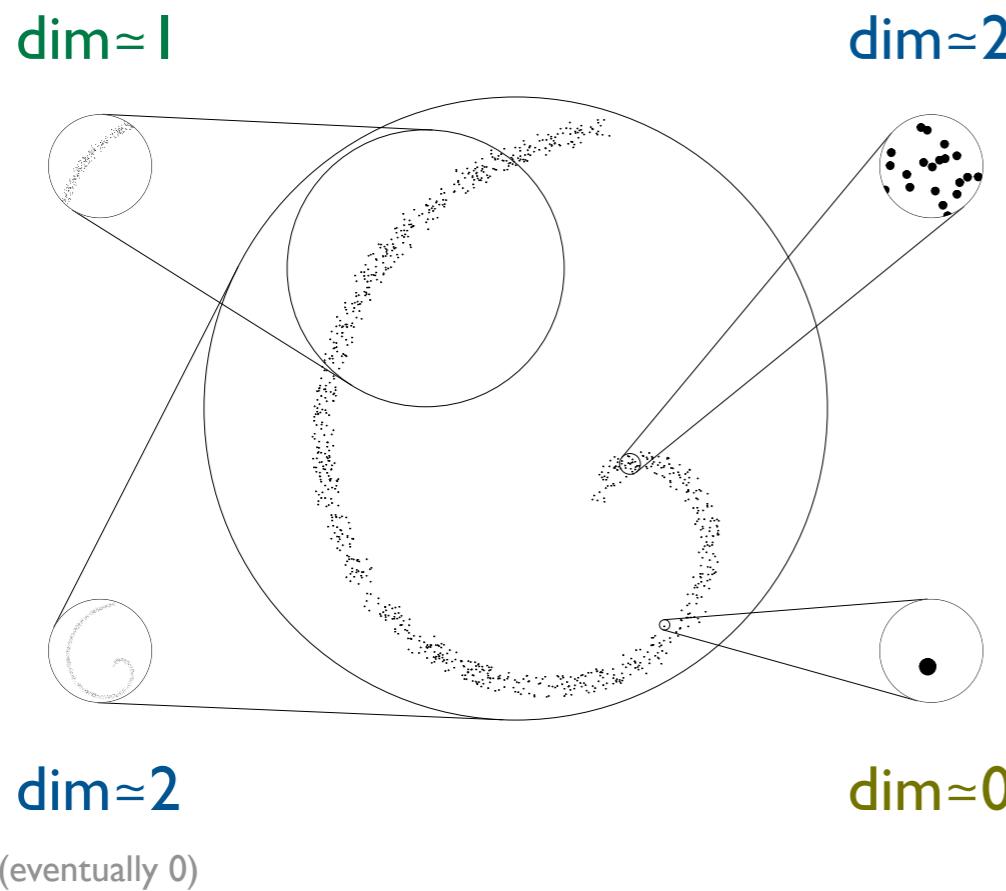
Dimensionality of Space of Jets



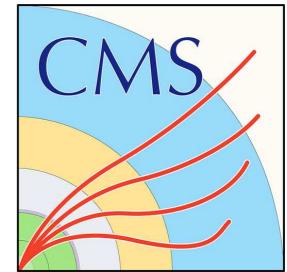
$$N_{\text{neighbors}}(r) \sim r^{\dim}$$

$$\Rightarrow \dim(r) \sim r \frac{\partial}{\partial r} \ln N_{\text{neighbors}}(r)$$

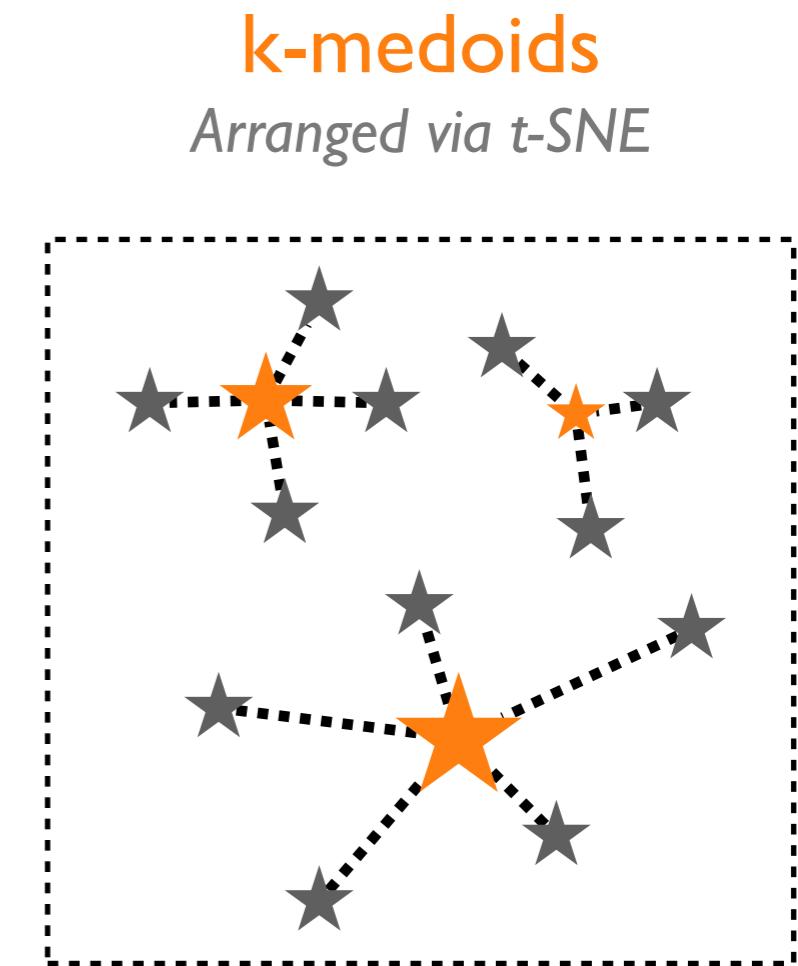
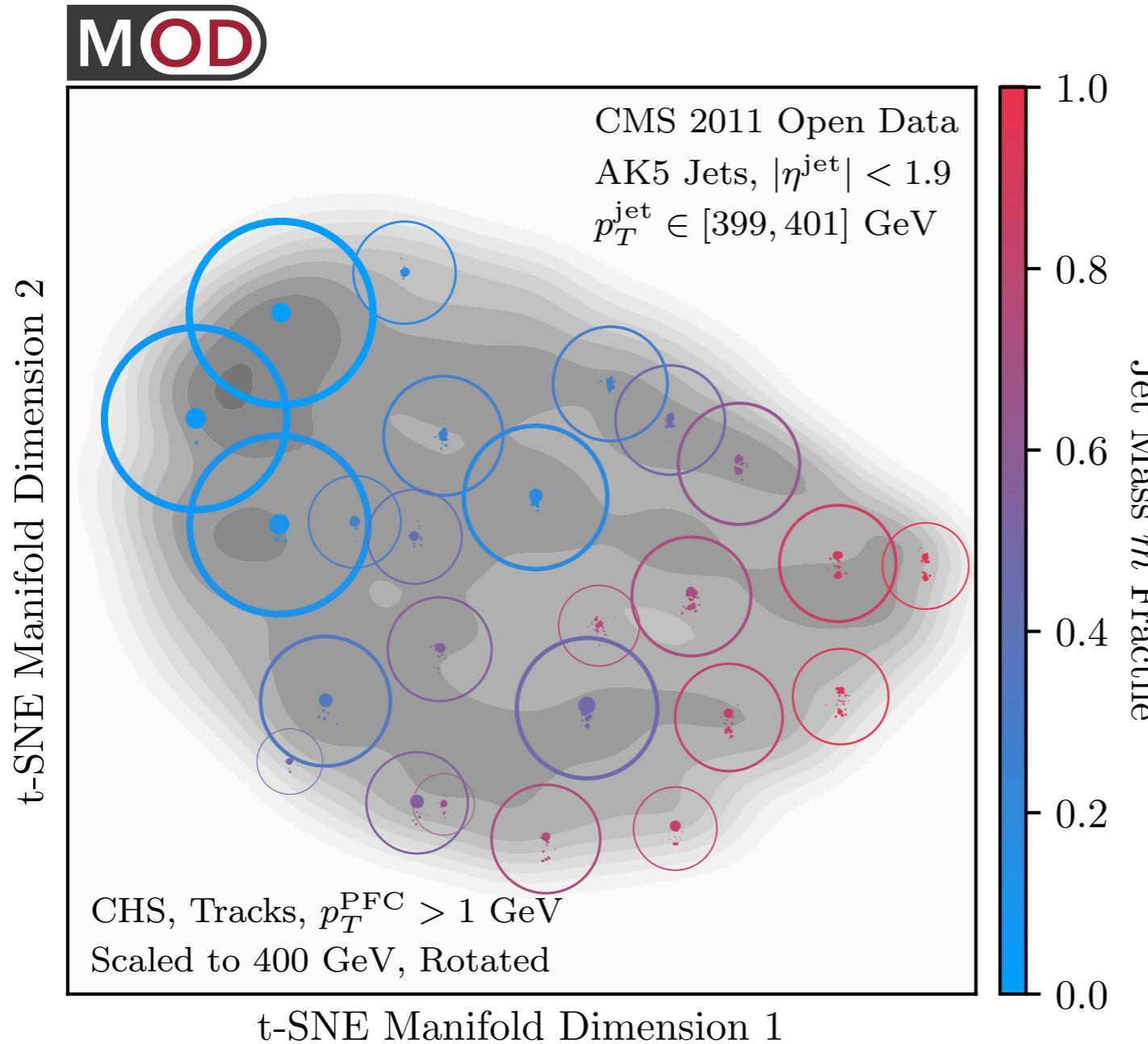
[Grassberger, Procaccia, [PRL 1983](#); Kégl, [NIPS 2002](#)]



Most Representative Jets

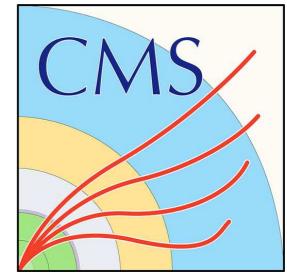


[<http://opendata.cern.ch/>]

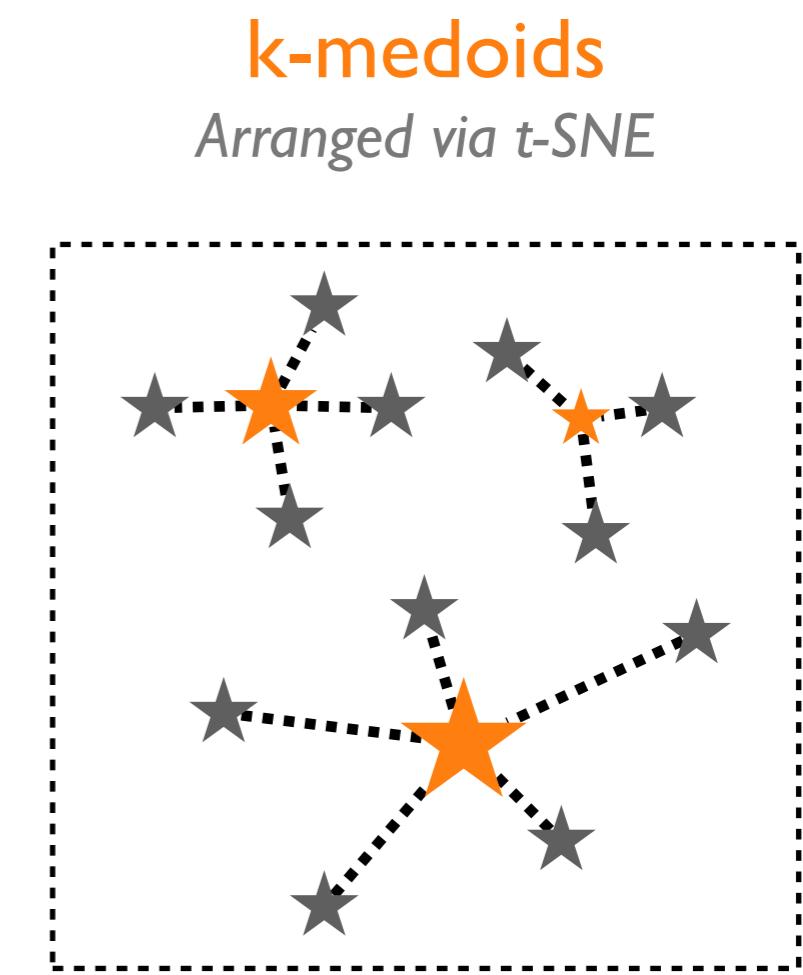
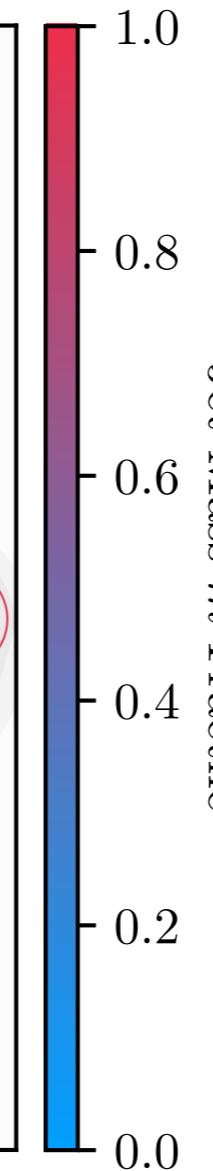
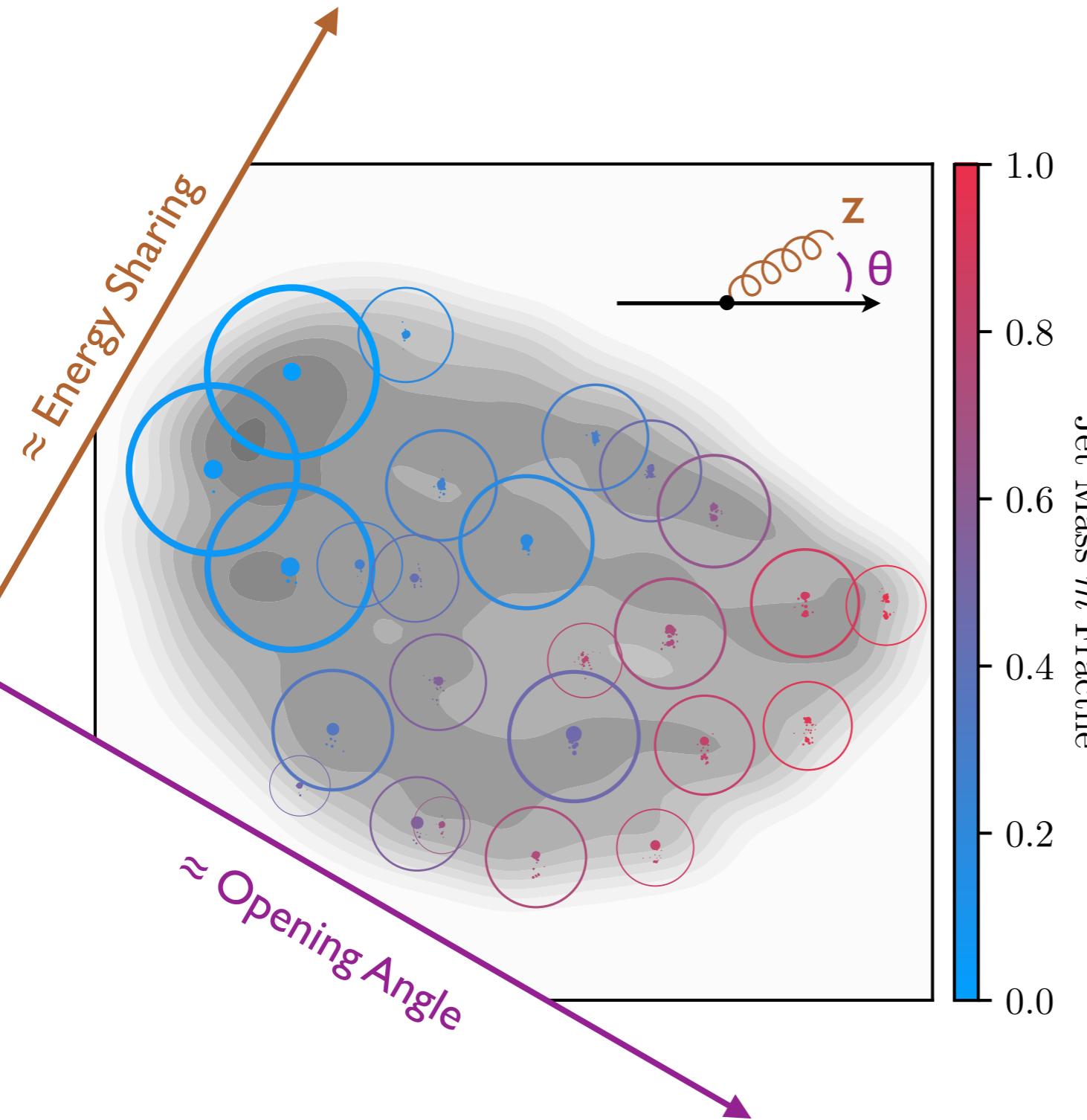


[Komiske, Mastandrea, Metodiev, Naik, JDT, PRD 2020; using van der Maaten, Hinton, JMLR 2008]

Most Representative Jets

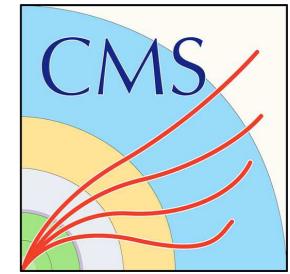


[<http://opendata.cern.ch/>]

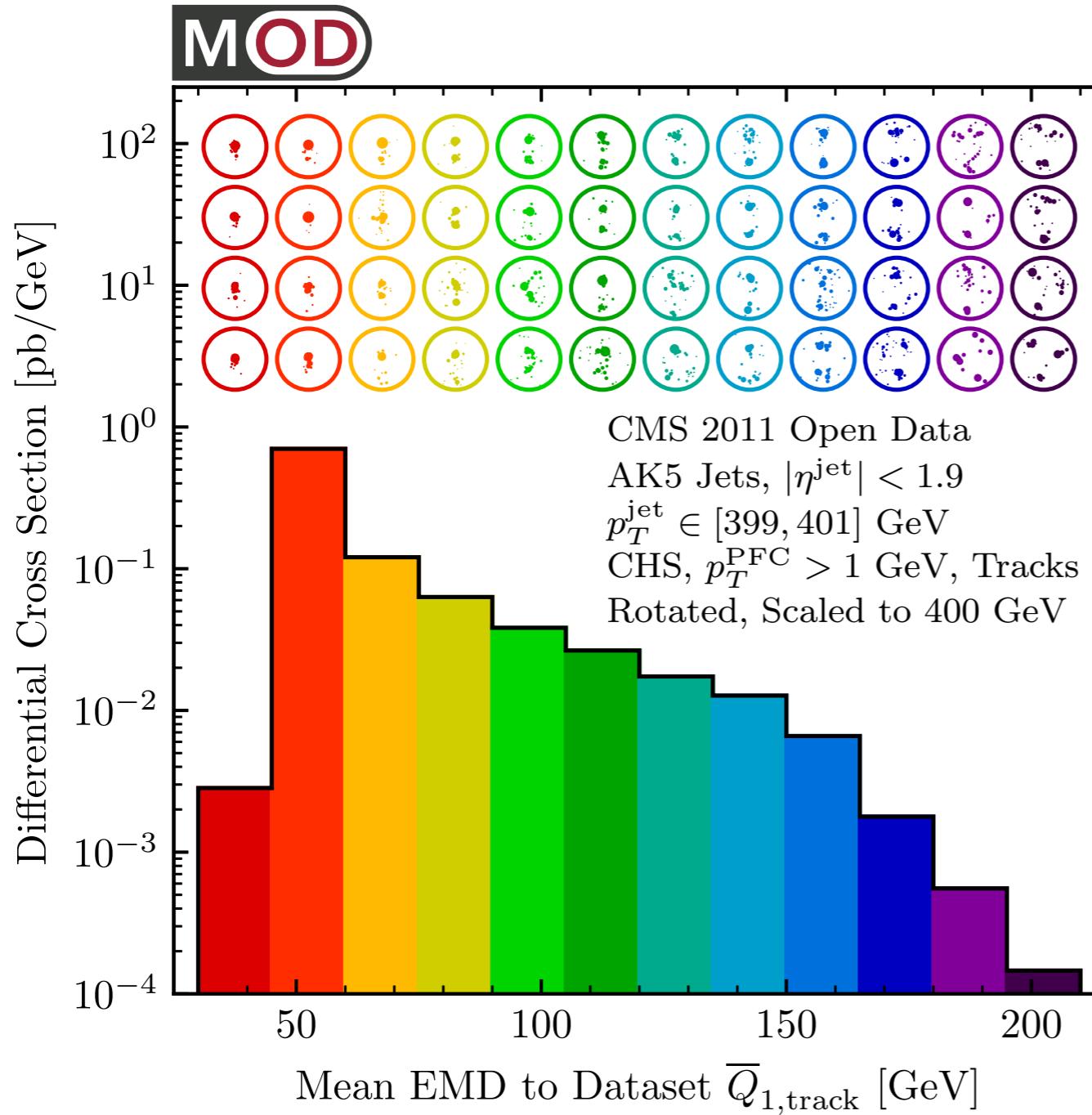


[Komiske, Mastandrea, Metodiev, Naik, JDT, PRD 2020; using van der Maaten, Hinton, JMLR 2008]

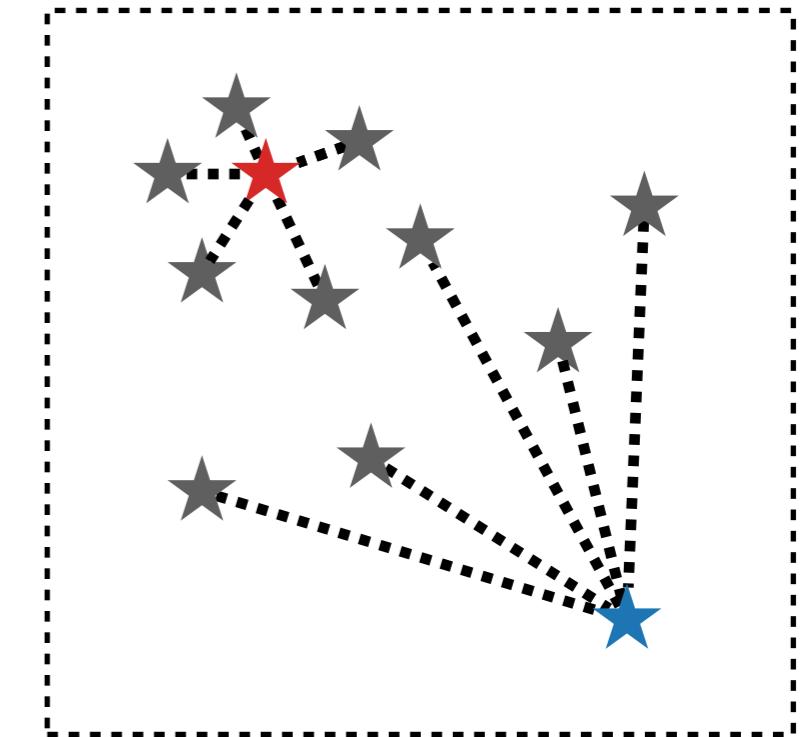
Least Representative Jets



[<http://opendata.cern.ch/>]

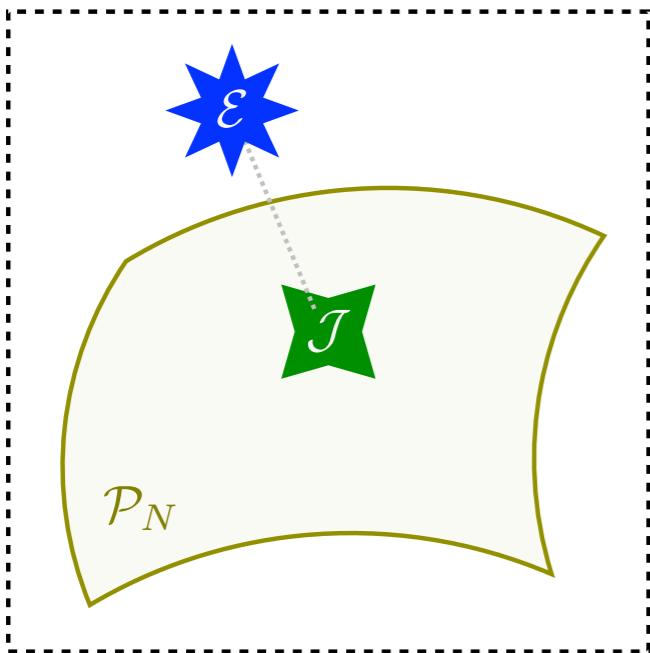


New Physics?
Or tails of QCD?



[Komiske, Mastandrea, Metodiev, Naik, JDT, [PRD 2020](#)]

More Fun with N-particle Manifolds



N-jettiness

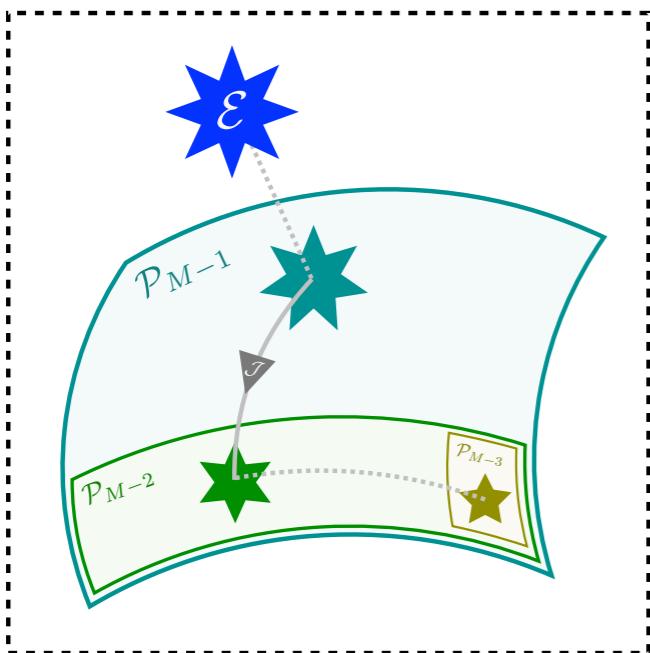
Distance of closest approach to N -particle manifold

[Brandt, Dahmen, [ZPC 1979](#); Stewart, Tackmann, Waalewijn, [PRL 2010](#)]

Exclusive Cone Jet Finding

Point of closest approach on N -particle manifold

[Stewart, Tackmann, JDT, Vermilion, Wilkason, [JHEP 2015](#)]



Sequential Jet Recombination

Iteratively stepping between various N -particle manifolds

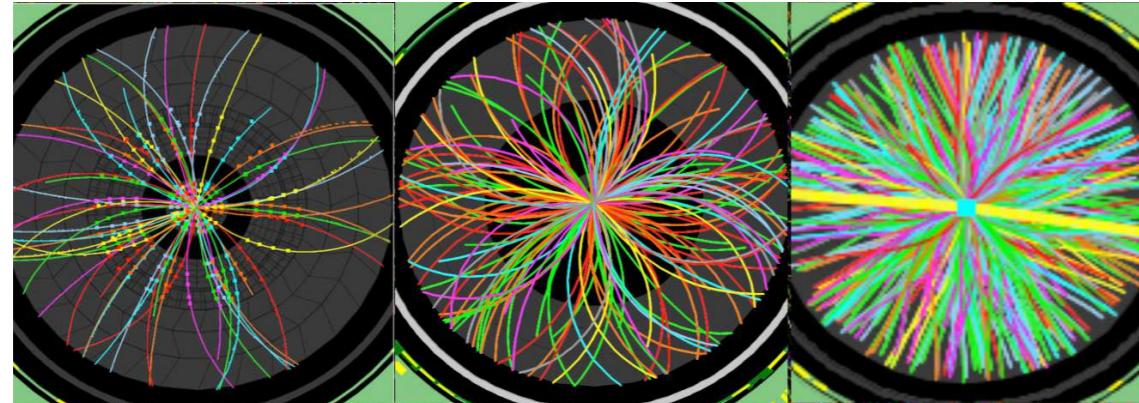
[Catani, Dokshitzer, Seymour, Webber, [NPB 1993](#); Ellis, Soper, [PRD 1993](#)]

[Dokshitzer, Leder, Moretti, Webber, [JHEP 1997](#); Wobisch, Wengler, [arXiv 1999](#)]

[Butterworth, Couchman, Cox, Waugh, [CPC 2003](#); Larkoski, Neill, JDT, [JHEP 2014](#)]

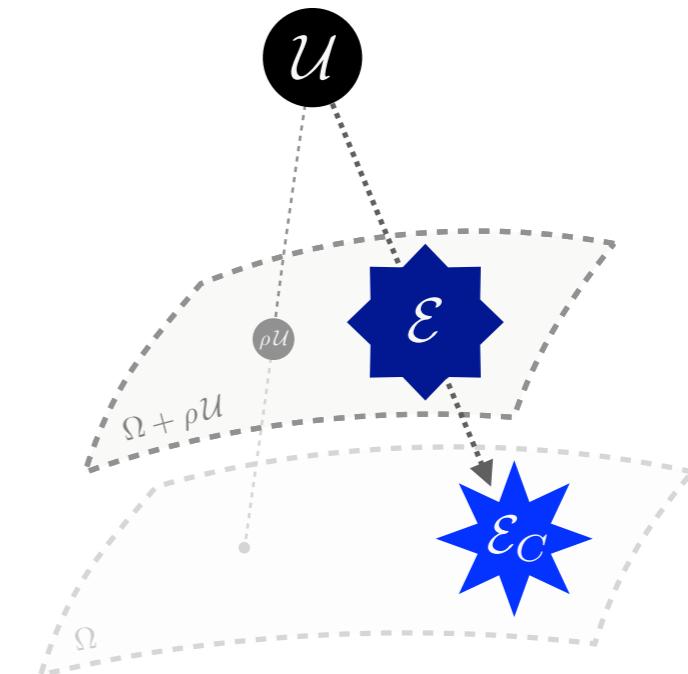
[Komiske, Metodiev, JDT, [JHEP 2020](#)]

Pileup Mitigation



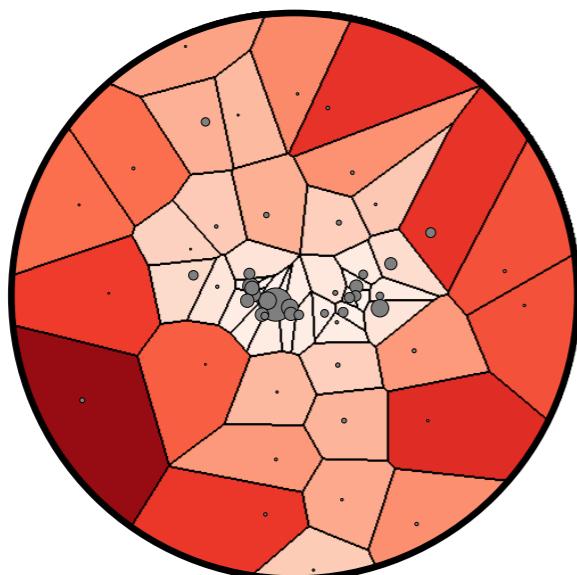
[see review in Soyez, PR 2019]

Uniform event contamination from overlapping proton-proton collisions



Pileup Mitigation:
“Move away” from uniform event

Voronoi



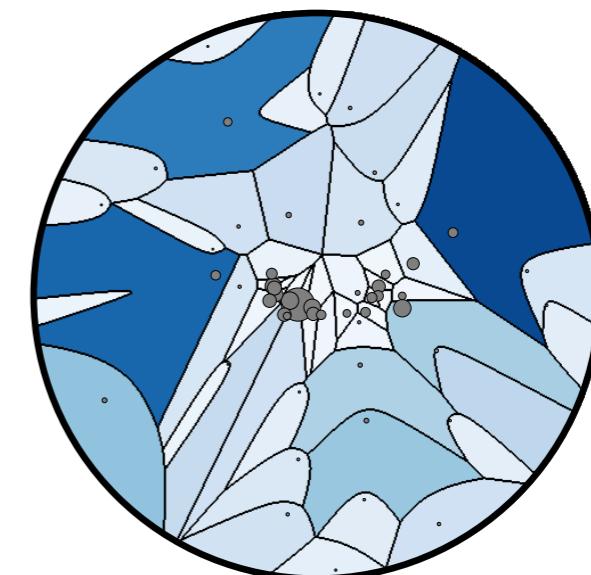
[Cacciari, Salam, Soyez, JHEP 2008]

Constituent Subtraction



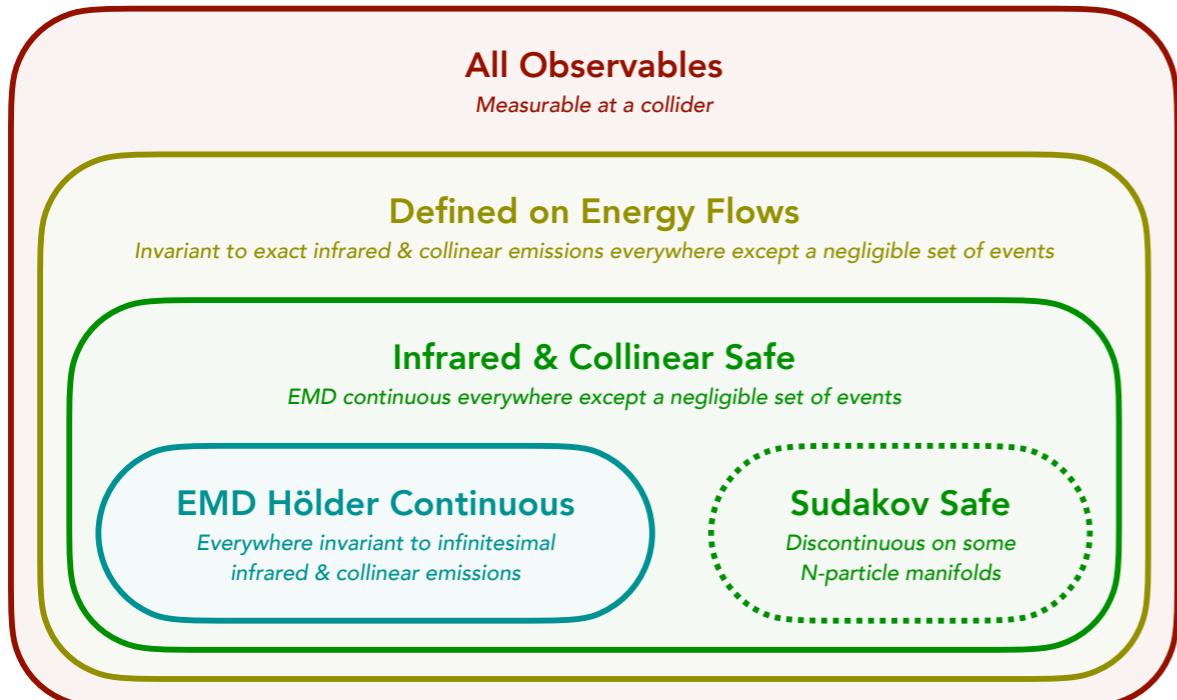
[Berta, Spousta, Miller, Leitner, JHEP 2014]

Apollonius



[Komiske, Metodiev, JDT, JHEP 2020]

Observable Taxonomy



All Observables	Comments
Multiplicity ($\sum_i 1$)	IR unsafe and C unsafe
Momentum Dispersion [65] ($\sum_i E_i^2$)	IR safe but C unsafe
Sphericity Tensor [66] ($\sum_i p_i^\mu p_i^\nu$)	IR safe but C unsafe
Number of Non-Zero Calorimeter Deposits	C safe but IR unsafe
Defined on Energy Flows	
Pseudo-Multiplicity ($\min\{N \mid \mathcal{T}_N = 0\}$)	Robust to exact IR or C emissions
Infrared & Collinear Safe	
Jet Energy ($\sum_i E_i$)	Disc. at jet boundary
Heavy Jet Mass [67]	Disc. at hemisphere boundary
Soft-Dropped Jet Mass [38, 68]	Disc. at grooming threshold
Calorimeter Activity [69] (N_{95})	Disc. at cell boundary
Sudakov Safe	
Groomed Momentum Fraction [39] (z_g)	Disc. on 1-particle manifold
Jet Angularity Ratios [37]	Disc. on 1-particle manifold
N -subjettiness Ratios [47, 48] (τ_{N+1}/τ_N)	Disc. on N -particle manifold
V parameter [36] (Eq. (2.11))	Hölder disc. on 3-particle manifold
EMD Hölder Continuous Everywhere	
Thrust [40, 41]	
Spherocity [42]	
Angularities [70]	
N -jettiness [44] (\mathcal{T}_N)	
C parameter [71–74]	Resummation beneficial at $C = \frac{3}{4}$
Linear Sphericity [72] ($\sum_i E_i n_i^\mu n_i^\nu$)	
Energy Correlators [36, 75–77]	
Energy Flow Polynomials [15, 17]	

[Komiske, Metodiev, JDT, [JHEP 2020](#); cf. Sterman, [PRD 1979](#); Banfi, Salam, Zanderighi, [JHEP 2005](#); Larkoski, Marzani, JDT, [PRD 2015](#)]