

The Hidden Geometry of Particle Collisions

Jesse Thaler



GGI Tea Breaks Seminar — June 30, 2021

The NSF AI Institute for Artificial Intelligence and Fundamental Interactions (IAIFI)

“eye-phi”



*Advance physics knowledge — from the smallest building blocks of nature
to the largest structures in the universe — and galvanize AI research innovation*



[<http://iaifi.org>, MIT News Announcement]

“Collision Course”

Robustness of Energy Flow

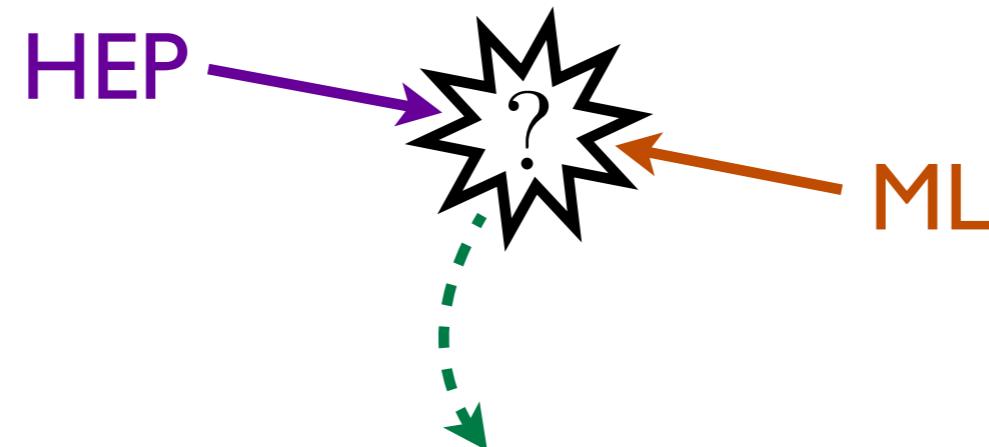
[Komiske, Metodiev, JDT, JHEP 2018]



Patrick Komiske

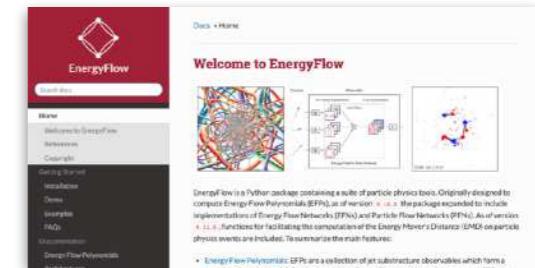


Eric Metodiev



Power of Point Cloud Learning

[Zaheer, Kottur, Ravanbakhsh, Poczos, Salakhutdinov, Smola, NIPS 2017]



Energy Flow Networks

<https://energyflow.network/>
[Komiske, Metodiev, JDT, JHEP 2019]

Machine Learning for
Particle Physics
21 June – 2 July 2021

<https://indico.mitp.uni-mainz.de/event/199>



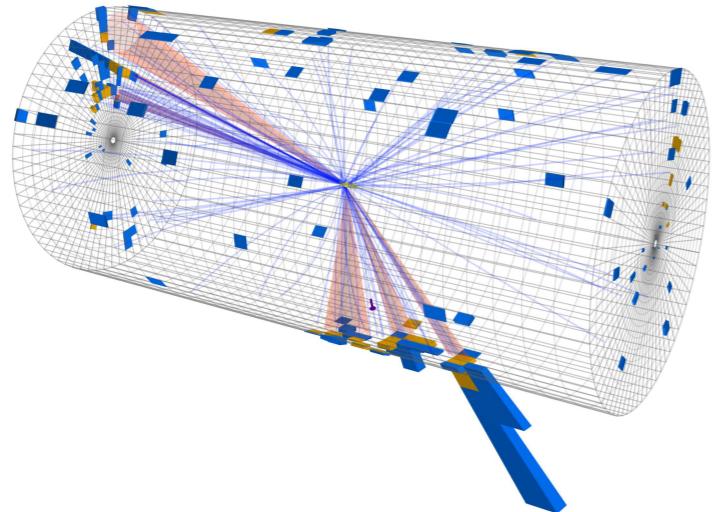
mitp
Mainz Institute for
Theoretical Physics

Optimal Transport for HEP?

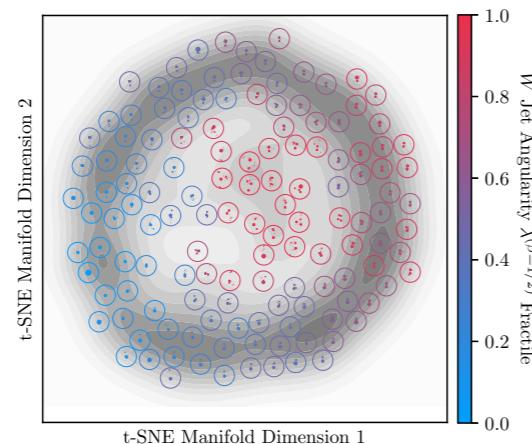


What does moving “earth”
have to do with collider physics?

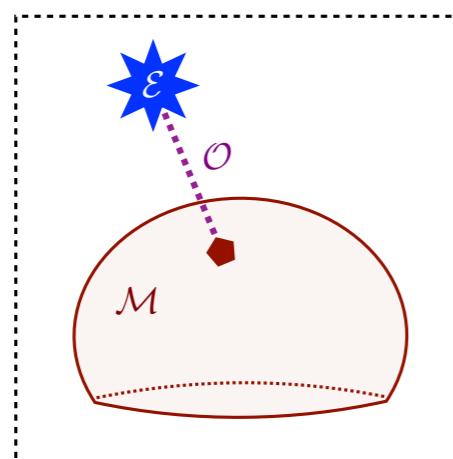
[image from [Wikipedia](#); see MITP talks last week by [Craig and Howard](#)]



From Manifest Geometry...

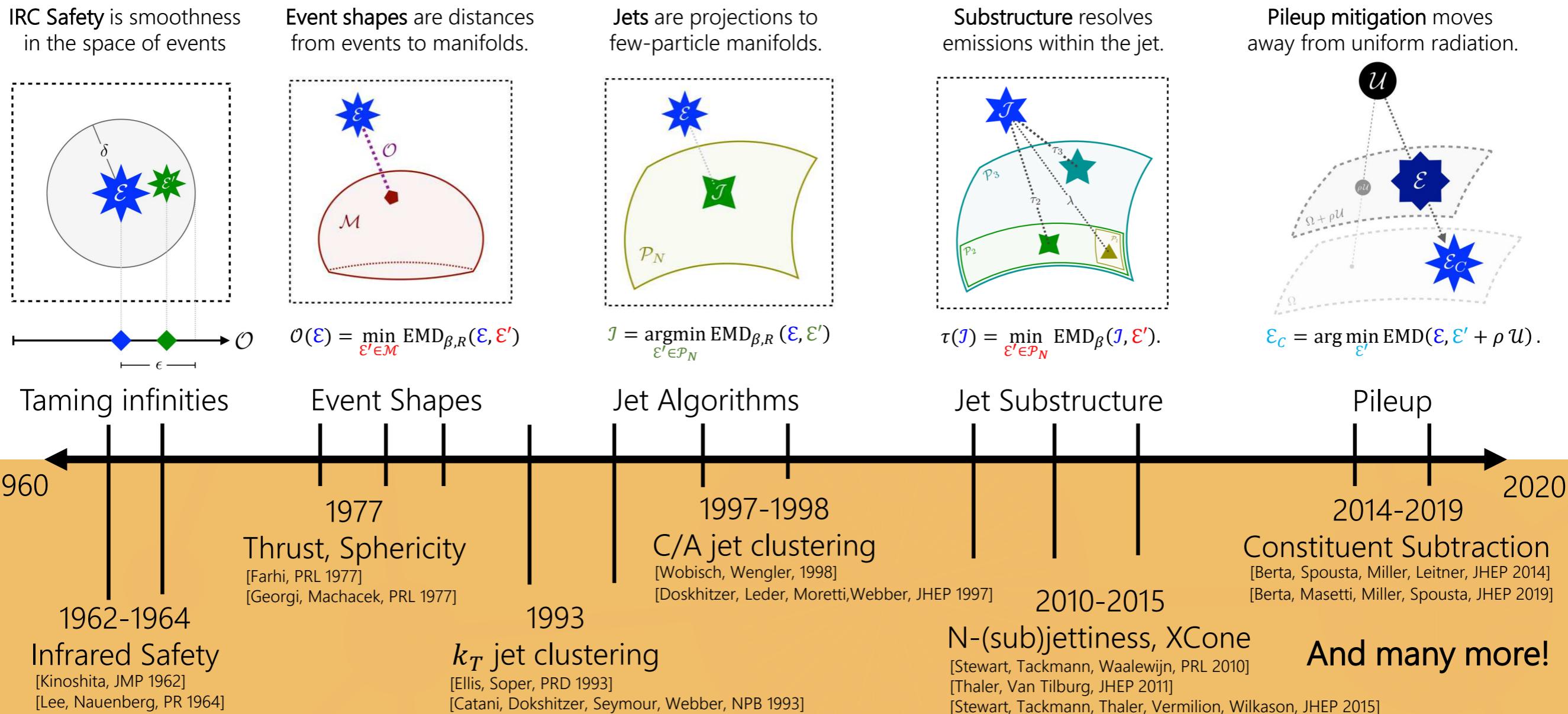


...to Emergent Geometry...



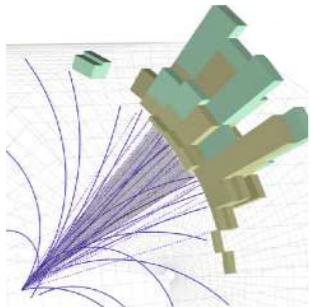
...to Hidden Geometry!

Six Decades of Collider Physics Translated into a New Geometric Language!

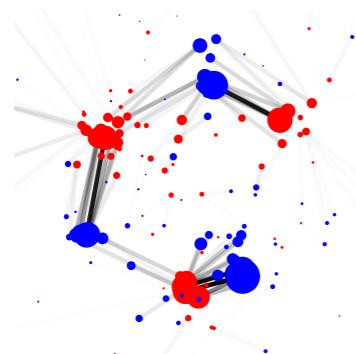


[timeline from Eric Metodiev]

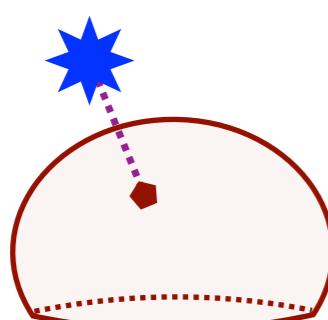
Outline



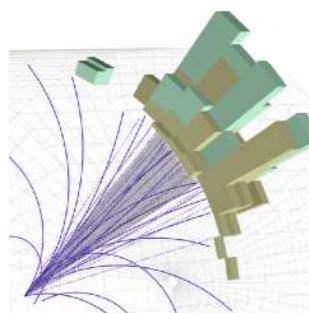
Going with the (Energy) Flow



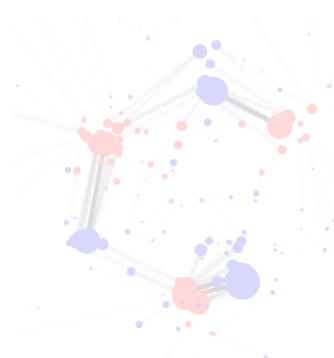
The Energy Mover's Distance



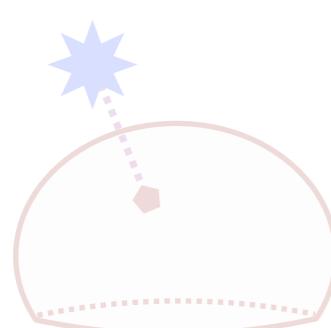
Revealing a Hidden Geometry



Going with the (Energy) Flow



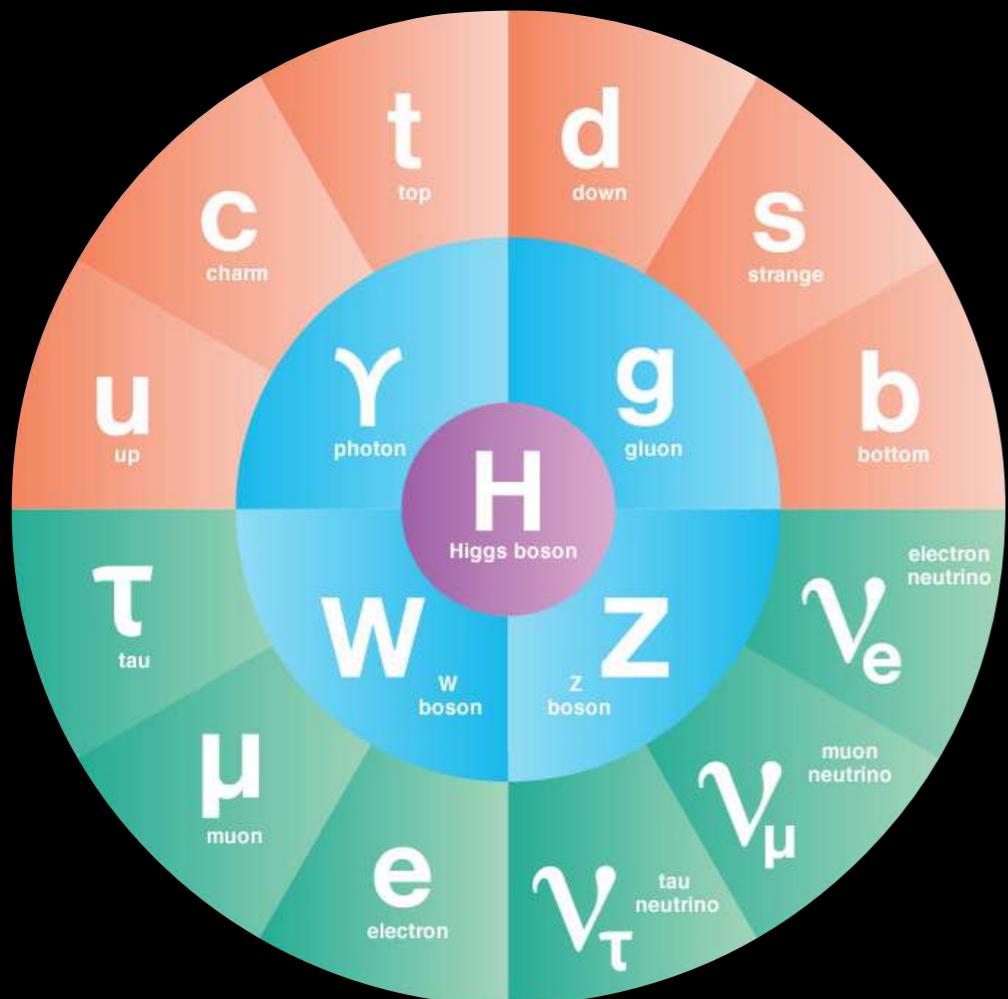
The Energy Mover's Distance



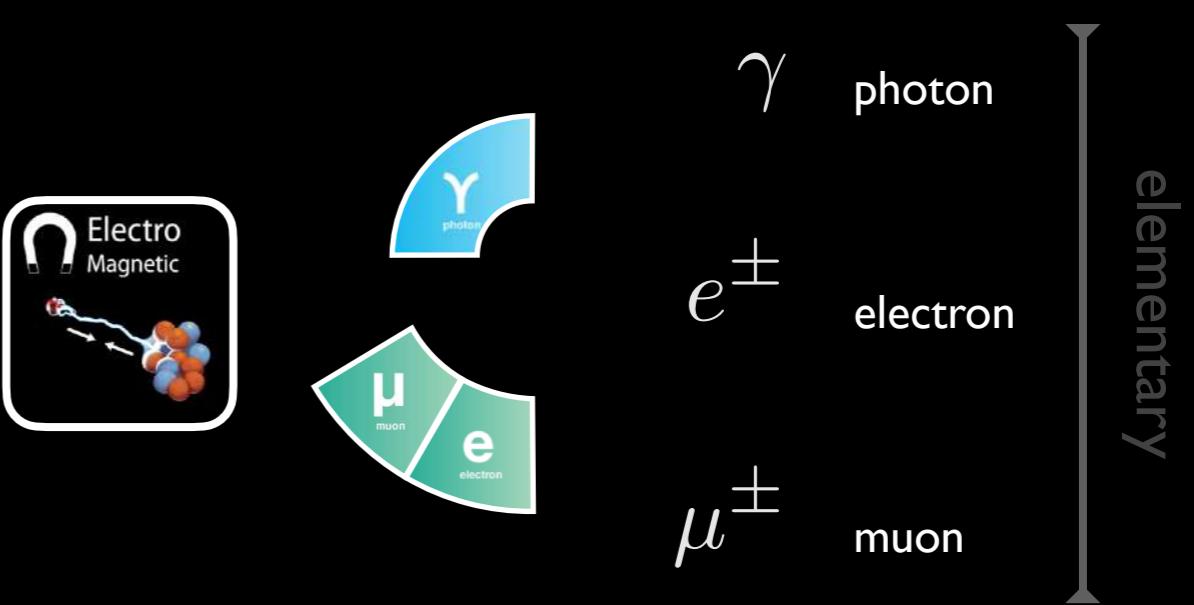
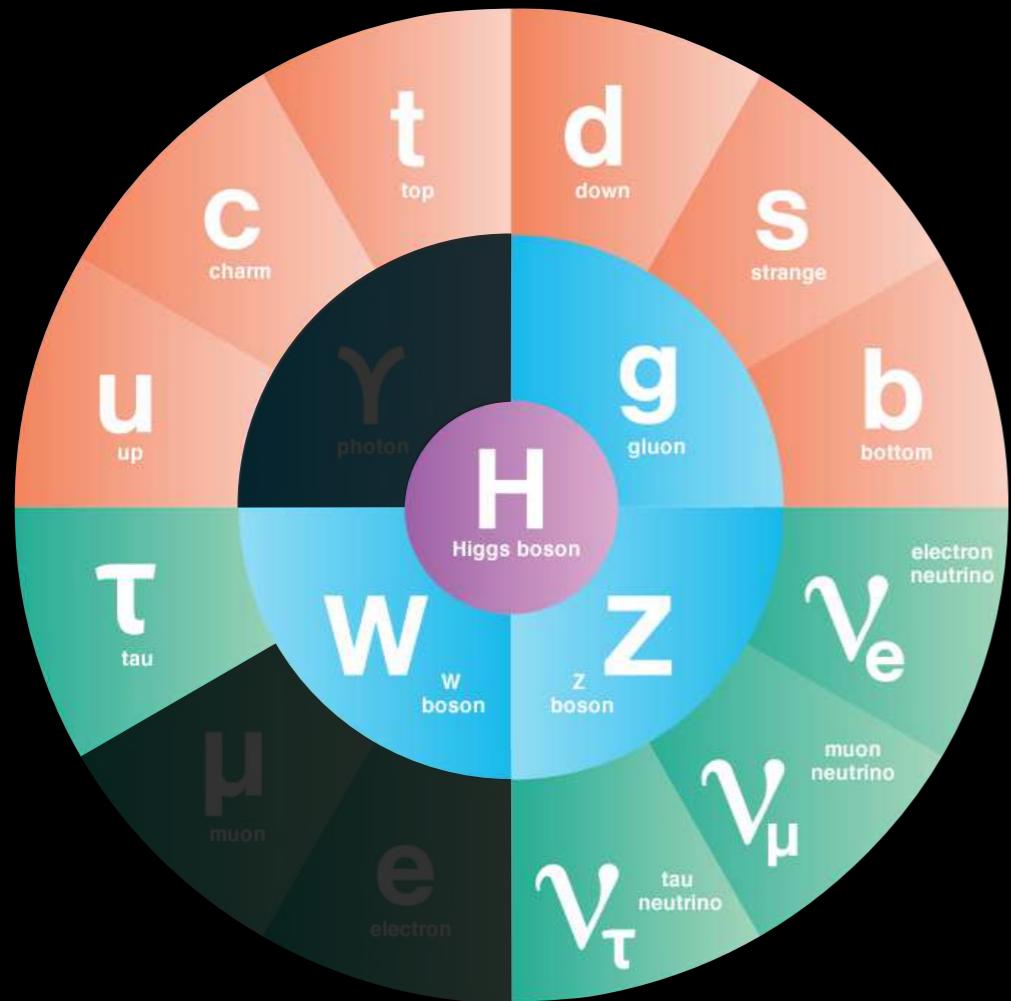
Revealing a Hidden Geometry

*Taking a step back to
particle physics 101...*

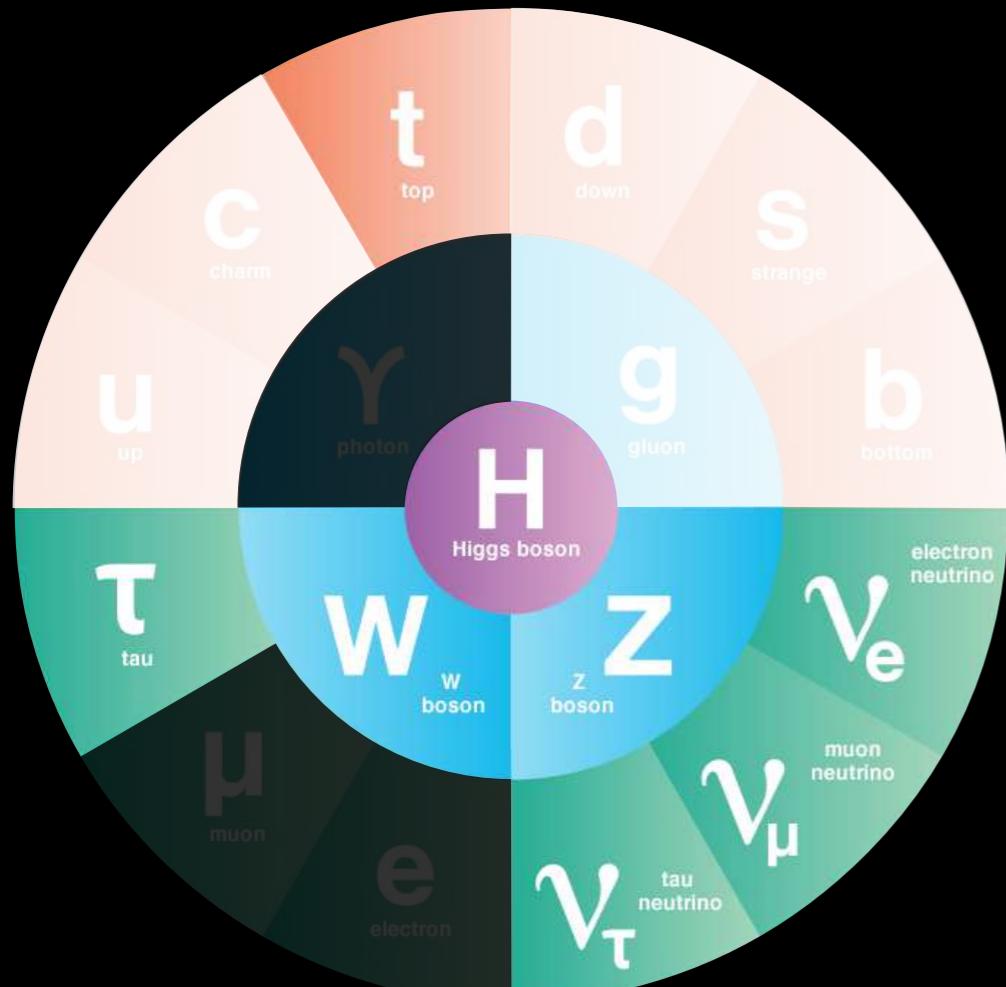
Particle Physics 101



Particle Physics 101

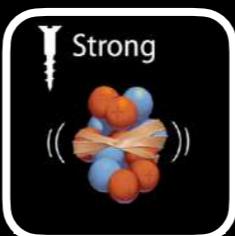


Particle Physics 101



QCD Confinement

Quarks
&
Gluons



γ
photon

e^+
electron

μ^+
muon

π^+
pion

K^+
kaon

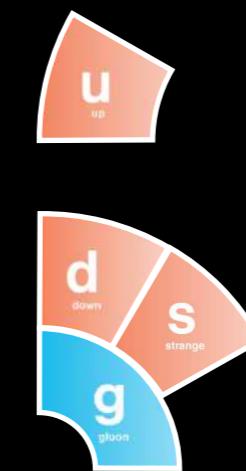
K_L^0
K-long

p/\bar{p}
proton

n/\bar{n}
neutron

elementary

composite



T E H M



γ

photon



e^+

electron



μ^+

muon



π^+

pion



K^+

kaon



K_L^0

K-long



p/\bar{p}

proton



n/\bar{n}

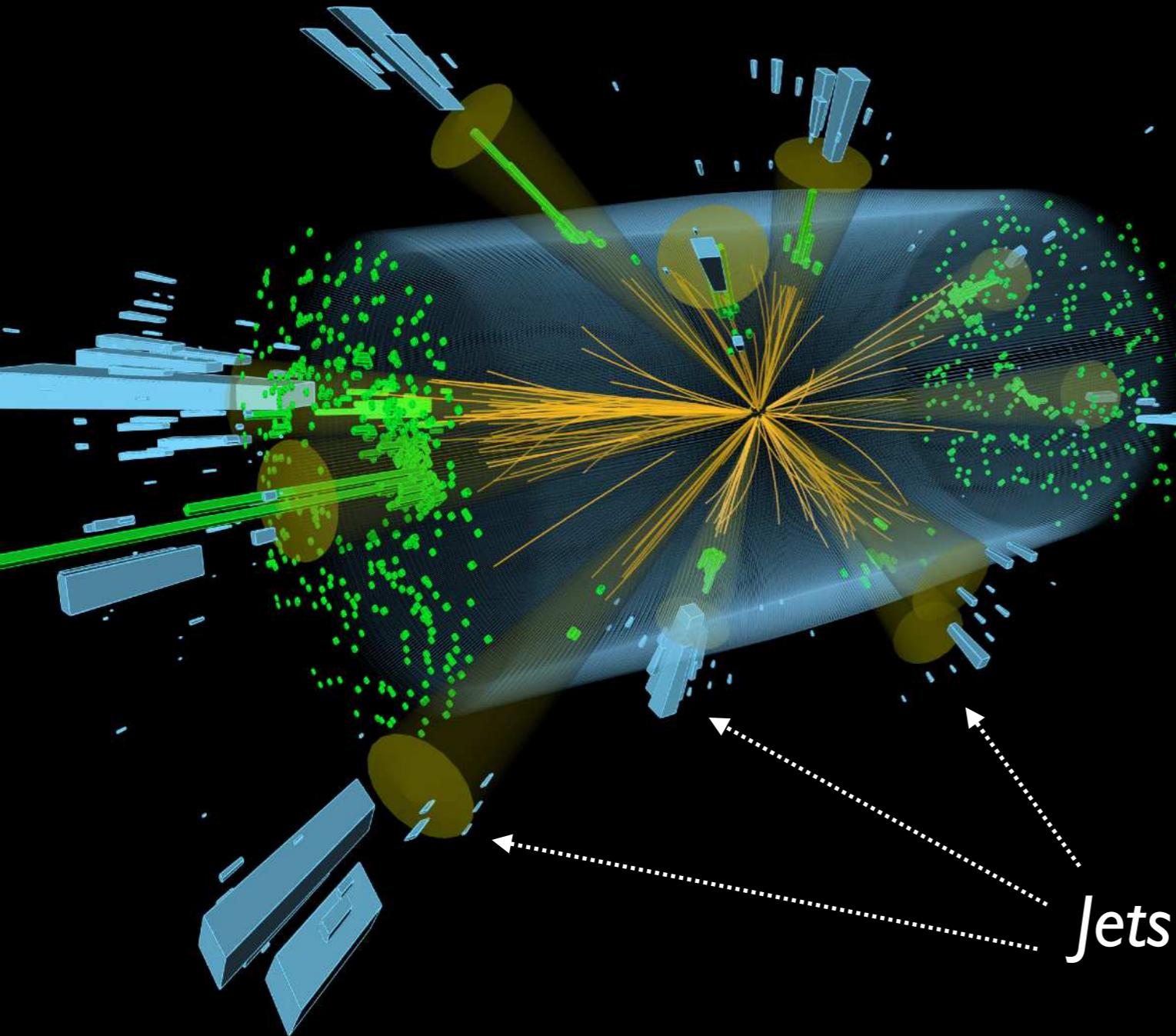
neutron

elementary

composite

Collider Event

Every 25 nanoseconds at the LHC



T E H M



γ

photon



e^+

electron



μ^+

muon



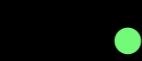
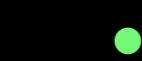
π^+

pion



K^+

kaon



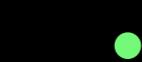
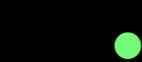
K_L^0

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p/\bar{p}

proton



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neutron

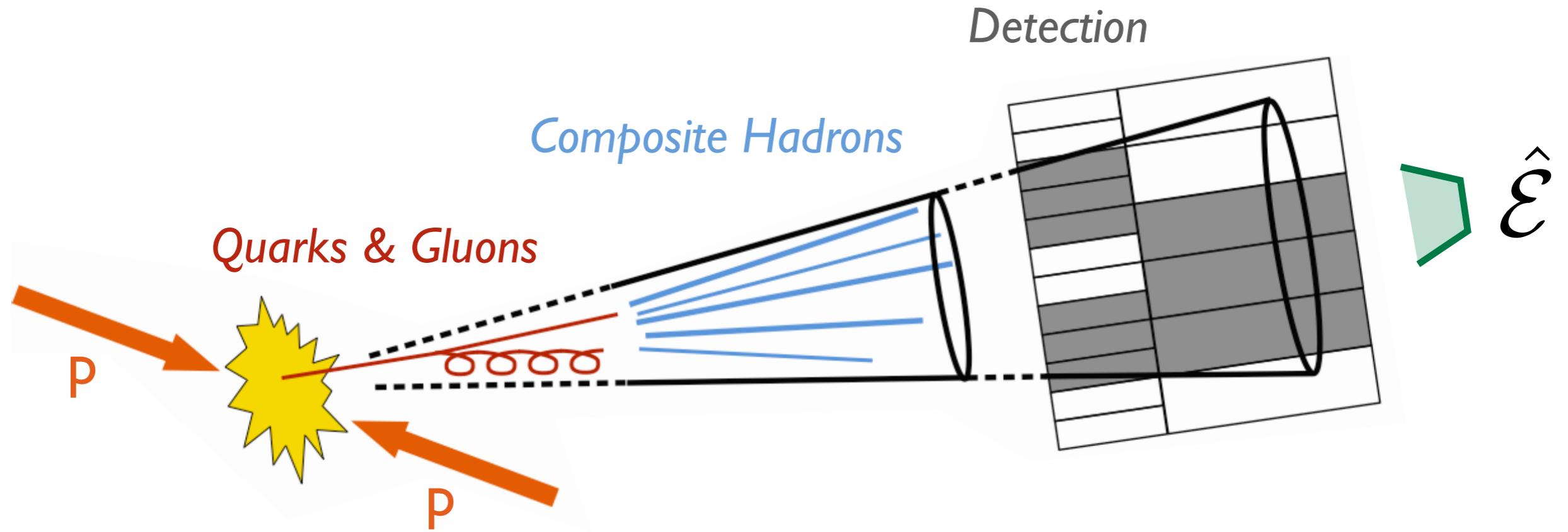
elementary

composite

Energy Flow Representation

Emphasizes *infrared and collinear safety*

Theory



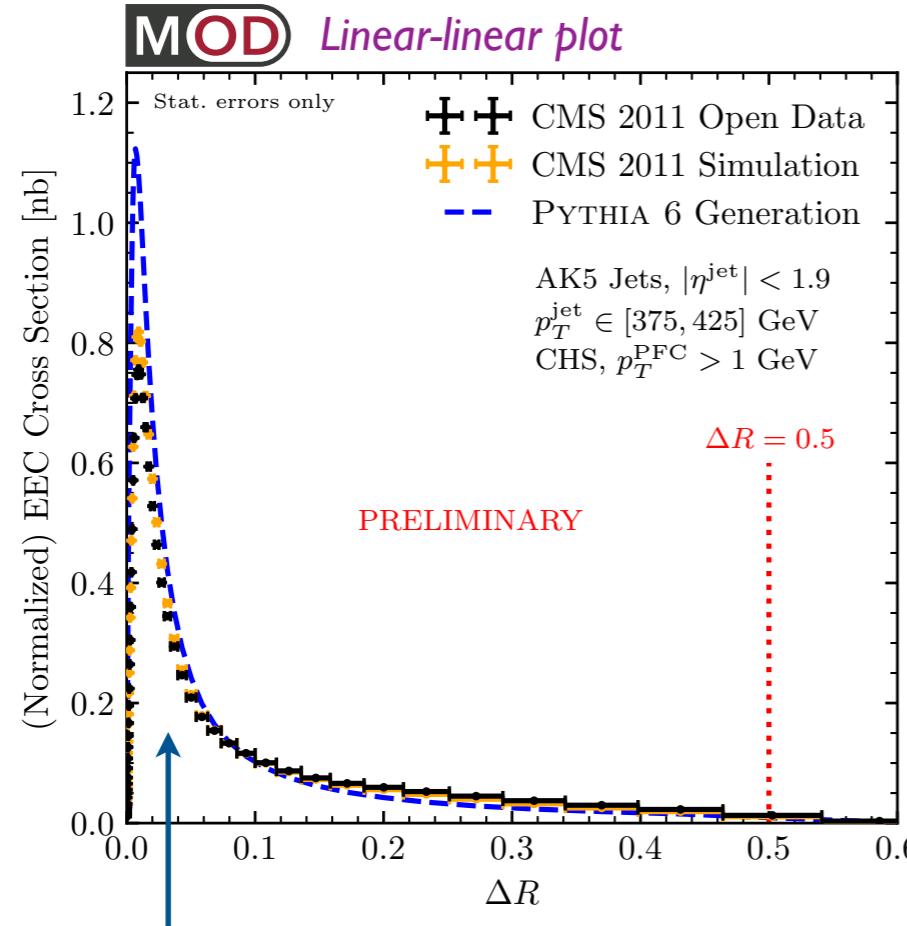
Energy Flow:

Robust to hadronization and detector effects
Well-defined for massless gauge theories

$$\hat{\mathcal{E}} \simeq \lim_{t \rightarrow \infty} \hat{n}_i T^{0i}(t, vt\hat{n})$$

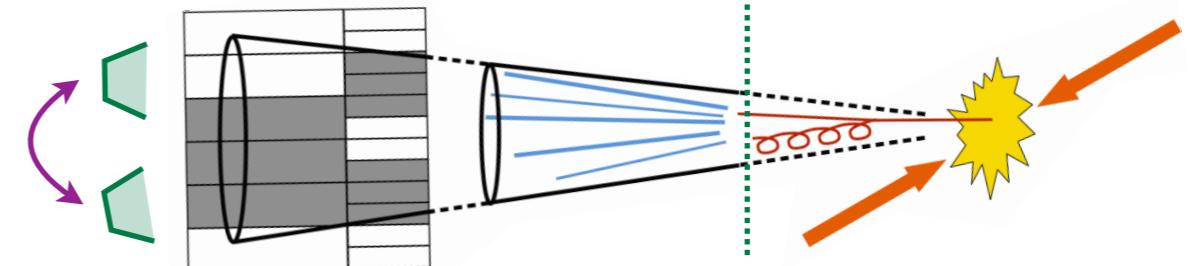
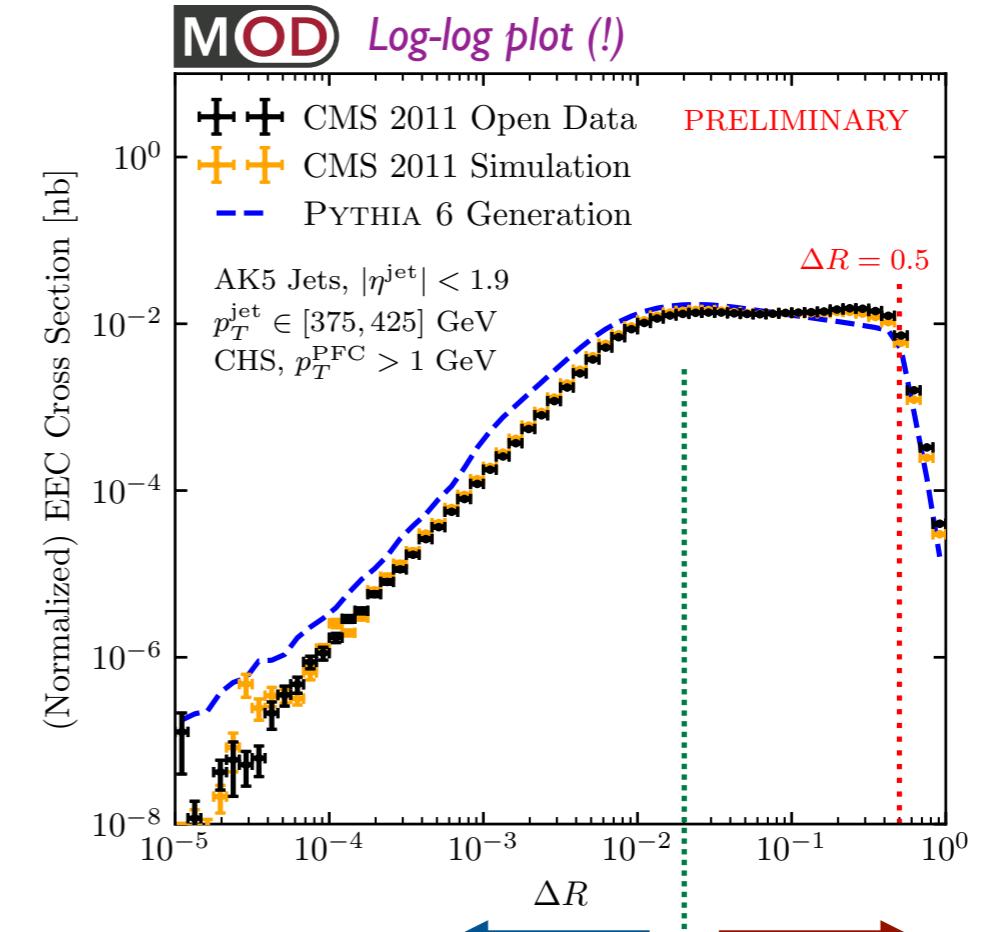
[see e.g. Sveshnikov, Tkachov, [PLB 1996](#); Hofman, Maldacena, [JHEP 2008](#); Mateu, Stewart, [JDT, PRD 2013](#); Belitsky, Hohenegger, Korchemsky, Sokatchev, Zhiboedov, [PRL 2014](#); Chen, Moult, Zhang, Zhu, [PRD 2020](#)]

QCD Phase Transition in Jets?



Probing the small angle limit of QCD

First Jet EEC Plot from the LHC (!)



[Komiske, Moult, JDT, Zhu, in progress;
based on Basham, Brown, Ellis, Love, [PRL 1978](#); see talks by Moult, [BOOST 2019](#), [BOOST 2020](#)]



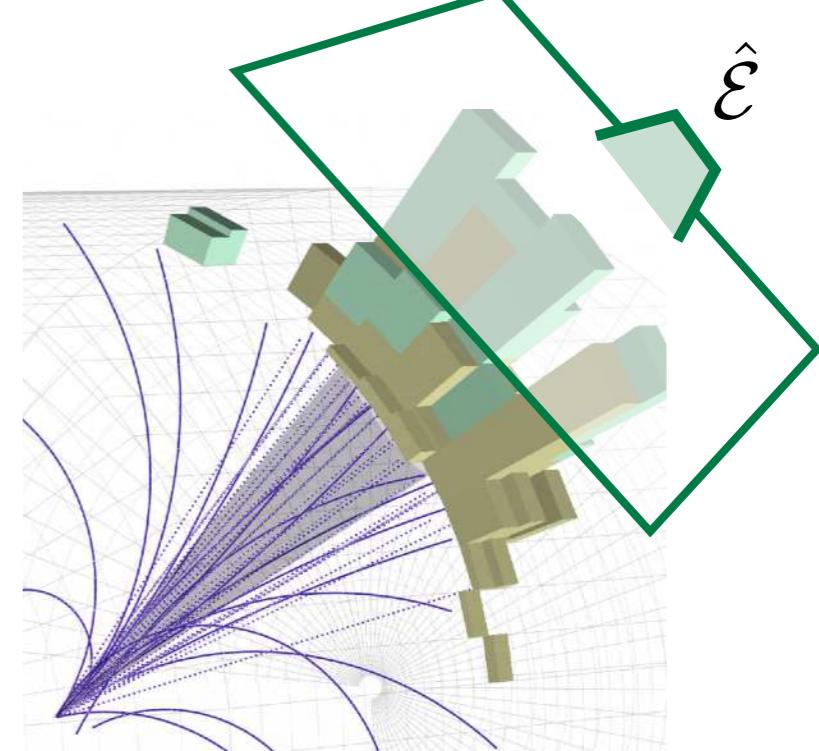
Jets as Weighted Point Clouds

- Energy-Weighted Directions

$$\vec{p} = \{E, \hat{n}_x, \hat{n}_y, \hat{n}_z\}$$

↑ |
Energy Direction

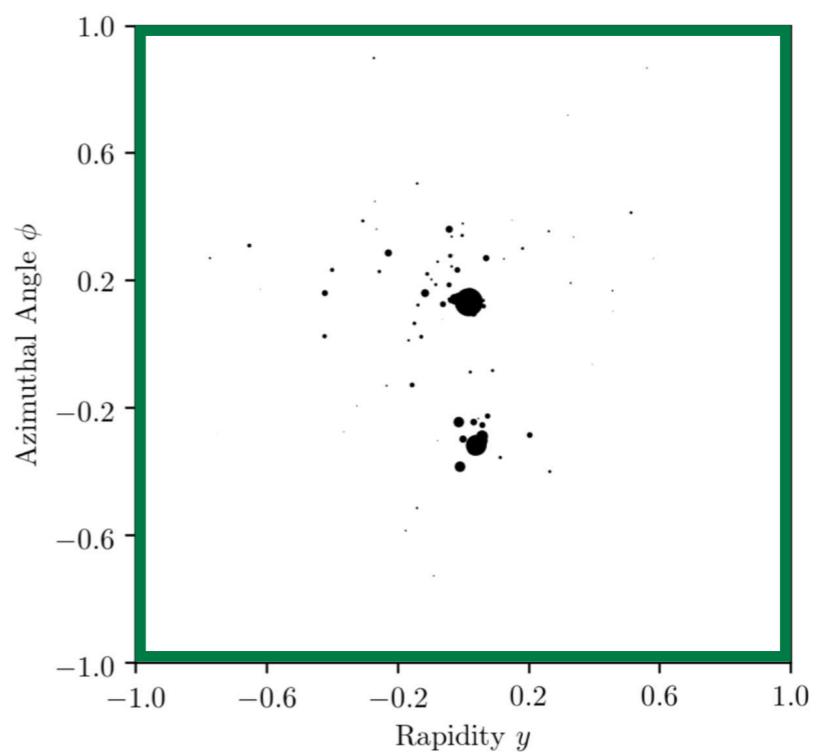
(suppressing “unsafe” charge/flavor information)

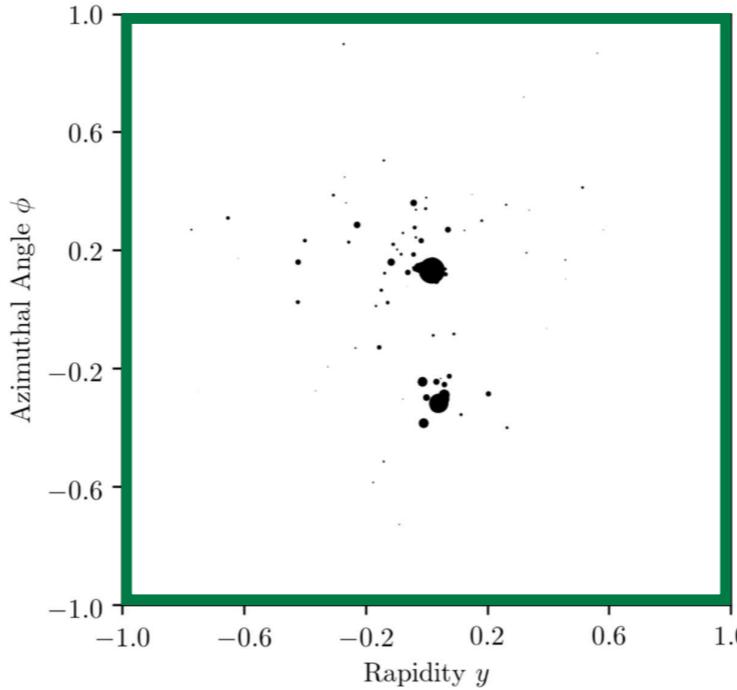


- Equivalently: Energy Density

$$\rho(\hat{n}) = \sum_{i \in \mathcal{J}} E_i \delta^{(2)}(\hat{n} - \hat{n}_i)$$

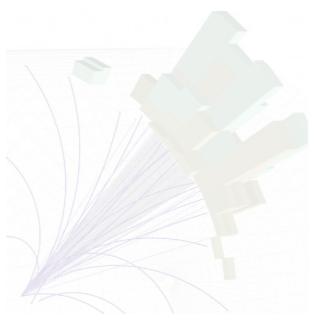
↑ ↑
Energy Direction



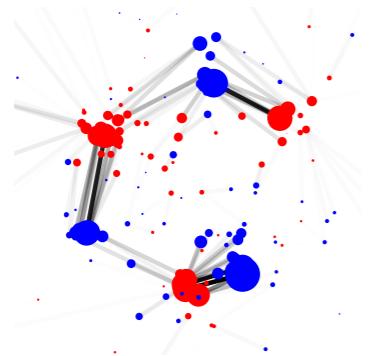


*When restricted to IRC safe information,
jets/events are naturally represented
as energy densities*

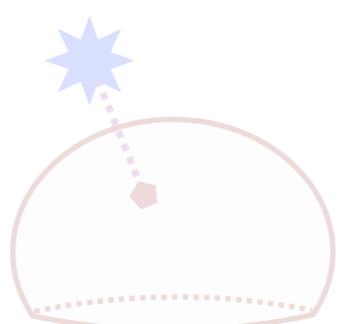
It is not obvious how to include unsafe information in this picture,
since flow of charge/flavor is theoretically delicate



Going with the (Energy) Flow



The Energy Mover's Distance



Revealing a Hidden Geometry

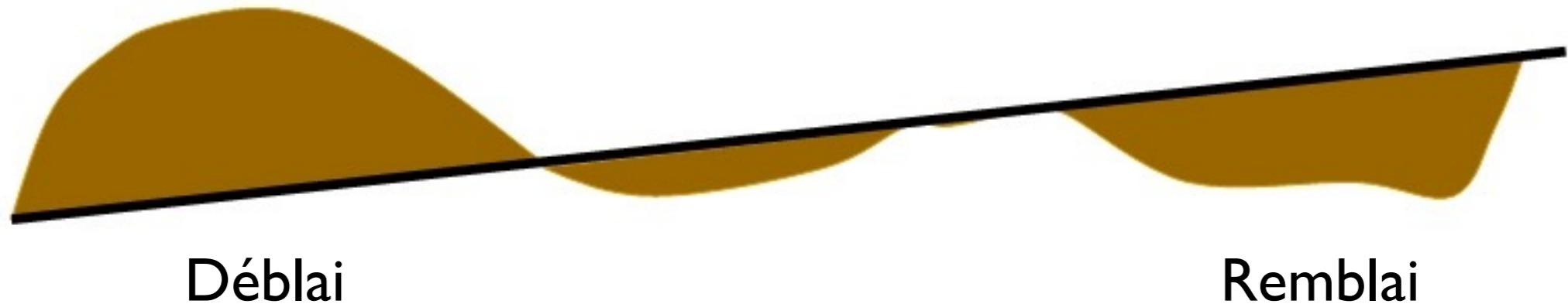
*If you ask your local computational geometry
expert how to process densities...*

The Earth Mover's Distance

Optimal Transport:

[Peleg, Werman, Rom, [IEEE 1989](#);
Rubner, Tomasi, Guibas, [ICCV 1998](#), [ICCV 2000](#);
Pele, Werman, [ECCV 2008](#); Pele Taskar, [GSI 2013](#)]

Minimum “work” (stuff \times distance) to make one distribution look like another distribution



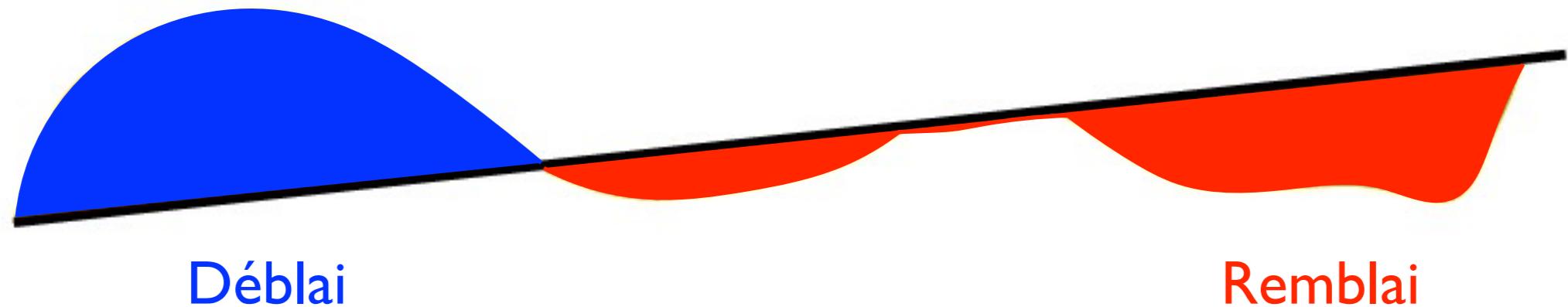
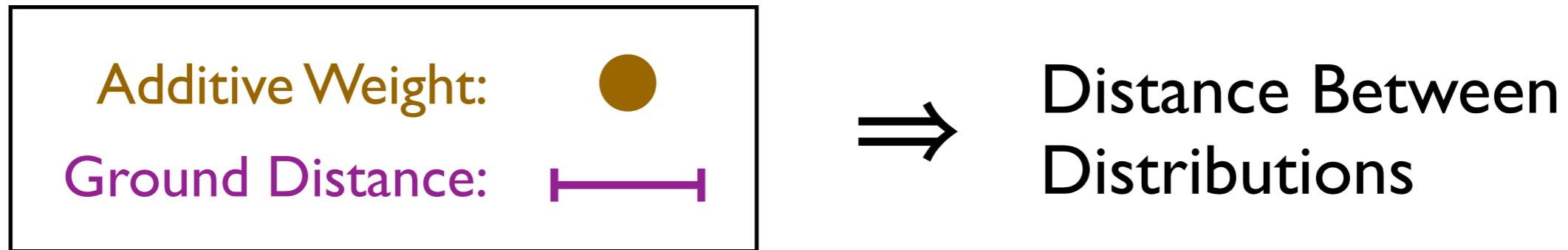
[h/t Niles-Weed, [ML4Jets 2020](#); Monge, 1781; Kantorovich, 1939; Vaserštejn, 1969; [Wikipedia](#)]

The Earth Mover's Distance

Optimal Transport:

[Peleg, Werman, Rom, [IEEE 1989](#);
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Minimum “work” (**stuff** × **distance**) to make
one distribution look like **another distribution**



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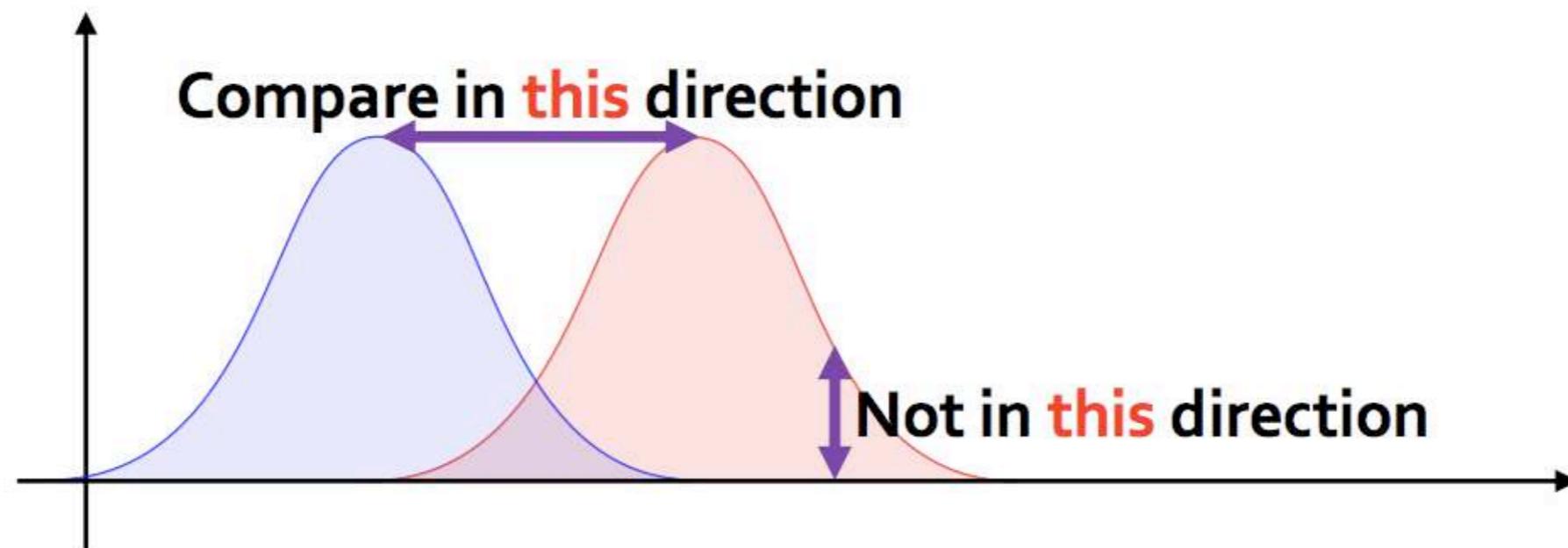
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Minimum “work” (**stuff \times distance**) to make
one distribution look like **another distribution**

“Horizontal” comparison (EMD) yields better
dynamic range than “vertical” comparison (e.g. KL)

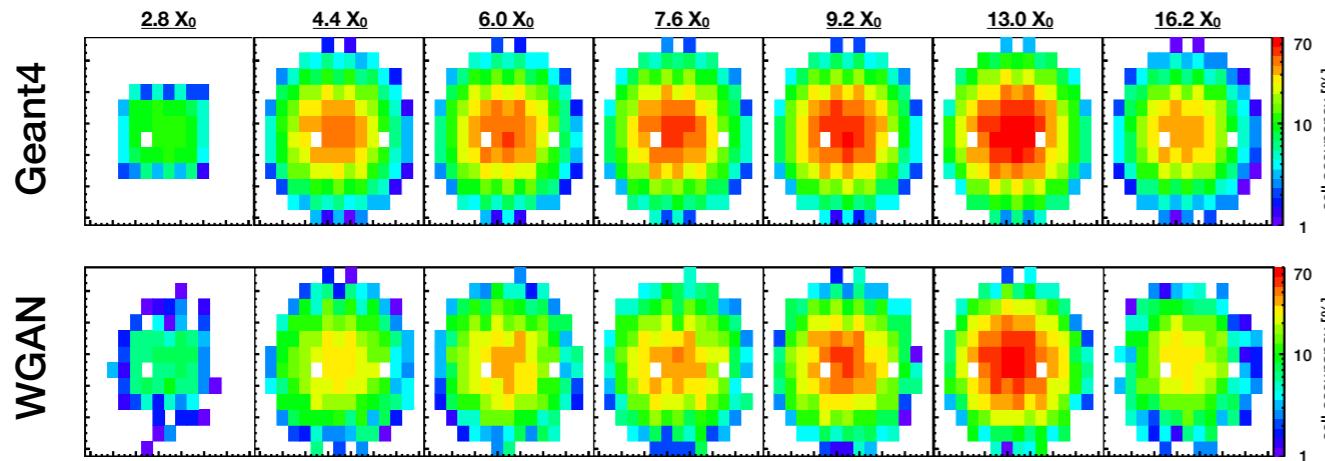


[figure from Kun, [Math n Programming](#)]

[h/t Niles-Weed, [ML4Jets 2020](#); Monge, 1781; Kantorovich, 1939; Vaserštejn, 1969; [Wikipedia](#)]

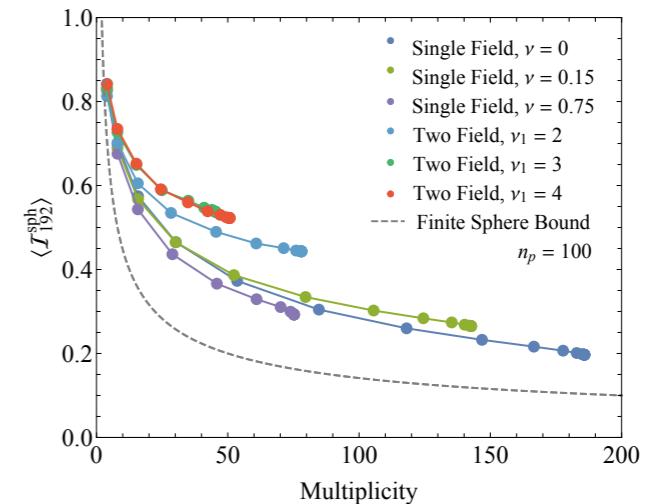
Wasserstein in HEP

Generative Modeling



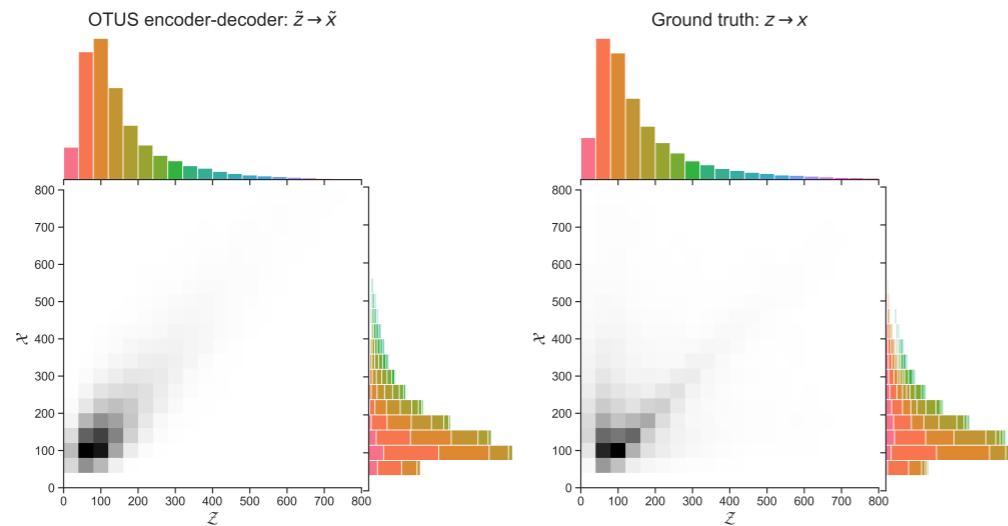
[Erdmann, Geiger, Glombitza, Schmidt, [CSBS 2018](#); Erdmann, Glombitza, Quast, [CSBS 2019](#);
Chekalina, Orlova, Ratnikov, Ulyanov, Ustyuzhanin, Zakharov, [CHEP 2018](#)]

BSM Characterization



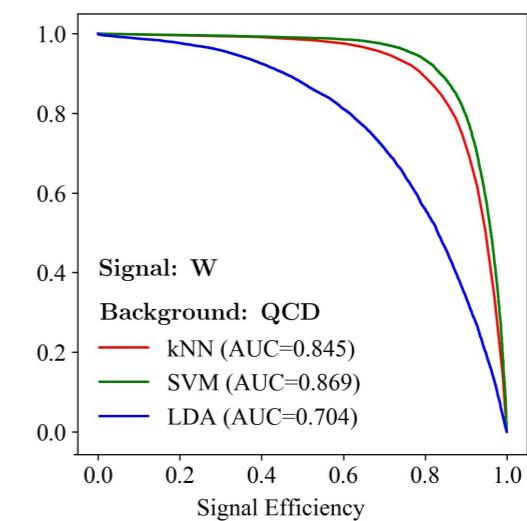
[Cesarotti, Reece, Strassler, [JHEP 2021](#), arXiv 2020]

Estimated Simulation/Unfolding



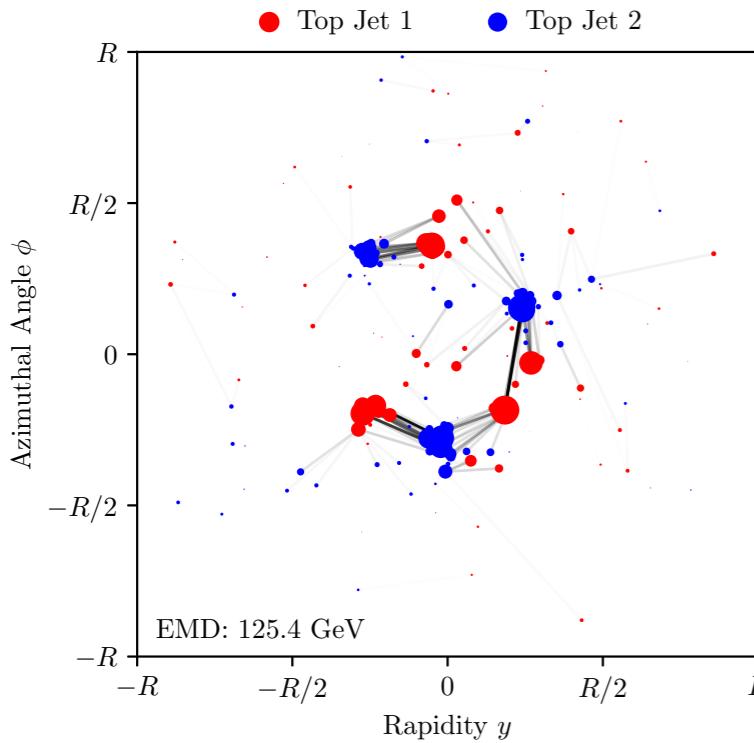
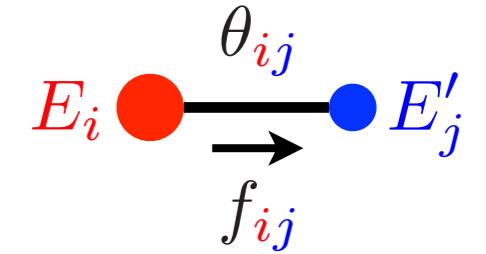
[Howard, Mandt, Whiteson, Yang, arXiv 2021]

Jet Classification



[Cai, Cheng, Craig, Craig, PRD 2020]

The Energy Mover's Distance



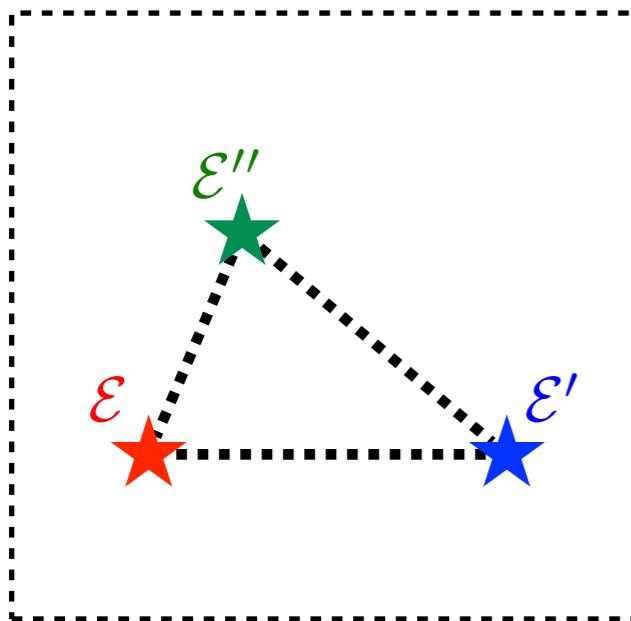
Optimal transport between energy flows...

$$\text{EMD}(\mathcal{E}, \mathcal{E}') = \min_{\{f\}} \sum_i \sum_j f_{ij} \frac{\theta_{ij}}{R} + \left| \sum_i E_i - \sum_j E'_j \right|$$

↑
in GeV

Cost to move energy

Cost to create energy



...defines a metric on the space of events

$$0 \leq \text{EMD}(\mathcal{E}, \mathcal{E}') \leq \text{EMD}(\mathcal{E}, \mathcal{E}'') + \text{EMD}(\mathcal{E}', \mathcal{E}'')$$

(assuming $R \geq \theta_{\max}/2$, i.e. $R \geq$ jet radius for conical jets)

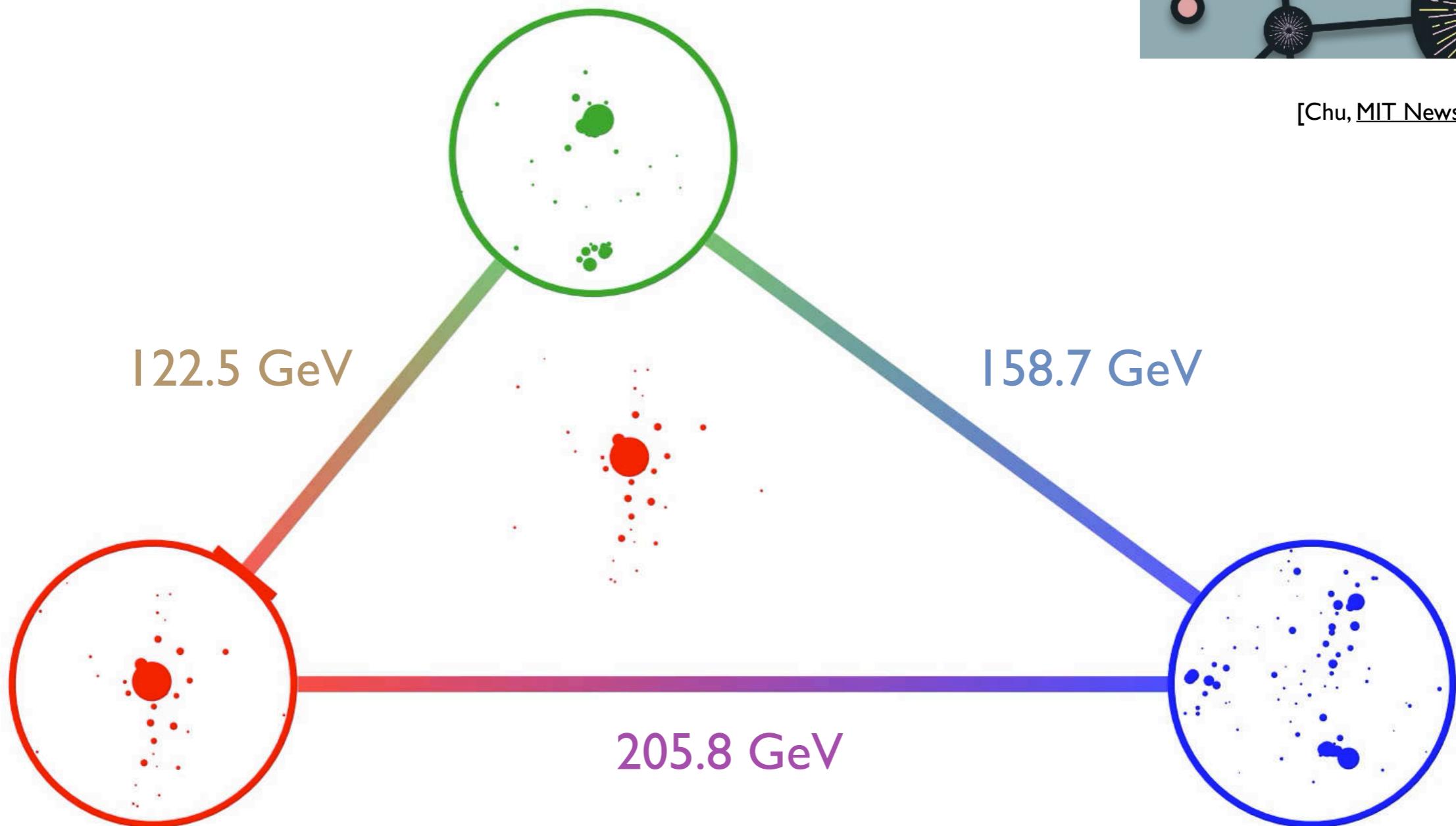
[Komiske, Metodiev, JDT, [PRL 2019](#)
 [see flavored variant in Crispim Romão, Castro, Milhano, Pedro, Vale, [EPJC 2021](#)]
 [see computational speed up in Cai, Cheng, Craig, Craig, [PRD 2020](#)]



Similarity of Three Energy Flows?



[Chu, MIT News July 2019]



[Komiske, Metodiev, JDT, PRL 2019; code at Komiske, Metodiev, JDT, [energyflow.network](#); see alternative graph network approach in Mullin, Pacey, Parker, White, Williams, JHEP 2021]

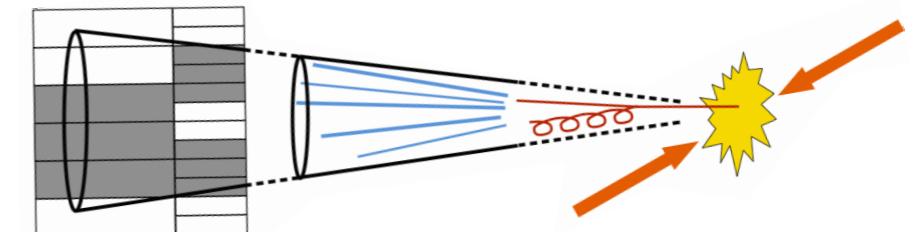
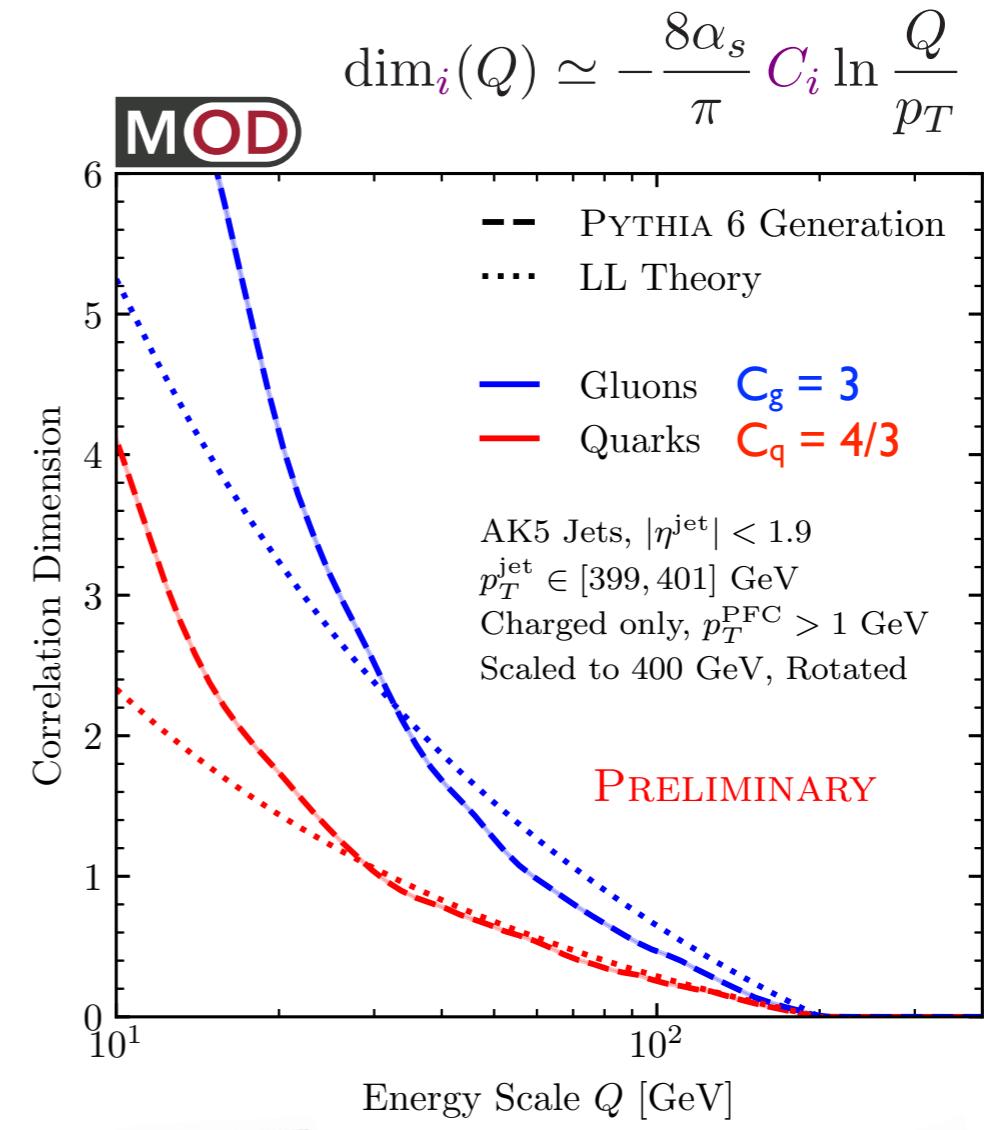
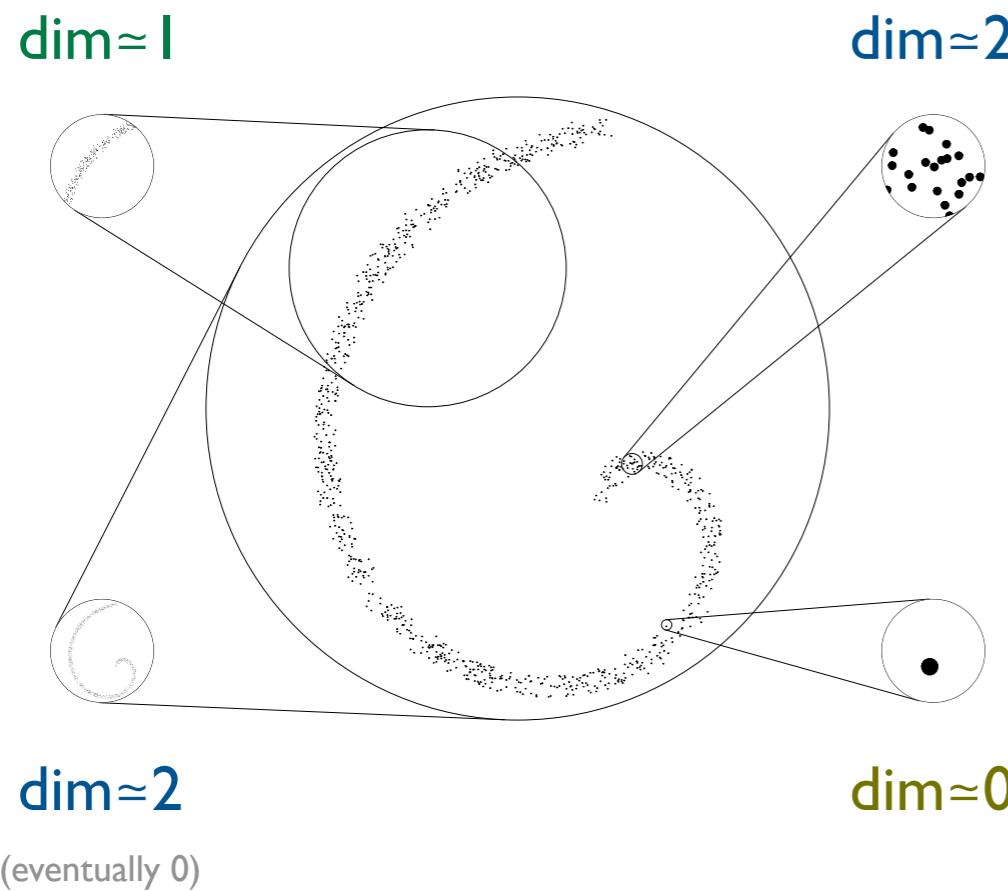
Dimensionality of Space of Jets



$$N_{\text{neighbors}}(r) \sim r^{\dim}$$

$$\Rightarrow \dim(r) \sim r \frac{\partial}{\partial r} \ln N_{\text{neighbors}}(r)$$

[Grassberger, Procaccia, [PRL 1983](#); Kégl, [NIPS 2002](#)]



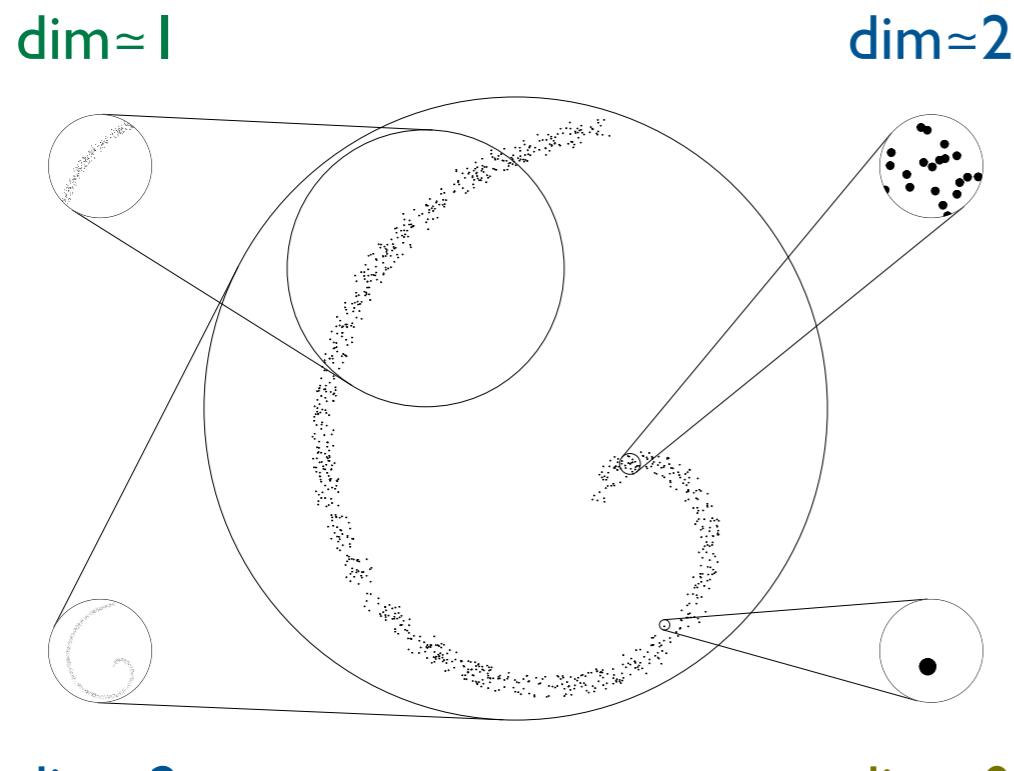
Dimensionality of Space of Jets



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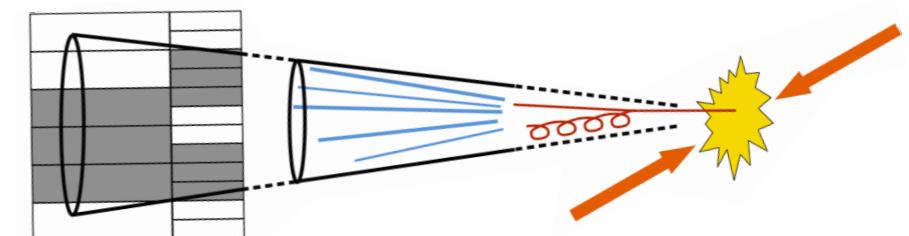
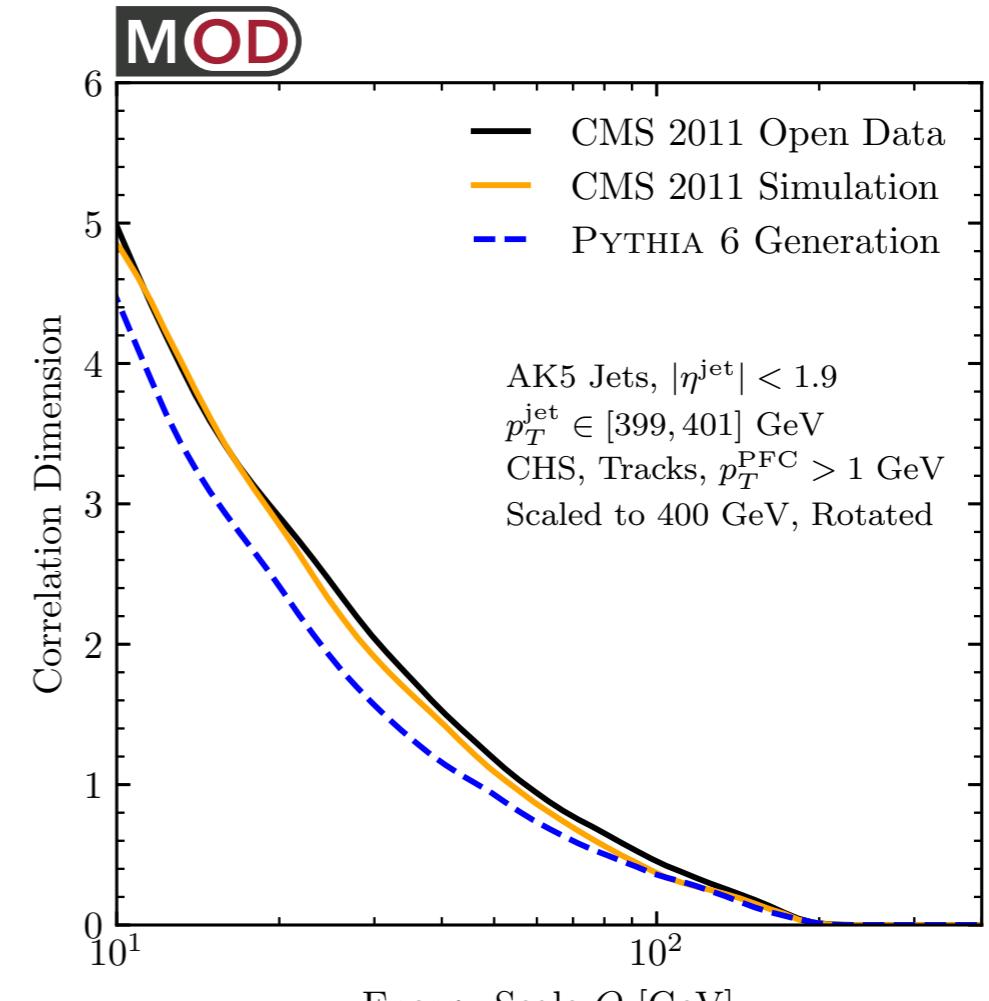
$$\Rightarrow \dim(r) \sim r \frac{\partial}{\partial r} \ln N_{\text{neighbors}}(r)$$

[Grassberger, Procaccia, [PRL 1983](#); Kégl, [NIPS 2002](#)]



(eventually 0)

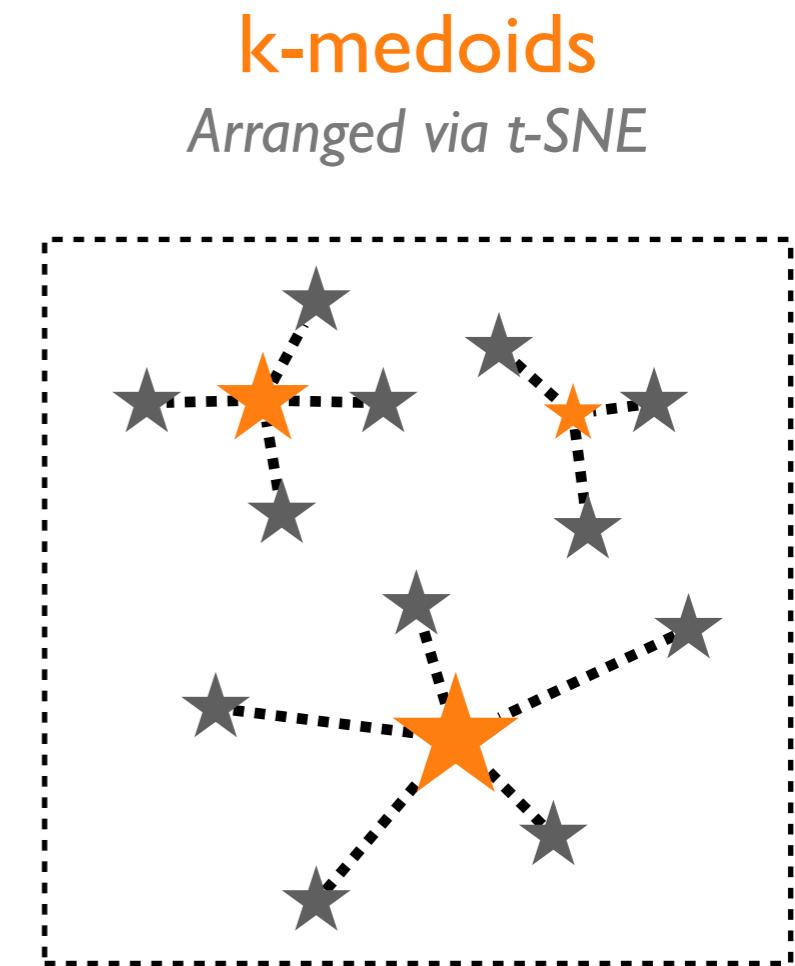
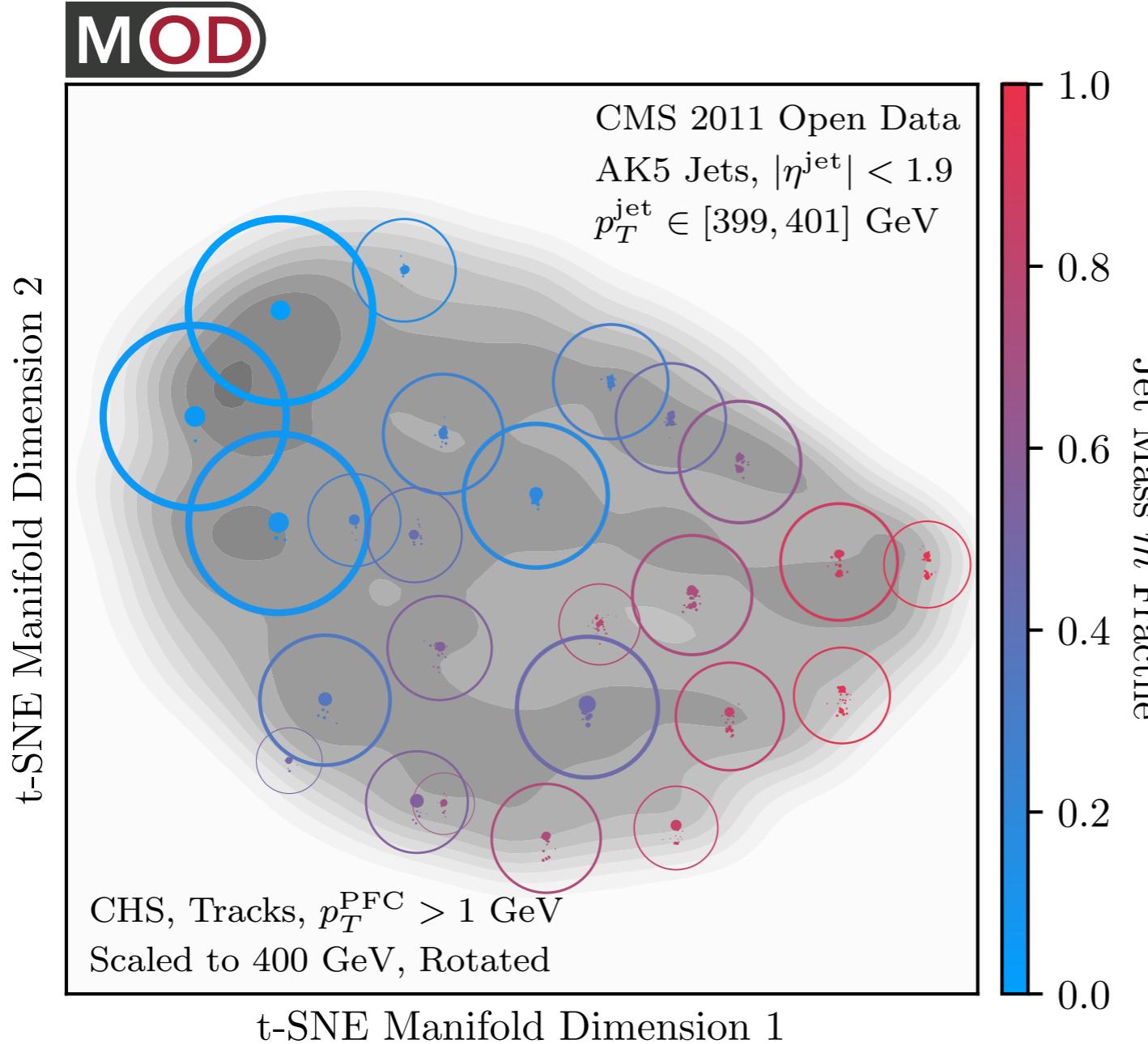
[Komiske, Mastandrea, Metodiev, Naik, [JDT, PRD 2020](#);
using [CMS Open Data](#)]



Most Representative Jets



[<http://opendata.cern.ch/>]

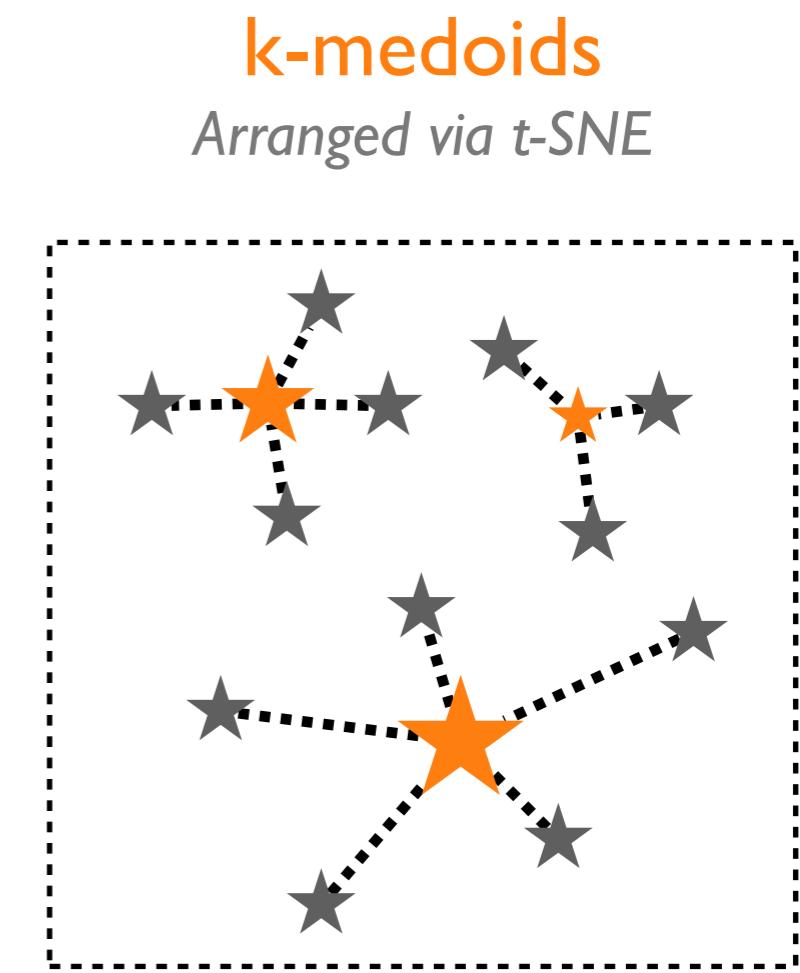
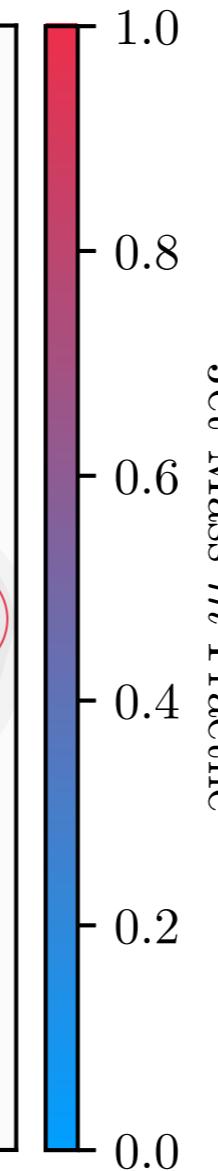
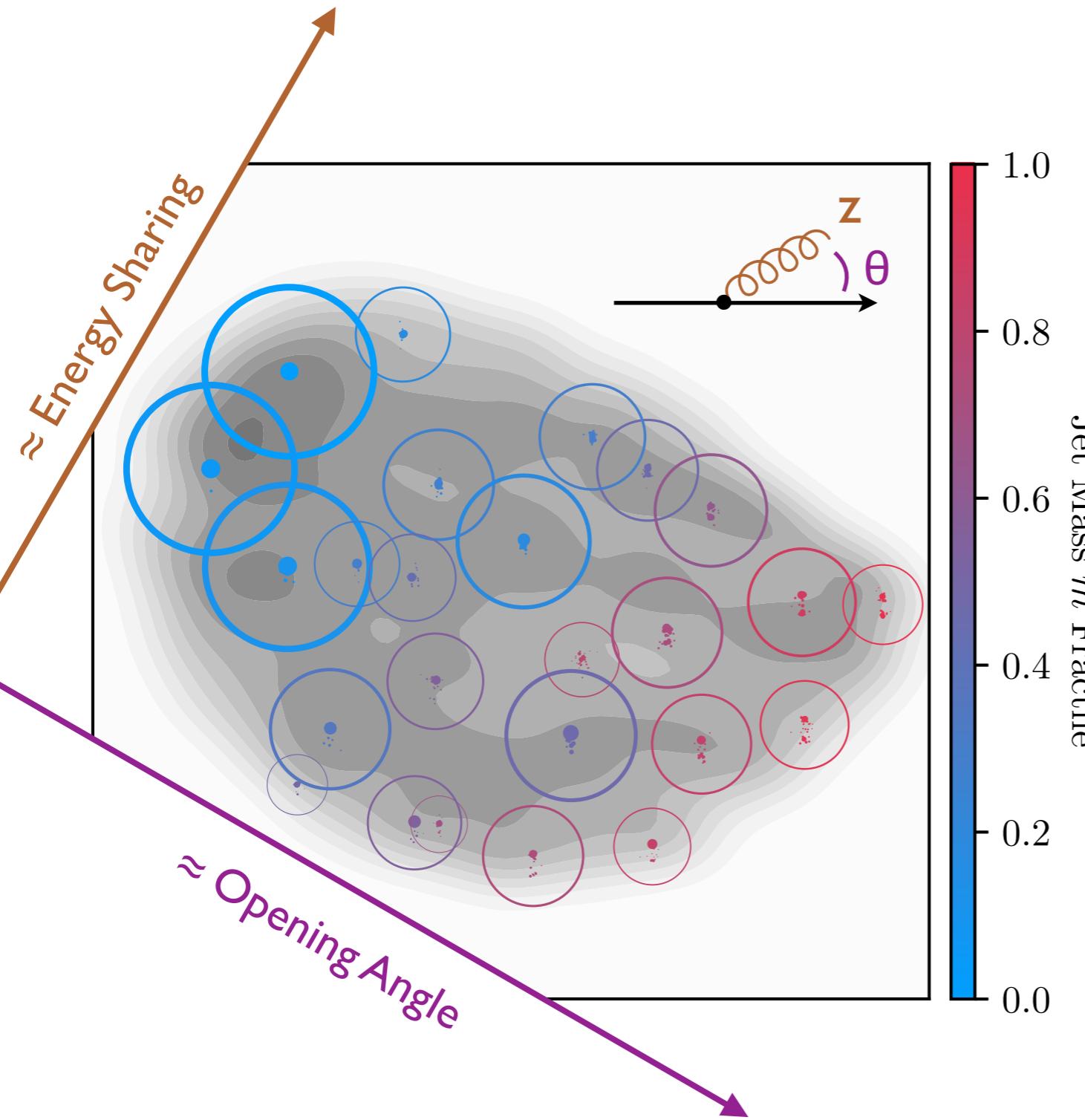


[Komiske, Mastandrea, Metodiev, Naik, JDT, [PRD 2020](#); using van der Maaten, Hinton, [JMLR 2008](#)]

Most Representative Jets

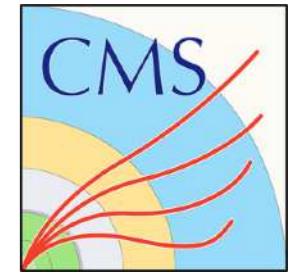


[<http://opendata.cern.ch/>]

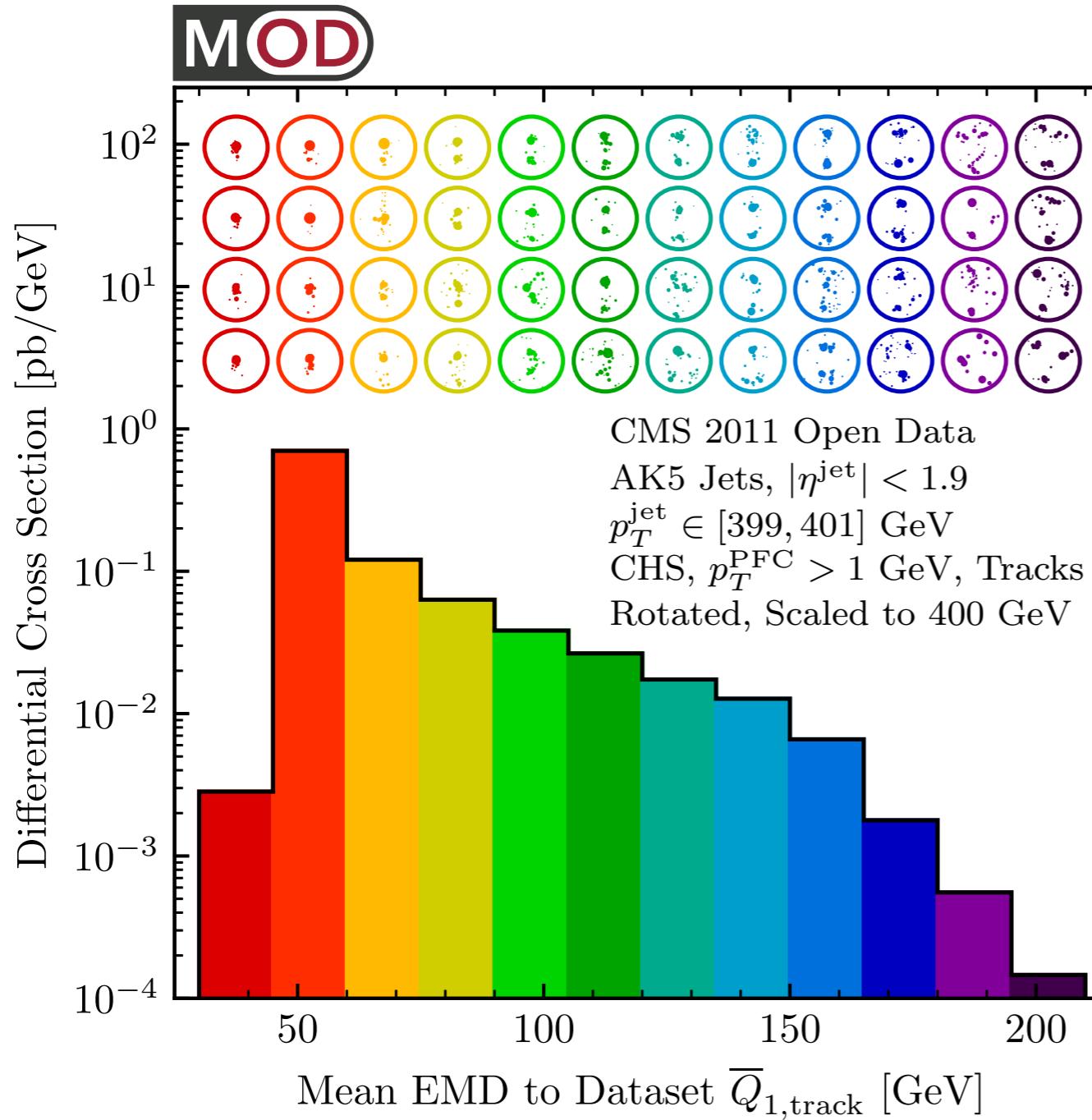


[Komiske, Mastandrea, Metodiev, Naik, JDT, *PRD* 2020; using van der Maaten, Hinton, *JMLR* 2008]

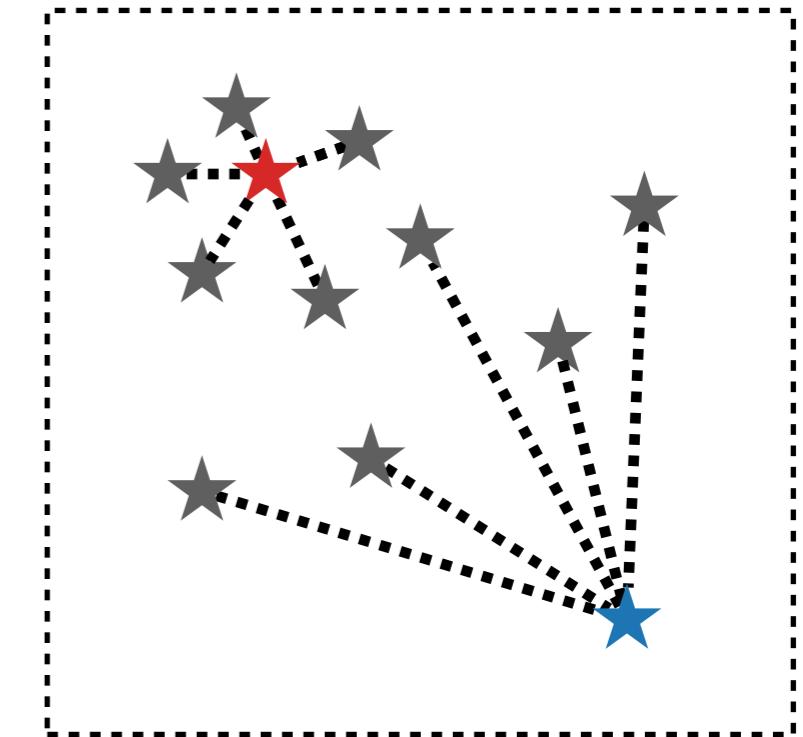
Least Representative Jets



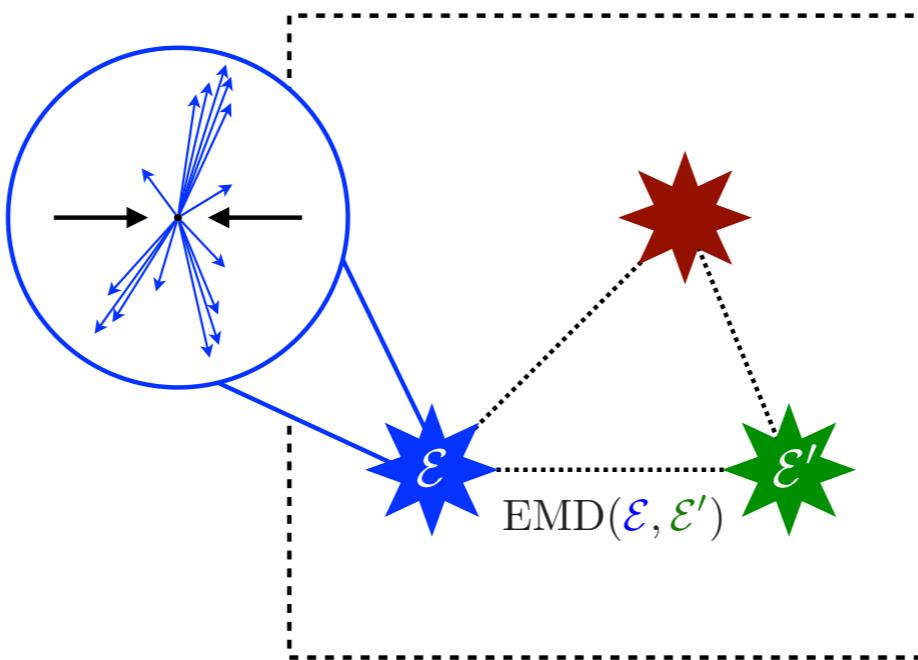
[<http://opendata.cern.ch/>]



New Physics?
Or tails of QCD?

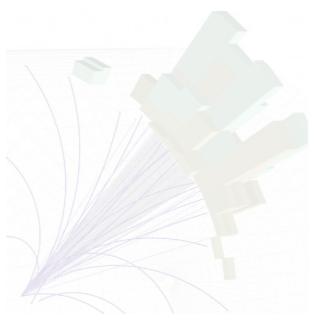


[Komiske, Mastandrea, Metodiev, Naik, JDT, [PRD 2020](#)]

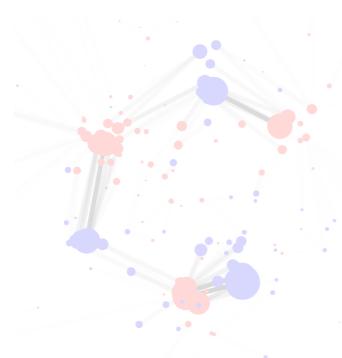


*Viewed through the data science lens,
the EMD unlocks a suite of
geometric analysis strategies*

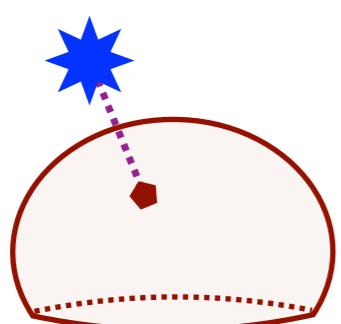
At this point, it should not be obvious why optimal transport distances should be particularly well-suited to collider applications



Going with the (Energy) Flow



The Energy Mover's Distance

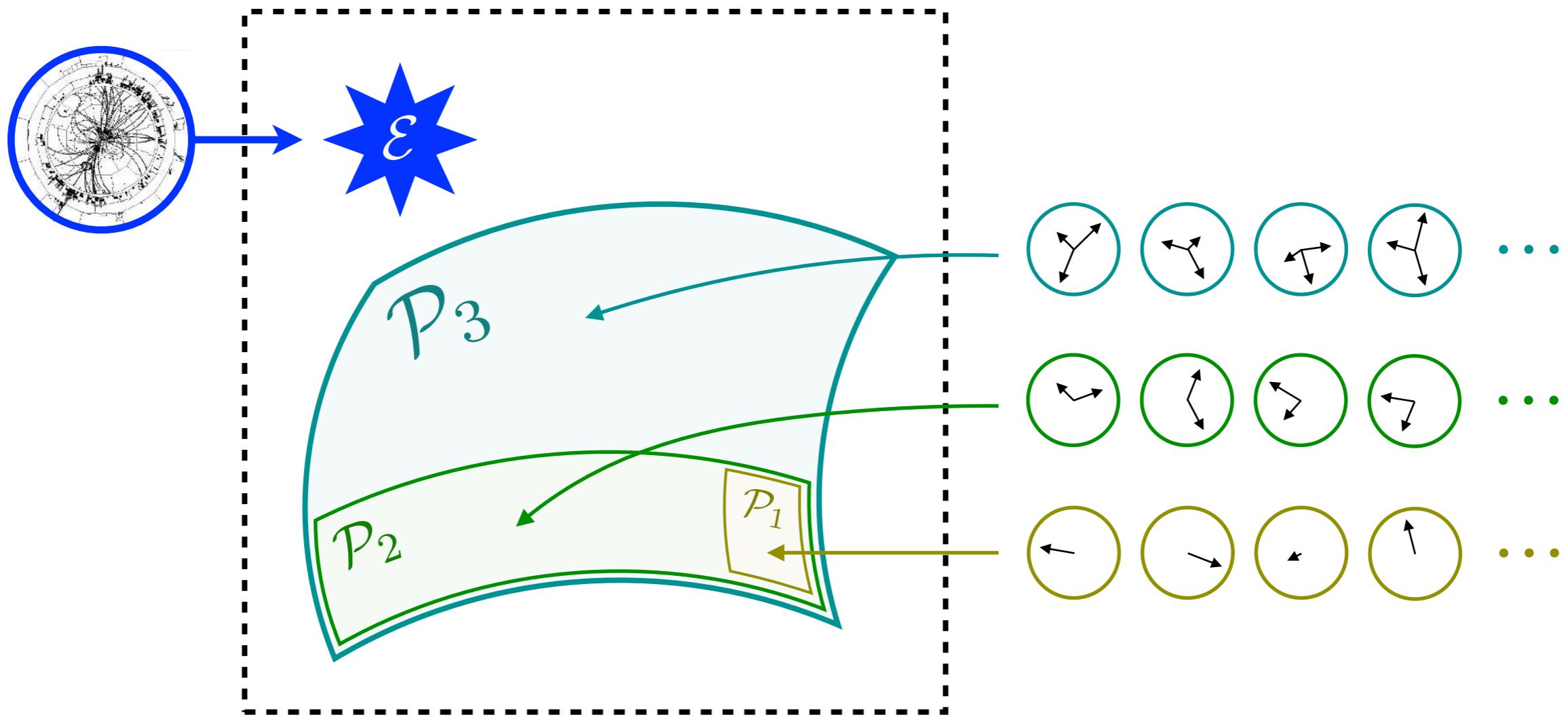


Revealing a Hidden Geometry

Given a metric space, the first geometric object you might think to construct is...

Introducing N-particle Manifolds

\mathcal{P}_N = set of all N-particle configurations



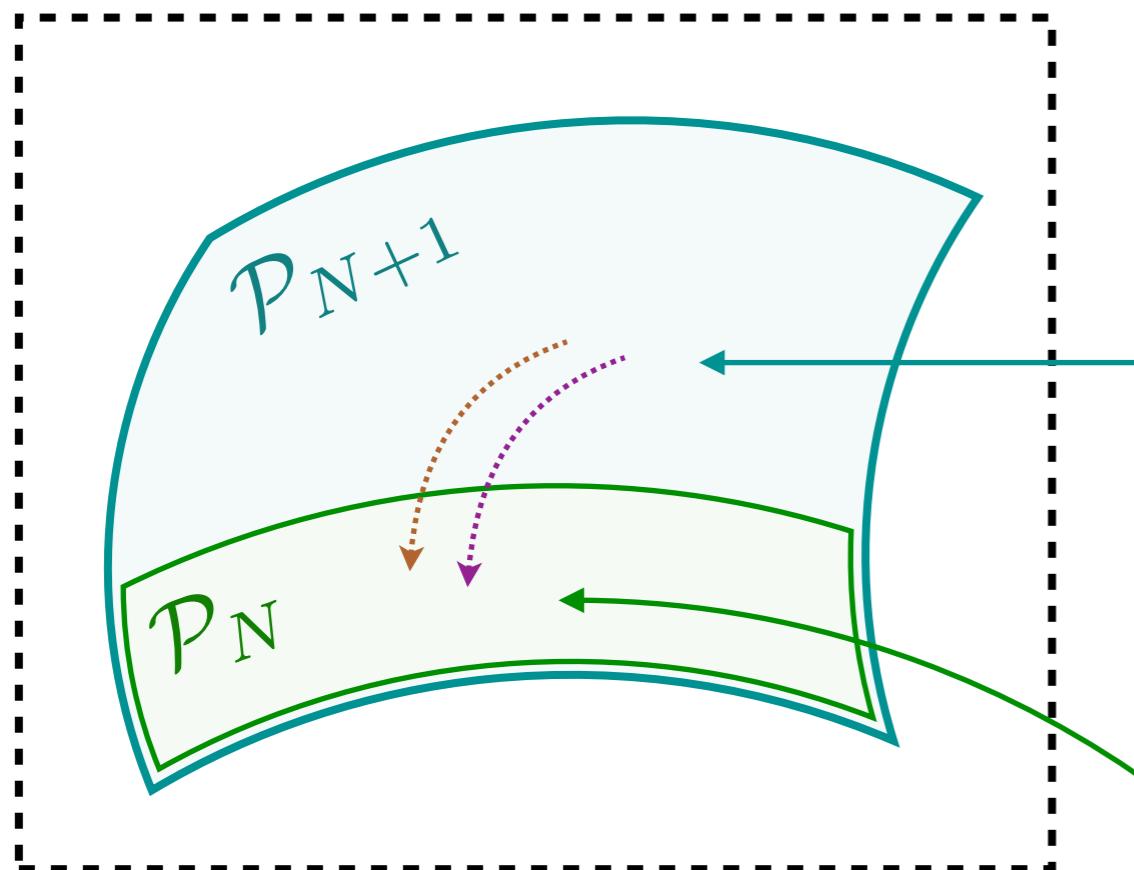
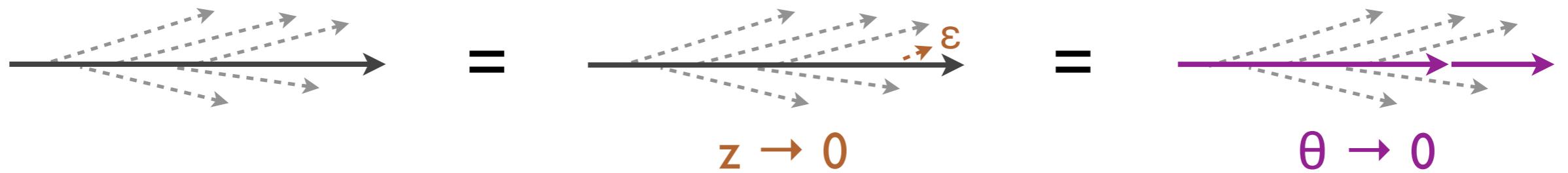
$\mathcal{P}_N \supset \mathcal{P}_{N-1} \supset \dots \supset \mathcal{P}_2 \supset \mathcal{P}_1$ by soft/collinear limits

[see related discussion in Larkoski, Melia, [PRD 2020](#)]

When are Two Events the Same?

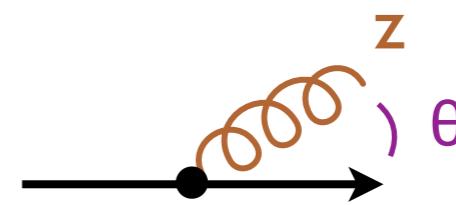
$$\mathcal{E}(\hat{n}) = \sum_i E_i \delta(\hat{n} - \hat{n}_i)$$

Energy Flow unchanged by infinitesimal soft/collinear emissions



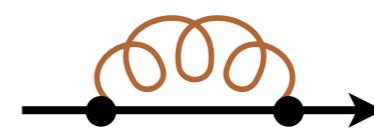
Infrared divergences “live” together!

Real:



$$dP_{i \rightarrow ig} \simeq \frac{2\alpha_s}{\pi} C_i \frac{dz}{z} \frac{d\theta}{\theta}$$

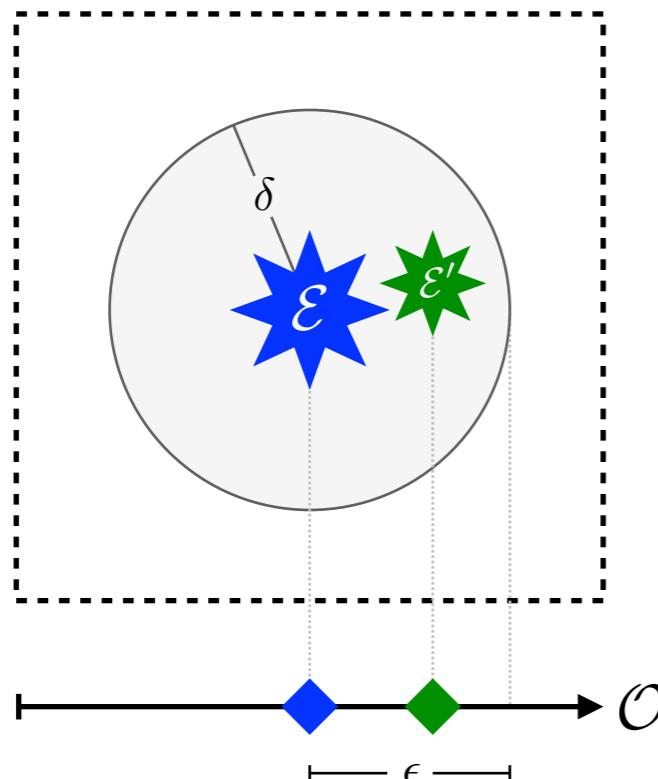
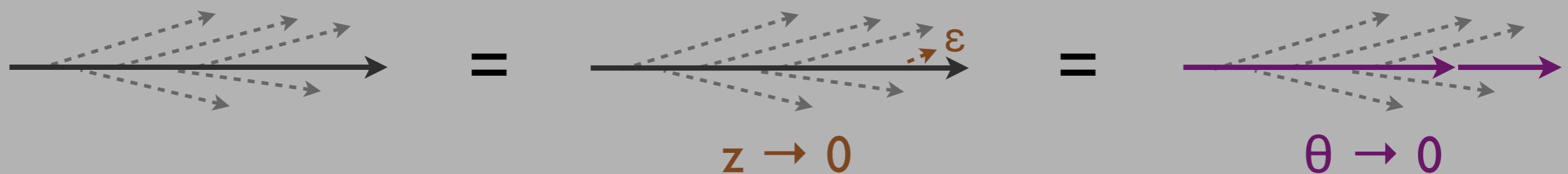
Virtual:



When are Two Events the Same?

$$\mathcal{E}(\hat{n}) = \sum_i E_i \delta(\hat{n} - \hat{n}_i)$$

Energy Flow unchanged by infinitesimal soft/collinear emissions



Infrared & Collinear Safety

≈ calculable in perturbative quantum field theory

*is** ← (see backup for subtleties)

Continuity in EMD Space

[Komiske, Metodiev, JDT, [JHEP 2020](#)]

[Sterman, Weinberg, [PRL 1977](#); Sterman, [PRD 1979](#)]

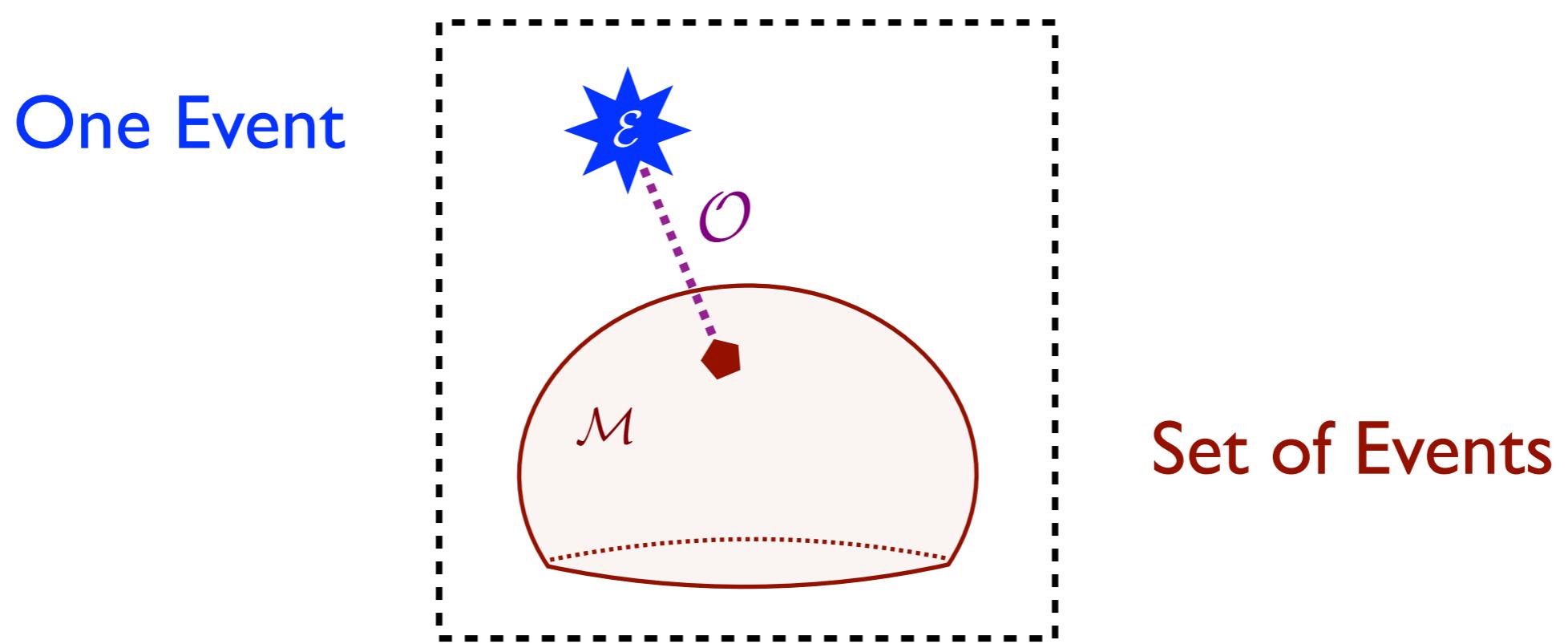
[see also Banfi, Salam, Zanderighi, [JHEP 2005](#); Larkoski, Marzani, JDT, [PRD 2015](#)]

The EMD seems to define the “natural” geometry for massless gauge theories

Open question: Can you define $|\mathcal{M}_{AB \rightarrow 12\dots n}|^2$ directly in this space?

What does it mean to “integrate” in this space?

Manifolds for Observables



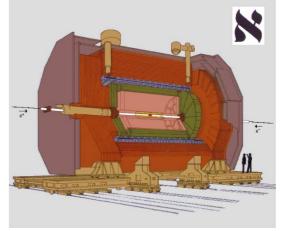
Distance of Closest Approach \Rightarrow Observable

$$O(\mathcal{E}) = \min_{\mathcal{E}' \in \mathcal{M}} \text{EMD}(\mathcal{E}, \mathcal{E}')$$

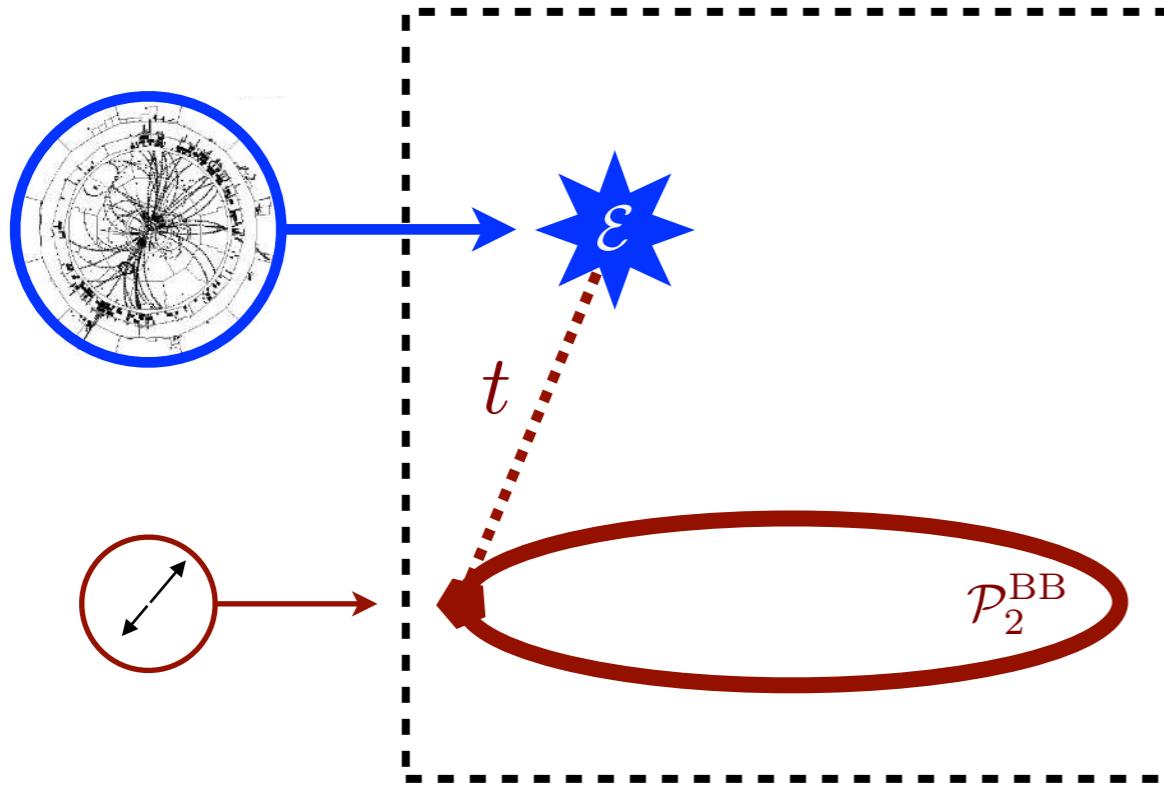
[Komiske, Metodiev, JDT, [JHEP 2020](#)]

E.g. Thrust

How dijet-like is an event?



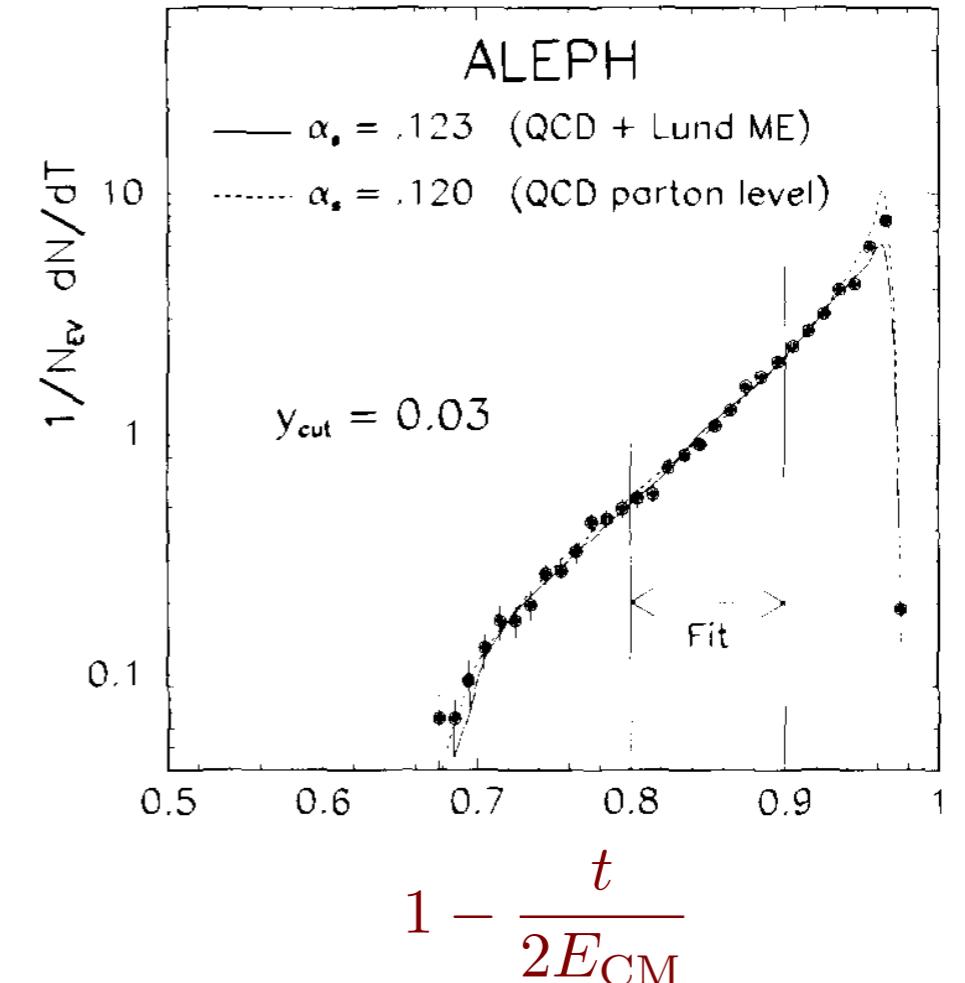
$$t(\mathcal{E}) = \min_{\mathcal{E}' \in \mathcal{P}_2^{\text{BB}}} \text{EMD}_2(\mathcal{E}, \mathcal{E}')$$



All Back-to-Back Two Particle Configurations

$$\mathcal{P}_2^{\text{BB}} = \left\{ \begin{array}{c} \text{red circles with two arrows pointing away from each other} \\ \cdots \end{array} \right\}$$

(using $\beta=2$ EMD variant)



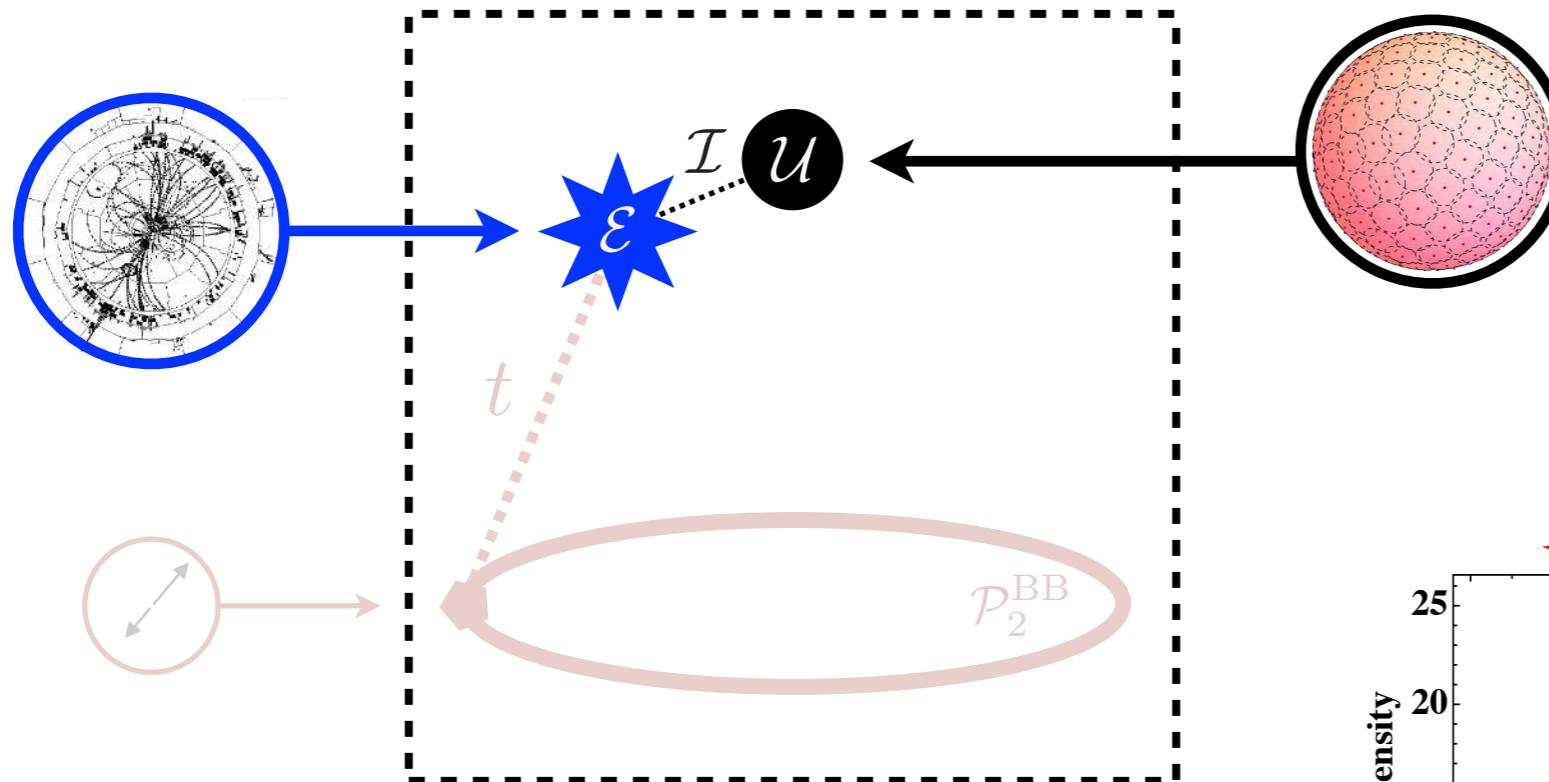
$$1 - \frac{t}{2E_{\text{CM}}}$$

$$\text{cf. } T(\mathcal{E}) = \max_{\hat{n}} \frac{\sum_i |\vec{p}_i \cdot \hat{n}|}{\sum_j |\vec{p}_j|}$$

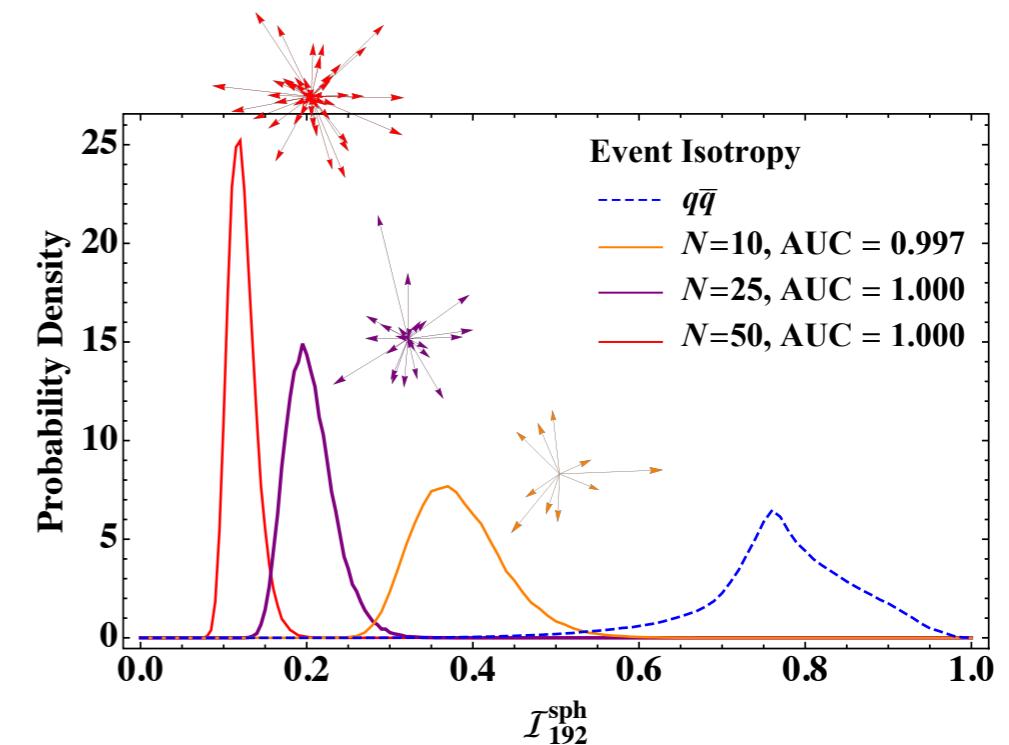
[Komiske, Metodiev, JDT, JHEP 2020]
 [Brandt, Peyrou, Sosnowski, Wroblewski, PL 1964; Farhi, PRL 1977; ALEPH, PLB 1991]

New! Event Isotropy

How isotropic is an event?



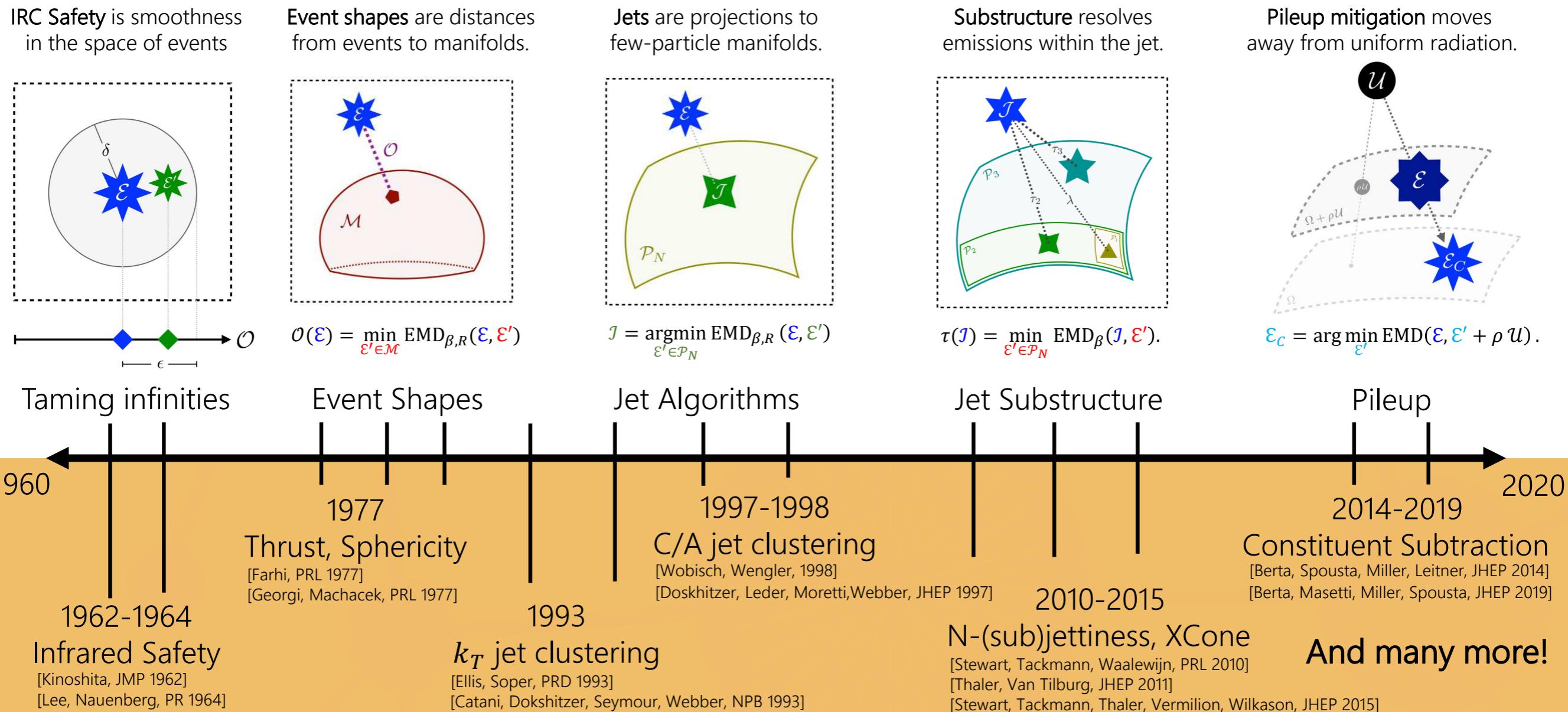
$$\mathcal{I}(\mathcal{E}) = \text{EMD}(\mathcal{E}, \mathcal{U})$$



[Cesarotti, JDT, [JHEP 2020](#);
see also Cesarotti, Reece, Strassler, [arXiv 2020](#)]



Six Decades of Collider Physics Translated into a New Geometric Language!

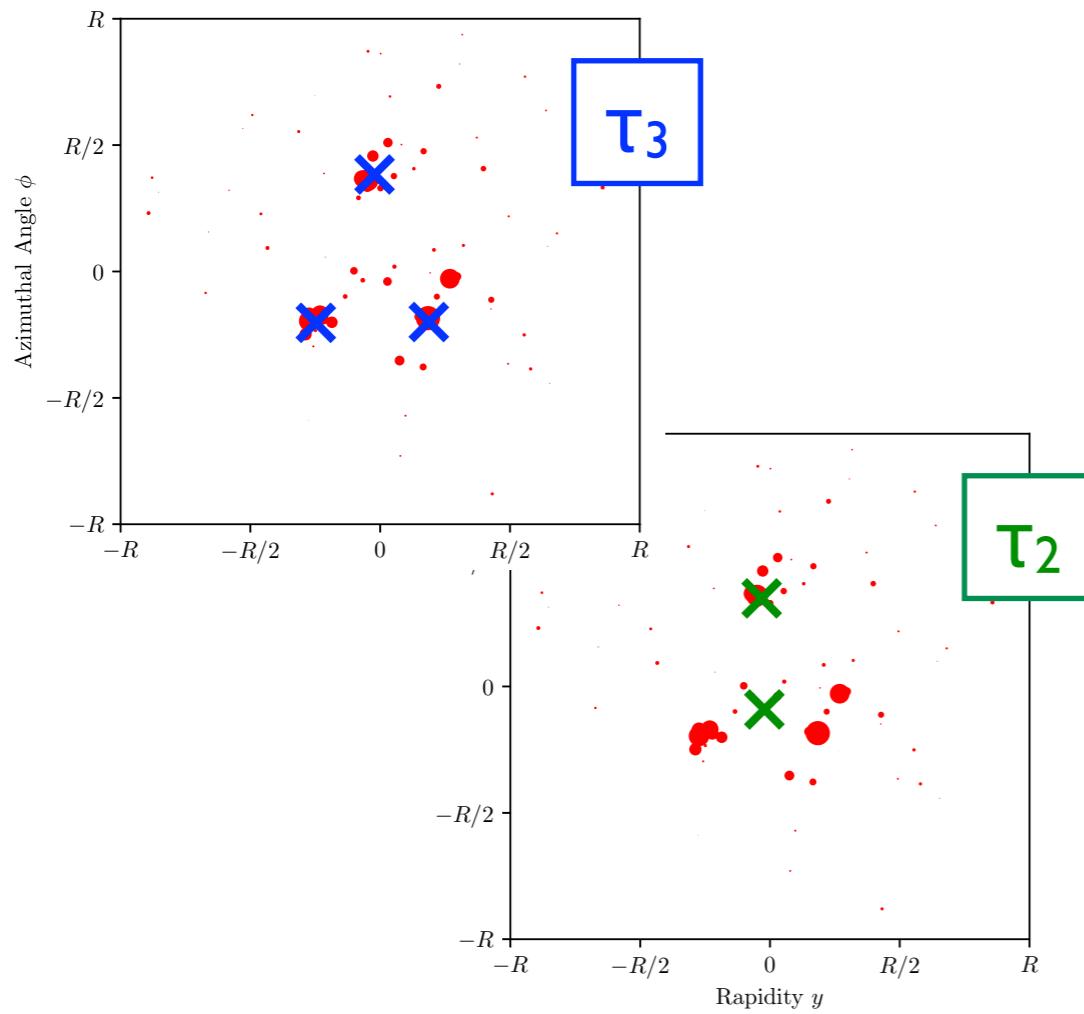


[timeline from Eric Metodiev]

N-subjettiness

Ubiquitous jet substructure observable used for almost a decade...

$$\tau_N(\mathcal{J}) = \min_{N \text{ axes}} \sum_i E_i \min \{\theta_{1,i}, \theta_{2,i}, \dots, \theta_{N,i}\}$$



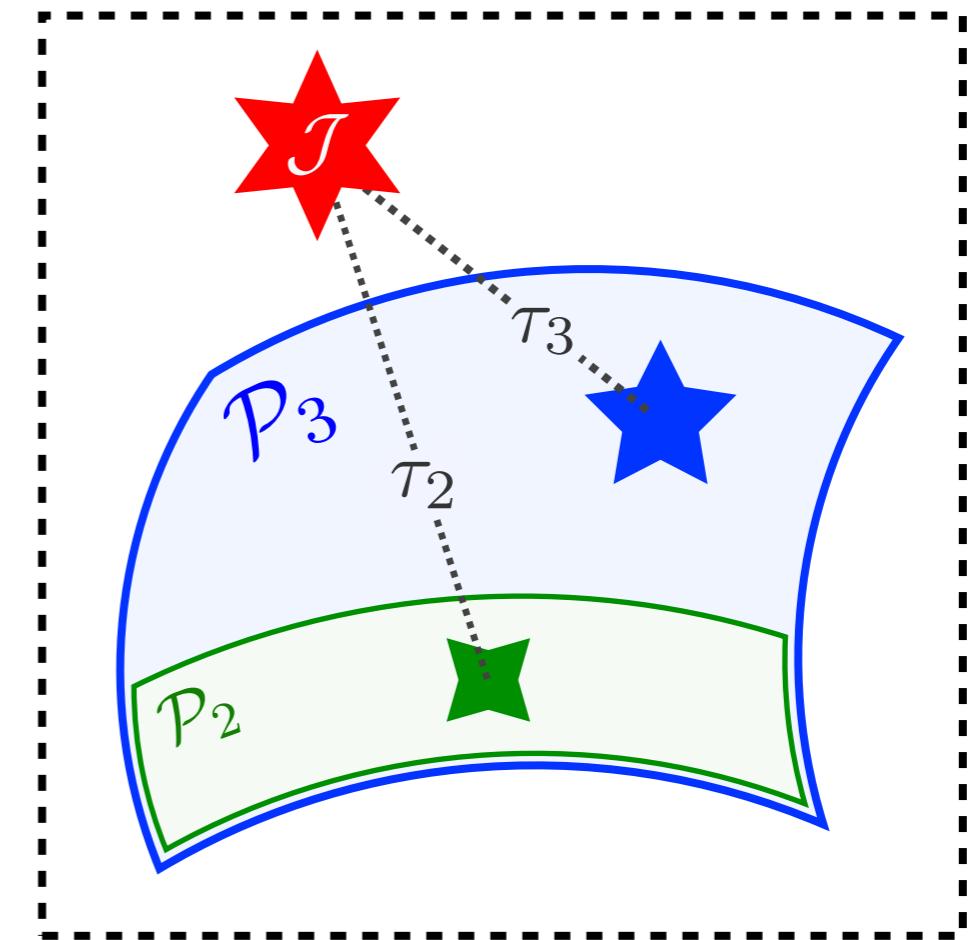
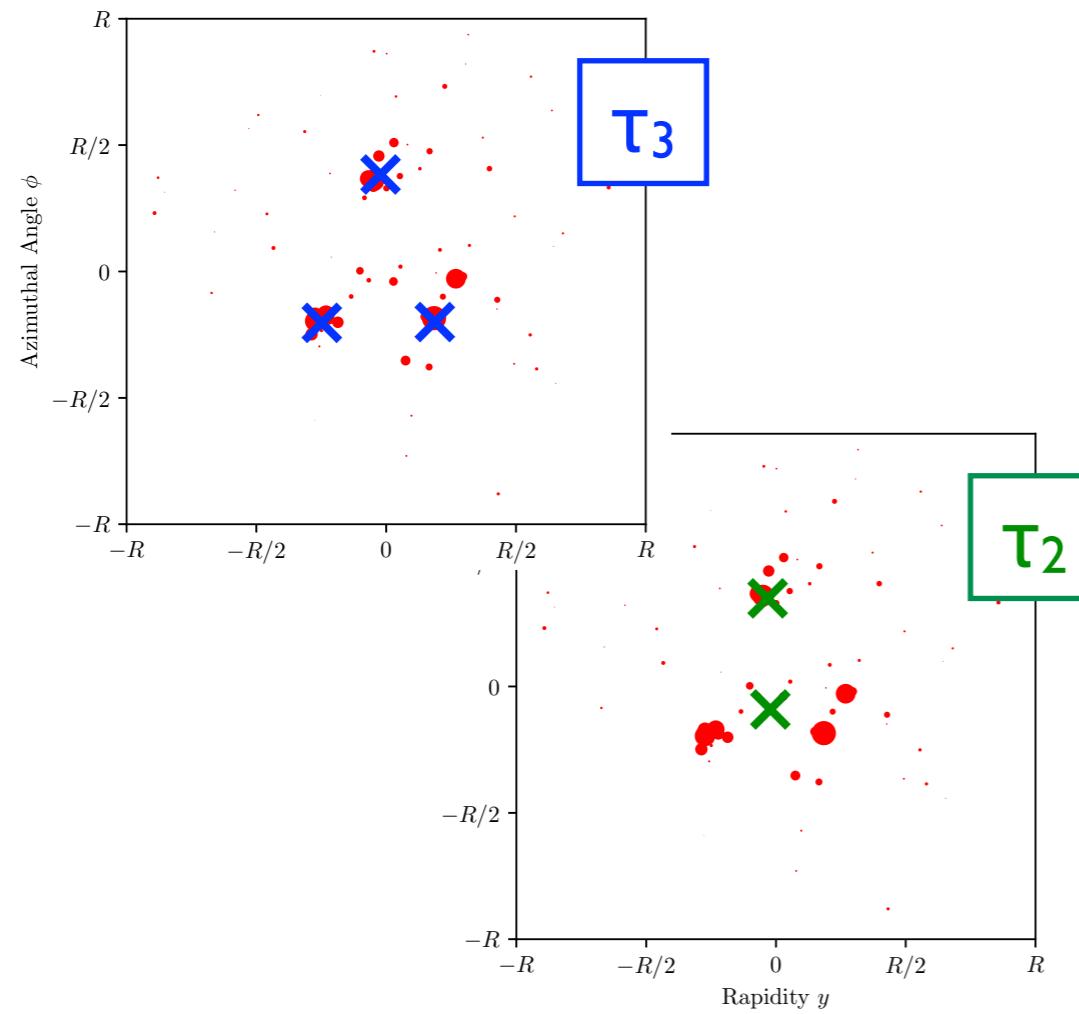
[JDT, Van Tilburg, [JHEP 2011](#), [JHEP 2012](#);
based on Brandt, Dahmen, [ZPC 1979](#); Stewart, Tackmann, Waalewijn, [PRL 2010](#)]



N-subjettiness = Point to Manifold EMD

...is secretly an optimal transport problem

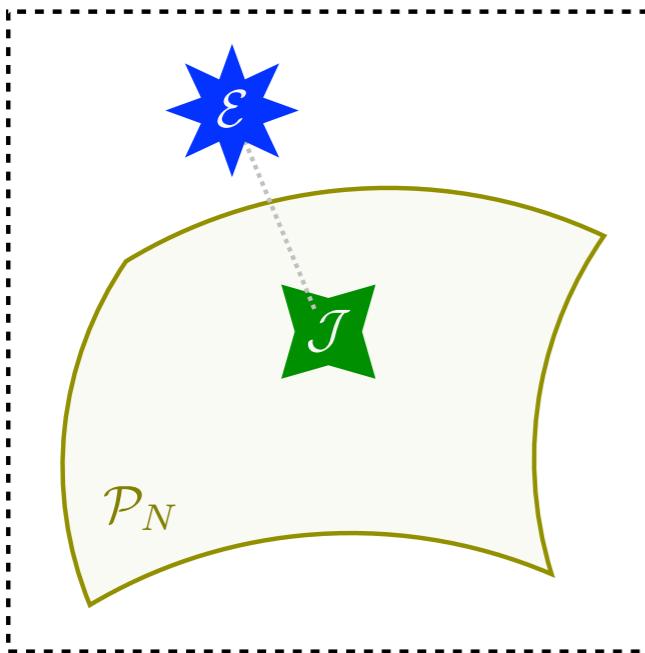
$$\tau_N(\mathcal{J}) = \min_{\mathcal{J}' \in \mathcal{P}_N} \text{EMD}(\mathcal{J}, \mathcal{J}')$$



[JDT, Van Tilburg, JHEP 2011, JHEP 2012;
rephrased in the language of Komiske, Metodiev, JDT, PRL 2019]



More Fun with N-particle Manifolds



N-jettiness

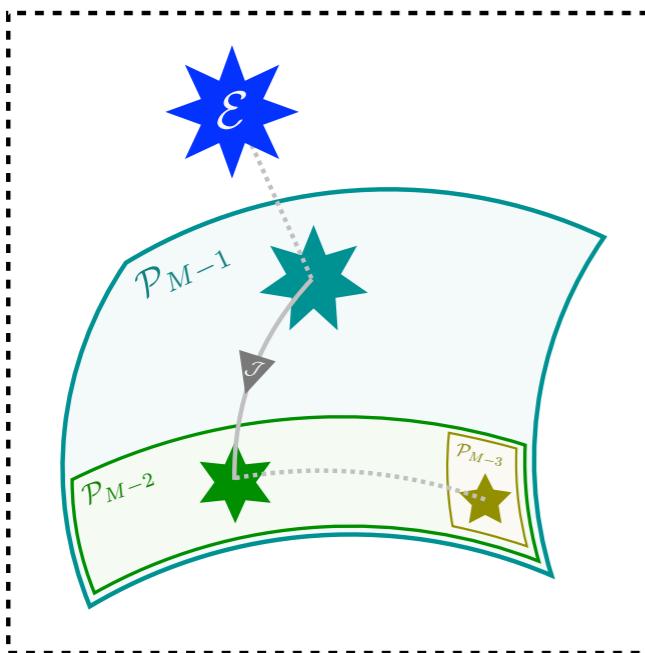
Distance of closest approach to N-particle manifold

[Brandt, Dahmen, [ZPC 1979](#); Stewart, Tackmann, Waalewijn, [PRL 2010](#)]

Exclusive Cone Jet Finding

Point of closest approach on N-particle manifold

[Stewart, Tackmann, JDT, Vermilion, Wilkason, [JHEP 2015](#)]



Sequential Jet Recombination

Iteratively stepping between various N-particle manifolds

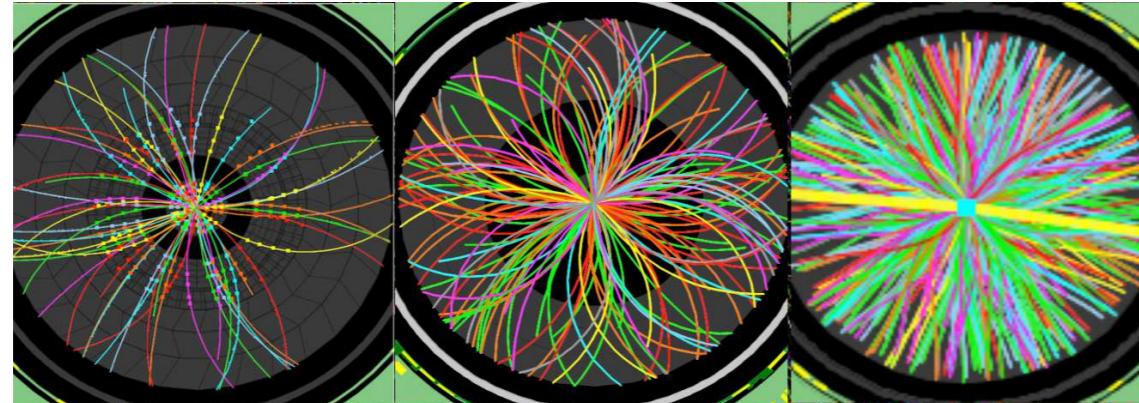
[Catani, Dokshitzer, Seymour, Webber, [NPB 1993](#); Ellis, Soper, [PRD 1993](#)]

[Dokshitzer, Leder, Moretti, Webber, [JHEP 1997](#); Wobisch, Wengler, [arXiv 1999](#)]

[Butterworth, Couchman, Cox, Waugh, [CPC 2003](#); Larkoski, Neill, JDT, [JHEP 2014](#)]

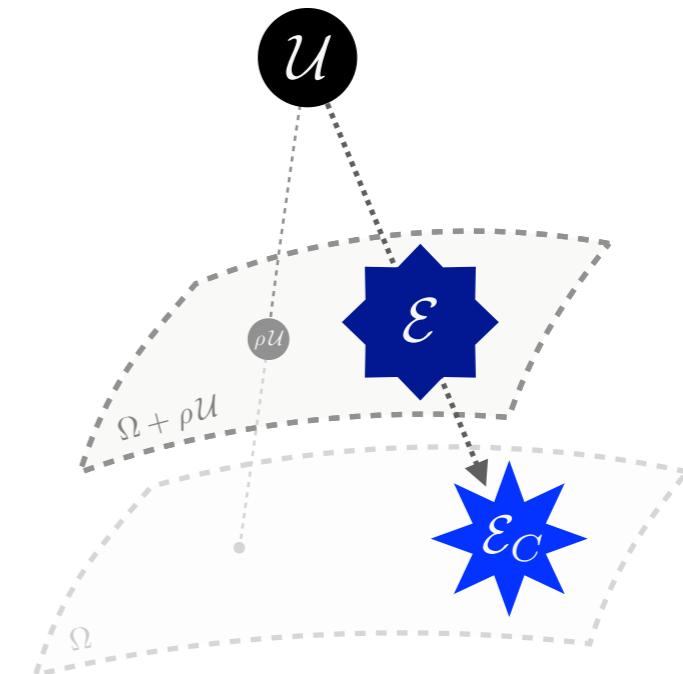
[Komiske, Metodiev, JDT, [JHEP 2020](#)]

Pileup Mitigation



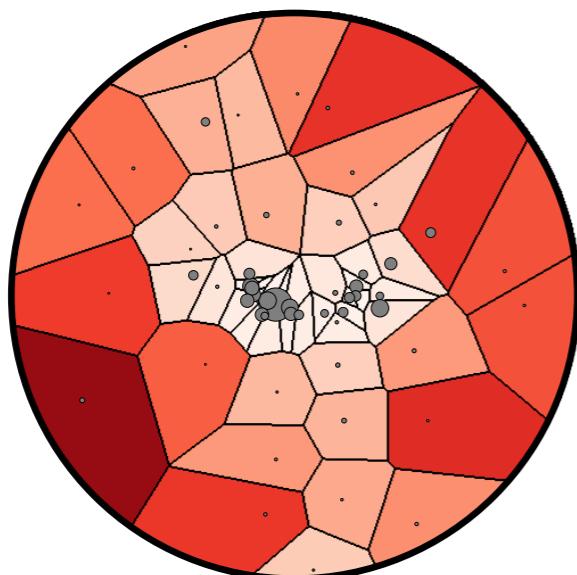
[see review in Soyez, PR 2019]

Uniform event contamination from overlapping proton-proton collisions



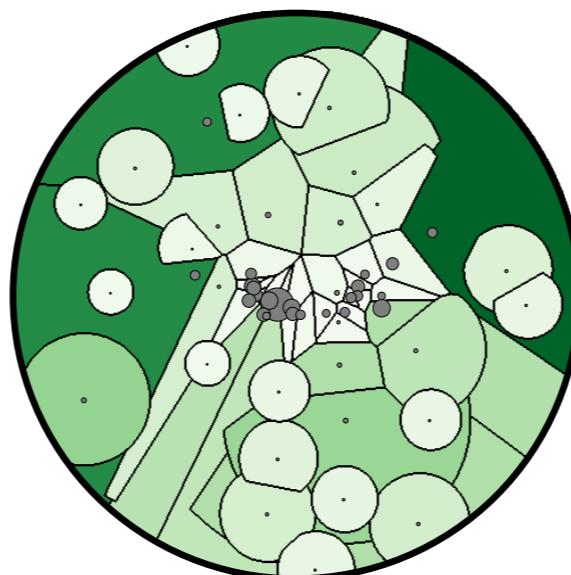
Pileup Mitigation:
“Move away” from uniform event

Voronoi



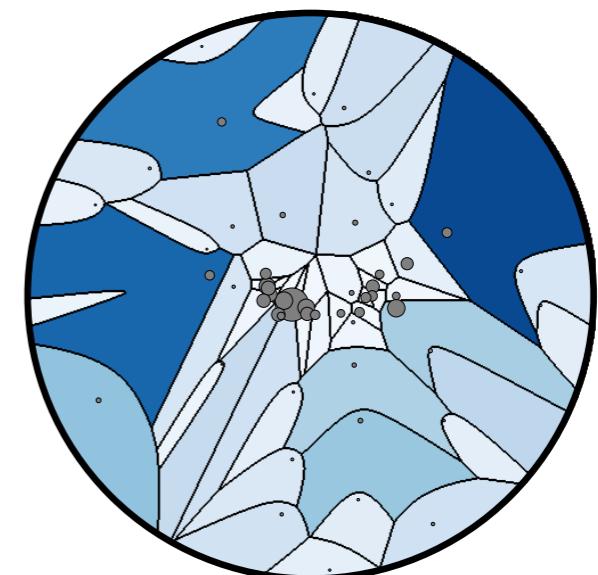
[Cacciari, Salam, Soyez, JHEP 2008]

Constituent Subtraction

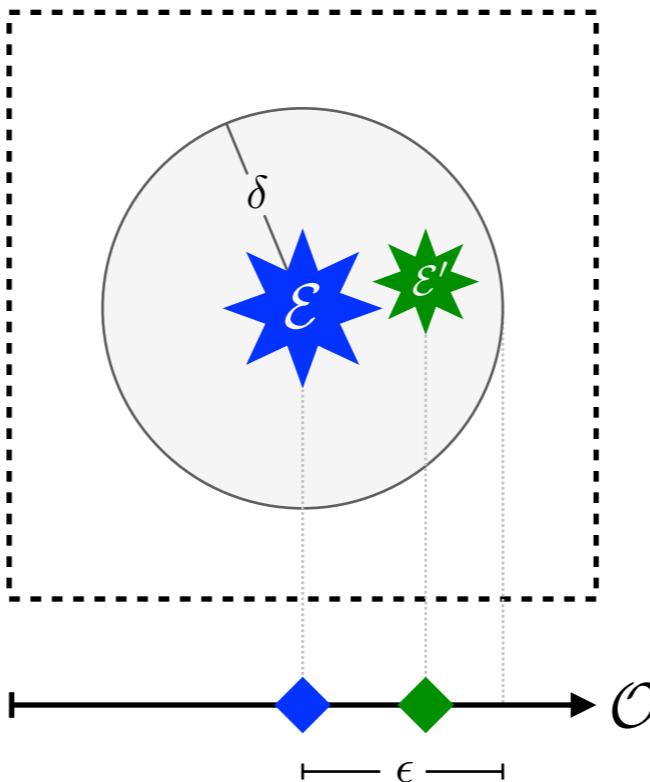


[Berta, Spousta, Miller, Leitner, JHEP 2014]

Apollonius



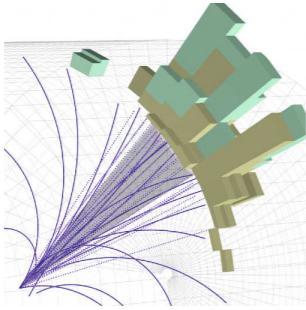
[Komiske, Metodiev, JDT, JHEP 2020]



We are just beginning to leverage the conceptual richness of optimal transport for high-energy physics application

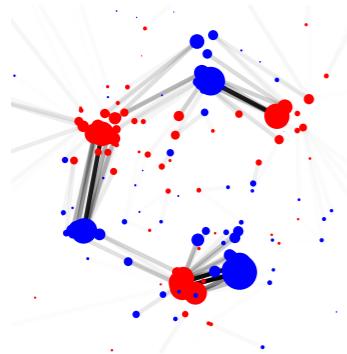
Ask me how far down this rabbit hole goes!

Summary



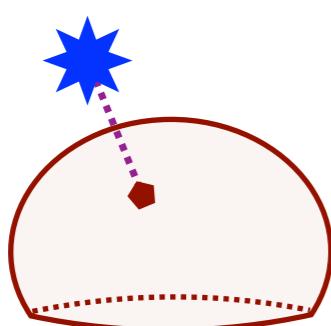
Going with the (Energy) Flow

*Restricting our attention to IRC safe information
is a theoretically motivated data analysis strategy*



The Energy Mover's Distance

*Optimal transport allows us to triangulate the space
of collider events and define an emergent geometry*



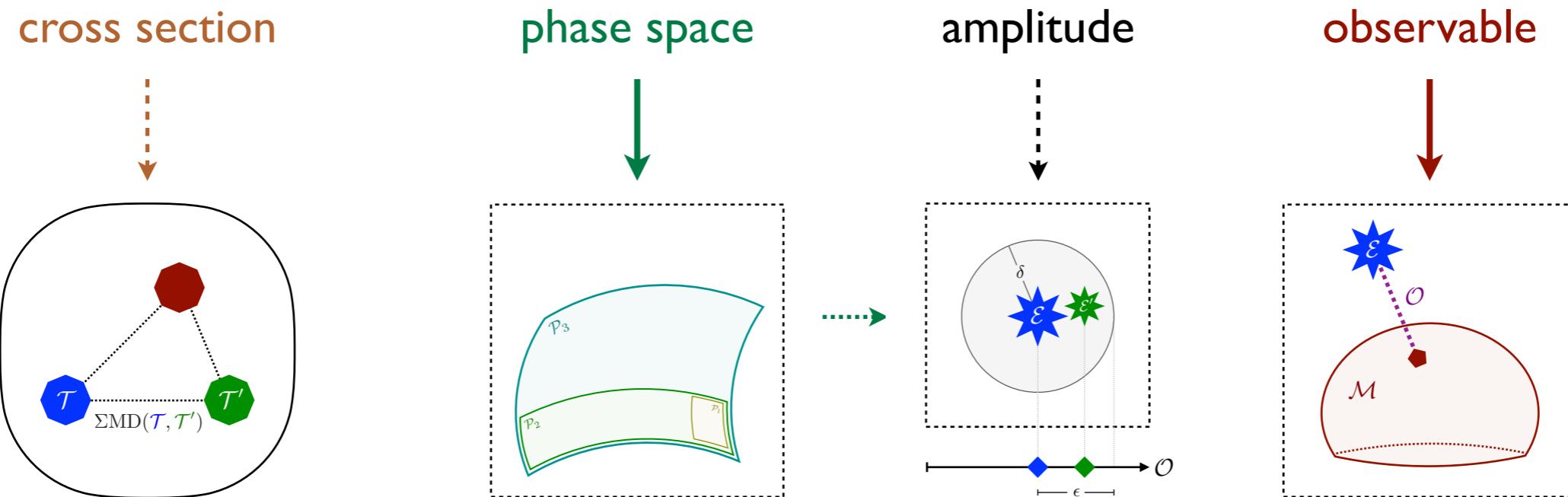
Revealing a Hidden Geometry

*We can gain new perspectives on concepts/techniques
in QFT and collider physics from the last half century*

How far down does this rabbit hole go?

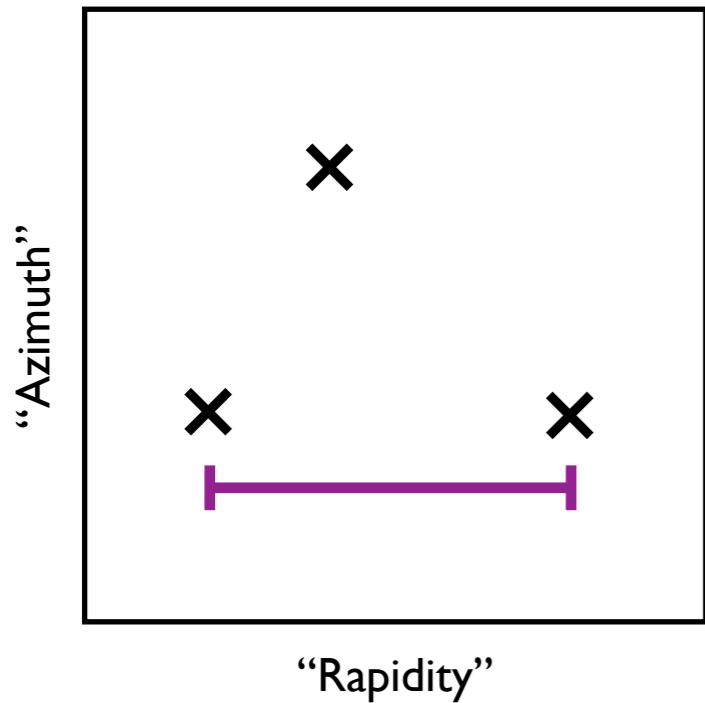
Master Formula for Collider Physics

$$\sigma_{\text{obs}} \simeq \frac{1}{2E_{\text{CM}}^2} \sum_{n=2}^{\infty} \int d\Phi_n |\mathcal{M}_{AB \rightarrow 12\dots n}|^2 f_{\text{obs}}(\Phi_n)$$



[Komiske, Metodiev, JDT, [JHEP 2020](#)]

Direction Space



x = Direction

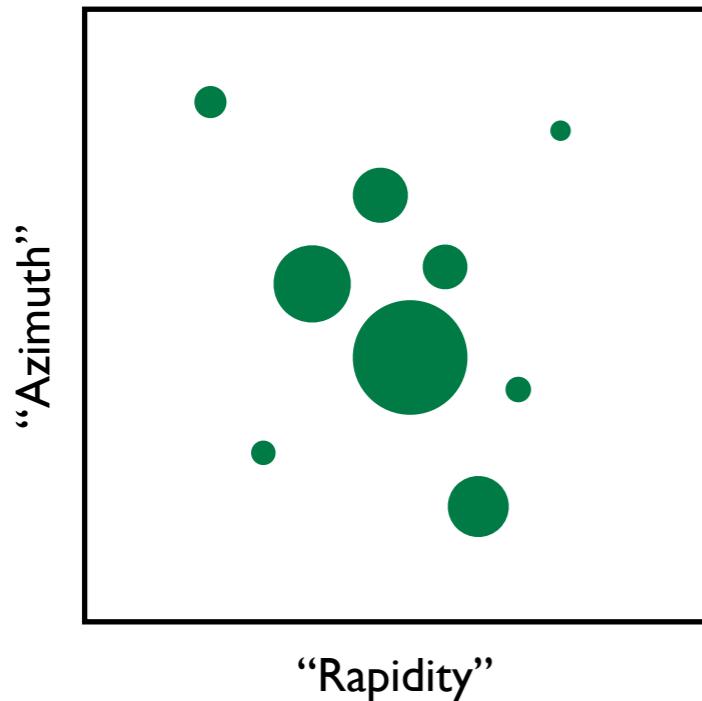
= Angular Distance

$$n_i^\mu = \frac{p_i^\mu}{E_i} = (1, \hat{n})^\mu$$

$$\theta_{ij} = \sqrt{2n_i^\mu n_{j\mu}}$$

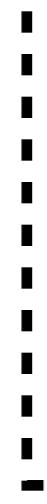
(for massless particles)

Direction Space Distribution



● = Weighted Direction

— = Angular Distance



★ = Event

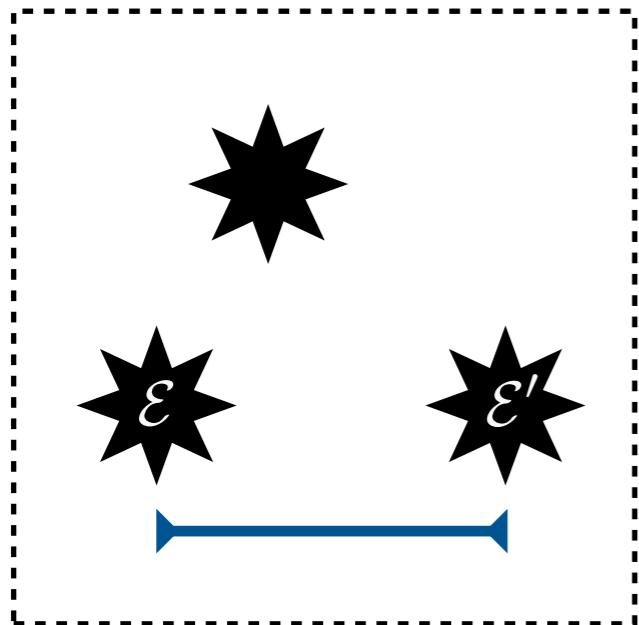
$$n_i^\mu = \frac{p_i^\mu}{E_i} = (1, \hat{n})^\mu$$

$$w_i = E_i$$

$$\theta_{ij} = \sqrt{2n_i^\mu n_{j\mu}}$$

(for massless particles)

Event Space



★ = Event
↔ = EMD
Energy Mover's Distance

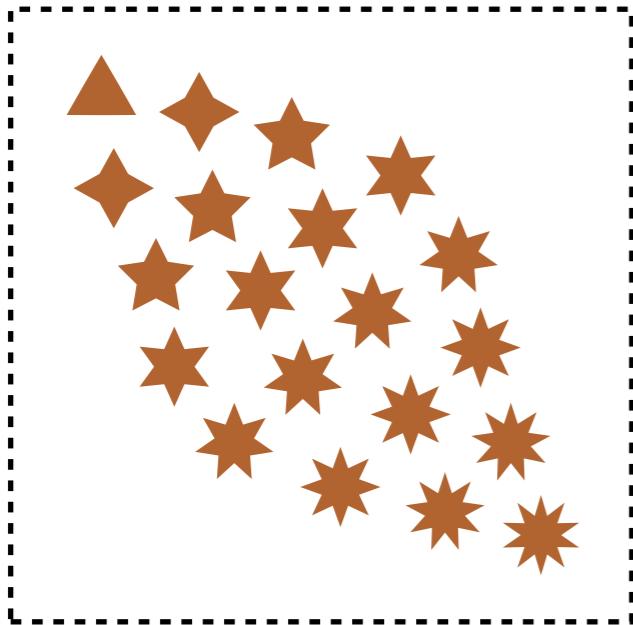
$$\mathcal{E}(\hat{n}) = \sum_i E_i \delta(\hat{n} - \hat{n}_i)$$

$$\text{EMD}(\mathcal{E}, \mathcal{E}') = \min_{\{f\}} \sum_i \sum_j f_{ij} \theta_{ij}$$

(for equal total energy)

[Komiske, Metodiev, JDT, PRL 2019]

Event Space Distribution



= Weighted Event

$$\mathcal{E}(\hat{n}) = \sum_i E_i \delta(\hat{n} - \hat{n}_i)$$

$$w_a = \sigma_a$$

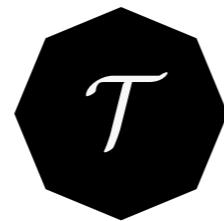
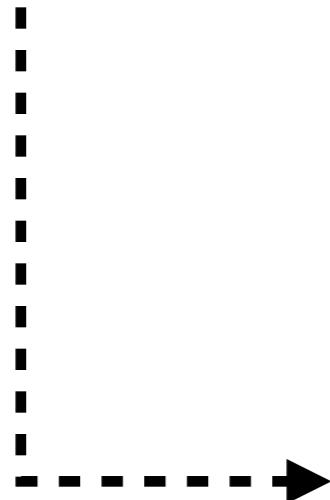


= EMD

Energy
Mover's Distance

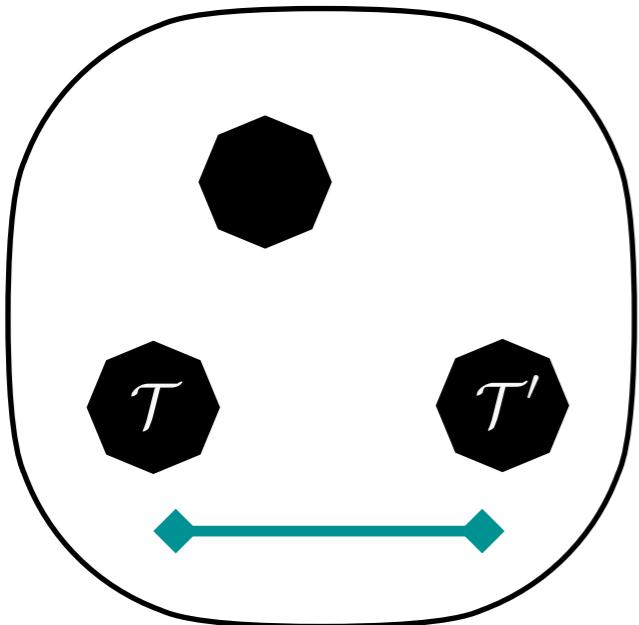
$$\text{EMD}(\mathcal{E}, \mathcal{E}') = \min_{\{f\}} \sum_i \sum_j f_{ij} \theta_{ij}$$

(for equal total energy)



= Theory

Theory Space



● = Theory

↔ = ΣMD
Cross-Section
Mover's Distance

$$\mathcal{T}(\mathcal{E}) = \sum_a \sigma_a \delta(\mathcal{E} - \mathcal{E}_a)$$

$$\Sigma\text{MD}(\mathcal{T}, \mathcal{T}') = \min_{\{\mathcal{F}\}} \sum_a \sum_b \mathcal{F}_{ab} \text{EMD}(\mathcal{E}_a, \mathcal{E}'_b)$$

(for equal total xsec)

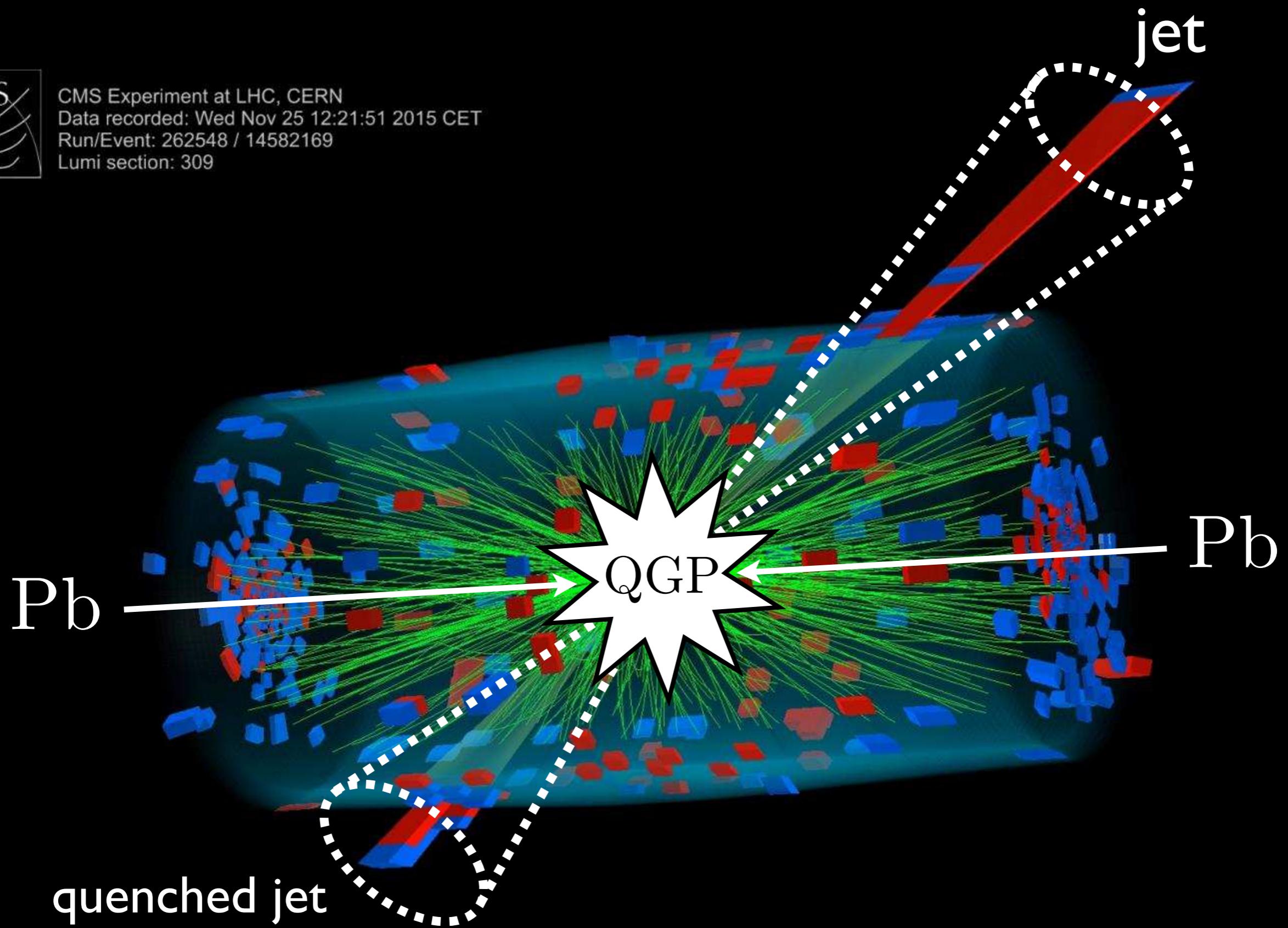
A distance between theories!

(e.g. EMD : N-jettiness :: ΣMD : k-eventiness)

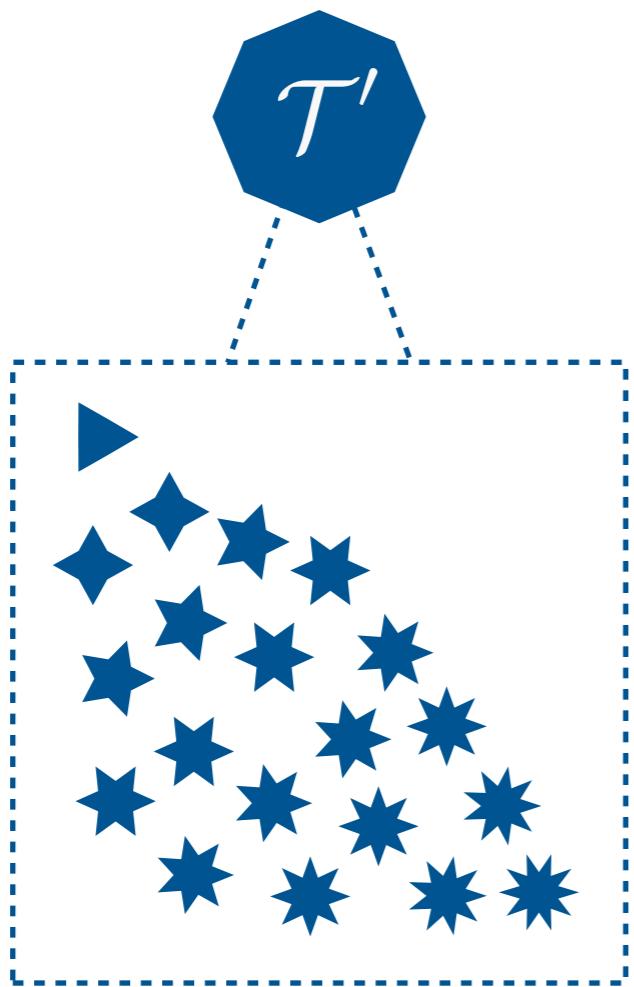
[Komiske, Metodiev, JDT, [JHEP 2020](#)]



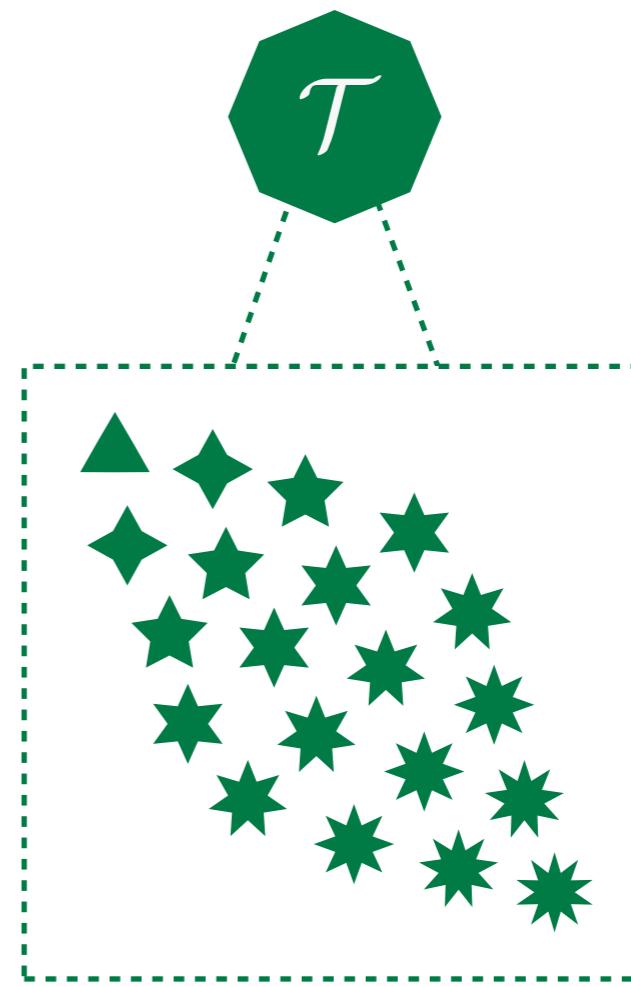
CMS Experiment at LHC, CERN
Data recorded: Wed Nov 25 12:21:51 2015 CET
Run/Event: 262548 / 14582169
Lumi section: 309



Theory Prime: In-Medium QCD

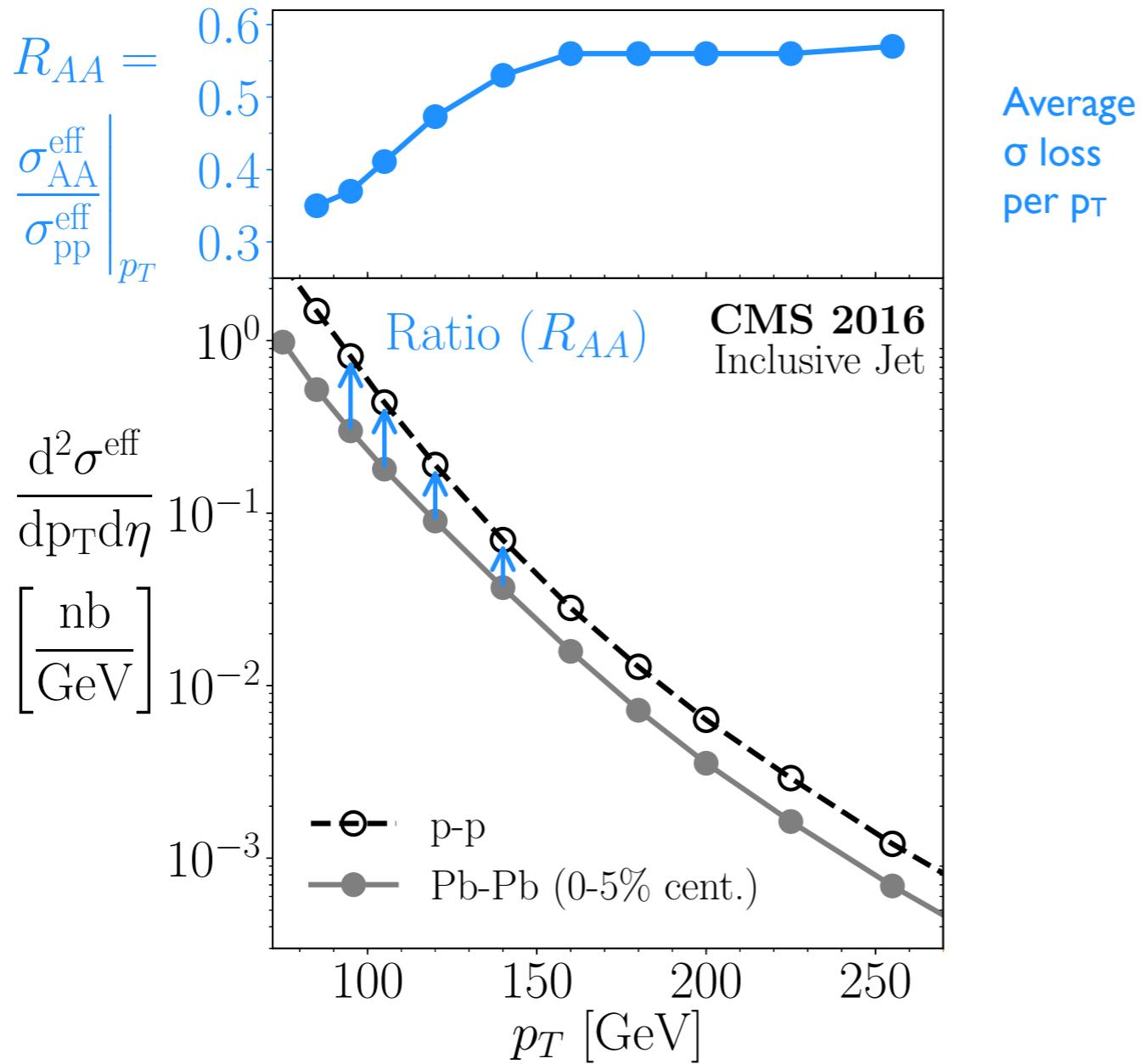


Theory: Vacuum QCD



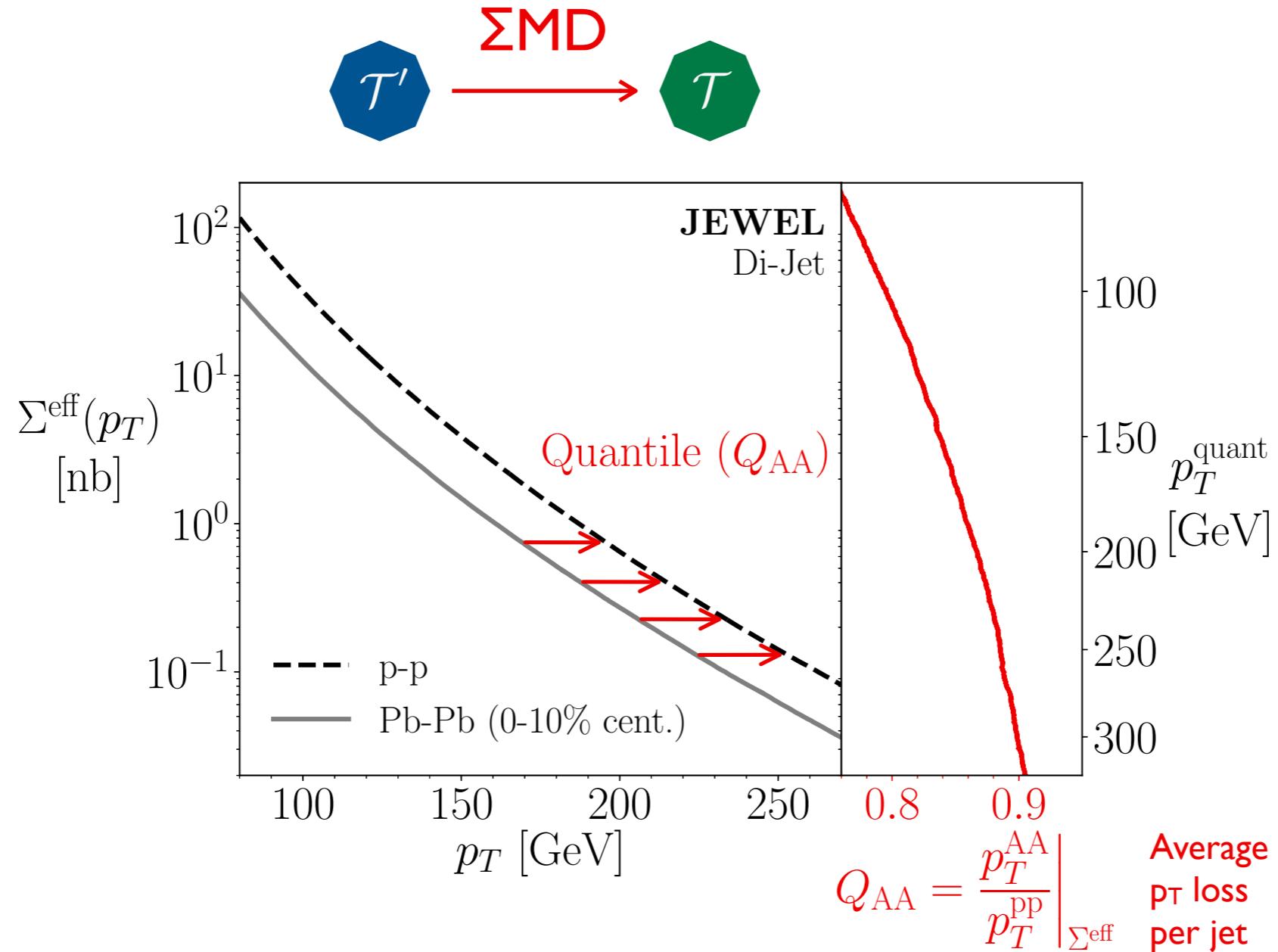
ΣMD
 \Updownarrow

*Optimal transportation plan defines mapping
between in-medium jets and vacuum jets!*



Jet Quenching via Quantile Matching

Equivalent to following a geodesic in theory space (!)



[Brewer, Milhano, JDT, PRL 2019]



Backup Slides

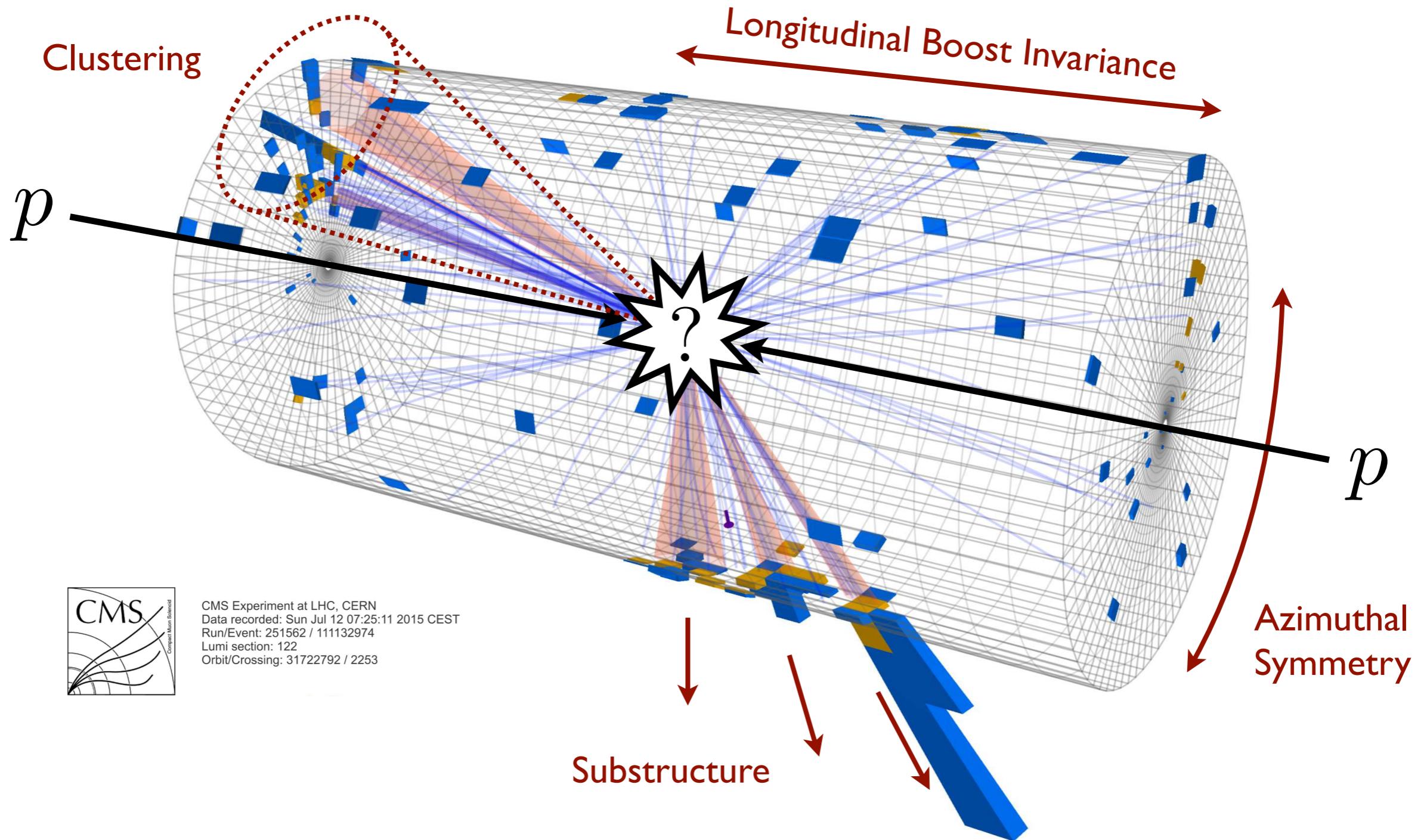
AI²: Ab Initio Artificial Intelligence



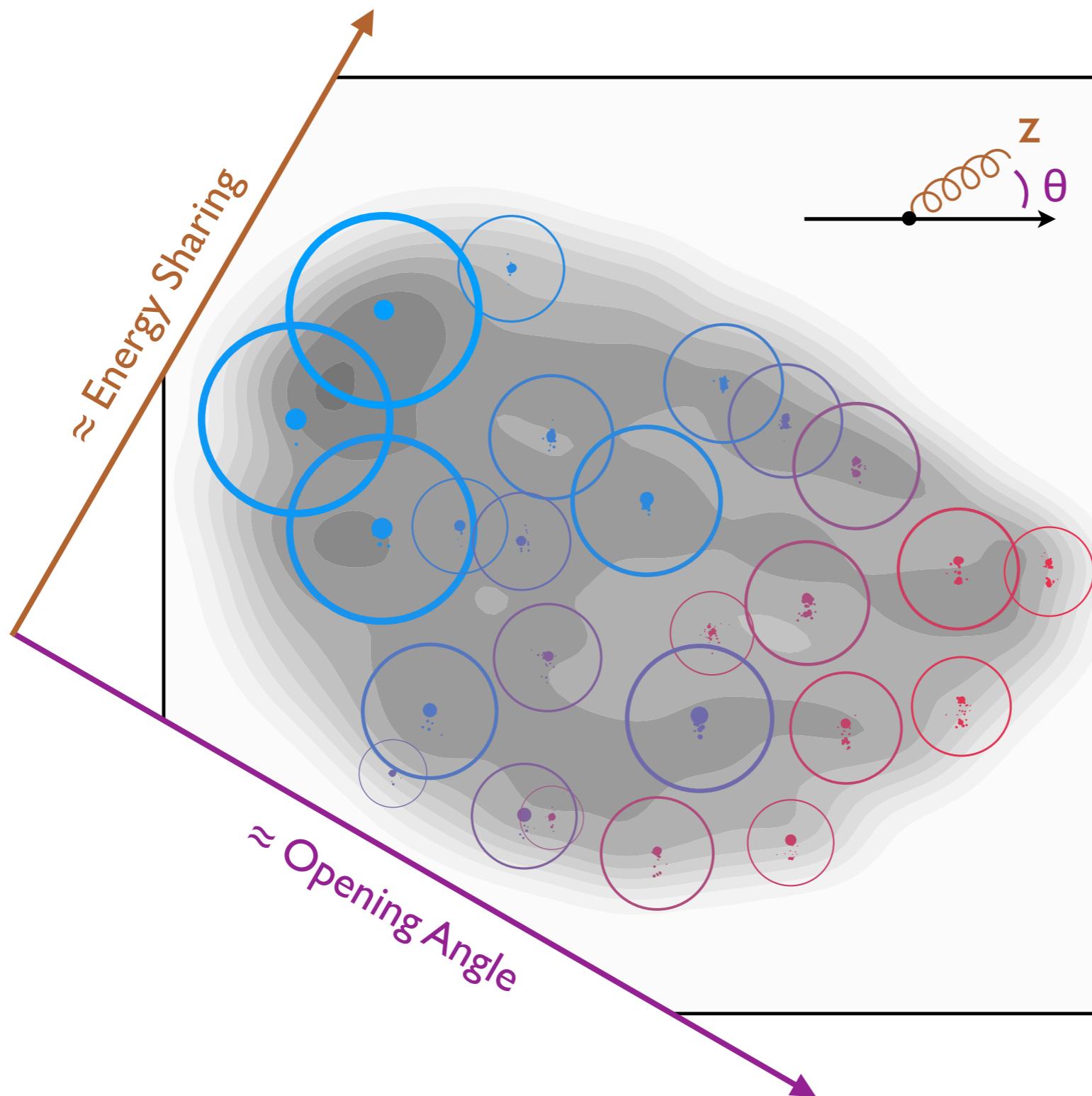
*Machine learning that incorporates
first principles, best practices, and domain knowledge
from fundamental physics*

*Symmetries, conservation laws, scaling relations, limiting behaviors, locality, causality,
unitarity, gauge invariance, entropy, least action, factorization, unit tests,
exactness, systematic uncertainties, reproducibility, verifiability, ...*

The Manifest Geometry of One Collision



The Emergent Geometry of Many Collisions

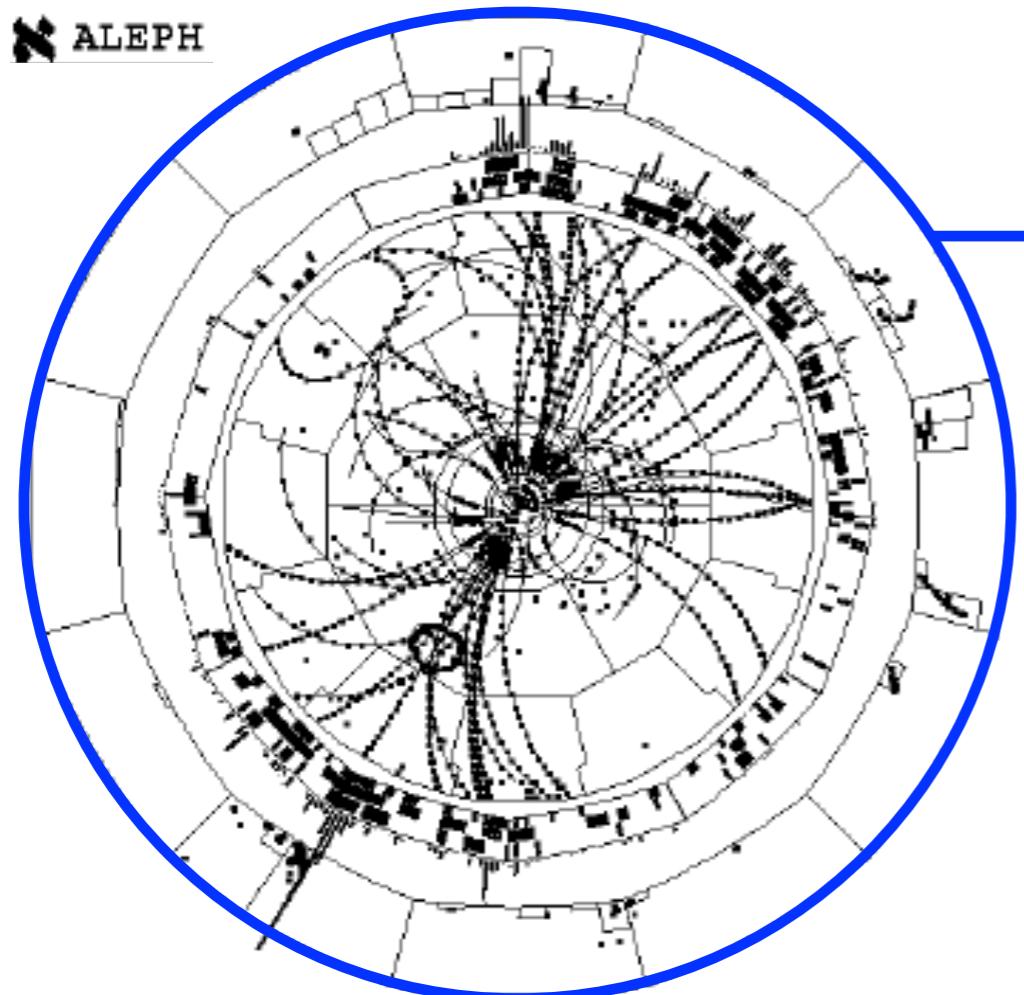


[Komiske, Mastandrea, Metodiev, Naik, JDT, [PRD 2020](#);
based on Komiske, Metodiev, JDT, [PRL 2019](#); using [EnergyFlow](#) and [CMS Open Data](#)]

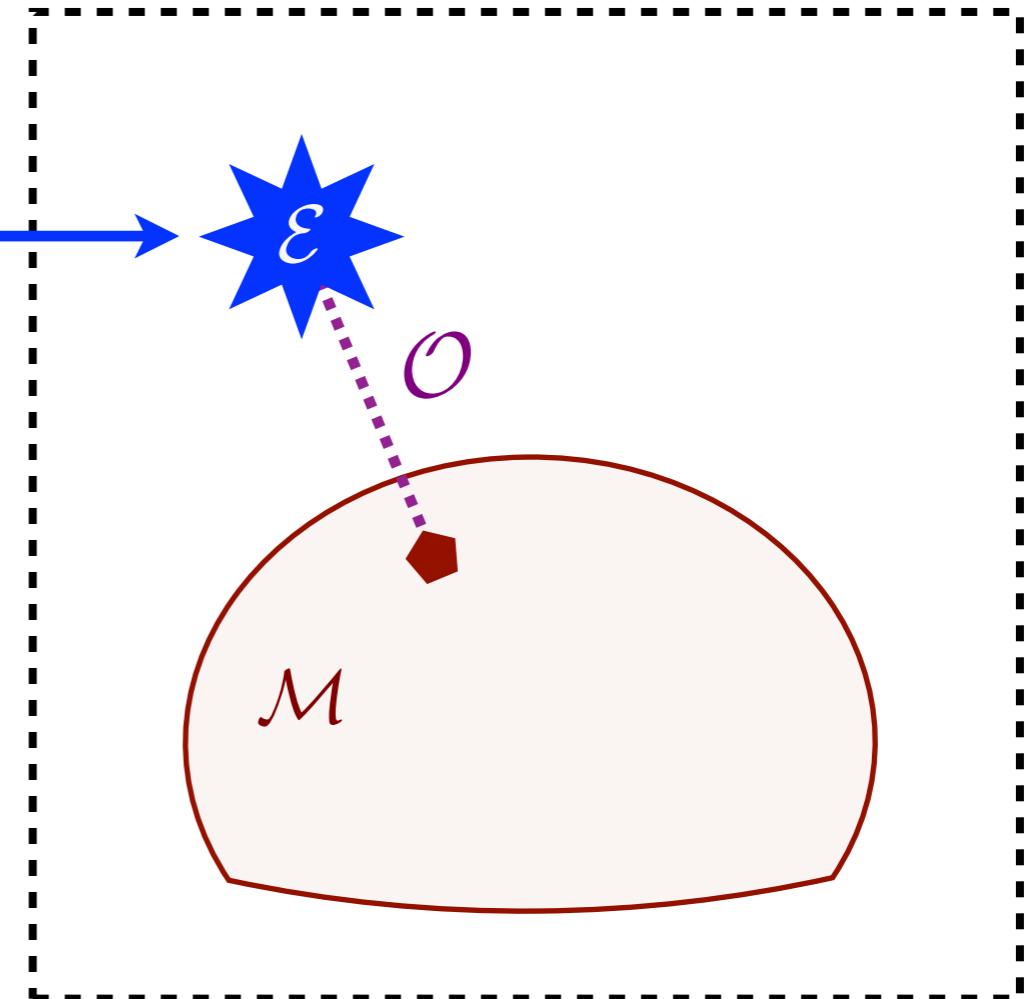


The Hidden Geometry of Particle Collisions

E.g. Classic QCD Event Shapes



One Electron-Positron Event



Distance to a Manifold in Event Space

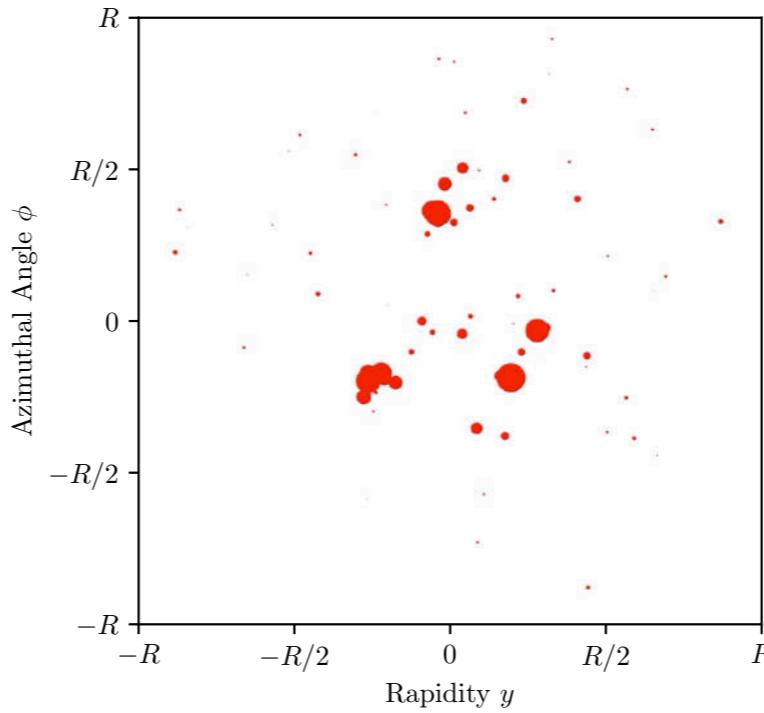
[Komiske, Metodiev, JDT, [JHEP 2020](#)]

[Brandt, Peyrou, Sosnowski, Wroblewski, [PL 1964](#); Farhi, [PRL 1977](#)]

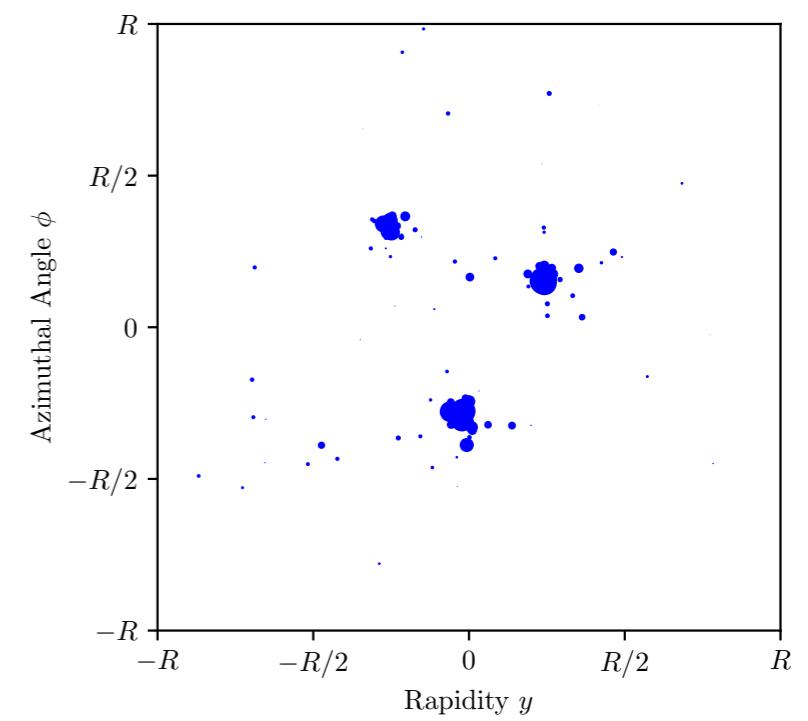
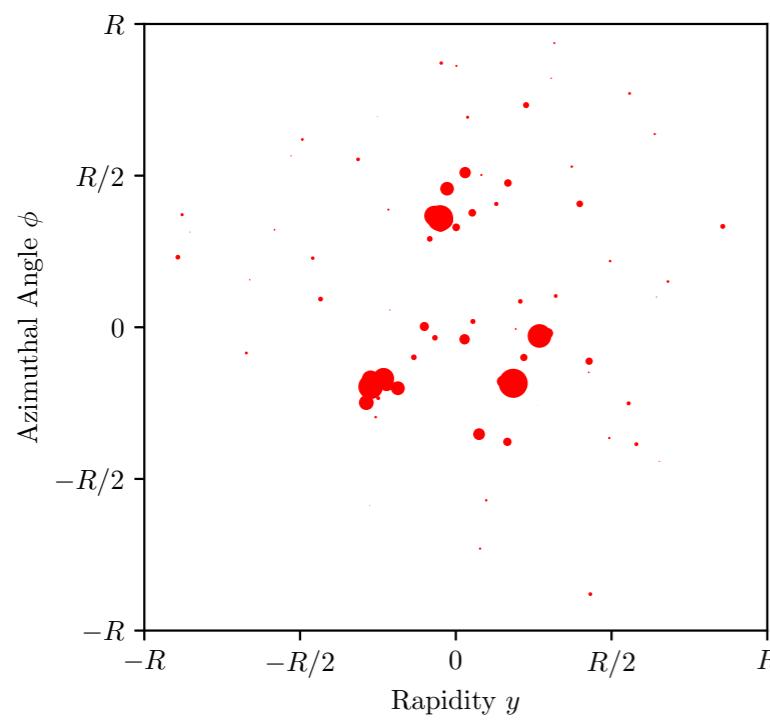


Similarity of Two Energy Flows?

$$\mathcal{E}(\hat{n}) = \sum_i E_i \delta(\hat{n} - \hat{n}_i)$$

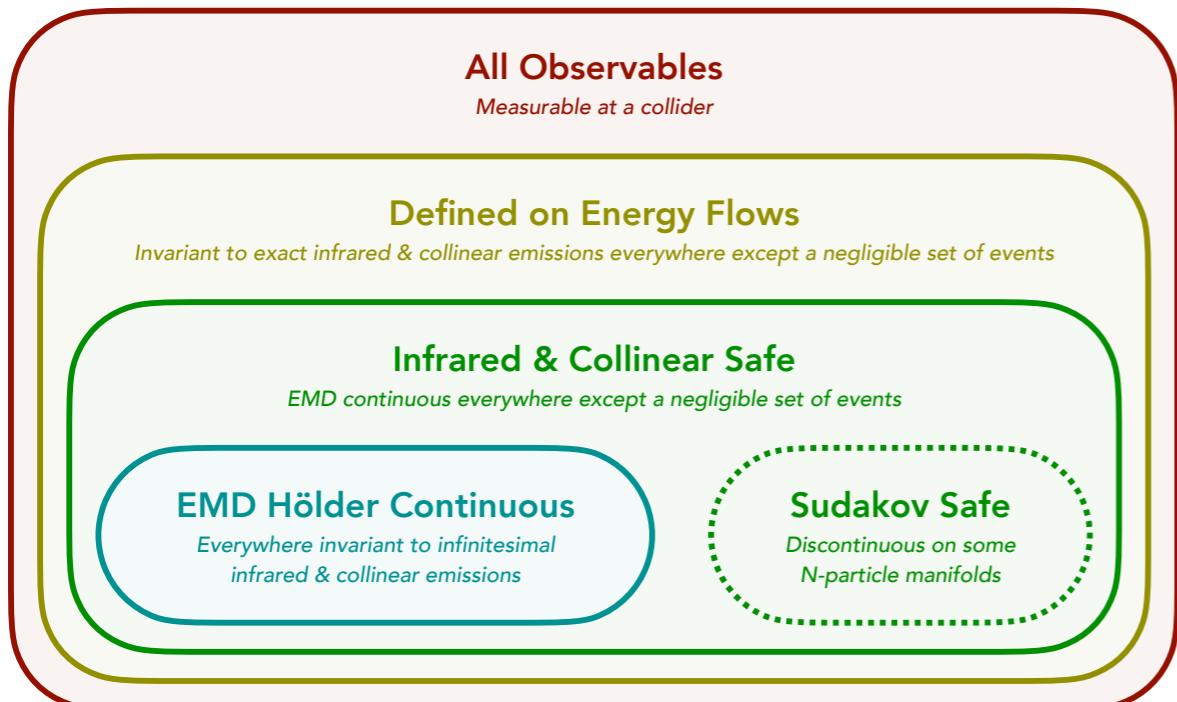


Optimal Transport:
Earth Mover's Distance
a.k.a. l -Wasserstein metric



[Komiske, Metodiev, JDT, PRL 2019; code at Komiske, Metodiev, JDT, [energyflow.network](#)]

Observable Taxonomy



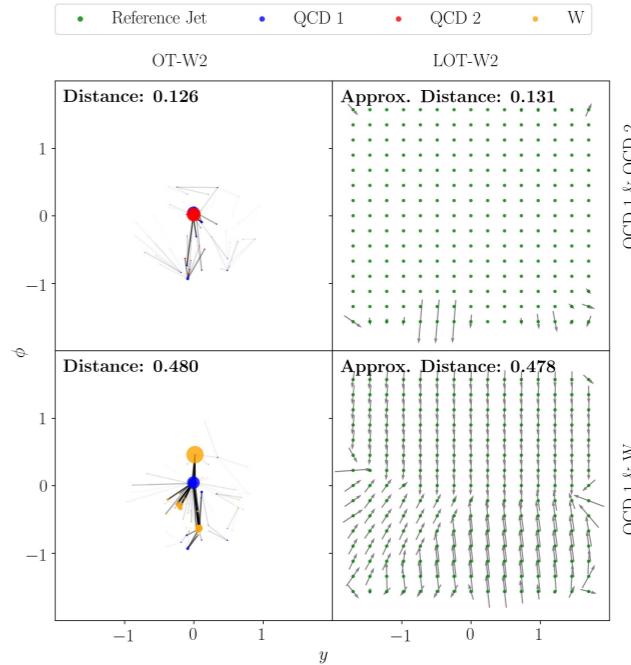
All Observables	Comments
Multiplicity ($\sum_i 1$)	IR unsafe and C unsafe
Momentum Dispersion [65] ($\sum_i E_i^2$)	IR safe but C unsafe
Sphericity Tensor [66] ($\sum_i p_i^\mu p_i^\nu$)	IR safe but C unsafe
Number of Non-Zero Calorimeter Deposits	C safe but IR unsafe
Defined on Energy Flows	
Pseudo-Multiplicity ($\min\{N \mid \mathcal{T}_N = 0\}$)	Robust to exact IR or C emissions
Infrared & Collinear Safe	
Jet Energy ($\sum_i E_i$)	Disc. at jet boundary
Heavy Jet Mass [67]	Disc. at hemisphere boundary
Soft-Dropped Jet Mass [38, 68]	Disc. at grooming threshold
Calorimeter Activity [69] (N_{95})	Disc. at cell boundary
Sudakov Safe	
Groomed Momentum Fraction [39] (z_g)	Disc. on 1-particle manifold
Jet Angularity Ratios [37]	Disc. on 1-particle manifold
N -subjettiness Ratios [47, 48] (τ_{N+1}/τ_N)	Disc. on N -particle manifold
V parameter [36] (Eq. (2.11))	Hölder disc. on 3-particle manifold
EMD Hölder Continuous Everywhere	
Thrust [40, 41]	
Spherocity [42]	
Angularities [70]	
N -jettiness [44] (\mathcal{T}_N)	
C parameter [71–74]	Resummation beneficial at $C = \frac{3}{4}$
Linear Sphericity [72] ($\sum_i E_i n_i^\mu n_i^\nu$)	
Energy Correlators [36, 75–77]	
Energy Flow Polynomials [15, 17]	

[Komiske, Metodiev, JDT, [JHEP 2020](#); cf. Sterman, [PRD 1979](#); Banfi, Salam, Zanderighi, [JHEP 2005](#); Larkoski, Marzani, JDT, [PRD 2015](#)]

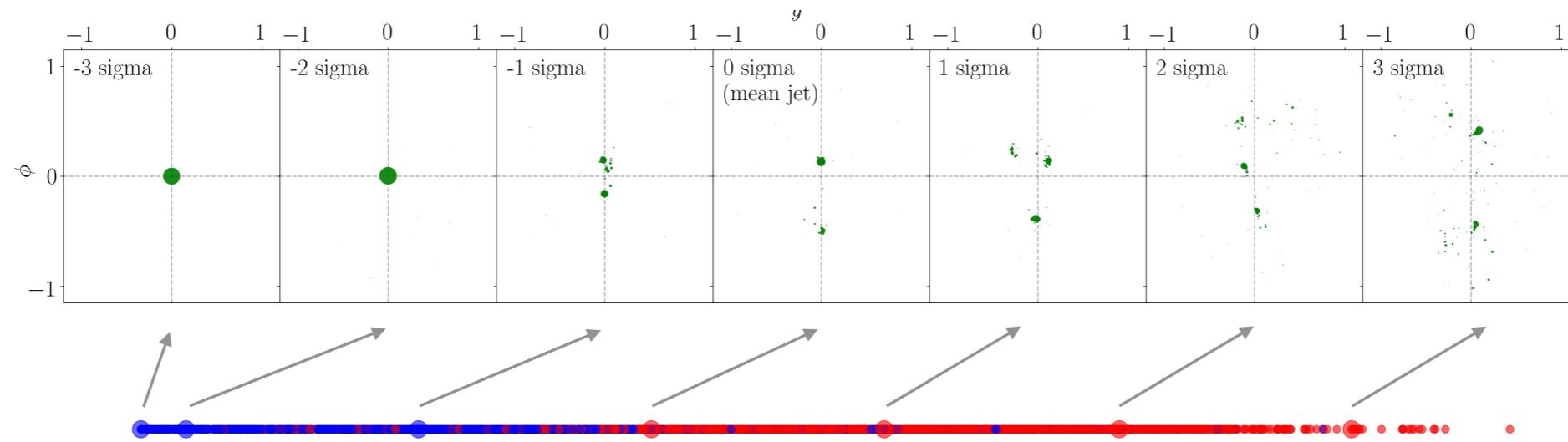
Linearized Optimal Transport

With the help of a reference event, transportation distances* can be efficiently mapped to Euclidean distances

* assuming the 2-Wasserstein measure



Enables coordinate-based techniques like Linear Discriminate Analysis



[Cai, Cheng, Craig, Craig, PRD 2020]