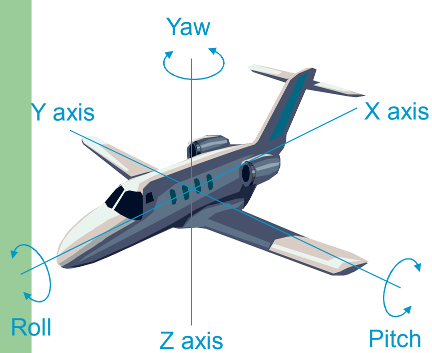
ENSF 545
Introduction to Virtual Reality

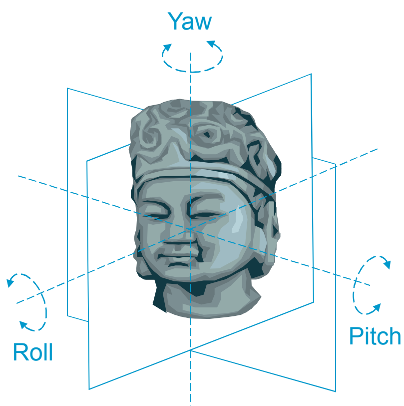
Basic Mathematics

Why Needs Mathematics in VR?

- Mathematics
  - Describing a scene
  - Performing operations on the scene
- Method
  - Coordinate systems (right- and left-handed) with 3 axes labelled x, y, z at right angles.

Coordinate Frame





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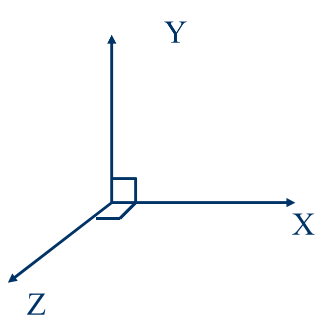
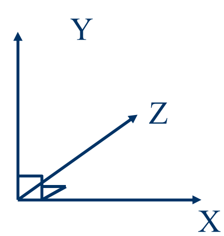
Co-ordinate Systems





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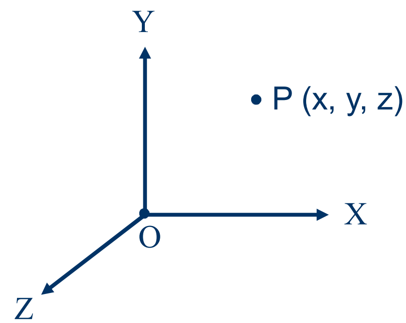
Points, P (x, y, z)

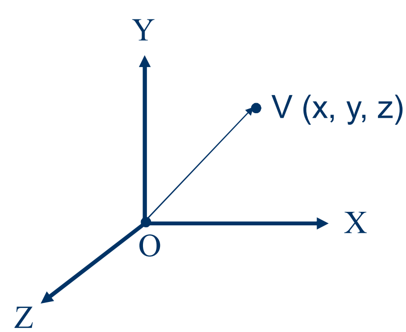
- Gives a position in relation to the origin (O) of a coordinate system



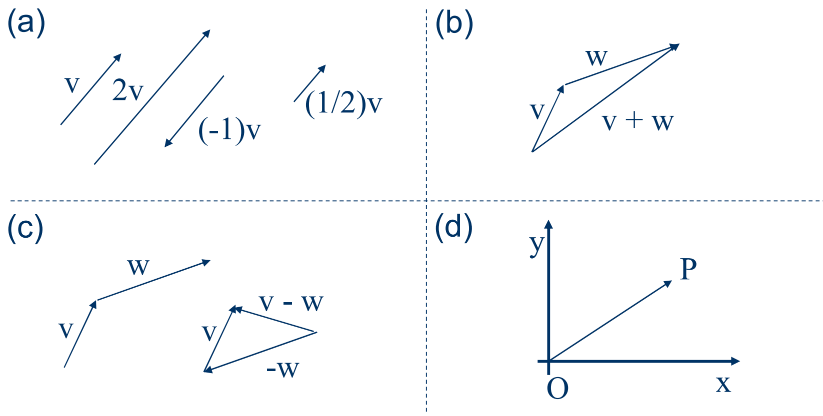
Vectors, V (x, y, z)

- Represent a *direction* (and magnitude) in 3D space

- Points != Vectors



Vectors, V (x, y, z)



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Vectors V

- Length (modulus) of a vector **V** (x, y, z)
- A unit vector:

  a vector can be *normalised* such that it retains its direction, but is scaled to have unit length.

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Dot Product

$\mathbf{u} \cdot \mathbf{v} = x_u \cdot x_v + y_u \cdot y_v + z_u \cdot z_v$

$\mathbf{u} \cdot \mathbf{v} = |u|\,|v| \cos\theta$

$\therefore \cos\theta = \mathbf{u} \cdot \mathbf{v} / |\mathbf{u}|\,|\mathbf{v}|$

Important: ($\mathbf{u} \cdot \mathbf{v}$) is a scalar number, not a vector

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Cross Product

- Important:

  ($\mathbf{u}$ x $\mathbf{v}$) is not a scalar but a vector, which is normal to the plane of the vectors **u** and **v**.
- Can be computed using the determinant of:

$$\begin{vmatrix} i & j & k \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix} \quad \begin{vmatrix} i & j & k \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix} \quad \begin{vmatrix} i & j & k \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix}$$

$\mathbf{u} \times \mathbf{v} = \mathbf{i}(y_u z_v - z_u y_v), -\mathbf{j}(x_u z_v - z_u x_v), \mathbf{k}(x_u y_v - y_u x_v)$

v

- $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}|\sin\theta$

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Parametric Equation of a Line (Ray)

Given two points $P_0 = (x_0, y_0, z_0)$ and $P_1 = (x_1, y_1, z_1)$, the line passing through them can be expressed as:

$$P(t) = P_0 + t(P_1 - P_0) = \begin{cases} x(t) = x_0 + t(x_1 - x_0) \\ y(t) = y_0 + t(y_1 - y_0) \\ z(t) = z_0 + t(z_1 - z_0) \end{cases}$$

With $-\infty < t < \infty$

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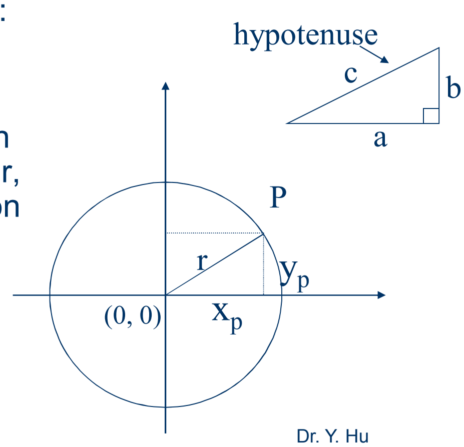
Equation of a Sphere

- Pythagoras Theorem:

$$a^2 + b^2 = c^2$$

- Given a circle through the origin with radius r, then for any point P on it we have:

$$x^2 + y^2 = r^2$$



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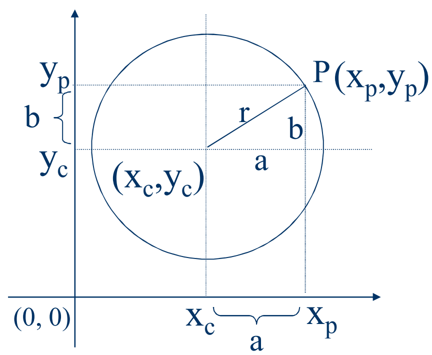
Equation of a Sphere (cont'd)

* If the circle is not centred on the origin:



We still have

$$a^2 + b^2 = r^2$$

but

$$a = x_p - x_c$$

$$b = y_p - y_c$$

So for the general case ????

Equation of a Sphere (cont'd)

* Pythagoras theorem generalises to 3D giving

$$a^2 + b^2 + c^2 = d^2$$

* The general equation of a sphere is:

???

Vectors and Matrices

- Matrix is an array of numbers with dimensions M (rows) by N (columns)

$$\begin{pmatrix} 3 & 0 & 0 & -2 & 1 & -2 \\ 1 & 1 & 3 & 4 & 1 & -1 \\ -5 & 2 & 0 & 0 & 0 & 1 \end{pmatrix}$$

- Vector can be considered a 1 x M matrix

$$v = \begin{pmatrix} x & y & z \end{pmatrix}$$

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Types of Matrix

- Identity - I

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- Symmetric

$$\begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix}$$

- Diagonal

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -4 \end{pmatrix}$$

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Operation on Matrices

- Addition

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix}$$

- Transpose

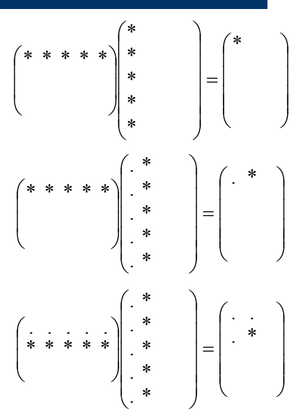
$$\begin{pmatrix} 1 & 4 & 9 \\ 5 & 2 & 8 \\ 6 & 7 & 3 \end{pmatrix}^T = \begin{pmatrix} 1 & 5 & 6 \\ 4 & 2 & 7 \\ 9 & 8 & 3 \end{pmatrix}$$

Operations on Matrices

- Multiplication
  - A is n by k , B is k by m; A x B is n by m
  - C = A x B defined by

$$c_{ij} = \sum_{l=1}^{k} a_{il} b_{lj}$$

  - B x A not necessarily equal to A x B



Example Multiplications

$$\begin{pmatrix} 2 & 3 & 1 \\ -1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \_\_ & \_\_ \\ \_\_ & \_\_ \end{pmatrix}$$

$$\begin{pmatrix} 2 & -2 & 3 \\ -3 & 1 & 0 \\ 1 & -1 & -1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} \_\_ & \_\_ & \_\_ \\ \_\_ & \_\_ & \_\_ \\ \_\_ & \_\_ & \_\_ \end{pmatrix}$$

Inverse

- If A x B = I and B x A = I then
  $A = B^{-1}$ and $B = A^{-1}$

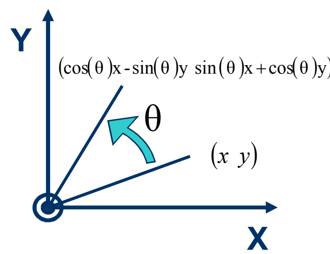
3D Transforms – Using 3 by 3 Matrices

- Scale
  - Use a diagonal matrix
  - Example:

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} = \begin{pmatrix} 6 \\ 4 \\ -10 \end{pmatrix}$$

- Rotation
  - Example: rotation about z axis

$$\begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



Rotation X, Y; Scale; Translation

- About X

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{pmatrix}$$

- About Y

- Scale (should look familiar)

- Translation

Homogenous Points

- Add 1D, but constrain that to be equal to 1
- Homogeneity means that any point in 3D space can be represented by an infinite variety of homogenous 4D points
- Why?
  - 4D allows as to include 3D translation in matrix form

$$\begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 \\ 3 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \\ 8 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 4.5 \\ 6 \\ 1.5 \end{pmatrix}$$

Homogenous Vectors

- Vectors != Points
- Remember points can not be added
- If A and B are points A-B is a vector

- Vectors have form $(x\ y\ z\ 1)^T$
- Addition makes sense

Translation in Homogenous Form

$$\begin{pmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = \begin{pmatrix} x+a \\ y+b \\ z+c \\ 1 \end{pmatrix}$$

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Putting it Together

$$\begin{pmatrix} R_1 & R_2 & R_3 & T_1 \\ R_4 & R_5 & R_6 & T_2 \\ R_7 & R_8 & R_9 & T_3 \\ 0 & 0 & 0 & 1 \end{pmatrix} = R.T$$

- R is rotation and scale components
- T is translation component

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Order Matters

- Composition order of transforms matters

$$\begin{pmatrix}1&0&0&0\\0&0&-1&0\\0&1&0&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}1&0&0&2\\0&1&0&3\\0&0&1&4\\0&0&0&1\end{pmatrix}\begin{pmatrix}X\\Y\\Z\\1\end{pmatrix}=\begin{pmatrix}1&0&0&2\\0&0&-1&-4\\0&1&0&3\\0&0&0&1\end{pmatrix}\begin{pmatrix}X\\Y\\Z\\1\end{pmatrix}=\begin{pmatrix}X+2\\-Z-4\\Y+3\\1\end{pmatrix}$$

$$\begin{pmatrix}1&0&0&2\\0&1&0&3\\0&0&1&4\\0&0&0&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&0&-1&0\\0&1&0&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}X\\Y\\Z\\1\end{pmatrix}=\begin{pmatrix}1&0&0&2\\0&0&-1&3\\0&1&0&4\\0&0&0&1\end{pmatrix}\begin{pmatrix}X\\Y\\Z\\1\end{pmatrix}=\begin{pmatrix}X+2\\-Z+3\\Y+4\\1\end{pmatrix}$$

Exercises

1. Calculate the following matrix: $\pi/2$ about X, then $\pi/2$ about Y, then $\pi/2$ about Z (remember "then" means multiply on the left). What is a simpler form of this matrix?
2. Compose the following matrix: translate 2 along X, then rotate π /2 about Y, then translate -2 along X. Draw a figure with a few points (you will only need 2D) and then its translation under this transformation.

Summary

- Rotation, Scale, Translation
- Composition of transforms
- The homogenous form