# STA 325 Case Study

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## Introduction

Understanding fluid motion and turbulence is one of most challenging problems that physicists face. This case study will examine the relationship between three key properties of particles (Reynolds Number, Stokes Number, and Froude Number) and the probability distribution of particle cluster volumes. Through exploratory data analysis and nonlinear regression models, we are able to make predictions about the distribution of particle cluster volumes given parameter settings in terms of their four moments as well as better understand the way in which each parameter affects distribution.

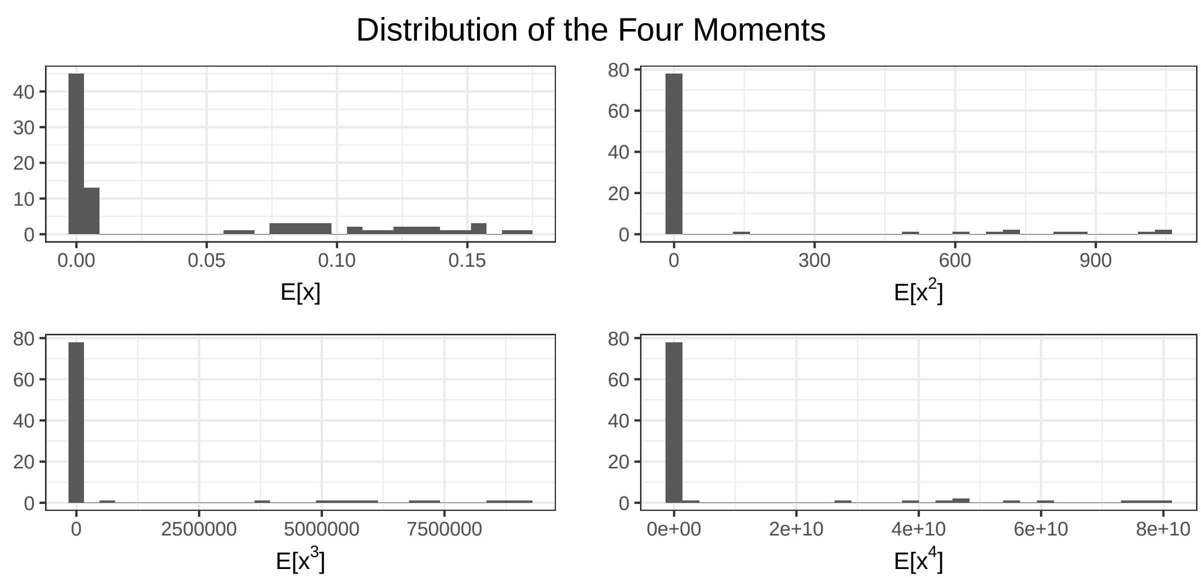
The dataset consists of 7 columns and 89 rows. Each row contains data from a simulation with different particle parameters. There are three predictor variables, or parameters:

**Reynolds Number**: a measure of the intensity of fluid turbulence of the particle. There are three values of Reynolds Number in the dataset: 90 (baseline value in regression), 224, and 338.

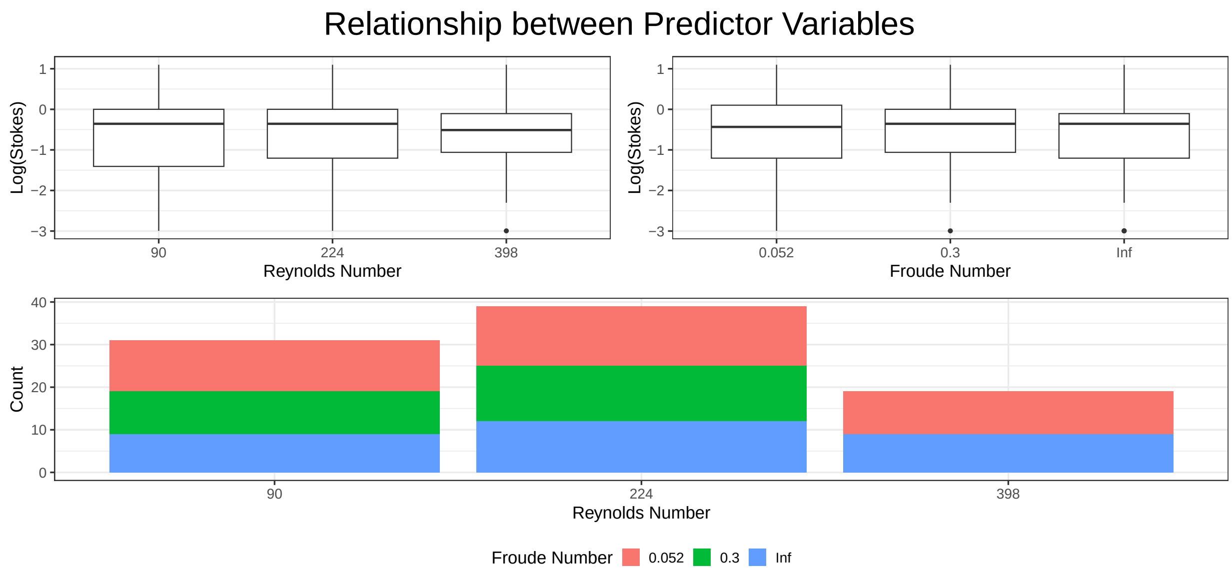
**Stokes Number**: a measure of the particle size and density. Values lie on the interval $[0, 3]$ in the dataset.

**Froude Number**: a measure of the gravitational acceleration of the particle. There are three values in the dataset: $\infty$, 0.3, and 0.052 (baseline value in regression).

There are four response variables, which are each of the four moments ($\mathbb{E}[x]$, $\mathbb{E}[x^2]$, $\mathbb{E}[x^3]$, $\mathbb{E}[x^4]$) of the probability distribution function of Voronoï volumes. We can examine the distributions through histograms.




We can see that all four moments are right skewed and not normally distributed. We also wanted to examine the relationship between the predictors to determine if there is value in including interaction effects. To do so, we create boxplots to illustrate the relationship between the continuous and categorical variables (i.e. the relationship between Stokes Number and Reynolds/Froude Number).




We notice potential relationships between all three combinations of predictor variables, with particular emphasis on the differences in range of log Stokes values based on different Reynolds and Froude numbers. We also notice that there are no particles with a Froude number of 0.3 and a Reynolds number of 398.

## Methodology

To examine the relationship between a particle's fluid turbulence, gravitational acceleration, and density on the four moments, we fit four nonlinear regression models with each moment as the response. We considered a number of transformations on each moment, such as a Box-Cox transformation, but ultimately decided to log transform each moment to have a clear interpretation of our subsequent regression coefficients. We also log transformed each particle's Stokes number as this predictor was far from normally distributed as well.

Regarding our predictors, we first converted both the Reynolds number and the Froude number of each particle to categorical predictors because there only existed three unique values for each parameter in the dataset and because the numerical differences between such values of Reynolds and Froude numbers were not easily interpretable. We posited from our background research that fitting three interaction, effects for all three predictors would better model the relationship between such predictors. For example, it is well established in existing research that fluid particle acceleration (Fr) is innately related to the turbulence of a flow (Re), in line with the Kolmogorov microscales. Therefore, we included all three potential interactions in our model (Stokes number interactions are log-transformed).

We also considered adding high order polynomials of the log of the Stokes number. We ran analysis of variance tests fitting models of varying degrees of log of the Stokes number and found that the quartic fit appeared to be reasonable for all moments. Therefore, our general model for each moment is as follows:

$$\begin{aligned} Y = {} & \beta_0 + \beta_1 log(Stokes) + \beta_2 log(Stokes)^2 + \beta_3 log(Stokes)^3 + \beta_4 log(Stokes)^4 + \beta_5 Reynolds \\ & + \beta_6 Froude + \beta_7 (log(Stokes) * Froude) + \beta_8 (log(Stokes) * Reynolds) + \beta_9 (Froude * Reynolds) \end{aligned}$$

Finally, we ran both stepwise forward and backward variable selections on each of our four models using AIC as our criteria. We personally care less about penalizing more complex models since we are given so few predictors to begin with anyways. Forward and backward selection did not remove or add any variables to the second, third, and fourth moment models, but did remove the interaction term between the Reynolds number and the log of the Stokes number for the first moment model. However, we still decided to include this interaction because we believe it is scientifically grounded.

## Results

Here we display the results for our four regression models.

Table 1: Model 1

| | Estimate | Std. Error | Pr(>\|t\|) |
|---|---|---|---|
| Intercept | -2.05 | 0.0199 | 7.05e-81 |
| Log(St) | 1.5 | 0.141 | 1.84e-16 |
| Log(St)^2 | 0.196 | 0.0698 | 0.0064 |
| Log(St)^3 | 0.471 | 0.0702 | 3.71e-09 |
| Log(St)^4 | -0.153 | 0.0706 | 0.0334 |
| Re 224 | -3.81 | 0.0288 | 9.75e-89 |
| Re 398 | -6.01 | 0.0311 | 1.22e-100 |
| Fr 0.3 | -0.237 | 0.031 | 5.94e-11 |
| Fr Inf | -0.272 | 0.0327 | 3.64e-12 |
| Fr 0.3 * Log(St) | 0.102 | 0.0184 | 4.7e-07 |
| Fr Inf * Log(St) | 0.0647 | 0.016 | 0.000123 |
| Re 224 * Log(St) | 0.0136 | 0.0155 | 0.382 |
| Re 398 * Log(St) | -0.0159 | 0.0203 | 0.436 |
| Re 224 * Fr 0.3 | 0.261 | 0.0401 | 8.57e-09 |
| Re 244 * Fr Inf | 0.381 | 0.0409 | 4.68e-14 |
| Re 398 * Fr Inf | 0.488 | 0.0446 | 4.81e-17 |

Table 3: Model 3

| | Estimate | Std. Error | Pr(>\|t\|) |
|---|---|---|---|
| Intercept | 14.6 | 0.252 | 9.39e-63 |
| Log(St) | 19.3 | 1.79 | 1.07e-16 |
| Log(St)^2 | -8.55 | 0.886 | 1.14e-14 |
| Log(St)^3 | 6.72 | 0.891 | 1.01e-10 |
| Log(St)^4 | -3.61 | 0.895 | 0.000133 |
| Re 224 | -11.5 | 0.365 | 3.31e-44 |
| Re 398 | -18.1 | 0.395 | 1.76e-55 |
| Fr 0.3 | -13 | 0.393 | 8.41e-46 |
| Fr Inf | -13 | 0.415 | 4.19e-44 |
| Fr 0.3 * Log(St) | 0.14 | 0.234 | 0.55 |
| Fr Inf * Log(St) | -0.622 | 0.202 | 0.00298 |
| Re 224 * Log(St) | -0.312 | 0.196 | 0.116 |
| Re 398 * Log(St) | -1.08 | 0.257 | 7.65e-05 |
| Re 224 * Fr 0.3 | 8.97 | 0.508 | 4.9e-28 |
| Re 244 * Fr Inf | 8.74 | 0.519 | 7.14e-27 |
| Re 398 * Fr Inf | 13.2 | 0.566 | 1.21e-35 |

Table 2: Model 2

| | Estimate | Std. Error | Pr(>\|t\|) |
|---|---|---|---|
| Intercept | 6.01 | 0.146 | 3.15e-52 |
| Log(St) | 12.1 | 1.04 | 2.4e-18 |
| Log(St)^2 | -5.01 | 0.514 | 7.86e-15 |
| Log(St)^3 | 4.12 | 0.517 | 1.58e-11 |
| Log(St)^4 | -2.16 | 0.52 | 8.74e-05 |
| Re 224 | -7.61 | 0.212 | 4.05e-48 |
| Re 398 | -12 | 0.229 | 1.28e-59 |
| Fr 0.3 | -6.74 | 0.228 | 2.24e-42 |
| Fr Inf | -6.76 | 0.241 | 7.56e-41 |
| Fr 0.3 * Log(St) | 0.108 | 0.136 | 0.428 |
| Fr Inf * Log(St) | -0.327 | 0.118 | 0.00694 |
| Re 224 * Log(St) | -0.136 | 0.114 | 0.236 |
| Re 398 * Log(St) | -0.627 | 0.149 | 7.48e-05 |
| Re 224 * Fr 0.3 | 4.75 | 0.295 | 1.01e-25 |
| Re 244 * Fr Inf | 4.68 | 0.301 | 7.22e-25 |
| Re 398 * Fr Inf | 6.99 | 0.329 | 5.15e-33 |

Table 4: Model 4

| | Estimate | Std. Error | Pr(>\|t\|) |
|---|---|---|---|
| Intercept | 23.2 | 0.348 | 3.92e-67 |
| Log(St) | 25.6 | 2.48 | 6.41e-16 |
| Log(St)^2 | -11.6 | 1.22 | 2.03e-14 |
| Log(St)^3 | 9.01 | 1.23 | 2.59e-10 |
| Log(St)^4 | -4.87 | 1.24 | 0.000187 |
| Re 224 | -15.4 | 0.504 | 2.92e-43 |
| Re 398 | -24.1 | 0.545 | 1.97e-54 |
| Fr 0.3 | -19.2 | 0.543 | 8.88e-48 |
| Fr Inf | -19.2 | 0.573 | 4.96e-46 |
| Fr 0.3 * Log(St) | 0.153 | 0.323 | 0.637 |
| Fr Inf * Log(St) | -0.901 | 0.28 | 0.0019 |
| Re 224 * Log(St) | -0.468 | 0.271 | 0.0888 |
| Re 398 * Log(St) | -1.46 | 0.355 | 0.000103 |
| Re 224 * Fr 0.3 | 13.1 | 0.702 | 1.85e-29 |
| Re 244 * Fr Inf | 12.7 | 0.717 | 3.12e-28 |
| Re 398 * Fr Inf | 19.4 | 0.782 | 2.6e-37 |

The adjusted R-Squared values for each model were .9991, .9814, .9765, and .9757 respectively, indicating that essentially all of the variation in the log of the four moments can be explained by the combination of the predictor variables that we chose. Using an alpha of .05 as a threshold for statistical significance, we find that every individual predictor is statistically significant in each of the four models, though both the sign and magnitude of the coefficients vary by moment. This makes intuitive sense; though we shrank the values of the individual moments through log transformations, we saw from the histograms of the moments distributions that the third and fourth moments had significantly larger values than the first and second moments. We also notice that most of the interaction terms are also statistically significant, with the two clear exceptions being the interaction between a Froude number of 0.3 and the log of the Stokes number and the interaction between a Reynolds number of 224 and the log of the Stokes number.

Recall that we kept both the Froude number and the Reynolds number as categorical variables despite having numeric values. We can still consider the implications of a high Froude number (in this case, a value of infinity) and a high Reynolds number (in this case, 338) on each moment.

Let's first begin with examining the Froude number. High Froude values indicate that the interial forces dominate the gravitational forces. With 0.052 as our baseline Froude number, we examine the coefficients of the Froude indicator variables and find that the coefficients for a Froude number of 0.3 and a Froude number of infinity are essentially the same in all four moments models. This coefficient is negative, indicating that higher values of Froude numbers, holding a particle's Reynolds number and Stokes number constnat, lead to a lower mean, variance, skew, and kurtosis of the distribution of particle cluster volumes. However, for the second, third, and fourth moment model, we find that the interaction coefficient between a Froude value of 0.3 and log(Stokes) is positive, but is negative for the interaction between a Froude value of infinity and log(Stokes). This suggests that