92.222.001 EXAM 2 Spring 2014 ___________________________________ Name: James Wood

1) Find a, b and c so that following set of vectors is orthogonal $\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} a \\ 1 \\ -1 \end{pmatrix}$, and $\begin{pmatrix} b \\ 3 \\ c \end{pmatrix}$. What has to be done to make the set orthonormal.

$$2(a) + 1(1) + (-1)(-1) = 0 \implies 2a + 2 = 0 \implies a = -1$$

$$2(b) + 1(3) + (-1)(c) = 0 \implies 2(3 - c) + 3 - c = 0 \implies 9 - 3c = 0 \implies c = 3$$

$$ab + 1(3) + (-1)(c) = 0 \implies -b + 3 - c = 0 \implies 3 - c = b \implies b = 0$$

$$\therefore a = -1 \ , \ b = 0 \ , \ c = 3$$

> $2(-1) + 1 + 1 = 0✓$
>
> $2(0) + 3 - 3 = 0✓$
>
> $(-1)(0) + 3 - (3) = 0✓$

$$v_1 = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}, \ v_2 = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}, \ v_3 = \begin{pmatrix} 0 \\ 3 \\ 3 \end{pmatrix}$$

$$u_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{2^2 + 1^2 + (-1)^2}} \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{6}} \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$$

$$u_2 = \frac{v_2}{\|v_2\|} = \frac{1}{\sqrt{(-1)^2 + 1^2 + (-1)^2}} \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{3}} \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$$

$$u_3 = \frac{v_3}{\|v_3\|} = \frac{1}{\sqrt{0^2 + 3^2 + (3^2}} \begin{pmatrix} 0 \\ 3 \\ 3 \end{pmatrix} = \frac{1}{3\sqrt{2}} \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$$

2) Is the following matrix orthogonal: $A = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{6}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{pmatrix}$? Justify your answer.

$$Q = A = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{6}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{pmatrix} \qquad Q^T = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{pmatrix}$$

$$Q^T Q = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{6}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} + \frac{1}{3} + \frac{1}{6} & 0 + \frac{1}{3} - \frac{2}{6} & -\frac{1}{2} + \frac{1}{3} + \frac{1}{6} \\ 0 + \frac{1}{3} - \frac{2}{6} & 0 + \frac{1}{3} + \frac{4}{6} & 0 + \frac{1}{3} - \frac{2}{6} \\ -\frac{1}{2} + \frac{1}{3} + \frac{1}{6} & 0 + \frac{1}{3} - \frac{2}{6} & \frac{1}{2} + \frac{1}{3} + \frac{1}{6} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

$$\therefore Q^T Q = I$$

Per theorem 5.5.5 A is an orthogonal matrix.

3) What are the eigenvalues and associated eigenvectors if the matrix $A = \begin{pmatrix} 2 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 4 \end{pmatrix}$

$\left(A - \lambda I\right) = \begin{pmatrix} 2-\lambda & 1 & 1 \\ 0 & -1-\lambda & 1 \\ 0 & 0 & 4-\lambda \end{pmatrix} = 0$ A is upper Triangular, so eigenvalues are on the diagonal.

$$\therefore \lambda_1 = 2 \qquad \lambda_2 = -1 \qquad \lambda_3 = 4$$

$$\lambda_1 = 2, \quad \begin{pmatrix} 2-(2) & 1 & 1 \\ 0 & -1-(2) & 1 \\ 0 & 0 & 4-(2) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & -3 & 1 \\ 0 & 0 & 2 \end{pmatrix} x_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \implies x_1 = N(A-\lambda_1) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\lambda_2 = -1, \quad \begin{pmatrix} 2-(-1) & 1 & 1 \\ 0 & -1-(-1) & 1 \\ 0 & 0 & 4-(-1) \end{pmatrix} = \begin{pmatrix} 3 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 5 \end{pmatrix} x_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \implies x_2 = N(A-\lambda_2) = \begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix}$$

$$\lambda_3 = 4, \quad \begin{pmatrix} 2-(4) & 1 & 1 \\ 0 & -1-(4) & 1 \\ 0 & 0 & 4-(4) \end{pmatrix} = \begin{pmatrix} -2 & 1 & 1 \\ 0 & -5 & 1 \\ 0 & 0 & 0 \end{pmatrix} x_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \implies x_3 = N(A-\lambda_3) = \begin{pmatrix} 3 \\ 1 \\ 5 \end{pmatrix}$$

> $$\begin{pmatrix} 0 & 1 & 1 \\ 0 & -3 & 1 \\ 0 & 0 & 2 \end{pmatrix} \to \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \implies \begin{matrix} x_2 = 0 \\ x_3 = 0 \end{matrix} \qquad x_1 = \alpha \qquad X = \begin{pmatrix} \alpha \\ 0 \\ 0 \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$
>
> $$\begin{pmatrix} 3 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 5 \end{pmatrix} \to \begin{pmatrix} 3 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \implies \begin{matrix} \frac{1}{3} x_2 = -x_1 \\ x_3 = 0 \end{matrix} \qquad x_1 = \alpha \qquad X = \begin{pmatrix} -\alpha \\ 3\alpha \\ 0 \end{pmatrix} = \alpha \begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix}$$
>
> $$\begin{pmatrix} -2 & 1 & 1 \\ 0 & -5 & 1 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -\frac{3}{5} \\ 0 & 1 & -\frac{1}{5} \\ 0 & 0 & 0 \end{pmatrix} \implies \begin{matrix} x_1 = \frac{3}{5} x_3 \\ x_2 = \frac{1}{5} x_3 \end{matrix} \qquad x_3 = \alpha \qquad X = \begin{pmatrix} \frac{3}{5}\alpha \\ \frac{1}{5}\alpha \\ \alpha \end{pmatrix} = \alpha \begin{pmatrix} 3 \\ 1 \\ 5 \end{pmatrix}$$

4) What is the diaginalization of the matrix $A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$. Using $X\,X^{-1}$ and D check you get A.

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & 1 & 0 \\ 0 & \frac{3}{2} & 1 \\ 0 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & 1 & 0 \\ 0 & 3 & 2 \\ 0 & 0 & 1 \end{pmatrix} \implies \lambda_1 = 2\,,\ \lambda_2 = 3\,,\ \lambda_3 = 1$$

$$\lambda_1 = 2 \quad (A - \lambda_1 I) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \implies z = -x,\ y = 0 \implies x_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$\lambda_2 = 3 \quad (A - \lambda_2 I) = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \implies y = x\,,\ z = 0 \implies x_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$\lambda_3 = 1 \quad (A - \lambda_3 I) = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \implies y = -x\,,\ z = 0 \implies x_3 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

$$X = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} = \begin{pmatrix} -1 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} -1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

$$\begin{pmatrix} -1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

$$D = X^{-1}AX = \begin{pmatrix} 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} -1 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} -2 & 3 & -1 \\ 0 & 3 & 1 \\ 2 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A = XDX^{-1} = \begin{pmatrix} -1 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} -1 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 2 \\ \frac{3}{2} & \frac{3}{2} & \frac{3}{2} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix} = A\checkmark$$

5) Let $z_1 = \begin{pmatrix} 1-i \\ 1+i \end{pmatrix}$ and $z_2 = \begin{pmatrix} -i \\ -1 \end{pmatrix}$

a. Show $\{z_1, z_2\}$ is an orthogonal set of vectors.
b. Using $\{z_1, z_2\}$ form an orthogonal set of vectors.
c. Write the vectors $Z = \begin{pmatrix} 1-2i \\ i \end{pmatrix}$ as a linear combination of U_1 and U_2.

$$\langle z_1 , z_2 \rangle = \overline{z_2}^T z_1 = \begin{pmatrix} 0+i & -1-0 \end{pmatrix} \begin{pmatrix} 1-i \\ 1+i \end{pmatrix} = 0$$

$$\overline{z_2}^T z_1 = 0 \therefore z_1 \perp z_2$$

$$\|z_1\| = \sqrt{z_1^H z_1} = \sqrt{\left|1-i\right|^2 + \left|1+i\right|^2} = \sqrt{4} = 2 \implies U_1 = \frac{1}{\|z_1\|} z_1 = \frac{1}{2} \begin{pmatrix} 1-i \\ 1+i \end{pmatrix}$$

$$\|z_2\| = \sqrt{z_2^H z_2} = \sqrt{\left|0-i\right|^2 + \left|-1+0i\right|^2} = \sqrt{2} \implies U_2 = \frac{1}{\|z_2\|} z_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} -i \\ -1 \end{pmatrix}$$

$$ZU = Z^H U_1 + Z^H U_2 = \begin{pmatrix} 1+2i & -i \end{pmatrix} \begin{pmatrix} \frac{1-i}{2} \\ \frac{1+i}{2} \end{pmatrix} + \begin{pmatrix} 1+2i & -i \end{pmatrix} \begin{pmatrix} \frac{-i}{2} \\ -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} i \\ \frac{1}{2} \end{pmatrix}$$

> $0(1) + 0(-i) + i(1) + i(-i) + (-1)(1) + (-1)i + 0(1) + 0(i)$
>
> $0 + 0 + i - i^2 - 1 - i = i - i + 1 - 1 = 0$
>
> $\begin{pmatrix} 1+2i & -i \end{pmatrix} \begin{pmatrix} \frac{1}{2} - \frac{1}{2} i \\ \frac{1}{2} + \frac{1}{2} i \end{pmatrix}$
>
> $(-\frac{1}{2} + \frac{1}{2} i) + (\frac{1}{2} + \frac{1}{2} i) = i$
>
> $\begin{pmatrix} 1+2i & -i \end{pmatrix} \begin{pmatrix} -\frac{1}{2} i \\ -\frac{1}{2} \end{pmatrix}$
>
> $(1 - \frac{1}{2} i) + (-\frac{1}{2} + \frac{1}{2} i) = \frac{1}{2}$

6) a) What are a and b so that the matrix A is Hermitian: $A = \begin{pmatrix} 1 & 1+2i & 1-i \\ a & 2 & b \\ 1+i & 4-3i & 3 \end{pmatrix}$

b) Find an orthogonal or unitary diagonalizing matrix for $A = \begin{pmatrix} 4 & 5i & 0 \\ -5i & 4 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.

a)

$$M^H = M = \overline{M}^T$$

$$A = M = \begin{pmatrix} 1 & 1+2i & 1-i \\ a & 2 & b \\ 1+i & 4-3i & 3 \end{pmatrix} \qquad M^H = \overline{M}^T = \begin{pmatrix} \overline{1} & \overline{a} & \overline{1+i} \\ \overline{1+2i} & \overline{2} & \overline{4-3i} \\ \overline{1-i} & \overline{b} & \overline{3} \end{pmatrix}^T$$

Let a = $1+2i$ and let b = $4-3i$ $\implies$

$$\overline{M}^T = \begin{pmatrix} \overline{1} & \overline{1+2i} & \overline{1+i} \\ \overline{1+2i} & \overline{2} & \overline{4-3i} \\ \overline{1-i} & \overline{4-3i} & \overline{3} \end{pmatrix}^T = \begin{pmatrix} 1 & 1+2i & 1-i \\ 1+2i & 2 & 4-3i \\ 1+i & 4-3i & 3 \end{pmatrix} = M = A$$

b)

$$A = \begin{pmatrix} 4 & 5i & 0 \\ -5i & 4 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$\lambda_1 = 9,\ \lambda_2 = 2,\ \lambda_3 = -2 \qquad x_1 = \begin{pmatrix} i & 1 & 0 \end{pmatrix}^T \quad x_2 = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^T \quad x_3 = \begin{pmatrix} -1 & 1 & 0 \end{pmatrix}^T$$

$$u_1 = \frac{1}{\|x_1\|} x_1 = \frac{1}{\sqrt{i^2 + 1^2}} \begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix} = 0$$

$$p = (x_2^T u_1) u_1 = \left(\begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right) \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = 0$$

$$x_2 - p = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$u_2 = \frac{1}{\|x_2 - p\|}(x_2 - p) = \frac{1}{\sqrt{0^2 + 0^2 + 1^2}} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$u_3 = \frac{1}{\|x_3\|} x_3 = \frac{1}{\sqrt{-1^2 + 1^2 + 0^2}} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

$$\therefore U = \begin{pmatrix} 0 & 0 & -\frac{1}{2} \\ 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{pmatrix}$$