Lecture Notes

Neads a title

Consider $L : V - V$, the dimension $(v) = n$ The matrix representation depends on an ordered Basis.

Ex)

> $L : \mathbb{R}^2 \to \mathbb{R}^2$ Were $L(x) = (x - x_1, 3x_2)^T L(e_1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $L(e_2) = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$
>
> The matrix representation with respect to $\{e_1, e_2\}$ is $\begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix}$
>
> Suppose we choose $u_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, u_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}, L(u_1) = Au_1 = \begin{pmatrix} 2 \\ -3 \end{pmatrix}, L(u_2) = Au_2 = \begin{pmatrix} -1 \\ 6 \end{pmatrix}$
>
> We need $L(u_1) = au_1 + cu_2, L(u_2) = bu_1 + du_2 \implies \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ the transform from $(u_1, u_2) \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$ the transform from $(e_1, e_2) \to [u_1, u_2]$ is $U^{-1} = \begin{pmatrix} \frac{2}{3} & \frac{-1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix}$
>
> $U^{-1} L(u_1) = U^{-1} + Au_1 = \begin{pmatrix} \frac{7}{3} \\ \frac{-1}{3} \end{pmatrix}, U^{-1} L(u_2) = U^{-1} + Au_2 = \begin{pmatrix} \frac{8}{3} \\ \frac{5}{3} \end{pmatrix}$
>
> $L(u_1) = \frac{7}{3} v_1 - \frac{1}{3} u_2 , L(u_2) = - \frac{8}{3} u_1 + \frac{5}{5} u_2$
>
> $\Rrightarrow matrix\ B = \begin{pmatrix} \frac{7}{3} & \frac{8}{3} \\ \frac{1}{3} & \frac{5}{3} \end{pmatrix}$

The Columns of B are $(U^{-1}Au_1, U^{-1}Au_2) = U^{-1}(Au_1, Au_2)$, $B = U^{-1}AU$

Definition:-

If A and B are $n \times n$ matrices, and S is not singular such that $B = S^{-1}AS$, then B is similar to A

Since $B = S^{-1}AS\ SBS^{-1} = SS^{-1}ASS^{-1} \implies A = SBS^{-1} \Rightarrow A$ is similar to B

So, if B is Similar to A, A is Similar to B

Ex)

Let D be the differential operation on P_3 Find B representing D W.R.T $[1, x, x^2]$ and A representing D W.R.T. $[3, 4x, 2x^2 + x]$

Since

$D(1) = 0 = 0 \cdot 1 + 0 \cdot x + 0 \cdot x^2$

$D(x) = 1 = 1 \cdot 1 + 0 \cdot x + 0 \cdot x^2$

$D(x^2) = 2x = 0 \cdot 1 + 2 \cdot x + 0 \cdot x^2$

$B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$

Transition from $[3, 4x, 2x^2 = X]$ to $[1, x, x^2]$ is,

$D(3) = 0 = 0 \cdot 3 + 0 \cdot 4x + 0 \cdot (2x^2 + x)$

$D(4x) = 4 = \frac{4}{3} \cdot 3 + 0 \cdot 4x + 0 \cdot (2x^2 + x)$

$D(3) = 4x + 1 = \frac{1}{3} \cdot 3 + 0 \cdot 4x + 0 \cdot (2x^2 + x)$

$A = \begin{pmatrix} 0 & \frac{4}{3} & \frac{1}{3} \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

To go from $[3, 4x, 2x^2 + x]$ to $[1, x, x^2]$, $S = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 2 \end{pmatrix}$, $S^{-1} = \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{4} & -\frac{1}{8} \\ 0 & 0 & \frac{1}{2} \end{pmatrix}$ and $S^{-1}BS = A$

A is a matrix representation W.R.T. $[1, x, x^2]$

B is a matrix representation W.R.T. $[3, 4x, 2x^2 + x]$

S is a transitional matrix from $[3, 4x, 2x^2 + x]$ to $[1, x, x^2]$.

Orthogonality in $\mathbb{R}^2$ and $\mathbb{R}^n$

2 vectors are orthogonal vectors at right angles ($\mathbb{R}^2$ and $\mathbb{R}^3$)

Suppose $X = (x_1, x_2, \dots, x_n)^T$, $Y = (y_1, y_2, \dots, y_n)^T$

From 1.5 the inner product $X^T Y$ is

$$X^T Y = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n = \sum_{i=1}^{n} x_i y_i$$

in $\mathbb{R}^2$ and $\mathbb{R}^3$. We call this the Dot Product.

Eg)

> $X = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, Y = \begin{pmatrix} 3 \\ 1 \\ -4 \end{pmatrix}$
>
> $X^T Y = 1 \cdot 3 + (-2) \cdot 1 + 1 \cdot (-4) = -3$
>
> and
>
> $X = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, Y = \begin{pmatrix} 4 \\ -1 \\ 1 \\ 2 \end{pmatrix} X^T Y = 1 \cdot 4 + 2 \cdot (-1) + 3 \cdot 1 + 4 \cdot 2 = 13$

Scalar Dot Product in $\mathbb{R}^2$ and $\mathbb{R}^3$

Since

$$X^T X = (x_1, x_2) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_1^2 + x_2^2$$

$\|x\| = \sqrt{X^T X}$ in $\mathbb{R}^3$, $\|X\| = \sqrt{x_1^2 + x_2^2 + x_3^2}$

For X, Y in $\mathbb{R}^2$ or $\mathbb{R}^3$ where θ is an angle between Z and Y

Law of Cosines $X^TY = ||X||||Y||cos\left(\theta\right)$

$$||X-Y||^2 = ||X||^2 + ||Y||^2 - 2||X||||Y||cos(\theta)$$

$$\implies ||X||||Y||cos(\theta) = \frac{1}{2}\left[||X||^2 + ||Y||^2 - ||X-Y||^2\right]$$

$$||X||||Y||cos(\theta) = \frac{1}{2}\left(X^TX + Y^TY - (X-Y)^T(X-Y)\right) = \frac{1}{2}\left(X^TX + Y^TY - X^TX - Y^TY + X^TY + Y^TX\right)$$

$$= \frac{1}{2}\left(X^TY + Y^TX\right)$$

Since $X^TY = Y^TX = \frac{1}{2}\left(X^TY + X^TY\right) = \frac{1}{2}\left(2X^TY\right) = X^TY$

A Unit vector in the X direction $U = \frac{X}{||X||}$

$$||U|| = \left\|\frac{X}{||X||}\right\| = \frac{||X||}{||X||} = 1$$

and

$$cos(\theta) = \frac{X}{||X||} = \frac{||X||}{||X||} = 1$$

$$u = \frac{X}{||X||}, v = \frac{Y}{||Y||}, cos(\theta) = \frac{X^TY}{||X||||Y||} = \left(\frac{X}{||X||}\right)^T\left(\frac{Y}{||Y||}\right)^T = U^TV$$

Cauchy - Schwartz Inequality

If X and Y are in $\mathbb{R}^2$ or $\mathbb{R}^3$ then

$$|X^TY| \leq \|X\|\|Y\|$$

with equality IFF $x = 0$ or $y = 0$ or $y = \alpha x$

$$theta = \frac{\pi}{2} \Rrightarrow cos(\theta) = \theta$$

2 vectors X and Y θ $\mathbb{R}^2$ or $\mathbb{R}^3$ are called Orthogonal IFF $X^TY = 0$

EX)

> $$\begin{pmatrix}1\\1\end{pmatrix}\begin{pmatrix}1\\-1\end{pmatrix} \implies (1,1)^T\begin{pmatrix}1\\-1\end{pmatrix} = 1-1 = 0$$

Scalar and vector projections let X and Y be in $\mathbb{R}^2$ and $\mathbb{R}^3$

$$\alpha = \|X\|cos(\theta) = \frac{X^TY}{\|Y\|}$$

is called Scalar Projection.

Projection vector

$$P = \alpha\frac{Y}{\|Y\|} = \left(\frac{X^TY}{Y^TY}\right)y$$

Orthoganalaty in $\mathbb{R}^n$

Let X in $\mathbb{R}^n$ define Euclidian norm (length).

$$\|X\|_2 = \sqrt{X^TX} = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$

We define the angle between 2 vectors X and Y

$$cos(\theta) = \frac{X^TY}{\|X\|\|Y\|}$$

$$0 \le \theta \le \pi$$

Define: 2 Vectors X , Y in $\mathbb{R}^n$ are called orthogonal if $X^TY = 0$,

Notation is $X \cdot Y$

EX)

> Find the point on $Y = 3X$, closest to the point $(2, 1)$
>
> $V = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad W = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$
>
> (When x is 1 y is 3)
>
> $Q = \left(\frac{Y^TW}{W^TW}\right)w = \frac{5}{10}\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{3}{2} \end{pmatrix}$

Ex)

> Find the equation of a plane containing point $(4, 1, -3)$, perpendicular to the normal $N = (1, 2, 4)P_0 = (4, 1, 3), P = (x, y, z)$
>
> $\overrightarrow{P_0P} = (x-4, y-1, 2-(-3))^T = (x-4, y-1, 2+3)$
>
> p in the plain
>
> $$\implies \overrightarrow{P_0P}^TN = 0 \implies (x-4, y-1, 2+3)\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} = 0 \implies (x-4) + 2(y-1) + 4(2+3) = 0$$

In general a plain contains the point (x_0, y_0, z_0) normal to $N = (a, b, c)$ is given by:

$$a(x - x_0) + b(y - t_0) + c(z - z_0) = 0$$

EX)

> $$X = \begin{pmatrix} 1 \\ 2 \\ 1 \\ -2 \end{pmatrix}, Y = \begin{pmatrix} 2 \\ 1 \\ 2 \\ 1 \end{pmatrix}$$
>
> $cos(\theta) = \frac{X^TY}{||X||\,||Y||} = \frac{2+2+2-2}{\sqrt{10}\sqrt{10}} = \frac{4}{10} = \frac{2}{3}$
>
> $cos(\theta) = \frac{2}{3} \quad \theta = acos\left(\frac{2}{3}\right)$

Were $0 \leq \theta \leq \pi$

Lecture notes 2014/01/29

Similar Matries

Consider $L : V - v\, d\,\text{in}\,(v) = n$

Matrix representation depends on an ordered basis

Ex)

> $L : \mathbb{R}^2 \implies \mathbb{R}^2$ were $L(x) = (x - x_1, 3x_2)^T$
>
> $L(e_1) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, L(e_2) = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$
>
> The matrix representation with respect to $\{e_1, e_2\}$ is $\begin{pmatrix} 1 & -1 \\ 0 & 3 \end{pmatrix}$
>
> Suppose we choose $u_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $u_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$
>
> $L(u_1) = Au_1 = \begin{pmatrix} 2 \\ -3 \end{pmatrix}, L(u_1) = Au_1 = \begin{pmatrix} -1 \\ 6 \end{pmatrix}$
>
> We need $L(u_1) = au_1 + cu_2, L(u_2) = bu_1 + du_2 \implies \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ the transistion form $[u_1, u_2] = [e_1, e_2]$ if $(v_1 v_2) = \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}$ the transform form $[e_1, e_2] \to [u_1, u_2]$ is $U^{-1} = \begin{pmatrix} \frac{2}{3} & \frac{-1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix}$
>
> $U_1^{-1} L(u_1) = U^{-1}Au_1 = \begin{pmatrix} \frac{7}{3} \\ \frac{-1}{3} \end{pmatrix}, U^{-1}L(u_2) = U^{-1}Au_2 = \begin{pmatrix} \frac{-8}{3} \\ \frac{5}{3} \end{pmatrix}$
>
> $L(u_1) = \frac{7}{3}\, v_1 - \frac{1}{3}\, v_2\, , L(u_2) = \frac{-8}{3}\, u_1 + \frac{5}{3}\, u_2$
>
> Matrix
>
> $B = \begin{pmatrix} \frac{7}{3} & \frac{-8}{3} \\ \frac{-1}{3} & \frac{5}{3} \end{pmatrix}$

The columns of B are $(U^{-1}Av_1,\ U^{-1}Au_2) = U^{-1}(Au_1,\ Au_2)\ \therefore$

$$B = U^{-1}AU$$

Definition :-

If A and B are two $n \times n$, and S is singular such that $B = SBS^{-1}$ then B is similar to A Since $B = S^{-1}ASSBS^{-1} = SS^{-1}ASS^{-1} \Rightarrow A = SBS^{-1} \implies A$ is similar to B So if B is similar to A and A is similar B

EX)

Let D be the differentiation operation on P_3 find B representing D W.R.T. $[\ 1,\ x,\ x^2\ ]$ and A represtnting D W.R.T. $[\ 3,\ 4x,\ 2x^2+x\ ]$ Since

$$D(1) = 0 = 0 * 1 + 0 * x + 0 * x^2$$

$$D(2) = 1 = 1 * 1 + 0 * x + 0 * x^2$$

$$D(3) = 2x = 0 * 1 + 2 * x + 0 * x^2$$

$$B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

The transition from $[\ 3,\ 4x,\ 2x^2+x\ ]$ to $[\ 1,\ x,\ x^2\ ]$ is.

$$D(3) = 0 = 0 * 3 + 0 * 4 + 0 * (2x^2 + x)$$

$$D(4x) = 4 = \frac{4}{3} + 0 * 4x + 0 * (2x^2 + x)$$

$$D(2x^2 + x) = 4x + 1 = \frac{1}{3} * 3 + 1 * 4x + 0(2x^2 + x)$$

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

B is the matrix representation W.R.T. $[\ 1,\ x\ x^2]$ A is the matrix representation W.R.T. $[\ 3,\ 4x,\ 2x^2+x\ ]$ S is the transition matrix from $[\ 3,\ 4x,\ 2x^2+x\ ]$ to $[\ 1,\ x,\ x^2\ ]$

Class Notes 20140212

Orthogonal Subspaces

Suppose A is $m \times n$ and $x \epsilon N(A) \implies Ax = 0$ the $a_{i1}x_1 + 2_{i2}x_2 + \cdots + a_{in}x_n = 0 \implies x$ is orthogonal to the i^{th} of A^T, so if x is orthogonal to the i^{th} columns of linear combinations of the columns of A we say $R(A^T)$ and $N(A)$ are orthogonal.

Definition :-

If X and Y are two subspaces of $\mathbb{R}^n$ we say X is orthogonal to Y if and only if $X^T Y = 0 \;\forall\, x \in X$ and $y \in Y$

EX)

> Suppose X is spanned by e_1 and Y is spanned by e_3
>
> $\implies x \in X$ if and only if $x = \alpha \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $y \in Y$ if and only if $Y = \beta \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$,$X^T Y = \alpha \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \cdot \beta$
> $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \alpha\,\beta \left[\begin{pmatrix} 1\,0\,0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right] = 0$

Definition :-

Let Y be a subspace of $\mathbb{R}^n$. The set of vectors in $\mathbb{R}^n$ orthogonal to Y are called the "Orthogonal" and "Compliment" of Y, denoted $Y^\perp$

EX)

> Concider $\mathbb{R}^5$ were $X = span(e_1, e_2))$, $Y = span(e_3, e_4)$ if $X \epsilon Y \implies (\alpha, \beta, 0, 0, 0)$ and $Y \epsilon Y \implies (0, 0, \delta, \gamma, 0)$
> $X^T Y = 0 \implies X$ and Y are orthogonal, but they are not Orthogonal Compiment
>
> $X^\perp = span(e_1, e_4, e_5)$, $\frac{1}{Y} = span(e_1, e_2, e_5) \neq Y \neq X$

A Big Deal *Fundamental Subspaces*:

Let A be an $m \times n$ matrix range $R(A) = \{b \in \mathbb{R}^n \mid b = Ax$ for some $X \in \mathbb{R}^n\} =$ Column space of A $(S \cdot S \cdot \mathbb{R}^m)$

$R(A) = \{y \in \mathbb{R}^n \mid Y = A^T$ for some $X = \mathbb{R}^n\}$

$S \cdot S$ of $\mathbb{R}^n$

$$N(A) = \{X \in \mathbb{R}^n \mid Ax = 0\}$$

$$N(A^T) = \{Y \in \mathbb{R}^m \mid A^T Y = 0\}$$

Therom 5.2.1

Find the subspace theorem:- *If A is any $m \times n$ Matrix*

$$N(A) = (R(A^T))^{\perp}$$

$$N(A^T) = (R(A))^{\perp}$$

Ex)

> Let $A = \begin{pmatrix} 4 & 0 \\ 1 & 0 \end{pmatrix}$ and the Column space $\alpha \begin{pmatrix} 4 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \alpha \begin{pmatrix} 4 \\ 1 \end{pmatrix}$
>
> $R(A) = \left\{ b \mid b = x_1 \begin{pmatrix} 4 \\ 1 \end{pmatrix} \right\}, N(A^T) = \left\{ Y \mid A^T Y = 0 \right\}$
>
> $\begin{pmatrix} 4 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} Y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow Y = \beta \begin{pmatrix} 1 \\ 4 \end{pmatrix}$
>
> $X^T Y = \alpha \begin{pmatrix} 4 & 1 \end{pmatrix} \cdot \beta \begin{pmatrix} 1 \\ -4 \end{pmatrix} = \alpha \, \beta \cdot 0 = 0 \Rightarrow N(A^T) = \left[R(A) \right]^{\perp}$ show true for $N(A) = \left[R(A^T) \right]^{\perp}$

Theorem 5.2.2

If S is a subspace of $\mathbb{R}^n$ then $Dim(S) + Dim(S^{\perp}) = n$. Also if $\{x_1, x_2, \ldots, x_n\}$ is a basis of S then $\{x_{n+1}, x_{n+2}, \ldots, x_n\}$ is a basis for S then $\{x_1, x_2, \ldots, x_n\}$ is a basis for $\mathbb{R}^n$

Definition:-

The direct sum $W = U \oplus V$: given 2 subspaces U and V of W such that $W = U + V$ for any $u = U$ and $v = V$

Ex)

> Let $W = \mathbb{R}^3, u = span(e_1), v = span(e^2)$ then $w \in W \iff W = u \oplus v$

Theorem 5.2.3

$w = \alpha \, e_1 \; + \; \beta \, e_2 = span(e_2, \; e_2)$ then S in a subspace of $\mathbb{R}^n$ then $\mathbb{R}^n = S \oplus S^{\perp}$

ex)

> $S = span(e_1), S^{\perp} = span(e_2, e_3)$ then $S \oplus S^{\perp} = span(e_1, e_2, e_3) = \mathbb{R}^3$

Theorem 5.2.4

If S is subspace of $\mathbb{R}^n$ then $(S^{\perp}) = S$

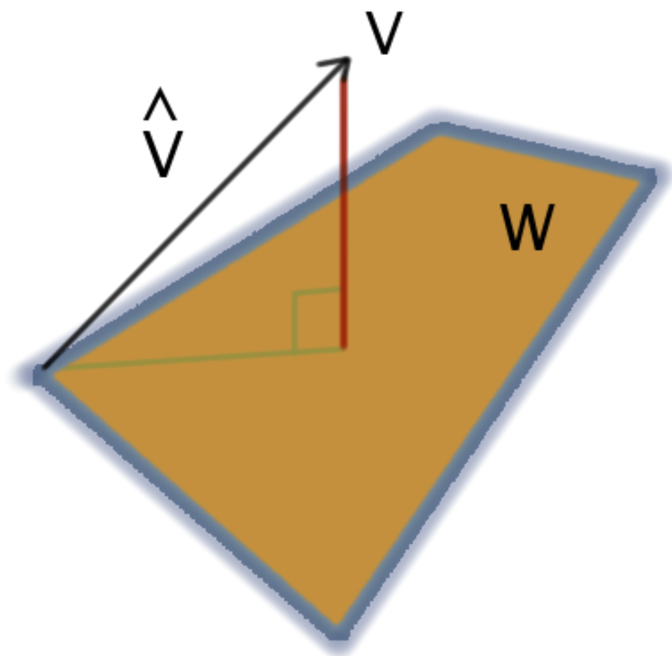
Ex)

> Let $A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 1 & 1 & -1 \end{pmatrix}$ find a basis for $N(A), R(A^T), N(A^T), R(A)$. To find $N(A)$ and $R(A^T)$ we have to row reduce A
>
> $$\Longrightarrow \begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$
>
> $N(A) \implies x_1 = 3x_3,\ x_2 = -2x_3$
>
> $x \in N(A) \iff X = \alpha \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$, $R(A^T)$ is spaned by $\begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$
>
> $Y \in R(A^T) Y = \beta \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix} + \delta \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$
>
> $X^T Y = 0$
>
> > $Dim(N(A)) = 1$
> >
> > $Dim(R(A^T)) = 2$
> >
> > $Dim(N(A)) + Dim(R(A^T)) = 3$
>
> $A^T = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 1 & 2 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$
>
> > So
>
> $X \in R(A) = \alpha \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$
>
> $Y \in N(A^T) = \delta \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$
>
> $X^T Y = 0 \implies N\left(A^T\right) = \left[R(A)\right]^{\perp}$

Class notes 20140226

Orthoganal Projectoions

Let W be a subspace of $\mathbb{R}^n$ and $dim W = k$, $W = span\{x_1, x_2, \dots, x_k\}$ $\hat{v} \in W$closest to v



Answer: Want $\hat{v} \in W$ so that $v - \hat{v} \in W^{\perp}$

$\hat{v}$ = is the Orthoganal projection of V into W We need $\hat{v}$ so that

$$V - \hat{v} \perp x_1 \implies (V - \hat{v}) \cdot x_1 = 0$$
$$V - \hat{v} \perp x_2 \implies (V - \hat{v}) \cdot x_2 = 0$$
$$\vdots$$
$$V - \hat{v} \perp x_n \implies (V - \hat{v}) \cdot X_n = 0$$

$$X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} Y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \quad X \cdot Y = X^T Y \quad = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \sum_{i=1}^{n} X_n \cdot Y_n$$

$$X_1^T (V - v^n) = 0 \implies x_1^T V - X_1^T \hat{v} = 0 \implies X_1^T V = X_1^{\hat{v}}$$
$$\vdots$$
$$X_k^T (V - v^n) = 0 \implies x_k^T V - X_k^T \hat{v} = 0 \implies X_k^T V = X_k^{\hat{v}}$$

$$\begin{pmatrix} x_1^T \\ x_2^T \\ \vdots \\ x_k^T \end{pmatrix} \begin{pmatrix} V \end{pmatrix} = \begin{pmatrix} x_1^T \\ x_2^T \\ \vdots \\ x_k^T \end{pmatrix} \begin{pmatrix} \hat{v} \end{pmatrix}$$

Let $A = \begin{pmatrix} X_1 & | & X_2 & | & \cdots & | & X_k \end{pmatrix}$ $A^T V = A^T \hat{v}$

col Space $(A) = W$ A^T is $n \times k$ and now invertable A is a $n \times k$ Matrix $\hat{v} \in W$ So $\hat{v} = c_1 x_1 + c_2 x_2 + \cdots c_k x_k$

$$c_1, c_2, \cdots, c_k \in \mathbb{R}$$

$$\hat{v} = \begin{pmatrix} X_1 & | & X_2 & | & \cdots & | & X_k \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{pmatrix} = AC \therefore A^T = A^T \hat{v} = A^T AC$$

$A^T A$ is $n \times k$ and $k \times n \implies A^T A \to k \times k$ and square, so invertible.

Prop

Since $x_1, \cdots, x_k$ are independent, $A^T A$ has an inversion.

$$A^T V = A^T AC \implies (A^T A)^{-1} A^T V = C$$

Formula For Projection

$W = span\{x_1, \cdots, x_k\}$ $X_1, \cdots x_k$ are independent. $V \in \mathbb{R}$ the projection of v onto $W = \hat{V} = A(A^TA)^{-1}A^TV$

Q_w = Projectio matrix onto W

URL for Lectures

If $\begin{pmatrix} A \\ \vdots \\ A^n \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = (?)$ an over determined system

If $Ax = d$ has no solutions, what's the best approximation for a solution? If $x_1A_1 + \cdots x_nA_n$ can't solve $Ax = b$, try $A^TAv = A^Tb$

set $x = (A^TA)^{-1}A^Tb$ The least squares approximation

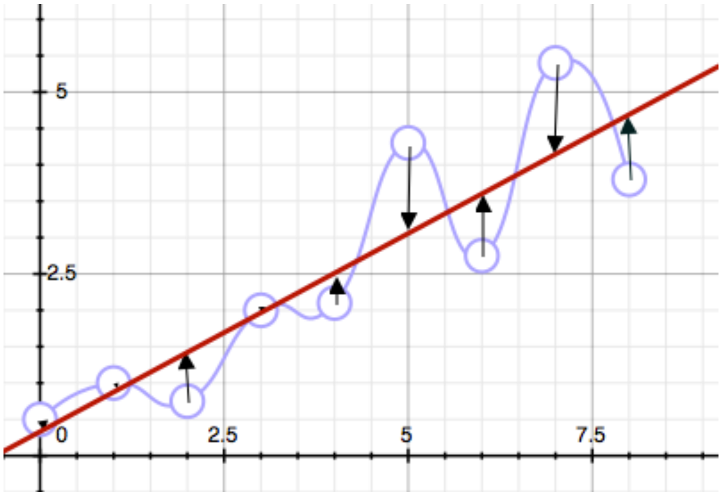
> **e.g.**
>
> Compute the projection matrix Q for the 2D subspace W of $\mathbb{R}^4$ spanned by $(1, 1, 0, 2)$ and $(-1, 0, 0, 1)$ What id the orthogonal projection of $(0, 2, 5, -1)$ onto W
>
> > $$A = \begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 0 & 0 \\ 2 & 1 \end{pmatrix}$$
> >
> > $$A^TA = \begin{pmatrix} 1 & 1 & 0 & 2 \\ -1 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 0 & 0 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 6 & 1 \\ 1 & 2 \end{pmatrix}$$
> >
> > $$(A^TA)^{-1} = \frac{1}{det(A)} = \frac{1}{12-1}\begin{pmatrix} 2 & -1 \\ -1 & 6 \end{pmatrix} = \begin{pmatrix} \frac{2}{11} & \frac{-1}{11} \\ \frac{-1}{11} & \frac{6}{11} \end{pmatrix} = \frac{1}{11}\begin{pmatrix} 2 & -1 \\ 1 & 6 \end{pmatrix}$$
> >
> > $$Q = \frac{1}{11}\begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 0 & 0 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ -1 & 6 \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 & 2 \\ -1 & 0 & 0 & 1 \end{pmatrix}$$
> > $$= \frac{1}{11}\begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 0 & 0 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 3 & 2 & 0 & 3 \\ -7 & 1 & 0 & 4 \end{pmatrix} = \frac{1}{11}\begin{pmatrix} 10 & 3 & 0 & -1 \\ 3 & 2 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ -1 & 3 & 0 & 10 \end{pmatrix}$$
>
> > b) $Q\begin{pmatrix} 0 \\ 2 \\ 5 \\ -1 \end{pmatrix} = \frac{1}{11}\begin{pmatrix} 7 \\ 1 \\ 0 \\ -4 \end{pmatrix}$

$$Q_wQ_w = A\overbrace{(A^TA)^{-1}A^TA}^{I}(A^TA)^{-1}A^T \implies Q_w^2A(A^TA)^{-1}A^T = Q_w$$

It is Indempotant.

Line fit with least squares

Fitting a line to data as the best fit we can.



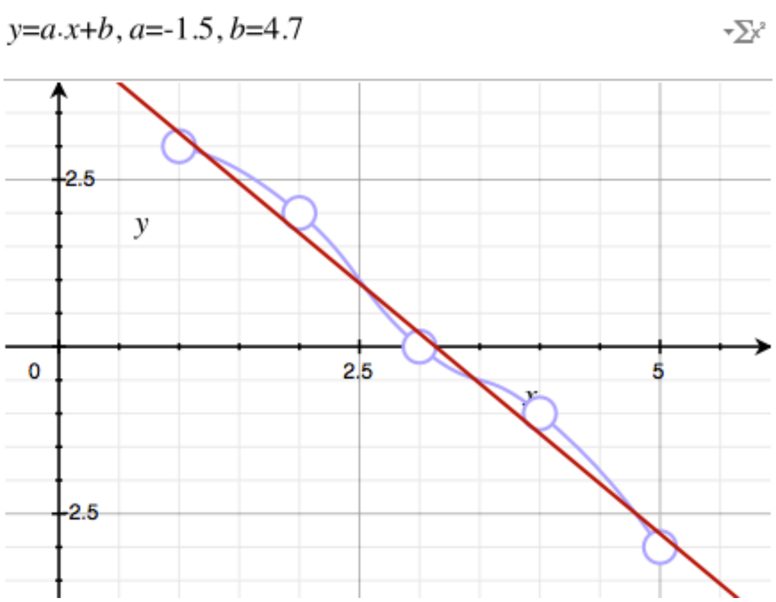
Wishful thinking, we need $y = ax + b$ as a best fit

$$\begin{aligned} y_1 &= mx_1 + b \\ y_2 &= mx_2 + b \\ &\vdots \\ y_n &= mx_n + b \end{aligned} \implies \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_3 \end{pmatrix} = \begin{pmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \vdots \\ x_n & 1 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix}$$

$Y = AC$ Has no solution so multiply both sides by A^T giving $A^TY = A^TAC \implies (A^TA)^{-1}A^TY = C = \begin{pmatrix} m \\ b \end{pmatrix}$ we have

$$A^TA = \begin{pmatrix} x_1 & \cdots & x_n \\ 1 & \cdots & 1 \end{pmatrix} \begin{pmatrix} x_1 & 1 \\ \vdots & \vdots \\ x_n & 1 \end{pmatrix} = \begin{pmatrix} \sum x_i^2 & \sum x_i \\ \sum x_i & n \end{pmatrix}$$

E.X.

> We have a data set $(1, 3), (2, 2), (3, 0), (4, -1), (5, -3)$ find the Best fit line.
>
> 
>
>
> $$\begin{pmatrix} 3 \\ 2 \\ 0 \\ -1 \\ -3 \end{pmatrix} \overset{?}{\underset{No!}{=}} \begin{pmatrix} 1 & 2 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} w \\ b \end{pmatrix} \longrightarrow \text{Least Squares.}$$

$$\begin{pmatrix} m \\ b \end{pmatrix} \approx (A^TA)^{-1}A^TY$$

$$A^TA = \begin{pmatrix} 5^2+4^2+3^2+2^2+1^2 & 5+4+3+2+1 \\ 5+4+3+2+1 & 5 \end{pmatrix} = \begin{pmatrix} 55 & 15 \\ 15 & 5 \end{pmatrix}$$

$$(A^TA)^{-1} = \frac{1}{5(55)-15^2}\begin{pmatrix} 5 & -15 \\ -15 & 55 \end{pmatrix} = \frac{1}{50}\begin{pmatrix} 5 & -15 \\ -15 & 55 \end{pmatrix} = \begin{pmatrix} \frac{1}{10} & \frac{-3}{10} \\ \frac{-3}{10} & \frac{11}{10} \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{10} & \frac{-3}{10} \\ \frac{-3}{10} & \frac{11}{10} \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \\ 0 \\ -1 \\ -3 \end{pmatrix} = \begin{pmatrix} 0.1 & -0.3 \\ -0.3 & 1.1 \end{pmatrix}\begin{pmatrix} -12 \\ 1 \end{pmatrix} = \begin{pmatrix} -1.2-0.3 \\ 3.6+1.1 \end{pmatrix} = \begin{pmatrix} -1.5 \\ 4.7 \end{pmatrix}$$

$$\therefore \begin{matrix} m = -1.5 \\ b = 4.7 \end{matrix} \implies y \approx -1.5x + 4.7$$

In polynomial form.

$y = ax^2 + bx + c$ we need

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} a_1x_1^2 + b_1x_1 + c \\ a_2x_2^2 + b_2x_2 + c \\ \vdots \\ a_nx_n^2 + b_nx_n + c \end{pmatrix} \implies \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ \vdots & \vdots & \vdots \\ x_n^2 & x_n & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$Y = AC \implies A^TY = A^TAC \implies (A^TA)^{-1}A^TY = C$$

e.x

$$x_1 + x_2 = 3,\ \ 2x_1 - 3x_2 = 1,\ \ 0x_1 + 0x_2 = 2$$

$$\begin{pmatrix} 1 & 1 \\ 2 & -3 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix},\quad A = \begin{pmatrix} 1 & 1 \\ 2 & -3 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 0 \\ 1 & -3 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 2 & -3 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 0 \\ 1 & -3 & 0 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 5 & -5 \\ -5 & 10 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \end{pmatrix} \implies \begin{pmatrix} 1 & -1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -2 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \hat{X}$$

e.x.

$$P = A\hat{X} = \begin{pmatrix} 1 & 1 \\ 2 & -3 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$$

$$r(\hat{X}) = b - p = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} - \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$$

$$A^T r = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -3 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\implies r \in N(A^T)$$

e.x.

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \\ -1 & -2 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \text{Solve } A^T A \hat{x} = A^T b$$

$$\begin{pmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 4 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$x_2 = \alpha, x_1 = 1 - 2x_2 = 1 - 2\alpha$$

$$\hat{x} \begin{pmatrix} 1 - 2\alpha \\ \alpha \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} -2 \\ 1 \end{pmatrix} \text{ has } \infty \text{ \# of solutions}$$

e.x.

$$\begin{pmatrix} 2 - 2\alpha \\ 1 - \alpha \\ \alpha \end{pmatrix} = \hat{x}$$

x	-1	0	1	2
y	0	1	3	9

$$\begin{pmatrix} 1 & x_1 \\ \vdots & \vdots \\ 1 & x_n \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \quad y = C_0 + C_1 x$$

$$\begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 3 \\ 9 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 3 \\ 9 \end{pmatrix}$$

$$\begin{pmatrix} 4 & 2 \\ 2 & 6 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} 13 \\ 21 \end{pmatrix}$$

$$\begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} 4 & 2 \\ 2 & 6 \end{pmatrix}^{-1} \begin{pmatrix} 13 \\ 21 \end{pmatrix}$$

$= \frac{1}{20}\begin{pmatrix} 6 & -2 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} 13 \\ 21 \end{pmatrix} =? = \begin{pmatrix} \frac{35}{20} \\ \frac{38}{20} \end{pmatrix}$

$y = C_0 + C_1 x = \begin{pmatrix} \frac{9}{5} \\ \frac{29}{10} \end{pmatrix}$

$y = \frac{9}{5} + \frac{29}{10}x$

If $(x_1, y_1) \dots (x_n, y_n)$ and $x = (\, x_1 \;\; x_2 \;\; \dots \;\; x_n \,)$ and $y = (\, y_1 \;\; y_2 \;\; \dots \;\; y_n \,)$. Then $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$ and $\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$

Let $y = C_0 + C_1 x$ be a liner function of best fit, show that if $\bar{x} = 0, C_0 = \bar{y}, C_1 = \frac{X^T Y}{X^T X}$

$$\begin{pmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{pmatrix}\begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_n \end{pmatrix}\begin{pmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{pmatrix}\begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_n \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$

$$\begin{pmatrix} n & \sum_{i=1}^{n} x_i \\ \sum_{i=1}^{n} x_i & \sum_{i=1}^{n} x_n^2 \end{pmatrix}\begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^{n} y_1 \\ \sum_{i=1}^{n} x_i y_i \end{pmatrix}$$

$$c_0 = \frac{1}{n}\sum_{i-1}^{n} y_1 = \bar{y}, \;\; c_1 = \frac{\sum_{i=1}^{n} x_i \, y_1}{\sum_{i=1}^{n} x_i \, x_i} = \frac{X^T Y}{X^T X}$$

$P = A(A^T A)^{-1} A^T$ for A is an $m \times n$ rank n Show $P^2 = P$ prove $P^k = P$.

$P^2 = [A\,(A^T A)^{-1}\,A^T][A\,(A^T A)^{-1}\,A^T]$

$= A\,(A^T A)^{-1}\,A^T A (A^T A)^{-1}\,A^T$

$= A\,(A^T A)^{-1} A^T = P$

Prove $P^k = P$

Assume $P^n = P$ show true for $m + 1$

$P\,P^m = P \cdot P$

$P^{m+1} = P^2 = P$

Show P is $sym \implies P^T = P$

$(A(A^T A)^{-1}\,A^T) = (A^T)^T [A^T A]^{-1}\,A^T = A[A^T A^{T^T}]^{-1} A^T = A[A^T A]^{-1} A^T = P$

$\implies P^T = P \implies P$ is Symmetrical

5.4 Inner Product Space

Def:- An inner product on a vector space V is an operation that assosiates a real number $\langle X, Y \rangle$ to each pair of vectors $X, Y \in V$ such that

1. $\langle X, Y\rangle = \langle Y, X\rangle$
2. $\langle X, X, \rangle \geq 0$ with equality if and only if $x = 0$
3. $\langle \alpha X + \beta Y, 2\rangle = \alpha\langle X, 2\rangle + \beta\langle Y, 2\rangle$

ex)

1. $\mathbb{R}^n$ $\langle X, y\rangle = X^T Y$
2. Weighted inner product
3. $\langle X, Y\rangle = \sum_{i=1}^n w_i$

1a)

> $A = \begin{pmatrix} 3 & 4 \\ 6 & 8 \end{pmatrix} \to \begin{pmatrix} 3 & 4 \\ 0 & 0 \end{pmatrix} \implies R(A^t)$ has a basis $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$
>
> $\begin{pmatrix} 3 & 4 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$
>
> $N(A)$ has the basis $\begin{pmatrix} 4 \\ -3 \end{pmatrix}$ $R(A)$ has the basis $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$
>
> $N(a)\begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies \begin{pmatrix} x_1 \\ x-2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$
>
> The Basis for $N(A^T)$ is $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ or $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$

1c)

> $A = \begin{pmatrix} 4 & -2 \\ 1 & 3 \\ 2 & 1 \\ 3 & 4 \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$ Basis for $R(A)$ is $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$
>
> $N(A)$ $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

ex)

> $e[a, b]$
>
> $\langle f, g \rangle = \int_a^b f \cdot g \, dx$ or weighted $\langle f, g \rangle = \int_a^b w f \cdot g \, dx$ where $w(x) > 0$

Define length or norm. $\|v\| = \sqrt{\langle v, v, \rangle}$

1. $\langle v, v\rangle = \sum v_i^2 \quad \|v\| = \sqrt{\sum v_i^2}$
2. $u \perp v \iff \langle u, v\rangle = 0$

Thm : Pythagorean Law

$$u \perp v \quad \|u + v\|^2 = \|u\|^2 + \|v\|^2$$

$$\|u+v\|^2 = \langle u+v, u+v\rangle = \langle u,u\rangle + \overbrace{\langle v,u\rangle + \langle u,v\rangle}^{=0} + \langle v,v\rangle = \|u\|^2 + \|v\|^2$$

ex)

> $e[-1,1]$
>
> $u = 1\,,\ \ v = x^2 \quad \langle 1, x^2\rangle = \int_{-1}^{1} 1\cdot x^2\ dx\ =\ 0$
>
> length $\|1\|^2 = \langle 1, 1\langle = \int_{-1}^{1} 1\cdot 1\ dx = 2$
>
> $\|x^3\|^2 = \langle x^3, x^3\rangle = \int_{-1}^{1} x^3\cdot x^2\ dx = \frac{2}{7}$
>
> So $\|1\| = \sqrt{2} \quad \|x^3\| = \sqrt{\frac{2}{7}}$
>
> One can say $\|1+x^2\| = \frac{16}{7} = 2 + \frac{2}{7}$

For $\mathbb{R}^{m\times n}$ Frobeio's Norm.

$\|\cdot\|_f$ where

$$\|A\|_f = \langle A, A\rangle^{\frac{1}{2}} = \left(\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}^2\right)^{\frac{1}{2}}$$

e.g.

$$A = \begin{pmatrix} 1 & 2 & 4 \\ -1 & 3 & 2 \end{pmatrix} \quad \|A\|_f = \left(1+4+16+1+9+4\right)^{\frac{1}{2}} = \sqrt{35}$$

Consider $P_n(x)$ polynomials of degree $<\ n$. Let $x_1,\ldots,x_n$ be n distinct numbers. Let $p(x),\ q(x)\in P_n(x)$

$$\langle p, q\rangle = \sum_{i=1}^{n} p(x_i)q(x_i)$$

is also the inner product.

1. $\langle P, p\rangle = \sum_{i=1}^{n} P^2(x_i) \geq 0$
2. Since $\sum_{i=1}^{n} p(x)q(x_i) = \sum_{i=1}^{n} Something$
3. $\langle \alpha p + \beta q, n\rangle = \sum_{i=1}^{n}(\ \alpha\, p(x_i) + \beta\, p\,(x_i)\,)\ r(x_i) = \sum_{i=1}^{n} \begin{matrix}\alpha\, p(x_i) & \cdots \\ \vdots & \end{matrix}$

e.x.

> $x_i\ =\ \frac{i-1}{2}\,,\ \ i = 1,2,3, \qquad x_1\ = 0,\ \ x_2\ =\ \frac{1}{2}\,,\ \ x_3\ = 1$
>
> $\|2x\| = \sqrt{\langle 2x, 2x\rangle} = (\sum_{i=1}^{n} 4x^2)^{\frac{1}{2}} = (0+1+4)^{\frac{1}{2}} = \sqrt{5}$

Def:

If $u_1 v_1\ \in V$ and $\surd \neq 0$ Then the scalar projection of u onto v is $\alpha = \frac{\langle u,v\rangle}{\|v\|}$ the vector projection

$$P = \alpha\left(\frac{v}{\|v\|}\right) = \left(\frac{\langle u,v\rangle}{\|v\|}\right)^{\frac{1}{2}}$$

The Couchy Schwartz Inequality

let u and v be the inner product space V, then $|\langle u, v\rangle| \leq \|u\|\|v\|$ with equality $\iff \quad u = \alpha\ v$ i.e. u and V are linearly independent.

Proof

If $v \equiv 0 \quad |\langle u, v\rangle| = \|u\| \cdot 0 = 0$

Let P be the vector projection of u onto v, since P and $U \cdot P$ are orthogonal

$$\|u-p\|^2 + \|p\|^2 = \|u\|^2$$

so

$$\|p\|^2 = \|u\|^2 - \|u-p\|^2 = \left[\frac{|\langle u, v\rangle|}{\|V\|^2} \cdot \|v\|^2\right]$$

$$\implies |\langle u, v\rangle| = \|v\|^2\left(\|u\|^2 - \|u-p\|^2\right) \leq \|v\|^2\|u\|^2$$

$$\implies |\langle U, V\rangle| \leq \|u\|\|v\|$$

equality $\iff \quad \|u-p\|^2 = 0 \implies u = p$ either $v = 0$ or u and v are in the same direction since.

$$|\langle u, v\rangle| \leq \|u\|\|v\| \leq \langle u, v\rangle \leq \|u\|\|v\| - 1 \leq \frac{\langle u, v\rangle}{\|u\|\|v\|} = \cos(\theta)$$

Def:-

Normalled linear vector space given a vector space V if each $v \in V$ has a real number …………

1. $\|v\| \geq 0 \quad \|v\| = 0 \iff v = 0$
2. $\|\alpha\| = |\alpha|\|v\| \Delta$ inequality.
3. $\|u+v\| \leq \|u\| + \|v\|$

Theorem.

If V is an inner ptroduct space, then

$$\|v\| = \langle v, v\rangle^{\frac{1}{2}} \ \forall \ v \in V$$

defines a norm on V

1. $\langle v, v\rangle^{\frac{1}{2}} \geq 0$ SInce $\langle v, v, \rangle \geq 0$
2. $\|\alpha v\| = \langle \alpha\ v, \alpha\ v\rangle^{\frac{1}{2}} = [\alpha^2\langle v, v\rangle]^{\frac{1}{2}} = \left(\alpha^2\right)^{\frac{1}{2}}\langle v, v\rangle^{\frac{1}{2}} = |\ \alpha\ |\langle v, v\rangle^{\frac{1}{2}} = |\ \alpha\ |\|v\|$
3. $\|u+v\|^2 = \langle u+v, u+v\rangle = \langle u, u\rangle + 2\langle u, v\rangle + \langle v, v\rangle \leq \|u\|^2 + 2\|u\|\|v\| + \|v\|^2 \leq \left(\|u\| + \|v\|\right)^2$

$$\|u+v\| \leq \|u\|\|v\|$$

Some other norms $V \equiv \mathbb{R}^n$

1. $\|x\|_1 = \sum_{i=1}^n |x_1|$
2. $\|x\|_\infty = \max_i |x_i|$

3. P-norm $\|x\|_p = \left[\sum_{i=1}^{n} |x_i|^P \right]^{\frac{1}{P}}$ for $P = 2 \implies$ Euclidian Norm. $\iff P \neq P_0$ Theorem won't hold.

Def:-

If $x, y \in V$ where V is a N.L.V.S, then $\|x - y\| \equiv$ to the distance between x and y

e.x

> Suppose $\langle f, g \rangle = \int_{-1}^{1} f \cdot g \, dx$ define $\| \cdot \| = \langle f, g \rangle^{\frac{1}{2}}$
>
> if $f = x$ and $g = x^2$
>
> $\| f - g \| = \| x - x^2 \| = \left[\int_{-1}^{1} (x - x^2)(x - x^2) dx \right]^{\frac{1}{2}}$ = distance between functions

5.5 Ortho Normal Spaces (sets)

Def:-

Let $\{v_1, v_2, \dots, v_n\}$ be non zero vectors in an inner product space V if $\langle v_i, v_i \rangle = 0$ whenever $i \neq i$ then the set is said to be orthogonal

e.g.

> $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ -10 \end{pmatrix}$ is an orthogonal set.

Thm:-

If $\{v_1, \dots, v_n\}$ is an orthogonal set of vectors in an inner product V, then $\{v_1, \dots, v_n\}$ is linearly independent. From

$$c_1 v_1 + c_2 v_2 + \dots + c_n v_n = 0$$

$$\langle c_1 v_1 + c_2 v_2 + \dots c_n v_n, v_i \rangle = 0$$

$$c_1 \langle v_1, v_i \rangle + c_2 \langle v_2, v_i \rangle + \dots + c_n \langle v_n, v_i \rangle = 0$$

$$\implies c_i \langle v_j, v_i \rangle = 0$$

$$c_i \|v_i\|^2 = 0$$

$$\implies c_i = 0$$

Def:-

An orthogonal set of vectors is an Orthogonal set of unit vectors. The set $\{u_1, u_2, \dots, u_n\}$ is orthogonal is $\langle u_i, u_i \rangle = \delta_i j = \begin{cases} 1, & \text{i = j} \\ 0, & \text{i} \neq \text{j} \end{cases}$ (Kronken Delta Function)

Def:-

Suppose $B = \{u_1, u_2, \dots, u_n\}$ is an orthonormal basis for S, then $S = span(u - 1, \dots, u_n)$

Perseval's Formula.

if $\{u_1, u_2, \cdots, u_n\}$ is an orthonormal basis for an inner product space V and $V = \sum_{i=1}^{n} c_i u_i$ then $\|u\|^2 = \sum c_i^2$

e.g.

> $u_1 = \left(\frac{1}{\sqrt{5}} , \frac{2}{\sqrt{5}} \right), \ u_2 = \left(\frac{-2}{\sqrt{5}} , \frac{1}{\sqrt{5}} \right)$
>
> if $X \in \mathbb{R}^2$ then $X^T u_1 = \dfrac{x_1 + 2x_2}{\sqrt{5}}$ and $X^T u_2 = \dfrac{-2x_1 + x_2}{\sqrt{5}}$
>
> $\|x\|^2 = \left(\dfrac{x_1 + 2x_2}{\sqrt{5}} \right) + \left(\dfrac{-2x_1 + x_2}{\sqrt{5}} \right) = x_1^2 + x_2^2$

Orthogonal Matrices

Def:-

An $n \times n$ matrix Q is called orthogonal if the Column vectors of Q form an orthogonal set in $\mathbb{R}^n$

Thm:-

An $n \times n$ matrix Q is orthogonal $\iff Q^T Q \cdot I = Q^{-1} = Q^T$

$$Q = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$

$$\left\| \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix} \right\| = \cos^2(\theta) + \sin^2(\theta) = 1$$

$$Q^T = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}$$

$$Q^T Q = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} =$$

$$\begin{pmatrix} \cos^2(\theta) + \sin^2(\theta) & -\cos(\theta)\sin(\theta) + \cos(\theta)\sin(\theta) \\ -\sin(\theta)\cos(\theta) + \sin(\theta)\cos(\theta) & \sin^2(\theta) + \cos^2(\theta) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

If Q is an $n \times n$ orthogonal matrix,

1. The Column vectors are orthogonal in $\mathbb{R}^n$
2. $Q^T Q = I$
3. $Q^{-1} = Q^T$
4. $\langle Q_x , Q_y \rangle = \langle x , y \rangle$
5. $\|Q_x\|_2 = \|x\|_2$

Least Squares.

Solve $Ax = b$ for $A : m \times n$ then $A^T A \hat{x} = A^T b$

Thm:-

If the columns of A form an orthoganal set on $\mathbb{R}^n$ then $A^TA = I$ and $\hat{x} = A^Tb$

e.x.

Find the best least squares approximation to $\cos(x)$ on $[0, 1]$ by a linear function.

Let S be the subspace of all linear functions in $C[0, 1]$, then $u_1 = 1$, $u_2 = \sqrt{12}\,(x - \frac{1}{2})$ is an orthogonal set of vectors.

Let

$$c_1 = \int_0^1 u_1 \cos(x)\, dx = 1$$

$$c_2 = \int_0^1 u_2 \cos(x)\, dx = \sqrt{12}\, \int_0^1 \left(x - \frac{1}{2}\right) cos(x)\, dx$$

$$= \left(\frac{1}{2} \sin(1) + \cos(-1) \right)x + \left(\frac{3}{4} \sin(1) - \frac{1}{2} \cos(1) + \frac{1}{2} \right)$$

Class Notes 03/26/2014

Let $L(x) = (x_2 + x_3 , x_1 + x_3 , x_1 + x_2)^T$ Determin the standard matrix.

$$L(e_1) = L\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\1\\1\end{pmatrix}$$

$$L(e_2) = L\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}1\\0\\1\end{pmatrix}$$

$$L(e_3) = L\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}1\\1\\0\end{pmatrix}$$

$$A = \begin{pmatrix}0 & 1 & 1\\1 & 0 & 1\\1 & 1 & 0\end{pmatrix}$$

$$L\begin{pmatrix}1\\3\\2\end{pmatrix} = A\begin{pmatrix}1\\3\\2\end{pmatrix} = \begin{pmatrix}0 & 1 & 1\\1 & 0 & 1\\1 & 1 & 0\end{pmatrix}\begin{pmatrix}1\\3\\2\end{pmatrix} = \begin{pmatrix}5\\3\\4\end{pmatrix}$$

Let A and B be similar matrices, λ is any scalar. Show $A - \lambda I$ and $B - \lambda I$ are similar.

$$A = S^{-1}BS$$

$$A - \lambda I = S^{-1}BS - \lambda I$$

$$A - \lambda I = S^{-1}BS - S^{-1}A(\lambda I) = S^{-1}BS - S^{-1}(\lambda I)S$$

$A - \lambda I = S^{-1}(B - \lambda I)S \implies A - \lambda I$ and $B - \lambda I$ are similar

Let A be $n \times n$, if there exists a nonzero vector x, so that $Ax = \lambda x$ for some scalar λ, then we call λ an Eigenvalue of A and x is the associated Eigenvector (Not Unique)

e.x.

> $A = \begin{pmatrix} 3 & -3 \\ -4 & 2 \end{pmatrix}$, $x = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$
>
> $Ax = \begin{pmatrix} 3 & -3 \\ -4 & 2 \end{pmatrix}\begin{pmatrix} -2 \\ 2 \end{pmatrix} = \begin{pmatrix} -12 \\ 12 \end{pmatrix} = 6\begin{pmatrix} -2 \\ 2 \end{pmatrix} = 6x$
>
> So $Ax = 6x$, $\lambda = 6$, $x = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$

If A is $n \times n$ and λ is an eigenvalue with eigenvectors x

a. λ is an eigenvalue of a.

b. $(A - \lambda I)x$ has nontrivial solutions.

c. $N(A - \lambda) \neq 0$.

d. $A - \lambda I$ is singular.

e. $\det(A - \lambda I) = 0$.

The last part $\det(A - \lambda I) \equiv n^{th}$ order polynomial in λ is called the Characteristic Ploynomial.

e.g.

FInd the eigenvalues and eigenvectors for $A = \begin{pmatrix} 1 & 5 \\ 2 & -2 \end{pmatrix}$

Form $(A - \lambda I) = \begin{pmatrix} 1-\lambda & 5 \\ 2 & -2-\lambda \end{pmatrix}$

We want $\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 5 \\ 2 & -2-\lambda \end{vmatrix} = 0$

$$(1-\lambda)(-2-\lambda) - 10 = 0$$

$$\lambda^2 + \lambda - 12 = 0 \implies (\lambda+4)(\lambda-3) = 0$$

$\lambda = -4$, $\lambda = 3$ to get the eigenvectors x_1, x_2 solve $(A - \lambda I)x = 0$

$$\lambda_1 = -4\,, \begin{vmatrix} 1-(-4) & 5 \\ 2 & -2-(-4) \end{vmatrix} = \begin{pmatrix} 5 & 5 \\ 2 & 2 \end{pmatrix} x_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies x_1 \alpha \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\lambda_2 = 3\,, \begin{vmatrix} 1-3 & 5 \\ 2 & -2-3 \end{vmatrix} = \begin{pmatrix} -2 & 5 \\ 2 & -5 \end{pmatrix} x_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies x_2 \alpha \begin{pmatrix} 5 \\ 2 \end{pmatrix}$$

eigenspace has basis vectors $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $\begin{pmatrix} 5 \\ 2 \end{pmatrix}$

$v \in$ eigenspace of $A \iff v = c_1\begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2\begin{pmatrix} 5 \\ 2 \end{pmatrix}$

In General

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$A - \lambda I = \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = (a-\lambda)(d-\lambda) - bc = 0$$

$$\implies \lambda^2 - \operatorname{tr}(A)\lambda + \det(A) = 0$$

Thm:-

If A and B are similar, they have the same characteristic polynomials. $\implies$ same eigenvalues. (eigenvectors will be diffwerent, possibly)

Complex Eigenvalues.

$Ax = \lambda x$ and A is Real and λ is complex,

$$\bar{A}\bar{x} = \bar{\lambda}\bar{x}$$

$$A\bar{x} = \bar{\lambda}\bar{x}$$

$\implies$ $\bar{x}$ is a eigenvector assosciated with $\bar{\lambda}$

$$\lambda = a - b\bar{\mathbb{i}}\ ,\ \bar{\lambda} = a - b\mathbb{i}$$

$$\lambda = 1 - 3\mathbb{i}\ ,\ \bar{\lambda} = 1 + 3\mathbb{i}$$

Systems of constant coefficient. linear differential equations.

In 1-D $y\prime = 3y$ is a 1st order differential equation.

$$y\ \alpha\ \mathbb{e}^{3t}\ ,\ y = c\mathbb{e}^{3t}$$

$$y\prime = c(3\mathbb{e}^{3t}) = 3(c\mathbb{e}^{3t}) = 3y$$

In general

$$y\prime = ay$$

assume

$$y(t) = \mathbb{e}^{\lambda t}$$

$$\lambda\mathbb{e}^{\lambda t} \implies \lambda = a$$

The solution is

$$y = c\mathbb{e}^{at}$$

First order systems

$$\begin{pmatrix} y_1\prime & = & a_{11}y_1 & + & a_{12}y_2 & + & \dots & + & a_{1n}y_n \\ y_2\prime & = & a_{21}y_1 & + & a_{22}y_2 & + & \dots & + & a_{2n}y_n \\ \vdots & & \vdots & & \vdots & & & & \vdots \\ y_n\prime & = & a_{n1}y_1 & + & a_{n2}y_2 & + & \dots & + & a_{nn}y_n \end{pmatrix}$$

In part 2-D

$$y_1\prime = ay_1 + by_2$$
$$y_2\prime = cy_1 + dy_2$$
$$A = \left(a_{ij}\right)$$
$$Y = \left(y_1, y_2, \dots y_n\right)^T$$
$$Y\prime = \left(y_1\prime, y_2\prime, \dots y_n\prime\right)^T$$

$Y\prime = AY$ looks like 1-D $y\prime = ay$ Look for solution to

$$Y = \begin{pmatrix} x_1 e^{\lambda t} \\ x_2 e^{\lambda t} \\ \vdots \\ x_n e^{\lambda t} \end{pmatrix} = e^{\lambda t} x$$

Where $x = \left(x_1, x_2, \dots, x_n\right)^T$ if $Y = e^{\lambda t} = \lambda e^{\lambda t} x$

$e^{\lambda t} \neq 0$ devide by it $Ax = \lambda x$ standard eigenvalue problem

In General we solve

$$Y\prime = AY$$

for initial condition $Y(0) = Y$

e.g.

$$y_1\prime = 2y_1 + 2y_2\ ,\ y_1(0) = 3$$
$$y_2\prime = 2y_1 + 5y_2\ ,\ y_2(0) = 1$$

So

$$Y' = \begin{pmatrix} 2 & 2 \\ 2 & 5 \end{pmatrix} Y$$

for $Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ and $A = \begin{pmatrix} 2 & 2 \\ 2 & 5 \end{pmatrix}$

The Characteristic polynomial is

$$\lambda^2 - 7\lambda + 6 = 0$$
$$(\lambda - 6)(\lambda - 1) = 0$$

$\lambda_1 = 1$ Solve $(A - \lambda I)x = 0$

$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} x_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}\ ,\ x_1\ \alpha \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$\lambda_2 = 6$ Solve $(A - \lambda I)x = 0$

$$\begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix} x_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}\ ,\ x_2\ \alpha \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

The solution is

$$Y = c_1 \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{6t}$$

$$t = 0\,,\ Y(0) = \begin{pmatrix} 2c_1 + c_2 \\ -c_1 + 2c_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \implies c_1 = c_2 = 1$$

$$Y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^{t} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{6t}$$

// insert a picture of tank mixing problem.

> Tank A initially has 40g of salt,
>
> Tank B initially has 100g of salt.
>
> Find the amount of salt in either tank at time (t)
>
> Let $y_1(t) =$ the amount of salt in tank A
>
> and $y_2(t) =$ the amount of salt in tank B
>
> Concentration in A is $\frac{y_1(t)}{100}$, B is $\frac{y_2(t)}{500}$
>
> $\frac{dy_1}{dt} \equiv$ of the rate of change of the salt in tank A
>
> $\frac{dy_2}{dt} \equiv$ of the rate of change of the salt in tank B
>
> $$\frac{dy_1}{dt} = -\left(\frac{y_1}{100}\right)(30) + \left(\frac{y_2}{500}\right)(10)$$
>
> $$\frac{dy_2}{dt} = \left(\frac{y_1}{100}\right)(30) - \left(\frac{y_2}{500}\right)(10) - \left(\frac{y_2}{500}\right)(20)$$
>
> $$\frac{dy_2}{dt} = -\left(\frac{y_1}{100}\right)(30) - \left(\frac{y_2}{500}\right)(30)$$
>
> $$Y_1(0) = 40g\,,\ y_2(0) = 100g$$
>
> Define $Y = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}$
>
> $$Y' = \begin{pmatrix} -\frac{3}{1}y_1 - \frac{1}{50}y_2 \\ \frac{3}{10}y_1 - \frac{3}{50}y_2 \end{pmatrix}$$
>
> $$Y' = AY\,,\ A = \begin{pmatrix} -\frac{3}{10} & \frac{1}{50} \\ \frac{3}{10} & -\frac{3}{50} \end{pmatrix}$$
>
> $$Y_0 = \begin{pmatrix} 40 \\ 100 \end{pmatrix}$$
>
> Assume $Y = xe^{\lambda t}$

$$\lambda^2+\tfrac{18}{50}\lambda-\tfrac{6}{500}=0$$

Charcteristic Polynomial is

$$\lambda^2+\tfrac{9}{25}\lambda-\tfrac{3}{250}=0 \implies \lambda_1\ ,\ \lambda_2$$

Lecture notes 2014/04/02

sec 5.5 homework solutions.

1. a) $\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}$, $\begin{pmatrix}1\\0\end{pmatrix}^T\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}$

$\left\|\begin{pmatrix}1\\0\end{pmatrix}\right\|=\sqrt{1^2+0^2}$

$\left\|\begin{pmatrix}1\\0\end{pmatrix}\right\|=1$

b) $\begin{pmatrix}\frac{3}{5}\\\frac{4}{5}\end{pmatrix}\begin{pmatrix}\frac{5}{13}\\\frac{12}{13}\end{pmatrix}$ These are a unit. But $\begin{pmatrix}\frac{3}{5} & \frac{4}{5}\end{pmatrix}\begin{pmatrix}\frac{5}{13}\\\frac{12}{13}\end{pmatrix}\neq 0$ So not orthogonal

c) $\begin{pmatrix}1\\-1\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}$ are not unit vectors, they are orthogonal.

d) $\begin{pmatrix}\frac{\sqrt{2}}{2}\\\frac{1}{2}\end{pmatrix}\begin{pmatrix}\frac{1}{2}\\\frac{\sqrt{2}}{2}\end{pmatrix}$

$\begin{pmatrix}\frac{\sqrt{2}}{2} & \frac{1}{2}\end{pmatrix}\begin{pmatrix}\frac{1}{2}\\\frac{\sqrt{2}}{2}\end{pmatrix}=-\frac{\sqrt{3}}{4}+\frac{sqrt3}{4}=0 \implies$ are orthoganal.

$\left\|\begin{pmatrix}\frac{\sqrt{2}}{2}\\\frac{1}{2}\end{pmatrix}\right\|=\sqrt{\left(\frac{\sqrt{3}}{2}\right)^2+\left(\frac{1}{2}\right)^2}=\sqrt{\frac{3}{4}+\frac{1}{4}}=1 \implies$ Unit vectors

Given a subspace spanned by orthogonal vectors, the projection matrix is UU^T were the collumns of U are.

$$u_2=\begin{pmatrix}\frac{2}{3}\\\frac{2}{3}\\\frac{1}{3}\end{pmatrix},\ u_3=\begin{pmatrix}\frac{1}{\sqrt{2}}\\-\frac{1}{\sqrt{2}}\\0\end{pmatrix}$$

$$U=\begin{pmatrix}u_1 & u_2\end{pmatrix}=\begin{pmatrix}\frac{2}{3} & \frac{1}{\sqrt{2}}\\\frac{2}{3} & -\frac{1}{\sqrt{2}}\\\frac{1}{3} & 0\end{pmatrix}$$

$$P=UU^T=\begin{pmatrix}\frac{17}{18} & -\frac{1}{18} & \frac{2}{9}\\-\frac{1}{18} & \frac{17}{18} & \frac{2}{9}\\\frac{2}{9} & \frac{2}{9} & \frac{1}{9}\end{pmatrix}$$

$$x = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

projection of x onto the subspace by u_2 and u_3 is,

$$p = Px = \begin{pmatrix} \frac{23}{18} \\ \frac{41}{18} \\ \frac{8}{9} \end{pmatrix}$$

$$p - x = \begin{pmatrix} \frac{5}{18} \\ \frac{5}{18} \\ -\frac{10}{9} \end{pmatrix}$$

$$(p-x)^T u_2 = \left(\frac{5}{27} + \frac{5}{27} - \frac{10}{9} \right) = 0$$

$$(p-x)^T u_3 = \left(\frac{5}{18\sqrt{2}} - \frac{5}{18\sqrt{2}} \right) = 0$$

1)

> c)
>
> $x_1 = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \ x_2 = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$
>
> $\left\| x_1 \right\| = \sqrt{\cos^2\theta + \sin^2\theta}, \ \left\| x_2 \right\| = \sqrt{\sin^2\theta + \cos^2\theta}$
>
> $X_1^T X_2 = -\cos\theta \sin\theta + \sin\theta \cos\theta = 0$
>
> b)
>
> $Y = c_1 x_1 + c_2 x_2$
>
> $c_1 = \langle y, x_1 \rangle, \ c_2 = \langle y, x_2 \rangle \implies \begin{aligned} c_1 &= y_1 \cos\theta + y_2 \sin\theta \\ c_2 &= -y_1 \sin\theta + y_2 \sin\theta \end{aligned}$
>
> $c_1^2 + c_2^2 = \|Y\| = y_1^2 + y_2^2$
>
> $= (y_1^2 \cos^2\theta + 2y_1 y_2 \cos\theta \sin\theta + y_2^2 \sin^2\theta) + (y_1^2 \sin^2\theta - 2y_1 y_2 \sin\theta \cos\theta + y_2^2 \cos^2\theta)$
>
> $= y_1^2(\cos^2\theta + \sin^2\theta) + 0 + y_2^2(\sin^2\theta + \cos^2\theta)$
>
> $= y_1^2 + y_2^2$

7)

> $x = c_1 u_1 + c_2 u_2 + c_3 u_3$
>
> $\|x\| = 5, \ \langle u_1, x \rangle = 4, \ x \perp u_2$
>
> $c_1 = \langle x, u_1 \rangle, \ c_2 = \langle x, u_2 \rangle, \ c_3 = \langle x, u_3 \rangle$
>
> $c_1^2 + c_2^2 + c_3^2 = \|x\|^2 = 25$
>
> $c_1 = 4, \ x \perp u_2 \implies c_2 = 0$

$16 + 0 + c_3^2 = 25 \implies c_3^2 = 9 \implies c_3 = \pm 3$

13)

Show $(Q^m)^{-1} = (Q^T)^m = (Q^m)^T$, Q is $n \times n$ orthogonal matrix.

a) $m = 1$, $Q^{-1} = Q^T = Q^T$ true for Q othogonal

$m = 2$, $(Q^2)^{-1} = (Q \cdot Q)^{-1} = Q^{-1} \cdot Q^{-1} = Q^T Q^T = (Q^T)^2$

Assume true for $m = k$

$(Q^k)^{-1} = (Q^T)^k$

Show true for all $m = k + 1$

$(Q^{k+1})^{-1} = (Q^k Q)^{-1} = Q^{-1}(Q^k)^{-1} = Q^T (Q^T)^k = (Q^T)^{k-1}$

$e[-1 , 1] ,]\langle f , g\rangle = \int_{-1}^{1} f \cdot g \, dx$

Show 1 and x are orthogonal

$\langle 1 , x\rangle = \int_{-1}^{1} 1 \cdot x \, dx = \int_{-1}^{1} x \, dx = \frac{x^2}{2} \Big|_{-1}^{1} = \frac{1}{2} - \frac{1}{2} = 0$

$\|1\| = \sqrt{\langle 1 , 1\rangle}$

$= \sqrt{\int_{-1}^{1} dx} = \sqrt{2}$

$\|x\| = \sqrt{\langle x , x\rangle}$

$\sqrt{\int_{-1}^{1} x^2 \, dx} = \sqrt{\frac{2}{3}} = \frac{\sqrt{6}}{3}$

Higher order systems.

Look at $Y' = AY$, $Y(0) = Y_0$ where A was 2×2

$$Y = x e^{\lambda t} \implies Ax = \lambda x$$

$y'' + 3y' - 4y = 0$

$y'' = 4y - 3y'$

Let $y = y_1$

$y' = y_2 = y_1'$

$y'' = y_2'$

$Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \implies Y' = \begin{pmatrix} 0 & 1 \\ 4 & -3 \end{pmatrix} Y$

$A = \begin{pmatrix} 0 & 1 \\ 4 & -3 \end{pmatrix}$, $\left| A - \lambda I \right| = \lambda^2 + 3\lambda - 4 = 0 \implies \lambda_1 = 1 , \lambda_2 = -4$

$\lambda_1 = 1$ Solve $(A - I)x_1 = 0$

$$\begin{pmatrix} -1 & 1 \\ 4 & -4 \end{pmatrix} x_1 = 0 \implies x_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$\lambda_2 = -4$ Solve $(A + 4I)x_2 = 0$

$$\begin{pmatrix} 4 & 1 \\ 4 & 1 \end{pmatrix} x_2 = 0 \implies x_2 = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$$

So $Y = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{-4t}$

$y = y_1 = c_1 e^t + c_2 e^{-4t}$, $Y(0) = \begin{pmatrix} 2, 7 \end{pmatrix}$ where $y(0) = 2$, $y'(0) = 7$

$\implies y = 3e^t - e^{-4t}$

In General.

$$Y'' = A_1 Y + A_2 Y'$$

can be translated into a 1st order problem

$$\begin{aligned} y_{n+1}(t) &= y_1' \\ y_{n+2}(t) &= y_2' \\ &\vdots \\ y_{2n}(t) &= y_n' \end{aligned}$$

Let

$$Y_1 = Y = (y_1, \ldots, y_n)^T$$

$$Y_2 = Y_1' = (y_1', \ldots, y_n')^T$$

$$\begin{aligned} \implies Y_1' &= 0 + Y^2 \\ Y_2' &= A_1 Y_1 + A_2 Y_2 \end{aligned}$$

$$\begin{pmatrix} Y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 0 & I \\ A_1 & A_2 \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$$

e.g.

$y_1'' = -2y_2$

$y_2'' = y_1 + 3y_2$

Let

$(y_3 = y_1')$, $y_3' = y_1'' = -2y_2$

$(y_4 = y_2')$, $y_4' = y_1 + 3y_2$

$$\implies \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} \implies Y' = AY \text{ for } A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -2 & 0 & 0 \\ 1 & 3 & 0 & 0 \end{pmatrix}$$

$$\left|A-\lambda I\right| = \lambda^4 - 3\lambda^2 + 2 = 0 \implies \lambda^2 = 1 \text{ or } \lambda^2 = 2$$

$$\lambda_1 = 1\ \lambda_2 = -1\ \lambda_3 = \sqrt{2}\ \ \lambda_4 = -\sqrt{2}$$

$$x_1 = \begin{pmatrix} -2 \\ 1 \\ -2 \\ 1 \end{pmatrix},\ x_2 = \begin{pmatrix} -2 \\ 1 \\ 2 \\ -1 \end{pmatrix},\ x_3 = \begin{pmatrix} 1 \\ -1 \\ \sqrt{2} \\ -\sqrt{2} \end{pmatrix},\ x_4 = \begin{pmatrix} 1 \\ -1 \\ -\sqrt{2} \\ \sqrt{2} \end{pmatrix}$$

$$Y(t) = c_1 x_1 e^t + c_2 x_2 e^{-t} + c_3 x_3 e^{\sqrt{2}t} + c_4 x_4 e^{-\sqrt{2}t}$$

6.3 Diagonalization

Thm:-

If $\lambda_1, \dots, \lambda_n$ are distinct eigenvalues of an $n \times n$ matrix, with corresponding eigenvectors $x_1,\ x_2,\ \dots,\ x_n$, then the eigenvectors are Linearly Independent.

Defiition:-

An $n \times n$ matrix A is diagonolizable $\iff$ there exsists a nonsingular matrix X and a diagonal matrix D such that

$$X^{-1}AX = D$$

We say that X Diagonalizes A

Thm:-

An $n \times n$ matrix A is diagonizable $\iff$ A has n Linear Independent eigenvectors.

1. X has collumns that are eigenvectors of A and $D = diag(\lambda_1 \lambda_2, \dots, \lambda_n)$
2. X is not unique (reorder eigenvalues)
3. If A is diagonal, then $A = XDX^{-1}$

Note :- If $A = XDX^{-1} \implies A^2 = (XDX^{-1})(XDX^{-1}) = XD^2X^{-1}$

In general,

$$A^k = XD^kX^{-1}$$

e.g.

$$A = \begin{pmatrix} 5 & 6 \\ -2 & -2 \end{pmatrix},\ \left|A-\lambda I\right| = (5-\lambda)(-2-\lambda) + 12 = 0$$

$$\implies \lambda^2 - 3\lambda + 2 = 0 \implies (\lambda-2)(\lambda-1) = 0$$

$$\lambda - 1 = 1\ ,\ \lambda_2 = 2$$

$$x_1 = \begin{pmatrix} 3 \\ -2 \end{pmatrix},\ x_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

$$X = \begin{pmatrix} 3 & -2 \\ -2 & 1 \end{pmatrix},\ D = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix},\ X^{-1} = \begin{pmatrix} -1 & -2 \\ -2 & -3 \end{pmatrix}$$

> Calc $XDX^{-1} = \begin{pmatrix} 3 & -2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} -1 & -2 \\ -2 & -3 \end{pmatrix}$
>
> $= \begin{pmatrix} 5 & 6 \\ -2 & -2 \end{pmatrix} = A$

Matrix Exponential

Power Series

$$e^a = 1 + a + \frac{a^2}{2!} + \frac{a^3}{3!} + \dots + \frac{a^n}{n!} + \dots$$

$$e^A = I + A + \frac{1}{2!} A^2 + \frac{1}{3!} A^3 + \dots + \frac{1}{n!} A^n + \dots$$

If A is diagonal

$$D = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \lambda_n \end{pmatrix}$$

$$e^D = \lim_{m\to\infty}(I + D + \frac{1}{2!} D^2 + \frac{1}{3!} D^3 + \dots + \frac{1}{m!} D^m$$

$$= \lim_{m\to\infty}\begin{pmatrix} \sum_{k=0}^{m} \frac{1}{k!} \lambda_1^k & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \sum_{k=0}^{m} \frac{1}{k!} \lambda_n^k \end{pmatrix}$$

$$= \begin{pmatrix} e^{\lambda_1} & 0 & 0 & 0 \\ 0 & e^{\lambda_2} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & e^{\lambda_n} \end{pmatrix}$$

If A is diagonal,

$$A^k = XD^kX^{-1}$$

$$e^A = X(I + D + \frac{1}{2!} D^2 + \dots)X^{-1} = Xe^DX^{-1}$$

Compute e^A for $A = \begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix}$

then Diagonal

> $A \quad \lambda_1 = 1 \quad x_1 = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$
>
> $\lambda_2 = -1 \quad x_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

> $X = \begin{pmatrix} 3 & 1 \\ -1 & -1 \end{pmatrix}$, $e^D = \begin{pmatrix} e^1 & 0 \\ 0 & e^{-}1 \end{pmatrix}$
>
> $e^A = Xe^DX^{-1} = \begin{pmatrix} 3 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} e^1 & 0 \\ 0 & e^{-}1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{3}{2} \end{pmatrix}$
>
> $e^A = \frac{1}{2} \begin{pmatrix} 3e - e^{-1} & 3e - 3e^{-1} \\ -e + e^{-1} & -e + 3e^{-1} \end{pmatrix}$

6.2

$$Y' = AY \, , \; \frac{d}{dt} e^{at} = ae^{at}$$

$$\frac{d}{dt} e^{tA} = Ae^{tA}$$

Let $Y(t) = e^{tA} Y(0)$ for $Y_0 = Y(0)$

$$Y'(t) = Ae^{tA} Y_0 = AY$$

If A is Diagonal,

$$e^{tA} = Xe^{tD}X_1 \implies Y(t) = xe^{tD}X^{-1} Y_0 = Xe^{tD}c \; for \; c = x^{-1} Y_0$$

$$e^{tD} = \begin{pmatrix} e^{t\lambda_1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & e^{t\lambda_n} \end{pmatrix}$$

$$Y(t) = c_1 x_1 e^{\lambda_1 t} + c_2 x_2 e^{\lambda_2 t} + \cdots + c_n x_n e^{\lambda_n t}$$

Class Notes 20140409

6.1

1a) $A = \begin{pmatrix} 3 & 2 \\ 4 & 1 \end{pmatrix} \left| A - \lambda I \right| = \begin{pmatrix} (3-\lambda) & 2 \\ 4 & (1-\lambda) \end{pmatrix}$

Characteristic Polynomial $\lambda^2 - 4\lambda - 5 = 0 = (3-\lambda)(1-\lambda) - 8 = 0$

for $2 \times 2 \quad \lambda^2 - Tr(A)\lambda + \det(A) = 0$

$$\lambda_1 = -1 \begin{pmatrix} 4 & 2 \\ 4 & 2 \end{pmatrix} x_1 = 0 \implies x_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$\lambda_2 = 5 \begin{pmatrix} -2 & 2 \\ 5 & -5 \end{pmatrix} x_2 = 0 \implies x_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

> for
>
> > λ_1 eigenspace $\alpha \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

λ_2 eigenspace $\alpha\begin{pmatrix}1\\1\end{pmatrix}$

1c)

$\begin{pmatrix}3 & -1\\1 & 1\end{pmatrix}$ Characteristic polynomial is $\lambda^2 - 4\lambda + 4 = 0$

$(\lambda - 2) = 0$ $\lambda_1 = \lambda_2 = 2$ Multiplicity of 2

$\begin{pmatrix}1 & -1\\1 & -1\end{pmatrix}x_1 = 0 \implies x_1 = \begin{pmatrix}1\\1\end{pmatrix}$ Called Defective.

1)a

$\begin{pmatrix}1 & 1 & 1\\0 & 2 & 1\\0 & 0 & 1\end{pmatrix}$ since upper triangular $\implies$ Eigenvalues are diagonal elements.

$\lambda_1 = \lambda_2 = 1$ $\lambda_3 = 2$

$$\lambda_1 = \lambda_2 = 1\begin{pmatrix}0 & 1 & 1\\0 & 1 & 1\\0 & 0 & 0\end{pmatrix}x = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$x = \begin{pmatrix}\alpha\\\beta\\-\beta\end{pmatrix} = \alpha\begin{pmatrix}1\\0\\0\end{pmatrix} + \beta\begin{pmatrix}0\\1\\-1\end{pmatrix}$ two linearly independent Eigenvectors

$$\lambda_3 = 2\begin{pmatrix}-1 & 1 & 1\\0 & 0 & 1\\0 &) & -1\end{pmatrix}x_3 = 0,$$

$$x_3 = \begin{pmatrix}a\\b\\c\end{pmatrix} \implies c = 0 \implies -a + b = 0 \implies a = b$$

$$x_3 = \begin{pmatrix}1\\1\\0\end{pmatrix}$$

1i)

$\begin{pmatrix}4 & -5 & 1\\1 & 0 & -1\\0 & 1 & -1\end{pmatrix}$ Characteristic Polynomial $\lambda(\lambda - 1)(\lambda - 2) = 0$

$\lambda_1 = 0,\ \lambda_2 = 1,\ \lambda_3 = 2$

$$\lambda_1 = 0\begin{pmatrix}4 & -5 & 1\\1 & 0 & -1\\0 & 1 & -1\end{pmatrix}x_1 = 0 \implies x_1 = \begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$\lambda_2 = 0\begin{pmatrix}3 & -5 & 1\\1 & -1 & -1\\0 & 1 & -2\end{pmatrix}x_2 = 1 \implies x_2 = \begin{pmatrix}3\\2\\1\end{pmatrix}$$

> $$\lambda_3 = 0 \begin{pmatrix} 2 & -5 & 1 \\ 1 & -2 & -1 \\ 0 & 1 & -3 \end{pmatrix} x_3 = 2 \implies x_3 = \begin{pmatrix} 7 \\ 3 \\ 1 \end{pmatrix}$$

1l)

> $$\lambda_1 = 1, \; x_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$
>
> $$\lambda_2 = \lambda_3 = 2, \; x_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$
>
> $$\lambda_4 = 3, \; x_3 = \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \end{pmatrix}$$
>
> Defective since $\lambda_2 = \lambda_3 = 2$ only has 1 eigenvector.

A is $n \times n$ $\det(A - 0 \cdot I) = 0 \implies \lambda = 0$ is eigenvalue.

If $\lambda = 0$ is an eigenvalue $\implies \det(A - 0 \cdot I) = 0 \implies \det(A) = 0 \implies A$ is singular

6)

> λ is eigenvalue for A with corresponding egienvector x Show λ^m is an eigenvalue od A^m with eigenvector x
>
> $m = 1$ $Ax = \lambda x$
>
> $m = 2$ $A^2 x = \lambda A x = \lambda \lambda x \implies A^2 x = \lambda^2 x \implies \lambda^2$ is an eigenvalue of A^2 with corresponding eigenvector x
>
> Assume true for $m = k$ $A^k x = \lambda^k x$ Show true for $m = k + 1$
>
> $A(A^k x) = A(\lambda^k x)$
>
> $A^{k+1} x = \lambda^k A x = \lambda^k \lambda x = \lambda^{k+1} x$

9)

> An $n \times n$ matrix A is nilpotent if $A^k = 0$ for some positive integer k. Show all eigenvalues are 0.
>
> If $A^k = 0 \implies$ all eigenvectors of A^k are 0. But eigenvalues of A^k are $\lambda^k \implies \lambda^k = 0 \implies \lambda = 0$

6.2

1a)

> $y_1' = y_1 + y_2$, $Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$
>
> $y_2' = -2y_1 + 4y_2$
>
> $Y' \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix} Y = AY$

$$A = \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix},\ \lambda^2 - 5\lambda + 6 = 0 \implies (\lambda - 3)(\lambda - 2) = 0$$

$$\lambda_1 = 2 \begin{pmatrix} -1 & 1 \\ -2 & 2 \end{pmatrix} x_1 = 0 \implies x_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\lambda_2 = 3 \begin{pmatrix} -2 & 1 \\ -2 & 1 \end{pmatrix} x_2 = 0 \implies x_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$Y = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t}$$

$$y_1 = c_1 e^{2t} + c_2 e^{3t}\ ,\ y_2 = c_1 e^{2t} + 2c_2 e^{3t}$$

1c)

$$Y' = AY\ ,\ A = \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix} \lambda^2 - 5\lambda = 0\ ,\ \lambda(\lambda - 5) = 0$$

$$\lambda_1 = 0\ ,\ \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix} x_1 = 0 \implies x_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\lambda_2 = 5\ ,\ \begin{pmatrix} -4 & -2 \\ -2 & -1 \end{pmatrix} x_2 = 0 \implies x_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

then $Y = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{0 \cdot t} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{5t}$

$$y_1 = 2c_1 + c_2 e^{5t}$$

$$y_2 = c_1 + 2c_2 e^{5t}$$

1f)

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 6 \\ 0 & 1 & 3 \end{pmatrix},\ Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$$Y' = AY \left| A - \lambda I \right| = (1 - \lambda)(\lambda^2 - 5\lambda) = (1 - \lambda)\lambda(\lambda - 5) = 0$$

$$\lambda_1 = 0\ ,\ \lambda_2 = 1\ ,\ \lambda_3 = 5$$

$$\lambda_1 = 0\ ,\ x_1 = \begin{pmatrix} -1 \\ -3 \\ 1 \end{pmatrix}$$

$$\lambda_2 = 1\ ,\ x_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\lambda_3 = 5\ ,\ x_3 = \begin{pmatrix} 1 \\ 8 \\ 4 \end{pmatrix}$$

$$Y = c_1 \begin{pmatrix} -1 \\ -3 \\ 1 \end{pmatrix} e^{0t} + c_2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{t} + c_3 \begin{pmatrix} 1 \\ 8 \\ 4 \end{pmatrix} e^{5t}$$

$$y_1 = -c_1 + c_2 e^t + c_3 e^{5t}$$

$$y_2 = -3c_1 + 8c_3 e^{5t}$$

$$y_3 = c_1 + 4c_3 e^{5t}$$

2a)

$$A = \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix}, \; \lambda^2 + 2\lambda - 3 = 0 \implies (\lambda+3)(\lambda-1) = 0$$

$$\lambda_1 = -3 \; , \; x_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\lambda_2 = 1 \; , \; x_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$Y(0) = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$Y = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-3t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t$$

$$Y(0) = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \implies c_1 = 1 \; , \; c_2 = 2$$

from $\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

Given $Y = c_1 x_1 e^{\lambda_1 t} + c_2 x_2 e^{\lambda_2 t} + \cdots + c_n x_n e^{\lambda_n t}$ is a solution to $Y' = AY \; , \; Y(0) = Y_0$

a)

$$t = 0 \implies c_1 x_1 + c_2 x_2 + \cdots + c_n x_n = Y_0 \implies \begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = Y_0$$

$$x = \begin{pmatrix} x_1 & x_2 \ldots x_n \end{pmatrix}, \;\; Xc = Y_0$$

$c = \begin{pmatrix} c_1 \\ \vdots \\ c_2 \end{pmatrix}$, x_i's Linear independent $\implies \det(x) = 0$

$\implies X^{-1}$ exists $\implies X^{-1}(Xc) = X^{-1} Y_0$ or $c = X^{-1} Y_0$

4)

Let $S_1 \equiv$ amount of salt in tank A and $S_2 \equiv$ amount of salt in tank B

$$\frac{ds_1}{dt} = -16\left(\frac{s_1}{100}\right) + 4\left(\frac{s_2}{100}\right)$$

$$\frac{ds_2}{dt} = 16\left(\frac{s_1}{100}\right) - 4 lpmatrix \frac{s_2}{100} - 12\left(\frac{s_3}{100}\right)$$

$$\implies S' = AS \text{ where } S = \begin{pmatrix} s_1 \\ s_2 \end{pmatrix}$$

$$A = \begin{pmatrix} -0.16 & +0.04 \\ 0.16 & -0.16 \end{pmatrix}$$

$$\lambda^2 + 0.32\lambda + 0.0192 = 0 \implies \lambda = -0.08\ ,\ \lambda = -0.24$$

$$\left.\begin{matrix} s_1(0) = 40 \\ s_2(0) = 20 \end{matrix}\right\} \implies S(0) = \begin{pmatrix} 40 \\ 20 \end{pmatrix}$$

$$\lambda_1 = -0.24\ ,\ x_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$\lambda_2 = -0.08\ ,\ x_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$S(t) = c_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-0.24t} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-0.008t}$$

$$S(0) \implies \ ,\ c_1 = 15\ ,\ c_2 = 25$$

$$s_1(t) = 15e^{-0.24t} + 25e^{-0/08t}$$

$$s_2(t) = -30e^{-0.24t} + 50e^{-0.08t}$$

$$\lim_{t\to\infty} s_1(t) = \lim_{t\to\infty} s_2(t) = 0$$

6)

$$y_1'' = -2y_2 + y_1' + 2y_2'$$

$$y_2'' = 2y_1 + 2y - y_2'$$

$$\text{Let } y_3 = y_1'\ ,\ y_4 = y_2'\ ,\ \text{ Let } Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} \implies Y' = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -2 & 1 & 2 \\ 2 & 0 & 2 & -1 \end{pmatrix} Y$$

$$Y = c_1 \begin{pmatrix} 1 \\ 2 \\ 1 \\ 2 \end{pmatrix} e^t + c_2 \begin{pmatrix} -2 \\ -1 \\ 2 \\ 1 \end{pmatrix} e^{-t} + c_3 \begin{pmatrix} 1 \\ 1 \\ 2 \\ 2 \end{pmatrix} e^{2t} + c_4 \begin{pmatrix} 1 \\ -1 \\ -2 \\ 2 \end{pmatrix} e^{-2t}$$

$$Y(0) = \begin{pmatrix} 1 \\ 0 \\ -3 \\ 2 \end{pmatrix} \implies c_1 = 1\ ,\ c_2 = 0\ ,\ c_3 = -1\ ,\ c_4 = 1$$

$$y_1 = e^t - e^2 t + e - 2t$$

$$y_2 = 2e^t - e^{2t} - e^{-2t}$$

6.4 Hermitian Matrices.

Consider complex numbers $a = bi$ Let C^n be n-tuples of complex numbers $v \in C^3$

$$v = \begin{pmatrix} a_1 + b_1 i \\ a_2 + b_2 i \\ a_3 + b_3 i \end{pmatrix}$$

For $1 - D$ if $\alpha = a + bi$ then $|\alpha| = \sqrt{\bar{\alpha}\alpha} = \sqrt{a^2 + b^2}$

$$\alpha = a + b\bar{i} \qquad \bar{\alpha} = a - bi$$

For $z \in C^n$

$$\left\|z\right\| = \left(\left|z_1\right|^2 + \left|z_2\right|^2 + \cdots + \left|z_n\right|^2\right)^{\frac{1}{2}}$$

or

$$\left\|z\right\| = \left(\overline{Z}^T Z\right)^{\frac{1}{2}}$$

Define

$$Z^H = \overline{Z}^T$$

Inner product space on V.

Def:

1. $\langle z, z, \rangle \geq 0 \qquad (= 0 \iff z = 0)$
2. $\langle z, w \rangle = \overline{\langle w, u \rangle} \quad \forall\, z\,,\ w\ \in\ V$
3. $\langle \alpha z + \beta w\ ,\ u \rangle = \alpha \langle z\ ,\ u \rangle + \beta \langle w\ ,\ u \rangle$

For an Orthoganal basis, $\left\{ w_1, w_2, \ldots, w_n \right\}$ and if

$$z = \sum_{i=1}^{n} c_i w_i$$

then

$$c_i = \langle z\ ,\ w_i \rangle$$

$$\overline{c_i} = \langle w_i\ ,\ z \rangle$$

and

$$\left\|z\right\|^2 = \sum_{i=1}^{n} c_i \overline{c_i}$$

We will define $\langle z\ ,\ w \rangle = W^H Z$

> If $Z = \begin{pmatrix} 1 + 2i \\ 4 - 3i \end{pmatrix} \qquad w = \begin{pmatrix} 4 - i \\ 2 + i \end{pmatrix}$
>
> $\langle z\ ,\ w \rangle = W^H Z = \begin{pmatrix} 4 + i & 2 - i \end{pmatrix} \begin{pmatrix} 1 + 2i \\ 4 - 3i \end{pmatrix} = 7 - 4i$

$$Z^H Z = \left|1+2i\right|^2 + \left|4-3i\right|^2 = (1+4)+(16+9) = 30$$

$$\left\|Z\right\| = \sqrt{Z^H Z} = \sqrt{30}$$

Hermitian Matrices

Let $M = (a_{ij} + ib_{ij})$ $\qquad i = \sqrt{-1}$ Assume a_{ij} and b_{ij} are real.

$$M = A + Bi$$

$$\overline{M} = A - Bi$$

$$M^H = \overline{M}^T$$

$$A^H A = \overline{A}^T A = A^T A$$

$$\left(A^H\right)^H = A$$

and

$$\left(\alpha A + \beta B\right)^H = \overline{\alpha} A^H + \overline{\beta} B^H = \overline{\alpha} A^T + \overline{\beta} B^T$$

$$\left(AC\right)^H = \left(\overline{AC}\right)^T = \left(\overline{A} \quad \overline{C}\right)^T = \overline{C}^T \overline{A}^T = C^H A^H$$

def:-

A matrix is called Hermitian if $M^H = M$

e.g.

$$M = \begin{pmatrix} 4 & 3+2i \\ 3-2i & 3 \end{pmatrix}$$

$$M^H = \overline{M}^T = \begin{pmatrix} 4 & 3-2i \\ 3+2i & 3 \end{pmatrix}^T = \begin{pmatrix} 4 & 3+2i \\ 3-2i & 3 \end{pmatrix}$$

The Diagonals are real. Matching offset diagonals are conjugates

If M is real $M^H = M^T$

If M is Hermitian $\implies M^T = M \implies M$ is symetric

Thm:-

All eigenvalues of a Hermitian matrix are real. Also eigenvectors belonging to distinct eigenvalues are orthogonal

e.g.

$$A = \begin{pmatrix} 2 & 1+i \\ 1-i & 3 \end{pmatrix}$$

$$\left|A - \lambda I\right| = \begin{vmatrix} 2-\lambda & 1+i \\ 1-i & 3-\lambda \end{vmatrix} = (2-\lambda)(3-\lambda) - (1+i)(1-i) = 0$$

$\implies \lambda^2 - 5\lambda + 6 - 2 = 0$

$\lambda^2 - 5\lambda + 4 = 0$

$\lambda_1 = 1 \qquad \lambda_2 = 4$

$\lambda_1 = 1 \qquad \begin{pmatrix} 1 & 1+i \\ 1-i & 2 \end{pmatrix} x_1 = 0 \implies x_1 = \begin{pmatrix} -1-i \\ 1 \end{pmatrix}$ or $x_1 = \begin{pmatrix} 1+i \\ -1 \end{pmatrix}$

$\lambda_2 = 4 \qquad \begin{pmatrix} -2 & 1+i \\ 1-i & -1 \end{pmatrix} x_2 = 0 \qquad x_2 = \begin{pmatrix} 1+i \\ 2 \end{pmatrix}$

$\langle x_2\ ,\ \rangle = x_1^H x_2 = \begin{pmatrix} 1-i & -1 \end{pmatrix} \begin{pmatrix} 1_i \\ 2 \end{pmatrix} = 2 - 2 = 0 \implies x_1 \perp x_2$ orthogonal

def:-

$U\ n \times n$ is called unitary if its collumn vectors form an orthogonal set in C^n

if U is unoitary $U^H U = I \qquad U^{-1} = U^H$

Thm:-

If the eigenvalues of a Hermitian matrix A are distinct, then there exists a unitary matrix that diagonalizes A

e.g.

$A = \begin{pmatrix} 1 & 1+i \\ 1-i & 3 \end{pmatrix}$ then let $u_1 = \dfrac{x_1}{\|x_1\|} \qquad u_2 = \dfrac{x_2}{\|x_2\|}$

$$U = \begin{pmatrix} \frac{-1-i}{\sqrt{3}} & \frac{1+i}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \end{pmatrix} = \frac{1}{\sqrt{3}} \begin{pmatrix} -1-i & \frac{1+i}{\sqrt{2}} \\ 1 & \sqrt{2} \end{pmatrix}$$

$$U^H = \frac{1}{\sqrt{3}} \begin{pmatrix} -1+i & 1 \\ \frac{1-i}{\sqrt{2}} & \sqrt{2} \end{pmatrix}$$

Show $U^H A U = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$

Normal Matrices

def:-

A Matrix A is normal if $AA^H = A^H A$

thm:-

A matrix A is normal $\iff$ it contains a complete orthonormal set of Eigenvalues.

$A = \begin{pmatrix} 2 & i & 0 \\ -i & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \qquad A^H = A$

$\implies AA^H = A^2 = A^H A$

e.g.

> $$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad A^H = A$$
>
> The Characteristic polynimial $\implies (1-\lambda)(\lambda^2 - 1) = 0 \implies \lambda_1 = -1 \quad \lambda_2 = \lambda_3 = 1$
>
> $$\lambda = -1 \quad x_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$
>
> $$\lambda_2 = \lambda_3 = 1 \quad x_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad x_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$
>
> $$U_1 = \frac{x_1}{\|x_1\|} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \quad u_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad u_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$
>
> $U = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ 1 & 0 & 1 \end{pmatrix}$ is Orthogonal