Algorithmique et programmation parallèle

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Part I

Theoretical models

Chapter 1

Sorting networks

The *simple comparator* is the basic element of a sorting network. It is modeled as a black box with two inputs and two outputs: $(a, b) \mapsto (\min(a, b), \max(a, b))$.

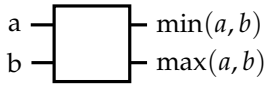


Figure 1.1: A simple comparator

We will build a network with these comparators, in order to sort efficiently a set.

1.1 Odd-Even merge sort

Divide and conquer paradigm.

1.1.1 Odd-Even merging network

Definition 1.

Let $\langle c_1, c_2, \ldots, c_n \rangle$ be some sequence. Then, $Sort(\langle c_1, \ldots, c_n \rangle)$ is the sorted sequence.

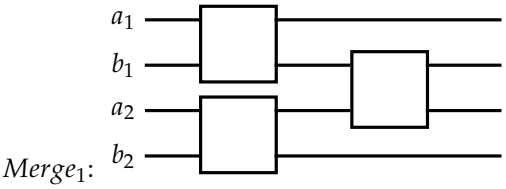
Definition 2.

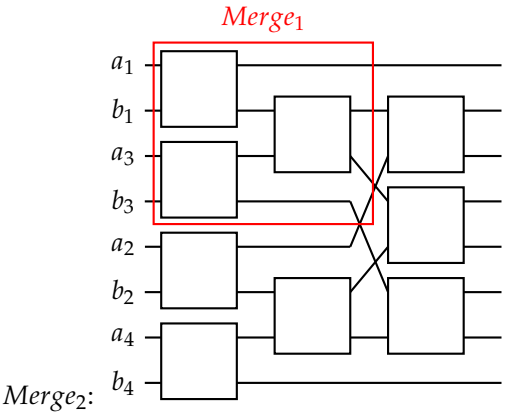
$Sorted(\langle c_1, \ldots, c_n \rangle)$ is true if and only if $c_1 \leqslant \ldots \leqslant c_n$.

Definition 3 (The merge operation).

If $Sorted(\langle a_1, \ldots, a_n \rangle) \wedge Sorted(\langle b_1, \ldots, b_n \rangle)$,
then $Merge(\langle a_1, \ldots, a_n \rangle, \langle b_1, \ldots, b_n \rangle) = Sort(\langle a_1, \ldots, a_n, b_1, \ldots, b_n \rangle)$.

$Merge_m$ is a sorting network for two sorted sequences of size 2^m. $Merge_0$ is a simple comparator.





$Merge_m$ is a network built with two networks $Merge_{m-1}$ and a comparator column.

Proposition 1.

Let the two sorted sequences $A = \langle a_1, \ldots, a_{2n} \rangle$ et $B = \langle b_1, \ldots, b_{2n} \rangle$
$\langle d_1, \ldots, d_{2n} \rangle = Merge \langle a_1, a_3, \ldots, a_{2n-1} \rangle, \langle b_1, b_3, \ldots, b_{2n-1} \rangle$
$\langle e_1, \ldots, e_{2n} \rangle = Merge \langle a_2, a_4, \ldots, a_{2n} \rangle, \langle b_2, b_4, \ldots, b_{2n} \rangle$
We have $Sorted(d_1, \min(d_2, e_1), \max(d_2, e_1), \ldots, \min(d_{2n}, e_{2n-1}), \max(d_{2n}, e_{2n-1}), e_{2n})$

Proof. Without loss of generality, let us suppose that all elements are different. $d_1 = \min(a_1, b_1)$ is therefore the global minimum. $e_{2n} = \max(a_{2n}, b_{2n})$ is the global maximum.

General case: for $2 \leqslant i \leqslant 2_n$, d_i and e_{i-1} are at index $2i-2$ or $2i-1$ (by the proposition), we will show that this assertion is true by showing that:

(i) d_i is greater than $2i-3$ elements

(ii) e_{i-1} is greater than $2i-3$ elements

(iii) d_i is lower than $4n-2i+1$ elements.

(iv) e_{i-1} is lower than $4n-2i+1$ elements.

Proof of (i): Without loss of generality, let us suppose that d_i is an element of A. Let k be the number of A elements in the sequence $\langle d_1, \ldots, d_i \rangle$.

This sequence contains $i-k$ elements from B: $d_i = a_{2k-1}$.

A is sorted, therefore d_i (i.e. a_{2k-1}) is greater than $2k-2$ elements from A. The greatest elements of B in $\langle d_1, \ldots, d_i \rangle$ is $b_{2(i-k)-1}$.

Thus, d_i is greater than $2(i-k)-1$ elements of B.

Therefore, d_i is greater than $(2k-2)+(2(i-k)-1) = 2i-3$ elements.

Proof of (ii): identical.

Proof of (iv): Without loss of generality, let us suppose that e_{i-1} belongs to B. Let k be the number of B elements belonging to $\langle e_1, \ldots, e_{i-1} \rangle$.

$e_{i-1} = b_{2k}$

Then, e_{i-1} is lower than b_{2k+2}, b_{2k+4}... thus by $2n - 2k$ elements from B.

$\langle e_1, e_2, \ldots, e_{i-1} \rangle$ contains $(i-1) - k$ elements of A.

The greatest one is $a_{2(i-1)-k}$.

Thus e_{i-1} is lower than $2n - [2i - 2k] + 1$ elements of A.

Therefore e_{i-1} is lower than $(2n - 2k) + (2n - 2i + 2k + 1) = 4n - 2i + 1$ elements.

Proof of (iii): identical. □

Lemma 2.

Computation time t_m of $Merge_m$ (merging of two sequences of 2^m elements).
Number of comparators: p_m
t_m = maximal number of comparators taken by a single value.
$t_m = t_{m-1} + 1$ with $t_0 = 1$. Thus $t_m = m + 1$. Therefore, as $n = 2^m$ we have $t_n = \mathcal{O}(\log n)$.
$p_m = 2p_{m-1} + (m-1)$ with $P_0 = 1$. Thus $p_m = 2^m m + 1$. Therefore $p_m = \mathcal{O}(n \log n)$.

Definition 4.

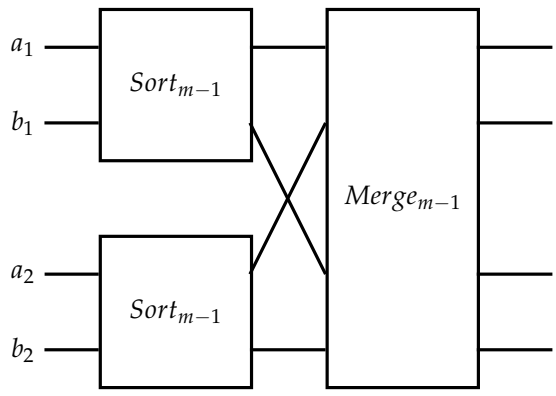
The *work* characterize the efficiency. It is the product of the number of actors by the computation time.

Lemma 3.

Parallel network: $w_n = t_n p_n = \mathcal{O}\left(n \log^2 n\right)$.
Sequential network: $w_n = \mathcal{O}(n)$.

1.1.2 Sorting network

We want to build a sorting network $Sort_m$ for 2^m elements. We build it using two networks $Sort_{m-1}$ and one network $Merge_{m-1}$.



Lemma 4.

Computation time t'_m:
$t'_1 = 1$, $t'_m = t'_{m-1} + t_{m-1}$.
$t'_m = \mathcal{O}\left(m^2\right)$
Number of comparators p'_m:
$p_1 = 1$, $p'_m = 2p'_{m-1} + p_{m-1} + 1$
$p'_m = \mathcal{O}\left(2^m m^2\right)$
$n = 2^m$.
Computation time: $\mathcal{O}\left(\log^2 n\right)$, number of comparators: $\mathcal{O}\left(n \log^2 n\right)$.
Work: $\mathcal{O}\left(n \log^4 n\right)$.
Sequentially:
Time: $\mathcal{O}\left(n \log n\right)$.
Work $\mathcal{O}\left(n \log n\right)$.

1.2 0-1 principle

Proposition 5.

A network is a sorting network for an arbitrary sequence if and only if it is a sorting network for 0-1 sequences (sequences containing only 0's and 1's).

Proof. Direct implication is trivial.

Other implication, by contraposition:

Let us suppose that there exists some sequence $x = \langle x_1, \ldots, x_n \rangle$ which is not sorted by the network R. Then, there exists an index k such that $R(x)_k > R(x)_{k+1}$

Let f be a non-decreasing function.

A comparator behaves in the same way on the input $\langle y_1, y_2 \rangle$ and on the input $\langle f(y_1), f(y_2) \rangle$.

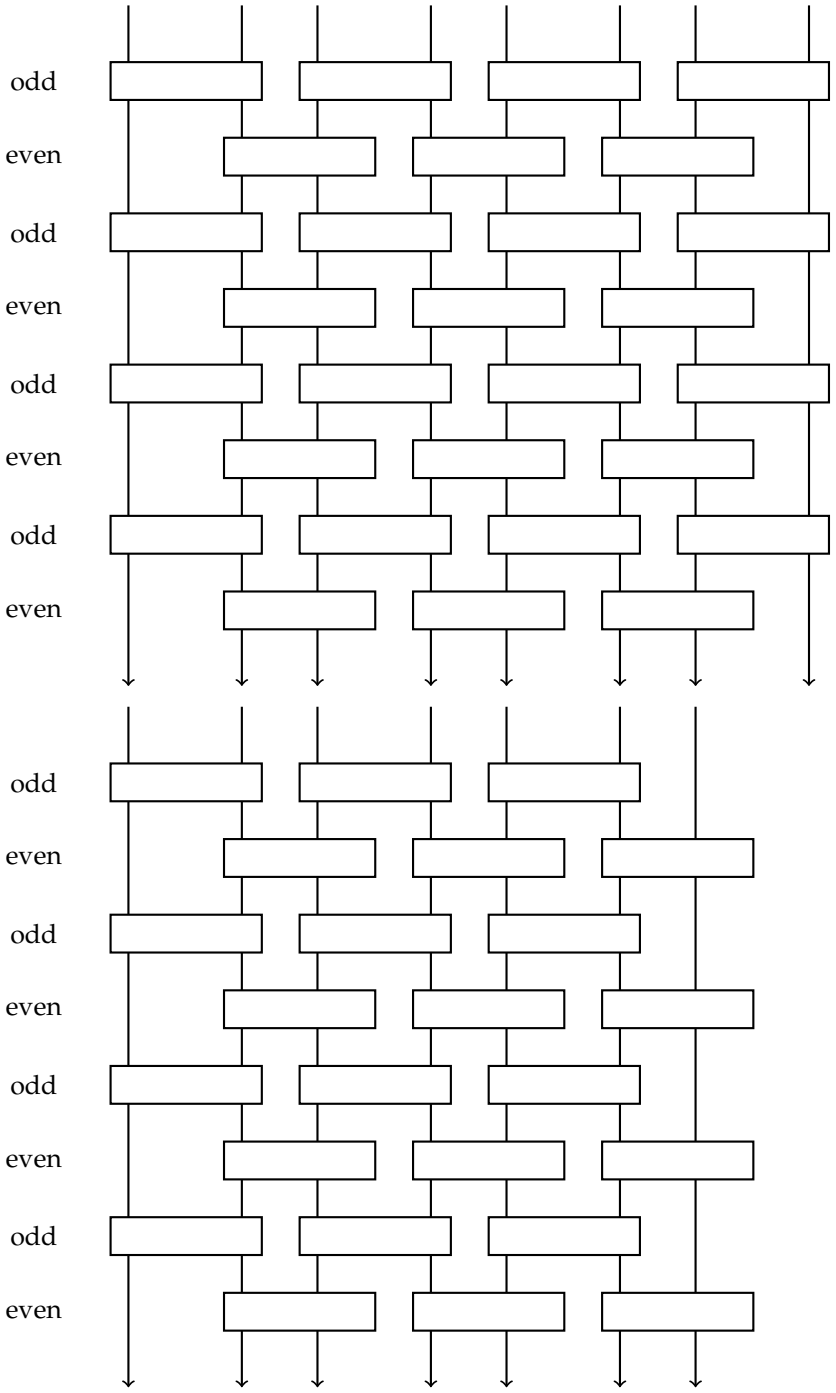
Let us define a non-decreasing function f: $f(y) = \begin{cases} 0 & \text{if } y < R(x)_k \\ 1 & \text{otherwise} \end{cases}$

Since $f(R(x)_k) = 1$ and $f(R(x)_{k+1}) = 0$, then $\langle f(x_1), \ldots, f(x_n) \rangle$ is a 0-1 sequence which is not sorted by R. □

Vocabulary: English vs French Positive ≡ *strictement* positif (> 0)
Non-negative ≡ positif ($\geqslant 0$)
Negative ≡ *strictement* negatif (< 0).
Non-positive ≡ negatif ($\leqslant 0$)
Pour une fonction f et $x < y$
Increasing ≡ *strictement* croissant ($f(x) < f(y)$)
Non-decreasing ≡ croissant ($f(x) \leqslant f(y)$)

1.3 Odd-even *transposition* sort

We still use the comparator.



n inputs, n comparator lines and $\frac{n(n-1)}{2}$ comparators.

Proposition 6.

The transposition network odd-even is a sorting network

Proof. We use the 0-1 principle. Let $\langle a_1, .., a_n \rangle$ be a $0-1$ sequence. Let j_0 be the index of the rightmost 1 in $\langle a_1, .., a_n \rangle$.

We have: $a_{j_0} = 1$ and $a_j = 0$ for $j > j_0$.

If j_0 is even, then this 1 does not move at the first step.
If j_0 is odd, then it moves by one position on the right at the first step.

Thus, after the first step, this 1 is at least at position 2.

Therefore, it moves gradually from left to right, ending at position n, since there is $n - 2 + 1 = n - 1$ steps to reach the final destination.

Let j_1 be the index of the second rightmost 1 of the input. From step 3, this 1 will move by one position on the right at each step (since a_{j_0} moves by one position on the right at each step, from step 2). This 1 will reach the $(n-1)$th position in at most $n-1$ steps.

The mth 1 take $n - m$ steps to reach its position $(n - m + 1)$. □

Fact 7.

The computation time of this network is $t_n = n$, its work is $\mathcal{O}\left(n^3\right)$.

1.4 Odd-even sorting on one dimension input

Idea We will use a line made of generic processors, to mimic the behaviour of the transposition networks.

We have n inputs, p processors, with $n \gg p$. For the sake of simplicity, we will suppose that $p|n$.

Algorithm 1: One line sort

Assign to each processor $\frac{n}{p}$ values
Each processor sorts its values
for p *times* **do**
  Each processor share its values with one of its two neighbours (one time the left one, one time the right one...we alternate).
  Both of the two processors merge the two sorted lists.
  The left processor keep the $\frac{n}{p}$ lowest values, the right processor keep the $\frac{n}{p}$ highest values.

Fact 8.

Time complexity: $\mathcal{O}\left(\frac{n}{p}\log\frac{n}{p} + p \times 2 \times \frac{n}{p}\right) = \mathcal{O}\left(n + \frac{n}{p}\log\frac{n}{p}\right)$
Work: $\mathcal{O}\left(p(n + \frac{n}{p}\log n\right) = \mathcal{O}\left(np + n\log n\right)$
If $p \leqslant \log n$ then work = $\mathcal{O}\left(n\log n\right) =$ is optimal (because a sorting algorithm by comparison requires $\mathcal{O}\left(n\log n\right)$ comparisons).

Chapter 2

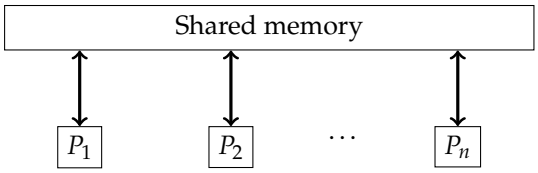
PRAM : Parallel Random Access Machine

Definition 5 (PRAM).

Parallel Random Access Machine

Remark 1.

The *M* of PRAM does not mean *memory* (mistake made in the original course).



PRAM is a central shared memory, that a set of processors (PUs, for processing units) can use.

All PUs run synchronously the same algorithm. They can work on different parts of the memory.

- The memory is infinite.
- The number of PUs is infinite.

Eventual problem several processors can try to use a same memory unit at the same time. There exists three different models for this problem.

CREW concurrent read, exclusive write. No limit for the number of PUs reading simultaneously a same memory unit. At most one PU can write on a single memory unit in one time unit.
Nothing is specified concerning a simultaneous read and write on a same memory unit. The behaviour of PRAM is undefined. We will try to avoid these situations.

At first, each processor will evaluate the condition. Then, all processors for which the condition holds will run the `then` branch. Finally, all processors for which the condition does not hold will run the `else` branch.

Algorithm 2: Synchronous run of the same algorithm

Array A
Processors P_i working on $A[i]$
forall $i \in$ *parallel* **do**
  if *A[i] > 0* **then**
    $A[i] \leftarrow A[i] \times 2$
  else
    $A[i] \leftarrow -A[i]$

EREW Exclusive Read, Exclusive Write. At most one processor per memory unit and time unit can read and write.

CRCW Concurrent Read, Concurrent Write. Several processors can read silmutaneously a same data unit. Several processors can write silmutaneously a same data unit.

Consistent mode All PUs wishing to write in a given data unit **must** write the same value.

Arbitrary mode One value among all available ones is choosen randomly.

Priority mode The choosen value is those of the highest priority PU (e.g. the lowest index PU).

Fusion Mode An associative and commutative operation (eg. max, sum, and, etc.) is applied on the fly to all values. The result is written in memory.

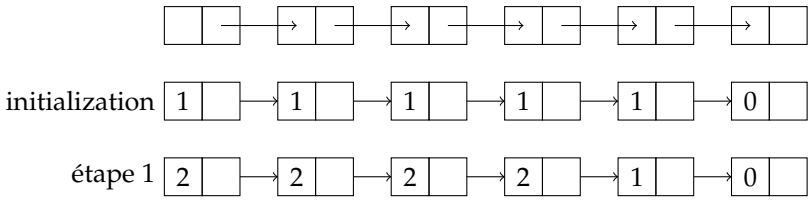
2.1 Pointer jumping

2.1.1 Linked list

We consider a linked list L containing n elements. For each element i, we would like to compute $d[i]$, its distance to the queue of the list.

$$d[i] = \begin{cases} 0 & \text{if } next[i] = \text{NIL} \\ 1 + d[next[i]] & \text{otherwise} \end{cases}$$

The sequential cost is linear (for simply linked lists, a first go through the list to count the number of elements, a second go to compute all distances... even easier for doubly linked lists).



Algorithm 3: Rank computation

forall *i in parallel* **do**
  if *next[i] = Nil* **then**
    d[i]=0
  else
    d[i]=1

while *there exists a node i s.t. next[i]* $\neq$ *Nil* **do**
  forall *i in parallel* **do**
    if *next[i]* $\neq$ *Nil* **then**
      d[i] $\leftarrow$ d[i] + d[next[i]];
      next[i] $\leftarrow$ next[next[i]];

We use n PUs. Each processor work on a single memory unit. Processors are indexed from 1 to n.

Condition evaluation

First idea We keep the number of elements such that $next[i] = \text{NIL}$. It can work for PRAM CRCW model, with fusion mode (addition).

Other idea to transform the **While** loop:
forall *i* **do**
  while *next[i]* $\neq$ *NIL* **do**
    ...

The loop is executed $\lceil \log_2 n \rceil$ times.

Algorithm 4: Loop conversion

for *s=1 to* $\lceil \log_2 n \rceil$ **do**
  for *i in parallel* **do**
    if *next[i]* $\neq$ *Nil* **then**
      ...

The list head is h.

while *next[h]* $\neq$ *Nil* **do**
  `/* Works on a EREW PRAM model */`
  forall *i in parallel* **do**
    head[i] $\leftarrow \top$
  forall *i in parallel* **do**
    if *next[i]*$\neq$ *Nil* **then**
      head[next[i]] $\leftarrow \bot$
  forall *i in parallel* **do**
    if *head[i]* = $\top$ **then**
      h $\leftarrow$ i

Transformation $d[i] \leftarrow d[i] + d[next[i]]$
by
$temp[i] \leftarrow d[next[i]]$
$d[i] \leftarrow d[i] + temp[i]$
In order to run this algorithm on a PRAM EREW model.

Complexity List of size n, n processors. Computation time: $\mathcal{O}(\log_2 n)$

2.1.2 Prefix computation

We consider a sequence:$x_1, x_2, \ldots, x_n$, stored as a simply linked list.
A binary associative operator $\otimes$

We wish to compute the sequence $y_1, \ldots, y_n$:

$$\begin{cases} y_1 & = 1 \\ y_k & = y_{k-1} \otimes x_k \\ & = x_1 \otimes x_2 \otimes \cdots \otimes x_k \end{cases}$$

Algorithm 5: Prefix computation

```
Data: list l
forall i in parallel do
    y[i] ← x[i]
while ∃i such that next[i] ≠ Nil do
    forall i in parallel do
        if next[i] ≠ Nil then
            y[next[i]] ← y[i] ⊗ y[next[i]]
            next[i] ← next[next[i]]
```

This algorithm runs in $\lceil \log_2 n \rceil$ steps, and with the very same transformations as for the rank computation, we can obtain a EREW algorithm.

2.2 Performances evaluation for PRAM algorithms

2.2.1 Cost, work, speedup and efficiency

Let P be some problem of size n.

Let $T_S(n)$ be the computation time of the best (known) sequential algorithm.

Let $T_P(p, n)$ be the PRAM computation time, using p processors.

Cost $C_p(n) = p.T_P(p, n)$

Work Sum of all processors computation time.

Remark 2.

$W_p(n) \leqslant C_p(n)$

Speedup $S_p(n) = \dfrac{T_S(n)}{T_P(p,n)}$

Efficiency $e_p(n) = \dfrac{S_p(n)}{p} = \dfrac{T_S(n)}{p \cdot T_P(p,n)} = \dfrac{T_S(n)}{C_p(n)}$

2.2.2 A simple result of simulation

Proposition 9.

Let A be an algorithm which has an execution time of t on a PRAM using p processors. A can be simulated on a PRAM of the same kind with $p' \leqslant p$ PUs in time $\mathcal{O}\left(\frac{tp}{p'}\right)$. The cost of the algorithm on the smallest PRAM is, at most, two times the cost on the biggest PRAM.

Proof. Each step of A is simulated in $\left\lceil \frac{p}{p'} \right\rceil$ steps with p' PUs.

$$\begin{aligned} C_{p'}(n) &= p' T_P(p', m) \\ &\leqslant p' \left\lceil \frac{p}{p'} \right\rceil T_P(p,n) \\ &\leqslant p' \left(\frac{p}{p'} + 1 \right) T_P(p,n) \\ &\leqslant (p + p') T_P(p,n) \\ &\leqslant 2p T_P(p,n) = 2C_p(n) \end{aligned}$$

□

Prefix calculation n elements, n PUs and a complexity of $\mathcal{O}(\log(n))$
According to the simulation theorem, with p PUs ($p \leqslant n$), we can simulate the computation in time $\mathcal{O}\left(\frac{n}{p} \log n\right)$.

A smarter solution We have $\frac{n}{p}$ elements for each PU.

1st phase prefix calculus on the set of $\frac{p}{n}$ elements of the PU. Done in $\mathcal{O}\left(\frac{n}{p}\right)$

2nd phase pointer-jumping on the prefix calculus on p processors with p values which have the entire prefix of the $\frac{n}{p}$ value of process.
(1th processor $x_1, \ldots, x_{n/p} \to x_1 \otimes \cdots \otimes x_{n/p}$)
Computation in $\mathcal{O}(\log p)$

3rd phase Combination of local prefixes and of the new ones computed with the pointer-jumping in $\mathcal{O}\left(\frac{n}{p}\right)$

Total complexity: $\mathcal{O}\left(\frac{n}{p} + \log p\right)$

2.2.3 BRENT's theorem

Theorem 10.

Let A be an algorithm which computes a total number of m operations and which halts in time t on a PRAM (with a fixed number of PUs).
A can be simulated in time $\mathcal{O}\left(\frac{m}{p}+t\right)$ on a PRAM of the same kind with p PUs.

Proof. $m[i]$: the number of operations computed by A at the step i.

The step i on p processors can be computed in time $\left\lceil \frac{m[i]}{p} \right\rceil$.

The running time of A for p processors:

$$\begin{aligned}\sum_{i=1}^{t}\left\lceil \frac{m[i]}{p}\right\rceil &= \sum_{i=1}^{t}\left(\frac{m[i]}{p}+1\right)\\ &= \sum_{i=1}^{t}\frac{m[i]}{p}+\sum_{i=1}^{t}1\\ &= \frac{m}{p}+t\end{aligned}$$

□

Example 1.

EREW PRAM: maximum of a list of n integers. Assume we have n PUs, the time complexity is $\mathcal{O}(\log n)$ (prefix computation or reduction time).
p processors, complexity ?
Number of operations: $m = n-1$, $t = \log n$.
According to the BRENT's theorem: $\mathcal{O}\left(\frac{n}{p}+\log n\right)$
If $p = \frac{n}{\log n}$, running time: $\mathcal{O}(\log n)$, which is the same complexity, but with less PUs!

Remark 3.

$S_p(n) = \frac{T_S(n)}{T_P(p,n)} \geqslant 1$
$e_{p(n)} \in [0,1]$
$p \geqslant S_p(n) \geqslant 1$

2.3 Comparisons of PRAM models

2.3.1 Model separation

2.3.1.1 CRCW vs. CREW

We're looking for a problem separating CRCW and CREW, ie that can be solved more efficiently when allowing concurrent write. The following problem has this property:

Maximum of n integers

CRCW

Algorithm 6: Maximum of n integers in a constant time using $n(n-1)$ PUs

```
forall i in parallel do
    isMaximum[i] ← ⊤
forall i,j, i ≠ j, in parallel do
    if A[i] < A[j] then
        isMaximum[i] ← ⊥
forall i in parallel do
    if isMaximum[i] then
        maximum ← i
```

This algorithm runs in constant time.

CREW Complexity: $\Omega(\log n)$.

This complexity cannot be improved, since at each timestep, a CREW algorithm can merge at most a constant amount of variables into one.

2.3.1.2 CREW vs. EREW

A separating problem between CREW and EREW is as follow:

Given a set S of values and an object e: does e belong to S? (We assume that each element of S is unique).

```
CREW  Found ← ⊥
forall i in parallel do
    if S[i]=e then
        Found = ⊤
```

There are parallel reads because each processor accesses e simultaneously. Since each element is unique, we ensure that only one processor will write in the variable Found, thus the algorithm is Exclusive Write.

This algorithm runs in constant time.

EREW At most, we can duplicate the numbers of copy of e at each step to allow processors to use it. The number of processors aware of the value of e double each time, so we need $\mathcal{O}(\log n)$ step to ensure that each and every processor has received it.

In the worst case, e is not in S, and in this case we have to wait that each processor knows the value of e to conclude $\rightarrow \Omega(\log n)$.

2.3.2 Simulation theorem

Theorem 11.

Any algorithm on a CRCW with p PUs has an running time of, at most, $\mathcal{O}(\log p)$ times lower than the running time of an algorithm for EREW with p PUs to solve the same problem.

Proof. We assume that the CRCW PRAM use the consistent mode (only the concurrent writing of the same value are allowed).

The method to simulate a concurrent write with exclusive read (symmetrically for reading):

We consider a step of the CRCW algorithm.

We use a temporary vector A of size p.

When the PU P_i write x_i in memory at the position l_i in CREW, P_i write (l_i, x_i) in $A[i]$.

We sort A in $\log p$ (we assume that we can sort p values in time $\log p$ using p PUs).

Each PU: P_i reads the two cells:
$A[i] = (l_j, x_j)$ and $A[i-1] = (l_k, x_k)$
if $l_j = l_k$ **then**
| p_i does nothing ($x_i = x_k$)

else
| p_i writes x_j at the position l_j

□

2.4 COLE's sort machine

1986 Algorithm CREW PRAM to sort n values in time $\mathcal{O}(\log n)$ with $\mathcal{O}(n)$ processors. The cost (and the work) of this algorithm is in $\mathcal{O}(n \log n)$, which is optimal.
(A EREW version with the same constraint was proposed by COLE in 1988.)

A merge sort on a binary tree with n values and of height $\log n$ is done in time $\log n$, the time to handle a tree level is $\mathcal{O}(n)$. We have to be able to merge two sorted lists of arbitrary size ($n/2$ at best) in constant time.

The underlying characteristics:

- computation pipelining
- pre-computation

2.4.1 The merge operation

If J and K are two sorted lists, $J|K$ is the result of the fusion.

The rank of the element x in the sequence J is the number of elements in J that are smaller than x.

$rank(x, J) = card\{y \in J | y < x\}$

The cross-rank of A in B:

$$\begin{aligned} R[A,B] : A &\to \mathbb{N} \\ e &\mapsto rank(e, B) \end{aligned}$$

Definition 6 (Good sampler (GS)).

A sequence L is called a good sampler of a sequence J if, for any $k \geqslant 1$, there are at most $2k+1$ elements of J between $k+1$ (arbitrary) consecutive elements of $\{-\infty\} \cup L \cup \{+\infty\}$

Example 2.

For $k = 1$, between 2 values of the GS, there are at most 3 values of J.

Example 3.

Using a good sampler, two sorted lists can be merged quickly.
Let

$$J = [2, 3, 7, 8, 10, 14, 17, 18, 21]$$
$$K = [1, 4, 6, 9, 11, 12, 13, 16, 19, 20]$$
$$L = [5, 10, 12, 17]$$

L is a GS of J and K. We can check it exhaustively:

$$k = 1 : (-\infty, 5); (5, 10); (10, 12); (12, 17); (17, \infty)$$
$$k = 2 : (-\infty, 10); (4, 12); (10, 17); (12, \infty)$$

The sets $odd(J)$ and $even(J)$ are good samplers of J ($odd(J)$ set of the elements of odd rank in J).

Deux listes triées J et K et un GS L de J et de K.

$\forall i, |J(i)| \leqslant J$ et $|K(j)| < 3$
On peut fusionner $J(i)$ et $K(i)$ en temps constant.

Algorithm 7: MERGEWITHHELP(J,K,L)

J et K sont partitionnés en $|L| + 1$ sous ensembles $J(i) = \{j \in J | l_{i-1} < j \leqslant l_i\}$ et
$K(i) = \{k \in K, l_{i-1} < k \leqslant l_i\}$
forall *i in parallel* **do**
  $res_i \leftarrow$ MERGE(J(i),K(i))
$J|K \leftarrow res_1 res_2 \dots res_{|i|+1}$

Lemma 12.

If L is a GS of the sorted list J et K and if the ranks $R[L, J], R[L, K], R[J, L] et R[K, L]$ are known, then MERGEWITHHELP is running in a constant time with $|J| + |K|$ PUs on a CREW PRAM.

Proof. 1) We need J PUs. Each P_j read $j \in J$, $rank(j, L)$ and insert j in $J(rank(j, l)$ (there are at most 3 PUs which write at the same time in the same array j, writing can be sequential).

2) We merge sequentially at most 2 sorted list containing 3 elements $\Rightarrow \mathcal{O}(1)$ and $|L|$ PUs.

3) We use $|J| + |K|$ PUs:
$rank(l, J|K) = rank(l, J) + rank(l, K)$

The value l which is the m^{th} element of $J(i)|K(i)$ must be stocked at the place $(rank(l_{i-1}|J|K) + m$ □

2.4.2 Sorting trees

We have $n = 2^m$ elements.

Algorithm 8: COLEMERGE

We receive $X(t+1)$ et $Y(t+1)$ respectively from the left and right child
merge de $val(t+1) \leftarrow$ MERGEWITHHELP$(X(t+1), Y(t+1), Val(t))$
reduction we send $Z(t+i) \leftarrow Reduce(Val(t+1)auparent)$
$(Reduce(z_1, z_2, \ldots, z_p) = \{z_4, z_8, z_{12}, \ldots\}$ (every fourth value)).

A node is complete when it has received all inputs.

A node at level k has 2^k inputs (a leaf is at level 0).

As soon as a node begin to receive data, its data quantity is doubled at each step (and for 2^k step).

if *if a node is complete as step t* **then**
- At step $t+1$, every fourth element of $val(t+1)$ is sent to its father.
- At step $t+2$, every other element of $val(t+1)$ is sent to its father.
- At step $t+3$ every element is sent to its father.
- At step $t+4$ and later no elements are sent.

Proof of correctness of COLE algorithm. is based on 3 invariants.

- The size of the inputof a node at level k doubles at each step from 1 to 2^k (from the time it receives its first input).
- If $X(t)$ is a good sampler of $X(t+1)$ and $Y(t)$ is a good sampler of $Y(t+1)$ then $Z(t)$ is a good sampler of $Z(t+1)$ ($Z(t) = Reduce(X(t), Y(t))$).
- We can compute $R[S(t+1), S(t)]$ for any sequence of input or output of a node.

2.4.3 Complexity and correctness

Lemma 13.

The pipelined sorting tee algorithm runs in time $\mathcal{O}(\log n)$ with $\mathcal{O}(n)$ PUs.

Proof. **Execution time**: A level k node is complete at step $t = 3k$ (by induction) that leads to a complexity in $O(\log n)$.

Number of processing units: At level k there are $\frac{n}{2^k}$ nodes. They all produce sorted list of size $\leqslant 2^k$ for which they need at most 2^k PUs.

So, $\frac{n}{2^k}$ nodes executing at most 2^k PUs. There is at most a total $O(n)$ PUs.

At time $t = 3k$ level k nodes handle input of size 2^{k-1} and use each 2^k PUs. Hence a total of n PUs.

Nodes at level $< k$ have stopped everything and use no PUs.

Nodes at level $k+1$: there are $\frac{n}{2^{k+1}}$ nodes, they handle input of size 2^{k-2} and use each 2^{k-1} PUs. So at most $\frac{n}{2^{k+1}} 2^{k-1} = \frac{n}{4}$ PUs.

Nodes at level $k+2$ use $\frac{n}{16}$ PUs.

Hence, the total number of PUs used at time $3k$ is $n + \frac{n}{4} + \frac{n}{16} + \cdots = n\frac{1}{1-\frac{1}{4}} = \frac{4}{3}n$. □

Lemma 14.

Lets$X$, X', Y, Y' 4 sorted sequences.
Si X is a GS of X', Y a GS of Y' then $Reduce(X|Y)$ is a GS of $Reduce(X'|Y')$.
$X|Y$ is not (necessary) a GS of $X'|Y'$.

Example 4.

$X = [2,7]$ $X' = [2,5,6,7]$
$Y = [1,8]$ $Y' = [1,3,4,8]$
$X|Y = [1,2,7,8]$ $X'|Y' = [1,2,3,4,5,6,7,8]$
$[2,7]$ is a set of 2 consecutive elements of $X|Y$.
There are 5 consecutive elements of $X'|Y'$ between two consecutive elements of $X|Y$ when the maximum number authorized is 3.

Proof. We have counter-example for $X|Y$ as GS of $X'|Y'$.

Lemma 15 (Intermediate result).

The are at most $2r+2$ elements of $X'|Y'$ between r consecutive elements of $X|Y$ (we assume $-\infty$ and $+\infty$ belong to $X|Y$).

Proof. Let us consider a sequence of r consecutive elements of $X|Y$: $e_1, \ldots, e_r$.

Let h_X (resp. h_Y) the number of elements of X (resp. Y) in $e1, \ldots, e_r$. We have $h_X + h_Y = r$. Without lost of generality, we assume that e_1 belongs to X.

2 cases:

$\underline{e_r \in X}$: As X is a good sample of X' there are at most $2(h_X - 1) + 1$ consecutive elements of X' between $(h_X - 1) + 1$consecutive elements of X.

As Y is a GS of Y'.

There are at most $2(h_Y + 1) + 1$ consecutive elements of Y' between $(h_Y + 1) + 1$ consecutive elements of Y.

At most there are $2h_X - 1 + 2h_y + 3 = 2r + 2$ elements of $X'|Y'$ between r consecutive elements of $X|Y$.

$\underline{e_r \in Y}$: Let us add $e_0 \in Y$ preceding e_1 and e_{r+1} succeeding e_r.

Elements of X′ and Y′ lying between $e_1, \ldots, e_r$ come from elements of X' (resp. Y') that lie between $e_1, \ldots, e_{r+1}$ (resp. $e_0, \ldots, e_r$) that is between $h_X + 1$ (resp. $h_Y + 1$) elements of X (resp. Y).

Overall, there are at most $2r + 2$ elements of $X|Y$ between r elements of $X'|Y'$. □

$Z = Reduce(X|Y)$ and $Z' = Reduce(X'|Y')$

We consider $k + 1$ consecutive elements of Z: $z_1, \ldots, z_{k+1}$.

By definition of the reduction operation, we have$z_1 = e_{4k}, z_2 = z_{4k+4}, \ldots$ where$X|Y = e_1, \ldots, e_n$.

There are $4k + 1$ elements of $X|Y$ between $z1, \ldots, z_{k+1}$.

Between these $r = 4k+1$ elements of $X|Y$, there are (because of the intermediate result) at most $2r+2 = 8k+4$ elements of $X'|Y'$.

Because the reduction operation takes every fourth element of its input in between $k+1$ consecutive elements of Z, there are, at most, $\frac{8k+4}{4} = 2k+1$ elements of Z'. □

In steady-state at node: receive $X(t+1)$ (resp. $Y(t+1)$) from its left (resp. right) child, computes $val(t+1) = \text{MERGEWITHHELP}(X(t+1), Y(t+1), Val(t))$, sends $Z(t+1) = Reduce(val(t+1))$ to its father.

The invariants are:

- $val(t+1 = X(t+1)|Y(t+1)$
- $X(t)$ is a GS of $X(t+1)$
- $Y(t)$ is a GS of $Y(t+1)$

Cross-ranks

At a given step, X (resp. Y) is a GS of X' (resp. Y').

$U = X|Y$

$Z = Reduce(u)$

We assume that the cross-rank $R[X', X]$ and $R[Y', Y]$ are known (induction hypothesis). To compute $U' = X'|Y'$ with MERGEWITHHELP we need to know the cross-ranks $R[X', U]$, $R[Y', U]$, $R[U, X']$, $R[U, Y']$ (we will then be able to compute $Z' = Reduce(U')$ and $R[Z', Z]$).

We assume that for any sorted sequence S, we know $R[S, S']$

Lemma 16.

if $S = [b_0, \ldots, b_k]$ is a sorted sequence the the rank of any element a in S can be computed in $\mathcal{O}(1)$ with $\mathcal{O}(k)$ PUs on a CREW PRAM.

Proof. $b_0 = -\infty$, $b_{k+1} = +\infty$

for $0 \leqslant i \leqslant k$ *in parallel* **do**
  if $b_i < a \leqslant b_{i+1}$ **then**
    $rank \leftarrow i$

□

Lemma 17.

Let S_1, S_2 and S be 3 sorted sequences with $S = S_1|S_2$ and $S_1 \cap S_2 = \emptyset$. Then, we can compute the cross-ranks $R[S_1, S_2]$, $R[S_2, S_1]$ in $\mathcal{O}(1)$ with $\mathcal{O}(|S|)$ PUs.

Proof. We assume we know $R[S_2, S_1]$, $R[S_1, S_2]$, $R[S, S]$.

$\forall a \in S_1 \subseteq S, rank(a, S_2) = rank(a, S) - rank(a, S_1)$ □

Lemma 18.

Let X, Y, X' and Y' be sorted sequences.
$U = X|Y$ and such that X is a GS of X' and Y is a GS of Y'. We assume we know the $R[X', X]$ and $R[Y', Y]$. Then we can compute in time $\mathcal{O}(1)$ using $\mathcal{O}(|X| + |Y|)$ PUs. The cross-rank $R[X', U]$, $R[Y, u]$, $R[U, X']$ and $R[U, Y']$.

Lemma 19.

If $Z = Reduce(u)$ and $Z' = Reduce(u')$ then one can compute $R[Z', Z]$ in $\mathcal{O}(1)$ using $\mathcal{O}(|X| + |Y|)$ PUs.

Theorem 20.

The COLE's merge sort algorithm can sort a list of size m in time $\mathcal{O}(\log m)$ using $\mathcal{O}(m)$ processors on a CREW PRAM.

Part II

Algorithms for rings and grids of processors

Chapter 3

Algorithms for rings of processors

3.1 Model

We have a model with distributed memory. Each processor have a piece of the memory and this is the only part of the memory that the processor can reach.

Data transfers are explicitly done by communication channels.

p processors in an unidirectional ring: processor P_i can only send messages to processor P_{i+1} and receive messages from processor P_{i-1}.

- No need to precise source or destination of a message.
- P_{i+1} should be understood modulo p.

Two functions:

- Number of processors in the ring: `NbProcs()`.
- Rank of a processor in the ring: `MyNum()`

Primitives of communication:

- `Send(addr, m)`: send a message stored at the address `addr` and which is of size `m`.
- `Receive(addr, m)`: receive a message of size `m` et write it at the address specified by `addr`.

Each `Send` (resp. `Receive`) must be matched by a `Receive` (resp. `Send`).

Communications can be blocking or non blocking. Most of the time, we will assume blocking `Receive` and non blocking `Send`.

Definition 7 (Blocking communication)**.**

The algorithm (or program) does not "return" from the call to a communication primitive (i.e. does not start to execute the next instruction) as long as the communication has not completed.

Definition 8 (Non-blocking communication).

The call of the communication primitive returns instantaneously. The communication itself takes place later on.

Most of the time, we will consider blocking `Receive` and non-blocking `Send` .

Usually, one needs to check that the communication has taken place before using the data "received" or reusing the memory location where the data was stored.

Definition 9 (Communication time).

Time needed to send or receive a message of length m is $L + mb$, where L is the latency (= communication time) and b is the inverse of the bandwidth (= maximum bit rate).

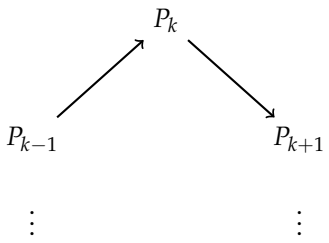
3.2 Macro communication

3.2.1 Broadcast

Consider a processor P_k.

P_k sends the same message of length m to every other processor.

We have a function `Broadcast(k,addr,m)` where `k` is the origin of the broadcast, `addr` is the location of the data, and `m` is the size of the message.



Algorithm 9: Broadcast

```
Data: k, addr, m
q ← myNum()
p ← nbProcs()
if q = k then
  | Send(addr,m)
else if q = k − 1 mod p then
  | Receive(addr,m)
else
  | Receive(addr,m)/* Blocking.                                  */
  |
  | Send(addr,m)/* Blocking or non blocking.                     */
```

Execution time $(p-1)(L+mb)$

3.2.2 Scatter

Processor P_k sends a different message to each other processor. Initially, processor P_k holds the message for processor P_q in `addr`$[q]$.

We assume that `addr`$[k]$ holds a message for P_k (in order to ease the writing).

At the end of the algorithm, each processor should have its own message stored at address `msg`.

Algorithm 10: Scatter

```
Data: k, msg, addr, m
q ← myNum()
p ← nbProcs()
if q = k then
    for i = 1 to p − 1 do
        Send(addr[k-i mod p], m)
else
    for i=1 to p-q+k-1 mod p do
        Receive(tempR, m) /* Blocking                                  */
        Send(tempR, m) /* Blocking                                     */
    Receive(msg,m)
```

Remark 4.

We can not perform non-blocking send in the above algorithm, because of the `tempR` variable (we might lose messages).

Assuming the messages are received in the same order ther are sent.

Execution time $2(p-2)+1$ communications $\Rightarrow$ $(2p-3)(L+mb)$

Algorithm 11: Parallel (blocking) communication version

```
Receive (tempR,m);
for i=1 to p-q+k-1 mod p do
    Send (tempR, m) || Receive (tempS, m) /* Parallel execution.     */
    swap(tempS,tempR)
```

Execution time $(p-1)(L+mB)$

Version where each processor sends a single message P_k sends a message of size $(p-1)m$ in time $L+(p-1)mb$

P_{k+1} sends a message of size $(p-2)m$ in time $L+(p-2)mb$

$\vdots$

P_{k-2} sends a message of size m in time $L+mb$

Hence, a total execution time of $L(p-1)+\frac{p-1}{2}pmb$.

3.2.3 All to all

Each processor in the ring sends the same message to every other processor. In other words, each processor is the source of a broadcast.

Naive solution: use p times the broadcast algorithm, for an execution time of $p(p-1)(L+mb)$.

Algorithm 12: allToAll

Data: myMsg, addr, m
$q \leftarrow$ myNum()
$p \leftarrow$ nbProcs()
addr[q] $\leftarrow$ myMsg
for *i = 1 **to** p-1* **do**
  `Send` (addr[q-i+1 mod p], m) || `Receive` (addr[q-i mod p], m)

Receive instruction has to be blocking.

Execution time $(p-1)(L+mb)$

3.2.4 Pipelined broadcast

- Shorter communications can enable processors to start forwarding data earlier.
- Splitting messages enables some parallelism.
- Splitting messages decreases the size of messages, and thus increases the impact of latencies.

Consider a message of size m. Split it in r same size pieces (assuming that r divides m).

The r pieces are stored in $addr[0], addr[1], \ldots, addr[r-1]$.

At the end, we want all pieces to be stored on each processor.

Algorithm 13: Broadcast

Data: k,addr,m
$q \leftarrow$ myNum()
$p \leftarrow$ nbProcs()
if *q=k* **then**
  for *i=0 **to** r-1* **do**
    Send(addr[i], $\frac{m}{r}$)
else if *q=k-1 mod p* **then**
  for *i=0 **to** r-1* **do**
    Receive(addr[i], $\frac{m}{r}$)
else
  Receive(addr[0], $\frac{m}{r}$)
  for *i=1 to r-1* **do**
    Send(addr[i-1],$\frac{m}{r}$) || Receive(addr[i],$\frac{m}{r}$)
  Send(addr[r-1],$\frac{m}{r}$)

Execution time There are two ways to compute the execution time.

1. • Time at which the first message arrives at P_{k-1}: $(p-1)\left(L+\frac{m}{r}b\right)$

- Time needed for the other messages to arrive at P_{k-1}: $(r-1)\left(L+\frac{m}{r}b\right)$

Hence a total execution time of $(p+r-2)(L+\frac{m}{r}b)$.

2. - Time at which the communication of the last message starts on P_k: $(r-1)\left(L+\frac{m}{r}b\right)$
   - Time needed for that message to arrive at P_{k-1}: $(p-1)\left(L+\frac{m}{r}b\right)$

   Hence a total execution time of $(p+r-2)(L+\frac{m}{r}b)$.

This is minimized for $r_{opt}=\sqrt{\frac{m(p-2)b}{L}}$, given an execution time of $\left(\sqrt{(p-2)L}+\sqrt{mb}\right)^2 \underset{m\to+\infty}{\sim} mb$.

3.3 Matrix-vector multiplication

Consider a $n \times n$ matrix A and a vector x of size n. We want to compute $y \leftarrow y + Ax$.

Algorithm 14: Sequential algorithm

```
for i = 0 to n − 1 do
    y_i ← 0
    for j = 0 to n − 1 do
        y_i ← y_i + A_ij × y_j
```

Each iteration of the loop executes a scalar product. The products are independant. If p divides n, give to each processor $r=\frac{n}{p}$ scalar products to compute. The complexity depends (in part) of the way data is distributed among the processors.

The matrices are large, one can not have a copy of A on each processor. We assume that A is distributed by blocks of rows. P_0 is holding the first rows of A, P_i is holding rows ri to $(r+1)i-1$ of A.

Assume that x is distributed the way A is distributed (processor P_i holds the components x_{ir} to $x_{(r+1)i-1}$ of x). We want output data to have the same distribution than input data, because after having computed $y=Ax$, we want to be able to directly compute $z=By$ (assuming that A and B are distributed the same way).

Obvious solution We give r values of y to be computed by each processor.
P_i compute $y_{(i-1)r},\ldots,y_{ir-1}$
P_i need the lines of index from $(i-1)r$ to $ir-1$ of the array A.
P_i need all values of x.
Each processor work independently: no communication is needed.

We won't use this way.

Usual assumptions All variables are distributed in the same direction. In the previous solution, that wasn't true. Only r components of y are given to each processor but all components of x are needed.

Initially P_i knows the line $(i-1)r$ to $ir-1$ of A and the components $(i-1)r$ to $ir-1$ of x and y.

Execution time $p \times \max\left(L+br, r^2\cdot w\right) \sim \frac{n^2 w}{p}$

Efficiency $\xrightarrow{\infty} 1$

Algorithm 15: matrixVectorProduct

Data: A, x, y, n, s
$q \leftarrow$ `myNum();`
$p \leftarrow$ `nbProcs();`
$r \leftarrow \frac{n}{p}$;
$tmpS \leftarrow$ `x // initialize y ← 0`
for $step = 0$ **to** $p - 1$ **do**
  `Send` $(tmpS, r)$;
  ‖`Receive` $(tmpR, r)$;
  ‖ **for** $i = 0$ **to** $r - 1$ **do**
    $y[i] \leftarrow y[j] + A\left[q - (step \mod p) \times r + j\right] \times tmpS\left[j\right]$;
  $tmpR \leftrightarrow tmpS$ `// swap pointers`

3.4 Matrix-matrix product

We have 3 square matrices A, B, C of size $n \times n$

We want to compute $C \leftarrow C + A \times B$.

Algorithm 16: Sequential algorithm

for *i=0* **to** *n-1* **do**
  for *j=0* **to** *n-1* **do**
    for *k=0* **to** *n-1* **do**
      $C_{i,j} \leftarrow C_{i,j} + A_{ik} B_{kj}$

n is divisible by p: $r = \frac{n}{p}$

The three matrices are distributed in the same way, as blocks of rows.

Processor P_q holds the rows of index $(q-1)r$ to $qr - 1$ of matrices A, B and C.

With the block notation, processor P_q holds the blocks $\hat{A}_{q,l}$, $\hat{B}_{q,l}$, $\hat{C}_{q,l}$ with $l \in [\![0, p-1]\!]$.

Processor P_q will compute completely the blocks $\hat{C}_{q,l}$.

Step 0 P_q holds $\hat{A}_{q,q}$ and all the $\hat{B}_{q,l}$
$\forall l \in [\![0, p-1]\!], \hat{C}_{q,l} \leftarrow \hat{C}_{q,l} + \hat{A}_{q,q} \hat{B}_{q,l}$
Processor P_q send all its blocks of B to processor P_{q+1}

Step 1 P_q holds $\hat{A}_{q,l}$, $\hat{C}_{q,l}$, $\hat{B}_{q-1,l}$
It can compute $\hat{C}_{q,l} \leftarrow \hat{C}_{q,l} + \hat{A}_{q,q-1} \hat{B}_{q,l}$
Sending $\hat{B}_{q-1,l}$ to P_{q+1} (receives $\hat{B}_{q-r,l}$ from P_{q-1})

Execution time $p \times \max\left(L + nrb, pr^3 w\right) = \max\left(pL + n^2 b, \frac{n^3}{p} w\right) \underset{n \to \infty}{\sim} \frac{n^3}{p} w$

Efficiency $\xrightarrow{\infty} 1$

Execution time for n matrix-vector multiplication
$np \max\left(L + br, \frac{n^2}{p^2} w\right) = \max\left(npL + n^2 b, \frac{n^3}{p} w\right)$

Algorithm 17: Matrix-matrix product

Data: A, B, C, n
$q \leftarrow$ `myNum()`
$p \leftarrow$ `myProcs()`
$r \leftarrow \frac{n}{p}$
$tmpS \leftarrow B$
for $step = 0$ **to** $p - 1$ **do**
  `Send` $(tmpS, n \times r)$
  ‖`Receive` $(tmpR, n \times r)$
  ‖**for** $l = 0$ **to** $p - 1$ **do**
    for $i = 0$ **to** $r - 1$ **do**
      for $j = 0$ **to** $r - 1$ **do**
        for $k = 0$ **to** $r - 1$ **do**
          $C[i, lr + j] \leftarrow C[i, lr + j] + A[i, r \times ((q - step) \mod q) + k] \times tmpS[k, lr + j]$
  $tmpR \leftrightarrow tmpS$

Asymptotic behaviour Same volume of communication but different latency.

3.5 Stencil computation

A discrete domain A made of cells. Iteratively, repeatedly, we update the content of these cells.

The update is made using a rule that defines the new value of the cell as a function of the previous values of the neighbourhood.

3.5.1 A sequential stencil

2-dimensional array (a grid) of size $n \times n$. Each cell has 8 neighbours (MOORE's neighbourhood).

NW	N	NE
W	C	E
SW	S	SE

But we don't use this neighbourhood. Only the new value of two neighbours are useful for the computation.

Update rule $C_{new} \leftarrow$ UPDATE $(C_{old}, W_{new}, N_{new})$

If no North neighbour, replace by NIL.

For the first line, the update rule is $C_{new} \leftarrow$ UPDATE (C_{old}, W_{new}, NIL)

The rule of this model use the new values of the neighbours. This is the main difference with a cellular automaton which use only old values of the neighbours to compute the new ones.

3.5.2 Algorithms

We have a ring of p processors. For the moment, we assume $p = n$. Each processor holds one row of A.

Greedy algorithm: Principle: a processor sends the cell it has computed as soon as it has completed them.

Algorithm 18: Greedy stencil

```
Data: A
q ←MyNum()
p ←NumProcs()
if q = 0 then
    for i = 0 to n − 1 do
        if i = 0 then
            A[0] ← UPDATE(A[0], Nil, Nil)
        else
            A[i] ← UPDATE(A[0], A[i − 1], Nil)
        Send (A[i], 1)
else if q = p − 1 then
    for i = 0 to n − 1 do
        Receive (tmpR, 1)
        if i = 0 then
            A[0] ← UPDATE(A[0], Nil, tmpR)
        else
            A[i] ← UPDATE(A[0], A[i − 1], tmpR)
        Send (A[i], 1)
else
    ...
```

But bof... because `Send` and `Receive` force the other parts to be too sequential.

So... new version!

Algorithm 19: Greedy update

```
Data: A
q ←MyNum()
p ←NumProcs()
if q = 0 then
    A[0] ← UPDATE(A[0], Nil, Nil)
    Send (A[0], 1)
else
    Receive (v, 1)
    A[0] ← UPDATE(A[0], Nil, v)
for j = 1 to p − 1 do
    if q = 0 then
        A[j] ← UPDATE(A[j], A[j − 1], Nil)
        Send (A[j], 1)
    else if q = p − 1 then
        Receive (v, 1)
        A[j] ← UPDATE(A[j], A[j − 1], v)
    else
        Receive (v, 1)‖Send (A[j − 1], 1)
        A[j] ← UPDATE(A[j], A[j − 1], v)
if q ≠ 0 ∧ q ≠ p − 1 then
    Send (A[p − 1], 1)
```

Remark 5.

The first and last processors are working together for one step.
$p \ll n : r = \frac{n}{p}$

Overall execution time : Focus on the last processor it starts working after $(p-1)$ steps. It works for p steps.

Length of a step: $L + b + w$.

Overall execution time: $(2p-1)(L+b+w)$.

General case: $p < n$ (we assume that p divides n). Distribution of data: cyclic distribution of rows. Row j is allocated to processor $P_{j \mod p}$.

Algorithm 20: CyclicUpdate

Data: A, n
for $i = 0$ **to** $r-1$ **do**
  if $q = 0 \wedge i = 0$ **then**
    $A[0][0] \leftarrow \text{UPDATE}(A[0][0], Nil, Nil)$
    `Send` $(A[0,0], 1)$
  else
    `Receive` $(v, 1)$
    $A[i][0] \leftarrow \text{UPDATE}(A[i][0], Nil, v)$
for $j = 1$ **to** $n-1$ **do**
  if $q = 0 \wedge i = 0$ **then**
    $A[i][j] \leftarrow \text{UPDATE}(A[i][j], A[i][j-1], Nil)$
    `Send` $(A[i][j], 1)$
  else if $q = p-1 \wedge i = \frac{n}{p} - 1$ **then**
    `Receive` $(v, 1)$ $A[i][j] \leftarrow \text{UPDATE}(A[i][j], A[i][j-1], v)$
  else
    `Receive` $(v, 1) \| $`Send` $(A[i][j-1])$
    $A[i][j] \leftarrow \text{UPDATE}(A[i][j], A[i][j-1], v)$
if $\neg(p = 0 \wedge i = 0) \wedge \neg(q = p-1 \wedge i = \frac{n}{p} - 1)$ **then**
  `Send` $(A[i][n-1])$

Overall execution time The last processor begin to work after $p-1$ steps. There are n^2 elements to update: $\frac{n^2}{p}$ elements assigned to each processor. $\rightarrow P_{p-1}$ not after $p - 1 + \frac{n^2}{p}$ step.

Overall execution time: $\left(p - 1 + \frac{n^2}{p}\right)\ (\underbrace{w}_{\text{update time}} + L + b)$

Efficiency $\frac{n^2 w}{p\left(p-1+\frac{n^2}{p}\right)(w+L+b)} \sim \frac{w}{w+L+b} < 1$ (bad efficiency)

$P_0 = \frac{n}{p}$ at the first line, $P_1 : \frac{n}{p}$ follows $\ldots \rightarrow$ decrease the volume and the number of communications.

But The last processor have to wait a lot before be able to work.

1 element per communication: too small $\ldots \rightarrow$ send k results per communication.

Other idea

- We send a message every k update ($k|n$).
- We assign r consecutive lines to a processor (a cyclic distribution for line bloc) $\rightarrow$ that decrease at the same time the volume and the number of communications.

Disadvantages

- $K > 1$: we add latency at the start of each processor(except for P_0).
- $r > 1$: the second processor have to wait that kr data are completed before being able to start.
- The time when P_0 can start to work on the second bloc of r columns:
  - If it had finished the computation of the r first columns: $T_{r,k}$:= the time needed to compute a bloc of $r \times k$ data and send the result $\geqslant \frac{n}{k} T_{r,k}$.
  - P_0 have to have received its first messages from $P_{p-1} \rightarrow$ at time $pT_{r,k}$.
  - No waiting time if $\frac{n}{k} \geqslant p$

We assume that $\frac{n}{k} \geqslant p$ and $pr|n$.

Execution time $\left((p-1) + \frac{n}{k} \times \frac{n}{rp})\right) T_{r,k}$
$(p-1)$: starting time of P_{p-1}
$\frac{n}{k}$: number of blocs in a column
$\frac{n}{rp}$: number of blocs in a column per processor.

Overall execution time $\left(p - 1\frac{n^2}{prl}\right)(rkw + L + kb)$

Asymptotic efficiency $\frac{n^2 w}{p(p-1+\frac{n^2}{prk})(rkw+L+bk)}$
The cyclic distribution algorithm has a asymptotic efficiency of $\frac{w}{w+b+L}$
The execution of functions is lowest for $k = k'(r) = n\sqrt{\frac{L}{p(p-1)+(rw+b)}}$
Then the optimal value (no waiting time) is $K^{opt}(r) = \min\left(\frac{n}{p}, K'(r)\right)$

Chapter 4

Matrix multiplication for a grid of processor

4.1 Topology

We have $p = q \times q$ processors organized in a grid: each processor has 4 neighbours (torus structure).

We assume that each link is bidirectional and full duplex:

- 1 port model: a processor can simultaneously send and receive messages, but at most one message is sent and receive at the same time.
- 4 ports model: the same except that 4 messages can be shared at the same time: one for each neighbour.

Number of connexions: p

4.2 Communication Primitives

The processors are indexed by $P_{i,j}$, with $0 \leqslant i, j \leqslant q-1$. The index are given by `myProcRow` and `myProcCol`.

`Send` (dest, addr, L), `Receive` (source, addr, L)

`BroadcastRow`(i, j, srcAddr, destAddr, L)

Every processor $P_{i,k}$ for $k \in [\![0, q-1]\!]$ is involved in this broadcast.

$P_{k,l}$ with $k \neq i$: nothing happens, return immediately.

`BroadcastCol`(i, j, srcAddr, destAddr, L)

Same idea.

4.3 Outer product algorithm

$(A, B, C) \in (\mathcal{M}_n)^3$

The algorithm 14 describe the sequential algorithm of matrix multiplication.

It's a inner multiplication algorithm.

The outer product algorithm does loops in this order:

for $k = 0$ **to** $n - 1$ **do**
  for $i = 0$ **to** $n - 1$ **do**
    for $j = 0$ **to** $n - 1$ **do**
      $c_{i,j} = c_{i,j} + a_{i,k} \cdot b_{k,j}$

Using the block notation:

for $k = 0$ **to** $q - 1$ **do**
  for $i = 0$ **to** $q - 1$ **do**
    for $j = 0$ **to** $q - 1$ **do**
      $\hat{C}_{i,j} = \hat{C}_{i,j} + \hat{A}_{i,k} \cdot \hat{B}_{k,j}$

For a given k: $P_{i,j}$ needs $A_{i,k}$ form $P_{i,k}$ and $B_{k,j}$ from $P_{k,j}$
$\rightarrow P_{k,j}$ broadcasts $B_{k,j}$ on the column j, and $P_{i,k}$ broadcasts $A_{i,k}$ on the row i.

Algorithm 21: outerMatrixProduct

Data: A,B,C,n
$q \leftarrow \sqrt{nbProcs}$
myRow, myCol $\leftarrow \ldots$
for $k = 0$ **to** $q - 1$ **do**
  for $i = 0$ **to** $q - 1$ **do**
    `BcastRow`(i, k, A, bufferA, $m \times m$)
  for $j = 0$ **to** $q - 1$ **do**
    `BcastCol`(k, j, B, bufferB, $m \times m$)
  if $myRow == k \wedge myCol == k$ **then**
    `MatrixMultiplyAdd`(C, A, B, m)
  else if $myRow == k$ **then**
    `MatricMultiplyAdd`(C ,bufferA, B, m)
  else if $myCol == k$ **then**
    `MatrixMultiplyAdd`(C, A, bufferB, m)
  else
    `MatrixMultiplyAdd`(C, bufferA, bufferB, m)

Time for a pipelined broadcast $T_{Bcast} = \left(\sqrt{(q-2)L} + \sqrt{m^2 b}\right)^2$

Overall execution time For a 1 port-model: $T_{n,p} = q \times (2T_{Bcast} + m^3 w)$
For a 4-part model: $T_{n,p} = q \times (T_{Bcast} + m^3 w)$

With less time? We could parallelize broadcasts and `MultAdd`:

Algorithm 22: Broadcast for $k = 0$

for $k = 0$ **to** $q - 2$ **do**
  Broadcast for $k + 1$
  Computation for k
Computation for $q - 1$

Execution time $2T_{Bcast} + (q-1)\max(2T_{Bcast}, m^3w) + m^3w$

$T_{Bcast} \sim qm^2b = \frac{n^2b}{q} = \frac{n^2b}{\sqrt{p}}$

$T(n,p) \sum qm^3w = \frac{n^3w}{p}$

4.4 The Common algorithms

Advantage all communication are done with ones neighbours.

Initially, each line (resp. column) block of A (resp. B) is shifted such that each processor in the first column (resp. row) holds a diagonal element of A (resp. B) (pre-skewing).

$$\begin{pmatrix} \hat{A}_{0,0} & \hat{A}_{0,1} & \hat{A}_{0,2} & \hat{A}_{0,3} \\ \hat{A}_{1,1} & \hat{A}_{1,2} & \hat{A}_{1,3} & \hat{A}_{1,0} \\ \hat{A}_{2,2} & \hat{A}_{2,3} & \hat{A}_{2,0} & \hat{A}_{2,1} \\ \hat{A}_{3,3} & \hat{A}_{3,0} & \hat{A}_{3,1} & \hat{A}_{3,2} \end{pmatrix}$$

$$\begin{pmatrix} \hat{B}_{0,0} & \hat{B}_{1,1} & \hat{B}_{2,2} & \hat{B}_{3,3} \\ \hat{B}_{1,0} & \hat{B}_{2,1} & \hat{B}_{3,2} & \hat{B}_{0,3} \\ \hat{B}_{2,0} & \hat{B}_{3,1} & \hat{B}_{0,2} & \hat{B}_{1,3} \\ \hat{B}_{3,0} & \hat{B}_{0,1} & \hat{B}_{1,2} & \hat{B}_{3,3} \end{pmatrix}$$

After the first step:

$$\begin{pmatrix} \hat{A}_{0,1} & \hat{A}_{0,2} & \hat{A}_{0,3} & \hat{A}_{0,0} \\ \hat{A}_{1,2} & \hat{A}_{1,3} & \hat{A}_{1,0} & \hat{A}_{1,1} \\ \hat{A}_{2,3} & \hat{A}_{2,0} & \hat{A}_{2,1} & \hat{A}_{2,2} \\ \hat{A}_{3,0} & \hat{A}_{3,1} & \hat{A}_{3,2} & \hat{A}_{3,3} \end{pmatrix}$$

$$\begin{pmatrix} \hat{B}_{1,0} & \hat{B}_{2,1} & \hat{B}_{3,2} & \hat{B}_{0,3} \\ \hat{B}_{2,0} & \hat{B}_{3,1} & \hat{B}_{0,2} & \hat{B}_{1,3} \\ \hat{B}_{3,0} & \hat{B}_{0,1} & \hat{B}_{1,2} & \hat{B}_{3,3} \\ \hat{B}_{0,0} & \hat{B}_{1,1} & \hat{B}_{2,2} & \hat{B}_{3,3} \end{pmatrix}$$

Algorithm 23: Common algorithm

horizontal pre-skewing of A
vertical pre-skewing of B
for $k = 0$ **to** $q-1$ **do**
  `MatrixMultiplyAdd`(C, A, B, m)
  Horizontal shift of A
  Vertical shift of B
horizontal post-skewing de A
vertical post-skewing de B

$$T_{skew}^{4\ ports} = \overbrace{2}^{\text{pre + post}} \times \left\lfloor \frac{q}{2} \right\rfloor (L + m^2b)$$

$$T_{skew}^{1p} = 2 \times T_{skew}^{4p}$$

$$T_{comp}^{4p} = q \times \max(m^3w, L + m^2b)$$

$$T_{comp}^{1p} = q \times \max(m^3w, 2L + 2m^2b)$$

Algorithm 24: `Fox` algorithm

for $k = 0$ **to** $q - 1$ **do**
  horizontal broadcast of the k^{th} diagonal of A
  `MatrixMultiplyAdd`(C, A, B, m)
  vertical shift of B

$T_{BCast} = (\sqrt{(q-2)L} + \sqrt{m^2 b})^2$

$T^{4p} = T_{Bcast} + (q-1)\max(m^3 w, T_{BCast}) + \max(m^3 w, L + m^2 b)$

$T^{1p} = T_{Bcast} + (q-1)\max(m^3 w, T_{BCast} + 4m^2 b) + \max(m^3 w, L + m^2 b)$

Part III

Tasks scheduling graphs

Chapter 5

Introduction to tasks graphs

$A \cdot x = b$ where

- $A \in GL_n(\mathbb{R})$ known lower triangular matrix
- $b \in \mathbb{R}^n$ known

Algorithm 25: Classical algorithm

for $i = 0$ **to** $n - 1$ **do**
  Task $T_{i,i} : x_i \leftarrow \frac{b_i}{a_{i,i}}$
  for $j = i + 1$ **to** $n - 1$ **do**
    Task $T_{i,j} : B_j \leftarrow b_j - a_{i,j} x_i$

The program is sequential, the entire computation is done by the program and a total order on tasks.

$<_{seq}$: the total order in the original program
$T <_{seq} T'$: T is computed before T' in the original program.

$T_{1,1} <_{seq} T_{1,2} <_{seq} T_{1,3} <_{seq} \cdots <_{seq} T_{1,n-1} <_{seq} T_{2,1} <_{seq} \cdots <_{seq} T_{n-1,n-1}$

$T_{i,j}$: $T_{1,1}$ is the first task and computes x_1, $T_{1,2}$ and $T_{2,3}$ reads x_1, they need to be computed after x_1.

$T_{1,2}$ updates b_2 and $T_{1;,3}$ updates b_3. $T_{1,2}$ and $T_{1,3}$ are independent and can be executed in any order.

Task T:
$in(T)$: set of variables read by task T.
$out(T)$: set of variables written by task T.

$T \perp T'$ if T and T' are not independent, iff they share some variable that is written by at least one of them.

$$T \perp T' \iff (Out(T) \cap Out(T') \neq \emptyset) \bigvee (Out(T) \cap In(T') \neq \emptyset) \bigvee (Out(T') \cap In(T) \neq \emptyset)$$

(BERNSTEIN conditions)

Definition 10 (Precedence relation $\prec$)**.**

If $T \perp T'$ and if $T \leqslant_{seq} T'$ then we have $T \prec T'$

$$\prec := (<_{seq} \cap \perp)^+$$

$\prec$: is a partial order (consistent with $<_{seq}$):
$<= (<_{seq} \cap \perp)^+$

$T_{2,4}$ and $T_{4,4}$:
$T_{2,4}$ updates b_4 and $T_{4,4}$ read $b_4 \;\Rightarrow\; T_{2,4} \perp T_{4,4}; T_{2,3} < T_{4,4}$

$T_{4,4}$ and $T_{4,5}$:
$T_{4,4}$ write x_4 and $T_{4,(}$ write $x_4 \;\Rightarrow\; T_{4,4} \perp T_{4,5}; T_{4,4} < T_{4,5}$

$T_{2,4}$ and $T_{4,5}$: $T_{2,4}$ and $T_{4,5}$ are independent $\Rightarrow$ we need to use the transitive closure of the dependency relation.

Representation in a directed graph $G = (V, E)$
V = set of tasks.
$e = (T, T') \in E \iff T < T'$

We do not include in this graph the transitive relationships: if $(T, T') \in E$ and $(T', T'') \in E$ there is no need to include (T, T'')

Chapter 6

Scheduling task graph

Definition 11.

A task system on a task graph is a directed graph which vertices have weight.
$G = (V, E, w)$

- V: the list of tasks
- The set of edges represent the constraints (or dependencies) of the procedure: $e = (u, v) \in E \iff u \prec v$
- $w : V \to \mathbb{N}^*$ gives the execution time.

Definition 12.

A *Scheduling* of a task system $G = (V, E, w)$ is a function $\sigma : V \to \mathbb{N}$ such that $\sigma(u) + w(v) \leqslant \sigma(x)$, moreover $u \prec v \iff (u, v) \in E$

If the number of available processors, p, is bounded ($p < \infty$), then $alloc(T)$ specifies on each processor the task T which is executed.

At a given time, only one task can be compute by each processor.

$alloc(T)$: the processor run the task T

$\forall T, T'$ such that $alloc(T) = alloc(T')$, then $\sigma(T) + w(T) \leqslant \sigma(T') \bigvee \sigma(T') + w(T') \leqslant \sigma(T)$

Theorem 21.

Let $G = (V, E, w)$ A task system. There exists a scheduling of G if and only if G does not contain cycles.

Proof. We assume that G contains a cycle $v_1 \to v_2 \to \cdots \to v_1$, then $v_1 \prec v_1$ and $\sigma(v_1) + w(v_1) \leqslant \sigma(v_1)$. Impossible because $w \in \mathbb{N}^*$

We assume that G does not contain cycles. We compute a topological sort of vertices (and build a scheduling). □

Task system $G = (V, E, w)$, σ a scheduling for G using p processors.

$MS(\sigma, p)$: lifetime of σ

$MS(\sigma, p) = \max_{v \in V} \{\sigma(v) + w(v)\} = \min_{u \in V} \sigma(v)$

Definition 13.

We name $Pb(p)$ the problem of finding a scheduling of minimal lifetime using at most p processors (we write $Pb(\infty)$ if the number of processor is not limited).

Definition 14.

$MS_{opt}(p)$ lifetime of a scheduling using at most p processors.
$MS_{opt}(p) = \min_{\sigma} MS(\sigma, p)$

Proposition 22.

$G(V, E, w)$ and a scheduling σ using p processors.
$MS(\sigma, p) = w(\phi)$ where ϕ is a path G
$\phi : v_1 \to v_2 \to \cdots \to v_k$
$w(\phi) = \sum_{i=1}^{k} w(v_i)$
$\forall i \in [1, k-1], \sigma(v_i) + w(v_i) \leqslant \sigma(v_{i+1})$
Per sum, $\sigma(v_1) + w(\phi) \leqslant \sigma(v_k)$, $w(\phi) \leqslant \sigma(v_k) - sigma(v_1) \leqslant MS(\sigma, p)$

The speedup get by scheduling σ using p processors.

$S(\sigma, p) = \frac{Seq}{MS(\sigma,p)}$, $Seq = \sum_{v \in V} w(v)$

Efficiency: $e(\sigma, p) : \frac{Seq}{pMS(\sigma,p)}$

Theorem 23.

$0 \leqslant e(\sigma, p) \leqslant 1$

Theorem 24.

$Seq = MS_{opt}(1) \Rightarrow MS_{opt}(2) \geqslant \ldots \geqslant MS_{opt}(|V|) = \cdots = MS_{opt}(\infty)$

Proof. Every scheduling with p processors can be scheduled on $p+1$ processors, the last one remains unemployed.

$MS'_{opt}(p)$: minimal lifetime get by scheduling using p processors (at worst, there is one task by processor).

If $p \leqslant |V|$, $MS'_{opt}(p) = MS_{opt}(p)$ (because there is no communication time). □

Chapter 7

Solve $Pb(\infty)$

$G = (V, E, w)$

$\forall v \in V$, $Pred(v)$ is the set of immediate predecessors of v. $Succ(v)$ is the set of immediate successors of v.

On input, a top vertex (or empty vertex) is a vertex v such that $Pred(v) = \emptyset$.

On output, a bottom vertex is a vertex v such that $Succ(v) = \emptyset$.

Definition 15.

$\forall \sigma \in V$, the top level of v, $tl(v)$, is the path of maximum weight from any entry vertex to vertex v without taking the weight of v into account.

Definition 16.

$\forall \sigma \in V$, the bottom level of v, $bl(v)$, is the path of maximum weight from v to an exit vertex with taking the weight of v into account.

$$tl(v) = \max_{u \in Pred(v)} (tl(u) + w(u)) \tag{7.1}$$

if v is not an entry point, else $tl(v) = 0$.

$$bl(v) = \max_{u \in Succ(v)} (bl(u) + w(v)) \tag{7.2}$$

if v is not an exit point, else $bl(v) = w(v)$.

Theorem 25.

Let $G = (V, E, w)$ be a task system. Let σ_{free} be defined as $\forall v \in V, \sigma_{free}(v) = tl(v)$
Then σ_{free} is a *optimal scheduling*.
σ_{free}:= as soon as possible (ASAP)

Proof. The scheduling is proved by the equation 7.1.

Optimality: each task is scheduled at the lower bound on its beginning time (induction on the vertices beginning with k nodes). □

$$MS_{opt}(\infty) = MS(\sigma_{free}, \infty) = \max_{v \in V} \{tl(v) + w(v)\}$$

Corollary 26.

For $G = (V, E, w)$, $Pb(\infty)$ can be solved in time $\mathcal{O}(|V| + |E|)$.

Proof. The scheduling as late as possible (ALAP, or procrastinator): $\sigma_{late}(v) = MS_{opt}(\infty) - bl(v)$

σ_{late} is another optimal scheduling. □

Chapter 8

Solve $Pb(p)$

8.1 NP-completeness of $Pb(p)$

Definition 17.

Knowing the tasks given by $G = (V, E, w)$, the number of processors $p \geqslant 1$ and a bound $k \in \mathbb{N}^*$, the problem $Dec(p)$ is: does it exist a scheduling σ of G using at most p processors such that $MS(\sigma, p) \leqslant k$.

When the tasks are independent $E = \emptyset$, the problem is named $Indep(p)$.

When p is fixed a priori, for instance $p = 2$, we note $Dec(2)$ or $Indep(2)$

Theorem 27.

$Indep(2)$ is NP-complete but can be solved by a pseudo polynomial algorithm.
$Indep(p)$ is strongly NP-complete.
$Dec(2)$ is strongly NP-complete.

Proof. $Indep(2)$ is exactly 2-partition with $k = \dfrac{\sum_{v \in V} w(v)}{2}$

$Indep(p)$: reduction from 3-partition.

Given $3n$ integers, $a_1, \ldots, a_{3n}$, and a bound B, assuming $\frac{B}{4} < a_i \leqslant \frac{B}{2}$ and such that $\sum_{i=1}^{3n} a_i \leqslant nB$.

Is there a partition of a_i in sets $I_1, \ldots, I_n$ such that $\sum_{a_j \leqslant I_i} w(a_j) = 3$ (each containers contains exactly 3 of the a_i).

$\forall (i, j), I_i \cap I_j = \emptyset$

The reduction: $p = n$, $k = 3$. $3n$ tasks and the tasks T_i have a weight a_i

$Dec(2)$: reduction from 3-partition.

We have 2 processors. We have:

- $3n$ independent tasks $T_1, \ldots, T_{3n}$ with $w(T_i) = a_i$
- $3n$ other tasks, all of weight B

$K = nB$

3-partition has a solution $I_1, \ldots, I_n$. σ is defined as follows: Processor P_1:

- executes task X_i at time $(i-1)2B$
- executes task Y_i at time $B + (i-1)2B = (2i-1)B$

Processor P_2:

- executes the tasks of I_i at time $(i-1)2B$
- executes the task Z_i at time $(2i-1)B$

We assume that ther exists a schedule $X_1 \to Y_1 \to X_2 \to Y_2 \to \cdots \to X_n \to Y_n$. Weight of this path is $2nB = K$. No freedom! Task X_i must be executed at time $2(i-1)B$. Task Y_i must be executed at time $(2i-1)B$. $X_1 \to Z_1 \to X_2 \to Z_2 \to \cdots \to X_n \to Z_n$. task Z_i must be executed at time $(2i-1)B$.

Without loss of generality, we assume that P_1 executes all the X_is and Y_is and processor P_2 executes all the Z_is and T_is.

Let I_i be the next of tasks T_j that are executed during the time interval $[2(i-1)B, (2i-1)B]$

$$\sum_{T_j \in I} w(T_j) \leqslant B$$

Because σ is a schedule, $\bigcup_{i=1}^{n} I_i = \{T_1, \ldots, T_{3n}\} \Rightarrow$ solution to 3-partition.

□

8.2 List scheduling heuristics

Principle : do not deliberately let a processor idle (if there is some tasks that can be scheduled).

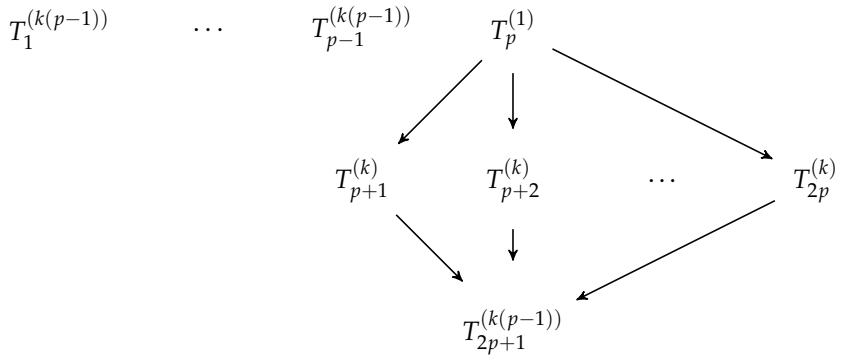
Vocabulary : a task is free at a time t if all its predecessors have completed at time t.

Theorem 28.

Let $G = (V, E, w)$ be a DAG and assume that there are p processors available. Let $MS_{opt}(p)$ be the optimal makespan when using at most p processors. Then, if σ is a list schedule, then $MS(\sigma, p) \leqslant (2 - \frac{1}{p}) MS_{opt}(p)$

Lemma 29.

There exists a path Φ in G, such that $Idle \leqslant (p-1)w(\Phi)$



Let T_{i_1} be one task where completion time is equal to the makespan. Let t_1 be the largest time, with $t_1 < \sigma(T_{i_1})$, such that at least one processor is idle at time t_1. Task T_{i_1} was not started at time t_1 besause one of its predecessors (may be d transitive predecessors) is executed at time t_1. Let T_{i_2} be such a predecessor. Let t_2 be the last time prior to $\sigma(T_{i_2})$ at which the procssor was idle. By induction we build a dependence path $\Phi : T_{ik} \to T_{i_{k-1}} \to \cdots \to T_{i_2} \to T_{i_1}$

Processors can only be idle while one task of Φ is executed : when one task of Φ is executed, at most $(p-1)$ processors are idle. $Idle \leqslant (p-1)w(\Phi)$

$$\begin{aligned} p \cdot MS(\sigma, p) &= Idle + Seq \\ &\leqslant (p-1)w(\Phi) + Seq \\ w(\Phi) &\leqslant MS_{opt}(p) \\ \frac{Seq}{p} &\leqslant MS_{opt}(p) \\ p \cdot MS(\sigma, p) &\leqslant (p-1)MS_{opt}(p) + pMS_{opt}(p) \\ &= (2p-1)MS_{opt}(p) \end{aligned}$$

Proposition 30.

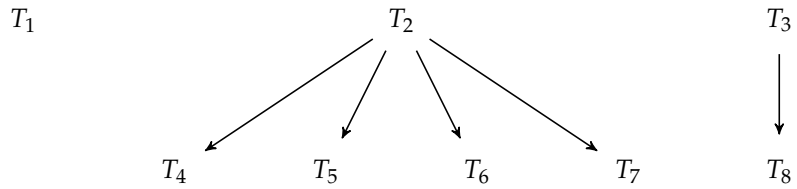
Let $MS_{list}(p)$ be the shortest possible makespan produced by a list scheduling algorithm. Then, the bound $MS_{list}(p) \leqslant (2 - \frac{1}{p})MS_{opt}(p)$ is tight.

$T_1 \dots T_{p-1}$ are tasks of size $K(p-1)$

Let σ be any list schedule. σ schedules tasks $T_1 \dots T_{p-1}$ and T_p at time 0. At time 1, T_p is completed and one processor, say P_1, si free. Then P_1 processes one task among $T_{p+1} \dots T_{2p}$ between times 1 and $k+1$, then a second of these tasks between time $k+1$ and $2k+1$. P_1 processes $p-1$ of the tasks T_{p+1} to T_{2p} between time 1 and $1+(p-1)k$ (if we assume $k \geqslant 2$). At time $k(p-1)$ all processors except P_1 are available, one of them computes the last of the $T_{p+1} \dots T_{2p}$ tasks. All these tasks are completed at time kp. Task T_{2p+1} is processed between times kp and $k(2p-1)$.

Optimal schedule

- At time 0 : process task T_p
- At time 1 : process task T_{p+1} to T_{2p}



- At time $k+1$: process task T_1 to T_{p-1} plus T_{2p+1}

The makespan is $k+1+k(p-1) = kp+1$ $MS(\sigma, p) = \frac{(2p-1)k}{kp+} MS_{opt}(p)$ on this instance. The quotient $\overset{k\to\infty}{\to} 2 - \frac{1}{p}$

8.3 Critical path scheduling

Intuition: the largest is the bottom-level of a task, the most urgent is the task.

Critical path scheduling: schedule tasks by non-increasing bottom-levels (break ties arbitrarily).

Chapter 9

Taking communications into account

Classical model : macro-dataflow Two tasks T and T′. Cos(T, T′) = 0 if alloc(T) = alloc(T′), $c(T, T')$ otherwise (independent of the choice of the two processors)

Assumptions:

- communication can start as early as task are completed
- no contention for network links*

A schedule must satisfy dependence constraints: if $(T, T') \in E$ if $alloc(T) = alloc(T')$, $\sigma(T') \geqslant \sigma(T) + w(t)$ and, otherwise, $\sigma(T') \geqslant \sigma(T) + w(T) + c(T, T')$

*https://en.wikipedia.org/wiki/Contention_(telecommunications)

Chapter 10

$Pb(\infty)$ with communications

Sequential case $MS_{opt}(1) = 13$. Assume we schedule each task on a different processor.

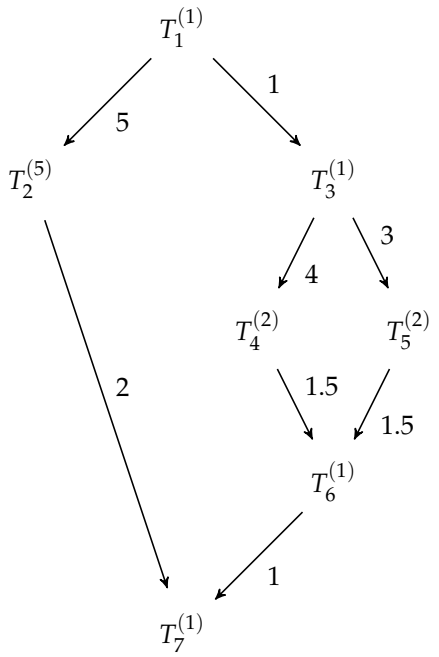


Figure 10.1:

Theorem 31.

$Pb(\infty)$ with communications is NP-complete.

Reduction from 2-partition. Let $a_1, \ldots, a_n$ be an instance of 2-partition with $\sum_{i=1}^{n} a_i = \alpha$ We build the following instance of $Pb(\infty)$ 10.2. All communications have a length of C : $C(T_0, T_i) = C(T_i, T_{i+1}) = C$ for $1 \leqslant i \leqslant n$ $w(T_0) = w(T_{n+1}) = A$ For $1 \leqslant i \leqslant n$, $w(T_i) = 2a_i$ C is an integer in the interval $]\alpha - \min_{1 \leqslant i \leqslant n} 2a_i, \alpha[$ -> interval of length at least 2; it contains at least one integer. $k = 2A + \alpha + C$ Is there a schedule whose makespan is $\leqslant \alpha$?

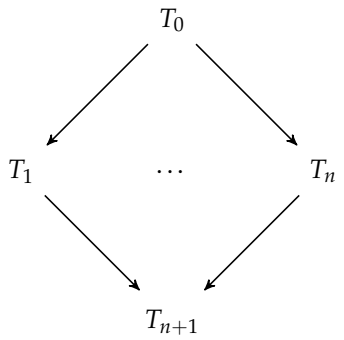


Figure 10.2:

Let us assume that there is a solution to the partition problem : $\exists I \sum_{i \in I} a_i = \sum_{i \min I} a_i$

Assume that we have a solution to the scheduling problem. Let σ be a schedule whose makespan is at most $k = 2A + \alpha + C$

Tasks T_0 and T_{n+1} are not executed on the same processor. Let us assume that T_0 and T_{n+1} are executed on the same processor. Let T_{i_0} be a task not executed on the same processor than T_0 and (or ?) T_{n+1}.

$$\begin{aligned} Makespan &\geqslant w(T_0) + c(T_0, T_{i_0}) + w(T_{i_0}) + c(T_{i_0}, T_{n+1}) + w(T_{n+1}) \\ &= A + C + w(T_{i_0}) + C + A \\ &= 2A + C + C + w(T_{i_0}) \\ &= k - \alpha + C + w(T_{i_0}) \\ &\geqslant k - \alpha + C + \min_{1 \leqslant i \leqslant n} 2a_i \end{aligned}$$

By definition, $C > \alpha - \min_{1 \leqslant i \leqslant n} 2a_i \; k - \alpha + C + \min_{1 \leqslant i \leqslant n} 2a_i > k$

Thus, if T_0 and T_{n+1} are executed on the same processor, all tasks are executed on the same processor.

$$\begin{aligned} Makespan &= \sum_{i=1}^{n+1} w(T_i) \\ &= 2A + 2\alpha \\ &= k + \alpha - C \end{aligned}$$

But, by definition, $C < \alpha$. Contradiction. Therefore, T_0 and T_{n+1} are executed on two different procs.

Any task is executed either on the same processor that T_0 on that T_{n+1}. Otherwise, let T_{i_1} be such a task.

$$\begin{aligned} Makespan &\leqslant w(T_0) + c(T_0, T_{i_1}) + w(T_{i_1}) + c(T_{i_1}, T_{n+1}) + w(T_{n+1}) \\ &\geqslant 2A + 2C + \min_{1 \leqslant i \leqslant n} 2a_i > k \end{aligned}$$

Let I be the set of tasks among $T_1 \dots T_n$ that are executed on the same processor than T_0 $\sigma(T_{n+1}) = \max(A + w(I) + C, A + C + 2d - w(I))$ The schedule satisfies the bound : $k = 2A + \alpha + C \geqslant A + \sigma(T_{n+1}) \geqslant 2A + C + w(I), 2A + C + 2\alpha - w(I)$ $\alpha \geqslant w(I)$ $w(I) \geqslant \alpha \Rightarrow w(I) = \alpha$

Chapter 11

List of heuristics for $Pb(\infty)$ with communications

How to take the notion of communications into account when defining critical paths. Conservative assumption considers that all communications are going to take place when computing the critical paths.

11.1 Naive critical path

$p = 3$ processors

11.2 Modified critical path scheduling

Map a task on the processor that enables to start it at the earliest.

Definition 18.

Let $G = (V, E, c, w)$ be a communication DAG. The granularity of G is the computation to communication ratio of G

$$g(G) = \frac{\min\limits_{T \in V} w(T)}{\max\limits_{T,T'} c(T, T')}$$

G is said to be coarse-grained if $g(G) \geqslant 1$.

Theorem 32.

Let $G = (V, E, c, w)$ be a communication DAG. Let $MS_{opt}(p)$be the optimal makespan. Then we can derive a schedule σ whose makespan satisfies

$$MS(\sigma, p) \leqslant \left(2 - \frac{1}{p}\right)\left(1 + \frac{1}{g(G)}\right) MS_{opt}(p)$$

if $g(G) > 0$.

Let σ be any list schedule for G when communications are not taken into account.

$$\overbrace{MS^{wc}(\sigma,p)}^{\text{without communication time}} \leqslant \left(2-\frac{1}{p}\right) MS^{wc}_{opt}(p)$$

11.3 Two Step clustering heuristics

1. Cluster the tasks (or partition them) as if an infinite number of processors was available.
2. Map the clusters on the processors (number of clusters at the end of step 1 can be dirrent than the number of processors).

Motivation : "if tasks are scheduled on the same processor, on the best possible architecture with unbounded number of processors, then that should be scheduled on the same processor in any other architecture"

11.4 KIM and BROWNE's linear clustering

1. Initially all edges are marked unexamined. Initial clustering $\mathcal{C}_0$ where each tasks is in a different cluster.
2. For each tasks v, compute, $bl(b,\mathcal{C}_i)$ and $tl(b,\mathcal{C}_i)$. Select a longest dependence path in the graph. Build $\mathcal{C}_{i+1}$ by grouping all tasks of the chosen dependence path. Mark as examined all the edges incident to these tasks.
3. While there remain unexamined edges in the graph, go to step 2.

Estimated parallel time for a clustering

$$EPT(\mathcal{C}) = \max\{tl(v)+bl(v) \mid v \in V\}$$

A new clustering is accepted only if it does not (strictly) increase the EPT.

SARKAR's greedy clustering.

1. Sort the edges by non-increasing communication costs
2. For each edge in this order, zero out the edge (merge the cluster containing the incident tasks) if the EPT does not increase.

11.5 Dominant sequence clustering

At each step, try to zero out one edge of the longest path (ie. dominant sequence).

1. Initially all the edges are marked unexamined. Initial clustering $\mathcal{C}_0$ compute the bottom and top levels of each task $EPT(\mathcal{C}_0)$ and a dominant sequence DS.
2. While (there remain unexamined edges)
   - Pick an unexamined edge of DS_i.
   - zero out the edge if the EPT does not increase.
   - Mark this edge as examined.
   - Compute the top and the bottom levels, $EPT(\mathcal{C}_{i+1})$, DS_{i+1}.

11.5.1 From clustering to scheduling

1. Cluster the tasks
2. (a) Assign clusters to processors
   (b) Order the execution of the tasks on the processors.

11.5.2 Cluster assignment

1. Compute the load ($\sum$ of weight of tasks) of each cluster.
2. Assign clusters by non increasing weight, to the best leaded processor/in a round robin way.

11.5.3 Final task scheduling

- Use in some way the critical paths (knowing which communications are going to take place).
- – At each step assign the free tasks of highest priority.
  – At each step start the ready tasks of highest priority (a task is ready when all its incomming communications have completed.)

Part IV

To go further

Chapter 12

Automatic parallelization: LAMPORT's hyperplane method

Ideal goal:

- Take an existing sequential program,
- Automatic analysis and transformation to obtain a parallel program that executes effectively on the target parallel platform.

12.1 Uniform loop nests

$$S_1 : a \leftarrow b + 1$$
$$S_2 : b \leftarrow a - 1$$
$$S_3 : a \leftarrow c - 2$$
$$S_4 : d \leftarrow c$$

We assume we deal with unaliased variables

$<_{text}$: textual order.

$<_{seq}$: sequential order.

Dependence analysis: find out which statements are dependent and which are independent, in order to determine which statements can be executed in parallel.

The output dependence from S_1 to S_3 is already obtained through transitive closure of $S_1 \overset{flow}{\longrightarrow} S_2$ and $S_2 \overset{anti}{\longrightarrow} S_3$

No dependences related to S_4 which is independent of the other statements.

---

for $i = 0$ **to** N **do**
  for $j = 0$ **to** N **do**
    $S_1(i,j) : a(i,j) = b(i,j-6) + d(i-1,j+3)$
    $S_2(i,j) : b(i+1,j-1) = c(i+2,j+5) + 1$
    $S_3(i,j) : c(i+3,j-1) = a(i,j+2)$
    $S_4(i,j) : d(i,j-1) = a(i,j-1) - 1$

A set of for loops and some statements. The loops are perfectly nested: each statement is surrounded by all the loop.

The nested loops surround a set of statements. An operation is the execution of a statement for a given values of the loop indices, that is for an iteration of the loops. The iteration vector is the vector of loop indices, here $\begin{pmatrix} i \\ j \end{pmatrix}$.

Iteration domain, set of values of the iteration vector.

$$\left\{ \begin{pmatrix} i \\ j \end{pmatrix} \middle| 0 \leqslant i, j \leqslant N \right\}$$

Let I be the iteration vector.

$$I = \begin{pmatrix} i \\ j \end{pmatrix}$$

The operations $S_k(I)$ are ordered in the loop nest by the $<_{seq}$ order.

$$S(I) <_{seq} T(J) \Leftrightarrow I <_{lex} J \vee (I = J \wedge S <_{text} T)$$

There is a dependence from $S(I)$ to $T(J)$ (ie. $T(J)$ depends on $S(I)$) if

- $S(I) <_{seq} T(J)$,
- Both $S(I)$ and $T(J)$ access the same memory location M, and at least one of the accesses is a write,
- The memory location M was not written between $S(I)$ and $T(J)$.

$$S(I) \longrightarrow T(J)$$

Dependence vector:

$$d_{S,I,T,J} = J - I$$

The loop nest is said to be uniform if $d_{S,I,T,J}$ is independent of I and of J. We then denote it $d_{S,T}$. All dependence vectors are lexicographically non negative (if all loop increments are positive).

$S_1(i,j)$ writes $a(i,j)$, $S_4(i,j)$ reads $a(i,j-1)$, $S_4(i,j+1)$ reads $a(i,j)$.

There is a flow dependence from $S_1(i,j)$ to $S_4(i,j+1)$. Dependence vector is $\begin{pmatrix} i \\ j+1 \end{pmatrix} - \begin{pmatrix} i \\ j \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$: constant independent of i and j: uniform dependence

Formally: Consider statements S_1 and S_3 that both access array a.

The question is: "Is there a dependence from S_1 to F_3 (because of array a)?". Is there an iteration vector I and an iteration vector J, such that $I \leqslant_{lex} J$ and such that

$$a(i,j) = a(i', j' + 2)$$

where $I = \begin{pmatrix} i \\ j \end{pmatrix}$ and $J = \begin{pmatrix} i' \\ j' \end{pmatrix}$

$$\begin{pmatrix} i \\ j \end{pmatrix} \leqslant_{lex} \begin{pmatrix} i' \\ j' \end{pmatrix}$$

with $i = i'$ and $j = j' + 2$

$$\begin{pmatrix} i \\ j \end{pmatrix} \leqslant_{lex} \begin{pmatrix} i \\ j - 2 \end{pmatrix}$$

which is impossible.

There does not exist any flows dependence for S_1 to S_3.

Is there a dependence from S_3 to S_1 because of a

$$I = \begin{pmatrix} i \\ j \end{pmatrix} \leqslant_{lex} J = \begin{pmatrix} i' \\ j' \end{pmatrix}$$

such that $S_3(i,j)$ access the same memory location than $S_1(i',j') \Leftrightarrow$

$$\begin{pmatrix} i \\ j+2 \end{pmatrix} \leqslant_{lex} \begin{pmatrix} i' \\ j' \end{pmatrix}$$

$$\left\{ \begin{array}{rcl} i' & = & i \\ j' & = & j+2 \end{array} \right. \Longrightarrow \begin{pmatrix} i \\ j \end{pmatrix} \leqslant_{lex} \begin{pmatrix} i' \\ j' \end{pmatrix}$$

There is an anti-dependence. from S_3 to S_1 because of array a, of dependence vector

$$\begin{pmatrix} i' \\ j' \end{pmatrix} - \begin{pmatrix} i \\ j \end{pmatrix} = \begin{pmatrix} i \\ j+2 \end{pmatrix} - \begin{pmatrix} i \\ j \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

uniform dependence

Dependence form S_2 to S_1 (flow dependence on b).

$$I = \begin{pmatrix} i \\ j \end{pmatrix} \leqslant_{lex} J = \begin{pmatrix} i' \\ j' \end{pmatrix}$$

and $S_2(I)$ and $S_1(J)$ acessing the same location

$$\begin{pmatrix} i+1 \\ j-1 \end{pmatrix} = \begin{pmatrix} i' \\ j'-6 \end{pmatrix}$$

with

$$\left\{ \begin{array}{l} i' = i+1 \\ j' = j+5 \end{array} \right.$$

$S_2 \xrightarrow{flow} S_1$ of dependence vector $\begin{matrix} 1 \\ 5 \end{matrix}$.

Dependence from S_4 to S_1(flow on d)

$$I = \begin{pmatrix} i \\ j \end{pmatrix} \leqslant_{lex} J = \begin{pmatrix} i' \\ j' \end{pmatrix}$$

$$\begin{pmatrix} i \\ j-1 \end{pmatrix} = \begin{pmatrix} i'-1 \\ j'+3 \end{pmatrix} \Leftrightarrow \left\{ \begin{array}{l} i' = i+1 \\ j' = j-4 \end{array} \right.$$

$S_4 \xrightarrow{flow} S_1$ of dependence vector $\begin{pmatrix} 1 \\ -4 \end{pmatrix}$.

Dependence from S_3 to S_2 (flow on c)

$$\begin{pmatrix} i+3 \\ j-1 \end{pmatrix} = \begin{pmatrix} i'+2 \\ j'+5 \end{pmatrix} \Leftrightarrow \left\{ \begin{array}{l} i' = i+1 \\ j' = j-6 \end{array} \right.$$

$S_4 \xrightarrow{flow} S_1$ of dependence vector $\begin{pmatrix} 1 \\ -6 \end{pmatrix}$.

12.2 LAMPORT's hyperplane method

Let Dom = the iteration domain.

Let D be the matrix of dependence vector in non decreasing lexicographic order.

Let $p_1 \in Dom$ and $p_2 \in Dom$. We write $p_1 \prec p_2$ if $\exists \alpha \in D$ such that $p_2 = p_1 + \alpha$.

Let σ be a schedule $\sigma : Dom \to \mathbb{Z}$. The schedule must satisfy the dependences.

$$\forall (p_1, p_2) \in Dom^2, p_1 \prec p_2 \Rightarrow \sigma(p_1) < \sigma(p_2)$$

Each iteration takes a unitary time.

The makespan of the schedule

$$T_\sigma = 1 + \max_{p \in Dom} \sigma(p) - \min_{p \in Dom} \sigma(p)$$

We assume we have an unlimited number of processors.

We could compute the free schedule by a traversal of the extended dependence graph, which is of size $\Omega\left(N^2\right)$. This could be as expensive in order of magnitude as executing the original sequential loop nest. We cant a compact schedule, computed once and for all values of N.

We restrict ourselves to linear schedules.

A linear schedule σ_π is defined by a vector π

$$\begin{aligned} \sigma_\pi : Dom &\to \mathbb{Z} \\ p &\mapsto \sigma_\pi(p) = \lfloor \pi \cdot p \rfloor \end{aligned}$$

---

for $time = time_{min}$ **to** $time_{max}$ **do**
  for *all* $p \in E(time)$ *in parallel* **do**
    $S_1(p)$
    $\vdots$
    $S_k(p)$

---

where

$$E(time) = \{p \in Dom | \lfloor \pi \cdot p \rfloor = time\}$$

$$p_1 \prec p_2 \Rightarrow \sigma_\pi(p_1) < \sigma_\pi(p_2)$$

$p_2 = p_1 + d$

$$\begin{aligned} \sigma_\pi(p_1) < \sigma_\pi(p_2) &\Leftrightarrow \lfloor \pi \cdot p_1 \rfloor < \lfloor \pi \cdot p_2 \rfloor \\ &\Leftrightarrow \lfloor \pi \cdot p_1 \rfloor + 1 \leqslant \lfloor \pi \cdot p_2 \rfloor \\ &\Leftrightarrow \lfloor \pi \cdot p_1 \rfloor + 1 \leqslant \lfloor \pi \cdot (p_1 + d) \rfloor \\ &\Leftarrow 1 < \pi \cdot d \end{aligned}$$

$$\begin{aligned} \pi \cdot d \geqslant 1 &\Rightarrow \pi d + \pi p_1 \geqslant 1 + \pi p_1 \\ &\Rightarrow \pi d + \pi p_1 \geqslant \lfloor 1 + \pi p_1 \rfloor \\ &\Rightarrow \lfloor \pi (p_1 + d) \rfloor \geqslant \lfloor \pi p_1 \rfloor + 1 \end{aligned}$$

LAMPORT's condition for the validity of a schedule.

$$\pi \cdot D \geqslant 1 \Leftrightarrow \forall d \in D, \pi \cdot d \geqslant 1$$

Let k_1 be the index of the first non null component of d_1. We let $\pi_{k_1+1} = \ldots = \pi_n = 0/$ We let $\pi_{k_1} = 1 \rightarrow \pi d_1 \geqslant 1$ Let d_a be the first vector of D whose first non null component is of rang $k_2 < k_1$. Let $\pi_{k+2} + 1 = \ldots = \pi_{k_1-1} = 0$.

We let $D_2 = \{d \in D | \text{the first non null component of } d \text{ is } k_2\}$. We want to have $\forall d \in D_2, \pi \cdot d \geqslant 1$.

$$\forall d \in D_2, \pi_{k_2} \geqslant 1 - \pi_{k_1+1} d_{k_2+1}$$
$$\vdots \quad \vdots$$
$$n \quad n$$

$$D = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 \\ 1 & 2 & -6 & -4 & 5 \end{pmatrix}$$

$$\pi = \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\begin{cases} b & \geqslant & 1 \\ 2b & \geqslant & 1 \end{cases} \Rightarrow b \geqslant 1$$

$$\begin{cases} a-6 & \geqslant & 1 \\ a-4 & \geqslant & 1 \\ a+5 & \leqslant & 1 \end{cases} \Leftrightarrow \begin{cases} a & \geqslant & 7 \\ a & \geqslant & 5 \\ a & \geqslant & -4 \end{cases} \Leftrightarrow a \geqslant 7$$

$$\pi = \begin{pmatrix} 7 \\ 1 \end{pmatrix}$$

is a solution

How to find a good solution?

$$T_{\sigma_\pi} = 1 + \max_{p \in Dom} \lfloor \pi p \rfloor - \min_{p \in Dom} \lfloor \pi p \rfloor$$

$$\begin{cases} a-6b & \geqslant & 1 \\ a-4b & \geqslant & 1 \\ a+5b & \geqslant & 1 \end{cases} \Leftrightarrow \begin{cases} a & \geqslant & 1+6b \\ a & \geqslant & 1+4b \\ a & \geqslant & 1-5b \end{cases} \Leftrightarrow \begin{cases} b & \geqslant & 1 \\ a & \geqslant & 1+6b \end{cases}$$

$\Leftarrow a, b$ are both non negative

$$\min_{p \in Dom} \lfloor \pi p \rfloor = 0$$

$$\max_{p \in Dom} \lfloor \pi p \rfloor = \lfloor aN + bN \rfloor$$

This is minimized for $a = 7$ and $b = 1$.

$$\pi = \begin{pmatrix} 7 \\ 1 \end{pmatrix}$$

time = $7i + j$

$$\begin{pmatrix} time \\ proc \end{pmatrix} = \begin{pmatrix} 7 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} i \\ j \end{pmatrix}$$

unimodular matrix = integral matrix of determinant 1 or -1.

$$\begin{pmatrix} i \\ j \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & -7 \end{pmatrix} \begin{pmatrix} time \\ proc \end{pmatrix}$$

$i = proc$

$0 \leqslant proc \leqslant N$

$j = time - 7proc$ $(0 \leqslant j \leqslant N)$

$$\left\lceil \frac{time - N}{7} \right\rceil \leqslant proc \leqslant \left\lfloor \frac{time}{7} \right\rfloor$$

$time = 7i + j$

$0 \leqslant time \leqslant 8N$

---

for $time = 0$ **to** $8N$ **do**
  for $proc = \max(0, \left\lceil \frac{time-N}{7} \right\rceil)$ **to** $\min(N, \left\lfloor \frac{time}{7} \right\rfloor)$ **do**
    $a(proc, time - 7proc) = b(proc, time - 7proc - 6) + d(proc - 1, time - 7proc + 3)$
    $\vdots$

Chapter 13

Algorithms for GPUs – An introduction

cf. ASR-1

GPU: graphical processing unit

GPGPU: graphical purpose GPU.

Motivation for a change in architecture

- memory wall: what is dictating the performance of an application is the cost of data movement and not the CPU processing power,
- energy wall: most powerful supercomputer on top 500 is using 26MW power for a performance of $\simeq$30 pétaflops. Objective 2018-2020:
  - exaflop ($\times$30),
  - energy budget $\leqslant$ 80MW.

Evolution:

- More parallelism. Single processor $\rightarrow$ multicore $\rightarrow$ manycore
- specialized hardware: GPU.

First underlying principle of GPUs simplification:

- Suppress big data cache,
- Suppress optimizing hardware: branch prediction etc..

Consequence: use many simple computing units in parallel.

The same operation is applied to different data at once.

There also exists in scientific computing: addition of vectors, matrix multiplication etc..: data parallelism.

Second principle: SIMD processing. SIMD: simple instruction many data.

SIMD processing $\neq$ SIMD instructions.

Data parallelism is not necessarily made explicitly in the program.

Chapter 14

Processeurs hétérogènes

Appendix A

Introduction à MPI

A.1 Parallel architectures

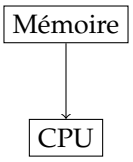


Figure A.1: Architecture séquentielle

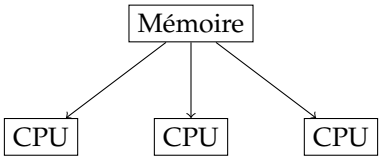


Figure A.2: Mémoire partagée

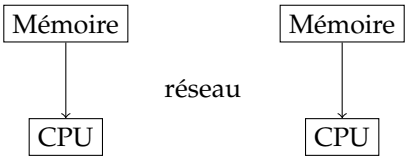


Figure A.3: Mémoire distribuée

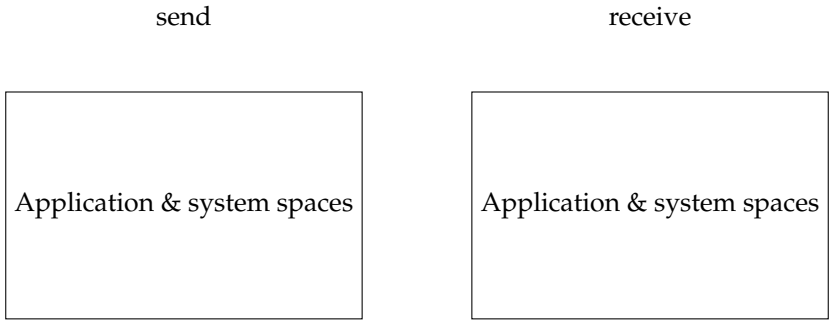
We also can have a hybrid version between shared and distributed memory.

MPI Message passing interface. It is not a program neither a library, It is a specification which describe to the programmer how to communicate with other process.

MPI is used by including the library MPI, writing sequential code, and, between the two functions `MPI_init` and `MPI_finalize`, the code is executed by each process.

A.2 Blocking and non blocking operations

Par exemple, imaginons qu'on ait 2 processus:



Dans un monde idéal, quand une information est envoyée, elle est reçue (communication synchrone). Or on peut avoir des pertes ou des canaux bloquants. Comme on veut éviter de mettre le processeur en attente, on met en place une communication asynchrone. On a donc des opérations bloquantes et non bloquantes : elles indiquent si le contenu du buffer peut être altéré par de nouvelles informations (bloquantes) ou non (non bloquantes).

On a donc 4 versions d'une même fonction (bloquante ou non, synchrone ou non): Send, Ssend, ISend, ISsend.

On peut de plus dire à MPI de placer le buffer système dans l'espace d'application pour pouvoir y toucher avec plus de précision. Ce qui rajoute entre 4 versions supplémentaires d'une fonction: Bsend, IBsend, Rsend, IRsend.

Dans la pratique, le nom des fonctions MPI commence par `MPI_`.

Avant de parler de la forme générale, nous allons nous examiner trois fonctions : wait, test (any, all, some) et prob. Qui traitent des opérations non bloquantes. Elles servent à vérifier si un message est arrivé ou non: par exemple Iprob pour effectuer d'autres calculs si le message n'est pas arrivé. Sinon prob permet d'attendre que le message est arrivé.

Si on veut vérifier si la mémoire dans laquelle a été stockée le message est sure pour pouvoir y écrire de nouvelles informations, on appelle test. Finalement wait attends tant que les opérations non bloquantes n'ont pas été effectuées.

A.3 MPI Functions Architecture

La fonction Isend : `MPI_Isend(&msg, nb_elt, size, dest, tag, comm, &req)`

tag : un identifiant du message.
comm : le communicateur, il désigne un groupe de processus MPI, où chaque process a un identifiant. Quand un programme démarre, il y a un communicateur : `MPI_Comm_World` qui désigne tous les processus. Si on veut subdiviser l'ensemble des processus pour qu'ils travaillent sur des calculs différents, on crée deux communicateurs, où les processus ont leur identifants propres. Cette notion est importante dans le cadre de communication/opérations collectives.
req : permet d'identifier l'opération pour vérifier par la suite si elle a terminée ou non.

Pour les communications collectives, nous avons :

- broadcast
- scatter
- gather
- reduce

`MPI_Comm_size` donne le nombre de processus existants dans le réseau MPI.
`MPI_Comm_rank` permet de récupérer son rang/identifiant.
`MPI_WTime` permet de synchroniser les horloges

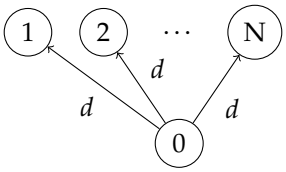


Figure A.4: MPI_Bcast : broadcast

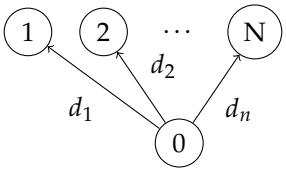


Figure A.5: Scatter

Bibliography

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