Nonstationary Time Series

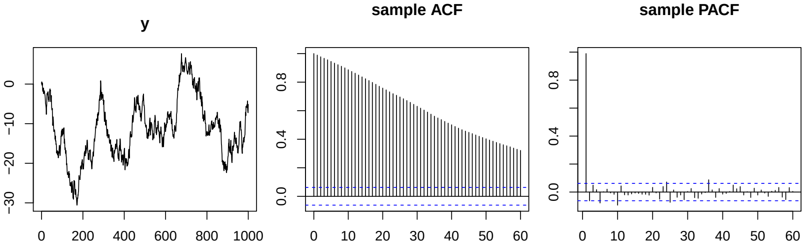
Nonstationary Time Series

A lot of series in economics and finance are not weakly stationary and instead
- show linear or exponential trend
- show stochastic trend - grow or fall over time or meander without a constant long-run mean
- show increasing variance over time

- GDP, personal consumption expenditures, investment, exports, imports, ...
- industrial production, retail sales, ...
- interest rates, foreign exchange rates, prices of assets, prices of commodities,...
- unemployment rate, labor force participation rate, ...
- loans, federal debt, ...

Nonstationary Time Series

A very slowly decaying ACF suggests nonstationarity and presence of deterministic or stochastic trend in the time series



Transformations

Log transformation - proper treatment if $\{y_t\}$ grows exponentially and shows increasing variability over time, or if we want to model growth rate not level

Detrending - regressing y_t on intercept and time trend - proper treatment id $\{y_t\}$ is trend stationary

Differencing - proper treatment id $\{y_t\}$ is difference stationary

Trend-Stationary Time Series

consider times series $\{y_t\}$ that follows

$$y_t = \alpha + \mu t + \varepsilon_t$$

where ε_t is a weakly stationary time series

- $E(y_t) = \alpha + \mu t$ and $Var(y_t) = Var(\varepsilon_t) = const.$
- since $E(y_t) \neq const.$ time series $\{y_t\}$ is not weakly stationary
- $\{y_t\}$ can hovewer be made stationary by removing time trend using a regression of y_t on constant and time
- $\{y_t\}$ is **trend stationary** time series

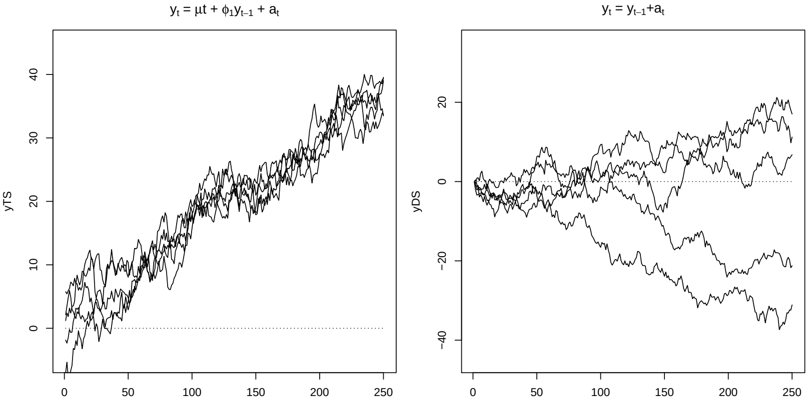
Difference-Stationary Time Series

Random Walk

- suppose a_t is white noise, consider a version of AR(1) model with $\phi_0 = 0$ and $\phi_1 = 1$
$$y_t = y_{t-1} + a_t$$
or, by repeated substituion
$$y_t = \alpha + \sum_{j=1}^{t} a_j$$
where $\alpha = y_0$
- $E(y_t) = \alpha$ and $Var(y_t) = Var(\sum_{j=1}^{t} a_j) = t\sigma_a^2$
- since $Var(y_t) \neq const.$ time series $\{y_t\}$ is not weakly stationary
- $\{y_t\}$ can hovewer be made stationary by differencing
- $\{y_t\}$ is **difference stationary** time series

Difference stationary series vs. Trend stationary series

five simulations of trend stationary time series vs random walk



Difference-Stationary Time Series

Random Walk with Drift

- suppose a_t is white noise, consider a version of AR(1) model with $\phi_1 = 1$

$$y_t = \mu + y_{t-1} + a_t$$

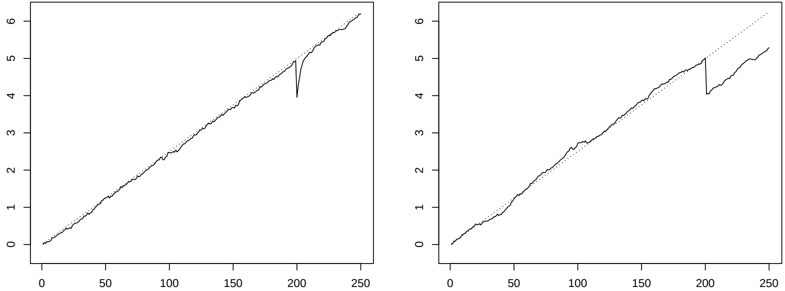
and by repeated substitution

$$y_t = \alpha + \mu t + \sum_{j=1}^{t} a_j$$

where $\alpha = y_0$
- $E(y_t) = \alpha + \mu t$ and $Var(y_t) = Var(\sum_{j=1}^{t} a_j) = t\sigma_a^2$
- $E(y_t) \neq const.$ and $Var(y_t) \neq const.$ so $\{y_t\}$ is not weakly stationary
- $\{y_t\}$ *can not* be made difference stationary by removing time trend using a regression of y_t on constant and time
- $\{y_t\}$ can hovewer be made stationary by differencing
- $\{y_t\}$ is **difference stationary** time series

Difference stationary series vs. Trend stationary series

It is important to be able to distinguish between the two cases:
- with trend stationary series shocks have **transitory effects**
- with difference stationary series shocks have **permanent effects**

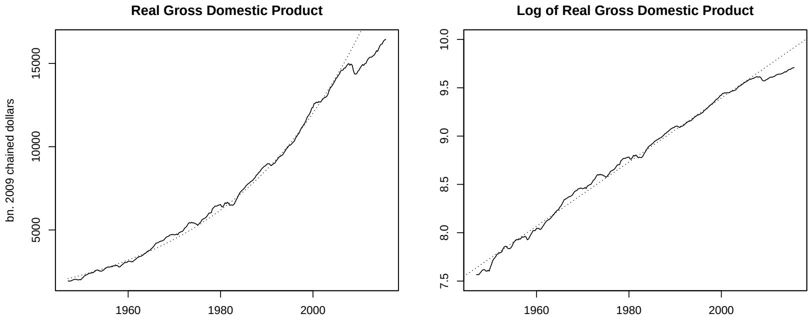


In addition, as we will see later additional issues arise with difference stationary series in the context of multivariate time series analysis

Difference stationary series vs. Trend stationary series

U.S. GDP and the effect of 2008-2009 recession

permanent effect or structural break?



Autoregressive integrated moving-average (ARIMA) models

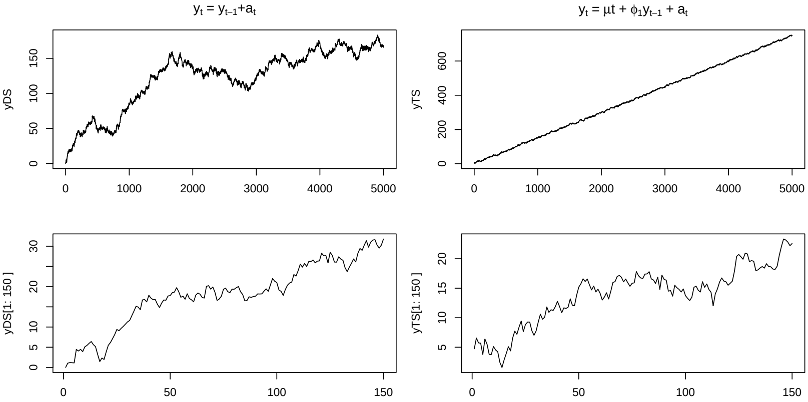
- non-stationary time series is said tp contain a unit root or to be integrated of order one, $I(1)$, if it can be made stationary by applying first differences
- time series $\{y_t\}$ follows an ARIMA$(p, 1, q)$ process if $\Delta y_t = (1 - B)y_t$ follows a stationary and invertible ARMA(p, q) process
- more generally, time series $\{y_t\}$ follows an ARIMA(p, d, q) process if $\Delta^d y_t = (1 - B)^d y_t$ follows a stationary and invertible ARMA(p, q) process
- pure random walk and randm walk with drift are special cases, ARIMA$(0, 1, 0)$

Example 1: Difference stationary series vs. Trend stationary series

it is often very hard to distinguish random walk and trend stationary model:

5000 vs. 150 observations of

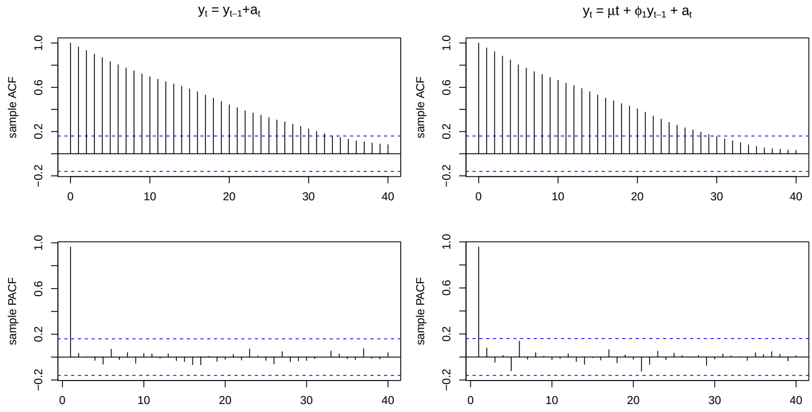
random walk vs. trend stationary AR(1) with $\mu = 0.15$, $\phi_1 = 0.95$



Example 1: Difference stationary series vs. Trend stationary series

ACF and PACF for 150 observations of y_t with

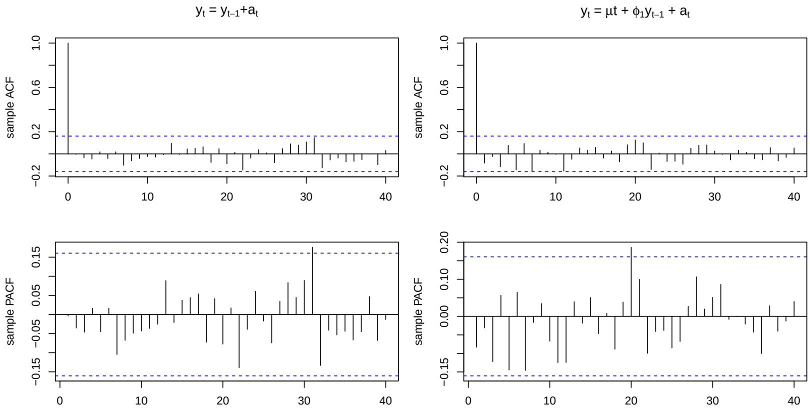
random walk vs. trend stationary AR(1) with $\mu = 0.15$, $\phi_1 = 0.95$



Example 1: Difference stationary series vs. Trend stationary series

ACF and PACF for 150 observations of Δy_t with

random walk vs. trend stationary AR(1) with $\mu = 0.15$, $\phi_1 = 0.95$



Example 1: Difference stationary series vs. Trend stationary series

random walk vs. trend stationary AR(1) with $\mu = 0.15$, $\phi_1 = 0.95$

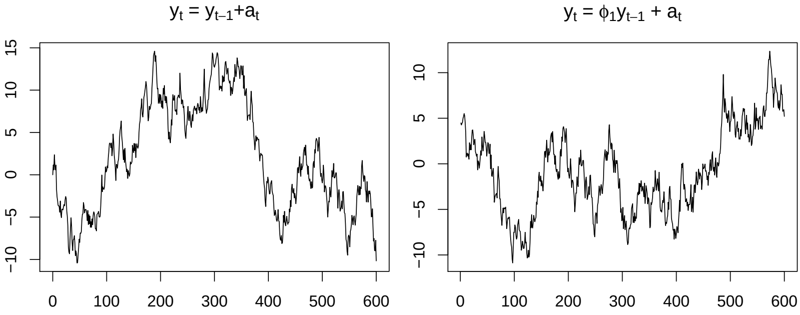
```
##
## Call:
## arima(x = yDS[1:T], order = c(1, 0, 0))
##
## Coefficients:
##          ar1  intercept
##       0.9971     16.279
## s.e.  0.0038     12.711
##
## sigma^2 estimated as 1.123:  log likelihood = -224.1,  aic = 454.19
```

```
##
## Call:
## arima(x = yTS[1:T], order = c(1, 0, 0))
##
## Coefficients:
##          ar1  intercept
##       0.9878    13.7733
## s.e.  0.0123     4.7683
##
## sigma^2 estimated as 1.051:  log likelihood = -218.44,  aic = 442.87
```

Example 2: Random Walk vs Highly Persistent AR(1)

also very hard to distinguish random walk model and highly persistent AR(1):

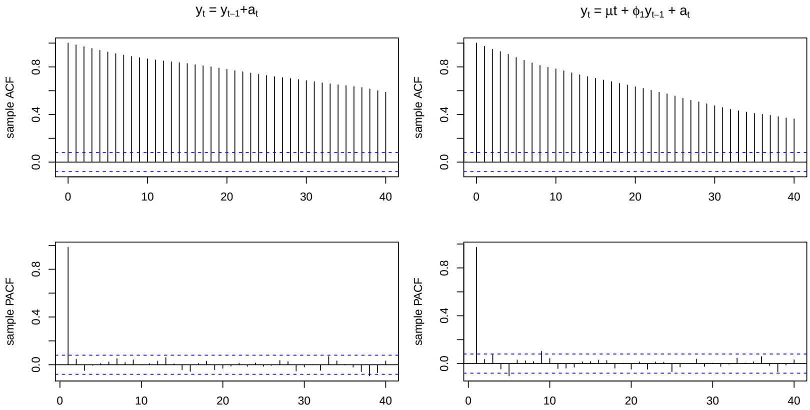
random walk I(1) vs. AR(1) with $\phi_1 = 0.98$



Example 2: Random Walk vs Highly Persistent AR(1)

ACF and PACF for y_t with

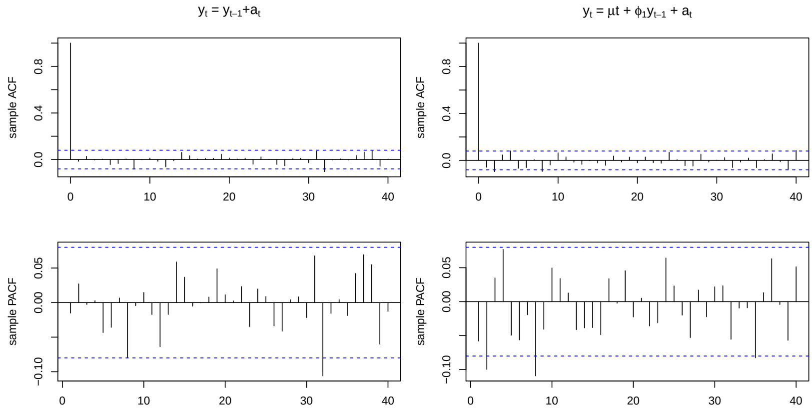
random walk vs. trend stationary AR(1) with $\phi_1 = 0.98$



Example 2: Random Walk vs Highly Persistent AR(1)

ACF and PACF for Δy_t with

random walk vs. trend stationary AR(1) with $\phi_1 = 0.98$



Example 2: Random Walk vs Highly Persistent AR(1)

random walk vs. trend stationary AR(1) with $\phi_1 = 0.98$

```
##
## Call:
## arima(x = yI1, order = c(1, 0, 0))
##
## Coefficients:
##          ar1  intercept
##       0.9885     0.4748
## s.e.  0.0060     3.2424
##
## sigma^2 estimated as 1.03:  log likelihood = -863.67,  aic = 1733.33
```

```
##
## Call:
## arima(x = yAR1, order = c(1, 0, 0))
##
## Coefficients:
##          ar1  intercept
##       0.9760    -0.2034
## s.e.  0.0087     1.6538
##
## sigma^2 estimated as 1.051:  log likelihood = -867.77,  aic = 1741.55
```

Unit Root and Stationarity Tests

- in general, the approach of the tests is to consider $\{y_t\}$ as a sum

$$y_t = d_t + z_t + \varepsilon_t$$

  where d_t is a deterministic component (time trend, seasonal component, etc.), z_t is a stochastic trend component and ε_t is a stationary process
- tests then investigate whether z_t is present
- two types of tests for nonstationarity
  - **unit root tests**: H_0 is difference stationarity, H_A is trend stationarity
  - **stationarity tests**: H_0 is trend stationary, H_A is difference stationarity

Augmented Dickey-Fuller (ADF) test

- main idea: suppose $\{y_t\}$ follows $AR(1)$

  $$y_t = \phi_1 y_{t-1} + a_t$$

  then

  $$\Delta y_t = \gamma y_{t-1} + a_t$$

  where $\gamma = \phi_1 - 1$
- if $\{y_t\}$ is $I(1)$ then $\gamma = 0$, otherwise $\gamma < 0$

Unit Root and Stationarity Tests

Augmented Dickey-Fuller (ADF) test

- unit root test
  H_0: time series $\{y_t\}$ has a unit root
  H_A: time series $\{y_t\}$ is stationary (with zero mean - model A), level stationary (with non-zero mean - model B) or trend stationary (stationary around a deterministic trend - model C)

$$\text{model A} \qquad \Delta y_t = \gamma y_{t-1} + \sum_{i=1}^{p-1} \rho_i \Delta y_{t-i} + e_t$$

$$\text{model B} \qquad \Delta y_t = \gamma y_{t-1} + \mu + \sum_{i=1}^{p-1} \rho_i \Delta y_{t-i} + e_t$$

$$\text{model C} \qquad \Delta y_t = \gamma y_{t-1} + \mu + \beta t + \sum_{i=1}^{p-1} \rho_i \Delta y_{t-i} + e_t$$

- if $\{y_t\}$ contains a unit root/is difference stationary, $\hat{\gamma}$ will be insignificant
- test $H_0 : \gamma = 0$ against $H_A : \gamma < 0$; if t-statistics for γ is lower than critical values we reject the null hypothesis of a unit root (one-sided left-tailed test)

Augmented Dickey-Fuller (ADF) test

If $\gamma < 0$ then
- under model A y_t fluctuates around zero
- under model B if $\mu \neq 0$ then y_t fluctuates around a non-zero mean - under model C if $\mu \neq 0$, $\beta \neq 0$ then y_t fluctuates around linear deterministic trend βt

If $\gamma = 0$ then
- under model A y_t contains stochastic trend only
- under model B if $\mu \neq 0$ then y_t contains both a linear deterministic trend μt and a stochastic trend
- under model C if $\mu \neq 0$, $\beta \neq 0$ then y_t contains a quadtratic deterministic trend βt^2 and a stochastic trend

Augmented Dickey-Fuller (ADF) test

- lags Δy_{t-i} used in the test are in order to control for the possible higher order autocorrelation
- number of lags can be chosen by a simple procedure where we start with some reasonably large number of lags p_{max} and check the significance of the coefficient on the highest lag with a t-test. If insignificant at the 10 % level, we reduce the number of lags by one and proceed in this way until achieving significance.
- an alternative approach would be to select the number of lags p to minimize AIC or BIC
- if p is too small errors will be serially correlated which will bias the test, if p is too large power of the test will suffer. It is better to err on the side of including too many lags.
- ADF has very low power against $I(0)$ alternatives that are close to being $I(1)$, it can't distinguish highly persistent stationary processes from nonstationary processes well

Augmented Dickey-Fuller (ADF) test

- including constant and trend in the regression also weakens the test (model C is thus the weakest on, model A the strongest one)
- if possible, we want to exclude the constant and/or the trend, but if they are incorrectly excluded, the test will be biased
- in addition to providing critical values to testing whether $\gamma = 0$, Dickey and Fuller also provide critical values for the following three F tests:
  - ϕ_1 statistic for model B to test $H_0 : \gamma = \mu = 0$
  - ϕ_2 statistic for model C to test $H_0 : \gamma = \mu = \beta = 0$
  - ϕ_3 statistic for model C to test $H_0 : \gamma = \beta = 0$
- these allow us to test whether we can restrict the test

Proposed Full Procedure for ADF test

Step 1. estimate model C, and use τ_3 statistic to test H_0: $\gamma = 0$

- if H_0 can not be rejected continue to Step 2
- if H_0 is rejected conclude that y_t is trend stationary

Step 2. use ϕ_3 statistic to test H_0: $\gamma = \beta = 0$

- if H_0 can not be rejected continue to step 3
- if H_0 is rejected estimate restricted model C, $\Delta y_t = \mu + \beta t + \sum_{i=1}^{p-1} \rho_i \Delta y_{t-i} + e_t$ and use t statistic to test $H_0 : \beta = 0$
  - if H_0 can not be rejected continue to Step 3
  - if H_0 is rejected conclude that y_t is difference stationary with quadratic trend

Step 3. estimate model B and use τ_2 statistic to test H_0: $\gamma = 0$

- if H_0 can not be rejected continue to Step 4
- if H_0 is rejected conclude that y_t is trend stationary

Step 4. use ϕ_1 statistic to test H_0: $\gamma = \mu = 0$

- if H_0 can not be rejected continue to step 5
- if H_0 is rejected estimate restricted model B, $\Delta y_t = \mu + \sum_{i=1}^{p-1} \rho_i \Delta y_{t-i} + e_t$ and use standard t statistic to test $H_0 : \mu = 0$
  - if H_0 can not be rejected continue to Step 5
  - if H_0 is rejected conclude that y_t is randwom walk with drift

Step 5. estimate model A and use τ_1 statistic to test H_0: $\gamma = 0$

- if H_0 can not be rejected conclude that y_t is random walk
- if H_0 is rejected conclude that y_t is trend stationary

Example 1: Difference stationary series vs. Trend stationary series contd.

```r
library(tseries)
adf.test(yTS)
```

```
## Warning in adf.test(yTS): p-value smaller than printed p-value
```

```
##
##  Augmented Dickey-Fuller Test
##
## data:  yTS
## Dickey-Fuller = -9.2144, Lag order = 17, p-value = 0.01
## alternative hypothesis: stationary
```

```r
adf.test(yDS)
```

```
##
##  Augmented Dickey-Fuller Test
##
## data:  yDS
## Dickey-Fuller = -2.6637, Lag order = 17, p-value = 0.2973
## alternative hypothesis: stationary
```

```r
adf.test(diff(yDS))
```

```
## Warning in adf.test(diff(yDS)): p-value smaller than printed p-value
```

```
##
##  Augmented Dickey-Fuller Test
##
## data:  diff(yDS)
## Dickey-Fuller = -16.259, Lag order = 17, p-value = 0.01
## alternative hypothesis: stationary
```

Example 1: Difference stationary series vs. Trend stationary series contd.

```
library(urca)
yTS.urdf <- ur.df(yTS, type="trend", selectlags="BIC")
summary(yTS.urdf)
```

```
##
## ###############################################
## # Augmented Dickey-Fuller Test Unit Root Test #
## ###############################################
##
## Test regression trend
##
##
## Call:
## lm(formula = z.diff ~ z.lag.1 + 1 + tt + z.diff.lag)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -3.6246 -0.6734 -0.0073  0.6816  4.3585
##
## Coefficients:
##               Estimate Std. Error t value Pr(>|t|)
## (Intercept)  0.2156769  0.0294252   7.330 2.68e-13 ***
## z.lag.1     -0.0562692  0.0047070 -11.954  < 2e-16 ***
## tt           0.0084263  0.0007048  11.955  < 2e-16 ***
## z.diff.lag   0.0119032  0.0141433   0.842      0.4
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 1.018 on 4994 degrees of freedom
## Multiple R-squared:  0.02808,    Adjusted R-squared:  0.02749
## F-statistic: 48.09 on 3 and 4994 DF,  p-value: < 2.2e-16
##
##
## Value of test-statistic is: -11.9543 83.6306 71.4597
##
## Critical values for test statistics:
##       1pct  5pct 10pct
## tau3 -3.96 -3.41 -3.12
## phi2  6.09  4.68  4.03
## phi3  8.27  6.25  5.34
```

Example 1: Difference stationary series vs. Trend stationary series contd.

```
library(tseries)
adf.test(yTS[1:150])
```

```
##
##  Augmented Dickey-Fuller Test
##
## data:  yTS[1:150]
## Dickey-Fuller = -2.0322, Lag order = 5, p-value = 0.563
## alternative hypothesis: stationary
```

```
adf.test(yDS[1:150])
```

```
##
##  Augmented Dickey-Fuller Test
##
## data:  yDS[1:150]
## Dickey-Fuller = -2.2927, Lag order = 5, p-value = 0.4545
## alternative hypothesis: stationary
```

```
adf.test(diff(yDS[1:150]))
```

```
## Warning in adf.test(diff(yDS[1:150])): p-value smaller than printed p-value
```

```
##
##  Augmented Dickey-Fuller Test
##
## data:  diff(yDS[1:150])
## Dickey-Fuller = -5.0307, Lag order = 5, p-value = 0.01
## alternative hypothesis: stationary
```

Unit Root and Stationarity Tests

Kwiatkowski-Phillips-Schmidt-Shin (KPSS) test

- stationarity test
  H_0: $\{y_t\}$ is stationary (either mean stationary or trend stationary)
  H_A: $\{y_t\}$ is difference stationary (has a unit root)
- main idea: decompose time series $\{y_t\}$ as

$$y_t = d_t + z_t + \varepsilon_t$$

  where d_t is the deterministic trend, z_t is random walk $z_t = z_{t-1} + a_t$, a_t is white noise (iid $E(a_t) = 0$, $Var(a_t) = \sigma_a^2$), and ε_t stationary error (i.e. $I(0)$ but not necessarily white noise)
- stationarity of $\{y_t\}$ depends on σ_a^2, we can run a test

$$H_0 : \sigma_a^2 = 0$$

  against

$$H_A : \sigma_a^2 > 0$$

  using Lagrange multiplier (LM) statistic

Unit Root and Stationarity Tests

Kwiatkowski-Phillips-Schmidt-Shin (KPSS) test

- to perform KPSS test we estimate

$$\begin{array}{ll} \text{model A} & y_t = \alpha + e_t \\ \text{model B} & y_t = \alpha + \mu t + e_t \end{array}$$

  model A is used if H_0 is mean stationarity, model B is used if H_0 is trend stationarity

- using residuals e_t we construct LM statistics η

$$\eta = \frac{1}{T^2} \frac{1}{s^2} \sum_{t=1}^{T} S_t^2$$

  where $S_t = \sum_{i=1}^{t} e_i$ is the partial sum process of the residuals e_t and s^2 is an estimator of the long-run variance of the residuals e_t.

- KPSS test is a one-sided right-tailed test: we reject H_0 at α% level if η is greater than $100(1-\alpha)$% percentile from the appropriate asymptotic distribution

Example 1: Difference stationary series vs. Trend stationary series contd.

```r
library(tseries)
kpss.test(yTS, null="Trend")
```

```
## Warning in kpss.test(yTS, null = "Trend"): p-value smaller than printed p-
## value
```

```
##
##  KPSS Test for Trend Stationarity
##
## data:  yTS
## KPSS Trend = 0.22978, Truncation lag parameter = 16, p-value =
## 0.01
```

```r
kpss.test(yDS, null="Trend")
```

```
## Warning in kpss.test(yDS, null = "Trend"): p-value smaller than printed p-
## value
```

```
##
##  KPSS Test for Trend Stationarity
##
## data:  yDS
## KPSS Trend = 3.6563, Truncation lag parameter = 16, p-value = 0.01
```

```r
kpss.test(diff(yDS), null="Level")
```

```
##
##  KPSS Test for Level Stationarity
##
## data:  diff(yDS)
## KPSS Level = 0.44454, Truncation lag parameter = 16, p-value =
## 0.05796
```

Example 1: Difference stationary series vs. Trend stationary series contd.

```r
library(urca)
yTS.urkpss <- ur.kpss(yTS, type="tau", lags="short")
summary(yTS.urkpss)
```

```
##
## #######################
## # KPSS Unit Root Test #
## #######################
##
## Test is of type: tau with 10 lags.
##
## Value of test-statistic is: 0.325
##
## Critical value for a significance level of:
##                  10pct  5pct 2.5pct  1pct
## critical values 0.119 0.146  0.176 0.216
```

Unit Root and Stationarity Tests

Phillips-Perron (PP) test

- an alternative to ADF test, estimates one of the models

$$\begin{array}{ll} \text{model A} & \Delta y_t = \gamma y_{t-1} + + e_t \\ \text{model B} & \Delta y_t = \gamma y_{t-1} + \mu + + e_t \\ \text{model C} & \Delta y_t = \gamma y_{t-1} + \mu + \beta t + e_t \end{array}$$

  and tests $H_0 : \gamma = 0$ against $H_A : \gamma < 0$
- unlike ADF uses non-parametric correction based on Newey-West heteroskedasticity and autocorrelation consistent (HAC) esrtimators to account for possible autocorrelation in e_t
- advantage over the ADF: PP tests are robust to general forms of heteroskedasticity and do not require to choose number of lags in the test regression
- assymptoticaly identical to ADF test, but likely inferior in small samples
- like ADF also not very powerful at distinguishing stationary near unit root series for unit root series

Elliot, Rothenberg and Stock (ERS) tests

- two efficient unit root tests with substantially higher power than the ADF or PP tests especially when ϕ_1 is close to 1
- P-test: optimal for point alternative $\phi_1 = 1 - \bar{c}/T$
- DF-GLS test: main idea - estimate test regression as in model A of ADF but on derended time series y_t

Example 1: Difference stationary series vs. Trend stationary series contd.

```r
library(urca)
yTS.urers1 <- ur.ers(yTS, type="P-test", model="trend")
summary(yTS.urers1)
```

```
##
## ###############################################
## # Elliot, Rothenberg and Stock Unit Root Test #
## ###############################################
##
## Test of type P-test
## detrending of series with intercept and trend
##
## Value of test-statistic is: 0.5048
##
## Critical values of P-test are:
##                 1pct 5pct 10pct
## critical values 3.96 5.62  6.89
```

Example 1: Difference stationary series vs. Trend stationary series contd.

```r
library(urca)
yTS.urers2 <- ur.ers(yTS, type="DF-GLS", model="trend")
summary(yTS.urers2)
```

```
##
## ###############################################
## # Elliot, Rothenberg and Stock Unit Root Test #
## ###############################################
##
## Test of type DF-GLS
## detrending of series with intercept and trend
##
##
## Call:
## lm(formula = dfgls.form, data = data.dfgls)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -3.5735 -0.7132 -0.0517  0.6432  4.2731
##
## Coefficients:
##               Estimate Std. Error t value Pr(>|t|)
## yd.lag       -0.041303   0.004285  -9.639  < 2e-16 ***
## yd.diff.lag1  0.003327   0.014217   0.234  0.81498
## yd.diff.lag2 -0.013141   0.014169  -0.927  0.35374
## yd.diff.lag3 -0.040292   0.014149  -2.848  0.00442 **
## yd.diff.lag4  0.002834   0.014147   0.200  0.84125
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 1.02 on 4990 degrees of freedom
## Multiple R-squared:  0.02337,    Adjusted R-squared:  0.02239
## F-statistic: 23.88 on 5 and 4990 DF,  p-value: < 2.2e-16
##
##
## Value of test-statistic is: -9.6387
##
## Critical values of DF-GLS are:
##                  1pct  5pct 10pct
## critical values -3.48 -2.89 -2.57
```