HW 1

1) $\vec{x}(t) = (t^5, t^{10})$

a)

$$\vec{x}'(t) = (5t^4, 10t^9)$$

$$\vec{x}'(t) = \vec{0} \Rightarrow 5t^4 = 10t^9 = 0 \Rightarrow t = 0$$

$\vec{x}$ is regular for all t except 0.

b)

$$\|\vec{x}'(t)\| = \sqrt{25t^8 + 100t^{18}}$$

$$\text{arclength} = \int_1^2 \sqrt{25t^8 + 100t^8}\, dt$$

c) No. Assume such $\vec{y}$ and $f(t)$ exists. $\Rightarrow \vec{y}(f) = \vec{x}(t(f))$
$\Rightarrow \vec{y}'(f) = \vec{x}'(t(f))\, t'(f)$. Since $\vec{x}$ not regular there exists t such that $\vec{x}'(t(f)) = 0 \Rightarrow y'(f) = 0 \Rightarrow \vec{y}$ not regular, Contradiction reached.

2) $\vec{y}(s) = \left(\left(\frac{s}{\sqrt{3}}+1\right)\cos\left(\ln\left(\frac{s}{\sqrt{3}}+1\right)\right), \left(\frac{s}{\sqrt{3}}+1\right)\sin\left(\ln\left(\frac{s}{\sqrt{3}}+1\right)\right), \frac{s}{\sqrt{3}}+1\right)$.

$\vec{y}$ parametrized by arclength $\Leftrightarrow \|\vec{y}'(s)\| = 1$ for all s.

$$\vec{y}'(s) = \left(\frac{1}{\sqrt{3}}\cos\left(\ln\left(\frac{s}{\sqrt{3}}+1\right)\right) - \frac{\left(\frac{s}{\sqrt{3}}+1\right)\sin\left(\ln\left(\frac{s}{\sqrt{3}}+1\right)\right)}{\left(\frac{s}{\sqrt{3}}+1\right)\frac{1}{\sqrt{3}}}, \frac{1}{\sqrt{3}}\sin\left(\ln\left(\frac{s}{\sqrt{3}}+1\right)\right) + \frac{\left(\frac{s}{\sqrt{3}}+1\right)\cos\left(\ln\left(\frac{s}{\sqrt{3}}+1\right)\right)}{\left(\frac{s}{\sqrt{3}}+1\right)\frac{1}{\sqrt{3}}}, \frac{1}{\sqrt{3}}\right)$$

$$= \frac{1}{\sqrt{3}}\left(\cos\left(\ln\left(\frac{s}{\sqrt{3}}+1\right)\right) - \sin\left(\ln\left(\frac{s}{\sqrt{3}}+1\right)\right), \sin\left(\ln\left(\frac{s}{\sqrt{3}}+1\right)\right) + \cos\left(\ln\left(\frac{s}{\sqrt{3}}+1\right)\right), 1\right)$$

$$= \frac{1}{\sqrt{3}}\begin{bmatrix} \cos\left(\ln\left(\frac{s}{\sqrt{3}}+1\right)+\frac{\pi}{4}\right) \\ \sin\left(\ln\left(\frac{s}{\sqrt{3}}+1\right)+\frac{\pi}{4}\right) \\ 1 \end{bmatrix}$$

$$\|y'(s)\| = \left(\frac{1}{3}\cos^2\left(\ln\left(\frac{s}{\sqrt{3}}+1\right)+\frac{\pi}{4}\right) + \frac{1}{3}\sin^2\left(\ln\left(\frac{s}{\sqrt{3}}+1\right)+\frac{\pi}{4}\right) + \frac{1}{3}\right)^{1/2}$$

$$= 1^{1/2} = 1$$

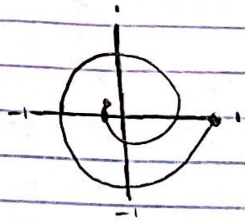
$\Rightarrow \vec{y}$ is parametrized by arclength.

3) $\vec{x}(t) = (e^{at}\cos(bt), e^{at}\sin(bt))$, $a,b>0$

a)

$a = 1/10$, $b=1$, $t=-10$ to 0

e^{-1} to e^{0}, $\frac{1}{e}$ to 1, $-10 \approx -3\pi$



$\vec{x}(-3\pi) \approx (-\frac{1}{e}, 0)$

$\vec{x}(0) = (1,0)$

make a little over 1.5 rotations.

b) arclength $= \int_c^0 ||\vec{x}'(t)||\,dt$

$\vec{x}'(t) = (ae^{at}\cos(bt) - be^{at}\sin(bt), ae^{at}\sin(bt) + be^{at}\cos(bt))$

$||\vec{x}'(t)|| = (a^2e^{2at}\cos^2(bt) - 2abe^{2at}\cos(bt)\sin(bt) + b^2e^{2at}\sin^2(bt)$
$+ a^2e^{2at}\sin^2(bt) + 2abe^{2at}\sin(bt)\cos(bt) + b^2e^{2at}\cos^2(bt))^{1/2}$

$= (a^2e^{2at} + b^2e^{2at})^{1/2} = e^{at}\sqrt{a^2+b^2}$

$$\int_c^0 \sqrt{a^2+b^2}\, e^{at}\,dt = \left[\frac{\sqrt{a^2+b^2}}{a} e^{at}\right|_c^0 = \frac{\sqrt{a^2+b^2}}{a} - \frac{\sqrt{a^2+b^2}}{a} e^{ac}$$

$$\lim_{c\to-\infty} \frac{\sqrt{a^2+b^2}}{a} - \frac{\sqrt{a^2+b^2}}{a} e^{ac} = \frac{\sqrt{a^2+b^2}}{a}$$

c) find unit speed reparametrization of $\vec{x}$.

$$s(t) = \int_0^t \sqrt{a^2+b^2}\, e^{au}\,du = \left[\frac{\sqrt{a^2+b^2}}{a} e^{au}\right|_0^t = \frac{\sqrt{a^2+b^2}}{a}(e^{at}-1)$$

$$\frac{a(s+1)}{\sqrt{a^2+b^2}} = e^{at}, \quad t(s) = \frac{\ln\left(\frac{a(s+1)}{\sqrt{a^2+b^2}}\right)}{a}$$

$$\vec{x}(t(s)) = \left(\ln\left(\frac{a(s+1)}{\sqrt{a^2+b^2}}\right)\cos\left(\frac{b\ln\left(\frac{a(s+1)}{\sqrt{a^2+b^2}}\right)}{a}\right), \ln\left(\frac{a(s+1)}{\sqrt{a^2+b^2}}\right)\sin\left(\frac{b\ln\left(\frac{a(s+1)}{\sqrt{a^2+b^2}}\right)}{a}\right)\right)$$

4) $\vec{x}:[a,b]\to\mathbb{R}^n$, continuously differentiable, $\vec{p}=\vec{x}(a)$, $\vec{q}=\vec{x}(b)$

a) $(\vec{q}-\vec{p})\cdot\vec{v}=(\vec{x}(b)-\vec{x}(a))\cdot\vec{v}=\int_a^b \vec{x}'(t)\,dt\cdot\vec{v}$

$=v_1\int_a^b x_1'(t)dt+v_2\int_a^b x_2'(t)\,dt+\dots+v_n\int_a^b x_n'(t)dt$

$=\int_a^b v_1x_1'(t)+v_2x_2'(t)+\dots+v_nx_n'(t)dt$

$=\int_a^b \vec{x}'(t)\cdot\vec{v}\,dt=\int_a^b \|\vec{x}'(t)\|\|\vec{v}\|\cos\theta\,dt$, where

θ is angle between $\vec{x}'(t)$ and $\vec{v}$

max of $\cos\theta$ is 1

meaning $\int_a^b \|\vec{x}'(t)\|\|\vec{v}\|\cos\theta\,dt\le\int_a^b\|\vec{x}'(t)\|\,dt$

b)

$\|\vec{q}-\vec{p}\|=\|\vec{q}-\vec{p}\|\,\|\vec{v}\|$ (where $\|\vec{v}\|=1$)

$\le\int_a^b\|\vec{x}'(t)\|\,dt$ (by part (a))

$=s(a,b)$ (definition of arclength

5) $\vec{x}(t) = (t, f(t))$

a) find arclength from a to b

$\vec{x}'(t) = (1, f'(t))$

$v = \|\vec{x}'(t)\| = \sqrt{1+(f'(t))^2}$

$s(a,b) = \int_a^b \sqrt{1+(f'(t))^2}\,dt$

b) $\vec{x}''(t) = (0, f''(t))$

$\kappa(t) = \dfrac{x_1'x_2'' - x_2'x_1''}{v^3} = \dfrac{f''(t)}{(1+(f'(t))^2)^{3/2}}$

c)

v is a strictly non-negative number

$\Rightarrow v^3 \geq 0$

for $\kappa(t)$ to be defined, $v^3 \neq 0 \Rightarrow v^3 > 0$

meaning $\kappa(t) > 0 \Leftrightarrow \frac{f''(t)}{v^3} > 0 \Leftrightarrow f''(t) > 0$ (*)

and $\kappa(t) < 0 \Leftrightarrow f''(t)\, v^3 < 0$ (**)

f concave at $t_0 \Leftrightarrow f''(t_0) < 0 \Leftrightarrow \kappa(t_0) < 0$ (**)

f convex at $t_0 \Leftrightarrow f''(t_0) > 0 \Leftrightarrow \kappa(t_0) > 0$ (*)

6) $\vec{x}(t) = (\cos(t) + \ln(\tan(t/2)), \sin(t))$

a)

$\vec{x}'(t) = \left(-\sin(t) + \frac{1}{2\cos(\frac{t}{2})\sin(\frac{t}{2})}, \cos t\right)$

$\cos(t)$ only 0 at $\frac{\pi}{2} + \pi k$, not 0 at all on $(\frac{\pi}{2}, \pi)$

$\Rightarrow \vec{x}(t)$ regular on $(\frac{\pi}{2}, \pi)$

$\vec{x}'(\frac{\pi}{2}) = \left(-1 + \frac{1}{2(\frac{\sqrt{2}}{2})(\frac{\sqrt{2}}{2})}, 0\right) = (0,0) = \vec{0}.$

b) note: $2\cos(\frac{t}{2})\sin(\frac{t}{2}) = \sin(t)$

$$\vec{x}'(t) = \left(\frac{1}{\sin(t)} - \sin(t), \cos t\right)$$

$$\|\vec{x}'(t)\| = \sqrt{\frac{1}{\sin^2(t)} - 2 + \sin^2 t + \cos^2 t} = \sqrt{\frac{1}{\sin^2 t} - 1} = \sqrt{\frac{1-\sin^2 t}{\sin^2 t}}$$

$$= \sqrt{\frac{\cos^2 t}{\sin^2 t}} = \left|\frac{\cos t}{\sin t}\right| = \frac{1}{-\tan t} \quad (\text{for } t \in (\tfrac{\pi}{2}, \pi))$$

$$\vec{T}(t) = \frac{\vec{x}'(t)}{\|\vec{x}'(t)\|} = -\tan t\left(\frac{1}{\sin t} - \sin t, \cos t\right)$$

$$= \left(-\frac{1}{\cos t} + \frac{\sin^2 t}{\cos t}, -\sin t\right) = \left(\frac{\cos^2 t}{\cos t}, \sin t\right)$$

$$= (-\cos t, -\sin t)$$

$$\vec{T}(t) = (-\cos t, -\sin t)$$

$$\vec{y}(t) = \vec{x}(t) + \vec{T}(t) = (\ln(\tan(t/2)), 0)$$

y component $= 0$, always lives on x-axis.

c)

$$x'(t) = \left(\frac{1-\sin^2 t}{\sin t}, \cos t\right) = \left(\frac{\cos^2 t}{\sin t}, \cos t\right) = (\cos t \cot t, \cos t)$$

$$x''(t) = \left(\frac{\cos t}{-\sin^2(t)} - \cos t, -\sin t\right)$$

$$\kappa(t) = \frac{x_1' x_2'' - x_2' x_1''}{v^3} = \frac{\cos^2 t + \frac{\cos^2 t}{\sin^2 t} + \cos^2 t}{-\tan^3 t}$$

$$= \frac{-\cos^3 t}{\sin^3 t}\left[\frac{2\cos^2 t\sin^2 t + \cos^2 t}{\sin^2 t}\right]$$

$$= \frac{-2\cos^5 t\sin^2 t - \cos^5 t}{\sin^5 t}$$

7) $\kappa(t) = 3t^2 - 4t$, $\vec{x}$ is unit speed

a) $\vec{x}'(0) = (0,1)$. Find formula for $\vec{x}'(t)$.

$\vec{x}'(t) = (\cos\phi(t), \sin\phi(t))$

$\vec{x}''(t) = (-\phi'(t)\sin\phi(t), \phi'(t)\cos\phi(t))$

$\kappa(t) = \phi'(t)\cos^2\phi(t) + \phi'(t)\sin^2\phi(t) = \phi'(t)$

$\phi'(t) = 3t^2 - 4t$, $\phi(t) = t^3 - 2t^2 + c$

$x'(t) = (\cos(t^3 - 2t^2 + c), \sin(t^3 - 2t^2 + c))$

$(0,1) = (\cos c, \sin c)$, $c = \frac{\pi}{2}$

$x'(t) = (\cos(t^3 - 2t^2 + \frac{\pi}{2}), \sin(t^3 - 2t^2 + \frac{\pi}{2}))$

b) $x(t) = \left(\int \cos(t^3 - 2t^2 + \frac{\pi}{2})\,dt + 2, \int \sin(t^3 - 2t^2 + \frac{\pi}{2})\,dt + 3\right)$

8)

a) arc length of C over $[a,b]$. $p = p(\theta)$

$\vec{c}(\theta) = (\theta, p(\theta))$, $\vec{c}'(\theta) = (1, p'(\theta))$

$\|\vec{c}'(\theta)\| = \sqrt{1 + (p'(\theta))^2}$, arclength $= \int_a^b \sqrt{1 + (p'(\theta))^2}\,d\theta$