UPLB Eliens - Pegaraw Notebook

Contents

1 Data Structures

1.1 Minimum Queue

```
ll get_minimum(stack<pair<ll, ll>> &s1, stack<pair<ll, ll>> &s2) {
  if (s1.empty() || s2.empty()) {
    return s1.empty() ? s2.top().second : s1.top().second;
  } else {
    return min(s1.top().second, s2.top().second);
  }
}
void add_element(ll new_element, stack<pair<ll, ll>> &s1) {
  ll minimum = s1.empty() ? new_element : min(new_element, s1.top().second);
  s1.push({new_element, minimum});
}
ll remove_element(stack<pair<ll, ll>> &s1, stack<pair<ll, ll>> &s2) {
  if (s2.empty()) {
    while (!s1.empty()) {
      ll element = s1.top().first;
      s1.pop();
      ll minimum = s2.empty() ? element : min(element, s2.top().second);
      s2.push({element, minimum});
    }
  }
  ll removed_element = s2.top().first;
  s2.pop();
  return removed_element;
}
```

1.2 Segment Tree 1

```
void build(vector<ll> &a, ll v, ll tl, ll tr) {
  if (tl == tr) {
    t[v] = a[tl];
  } else {
    ll tm = (tl + tr) / 2;
    build(a, v * 2, tl, tm);
    build(a, v * 2 + 1, tm + 1, tr);
    t[v] = 0;
  }
}
void update(ll v, ll tl, ll tr, ll l, ll r, ll add) {
  if (l > r) {
    return;
  }
  if (l == tl && r == tr) {
    t[v] += add;
  } else {
    ll tm = (tl + tr) / 2;
    update(v * 2, tl, tm, l, min(r, tm), add);
    update(v * 2 + 1, tm + 1, tr, max(l, tm + 1), r, add);
  }
}
ll query(ll v, ll tl, ll tr, ll pos) {
  if (tl == tr) {
    return t[v];
  }
  ll tm = (tl + tr) / 2;
  if (pos <= tm) {
    return t[v] + get(v * 2, tl, tm, pos);
  } else {
    return t[v] + get(v * 2 + 1, tm + 1, tr, pos);
  }
}
```

1.3 Segment Tree 2

```
void push(ll v) {
  if (marked[v]) {
    t[v * 2] = t[v * 2 + 1] = t[v];
    marked[v * 2] = marked[v * 2 + 1] = true;
    marked[v] = false;
  }
}
void update(ll v, ll tl, ll tr, ll l, ll r, ll new_val) {
  if (l > r) {
    return;
  }
  if (l == tl && tr == r) {
    t[v] = new_val;
    marked[v] = true;
  } else {
    push(v);
    ll tm = (tl + tr) / 2;
    update(v * 2, tl, tm, l, min(r, tm), new_val);
    update(v * 2 + 1, tm + 1, tr, max(l, tm + 1), r, new_val);
  }
}
ll get(ll v, ll tl, ll tr, ll pos) {
  if (tl == tr) {
    return t[v];
  }
  push(v);
  ll tm = (tl + tr) / 2;
  if (pos <= tm) {
    return get(v * 2, tl, tm, pos);
  } else {
    return get(v * 2 + 1, tm + 1, tr, pos);
  }
}
```

1.4 Sparse Table

```
ll log2_floor(ll i) {
  return i ? __builtin_clzll(1) - __builtin_clzll(i) : -1;
}
vector<vector<ll>> build_sum(ll N, ll K, vector<ll> &array) {
  vector<vector<ll>> st(K + 1, vector<ll>(N + 1));
  for (ll i = 0; i < N; i++) {
    st[0][i] = array[i];
  }
  for (ll i = 1; i <= K; i++) {
    for (ll j = 0; j + (1 << i) <= N; j++) {
      st[i][j] = st[i - 1][j] + st[i - 1][j + (1 << (i - 1))];
    }
  }
  return st;
```

```
}
ll sum_query(ll L, ll R, ll K, vector<vector<ll>> &st) {
  ll sum = 0;
  for (ll i = K; i >= 0; i--) {
    if ((1 << i) <= R - L + 1) {
      sum += st[i][L];
      L += 1 << i;
    }
  }
  return sum;
}
vector<vector<ll>> build_min(ll N, ll K, vector<ll> &array) {
  vector<vector<ll>> st(K + 1, vector<ll>(N + 1));
  for (ll i = 0; i < N; i++) {
    st[0][i] = array[i];
  }
  for (ll i = 1; i <= K; i++) {
    for (ll j = 0; j + (1 << i) <= N; j++) {
      st[i][j] = min(st[i - 1][j], st[i - 1][j + (1 << (i - 1))]);
    }
  }
  return st;
}
ll min_query(ll L, ll R, vector<vector<ll>> &st) {
  ll i = log2_floor(R - L + 1);
  return min(st[i][L], st[i][R - (1 << i) + 1]);
}
```

1.5 Union Find

```
class UF {
private: vector<ll> p;
public:
  UF(ll N) {p.assign(N, -1);}
  ll fs(ll i ) {
    return (p[i] < 0) ? i : (p[i] = fs(p[i]));
  }
  bool isSame(ll i, ll j) {
    return fs(i) == fs(j);
  }
  void join(ll i, ll j) {
    ll x = fs(i), y = fs(j);
    if (x != y){
      if (x < y) {
        p[x] += p[y];
        p[y] = x;
      }
      else {
        p[y] += p[x]; p[x] = y;
      }
    }
  }
};
```

2 Dynamic Programming

2.1 Divide And Conquer

```
ll m, n;
vector<ll> dp_before(n), dp_cur(n);
ll C(ll i, ll j);
void compute(ll l, ll r, ll optl, ll optr) {
  if (l > r) {
    return;
  }
  ll mid = (l + r) >> 1;
  pair<ll, ll> best = {LLONG_MAX, -1};
  for (ll k = optl; k <= min(mid, optr); k++) {
    best = min(best, {(k ? dp_before[k - 1] : 0) + C(k, mid), k});
  }
  dp_cur[mid] = best.first;
  ll opt = best.second;
  compute(l, mid - 1, optl, opt);
  compute(mid + 1, r, opt, optr);
}
ll solve() {
  for (ll i = 0; i < n; i++) {
    dp_before[i] = C(0, i);
  }
  for (ll i = 1; i < m; i++) {
    compute(0, n - 1, 0, n - 1);
    dp_before = dp_cur;
  }
  return dp_before[n - 1];
}
```

2.2 Edit Distance

```
ll edit_distance(string x, string y, ll n, ll m) {
  vector<vector<int>> dp(n + 1, vector<int>(m + 1, INF));
  dp[0][0] = 0;
  for (int i = 1; i <= n; i++) {
    dp[i][0] = i;
  }
  for (int j = 1; j <= m; j++) {
    dp[0][j] = j;
  }
  for (int i = 1; i <= n; i++) {
    for (int j = 1; j <= m; j++) {
      dp[i][j] = min({dp[i - 1][j] + 1, dp[i][j - 1] + 1, dp[i - 1][j - 1] + (x[i - 1] != y[j - 1])});
    }
  }
  return dp[n][m];
}
```

2.3 Knapsack

```
ll knapsack(ll W, vector<ll> &wt, vector<ll> &val, ll n) {
  vector<ll> dp(W + 1, 0);
  for (ll i = 1; i <= n; i++) {
    for (ll w = W; w >= 0; w--) {
      if (wt[i - 1] <= w) {
        dp[w] = max(dp[w], dp[w - wt[i - 1]] + val[i - 1]);
      }
    }
  }
  return dp[W];
}
```

2.4 Knuth Optimization

```
ll solve() {
  ll N;
  // read N and input
  vector<vector<ll>> dp(N, vector<ll>(N)), opt(N, vector<ll>(N));
  auto C = [&](ll i, ll j) {
    // Implement cost function C.
  };
  for (ll i = 0; i < N; i++) {
    opt[i][i] = i;
    ... // Initialize dp[i][i] according to the problem
  }
  for (ll i = N - 2; i >= 0; i--) {
    for (ll j = i + 1; j < N; j++) {
      ll mn = ll_MAX, cost = C(i, j);
      for (ll k = opt[i][j - 1]; k <= min(j - 1, opt[i + 1][j]); k++) {
        if (mn >= dp[i][k] + dp[k + 1][j] + cost) {
          opt[i][j] = k;
          mn = dp[i][k] + dp[k + 1][j] + cost;
        }
      }
      dp[i][j] = mn;
    }
  }
  cout << dp[0][N - 1] << '\n';
}
```

2.5 Longest Common Subsequence

```
ll LCS(string x, string y, ll n, ll m) {
  vector<vector<ll>> dp(n + 1, vector<ll>(m + 1));
  for (ll i = 0; i <= n; i++) {
    for (ll j = 0; j <= m; j++) {
      if (i == 0 || j == 0) {
        dp[i][j] = 0;
      } else if (x[i - 1] == y[j - 1]) {
        dp[i][j] = dp[i - 1][j - 1] + 1;
      } else {
        dp[i][j] = max(dp[i - 1][j], dp[i][j - 1]);
      }
    }
  }
  ll index = dp[n][m];
  vector<char> lcs(index + 1);
  lcs[index] = '\0';
  ll i = n, j = m;
  while (i > 0 && j > 0) {
    if (x[i - 1] == y[j - 1]) {
      lcs[index - 1] = x[i - 1];
      i--;
      j--;
      index--;
    } else if (dp[i - 1][j] > dp[i][j - 1]) {
      i--;
    } else {
      j--;
    }
  }
  return dp[n][m];
}
```

2.6 Longest Increasing Subsequence

```cpp
ll get_ceil_idx(vector<ll> &a, vector<ll> &T, ll l, ll r, ll x) {
  while (r - l > 1) {
    ll m = l + (r - l) / 2;
    if (a[T[m]] >= x) {
      r = m;
    } else {
      l = m;
    }
  }
  return r;
}
ll LIS(ll n, vector<ll> &a) {
  ll len = 1;
  vector<ll> T(n, 0), R(n, - 1);
  T[0] = 0;
  for (ll i = 1; i < n; i++) {
    if (a[i] < a[T[0]]) {
      T[0] = i;
    } else if (a[i] > a[T[len - 1]]) {
      R[i] = T[len - 1];
      T[len++] = i;
    } else {
      ll pos = get_ceil_idx(a, T, -1, len - 1, a[i]);
      R[i] = T[pos - 1];
      T[pos] = i;
    }
  }
  return len;
}
```

2.7 Subset Sum

```cpp
bool subset_sum(ll n, vector<ll> &arr, ll sum) {
  vector<vector<ll>> dp(n + 1, vector<ll>(sum + 1, false));
  dp[0][0] = true;
  for (ll i = 1; i <= n; i++) {
    for (ll j = 0; j <= sum; j++) {
      dp[i][j] = dp[i - 1][j];
      if (j >= arr[i]) {
        dp[i][j] |= dp[i - 1][j - arr[i]];
      }
    }
  }
  return dp[n][sum];
}
```

3 Geometry

3.1 Circle Line Intersection

```cpp
double r, a, b, c; // given as input
double x0 = -a * c / (a * a + b * b);
double y0 = -b * c / (a * a + b * b);
if (c * c > r * r * (a * a + b * b) + EPS) {
  puts ("no points");
} else if (abs (c  *c - r * r * (a * a + b * b)) < EPS) {
  puts ("1 point");
  cout << x0 << ' ' << y0 << '\n';
} else {
  double d = r * r - c * c / (a * a + b * b);
  double mult = sqrt (d / (a * a + b * b));
  double ax, ay, bx, by;
  ax = x0 + b * mult;
  bx = x0 - b * mult;
  ay = y0 - a * mult;
  by = y0 + a * mult;
  puts ("2 points");
  cout << ax << ' ' << ay << '\n' << bx << ' ' << by << '\n';
}
```

3.2 Convex Hull

```cpp
struct pt {
  double x, y;
};
ll orientation(pt a, pt b, pt c) {
  double v = a.x * (b.y - c.y) + b.x * (c.y - a.y) + c.x * (a.y - b.y);
  if (v < 0) {
    return -1;
  } else if (v > 0) {
    return +1;
  }
  return 0;
}
bool cw(pt a, pt b, pt c, bool include_collinear) {
  ll o = orientation(a, b, c);
  return o < 0 || (include_collinear && o == 0);
}
bool collinear(pt a, pt b, pt c) {
  return orientation(a, b, c) == 0;
}
void convex_hull(vector<pt>& a, bool include_collinear = false) {
  pt p0 = *min_element(a.begin(), a.end(), [](pt a, pt b) {
    return make_pair(a.y, a.x) < make_pair(b.y, b.x);
  });
  sort(a.begin(), a.end(), [&p0](const pt& a, const pt& b) {
    ll o = orientation(p0, a, b);
    if (o == 0) {
      return (p0.x - a.x) * (p0.x - a.x) + (p0.y - a.y) * (p0.y - a.y)
        < (p0.x - b.x) * (p0.x - b.x) + (p0.y - b.y) * (p0.y - b.y);
    }
    return o < 0;
  });
  if (include_collinear) {
    ll i = (ll) a.size()-1;
    while (i >= 0 && collinear(p0, a[i], a.back())) i--;
    reverse(a.begin()+i+1, a.end());
  }
  vector<pt> st;
  for (ll i = 0; i < (ll) a.size(); i++) {
    while (st.size() > 1 && !cw(st[st.size() - 2], st.back(), a[i], include_collinear)) {
      st.pop_back();
    }
    st.push_back(a[i]);
  }
  a = st;
}
```

3.3 Line Sweep

```cpp
const double EPS = 1E-9;
struct pt {
  double x, y;
};
struct seg {
  pt p, q;
  ll id;
  double get_y(double x) const {
    if (abs(p.x - q.x) < EPS) {
      return p.y;
    }
    return p.y + (q.y - p.y) * (x - p.x) / (q.x - p.x);
  }
};
bool intersect1d(double l1, double r1, double l2, double r2) {
  if (l1 > r1) {
    swap(l1, r1);
  }
  if (l2 > r2) {
    swap(l2, r2);
  }
  return max(l1, l2) <= min(r1, r2) + EPS;
}
ll vec(const pt& a, const pt& b, const pt& c) {
  double s = (b.x - a.x) * (c.y - a.y) - (b.y - a.y) * (c.x - a.x);
  return abs(s) < EPS ? 0 : s > 0 ? +1 : -1;
}
bool intersect(const seg& a, const seg& b) {
  return intersect1d(a.p.x, a.q.x, b.p.x, b.q.x) &&
         intersect1d(a.p.y, a.q.y, b.p.y, b.q.y) &&
         vec(a.p, a.q, b.p) * vec(a.p, a.q, b.q) <= 0 &&
         vec(b.p, b.q, a.p) * vec(b.p, b.q, a.q) <= 0;
}
bool operator<(const seg& a, const seg& b) {
    double x = max(min(a.p.x, a.q.x), min(b.p.x, b.q.x));
    return a.get_y(x) < b.get_y(x) - EPS;
}
struct event {
  double x;
  ll tp, id;
  event() {}
  event(double x, ll tp, ll id) : x(x), tp(tp), id(id) {}
  bool operator<(const event& e) const {
    if (abs(x - e.x) > EPS) {
      return x < e.x;
    }
    return tp > e.tp;
  }
};
set<seg> s;
vector<set<seg>::iterator> where;
set<seg>::iterator prev(set<seg>::iterator it) {
  return it == s.begin() ? s.end() : --it;
```

```
}
set<seg>::iterator next(set<seg>::iterator it) {
  return ++it;
}
pair<ll, ll> solve(const vector<seg>& a) {
  ll n = (ll) a.size();
  vector<event> e;
  for (ll i = 0; i < n; ++i) {
    e.push_back(event(min(a[i].p.x, a[i].q.x), +1, i));
    e.push_back(event(max(a[i].p.x, a[i].q.x), -1, i));
  }
  sort(e.begin(), e.end());
  s.clear();
  where.resize(a.size());
  for (size_t i = 0; i < e.size(); ++i) {
    ll id = e[i].id;
    if (e[i].tp == +1) {
      set<seg>::iterator nxt = s.lower_bound(a[id]), prv = prev(nxt);
      if (nxt != s.end() && intersect(*nxt, a[id])) {
        return make_pair(nxt->id, id);
      }
      if (prv != s.end() && intersect(*prv, a[id])) {
        return make_pair(prv->id, id);
      }
      where[id] = s.insert(nxt, a[id]);
    } else {
      set<seg>::iterator nxt = next(where[id]), prv = prev(where[id]);
      if (nxt != s.end() && prv != s.end() && intersect(*nxt, *prv)) {
        return make_pair(prv->id, nxt->id);
      }
      s.erase(where[id]);
    }
  }
  return make_pair(-1, -1);
}
```

3.4 Nearest Points

```
struct pt {
  ll x, y, id;
};
struct cmp_x {
  bool operator()(const pt & a, const pt & b) const {
    return a.x < b.x || (a.x == b.x && a.y < b.y);
  }
};
struct cmp_y {
  bool operator()(const pt & a, const pt & b) const {
    return a.y < b.y;
  }
};
ll n;
vector<pt> a;
double mindist;
pair<ll, ll> best_pair;
void upd_ans(const pt & a, const pt & b) {
  double dist = sqrt((a.x - b.x) * (a.x - b.x) + (a.y - b.y) * (a.y - b.y));
  if (dist < mindist) {
    mindist = dist;
    best_pair = {a.id, b.id};
  }
}
vector<pt> t;
void rec(ll l, ll r) {
  if (r - l <= 3) {
    for (ll i = l; i < r; ++i) {
      for (ll j = i + 1; j < r; ++j) {
        upd_ans(a[i], a[j]);
      }
    }
    sort(a.begin() + l, a.begin() + r, cmp_y());
    return;
  }
  ll m = (l + r) >> 1, midx = a[m].x;
  rec(l, m);
  rec(m, r);
  merge(a.begin() + l, a.begin() + m, a.begin() + m, a.begin() + r, t.begin(), cmp_y());
  copy(t.begin(), t.begin() + r - l, a.begin() + l);
  ll tsz = 0;
  for (ll i = l; i < r; ++i) {
    if (abs(a[i].x - midx) < mindist) {
      for (ll j = tsz - 1; j >= 0 && a[i].y - t[j].y < mindist; --j) {
        upd_ans(a[i], t[j]);
      }
      t[tsz++] = a[i];
    }
  }
}
t.resize(n);
sort(a.begin(), a.end(), cmp_x());
mindist = 1E20;
rec(0, n);
```

4 Graph Theory

4.1 Articulation Point

```
void APUtil(vector<vector<ll>> &adj, ll u, vector<bool> &visited,
vector<ll> &disc, vector<ll> &low, ll &time, ll parent, vector<bool> &isAP) {
  ll children = 0;
  visited[u] = true;
  disc[u] = low[u] = ++time;
  for (auto v : adj[u]) {
    if (!visited[v]) {
      children++;
      APUtil(adj, v, visited, disc, low, time, u, isAP);
      low[u] = min(low[u], low[v]);
      if (parent != -1 && low[v] >= disc[u]) {
        isAP[u] = true;
      }
    } else if (v != parent) {
      low[u] = min(low[u], disc[v]);
    }
  }
  if (parent == -1 && children > 1) {
    isAP[u] = true;
  }
}
void AP(vector<vector<ll>> &adj, ll n) {
  vector<ll> disc(n), low(n);
  vector<bool> visited(n), isAP(n);
  ll time = 0, par = -1;
  for (ll u = 0; u < n; u++) {
    if (!visited[u]) {
      APUtil(adj, u, visited, disc, low, time, par, isAP);
    }
  }
  for (ll u = 0; u < n; u++) {
    if (isAP[u]) {
      cout << u << " ";
    }
  }
}
```

4.2 Bellman Ford

```
void bellman_ford(vector<vector<ll>> &edges, ll n, ll m, ll src, vector<ll> &dis) {
  for (ll i = 0; i < n; i++) {
    dis[i] = INF;
  }
  for (ll i = 0; i < n - 1; i++) {
    for (ll j = 0; j < m; j++) {
      ll u = edges[j][0], v = edges[j][1], w = edges[j][2];
      if (dis[u] < INF) {
        dis[v] = min(dis[v], dis[u] + w);
      }
    }
  }
  for (ll i = 0; i < m; i++) {
    ll u = edges[i][0], v = edges[i][1], w = edges[i][2];
    if (dis[u] < INF && dis[u] + w < dis[v]) {
      cout << "The graph contains a negative cycle." << '\n';
    }
  }
}
```

4.3 Bridge

```
void bridge_util(vector<vector<ll>> &adj, ll u, vector<bool> &visited, vector<ll> &disc, vector<ll> &low, vector<ll> &parent) {
  static ll time = 0;
  visited[u] = true;
  disc[u] = low[u] = ++time;
  list<ll>::iterator i;
  for (auto v : adj[u]) {
    if (!visited[v]) {
      parent[v] = u;
      bridge_util(adj, v, visited, disc, low, parent);
      low[u] = min(low[u], low[v]);
      if (low[v] > disc[u]) {
        cout << u << ' ' << v << '\n';
      }
    } else if (v != parent[u]) {
      low[u] = min(low[u], disc[v]);
    }
```

```
    }
}
void bridge(vector<vector<ll>> &adj, ll n) {
  vector<bool> visited(n, false);
  vector<ll> disc(n), low(n), parent(n, -1);
  for (ll i = 0; i < n; i++) {
    if (!visited[i]) {
      bridge_util(adj, i, visited, disc, low, parent);
    }
  }
}
```

4.4 Dijkstra

```
void dijkstra(ll n, vector<vector<pair<ll, ll>>> &adj, vector<ll> &dis) {
  priority_queue<pair<ll, ll>, vector<pair<ll, ll>>, greater<pair<ll, ll>>> pq;
  for (int i = 0; i < n; i++) {
    dis[i] = INF;
  }
  dis[0] = 0;
  pq.push({0, 0});
  while (!pq.empty()) {
    auto p = pq.top();
    pq.pop();
    ll u = p.second;
    if (dis[u] != p.first) {
      continue;
    }
    for (auto x : adj[u]) {
      ll v = x.first, w = x.second;
      if (dis[v] > dis[u] + w) {
        dis[v] = dis[u] + w;
        pq.push({dis[v], v});
      }
    }
  }
}
```

4.5 Find Cycle

```
bool dfs(ll v) {
  color[v] = 1;
  for (ll u : adj[v]) {
    if (color[u] == 0) {
      parent[u] = v;
      if (dfs(u)) {
        return true;
      }
    } else if (color[u] == 1) {
      cycle_end = v;
      cycle_start = u;
      return true;
    }
  }
  color[v] = 2;
  return false;
}
void find_cycle() {
  color.assign(n, 0);
  parent.assign(n, -1);
  cycle_start = -1;
  for (ll v = 0; v < n; v++) {
    if (color[v] == 0 && dfs(v)) {
      break;
    }
  }
  if (cycle_start == -1) {
    cout << "Acyclic" << endl;
  } else {
    vector<ll> cycle;
    cycle.push_back(cycle_start);
    for (ll v = cycle_end; v != cycle_start; v = parent[v]) {
      cycle.push_back(v);
    }
    cycle.push_back(cycle_start);
    reverse(cycle.begin(), cycle.end());
    cout << "Cycle found: ";
    for (ll v : cycle) {
      cout << v << ' ';
    }
    cout << '\n';
  }
}
```

4.6 Floyd Warshall

```
void floyd_warshall(vector<vector<ll>> &dis, ll n) {
  for (ll i = 0; i < n; i++) {
    for (ll j = 0; j < n; j++) {
      dis[i][j] = (i == j ? 0 : INF);
    }
  }
  for (ll k = 0; k < n; k++) {
    for (ll i = 0; i < n; i++) {
      for (ll j = 0; j < n; j++) {
        if (dis[i][k] < INF && dis[k][j] < INF) {
          dis[i][j] = min(dis[i][j], dis[i][k] + dis[k][j]);
        }
      }
    }
  }
  for (ll i = 0; i < n; i++) {
    for (ll j = 0; j < n; j++) {
      for (ll k = 0; k < n; k++) {
        if (dis[k][k] < 0 && dis[i][k] < INF && dis[k][j] < INF) {
          dis[i][j] = -INF;
        }
      }
    }
  }
}
```

4.7 Hierholzer

```
void print_circuit(vector<vector<ll>> &adj) {
  map<ll, ll> edge_count;
  for (ll i = 0; i< adj.size(); i++) {
    edge_count[i] = adj[i].size();
  }
  if (!adj.size()) {
    return;
  }
  stack<ll> curr_path;
  vector<ll> circuit;
  curr_path.push(0);
  ll curr_v = 0;
  while (!curr_path.empty()) {
    if (edge_count[curr_v]) {
      curr_path.push(curr_v);
      ll next_v = adj[curr_v].back();
      edge_count[curr_v]--;
      adj[curr_v].pop_back();
      curr_v = next_v;
    } else {
      circuit.push_back(curr_v);
      curr_v = curr_path.top();
      curr_path.pop();
    }
  }
  for (ll i = circuit.size() - 1; i >= 0; i--) {
    cout << circuit[i] << ' ';
  }
}
```

4.8 Is Bipartite

```
bool is_bipartite(vector<ll> &col, vector<vector<ll>> &adj, ll n) {
  queue<pair<ll, ll>> q;
  for (ll i = 0; i < n; i++) {
    if (col[i] == -1) {
      q.push({i, 0});
      col[i] = 0;
      while (!q.empty()) {
        pair<ll, ll> p = q.front();
        q.pop();
        ll v = p.first, c = p.second;
        for (ll j : adj[v]) {
          if (col[j] == c) {
            return false;
          }
          if (col[j] == -1) {
            col[j] = (c ? 0 : 1);
            q.push({j, col[j]});
          }
        }
      }
    }
  }
  return true;
}
```

4.9 Is Cyclic

```
bool is_cyclic_util(int u, vector<vector<int>> &adj, vector<bool> &vis, vector<bool> &rec) {
  vis[u] = true;
  rec[u] = true;
  for(auto v : adj[u]) {
    if (!vis[v] && is_cyclic_util(v, adj, vis, rec)) {
      return true;
    } else if (rec[v]) {
      return true;
    }
  }
  rec[u] = false;
  return false;
```

```cpp
}
bool is_cyclic(int n, vector<vector<int>> &adj) {
  vector<bool> vis(n, false), rec(n, false);
  for (int i = 0; i < n; i++) {
    if (!vis[i] && is_cyclic_util(i, adj, vis, rec)) {
      return true;
    }
  }
  return false;
}
```

4.10 Kahn

```cpp
void kahn(vector<vector<ll>> &adj) {
  ll n = adj.size();
  vector<ll> in_degree(n, 0);
  for (ll u = 0; u < n; u++) {
    for (ll v: adj[u]) {
      in_degree[v]++;
    }
  }
  queue<ll> q;
  for (ll i = 0; i < n; i++) {
    if (in_degree[i] == 0) {
      q.push(i);
    }
  }
  ll cnt = 0;
  vector<ll> top_order;
  while (!q.empty()) {
    ll u = q.front();
    q.pop();
    top_order.push_back(u);
    for (ll v : adj[u]) {
      if (--in_degree[v] == 0) {
        q.push(v);
      }
    }
    cnt++;
  }
  if (cnt != n) {
    cout << -1 << '\n';
    return;
  }
  for (ll i = 0; i < (ll) top_order.size(); i++) {
    cout << top_order[i] << ' ';
  }
  cout << '\n';
}
```

4.11 Kruskal Mst

```cpp
struct Edge {
  ll u, v, weight;
  bool operator<(Edge const& other) {
    return weight < other.weight;
  }
};
ll n;
vector<Edge> edges;
ll cost = 0;
vector<ll> tree_id(n);
vector<Edge> result;
for (ll i = 0; i < n; i++) {
  tree_id[i] = i;
}
sort(edges.begin(), edges.end());
for (Edge e : edges) {
  if (tree_id[e.u] != tree_id[e.v]) {
    cost += e.weight;
    result.push_back(e);
    ll old_id = tree_id[e.u], new_id = tree_id[e.v];
    for (ll i = 0; i < n; i++) {
      if (tree_id[i] == old_id) {
        tree_id[i] = new_id;
      }
    }
  }
}
```

4.12 Lowest Common Ancestor

```cpp
struct LCA {
    vector<ll> height, euler, first, segtree;
    vector<bool> visited;
    ll n;
    LCA(vector<vector<ll>> &adj, ll root = 0) {
      n = adj.size();
      height.resize(n);
      first.resize(n);
      euler.reserve(n * 2);
      visited.assign(n, false);
      dfs(adj, root);
      ll m = euler.size();
      segtree.resize(m * 4);
      build(1, 0, m - 1);
    }
    void dfs(vector<vector<ll>> &adj, ll node, ll h = 0) {
      visited[node] = true;
      height[node] = h;
      first[node] = euler.size();
      euler.push_back(node);
      for (auto to : adj[node]) {
        if (!visited[to]) {
          dfs(adj, to, h + 1);
          euler.push_back(node);
        }
      }
    }
    void build(ll node, ll b, ll e) {
      if (b == e) {
        segtree[node] = euler[b];
      } else {
        ll mid = (b + e) / 2;
        build(node << 1, b, mid);
        build(node << 1 | 1, mid + 1, e);
        ll l = segtree[node << 1], r = segtree[node << 1 | 1];
        segtree[node] = (height[l] < height[r]) ? l : r;
      }
    }
    ll query(ll node, ll b, ll e, ll L, ll R) {
      if (b > R || e < L) {
        return -1;
      }
      if (b >= L && e <= R) {
        return segtree[node];
      }
      ll mid = (b + e) >> 1;
      ll left = query(node << 1, b, mid, L, R);
      ll right = query(node << 1 | 1, mid + 1, e, L, R);
      if (left == -1) return right;
      if (right == -1) return left;
      return height[left] < height[right] ? left : right;
    }
    ll lca(ll u, ll v) {
      ll left = first[u], right = first[v];
      if (left > right) {
        swap(left, right);
      }
      return query(1, 0, euler.size() - 1, left, right);
    }
};
```

4.13 Maximum Bipartite Matching

```cpp
bool bpm(ll n, ll m, vector<vector<bool>> &bpGraph, ll u, vector<bool> &seen, vector<ll> &matchR) {
  for (ll v = 0; v < m; v++) {
    if (bpGraph[u][v] && !seen[v]) {
      seen[v] = true;
      if (matchR[v] < 0 || bpm(n, m, bpGraph, matchR[v], seen, matchR)) {
        matchR[v] = u;
        return true;
      }
    }
  }
  return false;
}
ll maxBPM(ll n, ll m, vector<vector<bool>> &bpGraph) {
  vector<ll> matchR(m, -1);
  ll result = 0;
  for (ll u = 0; u < n; u++) {
    vector<bool> seen(m, false);
    if (bpm(n, m, bpGraph, u, seen, matchR)) {
      result++;
    }
  }
  return result;
}
```

4.14 Max Flow

```cpp
bool bfs(ll n, vector<vector<ll>> &r_graph, ll s, ll t, vector<ll> &parent) {
  vector<bool> visited(n, false);
  queue<ll> q;
  q.push(s);
  visited[s] = true;
  parent[s] = -1;
  while (!q.empty()) {
    ll u = q.front();
    q.pop();
    for (ll v = 0; v < n; v++) {
      if (!visited[v] && r_graph[u][v] > 0) {
        if (v == t) {
          parent[v] = u;
```

```
          return true;
        }
        q.push(v);
        parent[v] = u;
        visited[v] = true;
      }
    }
  }
  return false;
}
ll fordFulkerson(ll n, vector<vector<ll>> graph, ll s, ll t) {
  ll u, v;
  vector<vector<ll>> r_graph;
  for (u = 0; u < n; u++) {
    for (v = 0; v < n; v++) {
      r_graph[u][v] = graph[u][v];
    }
  }
  vector<ll> parent;
  ll max_flow = 0;
  while (bfs(n, r_graph, s, t, parent)) {
    ll path_flow = INF;
    for (v = t; v != s; v = parent[v]) {
      u = parent[v];
      path_flow = min(path_flow, r_graph[u][v]);
    }
    for (v = t; v != s; v = parent[v]) {
      u = parent[v];
      r_graph[u][v] -= path_flow;
      r_graph[v][u] += path_flow;
    }
    max_flow += path_flow;
  }
  return max_flow;
}
```

4.15 Prim Mst

```
vector<ll> prim_mst(ll n, vector<vector<pair<ll, ll>>> &adj) {
  priority_queue<pair<ll, ll>, vector<pair<ll, ll>>, greater<pair<ll, ll>>> pq;
  ll src = 0;
  vector<ll> key(n, INF), parent(n, -1);
  vector<bool> in_mst(n, false);
  pq.push(make_pair(0, src));
  key[src] = 0;
  while (!pq.empty()) {
    ll u = pq.top().second;
    pq.pop();
    if (in_mst[u]){
      continue;
    }
    in_mst[u] = true;
    for (auto p : adj[u]) {
      ll v = p.first, w = p.second;
      if (in_mst[v] == false && w < key[v]) {
        key[v] = w;
        pq.push(make_pair(key[v], v));
        parent[v] = u;
      }
    }
  }
  return parent;
}
```

4.16 Strongly Connected Component

```
void dfs(ll u, vector<vector<ll>> &adj, vector<bool> &visited) {
  visited[u] = true;
  cout << u + 1 << ' ';
  for (ll v : adj[u]) {
    if (!visited[v]) {
      dfs(v, adj, visited);
    }
  }
}
vector<vector<ll>> get_transpose(ll n, vector<vector<ll>> &adj) {
  vector<vector<ll>> res(n);
  for (ll u = 0; u < n; u++) {
    for (ll v : adj[u]) {
      res[v].push_back(u);
    }
  }
  return res;
}
void fill_order(ll u, vector<vector<ll>> &adj, vector<bool> &visited, stack<ll> &stk) {
  visited[u] = true;
  for(auto v : adj[u]) {
    if(!visited[v]) {
      fill_order(v, adj, visited, stk);
    }
  }
  stk.push(u);
}
void get_scc(ll n, vector<vector<ll>> &adj) {
  stack<ll> stk;
  vector<bool> visited(n, false);
  for (ll i = 0; i < n; i++) {
    if (!visited[i]) {
      fill_order(i, adj, visited, stk);
    }
  }
  vector<vector<ll>> transpose = get_transpose(n, adj);
  for (ll i = 0; i < n; i++) {
    visited[i] = false;
  }
  while (!stk.empty()) {
    ll u = stk.top();
    stk.pop();
    if (!visited[u]) {
      dfs(u, transpose, visited);
      cout << '\n';
    }
  }
}
```

4.17 Topological Sort

```
void dfs(ll v) {
  visited[v] = true;
  for (ll u : adj[v]) {
    if (!visited[u]) {
      dfs(u);
    }
  }
  ans.push_back(v);
}
void topological_sort() {
  visited.assign(n, false);
  ans.clear();
  for (ll i = 0; i < n; ++i) {
    if (!visited[i]) {
      dfs(i);
    }
  }
  reverse(ans.begin(), ans.end());
}
```

5 Miscellaneous

5.1 Gauss

```
const double EPS = 1e-9;
const ll INF = 2;
ll gauss(vector <vector<double>> a, vector<double> &ans) {
  ll n = (ll) a.size(), m = (ll) a[0].size() - 1;
  vector<ll> where (m, -1);
  for (ll col = 0, row = 0; col < m && row < n; ++col) {
    ll sel = row;
    for (ll i = row; i < n; ++i) {
      if (abs(a[i][col]) > abs(a[sel][col])) {
        sel = i;
      }
    }
    if (abs (a[sel][col]) < EPS) {
      continue;
    }
    for (ll i = col; i <= m; ++i) {
      swap(a[sel][i], a[row][i]);
    }
    where[col] = row;
    for (ll i = 0; i < n; ++i) {
      if (i != row) {
        double c = a[i][col] / a[row][col];
        for (ll j = col; j <= m; ++j) {
          a[i][j] -= a[row][j] * c;
        }
      }
    }
    ++row;
  }
  ans.assign(m, 0);
  for (ll i = 0; i < m; ++i) {
    if (where[i] != -1) {
      ans[i] = a[where[i]][m] / a[where[i]][i];
    }
  }
  for (ll i = 0; i < n; ++i) {
    double sum = 0;
    for (ll j = 0; j < m; ++j) {
      sum += ans[j] * a[i][j];
    }
    if (abs (sum - a[i][m]) > EPS) {
      return 0;
    }
  }
  for (ll i = 0; i < m; ++i) {
    if (where[i] == -1) {
      return INF;
    }
```

```
  }
  return 1;
}
```

5.2 Ternary Search

```
double ternary_search(double l, double r) {
  double eps = 1e-9;
  while (r - l > eps) {
    double m1 = l + (r - l) / 3;
    double m2 = r - (r - l) / 3;
    double f1 = f(m1);
    double f2 = f(m2);
    if (f1 < f2) {
      l = m1;
    } else {
      r = m2;
    }
  }
  return f(l);
}
```

6 Number Theory

6.1 Extended Euclidean

```
ll gcd_extended(ll a, ll b, ll &x, ll &y) {
  if (b == 0) {
    x = 1;
    y = 0;
    return a;
  }
  ll x1, y1, g = gcd_extended(b, a % b, x1, y1);
  x = y1;
  y = x1 - (a / b) * y1;
  return g;
}
```

6.2 Find All Solutions

```
bool find_any_solution(ll a, ll b, ll c, ll &x0, ll &y0, ll &g) {
  g = gcd_extended(abs(a), abs(b), x0, y0);
  if (c % g) {
    return false;
  }
  x0 *= c / g;
  y0 *= c / g;
  if (a < 0) {
    x0 = -x0;
  }
  if (b < 0) {
    y0 = -y0;
  }
  return true;
}
void shift_solution(ll & x, ll & y, ll a, ll b, ll cnt) {
  x += cnt * b;
  y -= cnt * a;
}
ll find_all_solutions(ll a, ll b, ll c, ll minx, ll maxx, ll miny, ll maxy) {
  ll x, y, g;
  if (!find_any_solution(a, b, c, x, y, g)) {
    return 0;
  }
  a /= g;
  b /= g;
  ll sign_a = a > 0 ? +1 : -1;
  ll sign_b = b > 0 ? +1 : -1;
  shift_solution(x, y, a, b, (minx - x) / b);
  if (x < minx) {
    shift_solution(x, y, a, b, sign_b);
  }
  if (x > maxx) {
    return 0;
  }
  ll lx1 = x;
  shift_solution(x, y, a, b, (maxx - x) / b);
  if (x > maxx) {
    shift_solution(x, y, a, b, -sign_b);
  }
  ll rx1 = x;
  shift_solution(x, y, a, b, -(miny - y) / a);
  if (y < miny) {
    shift_solution(x, y, a, b, -sign_a);
  }
  if (y > maxy) {
    return 0;
  }
  ll lx2 = x;
  shift_solution(x, y, a, b, -(maxy - y) / a);
  if (y > maxy) {
    shift_solution(x, y, a, b, sign_a);
  }
  ll rx2 = x;
  if (lx2 > rx2) {
    swap(lx2, rx2);
  }
  ll lx = max(lx1, lx2), rx = min(rx1, rx2);
  if (lx > rx) {
    return 0;
  }
  return (rx - lx) / abs(b) + 1;
}
```

6.3 Linear Sieve

```
void linear_sieve(ll N, vector<ll> &lowest_prime, vector<ll> &prime) {
  for (ll i = 2; i <= N; i++) {
    if (lowest_prime[i] == 0) {
      lowest_prime[i] = i;
      prime.push_back(i);
    }
    for (ll j = 0; i * prime[j] <= N; j++) {
      lowest_prime[i * prime[j]] = prime[j];
      if (prime[j] == lowest_prime[i]) {
        break;
      }
    }
  }
}
```

6.4 Miller Rabin

```
bool check_composite(u64 n, u64 a, u64 d, ll s) {
  u64 x = binpower(a, d, n);
  if (x == 1 || x == n - 1) {
    return false;
  }
  for (ll r = 1; r < s; r++) {
    x = (u128) x * x % n;
    if (x == n - 1) {
      return false;
    }
  }
  return true;
}
bool miller_rabin(u64 n) {
  if (n < 2) {
    return false;
  }
  ll r = 0;
  u64 d = n - 1;
  while ((d & 1) == 0) {
    d >>= 1;
    r++;
  }
  for (ll a : {2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37}) {
    if (n == a) {
      return true;
    }
    if (check_composite(n, a, d, r)) {
      return false;
    }
  }
  return true;
}
```

6.5 Modulo Inverse

```
ll mod_inv(ll a, ll m) {
  if (m == 1) {
    return 0;
  }
  ll m0 = m, x = 1, y = 0;
  while (a > 1) {
    ll q = a / m, t = m;
    m = a % m;
    a = t;
    t = y;
    y = x - q * y;
    x = t;
  }
  if (x < 0) {
    x += m0;
  }
  return x;
}
```

6.6 Pollard Rho Brent

```
ll mult(ll a, ll b, ll mod) {
  return (__int128_t) a * b % mod;
}
ll f(ll x, ll c, ll mod) {
  return (mult(x, x, mod) + c) % mod;
}
ll pollard_rho_brent(ll n, ll x0 = 2, ll c = 1) {
```

```
  ll x = x0, g = 1, q = 1, xs, y, m = 128, l = 1;
  while (g == 1) {
    y = x;
    for (ll i = 1; i < l; i++) {
      x = f(x, c, n);
    }
    ll k = 0;
    while (k < l && g == 1) {
      xs = x;
      for (ll i = 0; i < m && i < l - k; i++) {
        x = f(x, c, n);
        q = mult(q, abs(y - x), n);
      }
      g = __gcd(q, n);
      k += m;
    }
    l *= 2;
  }
  if (g == n) {
    do {
      xs = f(xs, c, n);
      g = __gcd(abs(xs - y), n);
    } while (g == 1);
  }
  return g;
}
```

6.7 Range Sieve

```
vector<bool> range_sieve(ll l, ll r) {
  ll n = sqrt(r);
  vector<bool> is_prime(n + 1, true);
  vector<ll> prime;
  is_prime[0] = is_prime[1] = false;
  prime.push_back(2);
  for (ll i = 4; i <= n; i += 2) {
    is_prime[i] = false;
  }
  for (ll i = 3; i <= n; i += 2) {
    if (is_prime[i]) {
      prime.push_back(i);
      for (ll j = i * i; j <= n; j += i) {
        is_prime[j] = false;
      }
    }
  }
  vector<bool> result(r - l + 1, true);
  for (ll i : prime) {
    for (ll j = max(i * i, (l + i - 1) / i * i); j <= r; j += i) {
      result[j - l] = false;
    }
  }
  if (l == 1) {
    result[0] = false;
  }
  return result;
}
```

6.8 Segmented Sieve

```
vector<ll> segmented_sieve(ll n) {
  const ll S = 10000;
  ll nsqrt = sqrt(n);
  vector<char> is_prime(nsqrt + 1, true);
  vector<ll> prime;
  is_prime[0] = is_prime[1] = false;
  prime.push_back(2);
  for (ll i = 4; i <= nsqrt; i += 2) {
    is_prime[i] = false;
  }
  for (ll i = 3; i <= nsqrt; i += 2) {
    if (is_prime[i]) {
      prime.push_back(i);
      for (ll j = i * i; j <= nsqrt; j += i) {
        is_prime[j] = false;
      }
    }
  }
  vector<ll> result;
  vector<char> block(S);
  for (ll k = 0; k * S <= n; k++) {
    fill(block.begin(), block.end(), true);
    for (ll p : prime) {
      for (ll j = max((k * S + p - 1) / p, p) * p - k * S; j < S; j += p) {
        block[j] = false;
      }
    }
    if (k == 0) {
      block[0] = block[1] = false;
    }
    for (ll i = 0; i < S && k * S + i <= n; i++) {
      if (block[i]) {
        result.push_back(k * S + i);
      }
    }
  }
  return result;
}
```

6.9 Tonelli Shanks

```
ll legendre(ll a, ll p) {
  return bin_pow_mod(a, (p - 1) / 2, p);
}
ll tonelli_shanks(ll n, ll p) {
  if (legendre(n, p) == p - 1) {
    return -1;
  }
  if (p % 4 == 3) {
    return bin_pow_mod(n, (p + 1) / 4, p);
  }
  ll Q = p - 1, S = 0;
  while (Q % 2 == 0) {
    Q /= 2;
    S++;
  }
  ll z = 2;
  for (; z < p; z++) {
    if (legendre(z, p) == p - 1) {
      break;
    }
  }
  ll M = S, c = bin_pow_mod(z, Q, p), t = bin_pow_mod(n, Q, p), R = bin_pow_mod(n, (Q + 1) / 2, p);
  while (t % p != 1) {
    if (t % p == 0) {
      return 0;
    }
    ll i = 1, t2 = t * t % p;
    for (; i < M; i++) {
      if (t2 % p == 1) {
        break;
      }
      t2 = t2 * t2 % p;
    }
    ll b = bin_pow_mod(c, bin_pow_mod(2, M - i - 1, p), p);
    M = i;
    c = b * b % p;
    t = t * c % p;
    R = R * b % p;
  }
  return R;
}
```

7 Strings

7.1 Hashing

```
ll compute_hash(string const& s) {
  const ll p = 31, m = 1e9 + 9;
  ll hash_value = 0, p_pow = 1;
  for (char c : s) {
    hash_value = (hash_value + (c - 'a' + 1) * p_pow) % m;
    p_pow = (p_pow * p) % m;
  }
  return hash_value;
}
```

7.2 Knuth Morris Pratt

```
vector<ll> prefix_function(string s) {
  ll n = (ll) s.length();
  vector<ll> pi(n);
  for (ll i = 1; i < n; i++) {
    ll j = pi[i - 1];
    while (j > 0 && s[i] != s[j]) {
      j = pi[j - 1];
    }
    if (s[i] == s[j]) {
      j++;
    }
    pi[i] = j;
  }
  return pi;
}
```

7.3 Rabin Karp

```
vector<ll> rabin_karp(string const& s, string const& t) {
  const ll p = 31, m = 1e9 + 9;
  ll S = s.size(), T = t.size();
  vector<ll> p_pow(max(S, T));
  p_pow[0] = 1;
  for (ll i = 1; i < (ll) p_pow.size(); i++) {
    p_pow[i] = (p_pow[i-1] * p) % m;
  }
  vector<ll> h(T + 1, 0);
  for (ll i = 0; i < T; i++) {
```

```
    h[i + 1] = (h[i] + (t[i] - 'a' + 1) * p_pow[i]) % m;
  }
  ll h_s = 0;
  for (ll i = 0; i < S; i++) {
    h_s = (h_s + (s[i] - 'a' + 1) * p_pow[i]) % m;
  }
  vector<ll> occurences;
  for (ll i = 0; i + S - 1 < T; i++) {
    ll cur_h = (h[i + S] + m - h[i]) % m;
    if (cur_h == h_s * p_pow[i] % m) {
      occurences.push_back(i);
    }
  }
  return occurences;
}
```

7.4 Suffix Array

```
vector<ll> sort_cyclic_shifts(string const& s) {
  ll n = s.size();
  const ll alphabet = 256;
  vector<ll> p(n), c(n), cnt(max(alphabet, n), 0);
  for (ll i = 0; i < n; i++) {
    cnt[s[i]]++;
  }
  for (ll i = 1; i < alphabet; i++) {
    cnt[i] += cnt[i - 1];
  }
  for (ll i = 0; i < n; i++) {
    p[--cnt[s[i]]] = i;
  }
  c[p[0]] = 0;
  ll classes = 1;
  for (ll i = 1; i < n; i++) {
    if (s[p[i]] != s[p[i - 1]]) {
      classes++;
    }
    c[p[i]] = classes - 1;
  }
  vector<ll> pn(n), cn(n);
  for (ll h = 0; (1 << h) < n; ++h) {
    for (ll i = 0; i < n; i++) {
      pn[i] = p[i] - (1 << h);
      if (pn[i] < 0) {
        pn[i] += n;
      }
    }
    fill(cnt.begin(), cnt.begin() + classes, 0);
    for (ll i = 0; i < n; i++) {
      cnt[c[pn[i]]]++;
    }
    for (ll i = 1; i < classes; i++) {
      cnt[i] += cnt[i - 1];
    }
    for (ll i = n-1; i >= 0; i--) {
      p[--cnt[c[pn[i]]]] = pn[i];
    }
    cn[p[0]] = 0;
    classes = 1;
    for (ll i = 1; i < n; i++) {
      pair<ll, ll> cur = {c[p[i]], c[(p[i] + (1 << h)) % n]};
      pair<ll, ll> prev = {c[p[i - 1]], c[(p[i - 1] + (1 << h)) % n]};
      if (cur != prev) {
        ++classes;
      }
      cn[p[i]] = classes - 1;
    }
    c.swap(cn);
  }
  return p;
}
vector<ll> build_suff_arr(string s) {
  s += (char) 0;
  vector<ll> sorted_shifts = sort_cyclic_shifts(s);
  sorted_shifts.erase(sorted_shifts.begin());
  return sorted_shifts;
}
```

7.5 Z Function

```
vector<ll> z_function(string s) {
  ll n = (ll) s.length();
  vector<ll> z(n);
  for (ll i = 1, l = 0, r = 0; i < n; ++i) {
    if (i <= r) {
      z[i] = min (r - i + 1, z[i - l]);
    }
    while (i + z[i] < n && s[z[i]] == s[i + z[i]]) {
      ++z[i];
    }
    if (i + z[i] - 1 > r) {
      l = i, r = i + z[i] - 1;
    }
  }
  return z;
}
```

$f(n) = O(g(n))$	iff $\exists$ positive c, n_0 such that $0 \le f(n) \le cg(n)\ \forall n \ge n_0$.
$f(n) = \Omega(g(n))$	iff $\exists$ positive c, n_0 such that $f(n) \ge cg(n) \ge 0\ \forall n \ge n_0$.
$f(n) = \Theta(g(n))$	iff $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$.
$f(n) = o(g(n))$	iff $\lim_{n\to\infty} f(n)/g(n) = 0$.
$\lim\limits_{n\to\infty} a_n = a$	iff $\forall \epsilon > 0$, $\exists n_0$ such that $\lvert a_n - a\rvert < \epsilon$, $\forall n \ge n_0$.
$\sup S$	least $b \in \mathbb{R}$ such that $b \ge s$, $\forall s \in S$.
$\inf S$	greatest $b \in \mathbb{R}$ such that $b \le s$, $\forall s \in S$.
$\liminf\limits_{n\to\infty} a_n$	$\lim\limits_{n\to\infty} \inf\{a_i \mid i \ge n, i \in \mathbb{N}\}$.
$\limsup\limits_{n\to\infty} a_n$	$\lim\limits_{n\to\infty} \sup\{a_i \mid i \ge n, i \in \mathbb{N}\}$.
$\binom{n}{k}$	Combinations: Size k subsets of a size n set.
$\left[{n \atop k}\right]$	Stirling numbers (1st kind): Arrangements of an n element set into k cycles.
$\left\{{n \atop k}\right\}$	Stirling numbers (2nd kind): Partitions of an n element set into k non-empty sets.
$\left\langle{n \atop k}\right\rangle$	1st order Eulerian numbers: Permutations $\pi_1\pi_2\ldots\pi_n$ on $\{1, 2, \ldots, n\}$ with k ascents.
$\left\langle\!\left\langle{n \atop k}\right\rangle\!\right\rangle$	2nd order Eulerian numbers.
C_n	Catalan Numbers: Binary trees with $n+1$ vertices.

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}, \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}, \qquad \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}.$$

In general:

$$\sum_{i=1}^{n} i^m = \frac{1}{m+1}\left[(n+1)^{m+1} - 1 - \sum_{i=1}^{n}\left((i+1)^{m+1} - i^{m+1} - (m+1)i^m\right)\right]$$

$$\sum_{i=1}^{n-1} i^m = \frac{1}{m+1}\sum_{k=0}^{m}\binom{m+1}{k} B_k n^{m+1-k}.$$

Geometric series:

$$\sum_{i=0}^{n} c^i = \frac{c^{n+1}-1}{c-1}, \quad c \ne 1, \qquad \sum_{i=0}^{\infty} c^i = \frac{1}{1-c}, \qquad \sum_{i=1}^{\infty} c^i = \frac{c}{1-c}, \quad |c| < 1,$$

$$\sum_{i=0}^{n} ic^i = \frac{nc^{n+2} - (n+1)c^{n+1} + c}{(c-1)^2}, \quad c \ne 1, \qquad \sum_{i=0}^{\infty} ic^i = \frac{c}{(1-c)^2}, \quad |c| < 1.$$

Harmonic series:

$$H_n = \sum_{i=1}^{n} \frac{1}{i}, \qquad \sum_{i=1}^{n} iH_i = \frac{n(n+1)}{2}H_n - \frac{n(n-1)}{4}.$$

$$\sum_{i=1}^{n} H_i = (n+1)H_n - n, \qquad \sum_{i=1}^{n}\binom{i}{m}H_i = \binom{n+1}{m+1}\left(H_{n+1} - \frac{1}{m+1}\right).$$

1. $\displaystyle\binom{n}{k} = \frac{n!}{(n-k)!k!}$, **2.** $\displaystyle\sum_{k=0}^{n}\binom{n}{k} = 2^n$, **3.** $\displaystyle\binom{n}{k} = \binom{n}{n-k}$,

4. $\displaystyle\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$, **5.** $\displaystyle\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$,

6. $\displaystyle\binom{n}{m}\binom{m}{k} = \binom{n}{k}\binom{n-k}{m-k}$, **7.** $\displaystyle\sum_{k=0}^{n}\binom{r+k}{k} = \binom{r+n+1}{n}$,

8. $\displaystyle\sum_{k=0}^{n}\binom{k}{m} = \binom{n+1}{m+1}$, **9.** $\displaystyle\sum_{k=0}^{n}\binom{r}{k}\binom{s}{n-k} = \binom{r+s}{n}$,

10. $\displaystyle\binom{n}{k} = (-1)^k\binom{k-n-1}{k}$, **11.** $\displaystyle\left\{{n \atop 1}\right\} = \left\{{n \atop n}\right\} = 1$,

12. $\displaystyle\left\{{n \atop 2}\right\} = 2^{n-1} - 1$, **13.** $\displaystyle\left\{{n \atop k}\right\} = k\left\{{n-1 \atop k}\right\} + \left\{{n-1 \atop k-1}\right\}$,

14. $\displaystyle\left[{n \atop 1}\right] = (n-1)!$, **15.** $\displaystyle\left[{n \atop 2}\right] = (n-1)!H_{n-1}$, **16.** $\displaystyle\left[{n \atop n}\right] = 1$, **17.** $\displaystyle\left[{n \atop k}\right] \ge \left\{{n \atop k}\right\}$,

18. $\displaystyle\left[{n \atop k}\right] = (n-1)\left[{n-1 \atop k}\right] + \left[{n-1 \atop k-1}\right]$, **19.** $\displaystyle\left\{{n \atop n-1}\right\} = \left[{n \atop n-1}\right] = \binom{n}{2}$, **20.** $\displaystyle\sum_{k=0}^{n}\left[{n \atop k}\right] = n!$, **21.** $\displaystyle C_n = \frac{1}{n+1}\binom{2n}{n}$,

22. $\displaystyle\left\langle{n \atop 0}\right\rangle = \left\langle{n \atop n-1}\right\rangle = 1$, **23.** $\displaystyle\left\langle{n \atop k}\right\rangle = \left\langle{n \atop n-1-k}\right\rangle$, **24.** $\displaystyle\left\langle{n \atop k}\right\rangle = (k+1)\left\langle{n-1 \atop k}\right\rangle + (n-k)\left\langle{n-1 \atop k-1}\right\rangle$,

25. $\displaystyle\left\langle{0 \atop k}\right\rangle = \begin{cases} 1 & \text{if } k = 0, \\ 0 & \text{otherwise} \end{cases}$ **26.** $\displaystyle\left\langle{n \atop 1}\right\rangle = 2^n - n - 1$, **27.** $\displaystyle\left\langle{n \atop 2}\right\rangle = 3^n - (n+1)2^n + \binom{n+1}{2}$,

28. $\displaystyle x^n = \sum_{k=0}^{n}\left\langle{n \atop k}\right\rangle\binom{x+k}{n}$, **29.** $\displaystyle\left\langle{n \atop m}\right\rangle = \sum_{k=0}^{m}\binom{n+1}{k}(m+1-k)^n(-1)^k$, **30.** $\displaystyle m!\left\{{n \atop m}\right\} = \sum_{k=0}^{n}\left\langle{n \atop k}\right\rangle\binom{k}{n-m}$,

31. $\displaystyle\left\langle{n \atop m}\right\rangle = \sum_{k=0}^{n}\left\{{n \atop k}\right\}\binom{n-k}{m}(-1)^{n-k-m}k!$, **32.** $\displaystyle\left\langle\!\left\langle{n \atop 0}\right\rangle\!\right\rangle = 1$, **33.** $\displaystyle\left\langle\!\left\langle{n \atop n}\right\rangle\!\right\rangle = 0 \quad \text{for } n \ne 0$,

34. $\displaystyle\left\langle\!\left\langle{n \atop k}\right\rangle\!\right\rangle = (k+1)\left\langle\!\left\langle{n-1 \atop k}\right\rangle\!\right\rangle + (2n-1-k)\left\langle\!\left\langle{n-1 \atop k-1}\right\rangle\!\right\rangle$, **35.** $\displaystyle\sum_{k=0}^{n}\left\langle\!\left\langle{n \atop k}\right\rangle\!\right\rangle = \frac{(2n)^{\underline{n}}}{2^n}$,

36. $\displaystyle\left\{{x \atop x-n}\right\} = \sum_{k=0}^{n}\left\langle\!\left\langle{n \atop k}\right\rangle\!\right\rangle\binom{x+n-1-k}{2n}$, **37.** $\displaystyle\left\{{n+1 \atop m+1}\right\} = \sum_{k}\binom{n}{k}\left\{{k \atop m}\right\} = \sum_{k=0}^{n}\left\{{k \atop m}\right\}(m+1)^{n-k}$, The Chinese remainder theorem: There exists a number C such that:

$$C \equiv r_1 \bmod m_1$$
$$\vdots \quad \vdots \quad \vdots$$
$$C \equiv r_n \bmod m_n$$

if m_i and m_j are relatively prime for $i \neq j$.

Euler's function: $\phi(x)$ is the number of positive integers less than x relatively prime to x. If $\prod_{i=1}^n p_i^{e_i}$ is the prime factorization of x then

$$\phi(x) = \prod_{i=1}^n p_i^{e_i-1}(p_i - 1).$$

Euler's theorem: If a and b are relatively prime then

$$1 \equiv a^{\phi(b)} \bmod b.$$

Fermat's theorem:

$$1 \equiv a^{p-1} \bmod p.$$

The Euclidean algorithm: if $a > b$ are integers then

$$\gcd(a, b) = \gcd(a \bmod b, b).$$

If $\prod_{i=1}^n p_i^{e_i}$ is the prime factorization of x then

$$S(x) = \sum_{d|x} d = \prod_{i=1}^n \frac{p_i^{e_i+1} - 1}{p_i - 1}.$$

Perfect Numbers: x is an even perfect number iff $x = 2^{n-1}(2^n - 1)$ and $2^n - 1$ is prime.
Wilson's theorem: n is a prime iff

$$(n-1)! \equiv -1 \bmod n.$$

Möbius inversion:

$$\mu(i) = \begin{cases} 1 & \text{if } i = 1. \\ 0 & \text{if } i \text{ is not square-free.} \\ (-1)^r & \text{if } i \text{ is the product of} \\ & r \text{ distinct primes.} \end{cases}$$

If

$$G(a) = \sum_{d|a} F(d),$$

then

$$F(a) = \sum_{d|a} \mu(d) G\left(\frac{a}{d}\right).$$

Prime numbers:

$$p_n = n \ln n + n \ln \ln n - n + n \frac{\ln \ln n}{\ln n} + O\left(\frac{n}{\ln n}\right),$$

$$\pi(n) = \frac{n}{\ln n} + \frac{n}{(\ln n)^2} + \frac{2!n}{(\ln n)^3} + O\left(\frac{n}{(\ln n)^4}\right).$$

Definitions:

Loop	An edge connecting a vertex to itself.
Directed	Each edge has a direction.
Simple	Graph with no loops or multi-edges.
Walk	A sequence $v_0 e_1 v_1 \ldots e_\ell v_\ell$.
Trail	A walk with distinct edges.
Path	A trail with distinct vertices.
Connected	A graph where there exists a path between any two vertices.
Component	A maximal connected subgraph.
Tree	A connected acyclic graph.
Free tree	A tree with no root.
DAG	Directed acyclic graph.
Eulerian	Graph with a trail visiting each edge exactly once.
Hamiltonian	Graph with a cycle visiting each vertex exactly once.
Cut	A set of edges whose removal increases the number of components.
Cut-set	A minimal cut.
Cut edge	A size 1 cut.
k-Connected	A graph connected with the removal of any $k-1$ vertices.
k-Tough	$\forall S \subseteq V, S \neq \emptyset$ we have $k \cdot c(G - S) \leq \|S\|$.
k-Regular	A graph where all vertices have degree k.
k-Factor	A k-regular spanning subgraph.
Matching	A set of edges, no two of which are adjacent.
Clique	A set of vertices, all of which are adjacent.
Ind. set	A set of vertices, none of which are adjacent.
Vertex cover	A set of vertices which cover all edges.
Planar graph	A graph which can be embeded in the plane.
Plane graph	An embedding of a planar graph.

$$\sum_{v \in V} \deg(v) = 2m.$$

If G is planar then $n - m + f = 2$, so

$$f \leq 2n - 4, \quad m \leq 3n - 6.$$

Any planar graph has a vertex with degree ≤ 5.

Notation:

$E(G)$	Edge set
$V(G)$	Vertex set
$c(G)$	Number of components
$G[S]$	Induced subgraph
$\deg(v)$	Degree of v
$\Delta(G)$	Maximum degree
$\delta(G)$	Minimum degree
$\chi(G)$	Chromatic number
$\chi_E(G)$	Edge chromatic number
G^c	Complement graph
K_n	Complete graph
K_{n_1,n_2}	Complete bipartite graph
$\mathrm{r}(k, \ell)$	Ramsey number

Geometry

Projective coordinates: triples (x, y, z), not all x, y and z zero.

$$(x, y, z) = (cx, cy, cz) \quad \forall c \neq 0.$$

Cartesian	Projective
(x, y)	$(x, y, 1)$
$y = mx + b$	$(m, -1, b)$
$x = c$	$(1, 0, -c)$

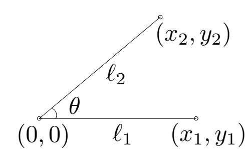
Distance formula, L_p and L_∞ metric:

$$\sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2},$$
$$\left[|x_1 - x_0|^p + |y_1 - y_0|^p\right]^{1/p},$$
$$\lim_{p \to \infty} \left[|x_1 - x_0|^p + |y_1 - y_0|^p\right]^{1/p}.$$

Area of triangle (x_0, y_0), (x_1, y_1) and (x_2, y_2):

$$\frac{1}{2} \operatorname{abs} \begin{vmatrix} x_1 - x_0 & y_1 - y_0 \\ x_2 - x_0 & y_2 - y_0 \end{vmatrix}.$$

Angle formed by three points:



$$\cos \theta = \frac{(x_1, y_1) \cdot (x_2, y_2)}{\ell_1 \ell_2}.$$

Line through two points (x_0, y_0) and (x_1, y_1):

$$\begin{vmatrix} x & y & 1 \\ x_0 & y_0 & 1 \\ x_1 & y_1 & 1 \end{vmatrix} = 0.$$

Area of circle, volume of sphere:

$$A = \pi r^2, \qquad V = \tfrac{4}{3} \pi r^3.$$

If I have seen farther than others, it is because I have stood on the shoulders of giants.
– Issac Newton

Taylor's series:

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2}f''(a) + \cdots = \sum_{i=0}^{\infty} \frac{(x-a)^i}{i!} f^{(i)}(a).$$

Expansions:

$$\begin{aligned}
\frac{1}{1-x} &= 1 + x + x^2 + x^3 + x^4 + \cdots &&= \sum_{i=0}^{\infty} x^i, \\
\frac{1}{1-cx} &= 1 + cx + c^2x^2 + c^3x^3 + \cdots &&= \sum_{i=0}^{\infty} c^i x^i, \\
\frac{1}{1-x^n} &= 1 + x^n + x^{2n} + x^{3n} + \cdots &&= \sum_{i=0}^{\infty} x^{ni}, \\
\frac{x}{(1-x)^2} &= x + 2x^2 + 3x^3 + 4x^4 + \cdots &&= \sum_{i=0}^{\infty} i x^i, \\
x^k \frac{d^n}{dx^n}\left(\frac{1}{1-x}\right) &= x + 2^n x^2 + 3^n x^3 + 4^n x^4 + \cdots &&= \sum_{i=0}^{\infty} i^n x^i, \\
e^x &= 1 + x + \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3 + \cdots &&= \sum_{i=0}^{\infty} \frac{x^i}{i!}, \\
\ln(1+x) &= x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 - \cdots &&= \sum_{i=1}^{\infty} (-1)^{i+1} \frac{x^i}{i}, \\
\ln \frac{1}{1-x} &= x + \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 + \tfrac{1}{4}x^4 + \cdots &&= \sum_{i=1}^{\infty} \frac{x^i}{i}, \\
\sin x &= x - \tfrac{1}{3!}x^3 + \tfrac{1}{5!}x^5 - \tfrac{1}{7!}x^7 + \cdots &&= \sum_{i=0}^{\infty} (-1)^i \frac{x^{2i+1}}{(2i+1)!}, \\
\cos x &= 1 - \tfrac{1}{2!}x^2 + \tfrac{1}{4!}x^4 - \tfrac{1}{6!}x^6 + \cdots &&= \sum_{i=0}^{\infty} (-1)^i \frac{x^{2i}}{(2i)!}, \\
\tan^{-1} x &= x - \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 - \tfrac{1}{7}x^7 + \cdots &&= \sum_{i=0}^{\infty} (-1)^i \frac{x^{2i+1}}{(2i+1)}, \\
(1+x)^n &= 1 + nx + \tfrac{n(n-1)}{2}x^2 + \cdots &&= \sum_{i=0}^{\infty} \binom{n}{i} x^i, \\
\frac{1}{(1-x)^{n+1}} &= 1 + (n+1)x + \tbinom{n+2}{2}x^2 + \cdots &&= \sum_{i=0}^{\infty} \binom{i+n}{i} x^i, \\
\frac{x}{e^x - 1} &= 1 - \tfrac{1}{2}x + \tfrac{1}{12}x^2 - \tfrac{1}{720}x^4 + \cdots &&= \sum_{i=0}^{\infty} \frac{B_i x^i}{i!}, \\
\frac{1}{2x}(1 - \sqrt{1-4x}) &= 1 + x + 2x^2 + 5x^3 + \cdots &&= \sum_{i=0}^{\infty} \frac{1}{i+1}\binom{2i}{i} x^i, \\
\frac{1}{\sqrt{1-4x}} &= 1 + x + 2x^2 + 6x^3 + \cdots &&= \sum_{i=0}^{\infty} \binom{2i}{i} x^i, \\
\frac{1}{\sqrt{1-4x}}\left(\frac{1-\sqrt{1-4x}}{2x}\right)^n &= 1 + (2+n)x + \tbinom{4+n}{2}x^2 + \cdots &&= \sum_{i=0}^{\infty} \binom{2i+n}{i} x^i, \\
\frac{1}{1-x} \ln \frac{1}{1-x} &= x + \tfrac{3}{2}x^2 + \tfrac{11}{6}x^3 + \tfrac{25}{12}x^4 + \cdots &&= \sum_{i=1}^{\infty} H_i x^i, \\
\frac{1}{2}\left(\ln \frac{1}{1-x}\right)^2 &= \tfrac{1}{2}x^2 + \tfrac{3}{4}x^3 + \tfrac{11}{24}x^4 + \cdots &&= \sum_{i=2}^{\infty} \frac{H_{i-1} x^i}{i}, \\
\frac{x}{1-x-x^2} &= x + x^2 + 2x^3 + 3x^4 + \cdots &&= \sum_{i=0}^{\infty} F_i x^i, \\
\frac{F_n x}{1 - (F_{n-1} + F_{n+1})x - (-1)^n x^2} &= F_n x + F_{2n} x^2 + F_{3n} x^3 + \cdots &&= \sum_{i=0}^{\infty} F_{ni} x^i.
\end{aligned}$$

Ordinary power series:

$$A(x) = \sum_{i=0}^{\infty} a_i x^i.$$

Exponential power series:

$$A(x) = \sum_{i=0}^{\infty} a_i \frac{x^i}{i!}.$$

Dirichlet power series:

$$A(x) = \sum_{i=1}^{\infty} \frac{a_i}{i^x}.$$

Binomial theorem:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k.$$

Difference of like powers:

$$x^n - y^n = (x-y) \sum_{k=0}^{n-1} x^{n-1-k} y^k.$$

For ordinary power series:

$$\begin{aligned}
\alpha A(x) + \beta B(x) &= \sum_{i=0}^{\infty} (\alpha a_i + \beta b_i) x^i, \\
x^k A(x) &= \sum_{i=k}^{\infty} a_{i-k} x^i, \\
\frac{A(x) - \sum_{i=0}^{k-1} a_i x^i}{x^k} &= \sum_{i=0}^{\infty} a_{i+k} x^i, \\
A(cx) &= \sum_{i=0}^{\infty} c^i a_i x^i, \\
A'(x) &= \sum_{i=0}^{\infty} (i+1) a_{i+1} x^i, \\
xA'(x) &= \sum_{i=1}^{\infty} i a_i x^i, \\
\int A(x)\,dx &= \sum_{i=1}^{\infty} \frac{a_{i-1}}{i} x^i, \\
\frac{A(x) + A(-x)}{2} &= \sum_{i=0}^{\infty} a_{2i} x^{2i}, \\
\frac{A(x) - A(-x)}{2} &= \sum_{i=0}^{\infty} a_{2i+1} x^{2i+1}.
\end{aligned}$$

Summation: If $b_i = \sum_{j=0}^{i} a_i$ then

$$B(x) = \frac{1}{1-x} A(x).$$

Convolution:

$$A(x)B(x) = \sum_{i=0}^{\infty} \left(\sum_{j=0}^{i} a_j b_{i-j} \right) x^i.$$

God made the natural numbers;
all the rest is the work of man.
– Leopold Kronecker