Appendix: Bessel functions

We may define them through the Laurent series expansion of the "generating function"

$$G(t,z) \equiv e^{\frac{1}{2}z(t-\frac{1}{t})} = \sum_{n=-\infty}^{\infty} t^n J_n(z) \qquad (A)$$

You may have seen generating functions in the context of Legendre polynomials $P_n(z)$:

$$\frac{1}{\sqrt{1+t^2-2zt}} = \sum_{n=0}^{\infty} t^n P_n(z)$$

(but here the expansion is a Taylor series).

One may recast (A) using $t = ie^{i\phi}$, so that $t - \frac{1}{t} = i(e^{i\phi}+e^{-i\phi}) = 2i\cos\phi$ and

$$e^{iz\cos\phi} = \sum_{n=-\infty}^{\infty} i^n e^{in\phi} J_n(z) \qquad (B)$$

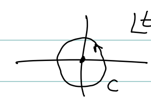
From the generating function we derive properties of J_m's.

- Parity: $e^{iz\cos(\phi+\pi)} = e^{-iz\cos\phi} \Rightarrow \sum_n i^n e^{in\pi} e^{in\phi} J_n(z) = \sum_n i^n e^{in\phi} J_n(-z) \Rightarrow J_n(-z) = (-1)^n J_n(z)$

  $e^{iz\cos(-\phi)} = e^{iz\cos\phi} \Rightarrow J_{-n}(z) = (-1)^n J_n(z)$

- Integral representation

$$J_n(z) = \frac{1}{2\pi i}\oint_C e^{\frac{1}{2}z(t-\frac{1}{t})} \frac{dt}{t^{n+1}}$$



Taking the ϕ rep and a circle

$$J_n(z) = \frac{(-i)^n}{2\pi}\int_0^{2\pi} e^{i(z\cos\phi - n\phi)}\, d\phi$$

which you can stick in a computer to get a numerical value.

- Small z expansion:

$$J_n(z) = \frac{(-i)^n}{2\pi}\int_0^{2\pi} d\phi\, e^{-in\phi} \sum_{k=0}^{\infty} \frac{i^k}{k!} z^k \left(\frac{e^{i\phi}+e^{-i\phi}}{2}\right)^k$$

The 1st non vanishing term has $|k| = n$, so

$$J_n(z) = \frac{(-i)^n}{2\pi}\cdot\left[2\pi \frac{i^n}{n!} z^n \frac{1}{2^n} + O(z^{n+1})\right] = \frac{1}{2^n n!} z^n + \dots$$

Exercise: get the whole series!

In particular $J_0(0) = 1$, $J_{n\neq 0}(0) = 0$.

• Large $|z|$. Use method of stationary phase.

Conditions $\frac{d}{d\phi}(z\cos\phi) = z\sin\phi = 0 \Rightarrow \phi = 0$ or π.

Expansion: $\cos\phi \simeq 1-\frac{1}{2}\phi^2$ and $\cos\phi = -1+\frac{1}{2}(\phi-\pi)^2$

$$J_n(z) \approx \frac{(-i)^n}{2\pi}\left[\int_{-\infty}^{\infty} d\phi\, e^{iz(1-\frac{1}{2}\phi^2)}\, e^{-in\phi} + \int_{-\infty}^{\infty} d\phi\, e^{-iz(1-\frac{1}{2}\phi^2)}\, e^{-in(\phi+\pi)}\right]$$

Use $\int_{-\infty}^{\infty} dx\, e^{i(ax^2+bx)} = \int_{-\infty}^{\infty} dx\, e^{ia(x+\frac{1}{2}b/a)^2 - ib^2/4a} = \frac{e^{-ib^2/4a}}{\sqrt{-ia}}\int_{-\infty}^{\infty} dx\, e^{-x^2} = \sqrt{\frac{\pi}{-ia}}\, e^{-ib^2/4a}$

So that

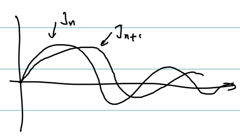
$$J_n(z) \approx \frac{(-i)^n}{2\pi}\left[\sqrt{\frac{\pi}{iz/2}}\, e^{-in^2/4(-z/2)}\, e^{iz} + (-1)^n\sqrt{\frac{\pi}{-iz/2}}\, e^{-in^2/4(z/2)}\, e^{-iz}\right]$$

$$= \frac{1}{2\pi}\, 2\mathrm{Re}\left[\sqrt{\frac{2\pi}{iz}}\, e^{i(z+\frac{n^2}{2z}-\frac{n\pi}{2})}\right]$$

or

$$J_n(z) \approx \sqrt{\frac{2}{\pi z}}\cos\left(z-\frac{n\pi}{2}-\frac{\pi}{4}\right)\left(1+O\left(\frac{1}{z}\right)\right)$$

So roughly



(and J_0 starting from $J_0(0)=1$)

Most importantly, $J_n(z)$ satisfies

$$\left(\frac{d^2}{dz^2}+\frac{1}{z}\frac{d}{dz}+1-\frac{n^2}{z^2}\right)J_n(z)=0 \qquad (C) \qquad \text{"Bessel equation"}$$

Recall in polar coordinates $ds^2 = d\rho^2+\rho^2 d\theta^2$, $\nabla^2 = \frac{1}{\sqrt{g}}\partial_i\sqrt{g}g^{ij}\partial_j = \frac{1}{\rho}\partial_\rho\rho\partial_\rho + \frac{1}{\rho}\partial_\theta\rho\frac{1}{\rho^2}\partial_\theta = \partial_\rho^2+\frac{1}{\rho}\partial_\rho+\frac{1}{\rho^2}\partial_\theta^2$

So J_n appears naturally in solutions to $\nabla^2\Psi = \lambda\Psi$ in polar (or cylindrical in 3D) coordinates.

To show (C), take derivatives of $G(t,z)$ in (A)

$$\frac{\partial G}{\partial z} = \frac{1}{2}\left(t-\frac{1}{t}\right)G \qquad \frac{\partial^2 G}{\partial z^2} = \left[\frac{1}{2}\left(t-\frac{1}{t}\right)\right]^2 G$$

$$t\frac{\partial G}{\partial t} = \frac{z}{2}\left(t+\frac{1}{t}\right)G \qquad \left(t\frac{\partial}{\partial t}\right)^2 G = \left\{\left[\frac{z}{2}\left(t+\frac{1}{t}\right)\right]^2 + \frac{z}{2}\left[t-\frac{1}{t}\right]\right\}G$$

So $\left(t\frac{\partial}{\partial t}\right)^2 G - z^2\frac{\partial^2}{\partial z^2}G = z^2 G + z\frac{\partial G}{\partial z}$

Now use the expansion of G in J_n's in (A) with $\left(t\frac{\partial}{\partial t}\right)^2 t^n = n^2 t^n$. This proves (C).

From Gradshteyn & Ryzhik: recursion formulas

$$n J_{n-1}(z) + z J_{n+1}(z) = 2n J_n(z)$$

$$J_{n-1}(z) - J_{n+1}(z) = 2\frac{d}{dz}J_n(z)$$

Exercise: show these.

Also from G&R, as consequence of the above

$$z\frac{d}{dz}J_n + nJ_n = zJ_{n-1}$$

$$z\frac{d}{dz}J_n - nJ_n = -zJ_{n+1}$$

$$\left(\frac{1}{z}\frac{d}{dz}\right)^m \left(z^n J_n(z)\right) = z^{n-m} J_{n-m}(z)$$

$$\left(\frac{1}{z}\frac{d}{dz}\right)^m \left(z^{-n} J_n(z)\right) = (-1)^m z^{-n-m} J_{n+m}(z)$$

From the last, with $n=0$, $m=1$ $\quad \frac{1}{z}\frac{d}{dz}J_0(z) = (-1)z^{-1}J_1(z) \Rightarrow J_1(z) = -J_0'(z)$.

- Independent solutions: Bessel equation, being second order and linear, admits two independent solutions. $J_n(z)$ is one. What about the other?

Consider the equation for non-integer n. One can find a solution in the form of a power expansion. Let's use ν for the general case

$$\left(\frac{d^2}{dz^2} + \frac{1}{z}\frac{d}{dz} + 1 - \frac{\nu^2}{z^2}\right) J_\nu = 0 \qquad (C')$$

so $\nu = n$, an integer, is the case studied so far. Assuming J_ν is regular near $z=0$ we write $J_\nu \sim z^p + \cdots$ close to $z=0$ then

$$p(p-1) + p - \nu^2 = 0 \quad \Rightarrow \quad p = \pm\nu$$

and for a regular solution $p = +\nu$ (assuming $\nu>0$). The regular solution so obtained coincides with $J_n(z)$ when $\nu = n$ an integer.

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+\nu+1)} \left(\frac{z}{2}\right)^{2m}$$

Taking $\nu \to -\nu$ when $\nu \neq$ integer gives a second, <u>independent</u> solution.

But for $\nu = n$, $J_{-n}(z) = (-1)^n J_n(z)$ as we have seen, and it is <u>not</u> independent

A 2^{nd} solution for ν = integer: the Neumann function, $N_\nu(z)$, is often used as the 2^{nd} independent solution (rather than $J_{-\nu}(z)$).

$$N_\nu(z) \equiv \frac{J_\nu(z)\cos(\nu\pi) - J_{-\nu}(z)}{\sin(\nu\pi)}$$

Since this solves (C') and has a limit as $\nu \to n$ (integer), this can be used as the 2^{nd} solution so that the general solution is

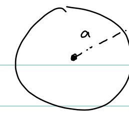
$$\psi(z) = a\, J_n(z) + b\, N_n(z) \qquad a, b = \text{constants}$$

Notice that $N_n(z)$ is not regular at the origin. For $n \geq 1$ clearly $N_n \sim z^{-n}$. For $n=0$, take $\nu = \epsilon$ and expand

$$N_0(z) \sim \frac{z^\epsilon \cos(\epsilon\pi) - z^{-\epsilon}}{\sin(\epsilon\pi)} = \frac{(1+\epsilon \ln z) - (1-\epsilon \ln z)}{\epsilon\pi} = \frac{2}{\pi}\ln z$$

For our purposes (presently) we can ignore $N_n(z)$: we are looking to solve $(\nabla^2 + \gamma^2)\psi = 0$ inside a circle that includes the origin.

Wave guide/cavity with circular cross section



We want to solve

$$(\nabla_\perp^2 + \gamma^2)\psi = 0$$

Use polar coordinates ρ, θ and try separation of variables:

$$\psi(\rho,\theta) = R(\rho)T(\theta)$$

$$\frac{1}{\psi}\nabla_\perp^2\psi = -\gamma^2 \quad \Rightarrow \quad \rho^2\frac{1}{R}\left(\frac{d^2}{d\rho^2}+\frac{1}{\rho}\frac{d}{d\rho}\right)R + \frac{1}{T}\frac{d^2T}{d\theta^2} = -\gamma^2\rho^2$$

Since $T(\theta+2\pi)=T(\theta)$ we take $T(\theta)=e^{in\theta}$ with n an integer $\Rightarrow$

$$\left(\frac{d^2}{d\rho^2}+\frac{1}{\rho}\frac{d}{d\rho} - \frac{n^2}{\rho^2}+\gamma^2\right)R = 0$$

Let $z=\gamma\rho$, so that as function of z, $R(z)$ satisfies Bessel's equation : $R(z) = J_n(z) = J_n(\gamma\rho)$.

Boundary conditions: $R(a)=0$ (for TM mode, $E_z|=0$) or $\left.\frac{dR}{d\rho}\right|_a = 0$ (for TE modes, $\left.\frac{\partial B_z}{\partial n}\right|_{\partial V}=0$).

(i) $R(a)=0 \Rightarrow J_n(\gamma a)=0$. We saw that J_n has an infinite number of positive zeroes. Let $0\le z_{n1}\le z_{n2}\le z_{n3}$ be the zeroes $J_n(z_{nm})=0$. The condition $J_n(\gamma a)$ gives

$$\gamma_{nm}a = z_{nm} \quad\Rightarrow\quad \gamma_{nm} = z_{nm}/a$$

The most general solution is

$$E_z = \sum_{n,m} c_{nm}e^{in\theta}J_n(\gamma_{nm}\rho) \qquad \left(\text{with } \left(\tfrac{\omega}{c}\right)^2 = k^2+\gamma_{nm}^2 = k^2+\frac{z_{nm}^2}{a^2}\right)$$

The zeroes of first few J_n's are tabulated. For example, $z_{01}=2.4$ $\quad z_{11}=3.8$ $\quad z_{02}=5.5$

(ii) $R'(a)=0 = J_n'(\gamma a)=0$. Since J_n is continuous and has ∞ number of zeros, it's derivative must also have an infinite number of zeros. We already know $J_0'(z)=-J_1(z)$ so the γ^2's of the $n=0$ TE modes are degenerate with those of the $n=1$ TM modes.

Let $0\le z_{n1}'\le z_{n2}'\le z_{n3}'\le\cdots$ be $J_n'(z_{nm}')=0$, then

$$B_z = \sum_{n,m} c_{nm}e^{in\theta}J_n(\gamma_{nm}\rho) \text{ with } \gamma_{nm}=\frac{z_{nm}'}{a}$$

And, eg, $z_{11}'=1.8$ $\quad z_{21}'=3.1$ $\quad z_{01}'=z_{11}=3.8$

See table attached.

For future reference

Inverting the series.

In many cases $\psi(\rho,\theta)$ is fixed (boundary value problems)

Then one needs to use this information to determine the coefficients of the expansion.

The idea is this: use completeness of the eigenfunctions,

i.e $\langle n|m\rangle = \delta_{nm}$ then $\sum_n |n\rangle\langle n| = \mathbb{1}$

or $$\int_0^a \rho d\rho \int_0^{2\pi} d\theta \left(J_n(z_{nm}\tfrac{\rho}{a})\, e^{in\theta}\right)^* \left(J_{n'}(z_{n'm'}\tfrac{\rho}{a})\, e^{in'\theta}\right) = N_{nm}\,\delta_{nn'}\,\delta_{mm'}$$

where the normalization factor is $N_{nm} = \int_0^a \rho d\rho \int_0^{2\pi} d\theta \left|J_n(z_{nm}\tfrac{\rho}{a})\, e^{in\theta}\right|^2$

or $$N_{nm} = 2\pi a^2 \int_0^1 dx \left[x J_n^2(z_{nm}x)\right]$$

ASIDE (Note that it matters but this integral can be done: take Bessel $\times$ $J' = \frac{dJ_n}{dz}$ (drop n's for now):

$$J'J'' + \frac{1}{z}(J')^2 + \left(1-\frac{n^2}{z^2}\right) J J' = 0$$

Now $J'J'' = \frac{1}{2}(J'^2)'$ so $\frac{1}{2z^2}\frac{d}{dz}\left(z^2 J'^2\right) = \frac{1}{2}(J'^2)' + \frac{1}{z}(J')^2$, i.e., the first two terms. Multiply by $2z^2$:

$$\frac{d}{dz}\left(z^2 J'^2\right) + (z^2-n^2)(J^2)' = 0$$

Now rewrite $(z^2-n^2)(J^2)' = \frac{d}{dz}\left[(z^2-n^2)J^2\right] - 2zJ^2$ and integrate

$$\int_0^{z_{nm}} dz\, zJ^2 = \frac{1}{2}\left[z^2 J'^2 + (z^2-n^2)J^2\right]_0^{z_{nm}}$$

Changing variables $z = x z_{nm}$, $\int_0^1 dx\, x J_n^2(x z_{nm}) = \frac{1}{2}\left[x^2 J'^2 + \left(x^2 - \frac{n^2}{z_{nm}^2}\right)J^2\right]\Big|_0^1 = \frac{1}{2}\left[J_n'(z_{nm})\right]^2$

And from $z\frac{dJ_n}{dz} - nJ_n = -zJ_{n+1}$ one can rewrite this as $\frac{1}{2}\left[J_{n+1}(z_{nm})\right]^2$).

So our orthonormal set is $|nm\rangle = \dfrac{e^{in\theta} J_n(z_{nm}\frac{\rho}{a})}{\sqrt{N_{nm}}}$

If $|\psi\rangle = \sum c_{nm}|nm\rangle \Rightarrow c_{nm} = \langle nm|\psi\rangle$ or

$$c_{nm} = \int_0^a \rho d\rho \int_0^{2\pi} d\theta\, \frac{e^{-in\theta} J_n(z_{nm}\rho/a)}{\sqrt{N_{nm}}}\, \psi(\rho,\theta)$$

Table of First 700 Zeros of Bessel Functions — $J_l(x)$ and $J'_l(x)$

By CURTIS L. BEATTIE

(Manuscript received August 20, 1957)

The zeros of the Bessel functions and Bessel function deriviatives are identified by standard waveguide notation which also serves as a code for more general mathematical applications.

The possibilities of low-loss transmission using the TE_{01} (circular electric) mode in circular cylindrical pipe of a diameter large compared to the wavelength has made the study of other modes of such a waveguide important. In order to find phase and attenuation constants of various modes for both solid and ring or helix walls, the zeros of the Bessel functions $J_l(x)$ and $J'_l(x)$ are essential.

In the table given here the first seven hundred roots of Bessel functions $J_l(x) = 0$ and $J'_l(x) = 0$ have been computed and arranged in the order of the magnitude of the arguments corresponding to the roots. In the table l is the order of the Bessel function and m is the serial number of the zero of either $J_l(x)$ or $J'_l(x)$, not counting $x = 0$. In waveguide applications the zeros of $J_l(x)$ correspond to transverse magnetic modes of propagation (TM modes) and those of $J'_l(x)$ to transverse electric modes (TE modes). The designations TM and TE appear in the table for the benefit of those who will use this table in waveguide research and serve as a code designating $J_l(x)$ and $J'_l(x)$ for those who are interested in a more general application of the mathematics.

The roots of the Bessel functions were calculated from the *Tables of the Bessel Functions of the First Kind of Orders, J_0 through J_{51}*, computed by the Staff of the Computation Laboratory of Harvard University, published by the Harvard University Press, 1946–1948.

This table was first formulated horizontally in the ascending order of the function and vertically in the ascending number of the root. Since the increments in each direction are of a predictable magnitude, the possibility of having neglected a root is virtually eliminated.

All 700 roots were calculated by means of a linear interpolation formula but checked and corrected for six-place accuracy for the first 300 roots. For arguments above 25, where the tabulated values were for each 0.01 only, the following Newton-Bessel formula was used:

$$J_n(x) = J_n(x_0 + h\mu) = J_n(x_0) + \mu B + \frac{\mu(\mu - 1)(C - A)}{4} + \cdots$$

where

$$\begin{aligned} \mu &= (x - x_0)/h \\ A &= J_n(x_0 + h) + J_n(x_0) \\ B &= J_n(x_0 + h) - J_n(x_0) \\ C &= J_n(x_0 + 2h) + J_n(x_0 - h). \end{aligned}$$

Spot checking was done with the first three terms of the Taylor's series expansion:

$$0 \cong J_n(x) + \frac{h}{2}[J_{n-1}(x) - J_{n+1}(x)] + \frac{h^2}{8}[J_{n-2}(x) - 2J_n(x) + J_{n+2}(x)] + \cdots.$$

BIBLIOGRAPHY

1. Wilson, I. G., Schramm, C. W. and Kinzer, J. P., High Q Resonant Cavities for Microwave Testing, B.S.T.J., **25**, July, 1946, p. 418.
2. Gray, A. and Mathews, G. B., *Treatise on Bessel Functions*, Macmillan, London, 1895. Table III. The first 40 roots of $J_0(x) = 0$, with the corresponding values of $J_1(x)$. (Ten decimal places.) Table IV. The first 50 roots of $J_1(x) = 0$, with the corresponding maximum or minimum values of $J_0(x)$. (Sixteen decimal places.)
3. Smith, D. B., Rodgers, L. M. and Traub, E. H., Zeros of Bessel Functions, Franklin Inst. Jnl., **237**, April, 1944, p. 301. Positive real roots between 0 and 25 for function and first derivative of Bessel functions of the first kind.
4. Bickley, W. G., Notes on the Evaluation of Zeros and Turning Values of Bessel Functions, Interpolation by Taylor Series, Phil. Mag., 7th Series, **36**, March, 1945, p. 200; Miller, J. C. P. and Jones, C. W., Notes on the Evaluation of Zeros and Turning Values of Bessel Functions, Phil. Mag., 7th Series, **36**, March, 1945, p. 206.
5. Schelkunoff, S. A., *Electromagnetic Waves*, D. Van Nostrand and Co., N. Y., 1943.
6. Southworth, G. C., *Principles and Applications of Waveguide Transmission*, D. Van Nostrand and Co., N. Y., 1950.

TABLE

	Mode*	l–m	Value†		Mode*	l–m	Value†
1	TE	1–1	1.841184	(48	TM	1–4	13.323692
2	TM	0–1	2.404826	(49	TE	0–4	13.323692
3	TE	2–1	3.054237	50	TM	9–1	13.354300
(4	TM	1–1	3.831706	51	TM	6–2	13.589290
(5	TE	0–1	3.831706	52	TE	12–1	13.878843
6	TE	3–1	4.201189	53	TE	5–3	13.987189
7	TM	2–1	5.135622	54	TE	8–2	14.115519
8	TE	4–1	5.317553	55	TM	4–3	14.372537
9	TE	1–2	5.331443	56	TM	10–1	14.475501
10	TM	0–2	5.520078	57	TE	3–4	14.585848
11	TM	3–1	6.380162	58	TM	2–4	14.795952
12	TE	5–1	6.415616	59	TM	7–2	14.821269
13	TE	2–2	6.706133	60	TE	1–5	14.863589
(14	TM	1–2	7.015587	61	TE	13–1	14.928374
(15	TE	0–2	7.015587	62	TM	0–5	14.930918
16	TE	6–1	7.501266	63	TE	6–3	15.268181
17	TM	4–1	7.588342	64	TE	9–2	15.286738
18	TE	3–2	8.015237	65	TM	11–1	15.589848
19	TM	2–2	8.417244	66	TM	5–3	15.700174
20	TE	1–3	8.536316	67	TE	4–4	15.964107
21	TE	7–1	8.577836	68	TE	14–1	15.975439
22	TM	0–3	8.653728	69	TM	8–2	16.037774
23	TM	5–1	8.771484	70	TM	3–4	16.223466
24	TE	4–2	9.282396	71	TE	2–5	16.347522
25	TE	8–1	9.647422	72	TE	10–2	16.447853
26	TM	3–2	9.761023	(73	TM	1–5	16.470630
27	TM	6–1	9.936110	(74	TE	0–5	16.470630
28	TE	2–3	9.969468	75	TE	7–3	16.529366
(29	TM	1–3	10.173468	76	TM	12–1	16.698250
(30	TE	0–3	10.173468	77	TM	6–3	17.003820
31	TE	5–2	10.519861	78	TE	15–1	17.020323
32	TE	9–1	10.711434	79	TM	9–2	17.241220
33	TM	4–2	11.064709	80	TE	5–4	17.312842
34	TM	7–1	11.086370	81	TE	11–2	17.600267
35	TE	3–3	11.345924	82	TM	4–4	17.615966
36	TM	2–3	11.619841	83	TE	8–3	17.774012
37	TE	1–4	11.706005	84	TE	3–5	17.788748
38	TE	6–2	11.734936	85	TM	13–1	17.801435
39	TE	10–1	11.770877	86	TM	2–5	17.959819
40	TM	0–4	11.791534	87	TE	1–6	18.015528
41	TM	8–1	12.225092	88	TE	16–1	18.063261
42	TM	5–2	12.338604	89	TM	0–6	18.071064
43	TE	4–3	12.681908	90	TM	7–3	18.287583
44	TE	11–1	12.826491	91	TM	10–2	18.433464
45	TE	7–2	12.932386	92	TE	6–4	18.637443
46	TM	3–3	13.015201	93	TE	12–2	18.745091
47	TE	2–4	13.170371	94	TM	14–1	18.899998

* TM designates a zero of $J_l(x)$; TE designates a zero of $J'_l(x)$; in each case l corresponds to the order of the Bessel function and m is the number of the root.

† $\bar{5}$ in last place indicates higher value and $\underline{5}$ indicates lower value in rounding off for fewer decimal places.

TABLE — *Continued*

	Mode* l–m	Value†		Mode* l–m	Value†
95	TM 5–4	18.980134	150	TM 4–6	24.019020
96	TE 9–3	19.004594	151	TE 3–7	24.144897
97	TE 17–1	19.104458	152	TM 9–4	24.23388$\bar{5}$
98	TE 4–5	19.196029	153	TM 15–2	24.269180
99	TM 3–5	19.40941$\bar{5}$	154	TM 2–7	24.270112
100	TE 2–6	19.512913	155	TE 22–1	24.289385
101	TM 8–3	19.554536	156	TE 1–8	24.311327
(102	TM 1–6	19.615859	157	TM 19–1	24.338250
(103	TE 0–6	19.615859	158	TM 0–8	24.352472
104	TM 11–2	19.615967	159	TE 17–2	24.381913
105	TE 13–2	19.883224	160	TM 12–3	24.49488$\bar{5}$
106	TE 7–4	19.941853	161	TE 8–5	24.587197
107	TM 15–1	19.994431	162	TM 7–5	24.934928
108	TE 18–1	20.144079	163	TE 14–3	25.001972
109	TE 10–3	20.223031	164	TE 11–4	25.008519
110	TM 6–4	20.320789	165	TE 6–6	25.18392$\bar{5}$
111	TE 5–5	20.575515	166	TE 23–1	25.322921
112	TM 12–2	20.789906	167	TM 16–2	25.417019
113	TM 9–3	20.807048	168	TM 20–1	25.417141
114	TM 4–5	20.826933	169	TM 5–6	25.430341
115	TE 3–6	20.972477	170	TE 18–2	25.495558
116	TE 14–2	21.015405	171	TM 10–4	25.509450
117	TM 16–1	21.085147	172	TE 4–7	25.589760
118	TM 2–6	21.116997	173	TM 13–3	25.705104
119	TE 1–7	21.164370	174	TM 3–7	25.748167
120	TE 19–1	21.182267	175	TE 2–8	25.826037
121	TM 0–7	21.211637	176	TE 9–5	25.891177
122	TE 8–4	21.229063	(177	TM 1–8	25.903672
123	TE 11–3	21.430854	(178	TE 0–8	25.903672
124	TM 7–4	21.641541	179	TE 15–3	26.177766
125	TE 6–5	21.93171$\bar{5}$	180	TE 12–4	26.246048
126	TM 13–2	21.956244	181	TM 8–5	26.26681$\bar{5}$
127	TM 10–3	22.046985	182	TE 24–1	26.355506
128	TE 15–2	22.142247	183	TM 21–1	26.493648
129	TM 17–1	22.17249$\bar{5}$	184	TE 7–6	26.545032
130	TM 5–5	22.217800	185	TM 17–2	26.559784
131	TE 20–1	22.21914$\bar{5}$	186	TE 19–2	26.605533
132	TE 4–6	22.401032	187	TM 11–4	26.773323
133	TE 9–4	22.501399	188	TM 6–6	26.820152
134	TM 3–6	22.582730	189	TM 14–3	26.907369
135	TE 12–3	22.629300	190	TE 5–7	27.010308
136	TE 2–7	22.671582	191	TE 10–5	27.182022
(137	TM 1–7	22.760084	192	TM 4–7	27.199088
(138	TE 0–7	22.760084	193	TE 3–8	27.310058
139	TM 8–4	22.945173	194	TE 16–3	27.347386
140	TM 14–2	23.115778	195	TE 25–1	27.387204
141	TE 21–1	23.254816	196	TM 2–8	27.420574
142	TM 18–1	23.256777	197	TE 1–9	27.457051
143	TE 16–2	23.264269	198	TE 13–4	27.474340
144	TE 7–5	23.268053	199	TM 0–9	27.493480
145	TM 11–3	23.275854	200	TM 22–1	27.567944
146	TM 6–5	23.586084	201	TM 9–5	27.583749
147	TE 10–4	23.760716	202	TM 18–2	27.697899
148	TE 5–6	23.803581	203	TE 20–2	27.712126
149	TE 13–3	23.819374	204	TE 8–6	27.889270

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† $\bar{5}$ in last place indicates higher value and 5 indicates lower value in rounding off for fewer decimal places.

TABLE — *Continued*

	Mode* l–m	Value†		Mode* l–m	Value†
205	TM 12–4	28.026710	256	TM 12–5	31.459960
206	TM 15–3	28.102416	257	TE 29–1	31.506195
207	TM 7–6	28.191189	258	TE 6–8	31.617876
208	TE 6–7	28.409776	259	TM 18–3	31.650118
209	TE 26–1	28.418072	260	TM 15–4	31.733414
210	TE 11–5	28.460857	261	TM 5–8	31.811717
211	TE 17–3	28.511361	262	TE 11–6	31.838425
212	TM 5–7	28.626619	263	TM 26–1	31.845888
213	TM 23–1	28.64018$\bar{5}$	264	TE 4–9	31.938540
214	TE 14–4	28.694271	265	TE 20–3	31.97371$\bar{5}$
215	TE 4–8	28.767836	266	TM 3–9	32.064853
216	TE 21–2	28.815590	267	TE 24–2	32.109320
217	TM 19–2	28.831731	268	TE 2–10	32.127327
218	TM 10–5	28.88737$\bar{5}$	(269	TM 1–10	32.189680
219	TM 3–8	28.908351	(270	TE 0–10	32.189680
220	TE 2–9	28.977673	271	TM 22–2	32.210587
(221	TM 1–9	29.046829	272	TM 10–6	32.211856
(222	TE 0–9	29.046829	273	TE 14–5	32.236970
223	TE 9–6	29.218564	274	TE 17–4	32.310894
224	TM 13–4	29.270631	275	TE 9–7	32.505248
225	TM 16–3	29.290871	276	TE 30–1	32.534220
226	TE 27–1	29.448163	277	TM 13–5	32.731053
227	TM 8–6	29.545660	278	TM 8–7	32.795800
228	TE 18–3	29.670147	279	TM 19–3	32.821803
229	TM 24–1	29.710509	280	TM 27–1	32.911154
230	TE 12–5	29.728978	281	TM 16–4	32.95366$\bar{5}$
231	TE 7–7	29.790749	282	TE 7–8	33.015179
232	TE 15–4	29.906591	283	TE 21–3	33.119162
233	TE 22–2	29.916147	284	TE 12–6	33.131450
234	TM 20–2	29.961604	285	TE 25–2	33.202272
235	TM 6–7	30.033723	286	TM 6–8	33.233042
236	TM 11–5	30.179061	287	TM 23–2	33.330177
237	TE 5–8	30.202849	288	TE 5–9	33.385444
238	TM 4–8	30.371008	289	TE 15–5	33.478449
239	TE 3–9	30.470269	290	TE 18–4	33.503929
240	TM 17–3	30.473280	291	TM 11–6	33.526364
241	TE 28–1	30.477523	292	TM 4–9	33.537138
242	TM 14–4	30.505951	293	TE 31–1	33.561634
243	TE 10–6	30.53450$\bar{5}$	294	TE 3–10	33.626949
244	TM 2–9	30.56920$\underline{5}$	295	TM 2–10	33.716520
245	TE 1–10	30.601923	296	TE 1–11	33.746183
246	TM 0–10	30.634607	297	TM 0–11	33.775821
247	TM 25–1	30.779039	298	TE 10–7	33.841966
248	TE 19–3	30.824148	299	TM 28–1	33.974930
249	TM 9–6	30.885379	300	TM 20–3	33.988703
250	TE 13–5	30.987394	301	TM 14–5	33.99319
251	TE 23–2	31.013998	302	TM 9–7	34.15438
252	TM 21–2	31.087805	303	TM 17–4	34.16727
253	TE 16–4	31.111945	304	TE 22–3	34.26077
254	TE 8–7	31.155327	305	TE 26–2	34.29300
255	TM 7–7	31.42279$\bar{5}$	306	TE 8–8	34.39663

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TABLE — *Continued*

	Mode* l–m	Value†		Mode* l–m	Value†
307	TE 13–6	34.4145$\underline{5}$	357	TE 10–8	37.11800
308	TM 24–2	34.44678	358	TM 31–1	37.15811
309	TE 32–1	34.58847	359	TE 18–5	37.16040
310	TM 7–8	34.63709	360	TM 9–8	37.40010
311	TE 19–4	34.69148	361	TM 14–6	37.40819
312	TE 16–5	34.71248	362	TM 23–3	37.46381
313	TE 6–9	34.81339	363	TE 29–2	37.55307
314	TM 12–6	34.82999	364	TE 8–9	37.62008
315	TM 5–9	34.98878	365	TE 25–3	37.66491
316	TM 29–1	35.03730	366	TE 35–1	37.66577
317	TE 4–10	35.10392	367	TM 17–5	37.73268
318	TM 21–3	35.15115	368	TM 20–4	37.77286
319	TE 11–7	35.16671	369	TM 27–2	37.78040
320	TM 3–10	35.21867	370	TE 13–7	37.78438
321	TM 15–5	35.24709	371	TM 7–9	37.83872
322	TE 2–11	35.27554	372	TE 6–10	37.99964
(323	TM 1–11	35.33231	373	TM 12–7	38.15638
(324	TE 0–11	35.33231	374	TM 5–10	38.15987
325	TM 18–4	35.37472	375	TE 16–6	38.21206
326	TE 27–2	35.38163	376	TM 32–1	38.21669
327	TE 23–3	35.39878	377	TE 22–4	38.22490
328	TM 10–7	35.49991	378	TE 4–11	38.26532
329	TM 25–2	35.56057	379	TM 3–11	38.37047
330	TE 33–1	35.6147$\underline{5}$	380	TE 19–5	38.37524
331	TE 14–6	35.68854	381	TE 2–12	38.42266
332	TE 9–8	35.76379	382	TE 11–8	38.46039
333	TE 20–4	35.87394	(383	TM 1–12	38.47477
334	TE 17–5	35.93963	(384	TE 0–12	38.47477
335	TM 8–8	36.02562	385	TM 24–3	38.61452
336	TM 30–1	36.09834	386	TE 30–2	38.63609
337	TM 13–6	36.12366	387	TM 15–6	38.68428
338	TE 7–9	36.22438	388	TE 36–1	38.6905$\bar{5}$
339	TM 22–3	36.30943	389	TM 10–8	38.76181
340	TM 6–9	36.42202	390	TE 26–3	38.79341
341	TE 28–2	36.46829	391	TM 28–2	38.88671
342	TE 12–7	36.4805$\underline{5}$	392	TM 21–4	38.96429
343	TM 16–5	36.49340	393	TM 18–5	38.96543
344	TE 24–3	36.53343	394	TE 9–9	39.00190
345	TE 5–10	36.56078	395	TE 14–7	39.07900
346	TM 19–4	36.5764$\bar{5}$	396	TM 8–9	39.2404$\underline{5}$
347	TE 34–1	36.64051	397	TM 33–1	39.27413
348	TM 26–2	36.67173	398	TE 23–4	39.39398
349	TM 4–10	36.69900	399	TE 7–10	39.42227
350	TE 3–11	36.78102	400	TE 17–6	39.46277
351	TM 11–7	36.83357	401	TM 13–7	39.46921
352	TM 2–11	36.86286	402	TE 20–5	39.58453
353	TE 1–12	36.88999	403	TM 6–10	39.60324
354	TM 0–12	36.91710	404	TE 37–1	39.71489
355	TE 15–6	36.95417	405	TE 31–2	39.71743
356	TE 21–4	37.05164	406	TE 5–11	39.73064

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TABLE — *Continued*

	Mode* l–m	Value†		Mode* l–m	Value†
407	TM 25–3	39.76179	458	TE 29–3	42.16260
408	TE 12–8	39.79194	459	TM 31–2	42.1927$\underline{5}$
409	TM 4–11	39.85763	460	TE 9–10	42.22464
410	TE 27–3	39.91909	461	TE 14–8	42.4258$\bar{5}$
411	TE 3–12	39.93311	462	TM 36–1	42.44014
412	TM 16–6	39.9525$\bar{5}$	463	TM 8–10	42.44389
413	TM 29–2	39.99080	464	TM 18–6	42.46781
414	TM 2–12	40.00845	465	TM 24–4	42.51168
415	TE 1–13	40.03344	466	TE 7–11	42.61152
416	TM 0–13	40.05843	467	TM 21–5	42.62870
417	TM 11–8	40.11182	468	TM 6–11	42.77848
418	TM 22–4	40.1510$\underline{5}$	469	TM 13–8	42.78044
419	TM 19–5	40.19210	470	TE 40–1	42.78537
420	TM 34–1	40.33048	471	TE 26–4	42.8785$\bar{5}$
421	TE 15–7	40.36510	472	TE 5–12	42.89627
422	TE 10–9	40.37107	473	TE 17–7	42.9141$\bar{5}$
423	TE 24–4	40.55913	474	TE 34–2	42.95218
424	TM 9–9	40.6285$\bar{5}$	475	TM 4–12	43.01374
425	TE 18–6	40.70680	476	TE 12–9	43.07549
426	TE 38–1	40.73879	477	TE 3–13	43.0836$\bar{5}$
427	TM 14–7	40.77283	478	TM 2–13	43.1534$\bar{5}$
428	TE 21–5	40.78864	479	TE 20–6	43.17654
429	TE 32–2	40.79718	480	TE 1–14	43.17663
430	TE 8–10	40.83018	481	TE 23–5	43.18255
431	TM 26–3	40.90580	482	TM 28–3	43.18477
432	TM 7–10	41.03077	483	TM 0–14	43.19979
433	TE 28–3	41.04211	484	TE 30–3	43.28071
434	TM 30–2	41.09278	485	TM 32–2	43.29081
435	TE 13–8	41.11351	486	TM 16–7	43.35507
436	TE 6–11	41.1788$\underline{5}$	487	TM 11–9	43.36836
437	TM 17–6	41.21357	488	TM 37–1	43.49352
438	TM 5–11	41.32638	489	TE 10–10	43.60677
439	TM 23–4	41.33343	490	TM 25–4	43.68603
440	TM 35–1	41.38580	491	TM 19–6	43.71571
441	TM 20–5	41.41307	492	TE 15–8	43.72963
442	TE 4–12	41.42367	493	TE 41–1	43.80808
443	TM 12–8	41.45109	494	TM 22–5	43.83932
444	TM 3–12	41.52072	495	TM 9–10	43.84380
445	TE 2–13	41.56894	496	TE 35–2	44.02758
(446	TM 1–13	41.61709	497	TE 8–11	44.03001
(447	TE 0–13	41.61709	498	TE 27–4	44.03321
448	TE 16–7	41.64331	499	TM 14–8	44.10059
449	TE 25–4	41.72059	500	TE 18–7	44.17813
450	TE 11–9	41.72863	501	TM 7–11	44.21541
451	TE 39–1	41.76228	502	TM 29–3	44.32003
452	TE 33–2	41.87540	503	TE 6–12	44.35258
453	TE 19–6	41.94459	504	TE 24–5	44.37290
454	TE 22–5	41.98788	505	TM 33–2	44.38706
455	TM 10–9	42.00419	506	TE 31–3	44.39653
456	TM 27–3	42.04674	507	TE 21–6	44.40300
457	TM 15–7	42.06792	508	TE 13–9	44.4124$\bar{5}$

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TABLE — *Continued*

	Mode* l–m	Value†		Mode* l–m	Value†
509	TM 5–12	44.48932	561	TE 20–7	46.68717
510	TM 38–1	44.54601	562	TM 16–8	46.71581
511	TE 4–13	44.57962	563	TE 26–5	46.74158
512	TM 17–7	44.63483	564	TE 10–11	48.82896
513	TM 3–13	44.66974	565	TE 23–6	46.8407$\bar{5}$
514	TE 2–14	44.7145$\bar{5}$	566	TE 44–1	46.87409
515	TM 12–9	44.72194	567	TM 9–11	47.04870
(516	TM 1–14	44.75932	568	TE 15–9	47.05946
(517	TE 0–14	44.75932	569	TM 19–7	47.17400
518	TE 42–1	44.83043	570	TM 28–4	47.1877$\bar{5}$
519	TM 26–4	44.85670	571	TE 8–12	47.22176
520	TM 20–6	44.95768	572	TE 38–2	47.24608
521	TE 11–10	44.97753	573	TM 7–12	47.39417
522	TE 16–8	45.02543	574	TM 14–9	47.4003$\bar{5}$
523	TM 23–5	45.04521	575	TM 22–6	47.42517
524	TE 36–2	45.10166	576	TM 25–5	47.4438$\bar{5}$
525	TE 28–4	45.18473	577	TE 30–4	47.47899
526	TM 10–10	45.23157	578	TE 6–13	47.52196
527	TM 15–8	45.41219	579	TE 18–8	47.59513
528	TE 9–11	45.43548	580	TM 5–13	47.64940
529	TE 19–7	45.43567	581	TM 36–2	47.66568
530	TM 30–3	45.45267	582	TE 13–10	47.6882$\bar{5}$
531	TM 34–2	45.48156	583	TM 41–1	47.69840
532	TE 32–3	45.51018	584	TM 32–3	47.7105$\bar{5}$
533	TE 25–5	45.55917	585	TE 34–3	47.73138
534	TM 39–1	45.59762	586	TE 4–14	47.73367
535	TE 22–6	45.62431	587	TM 3–14	47.81779
536	TM 8–11	45.63844	588	TE 2–15	47.85964
537	TE 14–9	45.74024	589	TE 45–1	47.89542
538	TE 7–12	45.79400	(590	TM 1–15	47.90146
539	TE 43–1	45.85243	(591	TE 0–15	47.90146
540	TM 18–7	45.90766	592	TE 27–5	47.92033
541	TM 6–12	45.94902	593	TE 21–7	47.93298
542	TM 27–4	46.02388	594	TM 12–10	47.97429
543	TE 5–13	46.05857	595	TM 17–8	48.01196
544	TM 13–9	46.06571	596	TE 24–6	48.05260
545	TM 4–13	46.1678$\bar{5}$	597	TE 11–11	48.21133
546	TE 37–2	46.17447	598	TE 39–2	48.31652
547	TM 21–6	46.19406	599	TM 29–4	48.34846
548	TE 3–14	46.23297	600	TE 16–9	48.37069
549	TM 24–5	46.24664	601	TM 20–7	48.43424
550	TM 2–14	46.29800	602	TM 10–11	48.4471$\bar{5}$
551	TE 17–8	46.31377	603	TE 31–4	48.62201
552	TE 1–15	46.31960	604	TE 9–12	48.63692
553	TE 29–4	46.33328	605	TM 26–5	48.63706
554	TE 12–10	46.33777	606	TM 23–6	48.65132
555	TM 0–15	46.34119	607	TM 15–9	48.72646
556	TM 35–2	46.57441	608	TM 42–1	48.74762
557	TM 31–3	46.58280	609	TM 37–2	48.75542
558	TM 11–10	46.60813	610	TM 8–12	48.82593
559	TE 33–3	46.62177	611	TM 33–3	48.83603
560	TM 40–1	46.64841	612	TE 35–3	48.83910

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TABLE—*Concluded*

	Mode* l–m	Value†		Mode* l–m	Value†
613	TE 19–8	48.86993	657	TE 6–14	50.68782
614	TE 46–1	48.91645	658	TM 5–14	50.80717
615	TE 7–13	48.97107	659	TM 44–1	50.84387
616	TE 14–10	49.02964	660	TE 4–15	50.88616
617	TE 28–5	49.09560	661	TE 33–4	50.90045
618	TM 6–13	49.11577	662	TM 39–2	50.93060
619	TE 22–7	49.17342	663	TE 48–1	50.93760
620	TE 5–14	49.21817	664	TM 22–7	50.93776
621	TE 25–6	49.26009	665	TE 13–11	50.94585
622	TM 18–8	49.30111	666	TM 3–15	50.96503
623	TM 4–14	49.32036	667	TE 18–9	50.97113
624	TM 13–10	49.33078	668	TE 2–16	51.00430
625	TE 3–15	49.38130	669	TM 28–5	51.01228
626	TE 40–2	49.38586	(670	TM 1–16	51.04354
627	TM 2–15	49.44216	(671	TE 0–16	51.04354
628	TE 1–16	49.46239	672	TE 37–3	51.04919
629	TM 0–16	49.48261	673	TM 35–3	51.0805$\bar{5}$
630	TM 30–4	49.50618	674	TM 25–6	51.0897$\underline{5}$
631	TE 12–11	49.58340	675	TM 12–11	51.21197
632	TE 17–9	49.67443	676	TM 17–9	51.35527
633	TM 21–7	49.68872	677	TE 21–8	51.40137
634	TE 32–4	49.76246	678	TE 11–12	51.43311
635	TM 43–1	49.79610	679	TE 30–5	51.43637
636	TM 77–5	49.82648	680	TE 42–2	51.5213$\bar{5}$
637	TM 11–11	49.83465	681	TE 24–7	51.63937
638	TM 38–2	49.84371	682	TM 10–12	51.6532$\bar{5}$
639	TM 24–6	49.87276	683	TE 27–6	51.66288
640	TE 47–1	49.93717	684	TE 16–10	51.68742
641	TE 36–3	49.94501	685	TM 32–4	51.81316
642	TM 34–3	49.95933	686	TE 9–13	51.83078
643	TE 10–12	50.04043	687	TM 20–8	51.86002
644	TM 16–9	50.04461	688	TM 45–1	51.8909$\bar{5}$
645	TE 20–8	50.13856	689	TE 49–1	51.97776
646	TM 9–12	50.24533	690	TM 8–13	52.00769
647	TE 29–5	50.26756	691	TM 40–2	52.0161$\underline{5}$
648	TE 15–10	50.36251	692	TM 15–10	52.01724
649	TE 8–13	50.40702	693	TE 34–4	52.03608
650	TE 23–7	50.40880	694	TE 7–14	52.1437$\bar{5}$
651	TE 41–2	50.45412	695	TE 38–3	52.15171
652	TE 26–6	50.46345	696	TM 23–7	52.18166
653	TM 7–13	50.56818	697	TM 29–5	52.1946$\underline{5}$
654	TM 19–8	50.58367	698	TM 36–3	52.19978
655	TM 31–4	50.66103	699	TE 19–9	52.26121
656	TM 14–10	50.67824	700	TM 6–14	52.2794$\bar{5}$

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