question 1

In spherical coordinate system: $u = (r \quad \theta \quad \phi)$ and $(h_i) = (1 \quad r \quad r\sin\theta)$

the expressions for $\Delta\psi, \Delta\frac{1}{r}, \Delta\frac{1}{r^\alpha}$:

we have:

1. $$\nabla\psi = \sum_i \mathbf{e_i}\frac{1}{h_i}\frac{\partial}{\partial u_i}\psi$$
2. $$\nabla\cdot\mathbf{A} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}(A_1h_2h_3) + \frac{\partial}{\partial u_2}(A_2h_3h_1) + \frac{\partial}{\partial u_3}(A_3h_1h_2)\right]$$

so

$$\Delta\psi = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial}{\partial u_1}\psi\right) + \frac{\partial}{\partial u_2}\left(\frac{h_3h_1}{h_2}\frac{\partial}{\partial u_2}\psi\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial}{\partial u_3}\psi\right)\right]$$

and for the spherial coordinate system:

$$\begin{aligned}\Delta\psi &= \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}\left(r^2\sin\theta\frac{\partial}{\partial r}\psi\right) + \frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\psi\right) + \frac{\partial}{\partial\phi}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\psi\right)\right] \\ &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2} \\ &= \frac{\partial^2}{\partial r^2}\psi + \frac{2}{r}\frac{\partial}{\partial r}\psi + \frac{1}{r^2}\cot\theta\frac{\partial}{\partial\theta}\psi + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}\end{aligned}$$

change ψ to $\frac{1}{r}$ and $\frac{1}{r^\alpha}$:

1. $$\Delta\frac{1}{r} = 0\,(r\neq 0)$$
2. $$\Delta\frac{1}{r^\alpha} = \frac{\alpha(\alpha-1)}{r^{\alpha+2}}\,(r\neq 0)$$

Then, we'll calculate them when r = 0, we calculate $\Delta\frac{1}{r^\alpha}$ and when $\alpha = 1$, it's $\Delta\frac{1}{r}$.

$$
\begin{aligned}
\int_V \Delta \frac{1}{r^\alpha}\, dx &= \lim_{a\to 0}\int_V \Delta \frac{1}{\left(r^\alpha + a^\alpha\right)^{\frac{1}{\alpha}}}\, dV \\
&= -\lim_{a\to 0}\int d\Omega \int_0^\infty \left(\frac{(\alpha+1)\, a^\alpha r^\alpha}{\left(r^\alpha + a^\alpha\right)^{\frac{1}{\alpha}+2}}\right) dr \\
&= -4\pi \int_0^\infty \frac{(\alpha+1)\,\rho^\alpha}{\left(\rho^\alpha+1\right)^{\frac{1}{\alpha}+2}}\, d\rho, \quad where \quad \rho = ar. \\
&= 4\pi \int_0^\infty \rho\, d\left(\frac{1}{\left(\rho^\alpha+1\right)^{\frac{1}{\alpha}+1}}\right) \\
&= 4\pi \left[\frac{\rho}{\left(\rho^\alpha+1\right)^{\frac{1}{\alpha}+1}}|_0^\infty - \int_0^\infty \frac{1}{\left(\rho^\alpha+1\right)^{\frac{1}{\alpha}+1}}\, d\rho\right] \\
&= -4\pi \frac{2}{\alpha}\int_0^{\frac{\pi}{2}} \sin^{\frac{2}{\alpha}-1}\theta \cos\theta\, d\theta, \quad where \quad \rho^\alpha = \tan^2\theta. \\
&= -4\pi \frac{1}{\alpha}\frac{\Gamma\left(\frac{1}{\alpha}\right)\Gamma(1)}{\Gamma\left(\frac{1}{\alpha}+1\right)} \\
&= -4\pi \frac{1}{\alpha}\frac{\Gamma\left(\frac{1}{\alpha}\right)\Gamma(1)}{\frac{1}{\alpha}\Gamma\left(\frac{1}{\alpha}\right)} \\
&= -4\pi
\end{aligned}
$$

so$\Delta \frac{1}{r} = -4\pi\delta(\mathbf{r})$,as $\Delta\frac{1}{r} = 0$, when $r \neq 0$.

But for others, the integral can be divided into two parts: $0 \to a, a \to \infty\ (a \to 0)$. When $\alpha \neq 0 or 1$, the notation of the answers is different, but they're all infinity. So the question may not be defined till now.

question 2

we have three vectors:

$(1 \quad 0.1 \quad 0.2)^\mathsf{T}$

$(-0.2 \quad 1 \quad 0.05)^\mathsf{T}$

$(0.1 \quad 0.1 \quad 1)^\mathsf{T}$

do these:

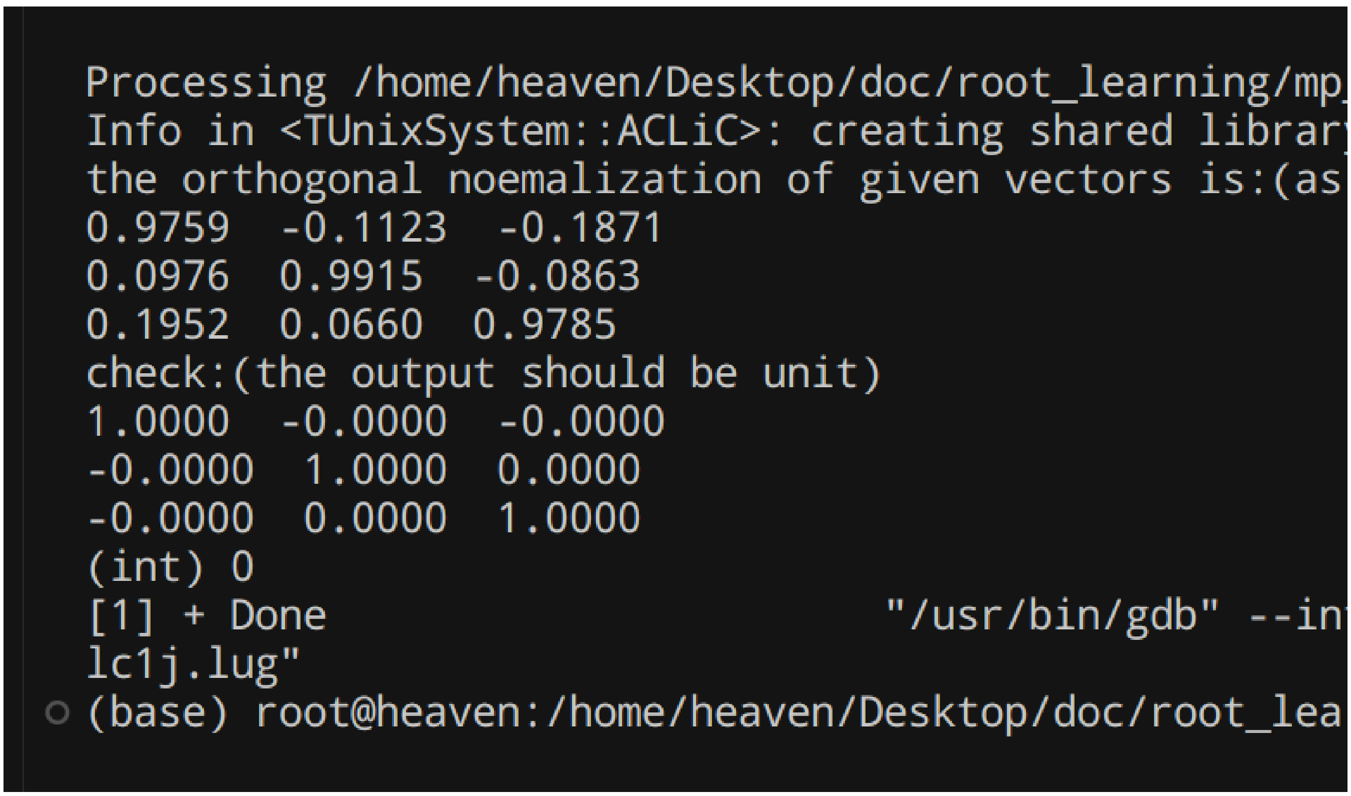
1. orthogonal normalization them
2. check them

the pseudocode is listing below:

```
BEGIN
  // Initialize matrix A with given values
  A = [ [1, -0.2, 0.1],
        [0.1, 1, 0.1],
        [0.2, 0.05, 1] ]

  // Initialize identity matrix I with zeros
  I = 3x3 matrix of zeros

  // Schmidt orthogonalization process
  FOR i FROM 1 TO N-1 DO
    FOR j FROM 0 TO i-1 DO
      // Calculate the dot product 'up' between vectors i and j
      up = dot(A[:, i], A[:, j])

      // Calculate the squared norm 'down' of vector j
      down = dot(A[:, j], A[:, j])

      // Update vector i to be orthogonal to vector j
      IF down != 0 THEN
        A[:, i] = A[:, i] - (up / down) * A[:, j]
      END IF
    END FOR
  END FOR

  // Normalize each vector in A
  FOR i FROM 0 TO N-1 DO
    // Calculate the norm 'mod' of vector i
    mod = sqrt(dot(A[:, i], A[:, i]))

    // Update vector i to be a unit vector
    A[:, i] = A[:, i] / mod
  END FOR

  // Output the orthogonal and normalized matrix A
  PRINT "Orthogonal and normalized matrix A:"
  PRINT A

  // Check: Calculate the dot products to form matrix I
```

```
  FOR i FROM 0 TO N-1 DO
    FOR j FROM 0 TO N-1 DO
      I[i, j] = dot(A[:, i], A[:, j])
    END FOR
  END FOR

  // Output matrix I, which should be close to the identity matrix
  PRINT "Check matrix I:"
  PRINT I
END
```

the result is:



question 3

draw the picture of $\delta\left(\mathbf{x}-\mathbf{x}'\right)=\frac{2}{l}\sum_{k=1}^{\infty}\sin\frac{k\pi}{l}x'\sin\frac{k\pi}{l}x.$

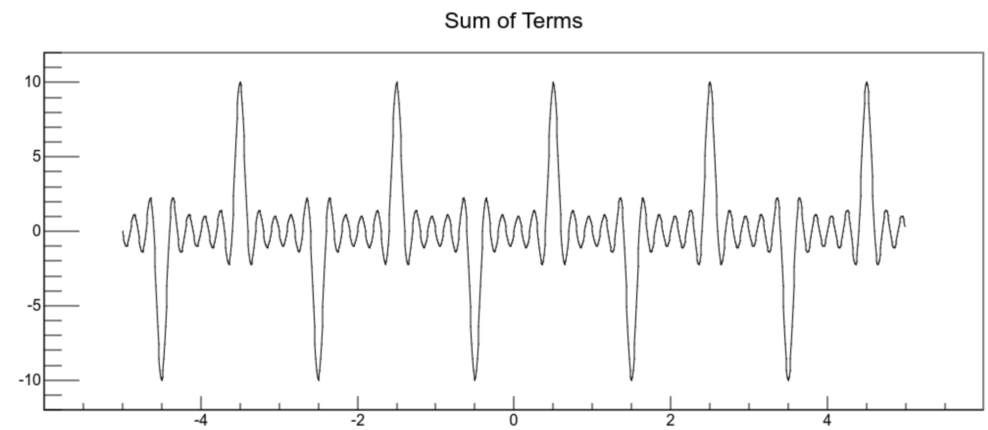
let a = 0.5 and l = 1

the pseudocode is listing below:

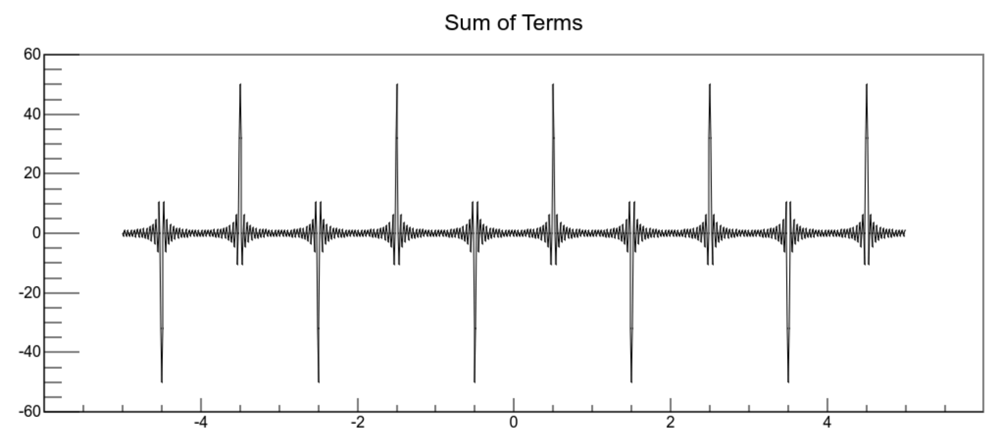
```
BEGIN
  FUNCTION calculateAndPlot(DOUBLE a)
    CONSTANT kMax = 50
    CONSTANT nPoints = 1000
    xMin = -5.0
    xMax = 5.0
    dx = (xMax - xMin) / nPoints
    DOUBLE x[nPoints]
    DOUBLE ySum[nPoints]

    FOR i FROM 0 TO nPoints - 1 DO
      x[i] = xMin + i * dx
      ySum[i] = 0.0
    END FOR

    FOR k FROM (1 + 50) TO (kMax + 100) DO
      FOR i FROM 0 TO nPoints - 1 DO
        DOUBLE term = 2.0 * SIN(k * PI() * a) * SIN(k * PI() * x[i])
        ySum[i] = ySum[i] + term
      END FOR
    END FOR

    CREATE TGraph* graph WITH nPoints, x, ySum
    graph.SetTitle("Sum of Terms")

    CREATE TCanvas* canvas WITH "canvas", "Sum of Terms", 800, 600
    graph.Draw("AL")

    STRING fileName = FORMAT("sum_graph_a_%.2f.root", a)
    CREATE TFile* outputFile WITH fileName, "RECREATE"
    graph.Write()
    outputFile.Close()

    DELETE graph
    DELETE canvas
  END FUNCTION

  FUNCTION delta()
    DOUBLE a = 0.5
```

```
    calculateAndPlot(a)
  END FUNCTION

  CALL delta()
END
```

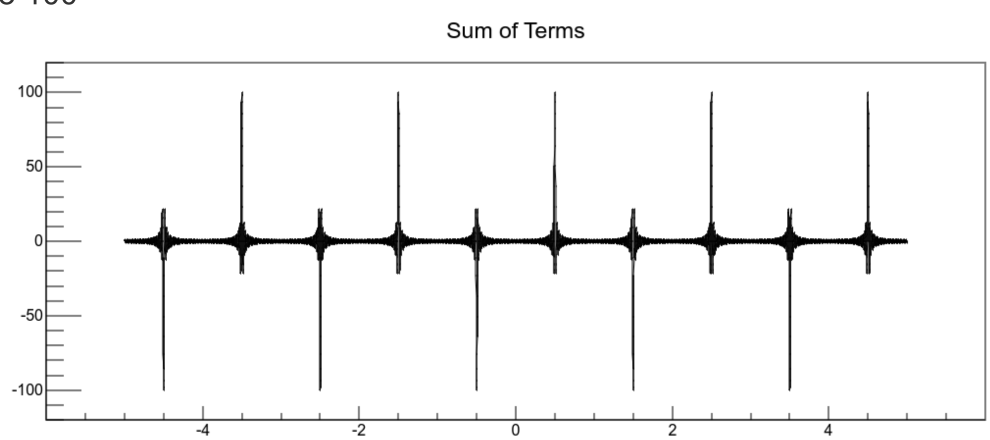
1. k: 1 to 10



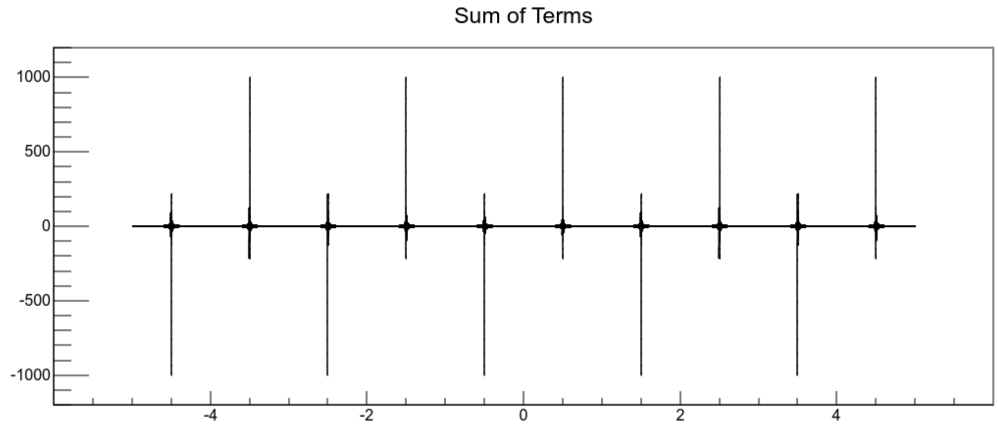
2. k: 1 to 50



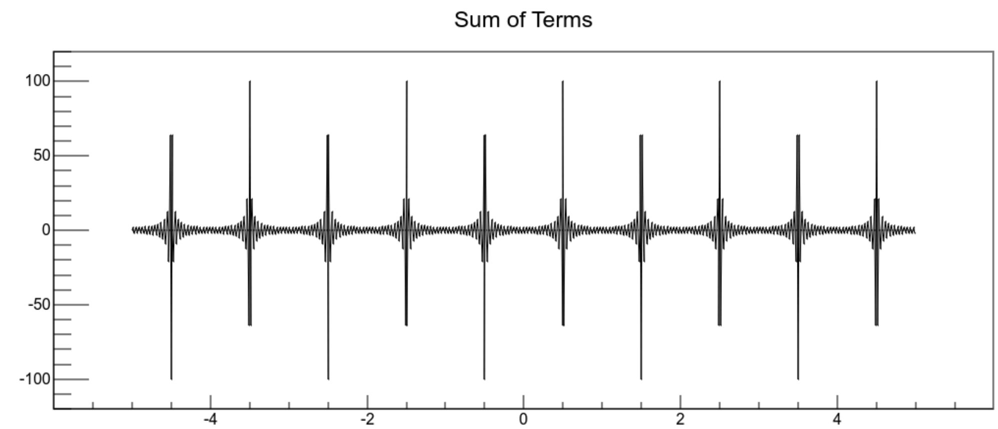
3. k: 1 to 100



4. k: 1 to 10



5. k: 50 to 100



6. k: 1000 to 2000