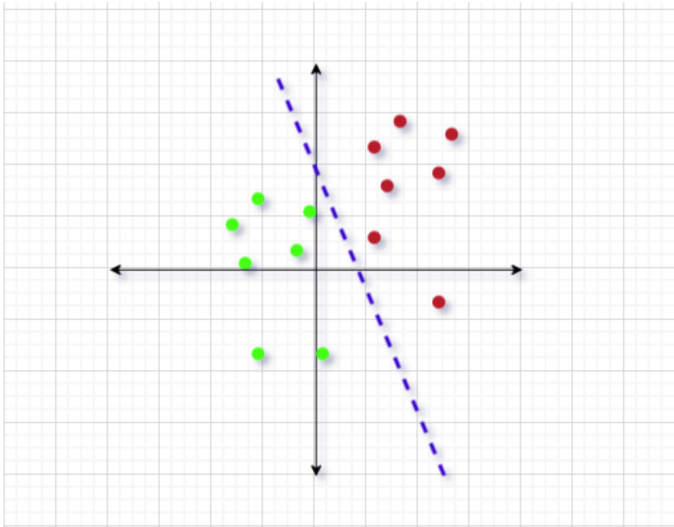
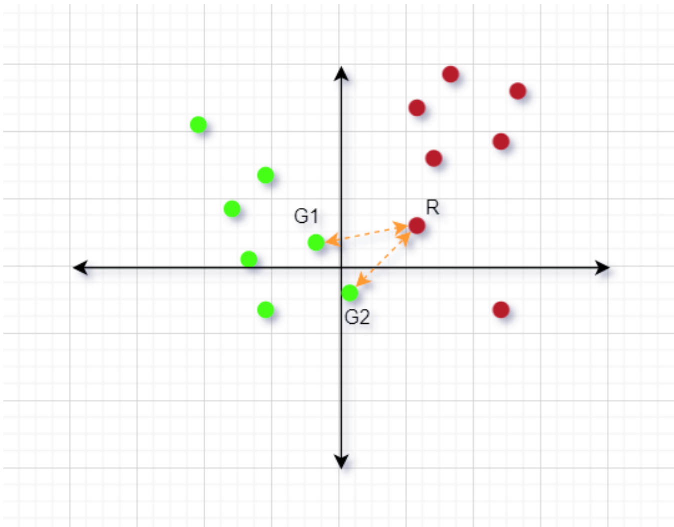
SVM Theory

$$SVM(Support\ Vector\ Machine)\ Theory$$



We will draw a line that will separate red and green points. These are two classes of a dataset.

- We will first calculate the distance between the **negative class and the positive class.**



- Here, the points G1, G2 and R are the closest for the two classes.

- For SVM, we consider the points of different closed classes to draw a boundary using them. These points are called **support vectors**.
- Here:
  - G1 ⇒ S1
  - G2 ⇒ S2
  - R ⇒ S3

Theoretical Steps:

- Let's say the coordinates of the support vectors are:

$$S_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \;\; S_2 = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \;\; S_3 = \begin{pmatrix} 4 \\ 0 \end{pmatrix}$$

- We will augment a 1 at the end for each vector.

$$\bar{S_1} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \;\; \bar{S_2} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \;\; \bar{S_3} = \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix}$$

- For 3 support vectors, we will have 3 parameters.

$$\alpha_1 \quad \alpha_2 \quad \alpha_3$$

- We will do the following calculations, using the matrix calculation:

$$\bar{S}\bar{S^T}\alpha$$

.

$$Where, \quad \bar{S} = \begin{pmatrix} \bar{S_1} \\ \bar{S_2} \\ \bar{S_3} \end{pmatrix} \;\; and \;\; \bar{S^T} = (\bar{S_1} \; \bar{S_2} \; \bar{S_3})$$

- The result will be as follows:

$$\tilde{S}\tilde{S}^T = \begin{bmatrix} \tilde{S}_1\tilde{S}_1 & \tilde{S}_1\tilde{S}_2 & \tilde{S}_1\tilde{S}_3 \\ \tilde{S}_2\tilde{S}_1 & \tilde{S}_2\tilde{S}_2 & \tilde{S}_2\tilde{S}_3 \\ \tilde{S}_3\tilde{S}_1 & \tilde{S}_3\tilde{S}_2 & \tilde{S}_3\tilde{S}_3 \end{bmatrix}$$
$$(3 \times 3)$$

- If the aforementioned is multiplied by alpha, then we will get the following:

$$\alpha_1\bar{S}_1\bar{S}_1 \;+\; \alpha_2\bar{S}_1\bar{S}_2 + \alpha_3\bar{S}_1\bar{S}_3 = -1$$
$$\alpha_1\bar{S}_2\bar{S}_1 + \alpha_2\bar{S}_2\bar{S}_2 + \alpha_3\bar{S}_2\bar{S}_3 = -1$$
$$\alpha_1\bar{S}_3\bar{S}_1 + \alpha_2\bar{S}_3\bar{S}_2 + \alpha_3\bar{S}_3\bar{S}_3 = +1$$

- We are assigning S1 and S2(Green) as the **negative class**, and S3 (Red) as the **positive class. That's why we are assigning -1 to the first two and +1 to the third one.**

- After putting the values of the augmented vectors:

$$\alpha_1\begin{pmatrix}2\\1\\1\end{pmatrix}\begin{pmatrix}2\\1\\1\end{pmatrix} + \alpha_2\begin{pmatrix}2\\1\\1\end{pmatrix}\begin{pmatrix}2\\-1\\1\end{pmatrix} + \alpha_3\begin{pmatrix}2\\1\\1\end{pmatrix}\begin{pmatrix}4\\0\\1\end{pmatrix} = -1$$
$$\alpha_1\begin{pmatrix}2\\-1\\1\end{pmatrix}\begin{pmatrix}2\\1\\1\end{pmatrix} + \alpha_2\begin{pmatrix}2\\-1\\1\end{pmatrix}\begin{pmatrix}2\\-1\\1\end{pmatrix} + \alpha_3\begin{pmatrix}2\\-1\\1\end{pmatrix}\begin{pmatrix}4\\0\\1\end{pmatrix} = -1$$
$$\alpha_1\begin{pmatrix}4\\0\\1\end{pmatrix}\begin{pmatrix}2\\1\\1\end{pmatrix} + \alpha_2\begin{pmatrix}4\\0\\1\end{pmatrix}\begin{pmatrix}2\\-1\\1\end{pmatrix} + \alpha_3\begin{pmatrix}4\\0\\1\end{pmatrix}\begin{pmatrix}4\\0\\1\end{pmatrix} = 1$$

- We know that, for matrix calculation:

$$if \quad A = \begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix} \quad and \quad B = \begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}$$
$$then,$$
$$A.B \;=\; a_1b_1 \;+\; a_2b_2 \;+\; a_3b_3$$

- Finally, after calculation, we have the following 3 equations:

$$
\begin{aligned}
6\alpha_1 \ + \ 4\alpha_2 \ + \ 9\alpha_3 \ &= \ -1 \ ----(i) \\
4\alpha_1 \ + \ 6\alpha_2 \ + \ 9\alpha_3 \ &= \ -1 \ ----(ii) \\
9\alpha_1 \ + \ 9\alpha_2 \ + \ 17\alpha_3 \ &= \ 1 \ ----(iii)
\end{aligned}
$$

- Now we solve for the alpha values and get the following:

$$
\alpha_1 = -3.25 \\
\alpha_2 = -3.25 \\
\alpha_3 = 3.5
$$

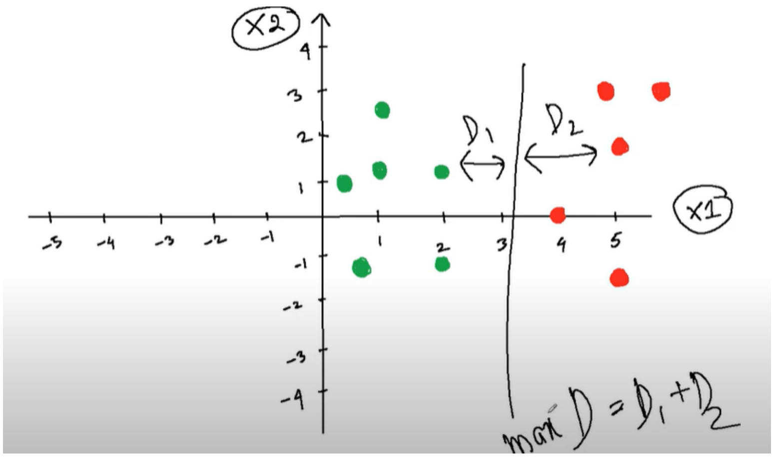
⇒ Since we are working with **Linear SVM**, the **decision boundary will be linear. Hence**

$$
y \ = \ mx \ + \ c
$$

- From the machine learning perspective, the equation becomes:

$$
y \ = \ WX \ + \ b
$$

- We have to **maximize** the margin D. Because we want to separate the classes using the decision boundary. If it's close to the classes, then there is a chance of overlap. So the distance of the **decision boundary** from the classes should be **as high as** possible.



- Finally, the value of Weight is as follows:

$$
\bar{W} \ = \sum \alpha_i \ \bar{S}_i
$$

$$
\begin{aligned}
\bar{W} &= \alpha_1\, \bar{S}_1 \;+\; \alpha_2\, \bar{S}_2 \;+\; \alpha_3\, \bar{S}_3 \\
&= -3.25 \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \;+\; -3.25 \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \;+\; -3.5 \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} -6.50 \\ -3.25 \\ -3.25 \end{pmatrix} + \begin{pmatrix} -6.50 \\ 3.25 \\ -3.25 \end{pmatrix} + \begin{pmatrix} 14.0 \\ 0 \\ 3.5 \end{pmatrix} \\
\bar{W} &= \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}
\end{aligned}
$$

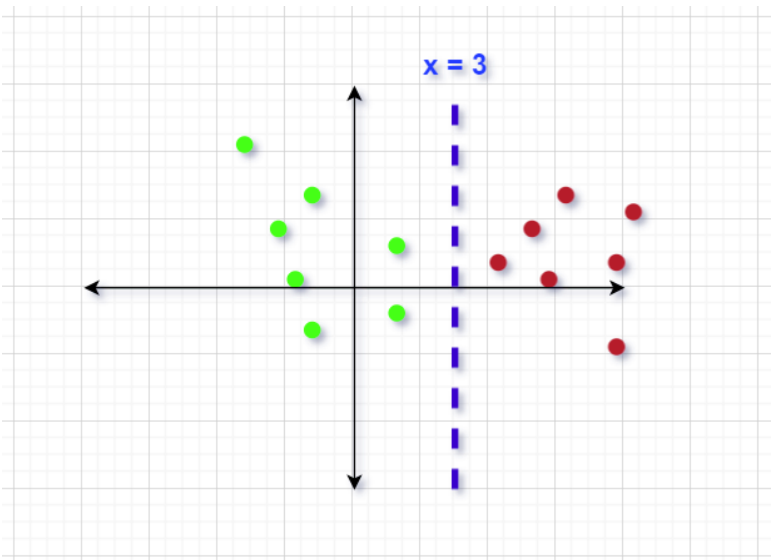
- This is the **augmented W**. The last value -3 was a bias. So finally:

$$
\bar{W} = \begin{pmatrix} 1 \\ 3 \\ -3 \end{pmatrix} \qquad So, \quad W = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \qquad and \qquad b = -3
$$

- The equation of the decision boundary becomes:

$$
\begin{aligned}
y &= W \;\; x \;\; - \;\; 3 \\
=> y &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \;\; x \;\; - \;\; 3
\end{aligned}
$$

- As b = -3, **x = 3, will be positive**. If b were positive, x would be negative. The decision boundary/hyperplane will be:



- For a new point (1, 0), the following will be the calculation to get the class:

$$
\begin{aligned}
y &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \ x \ - \ 3 \\
y &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \ - \ 3 \\
&= (1 + 0) - 3 \\
&= 1 - 3 \\
So, \ y &= -\ 2
\end{aligned}
$$

- As the value is negative, the new point belongs to the **negative class.** If the result were positive, the point would belong to the positive class.

Linear SVM also works with non-linear data, which is a great strength.