Isotopic incorporation and the temporal dynamics of foraging among individuals and within populations

Abstract.

1. Introduction

2. Time-dependent Ornstein-Uhlenbeck process

We can define the evolution of the concentration of a particular isotope in the predator tissue by the following Langevin equation,

$$\mathrm{d}X(t) = -\mathrm{f}\left(X(t) - \mu(t)\right)\mathrm{d}t + \mathrm{f}\sigma(t)\mathrm{d}W \tag{1}$$

where, X is the isotope-concentration, f is the tissue-dependent incorporation rate, $\mu(t)$ and $\sigma(t)$ are the time-dependent average and standard-deviations of isotope-concentration over the prey.

2.1. Expectation

We can find the expectation of $X(t)$ as follows,

$$\mathrm{d}X + \mathrm{f}\left(X(t) - \mu(t)\right)\mathrm{d}t = \mathrm{f}\sigma(t)\mathrm{d}W \tag{2}$$

Multiplying throughout by the integrating factor, $e^{\mathrm{f}t}$ and integrating both sides w.r.t t, we get

$$\int_{t=0}^{t} \mathrm{d}\left(e^{\mathrm{f}t}\left(X - \mu\right)\right) + \int_{t=0}^{t} e^{\mathrm{f}t'}\frac{\mathrm{d}\mu}{\mathrm{d}t'}\mathrm{d}t' = \int_{t=0}^{t} \mathrm{f}e^{\mathrm{f}t'}\sigma(t')\mathrm{d}W' \tag{3}$$

Taking the expectation of both sides, the RHS reduces to zero as $\mathbb{E}(\mathrm{d}W) = 0$ and we get,

$$\mathbb{E}(X(t)) = X_0 e^{-\mathrm{f}t} + \mathrm{f}e^{-\mathrm{f}t}\int_{t'=0}^{t} \mu(t')e^{\mathrm{f}t'}\mathrm{d}t' \tag{4}$$

2.2. Variance

Similarly for the variance, we square Eq. (3), to get

$$\left(e^{\mathrm{f}t}X - X_0 - \mathrm{f}\int_0^t \mu e^{\mathrm{f}t'}\mathrm{d}t'\right)^2 = \mathrm{f}^2\int_0^t\int_0^t e^{\mathrm{f}(t'+t'')}\sigma(t')\sigma(t'')\mathrm{d}W(t')\mathrm{d}W(t'') \tag{5}$$

Taking the expectation on both sides, in the RHS, using the property, $\mathbb{E}(\mathrm{d}W(t')\mathrm{d}W(t'')) = \delta(t'-t'')\mathrm{d}t'\mathrm{d}t''$, we get,

$$\mathbb{E}\left(e^{\mathrm{f}t}X - X_0 - \mathrm{f}\int_0^t \mu e^{\mathrm{f}t'}\mathrm{d}t'\right)^2 = \mathrm{f}^2\int_0^t e^{2\mathrm{f}t'}(\sigma(t'))^2\mathrm{d}t' \tag{6}$$

Let us call the integrals $\int_0^t \mu(t')e^{\mathrm{f}t'}\mathrm{d}t'$ and $\int_0^t e^{2\mathrm{f}t'}(\sigma(t'))^2\mathrm{d}t'$ as $\mathcal{U}$ and $\mathcal{S}^2$ and expanding the LHS, we get

$$e^{2\mathrm{f}t}\mathbb{E}(X^2) + X_0^2 + \mathrm{f}^2\mathcal{U}^2 - 2X_0\mathbb{E}(X)e^{\mathrm{f}t} - 2X_0\mathrm{f}\mathcal{U} - 2\mathrm{f}e^{\mathrm{f}t}\mathbb{E}(X)\mathcal{U} = \mathrm{f}^2\mathcal{S}^2 \tag{7}$$

And so the variance is given by,

$$\begin{aligned}\mathbb{V}(X) &= \mathbb{E}(X^2) - (\mathbb{E}(X))^2 \\ &= \mathrm{f}^2 e^{-2\mathrm{f}t}\mathcal{S}^2 - X_0^2 e^{-2\mathrm{f}t} - \mathrm{f}^2 e^{-2\mathrm{f}t}\mathcal{U}^2 + 2X_0\mathbb{E}(X)e^{-\mathrm{f}t} + 2X_0\mathrm{f}e^{-2\mathrm{f}t}\mathcal{U} + 2\mathrm{f}e^{-\mathrm{f}t}\mathbb{E}(X)\mathcal{U} - (\mathbb{E}(X))^2\end{aligned} \tag{8}$$

$$\mathbb{V}(X) = \mathrm{f}^2 e^{-2\mathrm{f}t}\int_0^t e^{2\mathrm{f}t'}(\sigma(t'))^2 \mathrm{d}t' \tag{9}$$

where to go from Eq. (**??**) to Eq. (9) we used $\mathbb{E}(X)$ given by Eq. (4).