

4.1

Adding more biological detail

$$\frac{dN}{dt} = rN \left(1 - \frac{N}{K}\right) - \mu N$$

$$0 = rN \left(1 - \frac{N}{K}\right) - \mu N$$

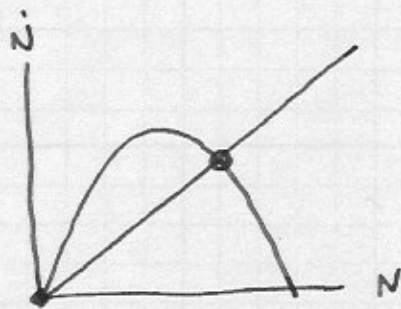
$$rN \left(1 - \frac{N}{K}\right) = \mu N$$

$$r \left(1 - \frac{N}{K}\right) = \mu$$

$$1 - \frac{N}{K} = \frac{\mu}{r}$$

$$\frac{N}{K} = 1 - \frac{\mu}{r}$$

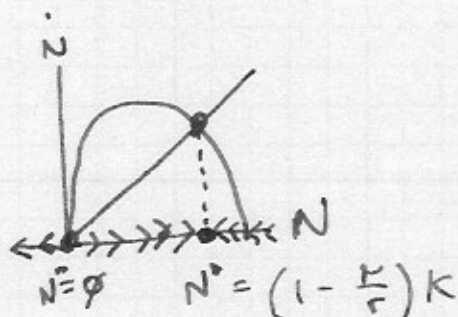
$$N^* = \left(1 - \frac{\mu}{r}\right)K$$

~ when $\mu = 0$, $N^* = K$ as before

F.P.

Stability:When $rN \left(1 - \frac{N}{K}\right) > \mu N$, $\frac{dN}{dt} > 0$ (growth)When $rN \left(1 - \frac{N}{K}\right) < \mu N$, $\frac{dN}{dt} < 0$ (decline)

So:

 $N^* = 0$ unstable $N^* = \left(1 - \frac{\mu}{r}\right)K$ stableRecall $\dot{N} = f(N)$

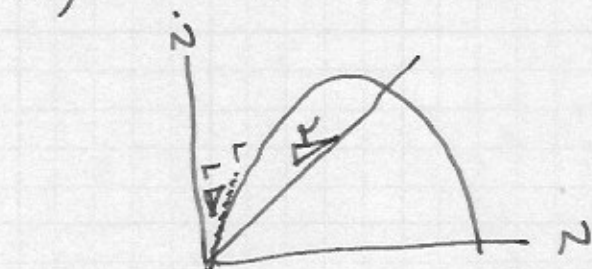
$$\dot{\eta} = f'(N^*)\eta \quad \text{so: } \eta(t) = \eta(0)e^{f'(N^*)t}$$

so: $f'(N^*)$ is the test functionif $f'(N^*) > 0$ then N^* is ~~unstable~~ unstableif $f'(N^*) < 0$ then N^* is stable

$$f(N) = rN \left(1 - \frac{N}{K}\right) - \mu N = rN - \frac{rN^2}{K} - \mu N$$

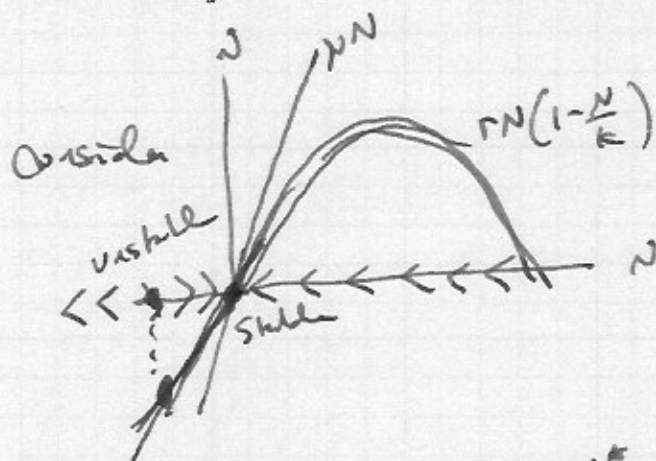
$$f'(N) = r - \frac{2rN}{K} - \mu$$

$$f'(N^* = 0) = r - 0 - \mu = r - \mu \quad (?)$$



$r \sim$ slope of parabola near $N=0$

$\mu \sim$ slope of mortality function



if $\mu > r$, then

$$N^* = \left(1 - \frac{\mu}{r}\right)K$$

Negative value

$$f'(N^* = (1 - \frac{\mu}{r})K) = r - \frac{2rN^*}{K} - \mu$$

$$= r - \frac{2r[(1 - \frac{\mu}{r})K]}{K} - \mu$$

$$= r - 2r(1 - \frac{\mu}{r}) - \mu$$

$$= r - 2r + \frac{2r\mu}{r} - \mu$$

$$= -r + \mu$$

$$= \mu - r$$

given we assume

$$\mu < r, f'(N^*) < 0$$

\downarrow stable.

4.4

$$\text{F.P. } N^* = \{\emptyset, A, K\}$$

$$\dot{N} = rN \left(\frac{N}{A} - 1 \right) \left(1 - \frac{N}{K} \right)$$

$$\frac{\partial \dot{N}}{\partial N} = \left(\frac{rN^2}{A} - rN \right) \left(1 - \frac{N}{K} \right)$$

$$= \frac{rN^2}{A} - \frac{rN^3}{AK} - rN + \frac{rN^2}{K}$$

$$= \frac{2rN}{A} - \frac{3rN^2}{AK} - r + \frac{2rN}{K}$$

$$= \frac{r}{AK} [2(A+K)N - 3N^2 - AK]$$

for $N^* = \emptyset$, $\frac{\partial \dot{N}}{\partial N} = \frac{r}{AK} (-AK) = -r$ stable

for $N^* = A$, $\frac{\partial \dot{N}}{\partial N} = \frac{r}{AK} [2A^2 + 2AK - 3A^2 - AK]$

$$= \frac{r}{AK} [-A^2 + AK]$$

$$= \frac{r}{AK} [-A^2 + AK] = \frac{r}{K} [K - A] = r - \frac{Ar}{K}$$

if $\frac{A}{K} < 1$, then $r - \frac{Ar}{K} > 0$

unstable

for $N^* = K$, $\frac{\partial \dot{N}}{\partial N} = \frac{r}{AK} [2AK + 2K^2 - 3K^2 - AK]$

$$= \frac{r}{AK} [AK - K^2] = \frac{r}{A} [A - K] = r - \frac{rK}{A}$$

if $\frac{K}{A} > 1$, then $r - \frac{rK}{A} < 0$ } Stable

Allee Effect

- growth rate is ~~less~~ negative @ low population densities
- Difficulty finding mates (mate limitation)
- Cooperative Defense: Increased energy devoted to predator-watching (less on mating) when $N \downarrow$
~~FOR~~ low efficacy of anti-predator herding behavior \uparrow predator rate
- Cooperative feeding (where foraging depends on pack-hunting for example)
- Engineering: where large populations can alter environment to advantage

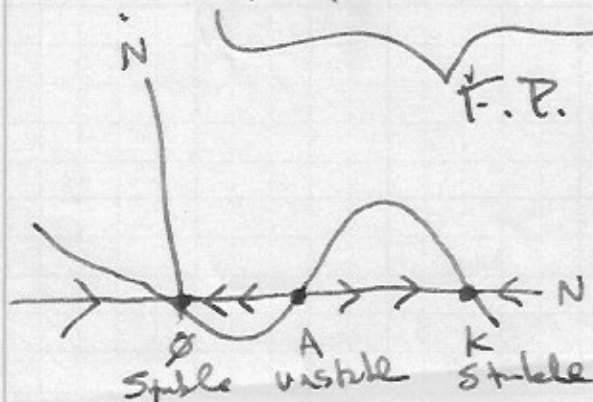
Imagine critical N $N_c = A$ below which $\frac{dN}{dt} < 0 \dots$

then: $\frac{dN}{dt} = rN \left(\frac{N}{A} - 1 \right) \left(1 - \frac{N}{K} \right)$
 if $N < A$, $0 < \phi$ and $\frac{dN}{dt} < 0$

$$\phi = rN \left(\frac{N}{A} - 1 \right) \left(1 - \frac{N}{K} \right)$$

$$N^* = 0 \quad N^* = A \quad N^* = K$$

F.P.



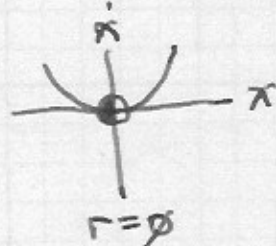
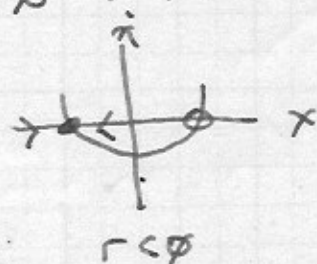
5.1 Bifurcations: Change in stability

Saddle-Node Bifurcations

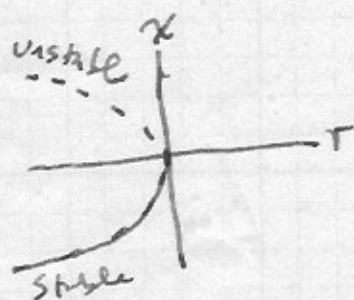
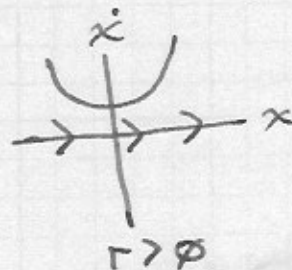
- Mechanism by which bifurcations are created & destroyed
- A parameter is varied, 2 F.P. move towards each other, collide, and annihilate

$$\dot{x} = r + x^2$$

r is a param



Bifurcation @ $r = 0$



vary r as a function of x

F.P. ~~the~~

$$\begin{aligned} 0 &= r + x^2 \\ x^2 &= -r \\ x^* &= \pm \sqrt{-r} \end{aligned}$$

When $r < 0$, sqrt is of a positive number

$$\text{Ex) } \dot{x} = r - x^2$$

$$\text{F.P.: } x^* = \pm \sqrt{r}$$

2 F.P. for $r \geq 0$

0 F.P. for $r < 0$

$$f'(x^*) = \frac{d}{dx}(r - x^2) = -2x^*$$

$$\begin{aligned} x^* &= +\sqrt{r} \rightarrow -2\sqrt{r} \\ x^* &= -\sqrt{r} \rightarrow 2\sqrt{r} \end{aligned}$$

$$\begin{aligned} &\rightarrow x^* \rightarrow +\sqrt{r} \rightarrow -2\sqrt{r} \\ &\rightarrow x^* \rightarrow -\sqrt{r} \rightarrow 2\sqrt{r} \end{aligned}$$

(stable)

(unstable)



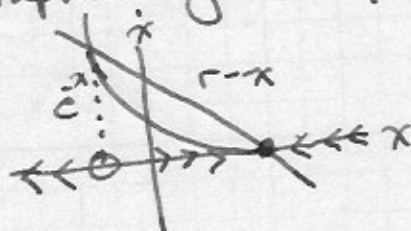
Ex) $\dot{x} = r - x - e^{-x}$

1) break up into pieces

$(r-x)$ is ~ line... how $x \uparrow$

$-e^{-x}$ ~ exponential... how $x \downarrow$

2) Graphically analyse it:



F.P. $r - x^* - e^{-x^*} = 0$
 $r - x^* = e^{-x^*}$

but can't find F.P. at a function of 'r'

VISUALLY, the bifurcation occurs when lines & tangents are eq -

$f'(x) \frac{df(x)}{dx} = -1 + e^{-x}$ where $\frac{d}{dx}(r-x) = \frac{d}{dx}e^{-x}$

- The critical value of x occurs @

defined by $\frac{df(x)}{dx} = 0$

- So! $-1 + e^{-x^*} = 0$

$e^{-x^*} = 1$

So! $x^* = 0$ @ Bifurcation occurs @ $x = 0$

F.P. @ Bifurcation:

$r_c - x^* - e^{-x^*} = 0$

$r_c = 1$