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Project 3

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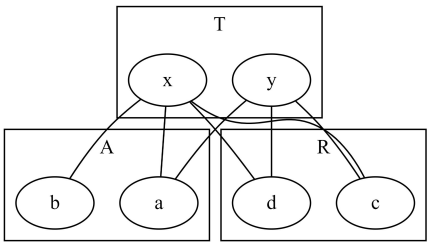
What is Balance?

Balanced Graph

A graph G is balanced if all cycles in G have a length that is a multiple of four. Balanced graphs have certain properties, such as, containing extended star cuts or 2-join, or being strongly balanced. Additionally balanced graphs are bipartite.[5]

For a graph G to contain an extended star cut, G must be split into 3 distinct sets (T, A, R) and a vertex $x \in T$ such that all vertices in $A \cup R$ are adjacent to x and the sets $T \cup R$ form a complete bipartite graph[5]. Here's an example:



To contain a 2-join a graph G must be partitioned into two sets, V_1, V_2 and for $i = 1, 2$ then there exists a disjoint non-empty $A_i, B_i \subseteq V_i$ that satisfy the following condition. Every vertex in A_1 is adjacent to A_2 and likewise for sets B_i. Also, sets V_1 and V_2 have no other edges between them. Lastly, V_1 and V_2 must contain 3 or more vertices each[4]. An example:

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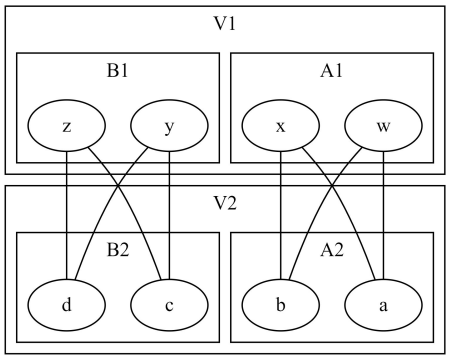


Strongly balanced graphs are balanced graphs that do not contain exactly one edge not part of a cycle but connects a two vertices in the cycle, also known as a chord.[5]

Theorem 1. *If G is a balanced graph, and not strongly balanced, then it contains an extended star cut or 2-join.*

Balanced Signed Graphs

A quick note on signed graphs is that they are weighted graphs, where weight is represented as pluses or minuses (or +1 or -1, respectively) instead of real number values. Plus-weighted edges refer to similarity, trust, friend between two vertices. Minus-weighted edges mean the opposite, dissimilarity, distrust, or ememy.[1]

Theorem 2. *A signed graph G is balanced if and only if G can be partitioned into sets V_1 and V_2 where all vertices in a set are connected by positive edges and sets V_1 and V_2 are connected by negative edges.*[1]

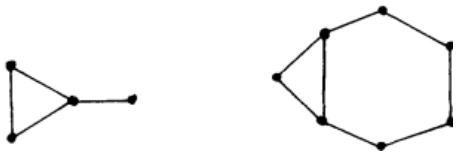
Proposition 1. *A signed graph G is balanced if and only if every cycle contains an even number of edges*

Balanced Matrix

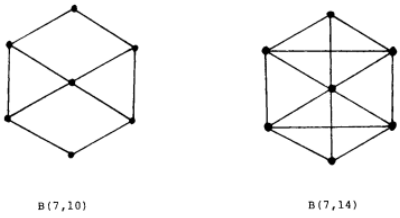
To create a (0,1) matrix M representation of a graph $G = (V_1 \cup V_2, E)$ where every row in M is a vertex in V_1 and every column in M is a vertex in V_2. The value of $m_{i,j} = 1$ if there is an edge that connects $i \in V_1$ and $j \in V_2$ and 0 otherwise.[5]

Theorem 3. *A matrix M is balanced if and only if M does not contain a square submatrix with odd order with two ones per row and column*

Examples



The two graphs above are balanced graphs, but are not strongly balanced graphs.



The two graphs above are strongly balanced graphs.

Applications

Applications of Balanced Graphs

Balanced graphs are sometimes hard to be applied to real world problems. Noisy data

can make it difficult or impossible to use. One way they are useful is by partitioning the graphs. Graph partitioning is when you separate vertices into a certain number of groups while minimizing the number of edges deleted.[2] This can be viewed in a real world setting by looking at the vertices as tasks and the edges as communications.

Applications of Balanced Signed Graphs

Signed graphs are applied in various sciences from STEM to social sciences. In social psychology, these graphs are used in conjunction with balance theory. The nodes represent people and edges represent relationships. Physics has an application for signed graphs when studying spin glasses. In biology, signed graphs are used to map population or ecosystems and used to study the behavior of both of the systems. Lastly, in computer science, signed graphs are used in correlation clustering.[7] Correlation clustering attempts to partition data in which data similar to each other is clustered together.[6]

Problems Balanced Graphs Simplify

3-Colorablity

Balanced graphs can be used to help simplify the problem of 3-Colorability. This is so when a balanced graph contains a star-cut. Within the graph each set (T,A,R) can be assigned a color. Then because no vertices within each set connect, the graph is 3-colorability.

Hamiltonian

We can use balanced graphs to tell if a graph is Hamaltonian. First we can use the balanced graph test 2-join. If a graph is a 2-join then the graph is balanced and is also hamiltonian because the graph is a cycle.

References

[1] *Signed graphs.*

[2] *Graph partition — Wikipedia, the free encyclopedia*, 2020. [Online; accessed 28-April-2020].

[3] Matthias Beck, *(enumeration results for) signed graphs.*

[4] Pierre Charbit, Michel Habib, Nicolas Trotignon, and Kristina Vušković, *Detecting 2-joins faster*, 2012.

[5] Michele Conforti, Gérard Cornuéjols, and M.R. Rao, *Decomposition of balanced matrices*, 1999.

[6] Wikipedia contributors, *Correlation clustering — Wikipedia, the free encyclopedia*, 2020. [Online; accessed 28-April-2020].

[7] ———, *Signed graph — Wikipedia, the free encyclopedia*, 2020. [Online; accessed 28-April-2020].