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Orbital Analysis of Passive Dynamic Walker with Dual Arms

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***Abstract*—Unlike traditional models that predominantly consider lower limb dynamics, this research introduces the study of the passive biped walking with dual arms. The system's dynamic movement is represented through an optimization formulation, subject to constraints that ensure periodicity and optimal behavior. With optimization, the passive walker could generate natural biped locomotion with dual arm swing motion. Next, the study extensively investigates the effects of the arm's dynamic parameters on the overall locomotion, stability, and energy demand of the entire system. The results clearly demonstrate the impact of these dynamic parameters on stability, energy consumption, and step characteristics of the biped locomotion. Moreover, the insights gained from this study provide valuable guidance for the future development of robotic systems that more closely mimic human locomotion.**

***Index Terms*—passive biped walker, optimization, dual arms**

I. Introduction

Passive walking systems, which utilize the natural dynamics and mechanical properties of their structures, have emerged as a focal point in robotics [1]–[3] and biomechanics [4], [5] due to their potential for achieving energy-efficient locomotion with minimal control inputs. These systems capitalize on gravity and momentum, allowing for movement without active actuation, which makes them an ideal model for studying the fundamentals of biped walking and the development of efficient robotic walkers [6], [7].

Traditionally, research in passive walking has primarily concentrated on the lower limbs. Studies such as those by Collins et al. [3] and Goswami et al. [8] have laid the groundwork by examining the dynamics of the legs and hips, offering insights into the optimization of these components for improved walking performance. Besides, it has been shown the foot shape is also critical and affects the behaviors of the system [9]. Study in [10] used asymmetric three-link model and obtained a natural biped locomotion. More complexity passive walkers have been studied using flexible legs [1], flat feet in knee-less models [11], and knee-equipped models [12]. However, these studies largely overlook the influence of upper limb dynamics, particularly arm movement, which plays a crucial role in human locomotion [4], [13].

Human gait studies have long established that arm swing is crucial for maintaining balance, enhancing momentum transfer, and improving overall gait efficiency [14]–[16]. The rhythmic motion of the arms not only counteracts the rotational forces generated by the legs but also stabilizes the body's center of mass (CoM) during walking. Further robotics research into upper body movements has provided additional insights into how these configurations can influence gait stability and efficiency [17]. Deng et al. proposed a passive walker with a torso coupled via torsional springs and the gait performances were significantly improved [18]. Bassel et al. [19] explored the optimization of biped locomotion with and without arm swing, finding that incorporating arm motion could significantly enhance the robot's stability and walking speed. Similarly, Sadeghian et al. [20] examined the effects of various arm parameters on passive walking dynamics, demonstrating that a damped arm configuration allowed the passive walker to quickly achieve a periodic gait, thereby underscoring the importance of arm dynamics in stabilizing bipedal locomotion. In general, compared with the research of lower limbs, the influence of upper limbs on bipedal walking still needs further exploration.

In the case of the passive walker derived from the compass-like robot [6], the dynamics of bipedal robots exhibit hybrid and highly nonlinear characteristics, even though the system's kinematics are relatively simple. The simplified model is a

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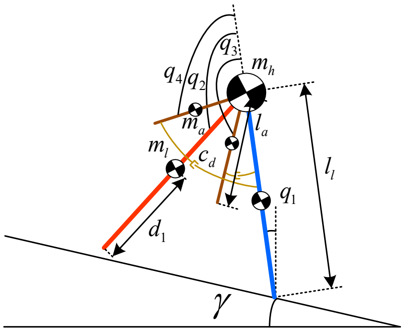


Fig. 1. A passive biped walker with dual stabilizing arms on a slope inclined at angle γ. The stance leg, swing leg, and arms are depicted in blue, red, and brown, respectively. Damping is applied between the arms and the stance leg.

widely used approach for analyzing bipedal walking [21], [22]. This paper examines the locomotion of a planar passive biped walker equipped with stabilizing arms, including the development of robotic models with dual-arm configurations. Additionally, the stability of periodic orbits, energy considerations, and step characteristics are thoroughly analyzed, with a discussion on the impact of the arms. Empirical trials will be conducted to observe the direct effects of arm movement on walking dynamics, while simulations will help predict the long-term outcomes of different arm configurations.

In this way, by understanding the role of arm movement in passive walking, this study aims to advance the design of more effective and adaptable robotic systems. The insights gained from this research could significantly contribute to the broader field of biomechanics by elucidating the complex interactions between upper and lower limb dynamics in passive locomotion. The novelty of this study includes:

- A new passive walker with dual arms is introduced.
- The effects of dual arms are discussed. The optimized parameters of the dual arms are also provided.

The remainder of this paper is structured in the following manner. In Sec. 2, the preliminaries including Poincaré return map are introduced. The simulation and analysis of the passive walker with free-swing arm are reported in Sec. 3. Section 4 provides the discussion and conclusion.

II. Passive dynamic walking

A. Dynamics of the Planar Passive Biped Walker

In this study, we analyze the dynamics of a planar biped walker, extending the traditional compass-like model to include dual damped arms. The walker consists of two rigid legs, namely, the stance leg and the swing leg, joined by a frictionless hip joint. The model operates on a slope with an inclination angle γ, as illustrated in Fig.1. Each leg has a length l_l and mass m_l. For simplicity, the CoM of each leg is assumed to be at its geometric center, and the leg is treated as a concentrated mass with negligible moment of inertia. The relative angles of the ankle and hip are denoted by q_1 and q_2, respectively. q_3 and q_4 represent the relative angles of the left and right shoulders. The physical parameters for these models are summarized in Table I.

TABLE I
Physical parameters of the passive walkers in this study.

Parameter	Symbol	Value
Upper body mass	m_u	0.240 kg
Leg mass	m_l	0.100 kg
Leg length	l_l	0.3 m
Slope angle	γ	2°
CoM of the leg	d_l	$l_l/2$
Arm mass	m_a	0.004 kg
Arm length	l_a	$l_l/2$
CoM of the arm	d_a	$l_a/2$
Hip mass	m_h	$m_u - 2m_a$
Damping coefficient	c_d	$0.0001\ \mathrm{N\cdot\ m\cdot\ s/rad}$

The dynamics of the biped walker are modeled as hybrid impulse systems. These systems combine continuous motion phases with discrete impact events. The model assumes that the impact between the swing leg and the ground is instantaneous, following the principles of angular momentum conservation. The impulsive forces lead solely to an abrupt shift and discontinuity in the robot's joint velocities, leaving the configuration unchanged. Additionally, the impact is modeled as perfectly inelastic, with sufficient friction to prevent sliding, ensuring no rebound or slip occurs.

The dynamics of the biped walker during the swing phase are described as follows,

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{G}(\boldsymbol{q}) = \boldsymbol{\tau_d}, \tag{1}$$

where $\boldsymbol{q}$ represents the generalized coordinates, $\boldsymbol{M}(\boldsymbol{q})$ denotes symmetric and positive-definite inertia matrix, $\boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})$ accounts for centrifugal and Coriolis forces, $\boldsymbol{G}(\boldsymbol{q})$ represents gravitational forces, and $\boldsymbol{\tau_d}$ is the damper torque applying to the biped walker. The damping coefficient is c_d.

According to the law of conservation of momentum, the impact dynamics are described as follows,

$$\boldsymbol{A}^{+}\left(\boldsymbol{q}^{+}\right)\dot{\boldsymbol{q}}^{+} = \boldsymbol{A}^{-}\left(\boldsymbol{q}^{-}\right)\dot{\boldsymbol{q}}^{-}, \tag{2}$$

where superscript $(.^{-})$and $(.^{+})$ denote the states immediately before and after the impact. The post-impact states can also be expressed as,

$$\left(\boldsymbol{q}^{+},\dot{\boldsymbol{q}}^{+}\right) = \boldsymbol{\Delta}\left(\boldsymbol{q}^{-},\dot{\boldsymbol{q}}^{-}\right). \tag{3}$$

Setting the state vector $\boldsymbol{x} = \left[\boldsymbol{q}^{\mathrm{T}},\dot{\boldsymbol{q}}^{\mathrm{T}}\right]^{\mathrm{T}}$, the hybrid dynamics in Eq. (1) and Eq. (3) can be expressed in state-space form as follows,

$$\Sigma : \begin{cases} \dot{\boldsymbol{x}} = f(\boldsymbol{x}), & \boldsymbol{x}^{-} \notin \boldsymbol{S} \\ \boldsymbol{x}^{+} = \boldsymbol{\Delta}\left(\boldsymbol{x}^{-}\right), & \boldsymbol{x}^{-} \in \boldsymbol{S}, \end{cases} \tag{4}$$

where $\boldsymbol{S}$ denotes the impact switching surface, which is triggered when the swing leg contacts the ground.

B. Poincaré Map and Periodic Orbit Stability Analysis

In the analysis of passive biped systems, we consider a model that transitions between a continuous swing phase and an instantaneous impact phase. Due to the cyclic nature of walking, the hybrid system described by Eq. 4 generates periodic orbits within its phase space. To assess the system's stability, we use the Poincaré return map. Specifically, we examine the Poincaré section immediately following each impact to analyze the return behavior of the system. The transition function between states is represented by $\boldsymbol{P}: \boldsymbol{S} \rightarrow \boldsymbol{S}$, defined as,

$$\boldsymbol{x}_{k+1} = \boldsymbol{P}\left(\boldsymbol{x}_k\right), \tag{5}$$

where this function maps the system's state from one impact to the next. A fixed point, $\boldsymbol{x}^*$, is a state where $\boldsymbol{x}^* = \boldsymbol{P}\left(\boldsymbol{x}^*\right)$.

To obtain a low-energy fixed point, we use an optimization function $G\left(\boldsymbol{x}\right)$ expressed as,

$$\min G\left(\boldsymbol{x}\right) = \alpha\left(P\left(\boldsymbol{x}\right) - \boldsymbol{x}\right) + \beta\left(\boldsymbol{x}^{\mathrm{T}}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{M}\boldsymbol{B}\boldsymbol{x}\right), \tag{6}$$

where α and β are the weight coefficients, and $\boldsymbol{B}$ is the selection matrix used to compute the generalized velocities $\dot{\boldsymbol{q}} = \boldsymbol{B}\boldsymbol{x}$. Due to the complexity of the Poincaré map $\boldsymbol{P}$, its explicit form is difficult to determine. Therefore, the stability of the fixed point is evaluated through linearization, via the Jacobian matrix of the Poincaré map,

$$\boldsymbol{\delta x}_{k+1} = \frac{\partial \boldsymbol{P}}{\partial \boldsymbol{x}}\boldsymbol{\delta x}_k, \tag{7}$$

where $\boldsymbol{J} = \frac{\partial \boldsymbol{P}}{\partial \boldsymbol{x}}$ characterizes how deviations evolve between impacts. Stability is assured if all eigenvalues of the Jacobian matrix have magnitudes less than one. The nonlinear equations governing the system are solved numerically using MATLAB's ODE45 function.

III. Passive Biped Walking with Arms

A. Periodic Orbits of the Passive Walking with the Dual Arms

Herein, to identify stable periodic orbits, we initially solve the optimization problem using a genetic algorithm [23]. The resulting feasible solution serves as the starting point for further refinement through the *Fmincon* toolbox in MATLAB [24]. The following equality and inequality constraints are used to find a feasible solution.

Equality constraint:

- For the walker at its initial instant, its two legs should land at the same time. According to geometry, we can get $q_{1i} + \frac{q_{2i}}{2} + \gamma = \frac{\pi}{2}$.

Inequality constraint:

- The initial velocity $\dot{q}_{1i}$ should be negative, $\dot{q}_{1i} < 0$, such that the walker could naturally walk down the slope.
- The initial angle, q_{1i}, should be within acceptable ranges ($q_{1i} > -0.08$) to give proper kinematic posture.

After optimization, the periodic orbit reaches a stable fixed point,

$$\begin{aligned}\boldsymbol{x}^* = [&0.1867, 2.6985, 2.8278, 2.8278,\\ &-1.6474, 0.8453, -2.3241, -2.3241]^{\mathrm{T}}.\end{aligned} \tag{8}$$

According to the fixed point, one could notice that the dual arms have the same trajectory, that is, the effect of dual arms is the same as that of one arm. To generate the human-like arm swing motion, a compass mechanism similar to that in [25] is used to keep the dual arms evenly spaced on either side of the middle line of the two legs, i.e., $q_4 = q_2 - q_3 + \pi$. In this view, the dimension of the system is reduced to three so that it is easier to find the orbit cycle. Then, the fixed point after optimization in this case is,

$$\boldsymbol{x}^* = [0.2052, 2.6614, 2.8578, -1.84410.8978, 0.1596]^{\mathrm{T}}. \tag{9}$$

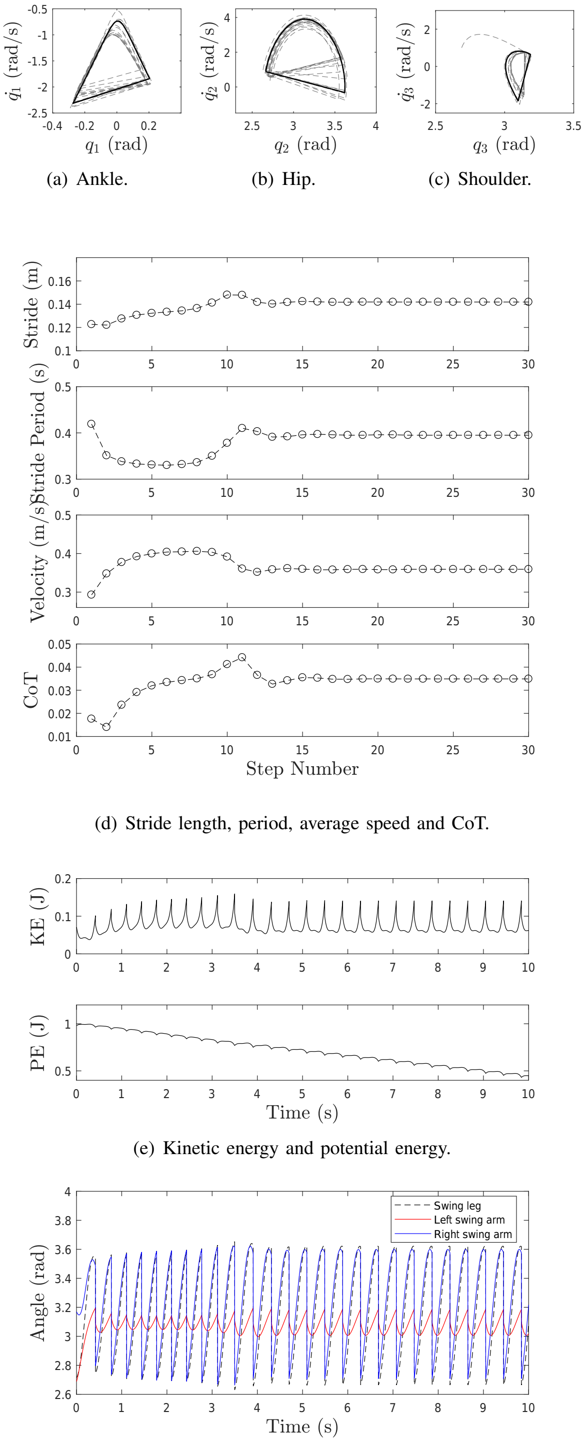
Figure. 2 shows the phase diagrams and the step characteristics including the stride length, step period, average velocity as well as the cost of transportation (CoT) [3] from the initial point $x_0 = [0.18, 2.72, 2.69, -1.61, 0.36, 1.01]^{\mathrm{T}}$. The system stabilizes to a periodic orbit within approximately 15 steps. The rhythmic motion of the dual arms is synchronized with the swing leg, as shown in Fig. 2 (f). The snapshots of the system from initial steps to normal step cycle is depicted in Fig. 3.

To further illustrate the arm dynamics effects on the system, the behavior of the passive walker with variable arm parameters is studied. Here, we introduce the initial kinetic energy (IKE) for walking to evaluate biped energy criterion. The results are shown in Fig. 4. Initially, the arm length is adjusted within a specified range, yielding corresponding periodic orbits. Numerical simulations are conducted with a fixed damping factor ($c_d = 0.0001\ \mathrm{N \cdot m \cdot s/rad}$) and constant arm mass ratio ($m_a/m_l = 0.05$). For each case, the orbit stability, stride length, and period are determined.

The results show that the step period and stride length remain nearly constant. When the length ratio is below 0.4, the stability value fluctuates slightly but not significantly. However, stability begins to deteriorate as the length increases further. Generally, IKE does not change significantly as shown in Fig. 4 (a).

In the subsequent tests, the arm mass is adjusted within a defined range, and the stability and walking characteristics are analyzed. Similarly, the simulations are performed with a constant damping factor ($c_d = 0.0001\ \mathrm{N \cdot m \cdot s/rad}$) and a constant arm length ($l_a/l_l = 0.5$). The findings are presented in Fig. 4 (b). As the mass increases, both the stride length and step period decrease, while the walking velocity remains nearly constant (0.34 m/s - 0.38 m/s). However, the overall trend of stability is gradually deteriorating with the increase of the mass. Stability changes quickly when the arm mass ratio is below 0.2. For IKE, it is shown that the lowest initial energy demand is around a mass ratio of 0.6.

To study the effects of the arm damper on the passive walking, the damping factor is adjusted within a defined range in the similar manner. The results are presented in Fig. 4 (c). Damping has little influences on the step period, stride length as well as required IKE. However, damping does help increase walking stability. With low or almost no damping, walking stability is poor, and even it is difficult to find a feasible periodic solution. As damping increases, walking stability gradually improves.



(a) Ankle. (b) Hip. (c) Shoulder.

(d) Stride length, period, average speed and CoT.

(e) Kinetic energy and potential energy.

(f) Trajectory of the swing leg and arms.

Fig. 2. The results of the passive walking with dual arms on the slope.

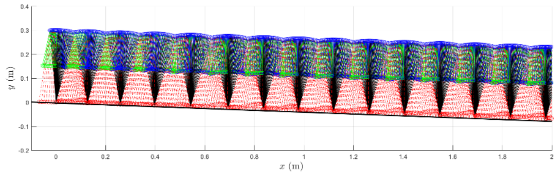


Fig. 3. Snapshots of the locomotion of the biped walker from initial state, with dual free-swing arms. The role of the stance and the swing leg changes after the impact.

IV. Conclusion

This paper explores the impact of arm dynamics on biped walking by examining the locomotion of a planar passive robot. The system consists of a continuous swing phase and an instantaneous impact phase. The Poincaré return map is used to analyze stability and identify feasible solutions. To create a rhythmic dual arm swing motion that mimics human movement, a compass mechanism is introduced to ensure that the arms remain evenly spaced on both sides of the midline between the legs. Next, the effects of arms on passive biped walking are discussed. The results show that the underactuated arms connected to the hip joints effectively synchronized their motion during locomotion, contributing significantly to the robot's stability and velocity with reasonable dual arms parameters.

Extensive numerical simulations demonstrate how various dynamic parameters of the arms, such as step length, period, and orbit stability, influence the overall gait. These findings provide mathematical confirmation of the essential role that human-like arm motion plays in improving the stability and efficiency of bipedal locomotion. We identified optimal physical parameters for both arms. The ideal arm length should not exceed 0.4 times the leg length. While increasing arm mass reduces walking stability, a mass around 0.6 times the leg mass minimizes initial walking energy requirements. Additionally, the damper significantly impacts walking stability, with even small damping greatly enhancing it.

This study primarily focuses on the planar dynamics of bipedal walking, as the majority of walking behavior occurs in the sagittal plane. However, the coordinated swing of arms and legs may also help counteract the yaw moment caused by lower limb movement. Future research will extend this analysis to three-dimensional passive walking to gain a more comprehensive understanding of the role of the upper body in bipedal locomotion.

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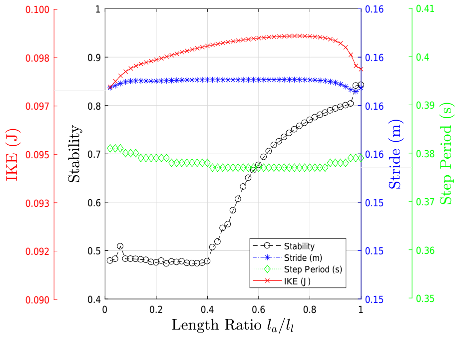
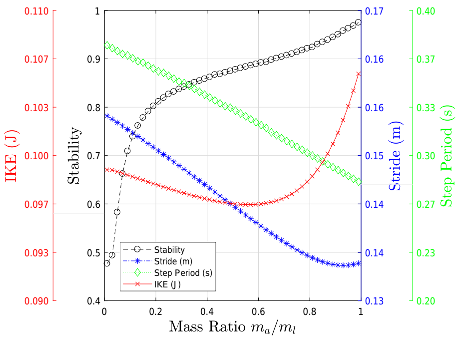
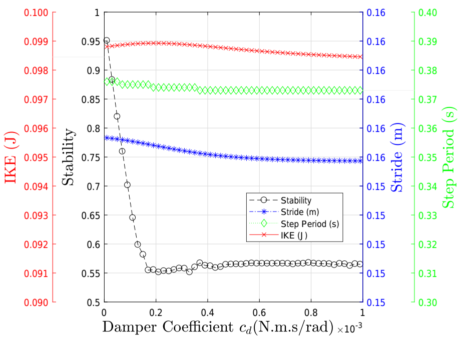


(a) $c_d = 0.0001$, $\frac{m_a}{m_l} = 0.05$.



(b) $c_d = 0.0001$, $\frac{l_a}{l_l} = 0.5$.



(c) $\frac{l_a}{l_l} = 0.5$, $\frac{m_a}{m_l} = 0.04$.

Fig. 4. The effects of the free-swing dual arms length ratio, mass ratio and damper coefficient on the stability, stride length, step period, and CoT of the biped walker.

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