Homework 12

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Exercise (10.2).

$$F(y) = \int_0^1 \{y^2 - yy'\}\,dx, \qquad y \in C^1([0,1]).$$

1. *Find all extremals.*
2. *If we require $y(0) = 0$, show by example that there is no minimum.*
3. *If we require that $y(0) = y(1) = 0$, show that the extremal is a minimum. (Hint: $yy' = \left(1/2y^2\right)'$).*

1. Let $f(x,y,y') = y^2 - yy'$. EL equation:

$$2y - y' = (-y)'$$

   gives the extremals $y = 0$.

2. For $y = 0$, $F(y) = 0$. However, the function $e^x - 1$ satisfies the boundary condition, and $F(e^x - 1) = 2 - e < 0$, so, $y = 0$ is not the minimizer. Since $y = 0$ is the only solution satisfying the EL equations, then we conclude that there is no minimum.

3. Now, we have

$$F(y) = \int_0^1 \{y^2 - (y^2/2)'\}\,dx = \int_0^1 y^2 dx \geq 0.$$

   And, the only function that attains zero is exactly $y = 0$, so it is the minimizer.

Exercise (10.4). *Minimize*

$$F(y) = \int_0^1 f(x, y(x), y'(x), y''(x))dx$$

over the set of $y \in C^2([0,1])$ such that $y(0) = \alpha$, $y'(0) = \beta$, $y(1) = \gamma$, and $y'(1) = \delta$. That is, with $C_0^2([0,1]) = \left\{u \in C^2([0,1]) : u(0) = u'(0) = u(1) = u'(1) = 0\right\}$, and $y \in C_0^2 + p(x)$ where p is the cubic polynomial that matches the BC.

1. *Find a differential equation, similar to the EL equation, that must be satisfied by the minimum (if it exists).*

2. *Apply your equation to find the extremal(s) of*

$$F(y) = \int_0^1 \left(y''(x)\right)^2 dx,$$

   where $y(0) = y'(0) = y'(1) = 0$, but $y(1) = 1$, and justify that each extremal is a (possibly non-strict) minimum.

We go through the exact same steps as in the text to derive the ordinary EL equations. Let A be defined as

$$Ah = \int_0^1 D_2 f h + D_3 f h' + D_4 f h''.$$

which is a bounded linear functional on C^2 with the norm $\|h\| = \max(\|h\|_\infty, \|h'\|_\infty, \|h''\|_\infty)$. We show now that A is indeed the Frechet derivative. We have

$$\begin{aligned}
|F(y+h) - F(y) - Ah| \leqslant & \int_0^1 \int_0^1 \left| D_2 f\left(x, y+th, y'+th', y''+th''\right) - D_2 f\left(x,y,y',y''\right)\right| dt dx \\
&+ \int_0^1 \int_0^1 \left| D_3 f\left(x, y+th, y'+th', y''+th''\right) - D_3 f\left(x,y,y',y''\right)\right| dt dx \\
&+ \int_0^1 \int_0^1 \left| D_4 f\left(x, y+th, y'+th', y''+th''\right) - D_4 f\left(x,y,y',y''\right)\right| dt dx
\end{aligned}$$

Since $D_2 f$, $D_3 f$, and $D_4 f$ are uniformly continuous on compact sets, the right hand side is $o(\|h\|)$ and so $DF(y) = A$.

To show that A is continuous, we use the uniform continuity of $D_2 f, D_3 f, D_4 f$ and for $\|k\| \leq 1$,

$$\begin{aligned}
|DF(y+h)k - DF(y)k| \leqslant & \int_0^1 \left| \left\{ D_2 f\left(x, y+h, y'+h', y''+h''\right) - D_2 f\left(x,y,y',y''\right)\right\} k\right| dx \\
&+ \int_0^1 \left| \left\{ D_3 f\left(x, y+h, y'+h', y''+h''\right) - D_3 f\left(x,y,y',y''\right)\right\} k'\right| dx \\
&+ \int_0^1 \left| \left\{ D_4 f\left(x, y+h, y'+h', y''+h''\right) - D_4 f\left(x,y,y',y''\right)\right\} k''\right| dx
\end{aligned}$$

which tends to zero as $\|h\| \to 0$.

Now we develop the ODE: for $h \in C_0^2$,

$$\begin{aligned}
Ah &= \int_0^1 D_2 f h + D_3 f h' + D_4 f h'' \\
&= \int_0^1 \left(D_2 f - (D_3 f)' + (D_4 f)''\right) h dx + \left[\left(D_3 f - (D_4 f)'\right) h\right]_0^1 + \left[D_3 f h'\right]_0^1
\end{aligned}$$

so, the ODE is

$$D_2 f - (D_3 f)' + (D_4 f)'' = 0.$$

If we apply this result to

$$F(y) = \int_0^1 \left(y''(x)\right)^2 dx,$$

we see that the ODE reduces to

$$y^{(iv)=0}$$

Thus, y is the cubic polynomial that matches the boundary conditions. (That is, $y=p$.) Elementary calculation gives

$$p=y=-2x^3+3x^2.$$

Moreover, we know that the extremal is a possibly non-strict minimum since F is convex (can be easily seen since x^2 is convex),

$$F(\lambda y+(1-\lambda)z)=\int_0^1 \left(\lambda y''+(1-\lambda)z''\right)^2 \leqslant \lambda F(y)+(1-\lambda)F(z)$$

Exercise (10.6). *Consider the functional*

$$\Phi(x,y,y')=\int_a^b F(x,y(x),y'(x))dx.$$

1. *If $F\in C^2$ and $F=F(y,y')$ only, and if we assume that $y\in C^2$, prove that in this case the EL equations reduce to*

$$\frac{d}{dx}\left(F-y'F_{y'}\right)=0.$$

2. *Among all C^2 curves $y(x)$ joining the points $(0,1)$ and $(1,\cosh(1))$, find the one which generates the minimum area when rotated about the x axis. This area is*

$$A=2\pi\int_0^1 y\sqrt{1+(y')^2}dx.$$

1. Direct computation.

$$\begin{aligned}\left(D_3Fy'-f\right)' &= D_3Fy''+(D_3F)'y'-f' \\ &= D_3Fy''+D_2Fy'-\left(D_2Fy'+D_3fy''\right) \\ &= 0\end{aligned}$$

2. Seek to minimize

$$A(y)=\int_0^1 2\pi y\sqrt{1+(y')^2}dx.$$

Note that $D_3^2 f\neq 0$ unless $y=0$, and clearly $y(x)>0$, so we have regular extremals. Now, we can use the Theorem when the integrand depends only on the second and third variables,

$$2\pi y\frac{y'^2}{\sqrt{1+(y')^2}}-2\pi y\sqrt{1+(y')^2}=D_3fy'-f=2\pi C$$

which implies

$$y' = \frac{1}{C}\sqrt{y^2 - C^2}.$$

Separate variables and integrate, (following the problem in the text) gives the solution

$$y(x) = C\cosh(x/C + \lambda).$$

The boundary conditons give the values for the two constants:

$$y(0) = C\cosh(\lambda) = 1$$
$$y(1) = C\cosh(1/C + \lambda) = \cosh 1$$

for which $C = 1$, $\lambda = 0$ is a solution.

Exercise (10.8). *Consider the problem of finding a C^1 curve that minimizes*

$$\int_0^1 (y'(t))^2 dt$$

subject to the conditions $y(0) = y(1) = 0$ and

$$\int_0^1 y^2 = 1.$$

1. *Remove the integral constraint by incorporating a Lagrange multiplier, and find the EL equations.*
2. *Find all extremals.*
3. *Find the solution.*
4. *Use your result to find the best constant C in the inequality*

   $$\|y\| \leq C\|y'\|$$

   for functions that satisfy $y(0) = y(1) = 0$.

1. We form the Lagrangian,

   $$\mathscr{L}(y,\lambda) = \int_0^1 (y'(t))^2 dt - \lambda\left[\int_0^1 \left\{y(t)^2 - 1\right\} dt\right] = \int_0^1 \left\{(y'(t))^2 - \lambda y(t)^2 + \lambda\right\} dt$$

   which we seek to minimize. To that end, define

   $$f(t,y,y') = (y'(t))^2 - \lambda y(t)^2 + \lambda.$$

   The EL equations are

   $$-2\lambda y = D_2 f = \frac{d}{dx}D_3 f = \frac{d}{dx}(2y') = 2y''.$$

   Hence, the resulting ODE is given simply by $-y'' = \lambda y$, accompanied with the BC $y(0) = y(1) = 0$.

2. Extremals can be found by solving the eigenvalue problem. To that end, we note that the operator $-\frac{d^2}{dx^2}$ accompanied with homogeneous boundary conditions is self-adjoint. Therefore, its eigenvalues are non-negative and real. We immediately see that zero cannot be an eigenvalue, for if $\lambda = 0$, then $y'' = 0$, so y is linear, and after incorporating BC, we see that $y \equiv 0$, which does not satisfy the original integral constraint. The general solution is
$$y(t) = A\cos\left(\sqrt{\lambda}t\right) + B\sin\left(\sqrt{\lambda}t\right).$$
BC imply $A = 0$ and
$$y(1) = B\sin\left(\sqrt{\lambda}\right) = 0 \implies \lambda = n^2\pi^2, \qquad n = 1, 2, \ldots.$$
The coefficient B is determined by satisfying the integral condition.
$$1 = \int_0^1 y^2 = B^2 \int_0^1 \sin^2(n\pi t)\, dt \implies B = \pm\sqrt{2}.$$

3. Make the observation
$$y(t) = \sin(n\pi t) \Rightarrow y'(t) = n\pi\cos(n\pi t).$$
So, no matter which n we pick, we will always be integrating $\cos^2(n\pi t)$, so the minimizer is the extremal with the smallest admissible n, that is, $n = 1$. So, the solution is $y(t) = \sqrt{2}\sin(\pi t)$.

4. The best constant is the one attaining the bound, that is, the smallest eigenvalue. This means $C = \pi^{-1}$.

Exercise (10.9). *Find the C^2 curve that minimizes the functional*
$$\int_0^1 \left\{y(t)^2 + y'(t)^2\right\} dt$$
subject to $y(0) = 0$, $y(1) = 1$ and the constraint
$$\int_0^1 y = 0$$

Lagrangian:
$$L(y, \lambda) = \int_0^1 y^2 + \left(y'\right)^2 - \lambda y$$
EL equations:
$$2y - \lambda = D_2 f = (D_3 f)' = \left(2y'\right)' = 2y''$$
whose solution is the function
$$y = -\frac{e^{-x}\left(-1+e^x\right)\left(-2e + \lambda e - \lambda e^2 - \lambda e^x - 2e^{1+x} + \lambda e^{1+x}\right)}{2\left(-1+e^2\right)}.$$
The integral constraint gives the value for the Lagrange multiplier,
$$\lambda = \frac{2(e-1)}{e-3}.$$

We check also when the derivatives of the constraints are zero, but this gives a trivial solution which does not satisfy the original problem.