Homework 4

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Problem 7.1:

Prove that for $f \in H^1\left(\mathbb{R}^d\right)$, $\|f\|_{H^1\left(\mathbb{R}^d\right)}$ is equivalent to

$$\left[\int_{\mathbb{R}^d} \left(1+|\xi|^2\right)|\hat{f}\left(\xi\right)|^2 d\xi\right]^{1/2}$$

Generalize to $H^k\left(\mathbb{R}^d\right)$?

Solution Make use of Plancherel theorem. Let $f \in H^1\left(\mathbb{R}^d\right)$. Then f and Df are L^2. The norm is

$$\|f\|_{H^1}^2 = \|f\|_{L^2}^2 + \|Df\|_{L^2}^2 = \|\hat{f}\|_{L^2}^2 + \|\left(Df\right)^\wedge\|_{L^2}^2 = \int |\hat{f}|^2 + |\left(Df\right)^\wedge|^2 = \int \left(1+|\xi|^2\right)|\hat{f}|^2.$$

The result also generalizes to H^k. In this case, propose a norm

$$\|f\|_{H^k\left(\mathbb{R}^d\right)} = \left[\int_{\mathbb{R}^d} \left(1+|\xi|^2\right)^k |\hat{f}\left(\xi\right)|^2 d\xi\right]^{1/2}.$$

We need to show that there exist constants $C_1, C_2 > 0$ that bound the two norms. But, this boils down to showing that there exists C_1, C_2 such that

$$C_1\left(1+x^2\right)^{k/2} \leq \sum_{r=0}^{k} x^r \leq C_2\left(1+x^2\right)^{k/2}.$$

Equivalently,

$$C_1\left(1+x^2\right)^{k} \leq \left(\sum_{r=0}^{k} x^r\right)^2 \leq C_2\left(1+x^2\right)^{k}.$$

Consider the function

$$f(x) = \frac{\left(\sum_{r=0}^{k} x^r\right)^2}{\left(1+x^2\right)^k} \in C^0\left([0,\infty)\right).$$

Since $f(0)=1$ and $\lim_{x\to\infty} f(x) = 1$, f has a maximum on $[0,\infty)$ which gives C_2. Similarly, $g(x) = 1/f(x)$ has a maximum, giving C_1.

Problem 7.2:

Prove that if $f \in H_0^1(0,1)$, then there exists $C > 0$ such that $\|f\|_{L^2(0,1)} \leq C\|f'\|_{L^2(0,1)}$. If instead $f \in \left\{g \in H^1(0,1) : \int_0^1 g(x)dx = 0\right\}$, provide a similar estimate.

Proof: Go by contradiction. Suppose to the contrary that there exists a sequence $f_n \in H_0^1$ with $\|f_n\| = 1$ and $\|Df_n\| \to 0$. From every bounded sequence in a Hilbert space, we can extract a weakly convergent subsequence, $f_{n_k} \rightharpoonup f \in H_0^1$. The weak convergence in H^1 implies weak convergence of the derivative $Df_{n_k} \rightharpoonup Df \in L^2(\Omega)$. Since the norm is continuous, then we would conclude that $Df = 0$, that is, f must be a constant. But in order to satisfy the boundary condition, then f is indentically zero.

On the other side, the Rellich-Kondrachov Theorem tells us that weak convergence in H^1 implies $f_{n_k} \to f \in L^2$, which implies convergence in the L^2 norm. But, that would mean that $\|f\| = \lim \|f_n\| = 1$, a contradiction to the above conclusion that $f = 0$.

Problem 7.3:

Prove that $\delta_0 \notin \left(H^1\left(\mathbb{R}^d\right)\right)^*$ for $d \geq 2$, but that $\delta_0 \in \left(H^1\left(\mathbb{R}\right)\right)^*$. You will need to define that δ_0 applied to $f \in H^1\left(\mathbb{R}\right)$ means.

Problem 7.4:

Prove that $H^1(0,1)$ is continuously embedded in $C_B(0,1)$, the set of bounded and continuous functions on $(0,1)$.

Solution Need to show that the inclusion map

$$H^1\left(0,1\right) \ni f \to f \in C_B\left(0,1\right)$$

is bounded, i.e. there exists $C > 0$ such that $\|f\|_\infty \leq C\|f\|_{H^1}$.