

# Homework 7

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**Exercise (8.1).** *If  $A$  is positive definite, show that its eigenvalues are positive. If  $A$  is symmetric and has positive eigenvalues, then  $A$  is positive definite.*

*Proof.* Suppose that  $A$  is positive definite. Let  $(\lambda, v)$  be an eigenpair for  $A$ . Then by the positive definiteness,

$$0 < (v, Av) = (v, \lambda v) = \lambda (v, v) = \lambda \|v\|^2.$$

Since this quantity is strictly positive, we must have  $\lambda > 0$ .

On the other side, assume now that  $A$  has all positive eigenvalues. Then,  $A$  admits a diagonalization  $A = U\Lambda U^*$  for some  $U^* = U^{-1}$  unitary matrix. Pick  $x \neq 0$ , and set  $y = U^*x \neq 0$ . Then

$$(x, Ax) = (x, U\Lambda U^*x) = (y, \Lambda y).$$

Expand  $y = \sum y_i e_i$ , then

$$(y, \Lambda y) = \sum_i \bar{y}_i y_i \lambda_i > 0$$

since  $\lambda_i > 0$  and  $\bar{y}_i y_i = |y_i|^2 \geq 0$  since  $y \neq 0$  by assumption. This shows that  $(x, Ax) = (y, \Lambda y) > 0$  and  $A$  is positive definite.

□

**Exercise (8.3).** Suppose we wish to find  $u \in H^2(\Omega)$  satisfying  $-\Delta u + cu^2 = f \in L^2(\Omega)$  for  $\Omega \subset \mathbb{R}^d$  is a bounded Lipschitz domain. For consistency, we would require that  $cu^2 \in L^2(\Omega)$ . Determine the smallest  $p$  such that if  $c \in L^p(\Omega)$ , then this holds. Depends on  $d$ .

*Proof.* We want  $cu^2 \in L^2(\Omega)$ . Suppose that  $u \in L^r(\Omega)$  and  $c \in L^p(\Omega)$ . Then by Generalized Hölder's, we have the estimate

$$\|cu^2\|_2^2 = \int_{\Omega} c^2 u^4 \leq \|c\|_p^2 \|u\|_r^4$$

provided that  $2/p + 4/r = 1$ . That is, given  $r$ , we determine  $p = \frac{2}{1-4/r}$ , and if  $c \in L^p$ , then we have  $cu^2 \in L^2(\Omega)$ .

We make use now of the Sobolev embedding theorem. We know  $u \in H^2(\Omega) = W^{2,2}(\Omega)$ . Let  $j = 0, m = 2, p = 2$ . Then we have

1. If  $4 < d$ , then  $W^{2,2}(\Omega) \hookrightarrow W^{0,r}(\Omega)$  for every  $r \leq \frac{2d}{d-4}$ . Let us first investigate when  $r = \frac{2d}{d-4}$ . Substituting in, we find  $p = \frac{2d}{8-d}$ . We immediately find that we have problems when  $d > 8$ , which is admissible in this case. Restricting to  $p \geq 1$ , we obtain  $r \geq 4$ , so we must have  $r \in [4, \frac{2d}{d-4}]$ .
2. If  $d < 4$ , then  $W^{2,2}(\Omega) \hookrightarrow C_B^0(\Omega) = C^0(\Omega) \cap L^\infty(\Omega)$ . In this case, we obtain  $r = \infty$ , and calculate  $p = 2$ , so  $c$  can be in  $L^2$ .

□

**Exercise (8.5).** Suppose that  $\Omega \subset \mathbb{R}^d$  is a smooth, bounded, connected domain. Let

$$H = \left\{ u \in H^2(\Omega) : \int_{\Omega} u = 0, \nabla u \cdot n = 0 \text{ on } \partial\Omega \right\}.$$

Show that  $H$  is a Hilbert space, and prove there exists  $C > 0$  such that

$$\|u\|_{H^1(\Omega)} \leq C \sum_{|\alpha|=2} \|D^{\alpha}u\|_{L^2(\Omega)}$$

for every  $u \in H$ .

*Proof.* 1. Recognize that the condition  $\nabla u \cdot n = 0$  is exactly  $\gamma_1 u = 0$ . We showed that  $\gamma_1$  is a continuous linear map. Furthermore, the averaging operator is continuous. Our space  $H$  then can be equivalently characterized as  $\mathcal{N}(A) \cap \mathcal{N}(\gamma_1)$ . Each set is closed, so  $H$  is closed. Therefore  $H$  is a Hilbert space equipped with the  $H^2(\Omega)$  inner product.

2. Proof by contradiction.

□

**Exercise (8.8).**  $\Omega \subset \mathbb{R}^d$  bounded, connected, Lipschitz domain. Let  $V \subset \Omega$  have positive measure. Let  $H = \{u \in H^1(\Omega) : u|_V = 0\}$ .

1. Why is  $H$  a Hilbert space?
2. Prove that there exists  $C > 0$  such that

$$\|u\| \leq C \|\nabla u\|$$

for every  $u \in H$ .

1.  $H$  is the null space of the restriction to  $V$  operator. Since restriction is continuous, then  $H$  is closed. As a closed subspace of Hilbert space  $H^1(\Omega)$ , then  $H$  is itself a Hilbert space.
2. Go by contradiction. Assume to the contrary that there exists a sequence  $u_n \in H$  such that

$$\|u_n\| = 1 \quad \text{and} \quad \|\nabla u_n\| \rightarrow 0.$$

Now, from every bounded sequence in a Hilbert space, we can extract a weakly convergent subsequence  $u_{n_k} \rightharpoonup u \in H$ . Weak convergence of  $u_{n_k}$  implies weak convergence of  $\nabla u_{n_k} \rightharpoonup \nabla u$  in  $L^2$ . Since the norm is continuous we pass to the limit and we have that  $\nabla u = 0$ , i.e.  $u$  is a constant. But if  $u$  vanishes on  $V$ , then the only possibility is that  $u \equiv 0$ .

On the other side, we know from the Rellich-Kondrachov Theorem that we have  $u_{n_k} \rightarrow u$  in  $L^2$ . Convergence in  $L^2$  also implies convergence in the norm, but since  $\|u_n\| = 1$  that means  $\|u\| = 1$ , a contradiction.