Homework 4

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Problem 7.1:

Prove that for $f \in H^1\left(\mathbb{R}^d\right)$, $\|f\|_{H^1\left(\mathbb{R}^d\right)}$ is equivalent to

$$\left[\int_{\mathbb{R}^d} \left(1+|\xi|^2\right) |\hat{f}(\xi)|^2 d\xi\right]^{1/2}$$

Generalize to $H^k\left(\mathbb{R}^d\right)$?

Solution Make use of Plancherel theorem. Let $f \in H^1\left(\mathbb{R}^d\right)$. Then f and Df are L^2. The norm is

$$\|f\|_{H^1}^2 = \|f\|_{L^2}^2 + \|Df\|_{L^2}^2 = \|\hat{f}\|_{L^2}^2 + \|(Df)^\wedge\|_{L^2}^2 = \int |\hat{f}|^2 + |(Df)^\wedge|^2 = \int \left(1+|\xi|^2\right)|\hat{f}|^2.$$

The result also generalizes to H^k. In this case, propose a norm

$$\|f\|_{H^k\left(\mathbb{R}^d\right)} = \left[\int_{\mathbb{R}^d} \left(1+|\xi|^2\right)^k |\hat{f}(\xi)|^2 d\xi\right]^{1/2}.$$

We need to show that there exist constants $C_1, C_2 > 0$ that bound the two norms. But, this boils down to showing that there exists C_1, C_2 such that

$$C_1\left(1+x^2\right)^{k/2} \le \sum_{r=0}^{k} x^r \le C_2\left(1+x^2\right)^{k/2}.$$

Equivalently,

$$C_1\left(1+x^2\right)^k \le \left(\sum_{r=0}^{k} x^r\right)^2 \le C_2\left(1+x^2\right)^k.$$

Consider the function

$$f(x) = \frac{\left(\sum_{r=0}^{k} x^r\right)^2}{\left(1+x^2\right)^k} \in C^0\left([0,\infty)\right).$$

Since $f(0)=1$ and $\lim_{x\to\infty} f(x) = 1$, f has a maximum on $[0,\infty)$ which gives C_2. Similarly, $g(x) = 1/f(x)$ has a maximum, giving C_1.

Problem 7.2:

Prove that if $f \in H_0^1(0,1)$, then there exists $C > 0$ such that $\|f\|_{L^2(0,1)} \leq C\|f'\|_{L^2(0,1)}$. Similarly, if $f \in \left\{ g \in H^1(0,1) : \int_0^1 g(x)dx = 0 \right\}$?

Proof: If $f \in H_0^1(0,1)$, then write f as $f(x) = \int\limits_0^x f'(s)\,ds$. Then

$$|f(x)| = \left| \int\limits_0^x f'(s)\,ds \right| \leqslant \int\limits_0^x |f'(s)|\,ds.$$

Taking supremum w.r.t. x, we have $\|f\|_\infty \leqslant \int\limits_0^1 |f'(s)|\,ds$. Squaring both sides,

$$\|f\|_\infty^2 \leqslant \left\{ \int\limits_0^1 |f'(s)|\,ds \right\}^2 \leqslant \int\limits_0^1 |f'(s)|^2 ds = \|f'\|_{L^2(0,1)}^2.$$

Finally,

$$\|f\|_2^2 = \int\limits_0^1 |f|^2 \leqslant \|f\|_\infty^2$$

which proves the result for $C = 1$. This merely a bound, not necessarily the tightest.

For the case of zero mean, we go by contradiction. We suppose to the contrary that there exists a sequence of functions $f_k \in H^1(0,1)$ with zero mean such that

$$\|f_k\|_{L^2} > k\|\nabla f_k\|_{L^2}.$$

Without loss of generality, let us assume that $\|f_k\|_{L^2} = 1$. Therefore,

$$\frac{1}{k} > \|\nabla f_k\|_{L^2} \to 0.$$

Now, from every bounded sequence in a Hilbert space, we can extract a weakly convergent subsequence, $f_{k_l} \rightharpoonup f$. Weak convergence in H^1 implies weak convergence of $\nabla f_k \rightharpoonup \nabla f$ in L^2. Passing to the limit, then $\nabla f = 0$, i.e. f must be a constant. But since f must have zero mean, then $f \equiv 0$. On the other side, the Rellich-Kondrachov Theorem tells us that weak convergence in H^1 implies strong convergence $f_{k_l} \to f$ in L^2, and therefore also in the norm. But convergence in the L^2 norm implies that $\|f\|_{L^2} = 1$, a contradiction.

Problem 7.3:

Prove that $\delta_0 \notin \left(H^1\left(\mathbb{R}^d\right)\right)^*$ for $d \geq 2$, but that $\delta_0 \in \left(H^1\left(\mathbb{R}\right)\right)^*$. You will need to define that δ_0 applied to $f \in H^1\left(\mathbb{R}\right)$ means.

Solution For $f \in H^1\left(\mathbb{R}\right)$, define the action of δ_0 as follows. First, identify a sequence $f_j \in \mathscr{D}$ converging to f. Then define the action of δ_0 using its definition on Schwartz functions, i.e. $\langle \delta_0, f \rangle = \lim_{j\to\infty} \left\langle \delta_0, f_j \right\rangle$. Indeed, for every f_j, we have

$$\left|\left\langle \delta_0, f_j \right\rangle\right| = \left|\left\langle \mathscr{F}^{-1}\mathscr{F}\delta_0, f_j \right\rangle\right| = \left|\left\langle \mathscr{F}\delta_0, \mathscr{F}f_j \right\rangle\right| = \left|\int_{\mathbb{R}^d} (2\pi)^{-d/2} \hat{f}_j(\xi)\, d\xi\right| = (2\pi)^{-d/2}\left|\int_{\mathbb{R}^d} \hat{f}_j(\xi)\, d\xi\right|$$

It follows from Cauchy-Schwarz that

$$\begin{aligned}
(2\pi)^{-1/2}\left|\int_{\mathbb{R}} \hat{f}_j(\xi)\, d\xi\right| &\leqslant (2\pi)^{-1/2}\int_{\mathbb{R}} \left|\hat{f}_j(\xi)\right| \sqrt{\frac{1+|\xi|^2}{1+|\xi|^2}}\, d\xi \\
&\leqslant (2\pi)^{-1/2}\left\{\int_{\mathbb{R}} \left(1+|\xi|^2\right)\left|\hat{f}_j(\xi)\right|^2 d\xi\right\}^{1/2}\left\{\int_{\mathbb{R}} \frac{1}{1+|\xi|^2}\, d\xi\right\}^{1/2}.
\end{aligned}$$

We showed earlier that the first integral is nothing more than $\|f_j\|_{H^1(\mathbb{R})}$ which converges to $\|f\|_{H^1(\mathbb{R})}$. By performing change of var. on the second integral, we see that

$$\int_{\mathbb{R}} \frac{1}{1+|\xi|^2}\, d\xi \sim d\omega_d \int_{\mathbb{R}} \frac{r^{d-1}}{1+r^2}\, dr$$

which converges only when $1-d+2>1$, i.e. when $d<2$.

To show that δ_0 is not a member of the dual space for $d>1$, we compute its norm in the dual space. First, the dual of $H^1\left(\mathbb{R}^d\right)$ is $H^{-1}\left(\mathbb{R}^d\right)$. Then, we have the norm

$$\|\delta_0\|_{H^{-1}\left(\mathbb{R}^d\right)} = \left\{\int_{\mathbb{R}^d} \left(1+|\xi|^2\right)^{-1}\left|\hat{\delta}_0\right|^2\right\}^{1/2} = (2\pi)^{-d/2}\left\{\int_{\mathbb{R}^d} \frac{1}{1+|\xi|^2}\, d\xi\right\}^{1/2}.$$

Performing the same change of variables, we again see that the integral is similar to

$$\int_{\mathbb{R}} \frac{1}{1+|\xi|^2}\, d\xi \sim d\omega_d \int_{\mathbb{R}} \frac{r^{d-1}}{1+r^2}\, dr$$

and also converges only when $d<2$.

Problem 7.4:

Prove that $H^1(0,1)$ is continuously embedded in $C_B(0,1)$, the set of bounded and continuous functions on $(0,1)$.

Solution Take a Cauchy sequence of smooth functions $f_j \in H^1(0,1) \cap C^\infty([0,1])$. Let now $g := f_j - f_k$. Notice that by Cauchy-Schwarz

$$\left|\int_0^1 g\right| \leqslant \sqrt{\int_0^1 1^2}\sqrt{\int_0^1 g^2} = 1\,\|g\|_{L^2}^2 \leqslant \|g\|_{H^1}^2 .$$

Furthermore, by the mean value theorem, g attains its average, $m = \int_0^1 g$ at some point x_0. Then

$$|g(x) - m| \leqslant \sqrt{|x - x_0|}\|g\|_{H^1} \leqslant \|g\|_{H^1}.$$

Finally, triangle inequality tells us

$$\|g\|_\infty \leqslant m + \|g\|_{H^1} \leqslant 2\|g\|_{H^1}.$$

So, if f_j is Cauchy in $H^1(0,1) \cap C^\infty([0,1])$, then it is Cauchy in $C_B(0,1)$ and therefore converges. The inclusion map is therefore sequentially continuous and therefore continuous.