UT Austin CSE 386D

Homework 6

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Exercise (7.10). Suppose $\Omega \subset \mathbb{R}^d$ is bounded with Lipschitz boundary and $f_j \rightharpoonup f$ and $g_j \rightharpoonup g$ weakly in $H^1(\Omega)$. Show that, for a subsequence, $\nabla (f_j g_j) \to \nabla (fg)$ as a distribution. Find all p in $[1, \infty]$ such that the convergence can be taken weakly in $L^p(\Omega)$.

Exercise (7.15). Suppose that $w \in L^1(\mathbb{R})$, w(x) > 0, and w is even. Moreover, for x > 0, $w \in C^2[0, infty)$, w'(x) < 0, and w''(x) > 0. Consider the equation

$$w * u - u'' = f \in L^2(\mathbb{R}).$$

- 1. Show that $\hat{w}(\xi) > 0$.
- 2. Fund a fundamental solution. Leave answer in terms of inverse Fourier transform.
- 3. Find the solution operator as a convolution operator.
- 4. Show that $u \in H^2(\mathbb{R})$.

Exercise (7.16). Let H be a Hilbert space and Z a closed linear subspace. Prove that H/Z has the inner product

$$(\hat{x}, \hat{y})_{H/Z} = \left(P_Z^{\perp} x, P_Z^{\perp} y\right)_H$$

and that H/Z is isomorphic to Z^{\perp} .

Exercise (7.17). Elliptic regularity theorey shows that if $\Omega \subset \mathbb{R}^d$ has a smooth boundary and $f \in H^s(\Omega)$, then $-\Delta u = f$ in Ω , u = 0 on $\partial \Omega$ has a unique solution $u \in H^{s+2}$. For what values of s will u be continuous? Can you be sure that a fundamental solution is continuous? Answer depends on d.

Exercise (7.19). Let $\Omega \subset \mathbb{R}^d$ be a domain with Lipschitz boundary. Let $w \in L^{\infty}(\Omega)$. Define

$$H_w\left(\Omega\right) = \left\{ f \in L^2(\Omega) : \nabla\left(wf\right) \in \left(L^2(\Omega)\right)^d \right\}$$

- 1. Give reasonable conditions on w so that $H_{w}\left(\Omega\right)=H^{1}\left(\Omega\right)$.
- 2. Prove that $H_w(\Omega)$ is a Hilbert space. What is the inner product?
- 3. Can you define the trace of f on $\partial\Omega$? First consider trace of wf.
- 4. Characterize $H_w(\mathbb{R})$ if w(x) is the Heaviside function.

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