Homework 8

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Unless otherwise stated, denote by $(\cdot,\cdot)$ as the L^2 inner product, and $\|\cdot\|$ as $\|\cdot\|_{L^2(\Omega)}$.

Exercise (2). *Suppose that the hypotheses of the Generalized Lax-Milgram theorem are satisfied. Suppose that $x_{0,1}$ and $x_{0,2}$ in $\mathcal{X}$ are such that the sets $X + x_{0,1} = X + x_{0,2}$. Prove that the solutions $u_1 \in X + x_{0,1}$ and $u_2 \in X + x_{0,2}$ of the abstract variational problem agree. What does this say about Dirichlet BVP?*

Assume that both u_1, u_2 satisfy the variational problem $B(u,v) = F(v)$ for every $v \in Y$. Then, $B(u_2 - u_1, v) = 0$ for every $v \in Y$. At the same time, B satisfies the inf-sup condition,

$$0 = \sup_{\|v\|=1} B\left(u_2 - u_1, v\right) \geqslant \gamma \left\|u_2 - u_1\right\|.$$

Thus, $\|u_2 - u_1\| = 0$ and we conclude that the solutions are the same. The result shows that the solution to Dirichlet BVP's is independent of how the boundary data is extended to the whole space.

Exercise (4). *BVP for $u(x,y) : \mathbb{R}^2 \to \mathbb{R}$. Write as a variational problem and show there exists a unique solution. Carefully define function spaces and identify where f must lie.*

$$\begin{cases} -u_{xx} + e^y u = f & (x,y) \in (0,1)^2 \\ u\left(0,y\right) = 0,\ u\left(1,y\right) = \cos y & y \in (0,1) \end{cases}$$

Multiply by test function v and integrate first in x only:

$$-\int_0^1 u_{xx} v dx + \int_0^1 e^y uv dx = \int_0^1 u_x v_x dx - u_x v|_0^1 + \int_0^1 e^y uv dx = \int_0^1 fv dx$$

if we impose $v(0,y) = v(1,y) = 0$, then

$$\int_0^1 u_x v_x dx + \int_0^1 e^y uv dx = \int_0^1 fv dx$$

Integrating in y,

$$\int_\Omega u_x v_x dx dy + \int_\Omega e^y uv dx dy = \int_\Omega fv dx dy.$$

Define the space

$$H = \left\{u \in L^2(\Omega) : u_x \in L^2(\Omega), u\left(0,y\right) = u\left(1,y\right) = 0\right\}.$$

Then, the VP reads:

$$\begin{cases} u \in H + x\cos(y), \\ (u_x, v_x) + (e^y u, v) = (f, v) \\ \quad \text{for every } v \in H. \end{cases}$$

We need f coming from the dual of H for this to make sense.

To show there exists a unique solution, first equip H with the norm $\|u\|_H^2 = \|u\|^2 + \|u_x\|^2$. Now, the RHS is continuious by definition, and the LHS is continuious since

$$B(u,v) \leq \|u_x\|\|v_x\| + e\|u\|\|v\| \leq (1+e)\,\|u\|_H\|v\|_H.$$

It is also coercive since

$$B\,(v,v) \geq \|v_x\|^2 + \|v\|^2 = \|v\|_H^2.$$

Therefore, there exists a unique solution.

Exercise (9). *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz boundary, $f \in L^2\,(\Omega)$, $\alpha > 0$. Consider the Robin problem*

$$\begin{cases} -\Delta u + u = f & x \in \Omega \\ \dfrac{\partial u}{\partial n} + \alpha u = 0 & x \in \partial\Omega \end{cases}$$

1. *Formulate a variational principle $B(u,v) = (f,v)$ for $v \in H^1\,(\Omega)$.*
2. *Show that this problem has a unique weak solution.*

Proof.

1. Variational Problem. Multiply by a test function v and integrate over Ω:

$$\left(-\nabla^2 u, v\right) + (u,v) = (f,v)\,.$$

Integrating the first term by parts,

$$-\left(\nabla^2 u, v\right) = (\nabla u, \nabla v) - \langle \nabla u \cdot n, v\rangle_{\partial\Omega} = (\nabla u, \nabla v) + \langle \alpha u, v\rangle_{\partial\Omega}$$

we arrive at

$$(\nabla u, \nabla v) + \langle \alpha u, v\rangle_{\partial\Omega} + (u,v) = (f,v)\,.$$

The VP reads:

$$\begin{cases} u \in H^1\,(\Omega) \\ (\nabla u, \nabla v) + \langle \alpha u, v\rangle_{\partial\Omega} + (u,v) = B\,(u,v) = F\,(v) = (f,v) \\ \qquad \text{for every } v \in H^1(\Omega). \end{cases}$$

2. Coercivity:

$$B\,(v,v) = \|\nabla v\|^2 + \langle \alpha v, v\rangle_{\partial\Omega} + \|v\|^2 \geqslant \|\nabla v\|^2 + \|v\|^2 = \|v\|_{H^1(\Omega)}^2\,.$$

Continuity:

$$\begin{aligned} |B\,(u,v)| &= |(\nabla u, \nabla v) + \langle \alpha u, v\rangle_{\partial\Omega} + (u,v)| \\ &\leqslant \|\nabla u\|\,\|\nabla v\| + \|u\|\,\|v\| + \alpha\|\gamma_0 u\|_{L^2(\partial\Omega)}\|\gamma_0 v\|_{L^2(\partial\Omega)} \\ &\leqslant C\|u\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)} \end{aligned}$$

Therefore, by Lax-Milgram, the VP admits a unique solution.

□

Exercise (13). *Suppose $\Omega \subset \mathbb{R}^d$ is a bounded Lipschitz domain. Consider the Stokes problem for vector u and scalar p:*

$$\begin{cases} -\Delta u + \nabla p = f & x \in \Omega \\ \nabla \cdot u = 0 & x \in \Omega \\ u = 0 & x \in \partial\Omega \end{cases}$$

where the first equation holds for each coordinate. This problem is a saddle-point problem in which we minimize some energy subject to the constraint $\nabla \cdot u = 0$. However, if we work over the constrained space, we can handle this problem. Let

$$H = \left\{ v \in \left(H_0^1(\Omega) \right)^d : \nabla \cdot v = 0 \right\}$$

1. *Show that H is a Hilbert space.*
2. *Determine an appropriate Sobolev space for f and formulate an appropriate VP for constrained Stokes.*
3. *Show that there is a unique solution to the VP.*

Proof.

1. H is clearly a linear subspace of Hilbert Space $\left(H_0^1(\Omega)\right)^d$. What remains is to show that H is closed. Let $u_n \in H$ be a Cauchy sequence. (That is, $\nabla \cdot u_n = 0$.) Assume that $u_n \to u$ in the H^1 sense. Then $\nabla u_n \to \nabla u$ in $(L^2(\Omega))^{d \times d}$, and hence each element $(\nabla u_n)_{ij} \to (\nabla u)_{ij}$ in $L^2(\Omega)$. Of course, this implies the diagonals converge and therefore that $0 = \nabla \cdot u_n \to \nabla \cdot u = 0$ in L^2, so $u \in H$ and H is a Hilbert space (equipped with the standard H^1 inner product).

2. Test with a vector function v and integrate by parts.

$$\begin{aligned} -\left(\nabla^2 u, v\right) + (\nabla p, v) &= (f, v) \\ (\nabla u, \nabla v) - \langle \nabla u \cdot n, v \rangle - (p, \nabla \cdot v) + \langle p, v \cdot n \rangle &= (f, v) \end{aligned}$$

   Now, if we restrict to functions $v \in H$, the VP reads

$$\begin{cases} u \in H, \\ (\nabla u, \nabla v) = (f, v) \quad \text{for every } v \in H. \end{cases}$$

   We should also have $f \in H^{-1}$.

3. Continuity of the right hand side is a result of Hölder's. Continuity on the bilinear form is also trivial.

   Coercivity:

$$B(v, v) = \|\nabla v\|^2 \geq \frac{1}{C_P^2 + 1} \|v\|_{H^1}^2.$$

   Therefore, by Lax-Milgram, the problem admits a unique solution for u. We conclude nothing regarding existence and uniqueness for p.

□

Exercise (14). *Let $\mathcal{H} = H_0^1(\Omega) \times H^1(\Omega)$ and consider the solution $(u, v) \in \mathcal{H}$ to the differential problem*

$$\begin{cases} -\Delta u = f + a(x)(v - u) & x \in \Omega \\ v - \Delta v = g + a(x)(u - v) & x \in \Omega \\ u = 0, \nabla v \cdot n = \gamma & x \in \partial\Omega \end{cases}$$

where $a \in L^\infty(\Omega)$.

1. *Develop an appropriate weak or variational form for the problem. In what Sobolev spaces should f, g, γ lie?*
2. *Prove that there exists a unique solution provided that $a \geq 0$.*

1. Test the first equation with w and the second with z:

$$\begin{cases} (a(x)(u - v), w) - \left(\nabla^2 u, w\right) = (f, w) \\ (a(x)(v - u), z) + (v, z) - \left(\nabla^2 v, z\right) = (g, z) \end{cases}$$

Integration by parts tells us

$$-\left(\nabla^2 u, w\right) = (\nabla u, \nabla w) - \langle w, \nabla u \cdot n\rangle = (\nabla u, \nabla w) \quad \text{if } w = 0 \text{ on } \partial\Omega$$

and

$$-\left(\nabla^2 v, z\right) = (\nabla v, \nabla z) - \langle \nabla v \cdot n, z\rangle = (\nabla v, \nabla z) - \langle \gamma, z\rangle.$$

The variational problem reads

$$\begin{cases} \mathsf{u} := (u, v) \in \mathcal{H}, \\ B(\mathsf{u}, \mathsf{w}) = F(\mathsf{w}) \quad \text{for every } \mathsf{w} := (w, z) \in \mathcal{H}, \end{cases}$$

where

$$\begin{aligned} B(\mathsf{u}, \mathsf{w}) &= (a(x)(u - v), w - z) + (\nabla u, \nabla w) + (\nabla v, \nabla z) + (v, z) \\ F(\mathsf{w}) &= (f, w) + (g, z) + \langle \gamma, z\rangle. \end{aligned}$$

We need $f \in H^{-1}$, $g \in \left(H^1\right)^*$, and $\gamma \in H^{-1/2}$.

2. Let us use the norm $\|\mathsf{u}\|_{\mathcal{H}}^2 = \|(u, v)\|_{\mathcal{H}}^2 = \|u\|_{H^1(\Omega)}^2 + \|v\|_{H^1(\Omega)}^2$.

Continuty of RHS is trivial under the assumption that f, g, γ come from the appropriate spaces.

$$F(\mathsf{w}) \leq \left(\|f\|_{H^{-1}} + \|g\|_{(H^1)^*} + C\|\gamma\|_{H^{-1/2}}\right)\|\mathsf{w}\|_{\mathcal{H}}.$$

Continuity of B:

$$\begin{aligned} |B(\mathsf{u}, \mathsf{w})| &\leq \|\nabla u\|\|\nabla w\| + \|\nabla v\|\|\nabla z\| + \|v\|\|z\| + \|a\|_\infty (\|u\| + \|v\|)(\|w\| + \|z\|) \\ &\leq \|u\|_{H^1}\|w\|_{H^1} + 2\|v\|_{H^1}\|z\|_{H^1} + \|a\|_\infty (\|u\|_{H^1} + \|v\|_{H^1})(\|w\|_{H^1} + \|z\|_{H^1}) \\ &\leq C\|\mathsf{u}\|_{\mathcal{H}}\|\mathsf{w}\|_{\mathcal{H}} \end{aligned}$$

Coercivity of B:

$$\begin{aligned} B\left(\mathsf{w},\mathsf{w}\right) &= (a(x)(w-z),(w-z)) + \|\nabla w\|^2 + \|\nabla z\|^2 + \|z\|^2 \\ &\geq \|\nabla w\|^2 + \|z\|_{H^1}^2 \\ &\geq C_P^{-2}\|w\|_{H^1}^2 + \|z\|_{H^1}^2 \\ &\geq \min\left\{1, C_P^{-2}\right\}\|\mathsf{w}\|_{\mathcal{H}}. \end{aligned}$$

Notice in particular that

$$(a(x)(w-z),(w-z)) = \int a(x)(w-z)^2 \geq 0.$$

By Lax-Milgram, the problem admits a unique solution.