Homework 5

Jonathan Zhang EID: jdz357

Problem 7.6:

Suppose $\Omega \subset \mathbb{R}^d$ is a possibly unbounded domain and $\{U_\alpha\}_{\alpha \in I}$ is a collection of open sets in $\mathbb{R}^d$ that cover Ω. Prove there exists a locally finite partition of unity in Ω subordinate to the cover.

Solution We start with a lemma:

Lemma 0.1. *If S is a countable dense subset of $\mathbb{R}^d$ with the collection of balls*

$$\mathcal{B} = \left\{ B_r(x) \subset \mathbb{R}^d | r \in \mathbb{Q}, x \in S, \text{ and } \exists \alpha \in I : B_r(x) \subset U_\alpha \right\},$$

then $\cup B_{r_j/2}(x_j)$ is an open cover of Ω.

Proof. Fix $r_j/2 > 0$ for all j and consider a point $x \in \Omega$. Suppose now to the contrary that there does not exists a j such that x belongs to $B_{r_j/2}(x_j)$. Then, since $\mathbb{Q}$ is dense in $\mathbb{R}$, for any $\varepsilon > 0$ there exists a rational q arbitrarily close to x. Since $q \in Q$, then also $q \in S$, and so it has a radius $r_{q/2}$ associated with it. If $\varepsilon \leq r_{q/2}$, then we are done. If not, select a new $\varepsilon_1 < \varepsilon$ and repeat the process, arriving at a new $q_1 \in \mathbb{Q}$ such that $x \in B_{\varepsilon_1}(q_1)$. Repeating, we keep looking with smaller ε_k, q_k, $r_{q_k/2}$. If we repeat infinitely many times, though, we have found some j such that $\varepsilon > r_j$ for every $\varepsilon > 0$, otherwise we would have stopped. The only way to satisfy this would be for $r_j = 0$, a contradiction. □

We now want to construct the functions that make up the partition of unity. Let $\varphi(x)$ denote Cauchy's infinitely differentiable function in 1D. We can easily translate and dilate this to form a bump function $\phi_j \in C_0^\infty(B_j)$ satisfying $0 \leq \phi_j \leq 1$ and $\phi_j = 1$ on $B_{r_j/2}$:

$$\phi_j(x) = \varphi\left(\frac{2}{r_j}\left(|x - x_j| + r_j\right)\right)\varphi\left(-\frac{2}{r_j}\left(|x - x_j| - r_j\right)\right).$$

Now let $\psi_1 = \phi_1$, $\psi_k = \prod_{j=1}^{k-1} \psi_j \psi_k$.

We can now demonstrate the desired properties. First of all, property (iii) is easy to see. Indeed, for every $j \in \mathcal{J}$ that indexes $\mathcal{B}$, there exists $\alpha \in I$ such that

$$\text{supp } \psi_j \subset \text{supp } \phi_j \subset \text{supp } B_j \subset \text{supp } U_\alpha.$$

Now we show that for every compact $K \subset \Omega$, all but finitely many ψ_j vanish. Observe that if $\psi_i = 0$, then all ψ_{i+k} vanish for $k > 0$. We have already shown that $B_{r_j/2}$ provide an open cover for Ω. Since K is compact, it admits a finite subcover, that is, $K \subset \cup_{j=1}^M B_{r_j/2}(x_j)$. Now for $x \in K$, there exists a $N \leq M$ such that $x \in B_{r_N/2}(x_N)$. We then have that $\phi_N(x) = 1$, and that $\psi_n(x) = 0$ for $n \geq N$. So, only finitely many ψ support on K. It is also easy to see that ψ_j range betewen 0 and 1. To show the summation property, go by induction. We need to show that

$$\sum_{j=1}^m \psi_j = 1 - \prod_{j=1}^m (1 - \phi_j).$$

This holds trivially for $m = 1$. Now, for the inductive step,

$$\begin{aligned}\sum_{j=1}^{m+1} \psi_j &= \psi_{m+1} + \sum_{j=1}^{m} \psi_j \\ &= \psi_{m+1} + 1 - \prod_{j=1}^{m} (1 - \phi_j) \\ &= \phi_{m+1} \prod_{j=1}^{m} (1 - \phi_j) + 1 - \prod_{j=1}^{m} (1 - \phi_j) \\ &= 1 - \prod_{j=1}^{m+1} (1 - \phi_j) \, .\end{aligned}$$

To show the summation property, we again use the fact that we have an open cover of half-radius balls. Using what we have just proved,

$$\sum_{j \in \mathcal{J}} \psi_j(x) = \sum_{j=1}^{m} \psi_j(x) = 1 - \prod_{j=1}^{m} (1 - \phi_j(x)) = 1 - (1 - \phi_m(x)) \prod_{j=1}^{m-1} (1 - \phi_j(x)) = 1.$$

Problem 7.7:

$u \in \mathcal{D}' \left(\mathbb{R}^d\right)$, $\phi \in \mathcal{D}$.

Solution

1. Define the functional
$$\psi : \mathbb{R} \ni t \to \psi(t) = \phi(x - ty) \in \mathbb{R}.$$
   Apply the fundamental theorem,

Problem 7.8:

Problem 7.11: