Homework 9

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Unless otherwise stated, denote by $(\cdot,\cdot)$ as the L^2 inner product, and $\|\cdot\|$ as $\|\cdot\|_{L^2(\Omega)}$.

Exercise (19). *Let $\Omega \subset \mathbb{R}^4$ be a bounded domain with smooth boundary, $f \in L^2(\Omega)$. Consider the BVP*

$$-\nabla^2 u + u^3 = f$$

with $u = 0$ on $\partial\Omega$. We attempt to solve iteratively from $u_0 = 0$ by computing for each $n = 1, 2, \ldots$ the solutions to the linear BVP

$$-\nabla^2 u_n + u_{n_1}^2 u_n = f, \qquad u_n = 0 \text{ on } \partial\Omega.$$

1. *Find appropraite variational problems for the linear BVP's and show that they are well deifned in $H_0^1\left(\mathbb{R}^4\right)$ provided that $u_{n-1} \in H_0^1\left(\mathbb{R}^4\right)$. (Hint: Sobolev Imbedding Theorem.)*
2. *Show that there is a unique solution $u_n \in H_0^1\left(\mathbb{R}^4\right)$ assuming that $u_{n-1} \in H_0^1(\mathbb{R}^4)$. Moreover, find a bound for the norm of u_n.*
3. *Show that the nonlinear BVP has a weak solution. Extract a subsequence of u_n that converges weakly to some u and show that u satisfied the weak form of the nonlinear BVP.*

Exercise (21). *$\Omega \subset \mathbb{R}^d$ bounded domain with Lipschitz $\partial\Omega$, define*

$$H\left(div, \Omega\right) = \left\{ v \in (L^2(\Omega))^d : \nabla \cdot v \in L^2(\Omega) \right\}.$$

1. *Show that $H\left(div,\Omega\right)$ is a Hilbert space with inner product*

$$(u, v)_{H(div,\Omega)} = (u, v) + (\nabla \cdot u, \nabla \cdot v).$$

2. *The trace Theorem does not imply that $\partial_\nu v = v \cdot \nu$ exists on $\partial\Omega$. Nevertheless, show that $\partial_\nu : H\left(div,\Omega\right) \to H^{-1/2}\left(\partial\Omega\right) = \left(H^{1/2}(\partial\Omega)\right)^*$ is a well defined bounded linear oeprator in the sense that*

$$\int_{\partial\Omega} v \cdot \nu \phi d\sigma(x) = \int_\Omega \nabla \cdot v \phi dx + \int_\Omega v \cdot \nabla\phi dx.$$

3. *Prove the following inf-sup condition: there exists $\gamma > 0$ such that*

$$\inf_{w \in L^2} \sup_{v \in H(div,\Omega)} \frac{(w, \nabla \cdot v)}{\|w\|\|v\|_{H(div,\Omega)}} \geq \gamma > 0.$$

Hint solve $\Delta\varphi = w$ in $H_0^1\left(\Omega\right)$ and consider $v = \nabla\varphi$.

Exercise (25). *Consider the finite element method.*

1. *Modify the method to account for nonhomogeneous Neumann conditions.*
2. *Modify the method to account for nonhomogeneous Dirichlet conditions.*

Exercise (27). *Suppose $u \in H^1(\Omega)$ where $\Omega \subset \mathbb{R}^d$ is bounded connected. Recall the H^1 seminorm is $|u|_{H^1} = \left\{ \sum_{|\alpha|=1} \|D^\alpha u\|^2 \right\}^{1/2}$.*

1. *Show that there is a constant C_Ω such that*

$$\inf_{c \in \mathbb{R}} \|u - c\| \leq C_\Omega |u|_{H^1}.$$

2. *Let $\Omega = (0,h)^d$ for $h > 0$. Show that there is a constant C independent of h and u such that*

$$\inf_{c \in \mathbb{R}} \|u - c\| \leq Ch|u|_{H^1}.$$

   Change var. to integrate over $(0,1)^d$ and then use previous.

3. *Let $\Omega = (0,1)^d$ and let P be the set of piecewise discontinuous constants over the grid of spacing $h = 1/N$ for some positive integer N. Show that there is a constant C independent of h, u, such that*

$$\inf_{p \in P} \|u - p\| \leq Ch|u|_{H^1}.$$

Exercise (29). *Consider the problem*

1. *Find the Green's Function.*
2. *Instead impose Neumann BC's, and find the Green's function. Recall we now require $-\partial^2/\partial x^2 G(x,y) = \delta_y(x) - 1$.*