Homework 2

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Problem 6.9:

Suppose that $f \in L^p\left(\mathbb{R}^d\right)$ for $p \in (1,2)$. Show that there exist $f_1 \in L^1\left(\mathbb{R}^d\right)$ and $f_2 \in L^2\left(\mathbb{R}^d\right)$ such that $f = f_1 + f_2$. Define $\hat{f} = \hat{f}_1 + \hat{f}_2$. Show that this definition is well defined, that is, independent of the choices f_1 and f_2.

Solution Write f as

$$f = f\chi_{[x:|f(x)<1]} + f\chi_{[x:|f(x)\geqslant 1]}$$

and define

$$f_2(x) = f\chi_{[x:|f(x)<1]}, \qquad f_1(x) = f\chi_{[x:|f(x)\geqslant 1]}.$$

Then,

$$\|f_1\|_1 = \int_{\mathbb{R}^d} |f(x)|\chi_{[x:|f(x)\geqslant 1]}dx = \int_{x:|f(x)|\geqslant 1} |f(x)| \leq \int_{x:|f(x)|\geqslant 1} |f(x)|^p \leq \|f\|_p^p.$$

$$\|f_2\|_2^2 = \int_{\mathbb{R}^d} |f(x)|^2\chi_{[x:|f(x)<1]}dx = \int_{x:|f(x)|<1} |f(x)|^2 \leq \int_{x:|f(x)|<1} |f(x)|^p \leq \|f\|_p^p.$$

Next, suppose that f admits two different decompositions, $f = f_1 + f_2 = g_1 + g_2$, where $f_i, g_i \in L^i\left(\mathbb{R}^d\right)$ for $i = 1, 2$. Then,

$$\begin{aligned}
\hat{f}_1 + \hat{f}_2 &= (2\pi)^{-d/2}\left(\int_{\mathbb{R}^d} f_1(x)e^{-ix\cdot\xi}dx + \int_{\mathbb{R}^d} f_2(x)e^{-ix\cdot\xi}dx\right) \\
&= (2\pi)^{-d/2}\int_{\mathbb{R}^d} \left(f_1(x) + f_2(x)\right)e^{-ix\cdot\xi}dx \\
&= (2\pi)^{-d/2}\int_{\mathbb{R}^d} \left(g_1(x) + g_2(x)\right)e^{-ix\cdot\xi}dx \\
&= \hat{g}_1 + \hat{g}_2.
\end{aligned}$$

So, the proposed definition of $\hat{f}$ is well-defined.

Problem 6.11:

Let the ground field be $\mathbb{C}$ and define $T : L^2\left(\mathbb{R}^d\right) \to L^2$ by

$$(Tf)(x) = \int e^{-|x-y|^2/2} f(y)dy.$$

Use the Fourier transform to show that T is a positive, injective but not surjective operator.

Solution Define $\varphi(x) = e^{-|x|^2/2}$. Observe that T is nothing more than convolution, i.e.

$$(Tf)(x) = (\varphi * f)(x) = \int\limits_{\mathbb{R}^d} \varphi(x-y) f(y) \, dy = \int\limits_{\mathbb{R}^d} e^{-|x-y|^2/2} f(y) \, dy.$$

If we consider now the Fourier transform,

$$\widehat{(Tf)} = \widehat{(\varphi * f)} = (2\pi)^{d/2} \hat{\varphi} \hat{f}.$$

The Fourier transform preserves inner product, so

$$(Tf, f) = \left(\widehat{(Tf)}, \hat{f}\right) = \int\limits_{\mathbb{R}^d} (2\pi)^{d/2} \hat{\varphi}(\xi) \left\{\hat{f}(\xi)\right\}^2 d\xi \geqslant 0$$

Notice this is the case since $\varphi = \hat{\varphi} > 0$. So we conclude that T is positive.

Since T is linear, it is sufficient to check $Tf = 0$. We can consider equivalently the Fourier transform, and using the Plancherel Theorem,

$$\widehat{Tf} = 0 = (2\pi)^{d/2} \hat{\varphi}(\xi) \hat{f}(\xi) = (2\pi)^{d/2} \varphi(\xi) \hat{f}(\xi) \Rightarrow \hat{f} = 0 \Rightarrow f = 0.$$

So T is injective.

Consider now solving the equation $Tf = \varphi$, or equivalently, the equation $\mathcal{F}^{-1}\left(\widehat{Tf}\right) = \varphi$. Since the Fourier transform is a unitary operator, then

$$\left(\widehat{Tf}\right)(\xi) = \varphi(\xi) = (2\pi)^{d/2} \varphi(\xi) \hat{f}(\xi).$$

However, since $\varphi > 0$, then it must be that

$$\hat{f}(\xi) = (2\pi)^{-d/2},$$

but this cannot be since the constant function $(2\pi)^{-d/2} \notin L^2\left(\mathbb{R}^d\right)$. Therefore, we cannot find a solution to this equation, so T is not surjective.

Problem 6.15:

Let $\varphi \in \mathcal{S}\left(\mathbb{R}^d\right)$, $\hat{\varphi}\left(0\right) = \left(2\pi\right)^{-d/2}$, and $\varphi_\varepsilon\left(x\right) = \varepsilon^{-d}\varphi\left(x/\varepsilon\right)$. Prove that $\varphi_\varepsilon \to \delta_0$ and $\hat{\varphi}_\varepsilon \to \left(2\pi\right)^{-d/2}$ as $\varepsilon \to 0^+$. In what sense do these convergences take place?

Notice first that

$$\hat{\varphi}\left(0\right) = \left(2\pi\right)^{-d/2} \int\limits_{\mathbb{R}^d} \varphi\left(x\right) dx \Rightarrow \int\limits_{\mathbb{R}^d} \varphi\left(x\right) dx = 1$$

We claim that $\varphi_\varepsilon \to \delta_0$ in $\mathcal{S}'$. Let $f \in \mathcal{S}$. Then

$$\begin{aligned}
\left\langle \varphi_\varepsilon, f \right\rangle &= \int\limits_{\mathbb{R}^d} \varepsilon^{-d}\varphi\left(x/\varepsilon\right) f\left(x\right) dx \\
&= \int\limits_{\mathbb{R}^d} \varphi\left(u\right) f\left(\varepsilon u\right) du \\
&\xrightarrow{\varepsilon\to 0^+} \int\limits_{\mathbb{R}^d} \varphi\left(u\right) f\left(0\right) du \\
&= f\left(0\right) \\
&= \left\langle \delta_0, f \right\rangle
\end{aligned}$$

where we have used the variable substitution $u = x/\varepsilon$, $du = \varepsilon^{-d}dx$ and the Lebesgue Dominated Convergence Theorem to justify passing the limit under the integral since $f(\varepsilon u) \to f(0)$ uniformly. Indeed, notice that for fixed u, the integrand converges pointwise to the desired result, and it is dominated by the measurable function $\|f\|_\infty \varphi(u)$.

Next,

$$\widehat{\varphi}_\varepsilon = \mathcal{F}\left(\varphi_\varepsilon\right) = \varepsilon^{-d}\mathcal{F}\left(\varphi\left(\frac{1}{\varepsilon}x\right)\right) = \varepsilon^{-d}\left(\frac{1}{\varepsilon}\right)^{-d}\hat{\varphi}\left(\varepsilon\xi\right) = \hat{\varphi}\left(\varepsilon\xi\right)$$

and so

$$\left\langle \widehat{\varphi}_\varepsilon, f \right\rangle = \int\limits_{\mathbb{R}^d} \hat{\varphi}\left(\varepsilon\xi\right) f\left(\xi\right) d\xi \xrightarrow{\varepsilon\to 0^+} \int\limits_{\mathbb{R}^d} \left(2\pi\right)^{-d/2} f\left(\xi\right) d\xi = \left\langle \left(2\pi\right)^{-d/2}, f \right\rangle$$

by the same dominated convergence argument.