

Homework 9

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Unless otherwise stated, denote by (\cdot, \cdot) as the L^2 inner product, and $\|\cdot\|$ as $\|\cdot\|_{L^2(\Omega)}$.

Exercise (19). Let $\Omega \subset \mathbb{R}^4$ be a bounded domain with smooth boundary, $f \in L^2(\Omega)$. Consider the BVP

$$-\nabla^2 u + u^3 = f$$

with $u = 0$ on $\partial\Omega$. We attempt to solve iteratively from $u_0 = 0$ by computing for each $n = 1, 2, \dots$ the solutions to the linear BVP

$$-\nabla^2 u_n + u_{n-1}^2 u_n = f, \quad u_n = 0 \text{ on } \partial\Omega.$$

1. Find appropriate variational problems for the linear BVP's and show that they are well defined in $H_0^1(\mathbb{R}^4)$ provided that $u_{n-1} \in H_0^1(\mathbb{R}^4)$. (Hint: Sobolev Imbedding Theorem.)
2. Show that there is a unique solution $u_n \in H_0^1(\mathbb{R}^4)$ assuming that $u_{n-1} \in H_0^1(\mathbb{R}^4)$. Moreover, find a bound for the norm of u_n .
3. Show that the nonlinear BVP has a weak solution. Extract a subsequence of u_n that converges weakly to some u and show that u satisfies the weak form of the nonlinear BVP.

Proof. 1. Assume that $u_{n-1} \in H_0^1(\mathbb{R}^4)$ is given. Multiply by a test function and integrate by parts the first term,

$$\begin{aligned} -(\nabla^2 u_n, v) + (u_{n-1}^2 u_n, v) &= (f, v) \\ (\nabla u_n, \nabla v) - \langle \nabla u_n \cdot n, v \rangle + (u_{n-1}^2 u_n, v) &= (f, v) \end{aligned}$$

For $v \in H_0^1(\Omega)$, we have

$$B(u_n, v) := (\nabla u_n, \nabla v) + (u_{n-1}^2 u_n, v) = (f, v) =: F(v).$$

The VP reads

$$\begin{cases} u_n \in H_0^1(\Omega), \\ (\nabla u_n, \nabla v) + (u_{n-1}^2 u_n, v) = (f, v) \\ \text{for every } v \in H_0^1(\Omega). \end{cases}$$

We need to check that the second integral is well-defined. We recall that for the zero spaces, we simply extend by zero to the whole space with the equality $\|f\|_{H_0^1(\Omega)} = \|f\|_{H_0^1(\mathbb{R}^4)}$.

Let $p = 2$, $j = 0$, $m = 1$, and $d = 4$. The Sobolev Embedding Theorem tells us that $H_0^1(\Omega) \equiv W_0^{1,2}(\Omega) \hookrightarrow W^{0,q}(\Omega) \equiv L^q(\Omega)$ for all finite $q \leq dp/(d - mp) = 4$. We will use exactly $q = 4$ for the estimate

$$\begin{aligned} |(u_{n-1}^2 u_n, v)| &\leq \|u_{n-1}\|_{L^4(\Omega)}^2 \|u_n\|_{L^4(\Omega)} \|v\|_{L^4(\Omega)} && \text{Hölder's} \\ &\leq C \|u_{n-1}\|_{H_0^1(\mathbb{R}^4)}^2 \|u_n\|_{H_0^1(\Omega)} \|v\|_{H_0^1(\Omega)} && \text{Sobolev Emb.} \\ &= C \|u_{n-1}\|_{H_0^1(\mathbb{R}^4)}^2 \|u_n\|_{H_0^1(\mathbb{R}^4)} \|v\|_{H_0^1(\mathbb{R}^4)} && \text{Bd. Ext. Op.} \end{aligned}$$

So, the problem is well-defined in the space $H^1(\mathbb{R}^4)$ in the sense that all terms are well-defined, and the solution obtained from solving the VP which lives presently in $H_0^1(\Omega)$ can be easily extended to $H^1(\mathbb{R}^4)$.

2. We have a symmetric functional setting, so Lax-Milgram is the tool. The right hand side is trivially continuous by Hölder's Inequality given $f \in L^2(\Omega)$. The left hand side is also continuous,

$$\begin{aligned} B(u_n, v) &= (\nabla u_n, \nabla v) + (u_{n-1}^2 u_n, v) \\ &\leq \|\nabla u_n\| \|\nabla v\| + C \|u_{n-1}\|_{H_0^1(\mathbb{R}^4)}^2 \|u_n\|_{H_0^1(\mathbb{R}^4)} \|v\|_{H_0^1(\mathbb{R}^4)} \\ &\leq \left(1 + C \|u_{n-1}\|_{H_0^1(\mathbb{R}^4)}^2\right) \|u_n\|_{H_0^1(\Omega)} \|v\|_{H_0^1(\Omega)}. \end{aligned}$$

The bilinear form is also coercive,

$$\begin{aligned} B(v, v) &= \|\nabla v\|^2 + \int_{\Omega} u_{n-1}^2 v^2 \\ &\geq \|\nabla v\|^2 \\ &\geq C_P^{-2} \|v\|_{H_0^1(\Omega)}^2. \end{aligned}$$

By Lax-Milgram, the problem admits a unique solution $u_n \in H_0^1(\Omega)$ which we can extend by zero to $u_n \in H_0^1(\mathbb{R}^4)$. Moreover, we have the estimate

$$\|u_n\|_{H_0^1(\Omega)} = \|u_n\|_{H_0^1(\mathbb{R}^4)} \leq C_P^2 \|f\|.$$

Note the bound is independent of n .

3. Notice that u_n has just been shown to be a bounded sequence in the Hilbert space $H_0^1(\Omega)$. Therefore, there is a subsequence $u_{n_k} \rightharpoonup u$ in $H_0^1(\Omega)$, furthermore, $u_{n_k} \rightarrow u$ in $L^2(\Omega)$ and $\nabla u_{n_k} \rightharpoonup \nabla u$ in $L^2(\Omega)^4$. We argue that u is indeed a weak solution to the nonlinear PDE. For every test function v , $(\cdot, \nabla v)$ is a linear functional on $L^2(\Omega)^4$, so $(\nabla u_{n_k}, \nabla v) \rightarrow (\nabla u, \nabla v)$. Additionally, $u_{n_k-1}^2 u_{n_k} \rightarrow u^3$, so $(u_{n_k-1}^2 u_{n_k}, v) \rightarrow (u^3, v)$. This implies

$$B(u_{n_k}, v) \rightarrow (\nabla u, \nabla v) + (u^3, v) = (f, v)$$

which is exactly the weak form of the original nonlinear problem. □

Exercise (21). $\Omega \subset \mathbb{R}^d$ bounded domain with Lipschitz $\partial\Omega$, define

$$H(\operatorname{div}, \Omega) = \left\{ v \in (L^2(\Omega))^d : \nabla \cdot v \in L^2(\Omega) \right\}.$$

1. Show that $H(\operatorname{div}, \Omega)$ is a Hilbert space with inner product

$$(u, v)_{H(\operatorname{div}, \Omega)} = (u, v) + (\nabla \cdot u, \nabla \cdot v).$$

2. The trace Theorem does not imply that $\partial_\nu v = v \cdot \nu$ exists on $\partial\Omega$. Nevertheless, show that $\partial_\nu : H(\operatorname{div}, \Omega) \rightarrow H^{-1/2}(\partial\Omega) = (H^{1/2}(\partial\Omega))^*$ is a well defined bounded linear operator in the sense that

$$\int_{\partial\Omega} v \cdot \nu \phi d\sigma(x) = \int_{\Omega} \nabla \cdot v \phi dx + \int_{\Omega} v \cdot \nabla \phi dx.$$

3. Prove the following inf-sup condition: there exists $\gamma > 0$ such that

$$\inf_{w \in L^2} \sup_{v \in H(\operatorname{div}, \Omega)} \frac{(w, \nabla \cdot v)}{\|w\| \|v\|_{H(\operatorname{div}, \Omega)}} \geq \gamma > 0.$$

Hint solve $\Delta\varphi = w$ in $H_0^1(\Omega)$ and consider $v = \nabla\varphi$.

Proof. 1. $H(\operatorname{div}, \Omega)$ is a linear subspace of the Hilbert space $L^2(\Omega)^d$. If $u, v \in H(\operatorname{div}, \Omega)$, then $u + v$ is also in $H(\operatorname{div}, \Omega)$ since $u + v \in L^2(\Omega)^d$ and $\nabla \cdot (u + v) = \nabla \cdot u + \nabla \cdot v \in L^2(\Omega)$. What remains to show is that $H(\operatorname{div}, \Omega)$ is closed with respect to $\|\cdot\|_{H(\operatorname{div}, \Omega)}$. Remark: the proposed inner product is indeed an inner product since it is a combination of inner products on $L^2(\Omega)^d$ and $L^2(\Omega)$. Let u_n be a Cauchy sequence in $H(\operatorname{div}, \Omega)$ with $u_n \rightarrow u$ in $L^2(\Omega)^d$. \square

Exercise (25). *Consider the finite element method.*

1. *Modify the method to account for nonhomogeneous Neumann conditions.*
2. *Modify the method to account for nonhomogeneous Dirichlet conditions.*

Exercise (27). Suppose $u \in H^1(\Omega)$ where $\Omega \subset \mathbb{R}^d$ is bounded connected. Recall the H^1 seminorm is $|u|_{H^1} = \left\{ \sum_{|\alpha|=1} \|D^\alpha u\|^2 \right\}^{1/2}$.

1. Show that there is a constant C_Ω such that

$$\inf_{c \in \mathbb{R}} \|u - c\| \leq C_\Omega |u|_{H^1}.$$

2. Let $\Omega = (0, h)^d$ for $h > 0$. Show that there is a constant C independent of h and u such that

$$\inf_{c \in \mathbb{R}} \|u - c\| \leq Ch |u|_{H^1}.$$

Change var. to integrate over $(0, 1)^d$ and then use previous.

3. Let $\Omega = (0, 1)^d$ and let P be the set of piecewise discontinuous constants over the grid of spacing $h = 1/N$ for some positive integer N . Show that there is a constant C independent of h, u , such that

$$\inf_{p \in P} \|u - p\| \leq Ch |u|_{H^1}.$$

Proof. 1. We have

$$\inf_{c \in \mathbb{R}} \|u - c\| \leq \|u - d\|$$

(definition of infimum). So, it is sufficient to demonstrate the result for a particular constant $d \in \mathbb{R}$ and then simply take infimum on both sides. □

Lemma 0.1. There exists $C > 0$ such that

$$\|u\|^2 \leq C \left\{ \left| \int_{\Omega} u \right|^2 + \|\nabla u\|^2 \right\}$$

for every $u \in H^1(\Omega)$.

Proof. (Standard proof of Poincaré-like inequality with contradiction.) Assume to the contrary that there exists a sequence u_n satisfying

$$\|u_n\| = 1, \quad \int_{\Omega} u_n \rightarrow 0, \quad \|\nabla u_n\| \rightarrow 0.$$

Then there exists a subsequence $u_{n_k} \rightharpoonup u$ in $H^1(\Omega)$. Weak convergence of u_{n_k} in $H^1(\Omega)$ implies $\nabla u_{n_k} \rightharpoonup \nabla u$ in $L^2(\Omega)$. But, by the lower weak sequential semicontinuity of the L^2 norm, then $\nabla u = 0$, that is, u is constant. At the same time, convergence of the average of u implies that u must be zero. By the Rellich-Kondrachov Theorem, however, we obtain that $u_{n_k} \rightarrow u$ in $L^2(\Omega)$ and $\|u\| = 1$, a contradiction. □

Exercise (29). *Consider the problem*

1. *Find the Green's Function.*
2. *Instead impose Neumann BC's, and find the Green's function. Recall we now require $-\partial^2/\partial x^2 G(x, y) = \delta_y(x) - 1$.*