Homework 10

Jonathan Zhang EID: jdz357

Exercise (9.3). *Let $X = C([0,1])$ be the space of bounded continuous functions on $[0,1]$ and, for $u \in X$, define $F(u)(x) = \int_0^1 K(x,y)f(u(y))dy$ where $K : [0,1] \times [0,1] \to \mathbb{R}$ is continuous and f is a C^1 mapping of $\mathbb{R}$ into $\mathbb{R}$. Find the Fréchet derivative $DF(u)$ of F at $u \in X$. Is the map $u \mapsto DF(u)$ continuous?*

$f \in C^1$ implies that f has a Fréchet derivative, denoted f'. Moreover, for $u \in X$, $f \circ u \in X$ since composition with a continuous function remains continuous. Define another operator $G : X \to X$ as

$$(Gv)(x) = \int_0^1 K(x,y)v(y)dy.$$

Since K is continuous, G is well defined, and $G \in B(X,X)$. We know then that the $DG(v) = G$. Furthermore, notice that $(Fu)(x)$ is nothing more than $(G \circ f)(x)$. So, using the chain rule,

$$DF(u) = D(G \circ f)(u) = DG(f(u))\, f'(u) = G(f(u))\, f'(u).$$

The mapping is continuous since f' is continuous, multiplication is continuous, and G is continuous.

Exercise (9.5). *Set up and apply contraction mapping principle to show that the problem*

$$-u_{xx} + u - \varepsilon u^2 = f(x), \quad x \in \mathbb{R}$$

has a smooth bounded solution if $\varepsilon > 0$ is small enough, where $f(x) \in \mathcal{S}(\mathbb{R})$.

Take the Foruier transform:

$$-u_{xx} + u = f + \varepsilon u^2 =: g \longrightarrow \left(1 + \xi^2\right)\hat{u} = \hat{g}$$

yields the solution

$$\hat{u} = \frac{\hat{g}}{1+\xi^2} \longrightarrow u = g * \frac{1}{2}e^{-|x|} = \left(f + \varepsilon u^2\right) * \frac{1}{2}e^{-|x|} = \frac{1}{2}\int_{\mathbb{R}} \left(f(y) + \varepsilon u^2(y)\right) e^{-|x-y|}dy.$$

It is natural then to regard the RHS as an operator acting on continuous function on some ball of radius r, say. Let $X = C(B_r(0)) \subset C(\mathbb{R})$ equipped with max norm, and let

$$(Gu)(x) = \frac{1}{2}\int_{\mathbb{R}} \left(f(y) + \varepsilon u^2(y)\right) e^{-|x-y|}dy.$$

In order to apply contraction mapping, we need to show that G sends X back into itself, and that G is a contraction. For $u \in X$, we have the estimate

$$
\begin{aligned}
\|Gu\|_\infty &= \sup_x \left| \frac{1}{2} \int_{\mathbb{R}} \left(f(y) + \varepsilon u^2(y)\right) e^{-|x-y|} dy \right| \\
&\leqslant \left(\|f\|_\infty + \varepsilon \|u\|_\infty^2\right) \left(\frac{1}{2} \sup_x \int_{\mathbb{R}} e^{-|x-y|} dy \right) \\
&\leqslant \|f\|_\infty + \varepsilon r^2 .
\end{aligned}
$$

The condition implies

$$
\|f\|_\infty + \varepsilon r^2 \leqslant r \Rightarrow \varepsilon \leqslant \frac{-\|f\|_\infty + r}{r^2}.
$$

Notably, we need $r > \|f\|_\infty$. To show that G is a contraction,

$$
\begin{aligned}
\|Gu_1 - Gu_2\|_\infty &= \left\| \frac{\varepsilon}{2} \int_{\mathbb{R}} \left(u_1^2(y) - u_2^2(y)\right) e^{-|x-y|} dy \right\|_\infty \\
&\leqslant \left\| \frac{\varepsilon}{2} \int_{\mathbb{R}} \left(u_1(y) + u_2(y)\right) e^{-|x-y|} dy \right\|_\infty \|u_1 - u_2\|_\infty \\
&\leqslant \varepsilon r \sup_x \int_{\mathbb{R}} e^{-|x-y|} dy \|u_1 - u_2\|_\infty \\
&= 2\varepsilon r \|u_1 - u_2\|_\infty
\end{aligned}
$$

Now G is a contraction if

$$
2\varepsilon r < 1 \Rightarrow \varepsilon < \frac{1}{2r}.
$$

So, for $\varepsilon < \min\left\{(2r)^{-1}, r^{-2}(-\|f\|_\infty + r)\right\}$, the problem admits a smooth and bounded solution.

Exercise (9.7). *Suppose that F is defined on a Banach space X, that $x_0 = F(x_0)$ is a fixed point of F, $DF(x_0)$ exists, and that 1 is not in the spectrum of $DF(x_0)$. Prove that x_0 is an isolated fixed point.*

Suppose to the contrary that x_0 is not an isolated fixed point. That is, there exists another fixed point, say x_1, within an ε ball of x_0. We consider the difference

$$
x_1 - x_0 = F(x_1) - F(x_0) = DF(x_0)(x_1 - x_0) + R(x_0, x_1 - x_0)
$$

whihc we can write as

$$
(I - DF(x_0))(x_1 - x_0) = R(x_0, x_1 - x_0).
$$

Since 1 is not in the spectrum of $DF(x_0)$, then the operator $I - DF(x_0)$ has a bounded inverse. Finally,

$$
1 = \lim_{x_1 \to x_0} \frac{\|x_1 - x_0\|}{\|x_1 - x_0\|} = \lim_{x_1 \to x_0} \frac{\left\| [I - DF(x_0)]^{-1} R(x_0, x_1 - x_0) \right\|}{\|x_1 - x_0\|} \lesssim \lim_{x_1 \to x_0} \frac{\|R(x_0, x_1 - x_0)\|}{\|x_1 - x_0\|} \to 0
$$

since $R(x_0, x_1 - x_0)$ is $o(\varepsilon)$. Contradiction.

Exercise (9.8). *Consider the ODE*

$$u'(t) + u(t) = \cos(u(t))$$

posed as an IVP for $t > 0$ with $u(0) = u_0$.

1. *Use contraction mapping theorem to show that there is exactly one solution u corresponding to any given $u_0 \in \mathbb{R}$.*
2. *Prove that there is a number ξ such that $\lim_{t\to\infty} u(t) = \xi$ for any solution u, independent of the value of u_0.*

1. ODE to integral equation:

$$u'(t) = \cos(u(t)) - u(t) \longrightarrow u(t) = u_0 + \int_0^t \{\cos(u(t)) - u(t)\}\, dt$$

Natural to consider $X = C([0,T])$ for some $T > 0$ and the map G defined by

$$(Gu)(t) = u_0 + \int_0^t \{\cos(u(t)) - u(t)\}\, dt.$$

We see that u solves the ODE only if it is a solution to the fixed point problem $Gu = u$. We also see that $G : X \to X$. (If u continuous, then fed through the continuous operations of cosine, integration, and addition produce another continuous function). Check that G is a contraction:

$$\begin{aligned}
\|Gu - Gv\|_\infty &= \left\| \int_0^t \{\cos(u(t)) - u(t)\}\, dt - \left\{ \int_0^t \{\cos(v(t)) - v(t)\}\, dt \right\} \right\|_\infty \\
&= \left\| \int_0^t (\cos u - \cos v)\, dt + \int_0^t (v - u)\, dt \right\|_\infty \\
&\leqslant \left\| \int_0^t (\cos u - \cos v)\, dt \right\|_\infty + \left\| \int_0^t (v - u)\, dt \right\|_\infty \\
&\leqslant \sup_{0\leqslant t\leqslant T} \int_0^t |\cos u - \cos v|\, dt + T\|u - v\|_\infty \\
&\leqslant \sup_{0\leqslant t\leqslant T} \int_0^t |-\sin(w(t))(u(t) - v(t))|\, dt + T\|u - v\|_\infty \\
&\leqslant 2T\|u - v\|_\infty
\end{aligned}$$

For $2T < 1 \Rightarrow T < 1/2$, G is a contraction, and we have a unique solution. In fact, the solution exists for all $t > 0$. Solve the initial equation up to time $T = 1/3$, say. Then repeat the process with a new initial condition given by the value of u at $t = 1/3$.

2. Let $\xi \in \mathbb{R}$ be the fixed point of cosine, that is $\xi = \cos \xi$. Indeed, ξ is unique. One can just plot the two functions and observe this, or argue with another contraction map argument on the interval $[-\pi/2, \pi/2]$, say. We now have the following situation:
   (a) If $u_0 = \xi$, then $u'(0) = \cos(u(0)) - u_0 = \xi - \xi = 0$, so $u = \xi$ is a constant.
   (b) If $u_0 < \xi$, then $u'(0) = \cos(u_0) - u_0 > 0$, so u is increasing. Moreover, $u'(t)$ remains positive as long as $u(t) < \xi$. However, since $\sin x - 1 \leq 0$ for all x, $u'(t)$ is increasing at a decreasing rate, and so $u'(t) \searrow 0$.
   (c) If $u_0 > \xi$, then $u'(0) = \cos(u_0) - u_0 < 0$, so u is decreasing. Moreover, $u'(t)$ remains negative as long as $u(t) > \xi$. However, since $\sin x - 1 \leq 0$ for all x, $u'(t)$ is decreasing at a decreasing rate, and so $u'(t) \nearrow 0$.

Exercise (9.10). *Consider the PDE*

$$u_t - u_{xxt} - \varepsilon u^3 = f, \quad x \in \mathbb{R}, t > 0$$

with $u(x,0) = g(x)$. Use the Fourier transform and a contraction mapping argument to show that there exists a solution for small enough ε, at least up to some time $T < \infty$. In what spaces should f and g lie?

Fourier transform in x since we are on the whole real line:

$$u_t - u_{xxt} = f + \varepsilon u^3 =: h \to \hat{u}_t + \xi^2 \hat{u}_t = \hat{h} \to \hat{u}_t = \frac{\hat{h}}{1+\xi^2}$$

which, after taking inverse (similar to Ex 9.5), yields

$$u_t(x,t) = \frac{1}{2} \int_{\mathbb{R}} \left(f(y,t) + \varepsilon u^3(y,t)\right) e^{-|x-y|} dy.$$

Integrating in time,

$$u(x,t) = g(x) + \frac{1}{2} \int_0^t \int_{\mathbb{R}} \left(f(y,\tau) + \varepsilon u^3(y,\tau)\right) e^{-|x-y|} dy d\tau.$$

Let $X = C(B_r(0)) \subset C(\mathbb{R} \times (0,T))$ equipped with the max norm. Define the map G by

$$(Gu)(x,t) = g(x) + \frac{1}{2} \int_0^t \int_{\mathbb{R}} \left(f(y,\tau) + \varepsilon u^3(y,\tau)\right) e^{-|x-y|} dy d\tau.$$

We want to show that G takes X into X and that G is a contraction. For now, take $f, g \in L^\infty(\Omega)$. We have the estimate

$$\begin{aligned}
\|Gu\|_\infty &= \left\| g(x) + \frac{1}{2} \int_0^t \int_{\mathbb{R}} \left(f(y,\tau) + \varepsilon u^3(y,\tau)\right) e^{-|x-y|} dy d\tau \right\|_\infty \\
&\leqslant \|g\|_\infty + \left(\|f\|_\infty + \varepsilon r^3\right) \left\| \frac{1}{2} \int_0^t \int_{\mathbb{R}} e^{-|x-y|} dy d\tau \right\|_\infty \\
&\leqslant \|g\|_\infty + \left(\|f\|_\infty + \varepsilon r^3\right) T
\end{aligned}$$

For $\|Gu\| \leq r$, we have

$$\|g\|_\infty + \left(\|f\|_\infty + \varepsilon r^3\right) T \leqslant r \Rightarrow \varepsilon \leqslant r^{-3} \left\{ \frac{r - \|g\|_\infty}{T} - \|f\|_\infty \right\}$$

and implicitly,

$$r > \|g\|_\infty, \qquad T < \frac{r - \|g\|_\infty}{\|f\|_\infty}.$$

For G to be a contraction, we estimate

$$\begin{aligned}
\|Gu_1 - Gu_2\| &= \left\| \frac{\varepsilon}{2} \int_0^t \int_{\mathbb{R}} \left(u_1^3(y,\tau) - u_2^3(y,\tau)\right) e^{-|x-y|} dy d\tau \right\| \\
&= \left\| \frac{\varepsilon}{2} \int_0^t \int_{\mathbb{R}} (u_1 - u_2)\left(u_1^2 + u_1 u_2 + u_2^2\right)(y,\tau)\, e^{-|x-y|} dy d\tau \right\| \\
&\leqslant \varepsilon 3 r^2 T \|u_1 - u_2\|
\end{aligned}$$

So, G is a contraction as long as $3\varepsilon r^2 T < 1$.

In summary, if $f, g \in L^\infty(\Omega)$, $r > \|g\|_\infty$, $T < \frac{r - \|g\|_\infty}{\|f\|_\infty}$, $\varepsilon < \min\left\{ r^{-3}\left\{ \frac{r - \|g\|_\infty}{T} - \|f\|_\infty \right\}, \frac{1}{3r^2 T} \right\}$, then the problem admits a unique solution $u \in B_r(0)$.