Homework 1

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Problem 6.2:

Compute the Fourier transform of $\exp\left(-a|x|^2\right)$, $a > 0$, directly for $x \in \mathbb{R}$.

Solution First of all, notice that the Gaussian is an L^1 function, so this integral makes sense. Computing directly,

$$\hat{f}\left(\xi\right) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{-ax^2 - ix\xi} dx$$

By completing the square,

$$\hat{f}\left(\xi\right) = (2\pi)^{-1/2} e^{-\xi^2/4a} \int_{-\infty}^{\infty} e^{-a(x+i\xi/2a)^2} dx.$$

We attempt to evaluate the integral by considering the corresponding complex integral over a rectangle. Disregarding the details for a moment to conclude the result, the original integral is equal to

$$\hat{f}\left(\xi\right) = (2\pi)^{-1/2} e^{-\xi^2/4a} \sqrt{\frac{\pi}{a}}.$$

To compute the integral, we consider the counter-clockwise countour in $\mathbb{C}$ given by the vertices $R, -R, R - i\xi/2a, -R - i\xi/2a$. Since the integrand is holomorphic, Cauchy's Theorem tells us the integral over this contour must be zero. Further, the integrals on the sides of the rectangles vanish as $R \to \infty$ as they are each bounded by e^{-aR^2}. (This can be seen by letting $z(t) = -R - i\xi t/2a$ for $t \in [0, 1]$ on the left edge, say. Then plug z into the integral, expand the exponent, and see that the integral in magnitude is dominated by e^{-aR^2}.) Denote the upper edge contour as Γ_1 and the lower as Γ_3. Cauchy's Theorem tells us

$$\int_{\Gamma_1} + \int_{\Gamma_3} = 0$$

In particular, since Γ_1 is oriented from right to left, we are actually interested in $-\int_{\Gamma_1} = \int_{\Gamma_3}$. On Γ_3, $z = t - i\xi/2a$ and so the integral is

$$\int_{\Gamma_3} e^{-a(z+i\xi/2a)^2} dz = \int_{-\infty}^{\infty} e^{-at^2} dt = \sqrt{\frac{\pi}{a}}.$$

Problem 6.3:

If $f \in L^1\left(\mathbb{R}^d\right)$ and $f > 0$, show that for every $\xi \neq 0$, $\left|\hat{f}\left(\xi\right)\right| < \hat{f}\left(0\right)$.

Solution By definition,

$$\hat{f}\left(0\right) = \left(2\pi\right)^{-d/2} \int\limits_{\mathbb{R}^d} f\left(x\right) dx.$$

The weak inequality follows by direct computation,

$$\begin{aligned}
\left|\hat{f}\left(\xi\right)\right| &= \left|\left(2\pi\right)^{-d/2} \int\limits_{\mathbb{R}^d} f\left(x\right) e^{-ix\cdot\xi} dx\right| \\
&\leqslant \left(2\pi\right)^{-d/2} \int\limits_{\mathbb{R}^d} \left|f\left(x\right)\right| dx \\
&\leqslant \left(2\pi\right)^{-d/2} \int\limits_{\mathbb{R}^d} f\left(x\right) dx \\
&= \hat{f}\left(0\right).
\end{aligned}$$

What remains is to show that the inequality is strict. We can decompose the real and imaginary parts of the fourier transform as

$$\hat{f}\left(\xi\right) = \left(2\pi\right)^{-d/2} \int f(x) \cos\left(x\cdot\xi\right) dx - \left(2\pi\right)^{-d/2} i \int f(x) \sin\left(x\cdot\xi\right) dx$$

Now, notice that since $f > 0$, $f(x)\cos(x\cdot\xi) \leq f(x)$, and so $\Re\left(\hat{f}\right)\left(\xi\right) \leq \int f(x)dx$. If $\xi \neq 0$, then $\cos(x\cdot\xi) < 1$ apart from a set of measure zero, so equality is reached only if $\|f\|_1 = 0$, but that would imply that $f = 0$ almost everywhere, a contradiction to the assumption that $f > 0$. So, this inequality is strict whenever $\xi \neq 0$.

Problem 6.4:

If $f \in L^1\left(\mathbb{R}^d\right)$ and $f(x) = g\left(|x|\right)$ for some g, show that $\hat{f}\left(\xi\right) = h\left(|\xi|\right)$ for some h. Can you relate g and h?

Solution We say that f is a radial function iff $f \circ T = f$ for every orthogonal transformation T on $\mathbb{R}^d$. We note that orthogonal transformations preserve inner products and thus their Jacobian is of unit magnitude. If $f \in L^1\left(\mathbb{R}^d\right)$ is radial, then for a fixed T orthogonal transformation,

$$\hat{f}\left(T\xi\right) = \int_{\mathbb{R}^d} f(x) e^{-T\xi\cdot x} = \int_{\mathbb{R}^d} f(Ts) e^{-T\xi\cdot Ts} ds = \int_{\mathbb{R}^d} f(s) e^{-i\xi\cdot s} ds = \hat{f}(\xi)$$

and thus $\hat{f}$ is radial and we may write $\hat{f}(\xi) = h(|\xi|)$. Furthermore, we see that g and h are related as h is the Fourier transform of g, although only if g and h are defined on $\mathbb{R}^+$, for if they are defined also on the entire $\mathbb{R}$, we cannot comment on their values for negative arguments.

Problem 6.6:

Show that the Fourier transform $\mathcal{F} : L^1\left(\mathbb{R}^d\right) \to C_v\left(\mathbb{R}^d\right)$ is not onto. Show that $\mathcal{F}\left(L^1\left(\mathbb{R}^d\right)\right)$ is dense in $C_v\left(\mathbb{R}^d\right)$.

Solution The general strategy is to attempt to find a function in C_v that cannot be the image of a function in L^1. It is sufficient to construct an odd function $g \in C_v$ such that

$$\left| \int_r^R \frac{g(t)}{t} dt \right| \to \infty \quad \text{as} \quad R \to \infty.$$

We consider the odd extension of the function

$$g(t) = \begin{cases} t/e & 0 \le t \le e \\ 1/\ln t & t > e \end{cases}$$

We see that

$$\int_e^R \frac{g(t)}{t} dt = \ln \ln R \to \infty.$$

To show that the image of L^1 is dense in C_v, we first show that if f is continuous with compact support and twice differentiable, then $\hat{f} \in L^1$. Indeed, for $\xi \neq 0$,

$$\hat{f}(\xi) = \int f(x) e^{-ix\xi} dx = \frac{1}{i\xi} \int f'(x) e^{-ix\xi} dx = \frac{1}{-\xi^2} \int f''(x) e^{-ix\xi} dx$$

with each integration by parts, the boundary terms disappear due to compact support. We see that the last expression is a C_v function times $1/\xi^2$ which shows that $\hat{f}$ is bounded by a constant times ξ^{-2}. Furthermore, around zero, we have already shown that $\hat{f}$ is bounded. Hence, we conclude that the range of $\mathcal{F}$ on L^1 is dense in C_v.