Homework 7

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Exercise (8.1). *If A is positive definite, show that its eigenvalues are positive. If A is symmetric and has positive eigenvalues, then A is positive definite.*

Proof. Suppose that A is positive definite. Let (λ, v) be an eigenpair for A. Then by the positive definiteness,

$$0 < (v, Av) = (v, \lambda v) = \lambda\,(v, v) = \lambda \|v\|^2.$$

Since this quantity is strictly positive, we must have $\lambda > 0$.

On the other side, assume now that A has all positive eigenvalues. Then, A admits a diagonalization $A = U\Lambda U^*$ for some $U^* = U^{-1}$ unitary matrix. Pick $x \neq 0$, and set $y = U^* x \neq 0$. Then

$$(x, Ax) = (x, U\Lambda U^* x) = (y, \Lambda y)\,.$$

Expand $y = \sum y_i e_i$, then

$$(y, \Lambda y) = \sum_i \overline{y_i} y_i \lambda_i > 0$$

since $\lambda_i > 0$ and $\overline{y_i} y_i = |y_i|^2 \geq 0$ since $y \neq 0$ by assumption. This shows that $(x, Ax) = (y, \Lambda y) > 0$ and A is positive definite.

□

Exercise (8.3). *Suppose we wish to find $u \in H^2(\Omega)$ satisfying $-\Delta u + cu^2 = f \in L^2(\Omega)$ for $\Omega \subset \mathbb{R}^d$ is a bounded Lipschitz domain. For consistency, we would require that $cu^2 \in L^2(\Omega)$. Determine the smallest p such that if $c \in L^p(\Omega)$, then this holds. Depends on d.*

Proof. We want $cu^2 \in L^2(\Omega)$. Suppose that $u \in L^r(\Omega)$ and $c \in L^p(\Omega)$. Then by Generalized Hölder's, we have the estimate

$$\|cu^2\|_2^2 = \int_\Omega c^2 u^4 \le \|c\|_p^2 \|u\|_r^4$$

provided that $2/p + 4/r = 1$. That is, given r, we determine $p = \frac{2}{1-4/r}$, and if $c \in L^p$, then we have $cu^2 \in L^2(\Omega)$.

We make use now of the Sobolev embedding theorem. We know $u \in H^2(\Omega) = W^{2,2}(\Omega)$. Let $j = 0$, $m = 2$, $p = 2$. Then we have

1. If $4 < d$, then $W^{2,2}(\Omega) \hookrightarrow W^{0,r}(\Omega)$ for every $r \le \frac{2d}{d-4}$. Let us first investigate when $r = \frac{2d}{d-4}$. Substituting in, we find $p = \frac{2d}{8-d}$. We immediately find that we have problems when $d > 8$, which is admissible in this case. Enforcing that $p \ge 1$, we obtain $r \ge 4$, so we must have $r \in [4, \frac{2d}{d-4}]$. However, we run into an issue when $d > 8$. Then, there are no values for r that work, and hence the argument breaks down.
2. If $d < 4$, then $W^{2,2}(\Omega) \hookrightarrow C_B^0(\Omega) = C^0(\Omega) \cap L^\infty(\Omega)$. In this case, we obtain $r = \infty$, and calculate $p = 2$, so c can be in L^2.

Therefore, we have the following result:

d	p
< 4	2
4	2^+
5	10/3
6	6
7	14
8	∞

$\square$

Exercise (8.5). *Suppose that $\Omega \subset \mathbb{R}^d$ is a smooth, bounded, connected domain. Let*

$$H = \left\{ u \in H^2(\Omega) : \int_\Omega u = 0, \nabla u \cdot n = 0 \text{ on } \partial\Omega \right\}.$$

Show that H is a Hilbert space, and prove there exists $C > 0$ such that

$$\|u\|_{H^1(\Omega)} \le C \sum_{|\alpha|=2} \|D^\alpha u\|_{L^2(\Omega)}$$

for every $u \in H$.

Proof. 1. H is the finite intersection of two null spaces. The null space of the averaging operator and the null space of the γ_1 operator. Therefore, H is closed. H also inherits the $H^2(\Omega)$ inner product.

2. Assume to the contrary that the inequality fails. That is, there exists a seqeuence $u_n \in H$ such that

$$\|u_n\|_{H^1} = 1 \qquad \text{and} \qquad \sum_{|\alpha|=2} \|D^\alpha u_n\|_{L^2(\Omega)} \to 0.$$

We remark that the second assumption implies that $D^\alpha u_n \to 0$ in L^2 for every $|\alpha| = 2$ multiindex.

On one hand, we have a bounded sequence in H^1 from which we can extract a weakly converging subsequence, denoted with the same symbol, $u_n \rightharpoonup u$ in H^1.

Now u_n is a bounded sequence in H since

$$\begin{aligned}\|u\|^2_{H^2(\Omega)} &= \|u\|^2 + \sum_{|\alpha|=1} \|D^\alpha u\|^2 + \sum_{|\alpha|=2} \|D^\alpha u\|^2 = \|u\|^2 + \|\nabla u\|^2 + \sum_{|\alpha|=2} \|D^\alpha u\|^2 \\ &\le \left(C_p^2 + 1\right) \|\nabla u\|^2 +\end{aligned}$$

So, from that we can extract a subsequence $u_{n_k} \rightharpoonup u$ in H. This means that $D^2 u_{n_k} \rightharpoonup D^2 u$ weakly in L^2. At the same time, $D^2 u_{n_k} \to 0$ in L^2. We would conclude then that $D^2 u = 0$ almost everywhere. Combined with the condition that $\gamma_1 u = 0$, we conclude that $\nabla u = 0$ almost everywhere.

On the other hand, $u_{n_k} \rightharpoonup u$ implies $u_{n_{k_\ell}} \to u$ in H^1, and $\nabla u_{n_{k_\ell}} \to \nabla u$ in L^2. Strong convergence in L^2 implies convergence in norm, so $1 = \|\nabla u_n\| \to \|\nabla u\|$ so $\|\nabla u\| = 1$. A contradiction. $\square$

Exercise (8.8). $\Omega \subset \mathbb{R}^d$ *bounded, connected, Lipschitz domain. Let* $V \subset \Omega$ *have positive measure. Let* $H = \left\{u \in H^1(\Omega) : u|_V = 0\right\}$.

1. *Why is* H *a Hilbert space?*
2. *Prove that there exists* $C > 0$ *such that*

$$\|u\| \leq C\|\nabla u\|$$

for every $u \in H$.

1. H is the null space of the restriction to V operator. Since restriction is continuous, then H is closed. As a closed subspace of Hilbert space $H^1(\Omega)$, then H is itself a Hilbert space.

2. Go by contradiction. Assume to the contrary that there exists a sequence $u_n \in H$ such that

$$\|u_n\| = 1 \qquad \text{and} \qquad \|\nabla u_n\| \to 0.$$

Now, from every bounded sequence in a Hilbert space, we can extract a weakly convergent subsequence $u_{n_k} \rightharpoonup u \in H$. Weak convergence of u_{n_k} implies weak convergence of $\nabla u_{n_k} \rightharpoonup \nabla u$ in L^2. Since the norm is continuous we pass to the limit and we have that $\nabla u = 0$, i.e. u is a constant. But if u vanishes on V, then the only possibility is that $u \equiv 0$.

On the other side, we know from the Rellich-Kondrachov Theorem that we have $u_{n_k} \to u$ in L^2. Convergence in L^2 also implies convergence in the norm, but since $\|u_n\| = 1$ that means $\|u\| = 1$, a contradiction.