Review

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1 Differential Calculus in Banach Spaces and the Calculus of Variations

1.1 Differentiation

Our goal will be to generalize the notion of the 1D derivative, in the context of results like the Mean Value Theorem and Taylor's Theorem. Let us start with what we know: the standard 1D definition of the derivative is

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}, \tag{1}$$

but it is not clear how $1/h$ will generalize in higher dimensions. Indeed, it makes no sense. We can, however, define a similar concept.

Definition 1.1 (Little o). *Let X, Y be NLS's, $f : X \to Y$. If*

$$\lim_{h\to 0} \frac{\|f(h)\|_Y}{\|h\|_X} = 0,$$

we say that f is "little oh" of h and write

$$\|f(h)\|_Y = o\left(\|h\|_X\right).$$

Here, we have generalized the notion of $1/h$ to multiple dimensions by looking only at the norm of the increment and of the output. We can now formulate a notion of the derivative.

Definition 1.2 (Fréchet Derivative). *Let X, Y NLS, $U \subset X$ open, $f : U \to Y$. We say that f is Fréchet differentiable (or strongly differentiable) at a point $x \in U$ if and only if there exists a bounded linear map $A \in B(X, Y)$ such that the remainder*

$$R(x,h) := f(x+h) - f(x) - Ah$$

is $o\left(\|h\|_X\right)$. That is,

$$\lim_{h\to 0} \frac{\|R(x,h)\|_Y}{\|h\|_X} = 0.$$

If A exists, we call A the Fréchet derivative of f at x, denoted $Df(x)$. Note that by construction, $A = Df(x)$ is a linear operator on h. The definition of the Fréchet derivative is consistent with the one dimensional analogue. For $f \in C^1(\mathbb{R})$,

$$R\left(x,h\right) = f\left(x+h\right) - f\left(x\right) - f'\left(x\right)h = \left[\frac{f\left(x+h\right) - f\left(x\right)}{h} - f'\left(x\right)\right]h, \tag{2}$$

and

$$\lim_{h\to 0}\frac{|R(x,h)|}{|h|}=\lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}-f'(x)\right]=0. \tag{3}$$

More generally, the map Df (as contrasted with $Df(x)$) can be regarded as a map $Df : U\times X \ni (x,h)\to Df(x)h\in Y$.

Proposition 1.1. *If f is Fréchet differentiable, then $Df(x)$ is unique, and f is continuous at x. Moreover, if g is Fréchet differentiable, then for every $\alpha,\beta\in\mathbb{F}$,*

$$D(\alpha f+\beta g)(x)=\alpha Df(x)+\beta Dg(x).$$

(That is to say, the operator D taking functions $f\mapsto Df$ is linear.)

Proof. Suppose that there are two derivatives at x, $A, B\in B(X,Y)$. Define the two remainders

$$\begin{aligned} R_A(x,h)&=f(x+h)-f(x)-Ah\\ R_B(x,h)&=f(x+h)-f(x)-Bh. \end{aligned}\tag{4}$$

By assumption, both $R_A(x,h)=o(\|h\|_X)$ and $R_B(x,h)=o(\|h\|_X)$. Now we show that in fact, $A=B$.

$$\begin{aligned} \|A-B\|_{B(X,Y)}&=\sup_{\|h\|=\varepsilon}\frac{\|Ah-Bh\|_Y}{\|h\|_X}\\ &=\sup_{\|h\|=\varepsilon}\frac{\|Ah-Bh\|_Y}{\|h\|_X}\\ &=\sup_{\|h\|=\varepsilon}\frac{\|R_A(x,h)-R_B(x,h)\|_Y}{\|h\|_X}\\ &\leqslant\sup_{\|h\|=\varepsilon}\frac{\|R_A(x,h)\|_Y}{\|h\|_X}+\sup_{\|h\|=\varepsilon}\frac{\|R_B(x,h)\|_Y}{\|h\|_X}. \end{aligned}\tag{5}$$

Is this the right definition of the norm on $B(X,Y)$? The RHS can be made small as $\varepsilon\to 0$. Indeed, the argument of the supremum goes to zero as $\|h\|\to 0$ by the $o(\|h\|_X)$ assumption. Thus, $A=B=Df(x)$ is unique. To prove that f is continuous at x, we have

$$\begin{aligned} \|f(x+h)-f(x)\|_Y&=\|Df(x)h+R(x,h)\|_Y\\ &\leqslant\|Df(x)h\|_Y+\|R(x,h)\|_Y\\ &\leqslant\|Df(x)\|_{B(X,Y)}\|h\|_X+\|R(x,h)\|_Y. \end{aligned}\tag{6}$$

The RHS tends to 0 as $h\to 0$ in X. Since $Df(x)$ is bounded, the first term on the right tends to zero trivially as $h\to 0$. Since $R(x,h)$ is $o(\|h\|)$, it means that $R(x,h)$ goes to zero faster than h goes to zero. This implies, however, that if $h\to 0$, then also $R(x,h)\to 0$. A somewhat obvious fact. □

Having differentiability, namely *strong* differentiability, at a point x is more than enough to conclude that f is continuous at x. A piecewise linear function may be continuous, but not differentiable at the interface points. Strong differentiability allows us to conclude more:

Lemma 1.1 (Local Lipschitz Property). *If $f : U \to Y$ is differentiable at $x \in U$, then for every $\varepsilon > 0$, there exists a $\delta = \delta(x, \varepsilon) > 0$ such that for every h with $\|h\|_X \leq \delta$,*

$$\|f(x+h) - f(x)\|_Y \leqslant \left(\|Df(x)\|_{B(X,Y)} + \varepsilon\right) \|h\|_X.$$

Proof. We have

$$f(x+h) - f(x) = Df(x)\,h + R(x,h)\,. \tag{7}$$

Since the remainder is $o(\|h\|)$, then for every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\frac{\|R(x,h)\|_Y}{\|h\|_X} < \varepsilon \tag{8}$$

whenever $\|h\|_X < \delta$. Definition of convergence of $\|R\|/\|h\| \to 0$. It follows then that

$$\|f(x+h) - f(x)\|_Y = \|Df(x)\,h + R(x,h)\| \leqslant \|Df(x)\|_{B(X,Y)}\|h\|_X + \varepsilon\|h\|_X. \tag{9}$$

□

Even if f is not strongly differentiable, it may still admit some kind of weaker derivative. The key here is that in the Fréchet derivative, we require the remainder goes to zero for all paths that take $\|h\|_X \to 0$. We now consider incrementing only in a specific direction h. Start with $f : U \to Y$, fix a direction $h \in X$, and consider the auxillary function

$$g : \mathbb{R} \ni t \to g(t) := f(x + th) \in Y. \tag{10}$$

Proposition 1.2.

$$f \textit{ is Fréchet differentiable at } x \implies f \textit{ is Gateaux differentiable at } x.$$

Theorem 1.2 (Chain Rule). *Let X, Y, Z be NLS's and $U \subset X$, $V \subset Y$ open, $f : U \to Y$ and $g : V \to Z$. Let $x \in U$ and $y = f(x) \in V$. Suppose that g is Fréchet differentiable at y and f is Gateaux (or Fréchet) differentiable at x. Then the composition $g \circ f$ is Gateaux (or Fréchet) differentiable at x with*

$$D(g \circ f)(x) = Dg(y) \circ Df(x)$$

Proposition 1.3 (Mean Value Theorem for Curves). *Let Y be an NLS and $\varphi : [a, b] \to Y$ be continuous, where $a < b$ are real numbers. Suppose $\varphi'(t)$ exists on (a, b) and $\|\varphi'(t)\|_{B(\mathbb{R},Y)} \leq M$. Then*

$$\|\varphi(b) - \varphi(a)\|_Y \leq M(b - a).$$

Proof. Use a compactness argument. □

Theorem 1.3 (Mean Value Theorem). *Let X, Y be NLS's, $U \subset X$ open. Let $f : U \to Y$ be Fréchet differentiable at all $x \in U$, and suppose that the line segment*

$$\ell = \{tx_2 + (1 - t)x_1 : t \in [0, 1]\}$$

is contained in U. Then

$$\|f(x_2) - f(x_1)\|_Y \leqslant \sup_{x \in \ell} \|Df(x)\|_{B(X,Y)} \|x_2 - x_1\|_X.$$

Proof. Use the MVT for curves applied to the function $\varphi : [0, 1] \to Y$,

$$\varphi(t) = f((1 - t)x_1 + tx_2) = f(x_1 + t(x_2 - x_1)) = f(\gamma(t)).$$

□

Definition 1.3. *Let $X = X_1 \oplus \cdots \oplus X_m$. Let $U \subset X$ open, $F : U \to Y$, Y NLS. Let $x = (x_1, \ldots, x_m) \in U$ and fix an integer $k \in [1, m]$. For z near x_k (k-th component) in space X_k, the point $(x_1, \ldots, x_{k-1}, z, x_{k+1}, \ldots, x_m) \in U$ since U is open. Define now*

$$f_k(z) = F(x_1, \ldots, x_{k-1}, z, x_{k+1}, \ldots, x_m).$$

Then f_k maps an open set in X_k to Y. If f_k has a Fréchet derivative at $z = x_k$, then we say that the original F has a k-th partial derivative at x and define $D_kF(x) := Df_k(x_k)$. Notice that $D_kF(x) = Df_k(x_k) \in B(X_k, Y)$.

Proposition 1.4. *Let $X = X_1 \oplus \cdots \oplus X_m$ be the direct sum of NLS's, $U \subset X$ open, and $F : U \to Y$, Y an NLS. Suppose that all partial derivatives $D_jF(x)$ exist for $x \in U$ and $j \in [1, m]$, and that these maps are continuous as a function of x at some $x_0 \in U$. Then, F is Fréchet differentiable at x_0 and for an increment $h = (h_1, \ldots, h_m) \in X$,*

$$DF(x_0)h = \sum_{j=1}^{m} D_jF(x_0)h_j.$$

1.2 Fixed Points and Contractive Maps

1.3 Nonlinear Equations

1.4 Higher Derivatives

1.5 Extrema

1.6 Euler-Lagrange Equations

1.7 Constrained Extrema and Lagrange Multipliers