

Homework 2

Jonathan Zhang EID: jdz357

Problem 6.9:

Suppose that $f \in L^p(\mathbb{R}^d)$ for $p \in (1, 2)$. Show that there exist $f_1 \in L^1(\mathbb{R}^d)$ and $f_2 \in L^2(\mathbb{R}^d)$ such that $f = f_1 + f_2$. Define $\hat{f} = \hat{f}_1 + \hat{f}_2$. Show that this definition is well defined, that is, independent of the choices f_1 and f_2 .

Solution Write f as

$$f = f\chi_{[x:|f(x)|<1]} + f\chi_{[x:|f(x)|\geq 1]}$$

and define

$$f_2(x) = f\chi_{[x:|f(x)|<1]}, \quad f_1(x) = f\chi_{[x:|f(x)|\geq 1]}.$$

Then,

$$\begin{aligned} \|f_1\|_1 &= \int_{\mathbb{R}^d} |f(x)|\chi_{[x:|f(x)|\geq 1]} dx = \int_{x:|f(x)|\geq 1} |f(x)| \leq \int_{x:|f(x)|\geq 1} |f(x)|^p \leq \|f\|_p^p. \\ \|f_2\|_2^2 &= \int_{\mathbb{R}^d} |f(x)|^2\chi_{[x:|f(x)|<1]} dx = \int_{x:|f(x)|<1} |f(x)|^2 \leq \int_{x:|f(x)|<1} |f(x)|^p \leq \|f\|_p^p. \end{aligned}$$

Next, suppose that f admits two different decompositions, $f = f_1 + f_2 = g_1 + g_2$, where $f_i, g_i \in L^i(\mathbb{R}^d)$ for $i = 1, 2$. Then,

$$\begin{aligned} \hat{f}_1 + \hat{f}_2 &= (2\pi)^{-d/2} \left(\int_{\mathbb{R}^d} f_1(x) e^{-ix \cdot \xi} dx + \int_{\mathbb{R}^d} f_2(x) e^{-ix \cdot \xi} dx \right) \\ &= (2\pi)^{-d/2} \int_{\mathbb{R}^d} (f_1(x) + f_2(x)) e^{-ix \cdot \xi} dx \\ &= (2\pi)^{-d/2} \int_{\mathbb{R}^d} (g_1(x) + g_2(x)) e^{-ix \cdot \xi} dx \\ &= \hat{g}_1 + \hat{g}_2. \end{aligned}$$

Problem 6.11:

Let the ground field be \mathbb{C} and define $T : L^2(\mathbb{R}^d) \rightarrow L^2$ by

$$(Tf)(x) = \int e^{-|x-y|^2/2} f(y) dy.$$

Use the Fourier transform to show that T is a positive, injective but not surjective operator.

Solution

Problem 6.15:

Let $\varphi \in \mathcal{S}(\mathbb{R}^d)$, $\hat{\varphi}(0) = (2\pi)^{-d/2}$, and $\varphi_\varepsilon(x) = \varepsilon^{-d}\varphi(x/\varepsilon)$. Prove that $\varphi_\varepsilon \rightarrow \delta_0$ and $\hat{\varphi}_\varepsilon \rightarrow (2\pi)^{-d/2}$ as $\varepsilon \rightarrow 0^+$. In what sense do these convergences take place?