Homework 2

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Problem 6.9:

Suppose that $f \in L^p\left(\mathbb{R}^d\right)$ for $p \in (1,2)$. Show that there exist $f_1 \in L^1\left(\mathbb{R}^d\right)$ and $f_2 \in L^2\left(\mathbb{R}^d\right)$ such that $f = f_1 + f_2$. Define $\hat{f} = \hat{f}_1 + \hat{f}_2$. Show that this definition is well defined, that is, independent of the choices f_1 and f_2.

Solution Write f as

$$f = f\chi_{[x:|f(x)<1]} + f\chi_{[x:|f(x)\geqslant 1]}$$

and define

$$f_2(x) = f\chi_{[x:|f(x)<1]}, \qquad f_1(x) = f\chi_{[x:|f(x)\geqslant 1]}.$$

Then,

$$\|f_1\|_1 = \int_{\mathbb{R}^d} |f(x)|\chi_{[x:|f(x)\geqslant 1]}dx = \int_{x:|f(x)|\geqslant 1} |f(x)| \leq \int_{x:|f(x)|\geqslant 1} |f(x)|^p \leq \|f\|_p^p.$$

$$\|f_2\|_2^2 = \int_{\mathbb{R}^d} |f(x)|^2\chi_{[x:|f(x)<1]}dx = \int_{x:|f(x)|<1} |f(x)|^2 \leq \int_{x:|f(x)|<1} |f(x)|^p \leq \|f\|_p^p.$$

Next, suppose that f admits two different decompositions, $f = f_1 + f_2 = g_1 + g_2$, where $f_i, g_i \in L^i\left(\mathbb{R}^d\right)$ for $i = 1, 2$. Then,

$$\begin{aligned}
\hat{f}_1 + \hat{f}_2 &= (2\pi)^{-d/2}\left(\int_{\mathbb{R}^d} f_1(x)e^{-ix\cdot\xi}dx + \int_{\mathbb{R}^d} f_2(x)e^{-ix\cdot\xi}dx\right) \\
&= (2\pi)^{-d/2}\int_{\mathbb{R}^d} \left(f_1(x) + f_2(x)\right)e^{-ix\cdot\xi}dx \\
&= (2\pi)^{-d/2}\int_{\mathbb{R}^d} \left(g_1(x) + g_2(x)\right)e^{-ix\cdot\xi}dx \\
&= \hat{g}_1 + \hat{g}_2.
\end{aligned}$$

Problem 6.11:

Let the ground field be $\mathbb{C}$ *and define* $T : L^2\left(\mathbb{R}^d\right) \to L^2$ *by*

$$\left(Tf\right)(x) = \int e^{-|x-y|^2/2} f(y)dy.$$

Use the Fourier transform to show that T *is a positive, injective but not surjective operator.*

Solution

Problem 6.15:

Let $\varphi \in \mathcal{S}\left(\mathbb{R}^d\right)$, $\hat{\varphi}(0) = (2\pi)^{-d/2}$, *and* $\varphi_\varepsilon(x) = \varepsilon^{-d}\varphi(x/\varepsilon)$. *Prove that* $\varphi_\varepsilon \to \delta_0$ *and* $\hat{\varphi}_\varepsilon \to (2\pi)^{-d/2}$ *as* $\varepsilon \to 0^+$. *In what sense do these convergences take place?*

Problem 6.18:

For $f \in \mathcal{S}(\mathbb{R})$, define the Hilbert transform of f as $Hf = PV\left(\frac{1}{\pi x}\right) * f$.

1. *Show that $PV(1/x) \in \mathcal{S}'$.*
2. *Show that $\mathcal{F}(PV(1/x)) = -i\sqrt{\pi/2}sgn(\xi)$. Recall that $xPV(1/x) = 1$.*
3. *Show that $\|Hf\|_2 = \|f\|_2$ and $HHf = -f$ for $f \in \mathcal{S}$.*
4. *Extend H to $L^2(\mathbb{R})$.*

Problem 6.20:

Compute the Fourier transforms of the following functions, considered as tempered distributions.

1. $f(x) = x^n$ for $x \in \mathbb{R}$, $\mathbb{N} \ni n \geq 0$.
2. $g(x) = e^{-|x|}$ for $x \in \mathbb{R}$.
3. $h(x) = e^{i|x|^2}$ for $x \in \mathbb{R}^d$.
4. $\sin x$ and $\cos x$ for $x \in \mathbb{R}$.