Exercises for Chapter 6

1. Compute the Fourier transform of $e^{-|x|}$ for $x \in \mathbb{R}$.

2. Compute the Fourier transform of $e^{-a|x|^2}$, $a > 0$, directly, where $x \in \mathbb{R}$. You will need to use the Cauchy Theorem.

3. If $f \in L^1(\mathbb{R}^d)$ and $f > 0$, show that for every $\xi \neq 0$, $|\hat{f}(\xi)| < \hat{f}(0)$.

4. If $f \in L^1(\mathbb{R}^d)$ and $f(x) = g(|x|)$ for some g, show that $\hat{f}(\xi) = h(|\xi|)$ for some h. Can you relate g and h?

5. Let $1 \leq p < \infty$ and suppose $f \in L^p(\mathbb{R})$. Let $g(x) = \displaystyle\int_x^{x+1} f(y)\,dy$. Prove that $g \in C_v(\mathbb{R})$.

6. Show that the Fourier transform $\mathcal{F} : L^1(\mathbb{R}^d) \to C_v(\mathbb{R}^d)$ is not onto. Show, however, that $\mathcal{F}(L^1(\mathbb{R}^d))$ is dense in $C_v(\mathbb{R}^d)$. [Hint: see Exercise 5.]

7. Consider the function $f(x) = |x|^{-1/2}e^{-|x|}$. Show that f is in $L^1(\mathbb{R})$ but not in $L^2(\mathbb{R})$. Also, find the Fourier transform of f.

8. Give an example of a function $f \in L^2(\mathbb{R}^d)$ which is not in $L^1(\mathbb{R}^d)$, but such that $\hat{f} \in L^1(\mathbb{R}^d)$. Under what circumstances can this happen?

9. Suppose that $f \in L^p(\mathbb{R}^d)$ for some p between 1 and 2.
   (a) Show that there are $f_1 \in L^1(\mathbb{R}^d)$ and $f_2 \in L^2(\mathbb{R}^d)$ such that $f = f_1 + f_2$.
   (b) Define $\hat{f} = \hat{f}_1 + \hat{f}_2$. Show that this definition is independent of the choice of f_1 and f_2.

10. The Young and Hausdorff-Young inequalities.
    (a) Prove the following generalization of Young's inequality. For $r \geq 1$, suppose $K(x, y)$ is continuous (measurable would be enough) on $\mathbb{R}^d \times \mathbb{R}^d$ and there is some constant $C > 0$ such that
    $$\int_{\mathbb{R}^d} |K(x,y)|^r\,dx \leq C^r \quad \text{and} \quad \int_{\mathbb{R}^d} |K(x,y)|^r\,dy \leq C^r.$$
    Let the operator T be defined by
    $$Tf(x) = \int_{\mathbb{R}^d} K(x,y)\,f(y)\,dy.$$
    If $p \geq 1$ and $q \geq 1$ satisfy $1/p + 1/r = 1/q + 1$, then $T : L^p(\mathbb{R}^d) \to L^q(\mathbb{R}^d)$ is a bounded linear map with norm $\|T\| \leq C$.
    (b) Use this result to show the following special case of the Hausdorff-Young inequality: The Fourier transform $\mathcal{F} : L^{4/3}(\mathbb{R}^d) \to L^4(\mathbb{R}^d)$. [Hint: note that $\|\hat{f}\|_{L^4}^2 = \|(\hat{f})^2\|_{L^2}$, and $(\hat{f})^2$ can be written as the Fourier transform of a convolution.]

11. Let the ground field be $\mathbb{C}$ and define $T : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ by
    $$Tf(x) = \int e^{-|x-y|^2/2} f(y)\,dy.$$
    Use the Fourier transform to show that T is a positive, injective operator, but that T is *not* surjective.

12. Suppose that f and g are in $L^2(\mathbb{R}^d)$. The convolution $f * g$ is in $L^\infty(\mathbb{R}^d)$ but it need not be in L^2. So it may not have a classically defined Fourier transform. Nevertheless, prove that

$f * g = (2\pi)^{d/2}(\hat{f}\hat{g})^\vee$ is well defined, and that the inverse Fourier transform is given by the usual integration formula.

13. Find the four possible eigenvalues of the Fourier transform: $\hat{f} = \lambda f$. For each possible eigenvalue, show that there is at least one eigenfunction. [Hint: when $d = 1$, consider $p(x)e^{-x^2/2}$, where p is a polynomial.]

14. Prove that if $\{f_j\}_{j=1}^\infty \subset \mathcal{S}$ and $f_j \xrightarrow{\mathcal{S}} f$, then for any $1 \le p \le \infty$, $f_j \xrightarrow{L^p} f$.

15. Let $\varphi \in \mathcal{S}(\mathbb{R}^d)$, $\hat{\varphi}(0) = (2\pi)^{-d/2}$, and $\varphi_\epsilon(x) = \epsilon^{-d}\varphi(x/\epsilon)$. Prove that $\varphi_\epsilon \to \delta_0$ and $\hat{\varphi}_\epsilon \to (2\pi)^{-d/2}$ as $\epsilon \to 0^+$. In what sense do these convergences take place?

16. Is it possible for there to be a continuous function f defined on $\mathbb{R}^d$ with the following two properties?
   (a) There is no polynomial P in d variables such that $|f(x)| \le P(x)$ for all $x \in \mathbb{R}^d$.
   (b) The distribution $\phi \mapsto \int \phi\, f\, dx$ is tempered.

17. When is $\displaystyle\sum_{k=1}^{\infty} a_k \delta_k \in \mathcal{S}'(\mathbb{R})$? (Here, δ_k is the point mass centered at $x = k$.)

18. For $f \in \mathcal{S}(\mathbb{R})$, define the Hilbert transform of f by $Hf = \mathrm{PV}\left(\dfrac{1}{\pi x}\right) * f$, where the convolution uses ordinary Lebesgue measure.
   (a) Show that $\mathrm{PV}(1/x) \in \mathcal{S}'$.
   (b) Show that $\mathcal{F}(\mathrm{PV}(1/x)) = -i\sqrt{\pi/2}\,\mathrm{sgn}(\xi)$, where $\mathrm{sgn}(\xi)$ is the sign of ξ. [Hint: recall that $x\mathrm{PV}(1/x) = 1$.]
   (c) Show that $\|Hf\|_{L^2} = \|f\|_{L^2}$ and $HHf = -f$, for $f \in \mathcal{S}(\mathbb{R})$.
   (d) Extend H to $L^2(\mathbb{R})$.

19. The gamma function is $\Gamma(s) = \displaystyle\int_0^\infty t^{s-1}e^{-t}\,dt$. Let $\phi \in \mathcal{S}(\mathbb{R}^d)$ and $0 < \alpha < d$.
   (a) Show that $|\xi|^{-\alpha} \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ and $|\xi|^{-\alpha}\hat{\phi} \in L^1(\mathbb{R}^d)$.
   (b) Let $c_\alpha = 2^{\alpha/2}\Gamma(\alpha/2)$. Show that

$$\left(|\xi|^{-\alpha}\hat{\phi}\right)^\vee(x) = \frac{c_{d-\alpha}}{(2\pi)^{d/2}c_\alpha}\int_{\mathbb{R}^d}|x-y|^{\alpha-d}\phi(y)\,dy.$$

   [Hint: first show that

$$c_\alpha|\xi|^{-\alpha} = \int_0^\infty t^{\alpha/2-1}e^{-|\xi|^2t/2}\,dt,$$

   and then recall that $e^{-|\xi|^2t/2} = t^{-d/2}\left(e^{-|x|^2/2t}\right)^\wedge$.]

20. Compute the Fourier transforms of the following functions, considered as tempered distributions.
   (a) $f(x) = x^n$ for $x \in \mathbb{R}$ and for integer $n \ge 0$.
   (b) $g(x) = e^{-|x|}$ for $x \in \mathbb{R}$.
   (c) $h(x) = e^{i|x|^2}$ for $x \in \mathbb{R}^d$.

(d) $\sin x$ and $\cos x$ for $x \in \mathbb{R}$.

21. Give a careful argument that $\mathcal{D}(\mathbb{R}^d)$ is dense in $\mathcal{S}$. Show also that $\mathcal{S}'$ is dense in $\mathcal{D}'$ and that distributions with compact support are dense in $\mathcal{S}'$.

22. Make an argument that there is no simple way to define the Fourier transform on $\mathcal{D}'$ in the way we have for $\mathcal{S}'$.

23. Let

$$H = \Big\{ \text{measurable functions } f : \mathbb{R} \to \mathbb{C} \,\Big|\, f \in \mathcal{S}',\ \hat{f} \text{ is a measurable function,} \\ \text{and } \|f\|_H^2 \equiv \int |\hat{f}(\xi)|^2 \, m(\xi)\, d\xi < \infty \Big\},$$

where $m(\xi) \geq m_* > 0$ is also measurable (but possibly unbounded). The space H is endowed with the inner-product

$$(f, g)_H \equiv \int \hat{f}(\xi)\, \overline{\hat{g}(\xi)}\, m(\xi)\, d\xi.$$

(a) Show carefully that H is complete (and thus a Hilbet space).

(b) Prove that H is continuously imbedded in $L^2(\mathbb{R})$.

(c) If $m(\xi) \geq \alpha|\xi|^2$, for some $\alpha > 0$, prove that for $f \in H$, the (tempered) distributional derivative $f' \in L^2(\mathbb{R})$.

24. Let $s \in \mathbb{R}$ and

$$H^s(\mathbb{R}^d) = \big\{ f \in L^2(\mathbb{R}^d) : (1 + |\xi|^2)^{s/2} |\hat{f}(\xi)| \in L^2(\mathbb{R}^d) \big\}.$$

(a) Show that there is some $s_0 \in \mathbb{R}$ such that for all $f \in H^s(\mathbb{R}^d)$, $\hat{f}(\xi) \in L^1(\mathbb{R}^d)$ for $s > s_0$.

(b) Apply the Riemann-Lebesgue Lemma to $\hat{f}(\xi)$ to show that, for $s > s_0$, there is some continuous function g such that $f = g$ almost everywhere.

25. Use the Fourier transform to find a solution to

$$u - \frac{\partial^2 u}{\partial x_1^2} - \frac{\partial^2 u}{\partial x_2^2} = e^{-x_1^2 - x_2^2}.$$

[Hint: write your answer in terms of a suitable inverse Fourier transform and a convolution.] Can you find a fundamental solution to the differential operator?

26. Consider the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad -\infty < x < \infty,\ 0 < y < \infty, \\ u(x, 0) = f(x) \text{ and } u(x, y) \to 0 \text{ as } x^2 + y^2 \to \infty,$$

for the unknown function $u(x, y)$, where f is a nice function.

(a) Find the Fourier transform of $e^{-|x|}$.

(b) Using your answer from (a), find the Fourier transform of $\dfrac{1}{1 + x^2}$.

(c) Find a function $g(x, y)$ such that $u(x, y) = f * g(x, y) = \displaystyle\int f(z)\, g(x - z, y)\, dz$.

27. Consider the *Telegrapher's equation*

$$u_{tt} + 2u_t + u = c^2 u_{xx} \quad \text{for } x \in \mathbb{R} \text{ and } t > 0,$$

where also

$$u(x,0) = f(x) \quad \text{and} \quad u_t(x,0) = g(x)$$

are given in $L^2(\mathbb{R})$.

(a) Use the Fourier transform (in x only) and its inverse to find an explicit representation of the solution.

(b) Justify that your representation is indeed a solution.

(c) Show that the solution can be viewed as a sum of two wave packets, one moving to the right with a given constant speed, and the other moving to the left with the same speed.

28. Use the Fourier transform and its inverse to find a representation of solutions to the *Klein-Gordon equation*

$$u_{tt} - \Delta u + u = 0, \quad x \in \mathbb{R}^d \text{ and } t > 0,$$

where $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$, $f, g \in L^2(\mathbb{R}^d)$, are given. Leave your answer in terms of an inverse Fourier transform.

29. Consider the problem $u'''' + u = f$ (4 derivatives). Up to calculating an integral, find a fundamental solution that is real. Using the fundamental solution, find a solution to the original problem.

30. Consider the linear problem

$$\Delta^2 u + u = f(x) \quad \text{for } x \in \mathbb{R}^d,$$

where $\Delta^2 u = \Delta\Delta u$.

(a) Suppose that $f \in \mathcal{S}'(\mathbb{R}^d)$. Use the Fourier Transform to find a solution $u \in \mathcal{S}'$ to the problem. Leave your answer as a multiplier operator, i.e., $u = \big(m(\xi)\hat{f}\big)^\vee$ for some $m(\xi)$, but justify each of your steps.

(b) Is your solution unique? Why or why not?

(c) Suppose now that $f \in \mathcal{S}(\mathbb{R}^d)$. Write the solution as a convolution operator (you may leave your answer in terms of an inverse Fourier Transform).

(d) Show that $\check{m} \in L^2(\mathbb{R}^d)$ for some range of d, and use this to extend your convolution solution to $f \in L^1(\mathbb{R}^d)$. In that case, in what $L^p(\mathbb{R}^d)$ space is u?

31. Consider $L^2(0,\infty)$ as a real Hilbert space. Let $A : L^2(0,\infty) \to L^2(-\infty,\infty)$ be defined by

$$Au(x) = u(x) + \int_0^\infty \omega(x-y)\, u(y)\, dy,$$

where $\omega \in L^1(0,\infty) \cap L^2(0,\infty) \cap C^2(0,\infty)$ is nonnegative, decreasing, convex, and extended to $\mathbb{R}$ as an even function.

(a) Show that A maps into $L^2(-\infty,\infty)$. [Hint: extend u by zero outside $(0,\infty)$ in the integral.]

(b) Show that A is symmetric on $L^2(0,\infty)$.

(c) Show that $\hat{\omega} \geq 0$.

(d) Use the Fourier transform to show that A is strictly positive definite on $L^2(0, \infty)$.

(e) Use the Fourier transform to solve $Au = f$ for $f \in L^2(-\infty, \infty)$.

(f) Why is the solution unique?

32. Let T be a multiplier operator mapping $L^2(\mathbb{R}^d)$ into itself, so there is a bounded measurable function $m(\xi)$ such that $\widehat{Tf}(\xi) = m(\xi)\hat{f}(\xi)$ for all $f \in L^2(\mathbb{R}^d)$. Show that T commutes with translation and $\|T\| = \|m\|_{L^\infty}$.

33. Let $u \in L^2(\mathbb{R}^d)$ and define the operator $T : L^1(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ by convolution with u. Show that T is a bounded linear operator, a multiplier operator, and translation invariant.

Exercises for Chapter 7

1. Prove that for $f \in H^1(\mathbb{R}^d)$, $\|f\|_{H^1(\mathbb{R}^d)}$ is equivalent to

$$\left\{ \int_{\mathbb{R}^d} (1+|\xi|^2)|\hat{f}(\xi)|^2\, d\xi \right\}^{1/2}.$$

Can you generalize this to $H^k(\mathbb{R}^d)$?

2. Prove that if $f \in H_0^1(0,1)$, then there is some constant $C > 0$ such that

$$\|f\|_{L^2(0,1)} \le C\|f'\|_{L^2(0,1)}.$$

If instead $f \in \left\{ g \in H^1(0,1) : \int_0^1 g(x)\, dx = 0 \right\}$, prove a similar estimate.

3. Prove that $\delta_0 \notin (H^1(\mathbb{R}^d))^*$ for $d \ge 2$, but that $\delta_0 \in (H^1(\mathbb{R}))^*$. You will need to define what δ_0 applied to $f \in H^1(\mathbb{R})$ means.

4. Prove that $H^1(0,1)$ is continuously embedded in $C_B(0,1)$. Recall that $C_B(0,1)$ is the set of bounded and continuous functions on $(0,1)$.

5. Suppose that $\Omega \subset \mathbb{R}^d$ is a bounded set and $\{U_j\}_{j=1}^N$ is a finite collection of open sets in $\mathbb{R}^d$ that cover the closure of Ω (i.e., $\bar{\Omega} \subset \displaystyle\bigcup_{j=1}^N U_j$). Prove that there exists a finite C^∞ partition of unity in Ω subordinate to the cover. That is, construct $\{\phi_k\}_{k=1}^M$ such that $\phi_k \in C_0^\infty(\mathbb{R}^d)$, $\phi_k \subset U_{j_k}$ for some j_k, and

$$\sum_{k=1}^M \phi_k(x) = 1.$$

6. Suppose that $\Omega \subset \mathbb{R}^d$ is a (possibly unbounded) domain and $\{U_\alpha\}_{\alpha \in \mathcal{I}}$ is a collection of open sets in $\mathbb{R}^d$ that cover Ω (i.e., $\Omega \subset \displaystyle\bigcup_{\alpha\in\mathcal{I}} U_\alpha$). Prove that there exists a locally finite partition of unity in Ω subordinate to the cover. That is, there exists a sequence $\{\psi_j\}_{j=1}^\infty \subset C_0^\infty(\mathbb{R}^d)$ such that:
   (i) For every K compactly contained in Ω, all but finitely many of the ψ_j vanish on K;
   (ii) Each $\psi_j \ge 0$ and $\displaystyle\sum_{j=1}^\infty \psi_j(x) = 1$ for every $x \in \Omega$;
   (iii) For each j, the support of ψ_j is contained in some U_{α_j}, $\alpha_j \in \mathcal{I}$.
   [Hint: let S be a countable dense subset of Ω (e.g., points with rational coordinates). Consider the countable collection of balls $\mathcal{B} = \{B_r(x) \subset \mathbb{R}^d : r$ is rational, $x \in S$, and $B_r(x) \subset U_\alpha$ for some $\alpha \in \mathcal{I}\}$. Order the balls and construct on $B_j = B_{r_j}(x_j)$ a function $\phi_j \in C_0^\infty(B_j)$ such that $0 \le \phi_j \le 1$ and $\phi_j = 1$ on $B_{r_j/2}(x_j)$. Then $\psi_1 = \phi_1$ and $\psi_j = (1-\phi_1)\cdots(1-\phi_{j-1})\phi_j$ should work.]

7. Let $u \in \mathcal{D}'(\mathbb{R}^d)$ and $\phi \in \mathcal{D}(\mathbb{R}^d)$. For $y \in \mathbb{R}^d$, the translation operator τ_y is defined by $\tau_y\phi(x) = \phi(x-y)$.

   (a) Show that

$$u(\tau_y\phi) - u(\phi) = \int_0^1 \sum_{j=1}^d y_j \frac{\partial u}{\partial x_j}(\tau_{ty}\phi)\, dt.$$

(b) Apply this to

$$f \in W^{1,1}_{\text{loc}}(\mathbb{R}^d) = \left\{ f \in L^1_{\text{loc}}(\mathbb{R}^d) : \frac{\partial f}{\partial x_j} \in L^1_{\text{loc}}(\mathbb{R}^d) \text{ for all } j \right\}$$

to show that

$$f(x+y) - f(x) = \int_0^1 y \cdot \nabla f(x+ty)\, dt.$$

(c) Let the locally Lipschitz functions be defined as

$$C^{0,1}_{\text{loc}}(\mathbb{R}^d) = \{ f \in C^0(\mathbb{R}^d) : \forall R > 0, \text{there is some } L_{R,f} \text{ depending on } R \text{ and } f \\ \text{such that } |f(x) - f(y)| \leq L_{R,f}|x-y| \ \forall x, y \in B_R(0)\}.$$

Conclude that $W^{1,\infty}_{\text{loc}}(\mathbb{R}^d) \subset C^{0,1}_{\text{loc}}(\mathbb{R}^d)$.

8. Counterexamples.

(a) No embedding of $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ for $1 \leq p < d$ and $q > dp/(d-p)$. Let $\Omega \subset \mathbb{R}^d$ be bounded and contain 0, and let $f(x) = |x|^r$. Find r so that $f \in W^{1,p}(\Omega)$ but $f \notin L^q(\Omega)$.

(b) No embedding of $W^{1,p}(\Omega) \hookrightarrow C^0_B(\Omega)$ for $1 \leq p < d$. Note that in the previous case, f is not bounded. What can you say about which (negative) Sobolev spaces the Dirac mass lies in?

(c) No embedding of $W^{1,p}(\Omega) \hookrightarrow L^\infty(\Omega)$ for $1 < p = d$. Let $\Omega \subset \mathbb{R}^d = B_R(0)$ and let $f(x) = \log(\log(4R/|x|))$. Show $f \in W^{1,p}(B_R(0))$.

(d) $C^\infty \cap W^{1,\infty}$ is not dense in $W^{1,\infty}$. Show that if $\Omega = (-1,1)$ and $u(x) = |x|$, then $u \in W^{1,\infty}$ but $u(x)$ is not the limit of C^∞ functions in the $W^{1,\infty}$-norm.

9. Suppose that $f_j \in H^2(\Omega)$ for $j = 1, 2, ...$, $f_j \rightharpoonup f$ weakly in $H^1(\Omega)$, and $D^\alpha f_j \rightharpoonup g_\alpha$ weakly in $L^2(\Omega)$ for all multi-indices α such that $|\alpha| = 2$. Show that $f \in H^2(\Omega)$, $D^\alpha f = g_\alpha$, and, for a subsequence, $f_j \to f$ strongly in $H^1(\Omega)$.

10. Suppose that $\Omega \subset \mathbb{R}^d$ is bounded with a Lipschitz boundary and $f_j \rightharpoonup f$ and $g_j \rightharpoonup g$ weakly in $H^1(\Omega)$. Show that, for a subsequence, $\nabla(f_j g_j) \to \nabla(fg)$ as a distribution. Find all p in $[1, \infty]$ such that the convergence can be taken weakly in $L^p(\Omega)$.

11. Interpolation inequalities.

(a) Show that for $f \in H^1(\mathbb{R}^d)$ and $0 \leq s \leq 1$, $\|f\|_{H^s(\mathbb{R}^d)} \leq \|f\|^s_{H^1(\mathbb{R}^d)} \|f\|^{1-s}_{L^2(\mathbb{R}^d)}$. Can you generalize this result to $f \in H^r(\mathbb{R}^d)$ for $r > 0$?

(b) If Ω is bounded and $\partial\Omega$ is smooth, show that there is a constant C such that for all $f \in H^1(\Omega)$, $\|f\|_{L^2(\partial\Omega)} \leq C\|f\|^{1/2}_{H^1(\Omega)} \|f\|^{1/2}_{L^2(\Omega)}$. [Hint: show for $d = 1$ on $(0,1)$ by considering

$$f(0)^2 = f(x)^2 - \int_0^x \frac{d}{dt} f(t)^2\, dt.$$

For $d > 1$, flatten out $\partial\Omega$ and use a $(d=1)$-type proof in the normal direction.]

12. Suppose that $\Omega \in \mathbb{R}^d$ is a bounded domain with Lipschitz boundary and $\{u_k\} \subset H^{2+\epsilon}(\Omega)$ is a bounded sequence, where $\epsilon > 0$. For this problem, you will need to use the fact that the Rellich-Kondrachov Theorem extends to fractional order spaces.

(a) Show that there is $u \in H^2(\Omega)$ such that, for a subsequence, $u_j \to u$ in $H^2(\Omega)$.

(b) Find all q and $s \geq 0$ such that, for a subsequence, $u_j \to u$ in $W^{s,q}(\Omega)$.

(c) For a subsequence, $|u_j|^r \nabla u_j \to |u|^r \nabla u$ in $L^2(\Omega)$ for certain $r \geq 1$. For fixed d, how big can r be? Justify your answer.

13. Prove that $H^s(\mathbb{R}^d)$ is continuously embedded in $C_B^0(\mathbb{R}^d)$ if $s > d/2$ by completing the following outline.

(a) Show that $\displaystyle\int_{\mathbb{R}^d} (1 + |\xi|^2)^{-s}\, d\xi < \infty$.

(b) If $\phi \in \mathcal{S}$ and $x \in \mathbb{R}^d$, write $\phi(x)$ as the Fourier inversion integral of $\hat{\phi}$. Introduce $1 = (1 + |\xi|^2)^{s/2}(1 + |\xi|^2)^{-s/2}$ into the integral and apply Hölder to obtain the result for Schwartz class functions.

(c) Use density to extend the above result to $H^s(\mathbb{R}^d)$.

14. Suppose $f \in L^2(\mathbb{R})$ and $\hat{\omega}(\xi) = \sqrt{|\xi|}$. Make sense of the definition $g = \omega * f$, and determine s such that $g \in H^s(\mathbb{R})$.

15. Suppose that $\omega \in L^1(\mathbb{R})$, $\omega(x) > 0$, and ω is even. Moreover, for $x > 0$, $\omega \in C^2[0, \infty)$, $\omega'(x) < 0$, and $\omega''(x) > 0$. Consider the following equation for u:

$$\omega * u - u'' = f \in L^2(\mathbb{R}).$$

(a) Show that $\hat{\omega}(\xi) > 0$. [Hint: use that ω is even, integrate by parts, and consider subintervals of size $2\pi/|\xi|$.]

(b) Find a fundamental solution to the differential equation (i.e., replace f by δ_0). You may leave your answer in terms of an inverse Fourier transform.

(c) For the original problem, find the solution operator as a convolution operator.

(d) Show that the solution $u \in H^2(\mathbb{R})$.

16. Let H be a Hilbert space and Z a closed linear subspace. Prove that H/Z has the inner-product

$$(\hat{x}, \hat{y})_{H/Z} = (P_Z^\perp x, P_Z^\perp y)_H,$$

and that H/Z is isomorphic to $Z^\perp$.

17. Elliptic regularity theory shows that if the domain $\Omega \subset \mathbb{R}^d$ has a smooth boundary and $f \in H^s(\Omega)$, then $-\Delta u = f$ in Ω, $u = 0$ on $\partial\Omega$, has a unique solution $u \in H^{s+2}$. For what values of s will u be continuous? Can you be sure that a fundamental solution is continuous? The answers depend on d.

18. Consider $x \in \mathbb{R}^{n\times n}$ with associated ℓ^2-norm $\|x\| = \left(\sum_{j=1}^n \sum_{k=1}^n |x_{j,k}|^2\right)^{1/2}$ and total-variation semi-norm

$$|x|_{TV} = \sum_{j=1}^n \sum_{k=1}^{n-1} |x_{j,k+1} - x_{j,k}| + \sum_{j=1}^{n-1} \sum_{k=1}^n |x_{j+1,k} - x_{j,k}|.$$

(a) Why is $|\cdot|_{TV}$ not a norm?

(b) Prove directly the following Sobolev inequality: if $x_{1,j} = x_{j,1} = 0$ for all $1 \leq j \leq n$, then

$$\|x\| \leq \frac{1}{2}\, |x|_{TV}.$$

[Hint: find two ways to write $x_{j,k}$ as a sum of discrete differences.]

(c) Does the result in (b) extend to ℓ^2 (assuming we reorder ℓ^2 to possess a two-dimensional indexing structure)?

19. Let $\Omega \subset \mathbb{R}^d$ be a domain with a Lipschitz boundary and let $w \in L^\infty(\Omega)$. Define

$$H_w(\Omega) = \{f \in L^2(\Omega) : \nabla(wf) \in (L^2(\Omega))^d\}.$$

(a) Give reasonable conditions on w so that $H_w(\Omega) = H^1(\Omega)$.

(b) Prove that $H_w(\Omega)$ is a Hilbert space. What is the inner-product?

(c) Can you define the trace of f on $\partial\Omega$? [Hint: first consider the trace of wf.]

(d) Characterize $H_w(\mathbb{R})$ if $w(x)$ is the Heaviside function.

Exercises for Chapter 8

1. If A is a positive definite matrix, show that its eigenvalues are positive. Conversely, prove that if A is symmetric and has positive eigenvalues, then A is positive definite.

2. Suppose that the hypotheses of the Generalized Lax-Milgram Theorem 8.10 are satisfied. Suppose also that $x_{0,1}$ and $x_{0,2}$ in $\mathcal{X}$ are such that the sets $X + x_{0,1} = X + x_{0,2}$. Prove that the solutions $u_1 \in X + x_{0,1}$ and $u_2 \in X + x_{0,2}$ of the abstract variational problem (8.16) agree (i.e., $u_1 = u_2$). What does this result say about Dirichlet boundary value problems?

3. Suppose that we wish to find $u \in H^2(\Omega)$ solving the nonlinear problem $-\Delta u + cu^2 = f \in L^2(\Omega)$, where $\Omega \subset \mathbb{R}^d$ is a bounded Lipschitz domain. For consistency, we would require that $cu^2 \in L^2(\Omega)$. Determine the smallest p such that if $c \in L^p(\Omega)$, you can be certain that this is true, if indeed it is possible. The answer depends on d.

4. Consider the boundary value problem for $u(x, y) : \mathbb{R}^2 \to \mathbb{R}$ such that

$$\begin{aligned} -u_{xx} + e^y u &= f, && \text{for } (x, y) \in (0, 1)^2, \\ u(0, y) = 0,\ u(1, y) &= \cos(y), && \text{for } y \in (0, 1). \end{aligned}$$

   Rewrite this as a variational problem and show that there exists a unique solution. Be sure to define your function spaces carefully and identify where f must lie.

5. Suppose that $\Omega \subset \mathbb{R}^d$ is a smooth, bounded, connected domain. Let

$$H = \left\{ u \in H^2(\Omega) : \int_\Omega u(x)\, dx = 0 \text{ and } \nabla u \cdot \nu = 0 \text{ on } \partial\Omega \right\}.$$

   Show that H is a Hilbert space, and prove that there exists $C > 0$ such that for any $u \in H$,

$$\|u\|_{H^1(\Omega)} \le C \sum_{|\alpha|=2} \|D^\alpha u\|_{L^2(\Omega)}.$$

6. Use the Lax-Milgram Theorem to show that, for $f \in L^2(\mathbb{R}^d)$, there exists a unique solution $u \in H^1(\mathbb{R}^d)$ to the problem

$$-\Delta u + u = f \qquad \text{in } \mathbb{R}^d.$$

   Be careful to justify integration by parts. [Hint: $\mathcal{D}$ is dense in $H^1(\mathbb{R}^d)$.]

7. The symmetric gradient of a vector function $v \in (H^1(\Omega))^d$ is a $d \times d$ matrix of the form

$$\epsilon(v)_{i,j} = \frac{1}{2}\Big(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\Big).$$

   It arises often in structural mechanics problems. Assuming that $\Omega \in \mathbb{R}^2$ is smooth and bounded, prove *Korn's inequality,* which says that there is some $C > 0$ such that

$$\|v\|_{H^1(\Omega)} \le C\|\epsilon(v)\|_{L^2(\Omega)} \quad \text{for all } v \in (H^1_0(\Omega))^2.$$

   [Hint: you will need to show that all *rigid motions,* i.e., vector fields of the form $v = \alpha(x_2, -x_1) + (\beta, \gamma)$, vanish in $(H^1_0(\Omega))^2$.]

8. Suppose $\Omega \subset \mathbb{R}^d$ is a bounded, connected Lipschitz domain and $V \subset \Omega$ has positive measure. Let $H = \{u \in H^1(\Omega) : u|_V = 0\}$.

(a) Why is H a Hilbert space?

(b) Prove the following Poincaré inequality: there is some $C > 0$ such that

$$\|u\|_{L^2(\Omega)} \leq C\|\nabla u\|_{L^2(\Omega)} \qquad \forall u \in H.$$

9. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with a Lipschitz boundary, $f \in L^2(\Omega)$, and $\alpha > 0$. Consider the Robin boundary value problem

$$\begin{cases} -\Delta u + u = f & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} + \alpha\, u = 0 & \text{on } \partial\Omega. \end{cases}$$

(a) For this problem, formulate a variational principle

$$B(u, v) = (f, v) \qquad \forall v \in H^1(\Omega).$$

(b) Show that this problem has a unique weak solution.

10. Let $\Omega = [0, 1]^d$, define

$$H^1_\#(\Omega) = \left\{ v \in H^1_{\text{loc}}(\mathbb{R}^d) \;:\; v \text{ is periodic of period 1 in each direction and } \int_\Omega v\, dx = 0 \right\},$$

and consider the problem of finding a periodic solution $u \in H^1_\#(\Omega)$ of

$$-\Delta u = f \quad \text{on } \Omega,$$

where $f \in L^2(\Omega)$.

(a) Define precisely what it means for $v \in H^1(\mathbb{R}^d)$ to be periodic of period 1 in each direction.

(b) Show that $H^1_\#(\Omega)$ is a Hilbert space.

(c) Show that there is a unique solution to the partial differential equation.

11. Consider

$$B(u, v) = (a\nabla u, \nabla v)_{L^2(\Omega)} + (bu, \nabla v)_{L^2(\Omega)} + (cu, v)_{L^2(\Omega)}$$

(a) Derive a condition on b to insure that B is coercive on $H^1(\Omega)$ when a is uniformly positive definite *and* c is uniformly positive.

(b) Suppose $b = 0$. If $c < 0$, is B not coercive? Show that this is true on $H^1(\Omega)$, but that by restricting how negative c may be, B is still coercive on $H^1_0(\Omega)$.

12. Suppose $\Omega \subset \mathbb{R}^d$ is a $C^{1,1}$ domain. Consider the biharmonic BVP

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega, \\ \nabla u \cdot \nu = g & \text{on } \partial\Omega, \\ u = u_D & \text{on } \partial\Omega, \end{cases}$$

wherein $\Delta^2 u = \Delta\Delta u$ is the application of the Laplace operator twice.

(a) Determine appropriate Sobolev spaces within which the functions u, f, g, and u_D should lie, and formulate an appropriate variational problem for the BVP. Show that the two problems are equivalent.

(b) Show that there is a unique solution to the variational problem. [Hint: use the Elliptic Regularity Theorem to prove coercivity of the bilinear form.]

(c) What would be the natural BC's for this partial differential equation?

(d) For simplicity, let u_D and g vanish and define the energy functional

$$J(v) = \int_\Omega \big(|\Delta v(x)|^2 - 2f(x)\,v(x)\big)\,dx.$$

Prove that minimization of J is equivalent to the variational problem.

13. Suppose $\Omega \subset \mathbb{R}^d$ is a bounded Lipschitz domain. Consider the Stokes problem for vector u and scalar p given by

$$\begin{cases} -\Delta u + \nabla p = f & \text{in } \Omega, \\ \nabla \cdot u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where the first equation holds for each coordinate (i.e., $-\Delta u_j + \partial p/\partial x_j = f_j$ for each $j = 1, ..., d$). This problem is not a minimization problem; rather, it is a *saddle-point problem,* in that we minimize some energy subject to the *constraint* $\nabla \cdot u = 0$. However, if we work over the constrained space, we can handle this problem by the ideas of this chapter. Let

$$H = \{v \in (H_0^1(\Omega))^d : \nabla \cdot u = 0\}.$$

(a) Verify that H is a Hilbert space.

(b) Determine an appropriate Sobolev space for f, and formulate an appropriate variational problem for the constrained Stokes problem.

(c) Show that there is a unique solution to the variational problem.

14. Let $\mathcal{H} = H_0^1(\Omega) \times H^1(\Omega)$ and consider the solution $(u, v) \in \mathcal{H}$ to the differential problem

$$\begin{aligned} -\Delta u &= f + a(v-u) && \text{in } \Omega, \\ v - \Delta v &= g + a(u-v) && \text{in } \Omega, \\ u = 0 \quad \text{and} \quad \nabla v \cdot \nu &= \gamma && \text{on } \partial\Omega, \end{aligned}$$

where $a \in L^\infty(\Omega)$.

(a) Develop an appropriate weak or variational form for the problem. In what Sobolev spaces should f, g, and γ lie?

(b) Prove that there exists a unique solution to the problem, provided that $a \geq 0$.

15. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with a smooth boundary. The equations of linear elasticity can be formulated for the displacement vector $u = (u_1, u_2)^T$ using the *symmetric gradient tensor* $\epsilon(u)$ defined by

$$\epsilon_{ij}(u) = \frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big)$$

as

$$\sum_i \frac{\partial}{\partial x_i}\big[2\mu\epsilon_{ij}(u) + \lambda\nabla\cdot u\,\delta_{ij}\big] = f_j \quad \text{in } \Omega,$$

where μ and λ are positive constants. We assume the boundary condition $u = 0$ on $\partial\Omega$.

(a) Develop an appropriate weak form for the equations that involve the integrals $(\nabla\cdot u, \nabla\cdot v)$ and $\sum_{i,j}\big(\epsilon_{ij}(u), \epsilon_{ij}(v)\big)$. For the latter, you will need to use the fact that $\epsilon(u)$ is indeed symmetric.

(b) In which spaces should u and $f = (f_1, f_2)^T$ lie?

(c) Show that there is a unique solution to the problem using the Lax-Milgram Theorem. You will need to use Korn's Inequality, which says that there is $\gamma > 0$ such that

$$\|\epsilon(u)\|_{(L^2(\Omega))^{2\times 2}} \geq \gamma\|u\|_{(H^1(\Omega))^2} \quad \forall u \in (H_0^1(\Omega))^2.$$

16. Let $\Omega \subset \mathbb{R}^2$ be a domain with a smooth boundary and consider the variational problem: Find $u \in V$ such that

$$(au, v) + (\nabla\cdot u, \nabla\cdot v) = (f, \nabla\cdot v) \quad \text{for all } v \in V,$$

where u and v are vectors in $\mathbb{R}^2$, $a \in L^\infty(\Omega)$, $a(x) \geq a_* > 0$ for some constant a_*, $(au, v) = \displaystyle\int_\Omega a(x)\big(u_1(x)\,v_1(x) + u_2(x)\,v_2(x)\big)\,dx$, and $\nabla\cdot u = \operatorname{div} u = \dfrac{\partial u_1}{\partial x_1} + \dfrac{\partial u_2}{\partial x_2}$.

(a) For the problem to make sense, define V and a space for f.

(b) State the Lax-Milgram theorem for Hilbert spaces.

(c) Show that the hypotheses of the Lax-Milgram theorem hold for this problem. What norm do we use for V?

(d) Define $p = f - \nabla\cdot u$ and determine the strong form of the equation represented by the variational problem. What boundary condition should you impose on p?

17. Let H and W be real Hilbert spaces and let $V \subset H$ be a linear subspace. Let $A : H \to H$ and $B : V \to W$ be bounded linear operators, where we give V the norm $\|v\|_V = \|v\|_H + \|Bv\|_W$. For any $f \in V$ and $0 \leq \delta < 1$, consider the problem: Find $(u, p) \in V \times W$ such that

$$\begin{aligned}&\langle Au, v\rangle_H - \langle B^*p, v\rangle_H + \langle Bu, w\rangle_W + \langle p, w\rangle_W + \delta\langle Bu, Bv\rangle_W + \delta\langle p, Bv\rangle_W\\ &\quad = \langle f, v\rangle_H \quad \forall (v, w) \in V \times W.\end{aligned}$$

Assume that A is coercive on V (i.e., there is $\alpha > 0$ such that $\alpha\|v\|_V^2 \leq \langle Av, v\rangle_V$).

(a) Assuming there is a solution, find a bound on the norm of the solution (u, p).

(b) Show that there is a unique solution for any $\delta \in (0, 1)$.

(c) Show that there is a unique solution for $\delta = 0$. [Hint: replace w by $w - \delta Bv$.]

18. Let $\Omega \subset \mathbb{R}^d$ be a domain and define the *potential* and *solenoidal* vector fields

$$H_{\text{pot}} = \{\nabla\phi : \phi \in H_0^1(\Omega)\} \quad \text{and} \quad H_{\text{sol}} = \big\{\psi \in (L^2(\Omega))^d : \nabla\cdot\psi = 0\big\}.$$

These are clearly linear subspaces of $(L^2(\Omega))^d$.

(a) Show that H_{pot} and H_{sol} are $(L^2(\Omega))^d$-orthogonal to each other.

(b) Show that both H_{pot} and H_{sol} are closed in $(L^2(\Omega))^d$.

(c) Prove that $(L^2(\Omega))^d = H_{\text{pot}} \oplus H_{\text{sol}}$, i.e., an L^2 vector can be written uniquely as the sum of a potential and solenoidal vector. This fact is called the *Helmholtz decomposition.*

(d) Show that for $u \in (L^2(\Omega))^d = \nabla\phi + \psi$,

$$\|u\|^2_{(L^2(\Omega))^d} = \|\nabla\phi\|^2_{(L^2(\Omega))^d} + \|\psi\|^2_{(L^2(\Omega))^d}.$$

(e) Suppose $u_j \in H_{\text{pot}}$ and $\nabla \cdot u_j = f_j \in L^2(\Omega)$ for all j. If

$$\|u_j\|_{(L^2(\Omega))^d} + \|\nabla \cdot u_j\|_{(L^2(\Omega))^d} \le M < \infty \quad \forall j,$$

show that there is some $u \in H_{\text{pot}}$ such that, for a subsequence, u_{j_k} converges strongly to u in $(L^2(\Omega))^d$.

19. Let $\Omega \subset \mathbb{R}^4$ be a bounded domain with a smooth boundary in 4 dimensions, and let $f \in L^2(\Omega)$. Consider the nonlinear BVP

$$-\Delta u + u^3 = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega.$$

We attempt to solve this iteratively from $u_0 = 0$ by computing for each $n = 1, 2, \dots,$ the solutions to the linear BVPs

$$-\Delta u_n + u_{n-1}^2 u_n = f \quad \text{in } \Omega, \quad u_n = 0 \quad \text{on } \partial\Omega.$$

(a) Find the appropriate variational problems (i.e., weak forms) for the linear BVPs and show that they are well defined in $H_0^1(\mathbb{R}^4)$ if $u_{n-1} \in H_0^1(\mathbb{R}^4)$. [Hint: Sobolev Imbedding Theorem.]

(b) Show that there is a unique solution $u_n \in H_0^1(\mathbb{R}^4)$, assuming $u_{n-1} \in H_0^1(\mathbb{R}^4)$. Moreover, find a bound for the norm of u_n.

(c) Show that the nonlinear BVP has a weak solution. [Hint: Extract a subsequence of $\{u_n\}_{n=0}^\infty$ that converges weakly to some u, and show that u satisfies the weak form of the nonlinear BVP.]

20. Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. Using the Lax-Milgram Theorem and the definition of the $H^{1/2}(\partial\Omega)$-norm (but *not* the Elliptic Regularity Theorem), show that there is some constant $C > 0$ such that

$$\|u\|_{H^1(\Omega)} \le C\big\{\|\Delta u\|_{H^{-1}(\Omega)} + \|u\|_{H^{1/2}(\partial\Omega)}\big\} \quad \text{for all } u \in H^1(\Omega).$$

21. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with a Lipschitz boundary, and define

$$H(\text{div}; \Omega) = \big\{v \in (L^2(\Omega))^d : \nabla \cdot v \in L^2(\Omega)\big\}.$$

(a) Show that $H(\text{div}; \Omega)$ is a Hilbert space with the inner-product

$$(u, v)_{H(\text{div};\Omega)} = (u, v)_{(L^2(\Omega))^d} + (\nabla \cdot u, \nabla \cdot v)_{L^2(\Omega)}.$$

(b) The trace theorem does not imply that $\partial_\nu v = v \cdot \nu$ exists on $\partial\Omega$. Nevertheless, show that $\partial_\nu : H(\text{div}; \Omega) \to H^{-1/2}(\partial\Omega) = (H^{1/2}(\partial\Omega))^*$ is a well defined bounded linear operator in the sense of integration by parts:

$$\int_{\partial\Omega} v \cdot \nu\, \phi\, d\sigma(x) = \int_\Omega \nabla \cdot v\, \phi\, dx + \int_\Omega v \cdot \nabla\phi\, dx.$$

(c) Prove the following inf-sup condition: there exists $\gamma > 0$ such that

$$\inf_{w\in L^2(\Omega)} \sup_{v\in H(\mathrm{div};\Omega)} \frac{(w, \nabla\cdot v)_{L^2(\Omega)}}{\|w\|_{L^2(\Omega)}\|v\|_{H(\mathrm{div};\Omega)}} \geq \gamma > 0.$$

[Hint: solve $\Delta\varphi = w$ in $H_0^1(\Omega)$ and consider $v = \nabla\varphi$.]

22. Let $s, t \geq 0$ and define

$$\langle f, g\rangle = \int_{\mathbb{R}^2} \hat{f}(\eta,\zeta)\,\overline{\hat{g}(\eta,\zeta)}(1+\eta^2)^s(1+\zeta^2)^t\,d\eta\,d\zeta,$$
$$H^{s,t} = \{f \in L^2(\mathbb{R}^2)\ :\ \|f\|^2 = \langle f, f\rangle < \infty\}.$$

(a) Show that $\langle\cdot,\cdot\rangle$ is an inner product on $H^{s,t}$, $H^{s,t}$ is continuously imbedded in $H^{\min(s,t)}(\mathbb{R}^2)$, and $H^{s+t}(\mathbb{R}^2)$ is continuously imbedded in $H^{s,t}(\mathbb{R}^2)$.

(b) Show that $H^{s,t}$ is complete and $\mathcal{S}(\mathbb{R}^2) \subset H^{s,t}$ is dense.

(c) Define the trace operator $\gamma_0 : C^2(\mathbb{R}^2) \to C^0(\mathbb{R})$ by $\gamma_0 f(x) = f(x,0)$. Find reasonable hypotheses on s and t as well as an r so that γ_0 extends to a bounded linear operator $\gamma_0 : H^{s,t} \to H^r(\mathbb{R})$.

23. Modify the statement of Theorem 8.14 to allow for nonhomogeneous essential boundary conditions, and prove the result.

24. Let $\Omega \in \mathbb{R}^d$ have a smooth boundary, V_n be the set of polynomials of degree up to n, for $n = 1, 2, ...,$ and $f \in L^2(\Omega)$. Consider the problem: Find $u_n \in V_n$ such that

$$(\nabla u_n, \nabla v_n)_{L^2(\Omega)} + (u_n, v_n)_{L^2(\Omega)} = (f, v_n)_{L^2(\Omega)} \quad \text{for all } v_n \in V_n.$$

(a) Show that there exists a unique solution for any n, and that

$$\|u_n\|_{H^1(\Omega)} \leq \|f\|_{L^2(\Omega)}.$$

(b) Show that there is $u \in H^1(\Omega)$ such that $u_n \rightharpoonup u$ weakly in $H^1(\Omega)$. Find a variational problem satisfied by u. Justify your answer.

(c) Show that $\|u - u_n\|_{H^1(\Omega)}$ decreases monotonically to 0 as $n \to \infty$.

(d) What can you say about u and $\nabla u \cdot \nu$ on $\partial\Omega$?

25. Consider the finite element method in Section 8.6.

(a) Modify the method to account for nonhomogeneous Neumann conditions.

(b) Modify the method to account for nonhomogeneous Dirichlet conditions.

26. Compute explicitly the finite element solution to (8.28) using $f(x) = x^2(1-x)$ and $n = 4$. How does this approximation compare to the true solution?

27. Suppose that $u \in H^1(\Omega)$, where $\Omega \subset \mathbb{R}^d$ is a bounded, connected domain. Recall that the $H^1(\Omega)$-seminorm is $|u|_{H^1(\Omega)} = \big\{\sum_{|\alpha|=1}\|D^\alpha u\|^2_{L^2(\Omega)}\big\}^{1/2}$.

(a) Show that there is some constant C_Ω, depending on Ω but not on u, such that

$$\inf_{c\in\mathbb{R}} \|u - c\|_{L^2(\Omega)} \leq C_\Omega |u|_{H^1(\Omega)}.$$

(b) Let $\Omega = (0,h)^d$ for some $h > 0$. Show that there is a constant C, independent of h and u, such that

$$\inf_{c \in \mathbb{R}} \|u - c\|_{L^2(\Omega)} \le C\, h\, |u|_{H^1(\Omega)}.$$

[Hint: change variables to integrate over $(0,1)^d$, and use (a).]

(c) Let $\Omega = (0,1)^d$ and let P be the set of piecewise discontinuous constants over the grid of spacing $h = 1/N$ for some positive integer N. Show that there is a constant C, independent of h and u, such that

$$\inf_{p \in P} \|u - p\|_{L^2(\Omega)} \le C\, h\, |u|_{H^1(\Omega)}.$$

28. Let H_h be the set of continuous piecewise linear functions defined on the grid $x_j = jh$, where $h = 1/n$ for some integer $n > 0$. Let the interpolation operator $\mathcal{I}_h : H_0^1(0,1) \to H_h$ be defined by

$$\mathcal{I}_h v(x_j) = v(x_j) \qquad \forall j = 1, 2, ..., n-1.$$

(a) Show that $\mathcal{I}_h$ is well defined, and that it is continuous. [Hint: use the Sobolev Embedding Theorem.]

(b) Show that there is a constant $C > 0$ independent of h such that

$$\|v - \mathcal{I}_h v\|_{H^1(x_{j-1},x_j)} \le C \|v\|_{H^2(x_{j-1},x_j)} h.$$

[Hint: change variables so that the domain becomes $(0,1)$, where the result is trivial by Poincaré's inequality Corollary 7.20 and Theorem 8.12.]

(c) Show that (8.30) holds.

29. Consider the problem (8.28).

(a) Find the Green's function.

(b) Instead impose Neumann BC's, and find the Green's function. [Hint: recall that now we require $-(\partial^2/\partial x^2)G(x,y) = \delta_y(x) - 1$.]

30. If $\mathcal{L}$ is self-adjoint on $L^2(\Omega)$ and has a Green's function G, show that $G(x,y) = G(y,x)$ for a.e. $x, y \in \Omega$.

Exercises for Chapter 9

1. Let X, Y_1, Y_2, and Z be normed linear spaces and $P : Y_1 \times Y_2 \to Z$ be a continuous bilinear map (so P is a "product" between Y_1 and Y_2).

   (a) Show that for $y_i, \hat{y}_i \in Y_i$,

   $$DP(y_1, y_2)(\hat{y}_1, \hat{y}_2) = P(y_1, \hat{y}_2) + P(\hat{y}_1, y_2).$$

   (b) If $f : X \to Y_1 \times Y_2$ is differentiable, show that for $h \in X$,

   $$D(P \circ f)(x)\, h = P(Df_1(x)\, h, f_2(x)) + P(f_1(x), Df_2(x)\, h).$$

2. Let X be a real Hilbert space and $A_1, A_2 \in B(X, X)$, and define $f(x) = (x, A_1 x)_X A_2 x$. Show that $Df(x)$ exists for all $x \in X$ by finding an explicit expression for it.

3. Let $X = C([0,1])$ be the space of bounded continuous functions on $[0,1]$ and, for $u \in X$, define $F(u)(x) = \displaystyle\int_0^1 K(x,y)\, f(u(y))\, dy$, where $K : [0,1] \times [0,1] \to \mathbb{R}$ is continuous and f is a C^1-mapping of $\mathbb{R}$ into $\mathbb{R}$. Find the Fréchet derivative $DF(u)$ of F at $u \in X$. Is the map $u \mapsto DF(u)$ continuous?

4. Suppose X and Y are Banach spaces, and $f : X \to Y$ is differentiable with derivative $Df(x) \in B(X,Y)$ being a compact operator for any $x \in X$. Prove that f is also compact.

5. Set up and apply the contraction mapping principle to show that the problem

   $$-u_{xx} + u - \epsilon u^2 = f(x), \quad x \in \mathbb{R},$$

   has a smooth bounded solution if $\epsilon > 0$ is small enough, where $f(x) \in \mathcal{S}(\mathbb{R})$.

6. Use the contraction-mapping theorem to show that the Fredholm Integral Equation

   $$f(x) = \varphi(x) + \lambda \int_a^b K(x,y) f(y)\, dy$$

   has a unique solution $f \in C([a,b])$, provided that λ is sufficiently small, wherein $\varphi \in C([a,b])$ and $K \in C([a,b] \times [a,b])$.

7. Suppose that F is defined on a Banach space X, that $x_0 = F(x_0)$ is a fixed point of F, $DF(x_0)$ exists, and that 1 is *not* in the spectrum of $DF(x_0)$. Prove that x_0 is an isolated fixed point.

8. Consider the first-order differential equation

   $$u'(t) + u(t) = \cos(u(t))$$

   posed as an initial-value problem for $t > 0$ with initial condition

   $$u(0) = u_0.$$

   (a) Use the contraction-mapping theorem to show that there is exactly one solution u corresponding to any given $u_0 \in \mathbb{R}$.

   (b) Prove that there is a number ξ such that $\lim_{t \to \infty} u(t) = \xi$ for any solution u, independent of the value of u_0.

9. Set up and apply the contraction mapping principle to show that the boundary value problem

$$\begin{aligned}&-u_{xx}+u-\epsilon u^2=f(x), \quad x\in(0,\infty),\\&u(0)=u(\infty)=0,\end{aligned}$$

has a smooth solution if $\epsilon>0$ is small enough, where $f(x)$ is a smooth compactly supported function on $(0,\infty)$.

10. Consider the partial differential equation

$$\begin{aligned}&\frac{\partial u}{\partial t}-\frac{\partial^3 u}{\partial t\partial x^2}-\epsilon u^3=f, \quad -\infty<x<\infty,\ t>0,\\&u(x,0)=g(x).\end{aligned}$$

Use the Fourier transform and a contraction mapping argument to show that there exists a solution for small enough ϵ, at least up to some time $T<\infty$. In what spaces should f and g lie?

11. Let $\phi(x)\in C(\mathbb{R})\cap L^\infty(\mathbb{R})$. Consider the nonlinear initial value problem (IVP)

$$\begin{aligned}&\frac{\partial u}{\partial t}+\frac{\partial u}{\partial x}+3u^2\frac{\partial u}{\partial x}-\frac{\partial^3 u}{\partial x^2\partial t}=0, \quad x\in\mathbb{R},\ t>0,\\&u(x,0)=\phi(x).\end{aligned}$$

(a) Use the Fourier transform in x to show that the (IVP) can be rewritten in the form

$$\begin{aligned}&\partial_t u=K*(u+u^3), \quad x\in\mathbb{R},\ t>0,\\&u(x,0)=\phi(x),\end{aligned}$$

for some $K(x)\in L^1(\mathbb{R})$.

(b) Set up and apply the contraction mapping principle to show that the initial value problem has a continuous and bounded solution $u=u(x,t)$, at least up to some time $T<\infty$.

12. Let X be a Banach space and $F:X\to X$ be a smooth map. Suppose that x_* is a simple root of F in the sense that $F(x_*)=0$ and the derivative $DF(x_*)$ is invertible. Given any starting point x_0, for $k\geq 0$ consider the full Newton-Kantorovich Method

$$x_{k+1}=G(x_k) \quad \text{where} \quad G(x)=x-DF(x)^{-1}F(x).$$

Complete the following outline to show that if x_0 is sufficiently close to x_*, then $x_k\to x_*$ as $k\to\infty$.

(a) Show that $G(x_*)=x_*$, $DG(x_*)=0$ and that there is a closed ball B about x_* such that $\|DG(x)\|\leq\frac{1}{2}$ for all $x\in B$. [Hint: you do *not* need to compute $DG(x)$, only $DG(x_*)$.]

(b) Show that $G(x)\in B$ for all $x\in B$.

(c) Show that $G:B\to B$ is a contraction.

(d) Prove that $x_k\to x_*$ as $k\to\infty$ for any $x_0\in B$.

(e) If $F\in C^3$, show that, for some $C<\infty$,

$$\|x_{k+1}-x_*\|\leq C\|x_k-x_*\|^2.$$

That is, if the Newton-Kantorovich iteration converges, it does so quadratically in the error. [Hint: note that $x_{k+1} - x_* = G(x_k) - G(x_*)$ and apply the Mean Value Theorem twice.]

13. Surjective Mapping Theorem: Let X and Y be Banach spaces, $U \subset X$ be open, $f : U \to Y$ be C^1, and $x_0 \in U$. If $Df(x_0)$ has a bounded right inverse, then $f(U)$ contains a neighborhood of $f(x_0)$.

    (a) Prove this theorem from the Inverse Function Theorem. [Hint: let R be the right inverse of $Df(x_0)$ and consider $g : V \to Y$ where $g(y) = f(x_0 + Ry)$ and $V = \{y \in Y : x_0 + Ry \in U\}$.]

    (b) Prove that if $y \in Y$ is sufficiently close to $f(x_0)$, there is at least one solution to $f(x) = y$.

14. Suppose $f \in C^0([0,1])$ and that we want to solve

$$\frac{1}{1+\epsilon u^2} u' = f(x), \quad x \in (0,1), \quad \text{and} \quad u(0) = 0.$$

    We note that there is a unique solution $u_0(x)$ when $\epsilon = 0$.

    (a) Use the Implicit Function Theorem to show that there is a continuously differentiable solution for ϵ small enough.

    (b) Use the Banach Contraction Mapping Theorem to show that there is a unique continuous solution in a closed ball about u_0 in an appropriate Banach space for ϵ sufficiently small.

15. Let X and Y be Banach spaces.

    (a) Let F and G take X to Y be C^1 on X, and let $H(x,\epsilon) = F(x) + \epsilon G(x)$ for $\epsilon \in \mathbb{R}$. If $H(x_0, 0) = 0$ and $DF(x_0)$ is invertible, show that there exists $x \in X$ such that $H(x,\epsilon) = 0$ for ϵ sufficiently close to 0.

    (b) Let $f \in C^0([0,1])$ and $\epsilon \in \mathbb{R}$. Use the previous result to show that, for sufficiently small $|\epsilon|$, there is a solution $w \in C^2([0,1])$ to

$$w'' = f + w + \epsilon w^2, \quad w(0) = w(1) = 0.$$

16. Prove that for sufficiently small $\epsilon > 0$, there is at least one solution to the functional equation

$$f(x) + \sin x \int_{-\infty}^{\infty} f(x-y)\, f(y)\, dy = \epsilon e^{-|x|^2}, \quad x \in \mathbb{R},$$

    such that $f \in L^1(\mathbb{R})$.

17. Let X and Y be Banach spaces, and let $U \subset X$ be open and convex. Let $F : U \to Y$ be an n-times Fréchet differentiable operator. Let $x \in U$ and $h \in X$. Prove that in Taylor's formula, the remainder is actually bounded as

$$\begin{aligned} \|R_{n-1}(x,h)\| &= \left\| F(x+h) - F(x) - DF(x)h + \cdots + \frac{1}{(n-1)!} D^{n-1}F(x)(h,\ldots,h) \right\| \\ &\le \sup_{0 \le \alpha \le 1} \|D^n F(x+\alpha h)\|\, \|h\|^n. \end{aligned}$$

18. Prove that if X is an NLS, U an open convex subset of X, and $f : U \to \mathbb{R}$ is strictly convex and differentiable, then, for $x, y \in U$, $x \neq y$,

$$f(y) > f(x) + Df(x)(y-x),$$

and $Df(x) = 0$ implies that f has a strict and therefore unique minimum.

19. Let $\Omega \subset \mathbb{R}^d$ have a smooth boundary, and let $g(x)$ be real with $g \in H^1(\Omega)$. Consider the BVP

$$\begin{cases} -\Delta u + u = 0 & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases}$$

(a) Write this as a variational problem.

(b) Define an appropriate energy functional $J(v)$ and find $DJ(v)$.

(c) Relate the BVP to a constrained minimization of $J(v)$.

20. Let $\Omega \subset \mathbb{R}^n$ have a smooth boundary, $A(x)$ be an $n \times n$ real matrix with components in $L^\infty(\Omega)$, and let $c(x)$, $f(x)$ be real with $c \in L^\infty(\Omega)$ and $f \in L^2(\Omega)$. Consider the BVP

$$\begin{cases} -\nabla \cdot A\nabla u + cu = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

(a) Write this as a variational problem.

(b) Assume that A is symmetric and uniformly positive definite and c is uniformly positive. Define the energy functional $J : H_0^1 \to \mathbb{R}$ by $J(v) = \dfrac{1}{2} \displaystyle\int_\Omega \big\{ |A^{1/2}\nabla v|^2 + c|v|^2 - 2fv \big\} dx$. Find $DJ(v)$.

(c) Prove that for $u \in H_0^1$, the following are equivalent: (i) u is the solution of the BVP; (ii) $DJ(u) = 0$; (iii) u minimizes $J(v)$.

21. Let X and Y be Banach spaces, $U \subset X$ an open set, and $f : U \to Y$ Fréchet differentiable. Suppose that f is compact, in the sense that for any $x \in U$, if $\overline{B_r(x)} \subset U$, then $f(B_r(x))$ is precompact in Y. If $x_0 \in U$, prove that $Df(x_0)$ is a compact linear operator.

22. Suppose we wish to find the surface $u(x, y)$ above the square $Q = [-1, 1]^2$, with $u = 0$ on ∂Q, that encloses the greatest volume, subject to the constraint that the surface area is fixed at $s > 4$.

(a) Formulate the problem, and reformulate it incorporating the constraint as a Lagrange multiplier. [Hint: the surface area is $\iint_Q \sqrt{1 + |\nabla u|^2}\, dx\, dy$.]

(b) Using the definition of the Fréchet derivative, find the conditions for a critical point.

(c) Find a partial differential equation that u must satisfy to be an extremal of this problem. [Remark: a solution of this differential equation, that also satisfies the area constraint, gives the solution to our problem.]

Exercises for Chapter 10

1. Let $F(u) = \displaystyle\int_{-1}^{5} [(u'(x))^2 - 1]^2\,dx$.

   (a) Find all extremals in $C^1([-1,5])$ such that $u(-1) = 1$ and $u(5) = 5$.

   (b) Decide if any extremal from (a) is a minimum of F. Consider $u(x) = |x|$.

2. Consider the functional
$$F(y) = \int_0^1 [(y(x))^2 - y(x)\,y'(x)]\,dx,$$
   defined for $y \in C^1([0,1])$.

   (a) Find all extremals.

   (b) If we require $y(0) = 0$, show by example that there is no minimum.

   (c) If we require $y(0) = y(1) = 0$, show that the extremal is a minimum. [Hint: note that $y\,y' = (\frac{1}{2}y^2)'$.]

3. Find all extremals of
$$\int_0^{\pi/2} \left[\, \big(y'(t)\big)^2 + \big(y(t)\big)^2 + 2\,y(t) \,\right] dt$$
   under the condition $y(0) = y(\pi/2) = 0$.

4. Suppose that we wish to minimize
$$F(y) = \int_0^1 f(x, y(x), y'(x), y''(x))\,dx$$
   over the set of $y(x) \in C^2([0,1])$ such that $y(0) = \alpha$, $y'(0) = \beta$, $y(1) = \gamma$, and $y'(1) = \delta$. That is, with $C_0^2([0,1]) = \{u \in C^2([0,1]) \,:\, u(0) = u'(0) = u(1) = u'(1) = 0\}$, $y \in C_0^2([0,1]) + p(x)$, where p is the cubic polynomial that matches the boundary conditions.

   (a) Find a differential equation, similar to the Euler-Lagrange equation, that must be satisfied by the minimum (if it exists).

   (b) Apply your equation to find the extremal(s) of
$$F(y) = \int_0^1 (y''(x))^2\,dx,$$
   where $y(0) = y'(0) = y'(1) = 0$ but $y(1) = 1$, and justify that each extremal is a (possibly nonstrict) minimum.

5. Prove the theorem: If f and g map $\mathbb{R}^3$ to $\mathbb{R}$ and have continuous partial derivatives up to second order, and if $u \in C^2([a,b])$, $u(a) = \alpha$ and $u(b) = \beta$, minimizes
$$\int_a^b f(x, u(x), u'(x))\,dx,$$
   subject to the constraint
$$\int_a^b g(x, u(x), u'(x))\,dx = 0,$$

then there is a nontrivial linear combination $h = \mu f + \lambda g$ such that $u(x)$ satisfies the Euler-Lagrange equation for h.

6. Consider the functional
$$\Phi(x, y, y') = \int_a^b F(x, y(x), y'(x))\, dx.$$

(a) If $F \in C^2$ and $F = F(y, y')$ only, and if we assume that $y \in C^2$, prove that in this case the Euler-Lagrange equations reduce to
$$\frac{d}{dx}(F - y'F_{y'}) = 0.$$

(b) Among all C^2 curves $y(x)$ joining the points $(0, 1)$ and $(1, \cosh(1))$, find the one which generates the minimum area when rotated about the x-axis. Recall that this area is
$$A = 2\pi \int_0^1 y\sqrt{1 + (y')^2}\, dx.$$

$\left[\text{Hint: } \int \frac{dt}{\sqrt{t^2 - C^2}} = \ln(t + \sqrt{t^2 - C^2}).\right]$

7. Consider the functional
$$J[x, y] = \int_0^{\pi/2} [(x'(t))^2 + (y'(t))^2 + 2x(t)y(t)]\, dt$$
and the boundary conditions
$$x(0) = y(0) = 0 \quad \text{and} \quad x(\pi/2) = y(\pi/2) = 1.$$

(a) Find the Euler-Lagrange equations for the functional.

(b) Find all extremals.

(c) Find a global minimum, if it exists, or show it does not exist.

(d) Find a global maximum, if it exists, or show it does not exist.

8. Consider the problem of finding a C^1 curve that minimizes
$$\int_0^1 (y'(t))^2\, dt$$
subject to the conditions that $y(0) = y(1) = 0$ and
$$\int_0^1 (y(t))^2\, dt = 1.$$

(a) Remove the integral constraint by incorporating a Lagrange multiplier, and find the Euler-Lagrange equations.

(b) Find all extremals to this problem.

(c) Find the solution to the problem.

(d) Use your result to find the best constant C in the inequality

$$\|y\|_{L^2(0,1)} \leq C\|y'\|_{L^2(0,1)}$$

for functions that satisfy $y(0) = y(1) = 0$.

9. Find the C^2 curve $y(t)$ that minimizes the functional

$$\int_0^1 \left[(y(t))^2 + (y'(t))^2\right] dt$$

subject to the endpoint constraints

$$y(0) = 0 \quad \text{and} \quad y(1) = 1$$

and the constraint

$$\int_0^1 y(t)\, dt = 0.$$

10. Find the form of the curve in the plane (*not* the curve itself), of minimal length, joining $(0,0)$ to $(1,0)$ such that the area bounded by the curve, the x and y axes, and the line $x = 1$ has area $\pi/8$.

11. Solve the constrained Brachistochrone problem: In a vertical plane, find a C^1-curve joining $(0,0)$ to (b,β), b and β positive and given, such that if the curve represents a track along which a particle slides without friction under the influence of a constant gravitational force of magnitude g, the time of travel is minimal. Note that this travel time is given by the functional

$$T(y) = \int_0^b \sqrt{\frac{1 + (y'(x))^2}{2g(\beta - y(x))}}\, dx.$$

12. Consider a stream between the lines $x = 0$ and $x = 1$, with speed $v(x)$ in the y-direction. A boat leaves the shore at $(0,0)$ and travels with constant speed $c > v(x)$. The problem is to find the path $y(x)$ of minimal crossing time, where the terminal point $(1,\beta)$ is unspecified.

(a) Find conditions on y so that it satisfies the Euler-Lagrange constraint. $\Big[$Hint: the crossing time is $t = \displaystyle\int_0^1 \frac{\sqrt{c^2(1 + (y')^2) - v^2} - vy'}{c^2 - v^2} dx.\Big]$

(b) What free endpoint constraint is required?

(c) If v is constant, find y.

13. For $y \in C^1([0,1])$ with $y(0) = 0$, consider minimizing

$$F(y) = \int_0^1 \left[xy(x) + \left(y'(x)\right)^2\right] dx.$$

(a) Show that $F(y)$ is strictly convex.

(b) Find the minimum if $y(1) = 0$.

(c) Find the minimum if $y(1) = 0$ and we add the constraint $\displaystyle\int_0^1 y(x)\, dx = 1$.

(d) Suppose we apply the same constraint, $\int_0^1 y(x)\, dx = 1$, but we do *not* constrain y at $x = 1$. First deriveÃ conditions for the minimum, including a free end condition, and then find the minimum.

14. Given $I = [0, b]$, consider the problem of finding $u : I \to \mathbb{R}$ such that

$$\begin{cases} u'(t) = g(t) f(u(t)), & \text{for a.e. } t \in I, \\ u(0) = \alpha, \end{cases}$$

where $\alpha \in \mathbb{R}$ is a given constant, $g \in L^p(I)$, $p \geq 1$, and $f : \mathbb{R} \to \mathbb{R}$ are given functions. We suppose that f is Lipschitz continuous and satisfies $f(0) = 0$. Consider the functional

$$F(u) = \alpha + \int_0^t g(s)\, f(u(s))\, ds.$$

(a) Show that F maps $C^0(I)$ into $C^0(I) \cap W^{1,p}(I)$. Moreover, show that $u \in C^0(I) \cap W^{1,p}(I)$ solves the problem if and only if it is a fixed point of F.

(b) Show that there exists b small enough, not depending on α, such that F has a unique fixed point in $C^0(I)$.

(c) Show that the problem has a unique solution $u \in C^0(I) \cap W^{1,p}(I)$ for any $g \in L^p(I)$ and $b > 0$.

15. Prove Proposition 10.11.