Homework 5

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Problem 7.6:

Suppose $\Omega \subset \mathbb{R}^d$ is a possibly unbounded domain and $\{U_\alpha\}_{\alpha\in I}$ is a collection of open sets in $\mathbb{R}^d$ that cover Ω. Prove there exists a locally finite partition of unity in Ω subordinate to the cover.

Solution We start with a lemma:

Lemma 0.1. *If S is a countable dense subset of $\mathbb{R}^d$ with the collection of balls*

$$\mathcal{B} = \left\{ B_r(x) \subset \mathbb{R}^d | r \in \mathbb{Q}, x \in S, \text{ and } \exists \alpha \in I : B_r(x) \subset U_\alpha \right\},$$

then $\cup B_{r_j/2}(x_j)$ is an open cover of Ω.

Proof. Fix $r_j/2 > 0$ for all j and consider a point $x \in \Omega$. Suppose now to the contrary that there does not exists a j such that x belongs to $B_{r_j/2}(x_j)$. Then, since $\mathbb{Q}$ is dense in $\mathbb{R}$, for any $\varepsilon > 0$ there exists a rational q arbitrarily close to x. Since $q \in Q$, then also $q \in S$, and so it has a radius $r_{q/2}$ associated with it. If $\varepsilon \le r_{q/2}$, then we are done. If not, select a new $\varepsilon_1 < \varepsilon$ and repeat the process, arriving at a new $q_1 \in \mathbb{Q}$ such that $x \in B_{\varepsilon_1}(q_1)$. Repeating, we keep looking with smaller ε_k, q_k, $r_{q_k/2}$. If we repeat infinitely many times, though, we have found some j such that $\varepsilon > r_j$ for every $\varepsilon > 0$, otherwise we would have stopped. The only way to satisfy this would be for $r_j = 0$, a contradiction. □

We now want to construct the functions that make up the partition of unity. Let $\varphi(x)$ denote Cauchy's infinitely differentiable function in 1D. We can easily translate and dilate this to form a bump function $\phi_j \in C_0^\infty(B_j)$ satisfying $0 \le \phi_j \le 1$ and $\phi_j = 1$ on $B_{r_j/2}$:

$$\phi_j(x) = \varphi\left(\frac{2}{r_j}\left(|x - x_j| + r_j\right)\right)\varphi\left(-\frac{2}{r_j}\left(|x - x_j| - r_j\right)\right).$$

Now let $\psi_1 = \phi_1$, $\psi_k = \prod_{j=1}^{k-1}\psi_j\psi_k$.

We can now demonstrate the desired properties. First of all, property (iii) is easy to see. Indeed, for every $j \in \mathcal{J}$ that indexes $\mathcal{B}$, there exists $\alpha \in I$ such that

$$\text{supp } \psi_j \subset \text{supp } \phi_j \subset \text{supp } B_j \subset \text{supp } U_\alpha.$$

Now we show that for every compact $K \subset \Omega$, all but finitely many ψ_j vanish. Observe that if $\psi_i = 0$, then all ψ_{i+k} vanish for $k > 0$. We have already shown that $B_{r_j/2}$ provide an open cover for Ω. Since K is compact, it admits a finite subcover, that is, $K \subset \cup_{j=1}^M B_{r_j/2}(x_j)$. Now for $x \in K$, there exists a $N \le M$ such that $x \in B_{r_N/2}(x_N)$. We then have that $\phi_N(x) = 1$, and that $\psi_n(x) = 0$ for $n \ge N$. So, only finitely many ψ support on K. It is also easy to see that ψ_j range betewen 0 and 1. To show the summation property, go by induction. We need to show that

$$\sum_{j=1}^m \psi_j = 1 - \prod_{j=1}^m (1 - \phi_j).$$

This holds trivially for $m = 1$. Now, for the inductive step,

$$\begin{aligned}\sum_{j=1}^{m+1} \psi_j &= \psi_{m+1} + \sum_{j=1}^{m} \psi_j \\ &= \psi_{m+1} + 1 - \prod_{j=1}^{m} (1 - \phi_j) \\ &= \phi_{m+1} \prod_{j=1}^{m} (1 - \phi_j) + 1 - \prod_{j=1}^{m} (1 - \phi_j) \\ &= 1 - \prod_{j=1}^{m+1} (1 - \phi_j) \,.\end{aligned}$$

To show the summation property, we again use the fact that we have an open cover of half-radius balls. Using what we have just proved,

$$\sum_{j \in \mathcal{J}} \psi_j(x) = \sum_{j=1}^{m} \psi_j(x) = 1 - \prod_{j=1}^{m} (1 - \phi_j(x)) = 1 - (1 - \phi_m(x)) \prod_{j=1}^{m-1} (1 - \phi_j(x)) = 1.$$

Problem 7.7:

$u \in \mathcal{D}' \left(\mathbb{R}^d\right)$, $\phi \in \mathcal{D}$.

Solution

1. Define the functional

   $$\psi : \mathbb{R} \ni t \to \psi(t) = \phi(x - ty) \in \mathbb{R}.$$

   Apply the fundamental theorem,

   $$\begin{aligned}\psi\left(1\right) - \psi\left(0\right) &= \int_0^1 \frac{d\psi}{dt} dt \\ \phi\left(x - y\right) - \phi\left(x\right) &= \int_0^1 \frac{d}{dt} \phi\left(x - ty\right) dt \\ \tau_y \phi - \phi\left(x\right) &= \int_0^1 -y \cdot \nabla \phi\left(x - ty\right) dt\end{aligned}$$

and apply $u \in \mathcal{D}'$. We have

$$\begin{aligned} u\left(\tau_y \phi - \phi\right) = u\left(\tau_y \phi\right) - u\left(\phi\right) &= u\left(\int_0^1 -y \cdot \nabla\left(\tau_{ty}\phi\right) dt\right) \\ &= -\int_0^1 u\left(\sum_{j=1}^d y_j \frac{\partial}{\partial x_j}\left(\tau_{ty}\phi\right)\right) dt \\ &= -\int_0^1 \sum_{j=1}^d y_j \left\langle u, \frac{\partial}{\partial x_j}\left(\tau_{ty}\phi\right)\right\rangle dt \\ &= \int_0^1 \sum_{j=1}^d y_j \left\langle \frac{\partial}{\partial x_j} u, \left(\tau_{ty}\phi\right)\right\rangle dt \\ &= \int_0^1 \sum_{j=1}^d y_j \frac{\partial u}{\partial x_j}\left(\tau_{ty}\phi\right) dt \end{aligned}$$

2. Let $f \in W^{1,1}_{loc}\left(\mathbb{R}^d\right)$. Then,

$$\Lambda_f\left(\tau_y \phi\right) - \Lambda_f \phi = \int_0^1 \sum_{j=1}^d y_j \frac{\partial f}{\partial x_j}\left(\tau_{ty}\phi\right) dt,$$

$$\tau_{-y}\Lambda_f \phi - \Lambda_f \phi = \int_{\mathbb{R}^d} \left(f\left(x+y\right) - f\left(x\right)\right)\phi\left(x\right) dx = \int_0^1 \sum_{j=1}^d y_j \int_{\mathbb{R}^d} \frac{\partial f}{\partial x_j}\left(x+ty\right)\phi\left(x\right) dx dt.$$

Exchanging order of integration, we have

$$\int_{\mathbb{R}^d} \left(f\left(x+y\right) - f\left(x\right)\right)\phi\left(x\right) dx = \int_{\mathbb{R}^d} \left\{\int_0^1 \sum_{j=1}^d y_j \frac{\partial f}{\partial x_j}\left(x+ty\right) dt\right\}\phi\left(x\right) dx.$$

Now, we see equality of the prefactors within the integrand. Finally, we need just recognize that $y \cdot \nabla f\left(x+ty\right)$ is exactly the expssion represented by the sum.

3. For $p \in (1, \infty]$, $L^p_{loc}\left(\mathbb{R}^d\right) \subset L^1_{loc}$. Thus, for $f \in W^{1,\infty}_{loc}\left(\mathbb{R}^d\right)$, $f \in W^{1,1}_{loc}$ and we can apply the result from the previous part. The previous part tells us that

$$\left|f\left(x + \left(y-x\right)\right) - f\left(x\right)\right| = \int_0^1 \left(y-x\right) \cdot \nabla f\left(x + t\left(y-x\right)\right) dt.$$

So then,

$$\begin{aligned}|f(y)-f(x)| &= \left|\int_0^1 (y-x)\cdot\nabla f(x+t(y-x))\,dt\right| \\ &\leqslant |y-x|\int_0^1 dt\|\nabla f\|_{L^\infty(B_R(0))} \\ &\leqslant C_{R,f}\,|y-x|\end{aligned}$$

Problem 7.8:

Counterexamples.

1. We know that for $f(x) = |x|^r$, that $f \in W^{1,p}(\Omega)$ if and only if $r > -(d-p)/p$. For $q > dp/(d-p)$, $f \notin L^q(\Omega)$ if, for some $B_r(0) \subset \Omega \cap B_1(0)$, $\|f\|_q^q \geq \int_{B_r(0)} |x|^{rq}dx$, that is, if $rq \leq -1$. Thus, we can find a counterexample for $r \in (-(d-p)/p, -(d-p)/dp]$.

2. The previous example is bot bounded and shows there is no embedding $W^{1,p}(\Omega) \hookrightarrow C_B^0$. However, from the Sobolev embedding theorem, if Ω is bounded and $mp > d$, then $W_0^{m,p}(\Omega) \hookrightarrow C_B^0(\Omega)$. In the case of the H^s spaces, though, we have $p = 2$ and thus $m > 1/2$, so $W_0^{s,2} \hookrightarrow C_B^0$ for $s > 1/2$. This implies that $\delta \in H^s$ for $s < -1/2$.

3. Notice immediately that f is not bounded at the origin. However,

$$\left|\log\log\left(\frac{4R}{|x|}\right)\right|^p \leqslant \left(4R|x|^{-1/2p}\right)^p$$

for every $x \in B_r(0)$. So then, $f \in L^p(\Omega)$. For the derivative, notice that

$$Df(x) = \frac{-x}{|x^2|\log(4R/|x|)}.$$

Define a similar sequence

$$Df_n = \begin{cases} Df(x) & x \in \Omega \setminus B_n \\ 0 & \text{otherwise}\end{cases}$$

and observe that Df_n converges monotonically to Df in $L^p(\Omega)$. Computation of the norm is done in spherical coordinates,

$$\|Df_n\|_p^p = \int_{\Omega\setminus B_n}\left|\frac{-x}{|x^2|\log(4R/|x|)}\right|^p dx = \int_{\Omega\setminus B_n}\left|\frac{1}{|x|\log(4R/|x|)}\right|^p dx \sim \int_{\varepsilon_n}^{R}\frac{1}{\rho\log(4R/\rho)}d\rho.$$

As $\rho \to 0$, the integrand is zero and is bounded across the rest of the domain. Thus, $f \in W^{1,p}$ as requested.

4. The distributional derivative of absolute value is

$$Du = \begin{cases} -1 & x < 0 \\ a & x = 0 \\ 1 & x > 0 \end{cases}$$

for any choice of $a \in \mathbb{R}$. Thus, $Du \in L^{\infty}(-1,1)$ and $u \in W^{1,\infty}(-1,1)$. Assume now to the contrary that there exists a sequence of smooth functions $f_n \in C^{\infty}$, $f_n \to u$ in $W^{1,\infty}(-1,1)$. If this is the case, then for the derivative, we have $Df_n \to Du$ in L^{∞}. Since f_n are smooth, then so are Df_n. For $\varepsilon = 1/2$, this implies in particular that $Df_n(x) \in (-3/2, -1/2)$ for $x < 0$ and $Df_n \in (1/2, 3/2)$ for $x > 0$. The contradiction is that the intermediate value theorem implies that there exists an interval such that $Df_n \in [-1/2, 1/2]$ in order to connect these two regions. So, $C^{\infty}(-1,1) \cap W^{1,\infty}(-1,1)$ is not dense in $W^{1,\infty}(-1,1)$.

Problem 7.11:

We have

$$\begin{aligned}
\|f\|_{H^s}^2 &= \int_{\mathbb{R}^d} \left(1+|\xi|^2\right)^s \left|\hat{f}(\xi)\right|^2 d\xi \\
&= \int_{\mathbb{R}^d} \left(1+|\xi|^2\right)^s \left|\hat{f}(\xi)\right|^{2s} \left|\hat{f}(\xi)\right|^{2(s-1)} d\xi \\
&= \int_{\mathbb{R}^d} \left\{\left(1+|\xi|^2\right)^{1/2} \left|\hat{f}(\xi)\right|\right\}^{2s} \left|\hat{f}(\xi)\right|^{2(s-1)} d\xi \\
&\leqslant \left\|\left(1+|\xi|^2\right)^{1/2} \left|\hat{f}(\xi)\right|\right\|^{2s}_{L^2\left(\mathbb{R}^d\right)} \left\|\left|\hat{f}\right|^{2(s-1)}\right\|_{L^2\left(\mathbb{R}^d\right)} \\
&\leqslant \left\|\left(1+|\xi|^2\right)^{1/2} \left|\hat{f}(\xi)\right|\right\|^{2s}_{L^2\left(\mathbb{R}^d\right)} \left\|\left|\hat{f}\right|\right\|^{2(1-s)}_{L^2\left(\mathbb{R}^d\right)}
\end{aligned}$$

and hence,

$$\|f\|_{H^s} \leqslant \left\|\left(1+|\xi|^2\right)^{1/2} \left|\hat{f}(\xi)\right|\right\|^{s}_{L^2\left(\mathbb{R}^d\right)} \left\|\left|\hat{f}\right|\right\|^{(1-s)}_{L^2\left(\mathbb{R}^d\right)} \Rightarrow \|f\|_{H^s} \leqslant \|f\|_{H^1}^s \|f\|_{L^2}^{1-s}$$