Homework 9

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Unless otherwise stated, denote by $(\cdot,\cdot)$ as the L^2 inner product, and $\|\cdot\|$ as $\|\cdot\|_{L^2(\Omega)}$.

Exercise (19). *Let $\Omega \subset \mathbb{R}^4$ be a bounded domain with smooth boundary, $f \in L^2(\Omega)$. Consider the BVP*

$$-\nabla^2 u + u^3 = f$$

with $u = 0$ on $\partial\Omega$. We attempt to solve iteratively from $u_0 = 0$ by computing for each $n = 1, 2, \ldots$ the solutions to the linear BVP

$$-\nabla^2 u_n + u_{n-1}^2 u_n = f, \qquad u_n = 0 \text{ on } \partial\Omega.$$

1. *Find appropraite variational problems for the linear BVP's and show that they are well defined in $H_0^1\left(\mathbb{R}^4\right)$ provided that $u_{n-1} \in H_0^1\left(\mathbb{R}^4\right)$. (Hint: Sobolev Imbedding Theorem.)*
2. *Show that there is a unique solution $u_n \in H_0^1\left(\mathbb{R}^4\right)$ assuming that $u_{n-1} \in H_0^1(\mathbb{R}^4)$. Moreover, find a bound for the norm of u_n.*
3. *Show that the nonlinear BVP has a weak solution. Extract a subsequence of u_n that converges weakly to some u and show that u satisfies the weak form of the nonlinear BVP.*

Proof. 1. Assume that $u_{n-1} \in H_0^1\left(\mathbb{R}^4\right)$ is given. Multiply by a test function and integrate by parts the first term,

$$\begin{aligned} -\left(\nabla^2 u_n, v\right) + \left(u_{n-1}^2 u_n, v\right) &= (f, v) \\ (\nabla u_n, \nabla v) - \langle \nabla u_n \cdot n, v\rangle + \left(u_{n-1}^2 u_n, v\right) &= (f, v) \end{aligned}$$

For $v \in H_0^1(\Omega)$, we have

$$B(u_n, v) := (\nabla u_n, \nabla v) + \left(u_{n-1}^2 u_n, v\right) = (f, v) =: F(v).$$

The VP reads

$$\begin{cases} u_n \in H_0^1(\Omega), \\ (\nabla u_n, \nabla v) + \left(u_{n-1}^2 u_n v\right) = (f, v) \\ \quad \text{for every } v \in H_0^1(\Omega). \end{cases}$$

We need to check that the second integral is well-defined. We recall that for the zero spaces, we simply extend by zero to the whole space with the equality $\|f\|_{H_0^1(\Omega)} = \|f\|_{H_0^1(\mathbb{R}^4)}$.

Let $p = 2$, $j = 0$, $m = 1$, and $d = 4$. The Sobolev Embedding Theorem tells us that $H_0^1(\Omega) \equiv W_0^{1,2}(\Omega) \hookrightarrow W^{0,q}(\Omega) \equiv L^q(\Omega)$ for all finite $q \le dp/(d - mp) = 4$. We will use exactly $q = 4$ for the estimate

$$\begin{aligned} \left|\left(u_{n-1}^2 u_n, v\right)\right| &\le \|u_{n-1}\|_{L^4(\Omega)}^2 \|u_n\|_{L^4(\Omega)} \|v\|_{L^4(\Omega)} && \text{Hölder's} \\ &\le C\|u_{n-1}\|_{H_0^1(\mathbb{R}^4)}^2 \|u_n\|_{H_0^1(\Omega)} \|v\|_{H_0^1(\Omega)} && \text{Sobolev Emb.} \\ &= C\|u_{n-1}\|_{H_0^1(\mathbb{R}^4)}^2 \|u_n\|_{H_0^1(\mathbb{R}^4)} \|v\|_{H_0^1(\mathbb{R}^4)} && \text{Bd. Ext. Op.} \end{aligned}$$

So, the problem is well-defined in the space $H^1(\mathbb{R}^4)$ in the sense that all terms are well-defined, and the solution obtained from solving the VP which lives presently in $H_0^1(\Omega)$ can be easily extended to $H^1(\mathbb{R}^4)$.

2. We have a symmetric functional setting, so Lax-Milgram is the tool. The right hand side is trivially continuous by Hölder's Inequality given $f \in L^2(\Omega)$. The left hand side is also continuous,

$$\begin{aligned} B(u_n, v) &= (\nabla u_n, \nabla v) + \left(u_{n-1}^2 u_n, v\right) \\ &\leq \|\nabla u_n\|\|\nabla v\| + C\|u_{n-1}\|_{H_0^1(\mathbb{R}^4)}^2 \|u_n\|_{H_0^1(\mathbb{R}^4)} \|v\|_{H_0^1(\mathbb{R}^4)} \\ &\leq \left(1 + C\|u_{n-1}\|_{H_0^1(\mathbb{R}^4)}^2\right) \|u_n\|_{H_0^1(\Omega)} \|v\|_{H_0^1(\Omega)}. \end{aligned}$$

B is also coercive,

$$\begin{aligned} B(v, v) &= \|\nabla v\|^2 + \int_\Omega u_{n-1}^2 v^2 \\ &\geq \|\nabla v\|^2 \\ &\geq C_P^{-2} \|v\|_{H_0^1(\Omega)}^2. \end{aligned}$$

By Lax-Milgram, there exists unique $u_n \in H_0^1(\Omega)$ which we can extend to $u_n \in H_0^1(\mathbb{R}^4)$. Moreover,

$$\|u_n\|_{H_0^1(\Omega)} = \|u_n\|_{H_0^1(\mathbb{R}^4)} \leq C_P^2 \|f\|.$$

3. Notice that u_n has just been shown to be a bounded sequence in the Hilbert space $H_0^1(\Omega)$. Therefore, there is a subsequence $u_{n_k} \rightharpoonup u$ in $H_0^1(\Omega)$, furthermore, $u_{n_k} \to u$ in $L^2(\Omega)$ and $\nabla u_{n_k} \rightharpoonup \nabla u$ in $L^2(\Omega)^4$. We argue that u is indeed a weak solution to the nonlinear PDE by arguing that $B(u_n - u, v) \to 0$. For every test function v, $(\cdot, \nabla v)$ is a linear functional on $L^2(\Omega)^4$, so $(\nabla u_{n_k}, \nabla v) \to (\nabla u, \nabla v)$. Additionally, sequence $u_{n_k-1}^2$ is bounded, and $u_{n_k} - u \to 0$, so we conclude $u_{n_k-1}^2(u_{n_k} - u) \to 0$. (Product of a bounded sequence and a sequence converging to zero converges to zero.) That is, $B(u_n - u, v) \to 0$, so u satisfies the weak form of the nonlinear PDE.

$\square$

Exercise (21). *$\Omega \subset \mathbb{R}^d$ bounded domain with Lipschitz $\partial\Omega$, define*

$$H\left(div, \Omega\right) = \left\{ v \in (L^2(\Omega))^d : \nabla \cdot v \in L^2(\Omega) \right\}.$$

1. *Show that $H\left(div, \Omega\right)$ is a Hilbert space with inner product*

$$(u, v)_{H(div,\Omega)} = (u, v) + (\nabla \cdot u, \nabla \cdot v).$$

2. *The trace Theorem does not imply that $\partial_\nu v = v \cdot \nu$ exists on $\partial\Omega$. Nevertheless, show that $\partial_\nu : H\left(div, \Omega\right) \to H^{-1/2}\left(\partial\Omega\right) = \left(H^{1/2}(\partial\Omega)\right)^*$ is a well defined bounded linear operator in the sense that*

$$\int_{\partial\Omega} v \cdot \nu \phi d\sigma(x) = \int_{\Omega} \nabla \cdot v \phi dx + \int_{\Omega} v \cdot \nabla \phi dx.$$

3. *Prove the following inf-sup condition: there exists $\gamma > 0$ such that*

$$\inf_{w \in L^2} \sup_{v \in H(div,\Omega)} \frac{(w, \nabla \cdot v)}{\|w\| \|v\|_{H(div,\Omega)}} \geq \gamma > 0.$$

Hint solve $\Delta\varphi = w$ in $H_0^1\left(\Omega\right)$ and consider $v = \nabla\varphi$.

Proof. 1. $H\left(\text{div}, \Omega\right) \subset L^2\left(\Omega\right)^d$ is clearly a linear subspace. What remains to show is that $H\left(\text{div}, \Omega\right)$ is closed with respect to $\|\cdot\|_{H(\text{div},\Omega)}$. Let u_n be a Cauchy sequence in $H\left(\text{div}, \Omega\right)$ with $u_n \to u$ in $L^2\left(\Omega\right)^d$. Convergence in norm? div is continuous in this norm?

2.

3. First consider $\nabla^2\varphi = w$. The VP is

$$\begin{cases} \varphi \in H_0^1\left(\Omega\right) \\ \left(\nabla\varphi, \nabla v\right) = -\left(w, v\right) \\ \text{for every } v \in H_0^1\left(\Omega\right). \end{cases}$$

Since $w \in L^2\left(\Omega\right)$, the RHS is continuous. The bilinear form is continuous with constant one and coercive by the Poincaré inequality. Therefore, the problem admits a unique solution in $H_0^1(\Omega)$. Moreover, by elliptic regularity, the solution $\varphi \in H^2\left(\Omega\right)$ and $\|\varphi\|_{H^2(\Omega)} \leq C\|w\|$. Now, let us consider $v = \nabla\varphi \implies \nabla \cdot v = \nabla \cdot \nabla\varphi = \Delta\varphi = w$. Then

$$\begin{aligned}
\frac{(w, \nabla \cdot (\nabla\varphi))}{\|w\| \|\nabla\varphi\|_{H(\text{div},\Omega)}} &= \frac{\|w\|^2}{\|w\| \|\nabla\varphi\|_{H(\text{div},\Omega)}} \\
&= \frac{\|w\|}{\|\nabla\varphi\|_{H(\text{div},\Omega)}} \\
&= \frac{\|w\|}{\sqrt{\|\nabla\varphi\|^2 + \|\nabla \cdot \nabla\varphi\|^2}} \\
&\geq \frac{\|w\|}{\|\varphi\|_{H^2(\Omega)}} \\
&\geq \frac{1}{C} > 0,
\end{aligned}$$

which implies the result.

□

Exercise (25). *Consider the finite element method.*

1. *Modify the method to account for nonhomogeneous Neumann conditions.*
2. *Modify the method to account for nonhomogeneous Dirichlet conditions.*

Proof. 1. Compatibilty condition:

2. Consider the nonhomogeneous Dirichlet boundary conditions

$$\begin{cases} -u'' = f & x \in (0,1) \\ u(0) = u_0 \\ u(1) = u_1. \end{cases}$$

Define the lift $u_D(x) = u_0 + (u_1 - u_0)x$, and write $u = w + u_D$.

□

Exercise (27). *Suppose $u \in H^1(\Omega)$ where $\Omega \subset \mathbb{R}^d$ is bounded connected. Recall the H^1 seminorm is $|u|_{H^1} = \left\{\sum_{|\alpha|=1} \|D^\alpha u\|^2\right\}^{1/2}$.*

1. *Show that there is a constant C_Ω such that*
$$\inf_{c\in\mathbb{R}} \|u - c\| \le C_\Omega |u|_{H^1}.$$
2. *Let $\Omega = (0,h)^d$ for $h > 0$. Show that there is a constant C independent of h and u such that*
$$\inf_{c\in\mathbb{R}} \|u - c\| \le Ch|u|_{H^1}.$$
Change var. to integrate over $(0,1)^d$ and then use previous.
3. *Let $\Omega = (0,1)^d$ and let P be the set of piecewise discontinuous constants over the grid of spacing $h = 1/N$ for some positive integer N. Show that there is a constant C independent of h, u, such that*
$$\inf_{p\in P} \|u - p\| \le Ch|u|_{H^1}.$$

Proof. 1. We always have
$$\inf_{c\in\mathbb{R}} \|u - c\| \le \|u - d\|,$$
so it is sufficient to demonstrate the result for a particular $d \in \mathbb{R}$. Let $d = |\Omega|^{-1}\int_\Omega u$. Then, $u - d$ has average zero, and the Poincare Inequality implies
$$\exists C_P > 0 : \|u - d\| \le C_P\|\nabla u\| = C_P|u|_{H^1(\Omega)}.$$

2. Use scaling arguments. Let $\xi \in \hat{\Omega} = (0,1)^d$ be the master element coordinates. Notice $x_i = h\xi_i$, $dx = h^d d\xi$. Then
$$\begin{aligned}
\|u - c\|^2_{L^2(\Omega)} &= \int_\Omega (u(x) - c)^2 dx \\
&= \int_{\hat{\Omega}} ((u\circ h)(\xi) - c)^2 h^d d\xi && (x = h\xi) \\
&= h^d \|u\circ h - c\|^2_{L^2(\hat{\Omega})} \\
&\le h^d C^2_{\hat{\Omega}} |u\circ h|^2_{H^1(\hat{\Omega})} && \text{(by previous)} \\
&= h^d C^2_{\hat{\Omega}} \|\hat{\nabla}(u\circ h)\|^2_{L^2(\hat{\Omega})^d} \\
&= h^d C^2_{\hat{\Omega}} \int_{\hat{\Omega}} (\hat{\nabla}(u\circ h)(\xi))^2 d\xi \\
&= C^2_{\hat{\Omega}} \int_\Omega h^2 (\nabla u)^2 dx \\
&= h^2 C^2_{\hat{\Omega}} |u|^2_{H^1(\Omega)}.
\end{aligned}$$
Finally, take infimum on both sides with respect to c.

3. Idea: consider each element separately, and apply the previous result. In 1D, for example, we have
$$\|u-p\|^2 = \int_0^1 (u(x)-p)^2 dx = \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} (u(x)-p)^2 dx.$$
On each interval, we can apply the previous result. Summing a finite number of times, we just increase the constant. In d dimensions, with each dimension split into N_i elements, we have $N_1 \times N_2 \times \cdots \times N_d$ small domains. Then we have
$$\begin{aligned}
\|u-p\|^2 &= \int_\Omega (u(x)-p)^2 dx \\
&= \sum_{i=1}^{N_1\times\cdots\times N_d} \int_{\Omega_i} (u(x)-p)^2 dx \\
&\le \sum_{i=1}^{N_1\times\cdots\times N_d} C^2 h^2 |u|^2_{H^1(\Omega_i)} && \text{(previous result on } \Omega_i\text{)} \\
&= C^2 h^2 |u|^2_{H^1(\Omega)}.
\end{aligned}$$
We remark that we have slightly glanced over the translation transformation required to shift an arbitrary element Ω_i to align with $(0,h)^d$. However, in the case where $\Omega = (0,1)^d$, this is merely a translation which the integral does not care about.

□

Exercise (29). *Consider the problem* $\Omega = (0,1) \subset \mathbb{R}$, $f \in L^2(0,1)$,

$$\begin{cases} -u'' = f & x \in \Omega \\ u(0) = u(1) = 0. \end{cases}$$

1. *Find the Green's Function.*
2. *Instead impose Neumann BC's, and find the Green's function.*
   Recall we now require $-\partial^2/\partial x^2 G(x,y) = \delta_y(x) - 1$.

Proof. 1.

2.

□