Homework 8

Jonathan Zhang EID: jdz357

Exercise (2). *Suppose that the hypotheses of the Generalized Lax-Milgram theorem are satisfied. Suppose that $x_{0,1}$ and $x_{0,2}$ in $\mathcal{X}$ are such that the ssets $X + x_{0,1} = X + x_{0,2}$. Prove that the solutions $u_1 \in X + x_{0,1}$ and $u_2 \in X + x_{0,2}$ of the abstract variational problem agree. What does this say about Dirichlet BVP?*

Exercise (4). *BVP for $u(x, y) : \mathbb{R}^2 \to \mathbb{R}$. Write as a variational problem and show there exists a unique solution. Carefully define function spaces and identify where f must lie.*

$$\begin{cases} -u_{xx} + e^y u = f & (x, y) \in (0,1)^2 \\ u\left(0, y\right) = 0,\ u\left(1, y\right) = \cos y & y \in (0,1) \end{cases}$$

Exercise (9). *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz boundary, $f \in L^2\left(\Omega\right)$, $\alpha > 0$. Consider the Robin problem*

$$\begin{cases} -\Delta u + u = f & x \in \Omega \\ \dfrac{\partial u}{\partial n} + \alpha u = 0 & x \in \partial\Omega \end{cases}$$

1. *Formulate a variational principle $B(u, v) = (f, v)$ for $v \in H^1\left(\Omega\right)$.*
2. *Show that this problem has a unique weak solution.*

Exercise (13). *Suppose $\Omega \subset \mathbb{R}^d$ is a bounded Lipschitz domain. Consider the Stokes problem for vector u and scalar p:*

$$\begin{cases} -\Delta u + \nabla p = f & x \in \Omega \\ \nabla \cdot u = 0 & x \in \Omega \\ u = 0 & x \in \partial\Omega \end{cases}$$

where the first equation holds for each coordinate. This problem is a saddle-point problem in which we minimize some energy subject to the constraint $\nabla \cdot u = 0$. However, if we work over the constrained space, we can handle this problem. Let

$$H = \left\{ v \in \left(H_0^1(\Omega)\right)^d : \nabla \cdot u = 0 \right\}$$

1. *Show that H is a Hilbert space.*
2. *Determine an appropriate Sobolev space for f and formulate an appropriate VP for constrained Stokes.*
3. *Show that there is a unique solution to the VP.*

Exercise (14). *Let* $\mathcal{H} = H_0^1(\Omega) \times H^1(\Omega)$ *and consider the solution* $(u, v) \in \mathcal{H}$ *to the differential problem*

$$\begin{cases} -\Delta u = f + a(v-u) & x \in \Omega \\ v - \Delta v = g + a(u-v) & x \in \Omega \\ u = 0, \nabla v \cdot n = \gamma & x \in \partial\Omega \end{cases}$$

where $a \in L^\infty(\Omega)$.

1. *Develop an appropriate weak or variational form for the problem. In what Sobolev spaces should* f, g, γ *lie?*

2. *Prove that there exists a unique solution provided that* $a \geq 0$.