Homework 10

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Exercise (9.3). *Let $X = C([0,1])$ be the space of bounded continuous functions on $[0,1]$ and, for $u \in X$, define $F(u)(x) = \int_0^1 K(x,y)f(u(y))dy$ where $K : [0,1] \times [0,1] \to \mathbb{R}$ is continuous and f is a C^1 mapping of $\mathbb{R}$ into $\mathbb{R}$. Find the Fréchet derivative $DF(u)$ of F at $u \in X$. Is the map $u \mapsto DF(u)$ continuous?*

Exercise (9.5). *Set up and apply contraction mapping principle to show that the problem*

$$-u_{xx} + u - \varepsilon u^2 = f(x), \quad x \in \mathbb{R}$$

has a smooth bounded solution if $\varepsilon > 0$ is small enough, where $f(x) \in \mathcal{S}(\mathbb{R})$.

Exercise (9.7). *Suppose that F is defined on a Banach space X, that $x_0 = F(x_0)$ is a fixed point of F, $DF(x_0)$ exists, and that 1 is not in the spectrum of $DF(x_0)$. Prove that x_0 is an isolated fixed point.*

Exercise (9.8). *Consider the ODE*

$$u'(t) + u(t) = \cos(u(t))$$

posed as an IVP for $t > 0$ with $u(0) = u_0$.

1. *Use contraction mapping theorem to show that there is exactly one solution u corresponding to any given $u_0 \in \mathbb{R}$.*
2. *Prove that there is a number ξ such that $\lim_{t\to\infty} u(t) = \xi$ for any solution u, independent of the value of u_0.*

Exercise (9.10). *Consider the PDE*

$$u_t - u_{xxt} - \varepsilon u^3 = f, \quad x \in \mathbb{R}, t > 0$$

with $u(x,0) = g(x)$. Use the Fourier transform and a contraction mapping argument to show that there exists a solution for small enough ε, at least up to some time $T < \infty$. In what spaces should f and g lie?