Some Notes and Theorems, Part 2

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1 Preliminaries

2 Normed Linear Spaces and Banach Spaces

3 Hilbert Spaces

4 Spectral Theory and Compact Operators

5 Distributions

Theorem 5.1. *Here, we use the following notation:* $(M^\alpha f)(x) = x^\alpha f(x)$.
If $T \in \mathcal{S}'$ *and* $f \in \mathcal{S}$, *then* $T * f$ *is smooth and polynomially bounded.*

6 The Fourier Transform

We motivate our discission on the Fourier Transform by observing that it converts the task of solving a PDE into the task of solving an ODE, and that we can represent functions in terms of harmonic functions.

The Fourier Transform can be defined naturally for functions in $L^1\left(\mathbb{R}^d\right)$, where we will start. Then, we can extend it to functions in $L^2\left(\mathbb{R}^d\right)$, and we shall see that it maps L^2 onto itself. Then, we will discuss the Fourier transform in context of distributions.

6.1 $L^1\left(\mathbb{R}^d\right)$ Theory

For (a point) $\xi \in \mathbb{R}^d$, the function

$$\varphi_\xi(x) = e^{-ix\cdot\xi} = \cos(x\cdot\xi) - i\sin(x\cdot\xi)$$

for $x \in \mathbb{R}^d$ defines a wave in the direction ξ. Moreover, the period of φ_ξ in the j-th direction is $2\pi/\xi_j$. Consider the simplest nontrivial example, say when we are in $\mathbb{R}^2$ and we let $\xi = (1,1)$. Then, the function $\varphi_\xi(x) = e^{-i(x+y)} = \cos(x+y) - i\sin(x+y)$ defines a wave that propagates in the direction $(1,1)$. Look at the real part by itself, for example. It is clear that the period in the x direction is 2π, indeed as it should be. The same can be easily seen with the imaginary part. These harmonic functions have certain nice properties:

Proposition 6.1. *1.* $|\varphi_\xi| = 1$, *and* $\overline{\varphi_\xi} = \varphi_{-\xi}$ *for every* $\xi \in \mathbb{R}^d$,

2. $\varphi_\xi(x+y) = \varphi_\xi(x)\,\varphi_\xi(y)$ *for every* $x, y, \xi \in \mathbb{R}^d$,

3. $-\nabla^2\varphi_\xi = |\xi|^2\varphi_\xi$ *for every* $\xi \in \mathbb{R}^d$.

Proof. 1. We clearly have

$$|\varphi_\xi|^2 = \cos^2(x\cdot\xi) + \sin^2(x\cdot\xi) = 1,$$

and

□

6.2 $\mathcal{S}\left(\mathbb{R}^d\right)$ Theory

6.3 $L^2\left(\mathbb{R}^d\right)$ Theory

6.4 $\mathcal{S}'$ Theory

6.5 Applications

7 Sobolev Spaces

7.1 Definitions, Basic Properties

7.2 Extensions from Ω to $\mathbb{R}^d$

7.3 Sobolev Imbedding Theorem

7.4 Compactness

7.5 The Spaces H^s

7.6 Trace Theorem

7.7 The Spaces $W^{s,p}(\Omega)$

8 Boundary Value Problems

8.1 Second Order Elliptic PDEs

8.2 A Variational Problem and Minimization of Energy

8.3 Closed Range Theorem, Operators Bounded Below

8.4 The Lax-Milgram Theorem

8.5 Applications to Second Order Elliptic Equations

8.6 Galerkin Approximations

8.7 Green's Functions

9 Differential Calculus in Banach Spaces

10 Calculus of Variations