Homework 7

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Exercise (8.1). *If A is positive definite, show that its eigenvalues are positive. If A is symmetric and has positive eigenvalues, then A is positive definite.*

Proof. Suppose that A is positive definite. Let (λ, v) be an eigenpair for A. Then by the positive definiteness,

$$0 < (v, Av) = (v, \lambda v) = \lambda\,(v, v) = \lambda \|v\|^2.$$

Since this quantity is strictly positive, we must have $\lambda > 0$.

On the other side, assume now that A has all positive eigenvalues. Then, a admits a diagonalization $A = U\Lambda U^*$ for some $U^* = U^{-1}$ unitary matrix. Pick $x \neq 0$, and set $y = U^* x \neq 0$. Then

$$(x, Ax) = (x, U\Lambda U^* x) = (y, \Lambda y)\,.$$

Expand $y = \sum y_i e_i$, then

$$(y, \Lambda y) = \sum_i \overline{y_i} y_i \lambda_i > 0$$

since $\lambda_i > 0$ and $\overline{y_i} y_i = |y_i|^2 \geq 0$ since $y \neq 0$ by assumption. This shows that $(x, Ax) = (y, \Lambda y) > 0$ and a is positive definite.

□

Exercise (8.3).

Exercise (8.5).

Exercise (8.8). $\Omega \subset \mathbb{R}^d$ *bounded, connected, Lipschitz domain. Let* $V \subset \Omega$ *have positive measure. Let* $H = \left\{ u \in H^1(\Omega) : u|_V = 0 \right\}$.

1. *Why is* H *a Hilbert space?*
2. *Prove that there exists* $C > 0$ *such that*

$$\|u\| \leq C \|\nabla u\|$$

for every $u \in H$.

1. H is the null space of the restriction to V operator. Since restriction is continuous, then H is closed. As a closed subspace of Hilbert space $H^1(\Omega)$, then H is itself a Hilbert space.
2. Go by contradiction. Assume to the contrary that there exists a sequence $u_n \in H$ such that

$$\|u_n\| = 1 \qquad \text{and} \qquad \|\nabla u_n\| \to 0.$$

Now, from every bounded sequence in a Hilbert space, we can extract a weakly convergent subsequence $u_{n_k} \rightharpoonup u \in H$. Weak convergence of u_{n_k} implies weak convergence of $\nabla u_{n_k} \rightharpoonup \nabla u$ in L^2. Since the norm is continuous we pass to the limit and we have that $\nabla u = 0$, i.e. u is a constant. But if u vanishes on V, then the only possibility is that $u \equiv 0$.

On the other side, we know from the Rellich-Kondrachov Theorem that we have $u_{n_k} \to u$ in L^2. Convergence in L^2 also implies convergence in the norm, but since $\|u_n\| = 1$ that means $\|u\| = 1$, a contradiction.