Homework 6

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Exercise (7.10). *Suppose $\Omega \subset \mathbb{R}^d$ is bounded with Lipschitz boundary and $f_j \rightharpoonup f$ and $g_j \rightharpoonup g$ weakly in $H^1(\Omega)$. Show that, for a subsequence, $\nabla(f_j g_j) \to \nabla(fg)$ as a distribution. Find all p in $[1, \infty]$ such that the convergence can be taken weakly in $L^p(\Omega)$.*

Exercise (7.15). *Suppose that $w \in L^1(\mathbb{R})$, $w(x) > 0$, and w is even. Moreover, for $x > 0$, $w \in C^2[0, \infty)$, $w'(x) < 0$, and $w''(x) > 0$. Consider the equation*

$$w * u - u'' = f \in L^2(\mathbb{R}).$$

1. *Show that $\hat{w}(\xi) > 0$.*
2. *Fund a fundamental solution. Leave answer in terms of inverse Fourier transform.*
3. *Find the solution operator as a convolution operator.*
4. *Show that $u \in H^2(\mathbb{R})$.*

Exercise (7.16). *Let H be a Hilbert space and Z a closed linear subspace. Prove that H/Z has the inner product*

$$(\hat{x}, \hat{y})_{H/Z} = \left(P_Z^{\perp} x, P_Z^{\perp} y\right)_H$$

and that H/Z is isomorphic to $Z^{\perp}$.

Exercise (7.17). *Elliptic regularity theorey shows that if $\Omega \subset \mathbb{R}^d$ has a smooth boundary and $f \in H^s(\Omega)$, then $-\Delta u = f$ in Ω, $u = 0$ on $\partial\Omega$ has a unique solution $u \in H^{s+2}$. For what values of s will u be continuous? Can you be sure that a fundamental solution is continuous? Answer depends on d.*

Exercise (7.19). *Let $\Omega \subset \mathbb{R}^d$ be a domain with Lipschitz boundary. Let $w \in L^\infty(\Omega)$. Define*

$$H_w(\Omega) = \left\{ f \in L^2(\Omega) : \nabla(wf) \in \left(L^2(\Omega)\right)^d \right\}$$

1. *Give reasonable conditions on w so that $H_w(\Omega) = H^1(\Omega)$.*
2. *Prove that $H_w(\Omega)$ is a Hilbert space. What is the inner product?*
3. *Can you define the trace of f on $\partial\Omega$? First consider trace of wf.*
4. *Characterize $H_w(\mathbb{R})$ if $w(x)$ is the Heaviside function.*