Homework 2

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Problem 6.9:

Suppose that $f \in L^p(\mathbb{R}^d)$ for $p \in (1,2)$. Show that there exist $f_1 \in L^1(\mathbb{R}^d)$ and $f_2 \in L^2(\mathbb{R}^d)$ such that $f = f_1 + f_2$. Define $\hat{f} = \hat{f}_1 + \hat{f}_2$. Show that this definition is well defined, that is, independent of the choices f_1 and f_2.

Solution Write f as

$$f = f\chi_{[x:|f(x)<1]} + f\chi_{[x:|f(x)\geqslant 1]}$$

and define

$$f_2(x) = f\chi_{[x:|f(x)<1]}, \qquad f_1(x) = f\chi_{[x:|f(x)\geqslant 1]}.$$

Then,

$$\|f_1\|_1 = \int_{\mathbb{R}^d} |f(x)|\chi_{[x:|f(x)\geqslant 1]}dx = \int_{x:|f(x)|\geqslant 1} |f(x)| \leq \int_{x:|f(x)|\geqslant 1} |f(x)|^p \leq \|f\|_p^p.$$

$$\|f_2\|_2^2 = \int_{\mathbb{R}^d} |f(x)|^2\chi_{[x:|f(x)<1]}dx = \int_{x:|f(x)|<1} |f(x)|^2 \leq \int_{x:|f(x)|<1} |f(x)|^p \leq \|f\|_p^p.$$

Next, suppose that f admits two different decompositions, $f = f_1 + f_2 = g_1 + g_2$, where $f_i, g_i \in L^i(\mathbb{R}^d)$ for $i = 1, 2$. Then,

$$\begin{aligned}
\hat{f}_1 + \hat{f}_2 &= (2\pi)^{-d/2}\left(\int_{\mathbb{R}^d} f_1(x)e^{-ix\cdot\xi}dx + \int_{\mathbb{R}^d} f_2(x)e^{-ix\cdot\xi}dx\right) \\
&= (2\pi)^{-d/2}\int_{\mathbb{R}^d} \left(f_1(x) + f_2(x)\right)e^{-ix\cdot\xi}dx \\
&= (2\pi)^{-d/2}\int_{\mathbb{R}^d} \left(g_1(x) + g_2(x)\right)e^{-ix\cdot\xi}dx \\
&= \hat{g}_1 + \hat{g}_2.
\end{aligned}$$

Problem 6.11:

Let the ground field be $\mathbb{C}$ and define $T : L^2\left(\mathbb{R}^d\right) \to L^2$ by

$$(Tf)(x) = \int e^{-|x-y|^2/2} f(y)dy.$$

Use the Fourier transform to show that T is a positive, injective but not surjective operator.

Solution Define $\varphi(x) = e^{-|x|^2/2}$. Observe that T is nothing more than convolution, i.e.

$$(Tf)(x) = (\varphi * f)(x) = \int\limits_{\mathbb{R}^d} \varphi(x-y)\, f(y)\, dy = \int\limits_{\mathbb{R}^d} e^{-|x-y|^2/2} f(y)\, dy.$$

If we consider now the Fourier transform,

$$\widehat{(Tf)} = \widehat{(\varphi * f)} = (2\pi)^{d/2} \hat{\varphi} \hat{f}.$$

The Fourier transform preserves inner product, so

$$(Tf, f) = \left(\widehat{(Tf)}, \hat{f}\right) = \int\limits_{\mathbb{R}^d} (2\pi)^{d/2} \hat{\varphi}(\xi) \left\{\hat{f}(\xi)\right\}^2 d\xi \geqslant 0$$

Notice this is the case since $\varphi = \hat{\varphi} > 0$. So we conclude that T is positive.

show T is injective?

Problem 6.15:

Let $\varphi \in \mathcal{S}\left(\mathbb{R}^d\right)$, $\hat{\varphi}(0) = (2\pi)^{-d/2}$, and $\varphi_\varepsilon(x) = \varepsilon^{-d}\varphi(x/\varepsilon)$. Prove that $\varphi_\varepsilon \to \delta_0$ and $\hat{\varphi}_\varepsilon \to (2\pi)^{-d/2}$ as $\varepsilon \to 0^+$. In what sense do these convergences take place?

We claim that $\varphi_\varepsilon \to \delta_0$ in $\mathcal{S}'$.