Homework 12

Jonathan Zhang EID: jdz357

Exercise (10.2).

$$F(y) = \int_0^1 \left\{y^2 - yy'\right\} dx, \qquad y \in C^1([0,1]).$$

1. *Find all extremals.*
2. *If we require $y(0) = 0$, show by example that there is no minimum.*
3. *If we require that $y(0) = y(1) = 0$, show that the extremal is a minimum. (Hint: $yy' = \left(1/2y^2\right)'$).*

Exercise (10.4). *Minimize*

$$F(y) = \int_0^1 f(x, y(x), y'(x), y''(x)) dx$$

over the set of $y \in C^2([0,1])$ such that $y(0) = \alpha$, $y'(0) = \beta$, $y(1) = \gamma$, and $y'(1) = \delta$. That is, with $C_0^2([0,1]) = \left\{u \in C^2([0,1]) : u(0) = u'(0) = u(1) = u'(1) = 0\right\}$, and $y \in C_0^2 + p(x)$ where p is the cubic polynomial that matches the BC.

1. *Find a differential equation, similar to the EL equation, that must be satisfied by the minimum (if it exists).*
2. *Apply your equation to find the extremal(s) of*

   $$F(y) = \int_0^1 \left(y''(x)\right)^2 dx,$$

   where $y(0) = y'(0) = y'(1) = 0$, but $y(1) = 1$, and justify that each extremal is a (possibly non-strict) minimum.

Exercise (10.6). *Consider the functional*

$$\Phi(x, y, y') = \int_a^b F(x, y(x), y'(x)) dx.$$

1. *If $F \in C^2$ and $F = F(y, y')$ only, and if we assume that $y \in C^2$, prove that in this case the EL equations reduce to*

   $$\frac{d}{dx}\left(F - y'F_{y'}\right) = 0.$$

2. *Among all C^2 curves $y(x)$ joining the points $(0,1)$ and $(1, \cosh(1))$, find the one which generates the minimum area when rotated about the x axis. This area is*

   $$A = 2\pi \int_0^1 y\sqrt{1 + (y')^2} dx.$$

1. Direct computation.

$$\begin{aligned}\left(D_3Fy' - f\right)' &= D_3Fy'' + (D_3F)'y' - f' \\ &= D_3Fy'' + D_2Fy' - \left(D_2Fy' + D_3fy''\right) \\ &= 0\end{aligned}$$

Exercise (10.8). *Consider the problem of finding a C^1 curve that minimizes*

$$\int_0^1 (y'(t))^2 dt$$

subject to the conditions $y(0) = y(1) = 0$ and

$$\int_0^1 y^2 = 1.$$

1. *Remove the integral constraint by incorporating a Lagrange multiplier, and find the EL equations.*
2. *Find all extremals.*
3. *Find the solution.*
4. *Use your result to find the best constant C in the inequality*

$$\|y\| \le C\|y'\|$$

for functions that satisfy $y(0) = y(1) = 0$.

1. We form the Lagrangian,

$$\mathscr{L}(y,\lambda) = \int_0^1 \left(y'(t)\right)^2 dt - \lambda\left[\int_0^1 \left\{y(t)^2 - 1\right\} dt\right] = \int_0^1 \left\{\left(y'(t)\right)^2 - \lambda y(t)^2 + \lambda\right\} dt$$

which we seek to minimize. To that end, define

$$f(t,y,y') = \left(y'(t)\right)^2 - \lambda y(t)^2 + \lambda.$$

The EL equations are

$$-2\lambda y = D_2 f = \frac{d}{dx} D_3 f = \frac{d}{dx}\left(2y'\right) = 2y''.$$

Hence, the resulting ODE is given simply by $-y'' = \lambda y$, accompanied with the BC $y(0) = y(1) = 0$.

2. Extremals can be found by solving the eigenvalue problem. To that end, we note that the operator $-\frac{d^2}{dx^2}$ accompanied with homogeneous boundary conditions is self-adjoint. Therefore, its eigenvalues are non-negative and real. We immediately see that zero cannot be an eigenvalue, for if $\lambda = 0$, then $y'' = 0$, so y is linear, and after incorporating BC, we see that $y \equiv 0$, which does not satisfy the original integral constraint. The general solution is

$$y(t) = A\cos\left(\sqrt{\lambda}t\right) + B\sin\left(\sqrt{\lambda}t\right).$$

BC imply $A = 0$ and

$$y(1) = B\sin\left(\sqrt{\lambda}\right) = 0 \implies \lambda = n^2\pi^2, \qquad n = 1, 2, \ldots.$$

The coefficient B is determined by satisfying the integral condition.

$$1 = \int_0^1 y^2 = B^2 \int_0^1 \sin^2(n\pi t)\, dt \implies B = \pm\sqrt{2}.$$

3. Make the observation

$$y(t) = \sin(n\pi t) \Rightarrow y'(t) = n\pi\cos(n\pi t).$$

So, no matter which n we pick, we will always be integrating $\cos^2(n\pi t)$, so the minimizer is the extremal with the smallest admissible n, that is, $n = 1$. So, the solution is $y(t) = \sqrt{2}\sin(\pi t)$.

4. The best constant is the one attaining the bound, that is, the smallest eigenvalue. This means $C = \pi^2$.

Exercise (10.9). *Find the C^2 curve that minimizes the functional*

$$\int_0^1 \left\{y(t)^2 + y'(t)^2\right\} dt$$

subject to $y(0) = 0$, $y(1) = 1$ *and the constraint*

$$\int_0^1 y = 0$$