Some Notes and Theorems, Part 2

Jonathan Zhang

1 Preliminaries

2 Normed Linear Spaces and Banach Spaces

3 Hilbert Spaces

4 Spectral Theory and Compact Operators

5 Distributions

Theorem 5.1. *Here, we use the following notation:* $(M^\alpha f)(x) = x^\alpha f(x)$.
If $T \in \mathcal{S}'$ *and* $f \in \mathcal{S}$, *then* $T * f$ *is smooth and polynomially bounded.*

6 The Fourier Transform

We motivate our discission on the Fourier Transform by observing that it converts the task of solving a PDE into the task of solving an ODE, and that we can represent functions in terms of harmonic functions.

The Fourier Transform can be defined naturally for functions in $L^1\left(\mathbb{R}^d\right)$, where we will start. Then, we can extend it to functions in $L^2\left(\mathbb{R}^d\right)$, and we shall see that it maps L^2 onto itself. Then, we will discuss the Fourier transform in context of distributions.

6.1 $L^1\left(\mathbb{R}^d\right)$ Theory

For (a point) $\xi \in \mathbb{R}^d$, the function

$$\varphi_\xi : x \mapsto e^{-ix\cdot\xi} = \cos\left(x\cdot\xi\right) - i\sin\left(x\cdot\xi\right)$$

for $x \in \mathbb{R}^d$ defines a wave in the direction ξ. Moreover, the period of φ_ξ in the j-th direction is $2\pi/\xi_j$. Consider the simplest nontrivial example, say when we are in $\mathbb{R}^2$ and we let $\xi = (1, 1)$. Then, the function $\varphi_\xi(x) = e^{-i(x+y)} = \cos\left(x+y\right) - i\sin\left(x+y\right)$ defines a wave that propagates in the direction $(1, 1)$. Look at the real part by itself, for example. It is clear that the period in the x direction is 2π, indeed as it should be. The same can be easily seen with the imaginary part. These harmonic functions have certain nice properties:

Proposition 6.1. *1. $|\varphi_\xi| = 1$, and $\overline{\varphi_\xi} = \varphi_{-\xi}$ for every $\xi \in \mathbb{R}^d$,*

2. $\varphi_\xi\left(x+y\right) = \varphi_\xi\left(x\right)\varphi_\xi\left(y\right)$ for every $x, y, \xi \in \mathbb{R}^d$,

3. $-\nabla^2\varphi_\xi = |\xi|^2\varphi_\xi$ for every $\xi \in \mathbb{R}^d$.

Proof. 1. We clearly have

$$|\varphi_\xi|^2 = \cos^2\left(x\cdot\xi\right) + \sin^2\left(x\cdot\xi\right) = 1,$$

and

$$\overline{\varphi_\xi} = \cos\left(x\cdot\xi\right) + i\sin\left(x\cdot\xi\right) = \cos\left(-x\cdot\xi\right) - i\sin\left(-x\cdot\xi\right) = \varphi_{-\xi}.$$

2.

$$\varphi_\xi\left(x+y\right) = e^{-i(x+y)\cdot\xi} = e^{-ix\cdot\xi - iy\cdot\xi} = e^{-ix\cdot\xi}e^{-iy\cdot\xi} = \varphi_\xi\left(x\right)\varphi_\xi\left(y\right).$$

3.

$$-\nabla^2\varphi_\xi = -\left(-i\xi\right)^2 e^{-ix\cdot\xi} = |\xi|^2\varphi_\xi.$$

We remark that this property demonstrates that functions φ_ξ are eigenfunctions of the Laplacian operator with eigenvalues $-|\xi|^2$.

$\square$

We now define the Fourier Transform:

Definition 6.1. *For $f \in L^1\left(\mathbb{R}^d\right)$, the Fourier Transform of f is*

$$(\mathcal{F}f)(\xi) = \hat{f}(\xi) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} f(x) e^{-ix\cdot\xi} dx.$$

At this moment, it is not immediately clear what function spaces the Fourier Transform connects. However, for $f \in L^1$, the Fourier transform is well defined since

$$\left| f(x) e^{-ix\cdot\xi} \right| = |f(x)|.$$

What we mean when we say well defined is that the integral defining $\hat{f}$ converges whenever $f \in L^1$. Indeed, $f \in L^1$ iff $\int |f(x)| dx < \infty$, but as we just showed, $\int |f(x)| dx = \int \left| f(x) e^{-ix\cdot\xi} \right| dx$, so the integral converges.

Additionally, we remark that it is possible (and equivalent) to define the Fourier transform in different ways, but that the end results are very similar and it is merely a matter of preference.

We preface some additional discussion with a motivating example to show that we need to take care when talking about function spaces:

Example. *On $\mathbb{R}$, let $f : x \mapsto \chi_{[-1/2,1/2]}(x)$, the characteristic function on the interval $[-1/2, 1/2]$. Obviously, f is an $L^1(\mathbb{R})$ function. Now for every $\xi \neq 0$,*

$$\begin{aligned} \hat{f}(\xi) &= (2\pi)^{-1/2} \int_{\mathbb{R}} f(x) e^{-ix\cdot\xi} dx \\ &= (2\pi)^{-1/2} \int_{-1/2}^{1/2} e^{-ix\cdot\xi} dx \\ &= (2\pi)^{-1/2} \frac{2}{k} \sin\left(\frac{k}{2}\right). \end{aligned}$$

This is fine, but notice that $\hat{f}$ is no longer integrable! So, just because $f \in L^1$, we have no reason to believe that $\hat{f}$ is also in L^1.

We have now the following proposition:

Proposition 6.2. *The Fourier Transform viewed in the spaces*

$$\mathcal{F} : L^1\left(\mathbb{R}^d\right) \to L^\infty\left(\mathbb{R}^d\right)$$

is a bounded linear map, and

$$\left\| \hat{f} \right\|_{L^\infty\left(\mathbb{R}^d\right)} \leq (2\pi)^{-d/2} \|f\|_{L^1\left(\mathbb{R}^d\right)}.$$

Proof. Clearly, $\mathcal{F}$ is linear since the integral is a linear operator. Furthermore,

$$\left\|\hat{f}\right\|_{L^\infty} = \max_{\xi\in\mathbb{R}^d} (2\pi)^{-d/2} \left| \int\limits_{\mathbb{R}^d} f(x)\, e^{-ix\cdot\xi} dx \right| \leqslant (2\pi)^{-d/2} \int\limits_{\mathbb{R}^d} \left| f(x)\, e^{-ix\cdot\xi} \right| dx = (2\pi)^{-d/2} \|f\|_{L^1}$$

□

Example. *Consider the characteristic function on the interval* $[-1,1]^d$,

$$f : x \mapsto \chi_{[-1,1]^d}(x)$$

Certainly, f *is an* L^1 *function. Its Fourier Transform is*

$$\begin{aligned}
\hat{f}(\xi) &= (2\pi)^{-d/2} \int\limits_{\mathbb{R}^d} f(x)\, e^{-ix\cdot\xi} dx \\
&= (2\pi)^{-d/2} \int\limits_{-1}^{1} \int\limits_{-1}^{1} \dots \int\limits_{-1}^{1} e^{-ix\cdot\xi} dx_1 dx_2 \dots dx_d \\
&= (2\pi)^{-d/2} \prod_{j=1}^{d} \int\limits_{-1}^{1} e^{-ix_j\xi_j} dx_j \\
&= (2\pi)^{-d/2} \prod_{j=1}^{d} \frac{2\sin\xi_j}{\xi_j}
\end{aligned}$$

We can easily verify that $\hat{f}$ *is bounded, since each term is bounded by* $2/\xi_j$ *and so the finite product is bounded.*

Additional properties include

Proposition 6.3. *For* $f \in L^1\left(\mathbb{R}^d\right)$ *and a translation operator* $\mathcal{T}_y$*, i.e.* $\mathcal{T}_y f(x) = f(x-y)$*,*

1. $\widehat{(\mathcal{T}_y f)}(\xi) = e^{-iy\cdot\xi} \hat{f}(\xi)$ *for every* $y \in \mathbb{R}^d$*, that is, the Fourier transform takes translations into multiplication.*
2. $\widehat{(e^{ix\cdot y} f)}(\xi) = \mathcal{T}_y \hat{f}(\xi)$ *for every* $y \in \mathbb{R}^d$*, that is, the Fourier transform takes multiplication by a harmonic function to translation.*
3. *For any* $r > 0$, $\widehat{f(rx)}(\xi) = r^{-d} \hat{f}(r^{-1}\xi)$*, that is, the Fourier transform takes scaling into premultplying and scaling.*
4. $\widehat{\overline{f}}(\xi) = \overline{\hat{f}(-\xi)}$*, that is, the Fourier transform of complex conjugate equals complex conjugate of the Fourier transform, reflected.*

Proof. 1.

$$
\begin{aligned}
\mathcal{F}f(x-y)(\xi) &= (2\pi)^{-d/2}\int\limits_{\mathbb{R}^d} f(x-y)\,e^{-ix\cdot\xi}dx \\
&= (2\pi)^{-d/2}\int\limits_{\mathbb{R}^d} f(u)\,e^{-i(u+y)\cdot\xi}du \\
&= e^{-iy\cdot\xi}\mathcal{F}f(x)(\xi)
\end{aligned}
$$

2.

$$
\begin{aligned}
\mathcal{F}\left(e^{ix\cdot y}f(x)\right)(\xi) &= (2\pi)^{-d/2}\int\limits_{\mathbb{R}^d} e^{ix\cdot y}f(x)\,e^{-ix\cdot\xi}dx \\
&= (2\pi)^{-d/2}\int\limits_{\mathbb{R}^d} f(x)\,e^{-ix\cdot\xi+ix\cdot y}dx \\
&= (2\pi)^{-d/2}\int\limits_{\mathbb{R}^d} f(x)\,e^{-ix\cdot(\xi-y)}dx \\
&= \mathcal{F}f(x)(\xi-y) \\
&= \mathcal{T}_y\hat{f}(\xi)
\end{aligned}
$$

3.

$$
\begin{aligned}
\mathcal{F}f(rx)(\xi) &= (2\pi)^{-d/2}\int\limits_{\mathbb{R}^d} f(rx)\,e^{-ix\cdot\xi}dx \\
&= (2\pi)^{-d/2}\int\limits_{\mathbb{R}^d} f(u)\,e^{-i\frac{u}{r}\cdot\xi}\frac{du}{r^d} \\
&= r^{-d}(2\pi)^{-d/2}\int\limits_{\mathbb{R}^d} f(u)\,e^{-i\frac{u}{r}\cdot\xi}du \\
&= r^{-d}(2\pi)^{-d/2}\hat{f}\left(r^{-1}\xi\right)
\end{aligned}
$$

4.

$$
\begin{aligned}
\mathcal{F}\bar{f}(\xi) &= (2\pi)^{-d/2} \int\limits_{\mathbb{R}^d} \bar{f}(x)\, e^{-ix\cdot\xi} dx \\
&= (2\pi)^{-d/2} \int\limits_{\mathbb{R}^d} \overline{f(x)\, \overline{e^{-ix\cdot\xi}}} dx \\
&= (2\pi)^{-d/2} \int\limits_{\mathbb{R}^d} \overline{f(x)\, e^{-ix\cdot(-\xi)}} dx \\
&= (2\pi)^{-d/2} \overline{\int\limits_{\mathbb{R}^d} f(x)\, e^{-ix\cdot(-\xi)} dx} \\
&= \overline{\hat{f}(-\xi)}
\end{aligned}
$$

Convince yourself that we can pull the complex conjugate out, or let $f(x) = a(x)+ib(x)$, and

$$
\begin{aligned}
\mathcal{F}\bar{f}(\xi) &= (2\pi)^{-d/2} \int\limits_{\mathbb{R}^d} \bar{f}(x)\, e^{-ix\cdot\xi} dx \\
&= (2\pi)^{-d/2} \left[\int\limits_{\mathbb{R}^d} a(x)\, e^{-ix\cdot\xi} dx - i \int\limits_{\mathbb{R}^d} b(x)\, e^{-ix\cdot\xi} dx \right] \\
&= (2\pi)^{-d/2} \overline{\int\limits_{\mathbb{R}^d} f(x)\, e^{-ix\cdot\xi} dx}
\end{aligned}
$$

by linearity of the integral.

□

While the Fourier transform maps L^1 into L^∞, it most definitely does not map onto the entire space L^∞. Let us attempt to better understand the range of $\mathcal{F}$. We do know that it is contained fully in a subspace of L^∞ we denote $C_v\left(\mathbb{R}^d\right)$.

Definition 6.2. *The space $C_v\left(\mathbb{R}^d\right)$ is the space of continuous functions that vanish at infinity, i.e.*

$$C_v\left(\mathbb{R}^d\right) = \{f \in C_0\left(\mathbb{R}^d\right) : f \text{ vanishes at } \infty\}.$$

We say that a continuous function f vanishes at infinity if and only if for every $\varepsilon > 0$, there exists a pre-compact $K \subset \mathbb{R}^d$ such that

$$|f(x)| < \varepsilon$$

whenever $x \notin K$. Compare this definition with that of compact support. Notice that this is a slightly weaker definition, and certainly all continuous functions with compact support belong to $C_v\left(\mathbb{R}^d\right)$.

Proposition 6.4. *$C_v\left(\mathbb{R}^d\right)$ is a closed linear subspace of $L^\infty\left(\mathbb{R}^d\right)$.*

Proof. The central task is to show that C_v is closed. Let f_n be a Cauchy sequence converging to f in L^∞. We need to demonstrate that $f \in C_v$ as well. Let us first prove that f is continuous.

Sub-Lemma 1. *The uniform limit of continuous functions is a continuous function.*

Proof. With the above setup, since f_n converges uniformly to f, there exists an N such that

$$|f(x) - f_n(x)| < \frac{\varepsilon}{3}$$

whenever $n > N$. Additionally, f_n is continuous for every n, so there exists a δ such that

$$|f_n(x) - f_n(y)| < \frac{\varepsilon}{3}$$

whenever

$$|x - y| < \delta.$$

Therefore, for $n > N$ and $|x-y| < \delta$,

$$|f(x) - f(y)| < |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| < \varepsilon$$

which proves that f is indeed continuous. □

With the knowledge that f is continuous, it remains to show that f vanishes at infinity. Since $f_n \to f$, given $\varepsilon > 0$, there exists an N such that

$$|f(x) - f_n(x)| < \frac{\varepsilon}{2}$$

whenever $n > N$. Additionally, since $f_n \in C_v$, there exists a precompact K such that for every $x \in X \setminus K$,

$$|f_n(x)| < \frac{\varepsilon}{2}.$$

Thus, for $x \in X \setminus K$ and $n > N$,

$$|f(x)| \leqslant |f(x) - f_n(x)| + |f_n(x)| \leqslant \varepsilon,$$

and hence, $f \in C_v$.

□

Lemma 6.1 (Reimann-Lebesgue Lemma). *The Fourier Transform maps*

$$\mathcal{F} : L^1\left(\mathbb{R}^d\right) \to C_v\left(\mathbb{R}^d\right).$$

That is to say, the Fourier Transform of an L^1 function is a C_v function. *Thus, for $f \in L^1\left(\mathbb{R}^d\right)$,*

$$\hat{f} \in C_0\left(\mathbb{R}^d\right) \qquad \textit{and} \qquad \lim_{|\xi|\to\infty} \left|\hat{f}(\xi)\right| = 0.$$

(Merely the definition of $\hat{f} \in C_0$)

Proof. See Hans' homework problem. Suppose $f \in L^1\left(\mathbb{R}^d\right)$. What we want to show is that $\hat{f} \in C_v\left(\mathbb{R}^d\right)$.

Sub-Lemma 2. *Simple functions are dense in* $L^1\left(\mathbb{R}^d\right)$*.*

Proof. need to show... □

Then, there exists a sequence of simple functions f_n converging to f in $L^1\left(\mathbb{R}^d\right)$. If we could show that $\hat{f}_n \in C_v$, then we could conclude immediately that $\hat{f} \in C_v$. Why? Well, by previous proposition, the Fourier Transform is a bounded and this continuous map when viewed as $\mathcal{F} : L^1 \to L^\infty$. This means that if $f_n \to f$ in L^1, then $\mathcal{F}f_n \to \mathcal{F}f$ in L^∞. However, since C_v is a closed subspace of L^∞, so if every $\hat{f}_n \in C_v$, then their limit must also belong to C_v. Remember that $\hat{f}_n$ are Fourier Transforms of simple functions, i.e. finite linear combinations of characteristic functions. We showed before that the Fourier transform of the characteristic function in $\mathbb{R}^d$ is

$$(2\pi)^{-d/2} \prod_{j=1}^{d} \frac{2\sin\xi_j}{\xi_j} = \left(\frac{2}{\pi}\right)^{d/2} \prod_{j=1}^{d} \frac{\sin\xi_j}{\xi_j}$$

which is a member of C_v. To see this, notice first that as the product of sines, $\mathcal{F}\chi$ is continuous. Furthermore, notice that as $|\xi| \to \infty$, the Fourier transform tends to zero, since sine oscillates between zero and one. Therefore, $\mathcal{F}\chi \in C_v$. By previous proposition, the translation and scaling of this characteristic function continue to be members of C_v. Therefore, any finite linear combination is also in C_v, and therefore every $\mathcal{F}f_n \in C_v$. Thus, since $\hat{f}_n \to \hat{f}$ and C_v is closed, then $\hat{f} \in C_v$.

□

We have the following proposition,

Proposition 6.5. *For* $f, g \in L^1\left(\mathbb{R}^d\right)$*,*

1. $$\int_{\mathbb{R}^d} \hat{f}(x)\, g(x)\, dx = \int_{\mathbb{R}^d} f(x)\, \hat{g}(x)\, dx$$

2. $f * g \in L^1\left(\mathbb{R}^d\right)$, $\quad \widehat{f * g} = (2\pi)^{d/2} \hat{f}\hat{g}$, *where the convolution of two functions* f *and* g *is defined as*

$$(f * g)(x) = \int_{\mathbb{R}^d} f(x-y)\, g(y)\, dy$$

which is defined for almost all $x \in \mathbb{R}^d$*. Why almost all* x*?*

6.2 $\mathcal{S}\left(\mathbb{R}^d\right)$ Theory

6.3 $L^2\left(\mathbb{R}^d\right)$ Theory

6.4 $\mathcal{S}'$ Theory

6.5 Applications

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