Homework 3

Jonathan Zhang EID: jdz357

Problem 6.18:

For $f \in \mathcal{S}(\mathbb{R})$, define the Hilbert transform of f by $Hf = PV\left(\frac{1}{\pi x}\right) * f$.

1. Show that $PV(1/x) \in \mathcal{S}'$.
2. Show that $\mathcal{F}\left(PV\left(1/x\right)\right) = -i\sqrt{\pi/2}sgn\left(\xi\right)$.
3. Show that $\|Hf\|_{L^2} = \|f\|_{L^2}$ and $HHf = -f$ for $f \in \mathcal{S}\left(\mathbb{R}\right)$.
4. Extend H to $L^2\left(\mathbb{R}\right)$.

Solution

1. asdf
2. Recall $xPV\left(1/x\right) = 1$. Taking Fourier transform on both sides yields

$$\mathcal{F}\left(xPV\left(1/x\right)\right) =$$

Problem 6.20:

Compute the Fourier transforms of the following functions considered as tempered distributions.

1. x^n for $x \in \mathbb{R}$ and integer $n \geq 0$
2. $e^{-|x|}$ for $x \in \mathbb{R}$
3. $e^{i|x|^2}$ for $x \in \mathbb{R}^d$
4. $\sin x$ and $\cos x$ for $x \in \mathbb{R}$.

Solution

1.

$$
\begin{aligned}
\langle \hat{x}^n, \phi \rangle &= \left\langle x^n, \hat{\phi} \right\rangle \\
&= \left\langle 1, x^n \hat{\phi} \right\rangle \\
&= (i)^{-n} \left\langle 1, (ix)^n \hat{\phi} \right\rangle \\
&= (i)^{-n} \left\langle 1, (D^n \phi)^\wedge \right\rangle \\
&= (i)^{-n} \left\langle \hat{1}, (D^n \phi) \right\rangle \\
&= (i)^{-n} \left\langle (2\pi)^{1/2} \delta_0, (D^n \phi) \right\rangle \\
&= (2\pi)^{1/2} (i)^{-n} \left\langle (-1)^n D^n \delta_0, \phi \right\rangle \\
&= \left\langle (2\pi)^{1/2} (i)^n D^n \delta_0, \phi \right\rangle
\end{aligned}
$$

2. In this case, $e^{-|x|}$ is actually an L^1 function. Therefore, we can equivalently just compute its Fourier transform in the usual way.

$$
\begin{aligned}
\mathcal{F}\left(e^{-|x|}\right) &= (2\pi)^{-1/2} \int_{\mathbb{R}} e^{-|x| - ix\xi} dx \\
&= (2\pi)^{-1/2} \left[\int_{-\infty}^{0} e^{x - ix\xi} dx + \int_{0}^{\infty} e^{-x - ix\xi} dx \right] \\
&= (2\pi)^{-1/2} \left[\frac{1}{1 - i\xi} + \frac{1}{1 + i\xi} \right] \\
&= (2\pi)^{-1/2} \frac{2}{1 + \xi^2}
\end{aligned}
$$

so

$$
\left\langle \left(e^{-|x|}\right)^\wedge, \phi \right\rangle = \left\langle (2\pi)^{-1/2} \frac{2}{1 + \xi^2}, \phi \right\rangle .
$$

3. asdf asdfj weird pie slice integral thing

4. Follows from the Fourier transform of x^n by linearity.

$$
\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \Rightarrow (\cos x)^\wedge = \sum_{n=0}^{\infty} \frac{(-1)^n \left(x^{2n}\right)^\wedge}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} (2\pi)^{1/2} i^{2n} D^{2n} \delta_0
$$

and

$$
\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \Rightarrow (\sin x)^\wedge \sum_{n=0}^{\infty} \frac{(-1)^n \left(x^{2n+1}\right)^\wedge}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (2\pi)^{1/2} i^{2n+1} D^{2n+1} \delta_0
$$

Problem 6.23:

Define the space H, endowed with inner product $(f,g)_H$.

1. Show that H is complete.
2. Prove that H is continuously imbedded in $L^2(\mathbb{R})$.
3. If $m(\xi) \geq \alpha|\xi|^2$ for some $\alpha > 0$, prove that for $f \in H$, the tempered distributional derivative $f' \in L^2(\mathbb{R})$.

Solution Notice first that the norm on H is merely a weighted L^2 norm on Fourier space.

1. Let $f_n \in H$ be a Cauchy sequence. Since the Fourier transform preserves norm, then $\hat{f}_n$ is also a Cauchy sequence in $L^2_m(\mathbb{R})$. Since m is a non-negative measure, L^2_m is complete, so the sequence posesses a limit $\hat{f}_n \to g \in L^2_m$. Now, g, as the limit of measurable functions $\hat{f}_n$, is also measurable. Furthermore, $g \in L^2$ since $||g||_{L^2_m} \geq m^*\|g\|_{L^2}$. If we define now $f := \mathcal{F}^{-1}(g)$ using the L^2 Fourier inverse, we uncover the remaining properties we need. As the limit of measurable functions f_n, f is measurable. Additionally, $f \in L^2$ and so it immediately generates a tempered distribution, so $f \in \mathcal{S}'(\mathbb{R})$. Finally, $\|f\|_H = \|\mathcal{F}^{-1}(g)\|_H = \|g\|_{L^2} < \infty$.

2. The inclusion map
$$H \ni f \to i(f) = f \in L^2$$
is bounded, since
$$\|f\|_H = \left\|\hat{f}\right\|_{L^2_m} \geqslant m^*\left\|\hat{f}\right\|_{L^2} = m^*\|f\|_{L^2}$$
i.e. the continuity constant is $1/m^*$.

3. We have
$$\begin{aligned}\|f\|_H^2 &= \int \left|\hat{f}(\xi)\right|^2 m(\xi)\, d\xi \\ &\geqslant \alpha \int \left|\xi \hat{f}(\xi)\right|^2 d\xi \\ &= \alpha \int \left|(Df)^\wedge(\xi)\right|^2 d\xi \\ &= \alpha \left\|(Df)^\wedge\right\|^2 \Rightarrow f' \in L^2.\end{aligned}$$

Problem 6.25:

Use the Fourier transform to find a solution to

$$u - \frac{\partial^2 u}{\partial x_1^2} - \frac{\partial^2 u}{\partial x_2^2} = e^{-x_1^2 - x_2^2}$$

Can you find a fundamental solution?

Solution If we can find a fundamental solution, then we have solved the given right hand side. Note $d = 2$. Let us attempt to solve

$$u_0 - \frac{\partial^2 u_0}{\partial x_1^2} - \frac{\partial^2 u_0}{\partial x_2^2} = \delta_0.$$

Taking Fourier transform,

$$\hat{u}_0 - (i\xi_1)^2 \hat{u}_0 - (i\xi_2)^2 \hat{u}_0 = \hat{u}_0 \left(1 + |\xi_1|^2 + |\xi_2|^2\right) = (2\pi)^{-1}$$

so, in the Fourier domain, the fundamental solution is

$$\hat{u}_0 = (2\pi)^{-1} \frac{1}{1 + |\xi_1|^2 + |\xi_2|^2} \Rightarrow u_0 = \left[(2\pi)^{-1} \frac{1}{1 + |\xi_1|^2 + |\xi_2|^2}\right]^{\vee}.$$

Now for any right hand side f, the solution is given by $u = u_0 * f$. In this case,

$$u = \left[(2\pi)^{-1} \frac{1}{1 + |\xi_1|^2 + |\xi_2|^2}\right]^{\vee} * e^{-x_1^2 - x_2^2}$$

Problem 6.27:

Telegrapher's equation

$$u_{tt} + 2u_t + u = c^2 u_{xx}$$

for $x \in \mathbb{R}$ and $t > 0$, and $u(x, 0) = f(x)$ and $u_t(x, 0) = g(x)$ are given in L^2.

1. Use Fourier transform in x and its inverse to find an explicit representation of the solution.
2. Justify your representation is a solution.
3. Show that the solution can be viewed as the sum of two wave packets, one moving to the right with a constant speed, and one moving to the left with the same speed.

Problem 6.30:

Consider

$$\Delta^2 u + u = f(x)$$

for $x \in \mathbb{R}^d$ and $\Delta^2 = \Delta\Delta$.

1. Suppose that $f \in \mathcal{S}'\left(\mathbb{R}^d\right)$. Use the Fourier transform to find a solution $u \in \mathcal{S}'$. Leave answer as a multiplier operator.
2. Is the solution unique?
3. Suppose that $f \in \mathcal{S}\left(\mathbb{R}^d\right)$. Write the solution as a convolution operator.
4. Show that $\check{m} \in L^2\left(\mathbb{R}^d\right)$ for some range of d and use this to extend the convolution solution to $f \in L^1\left(\mathbb{R}^d\right)$. In that case, in what L^p space is u?