Homework 2

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Problem 6.9:

Suppose that $f \in L^p\left(\mathbb{R}^d\right)$ *for* $p \in (1,2)$. *Show that there exist* $f_1 \in L^1\left(\mathbb{R}^d\right)$ *and* $f_2 \in L^2\left(\mathbb{R}^d\right)$ *such that* $f = f_1 + f_2$. *Define* $\hat{f} = \hat{f}_1 + \hat{f}_2$. *Show that this definition is well defined, that is, independent of the choices* f_1 *and* f_2.

Problem 6.11:

Suppose that f *and* g *are in* $L^2\left(\mathbb{R}^d\right)$. *The convolution* $f * g \in L^\infty\left(\mathbb{R}^d\right)$, *so it may not have a Fourier Transform. Prove that* $f * g = (2\pi)^{d/2}\mathcal{F}^{-1}\left(\hat{f}\hat{g}\right)$ *is well defined, using the natural Fourier inverse integration formula.*

Problem 6.15:

Is it psosible for there to be a continuous function f *defined on* $\mathbb{R}^d$ *with the following properties?*

1. *There is no polynomial* P *in* d *variables such that* $|f(x)| \le P(x)$ *for all* $x \in \mathbb{R}^d$.
2. *The distribution* $\phi \mapsto \int \phi f$ *is tempered.*

Problem 6.18:

The gamma function is $\Gamma(s) = \int\limits_0^\infty t^{s-1}e^{-t}dt$. *Let* $\phi \in \mathcal{S}\left(\mathbb{R}^d\right)$ *and* $0 < \alpha < d$.

1. *Show that* $|\xi|^\alpha \in L^1_{loc}\left(\mathbb{R}^d\right)$ *and* $|\xi|^\alpha\hat{\phi} \in L^1\left(\mathbb{R}^d\right)$.
2. *Let* $c_\alpha = 2^{\alpha/2}\Gamma\left(\alpha/2\right)$. *Show that*

$$\mathcal{F}^{-1}\left(|\xi|^{-\alpha}\hat{\phi}\right)(x) = \frac{c_{d-\alpha}}{(2\pi)^{d/2}c_\alpha}\int\limits_{\mathbb{R}^d}|x-y|^{\alpha-d}\phi(y)dy$$

Hint: first show that $c_\alpha|\xi|^{-\alpha} = \int\limits_0^\infty t^{\alpha/2-1}e^{-|xi|^2t/2}dt$ and recall $e^{-|\xi|^2t/2} = t^{-d/2}\widehat{\left(e^{-|x|^2/2t}\right)}$.

Problem 6.20:

Argue that $\mathcal{D}\left(\mathbb{R}^d\right)$ *is dense in* $\mathcal{S}$. *Show also that* $\mathcal{S}'$ *is dense in* $\mathcal{D}'$ *and that distributions with compact support are dense in* $\mathcal{S}'$.