Homework 3

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Problem 6.18:

For $f \in \mathcal{S}(\mathbb{R})$, define the Hilbert transform of f by $Hf = PV\left(\frac{1}{\pi x}\right) * f$.

1. Show that $PV(1/x) \in \mathcal{S}'$.
2. Show that $\mathcal{F}\left(PV\left(1/x\right)\right) = -i\sqrt{\pi/2}sgn\left(\xi\right)$.
3. Show that $\|Hf\|_{L^2} = \|f\|_{L^2}$ and $HHf = -f$ for $f \in \mathcal{S}\left(\mathbb{R}\right)$.
4. Extend H to $L^2\left(\mathbb{R}\right)$.

Solution

1. asdf
2. Recall $xPV\left(1/x\right) = 1$. Taking Fourier transform on both sides yields
$$\mathcal{F}\left(xPV\left(1/x\right)\right) =$$

Problem 6.20:

Compute the Fourier transforms of the following functions considered as tempered distributions.

1. x^n for $x \in \mathbb{R}$ and integer $n \geq 0$
2. $e^{-|x|}$ for $x \in \mathbb{R}$
3. $e^{i|x|^2}$ for $x \in \mathbb{R}^d$
4. $\sin x$ and $\cos x$ for $x \in \mathbb{R}$.

Problem 6.23:

Define the sapce H, endowed with inner product $(f, g)_H$.

1. Show that H is complete.
2. Prove that H is continuously imbedded in $L^2\left(\mathbb{R}\right)$.
3. If $m(\xi) \geq \alpha|\xi|^2$ for some $\alpha > 0$, prove that for $f \in H$, the tempered distributional derivative $f' \in L^2(\mathbb{R})$.

Problem 6.25:

Use the Fourier transform to find a solution to

$$u - \frac{\partial^2 u}{\partial x_1^2} - \frac{\partial^2 u}{\partial x_2^2} = e^{-x_1^2 - x_2^2}$$

Can you find a fundamental solution?

Solution If we can find a fundamental solution, then we have solved the given right hand side.

Problem 6.27:

Telegrapher's equation

$$u_{tt} + 2u_t + u = c^2 u_{xx}$$

for $x \in \mathbb{R}$ and $t > 0$, and $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$ are given in L^2.

1. Use Fourier transform in x and its inverse to find an explicit representation of the solution.
2. Justify your representation is a solution.
3. Show that the solution can be viewed as the sum of two wave packets, one moving to the right with a constant speed, and one moving to the left with the same speed.

Problem 6.30:

Consider

$$\Delta^2 u + u = f(x)$$

for $x \in \mathbb{R}^d$ and $\Delta^2 = \Delta\Delta$.

1. Suppose that $f \in \mathcal{S}'\left(\mathbb{R}^d\right)$. Use the Fourier transform to find a solution $u \in \mathcal{S}'$. Leave answer as a multiplier operator.
2. Is the solution unique?
3. Suppose that $f \in \mathcal{S}\left(\mathbb{R}^d\right)$. Write the solution as a convolution operator.
4. Show that $\check{m} \in L^2\left(\mathbb{R}^d\right)$ for some range of d and use this to extend the convolution solution to $f \in L^1\left(\mathbb{R}^d\right)$. In that case, in what L^p space is u?