Homework 1

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Problem 6.2:

Compute the Fourier transform of $\exp\left(-a|x|^2\right)$, $a > 0$, directly for $x \in \mathbb{R}$.

Solution First of all, notice that the Gaussian is an L^1 function. Computing directly,

$$\begin{aligned}
\hat{f}(\xi) &= (2\pi)^{-d/2} \int\limits_{\mathbb{R}^d} e^{-a|x|^2 - ix\cdot\xi} dx \\
&= (2\pi)^{-d/2} \int\limits_{\mathbb{R}^d} e^{-a\left(x + \frac{i\xi}{2a}\right)^2 - \frac{\xi^2}{4a}} dx \\
&= (2\pi)^{-d/2} e^{-\frac{\xi^2}{4a}} \int\limits_{\mathbb{R}^d} e^{-a\left(x + \frac{i\xi}{2a}\right)^2} dx.
\end{aligned}$$

To compute the remaining integral, we first observe that the integrand is a holomorphic function, so Cauchy's Theorem implies that the integral over any closed contour is zero. We choose the rectangle prescribed by vertices $-R, R, -R - \frac{i\xi}{2a}, R - \frac{i\xi}{2a}$.

Problem 6.3:

If $f \in L^1\left(\mathbb{R}^d\right)$ and $f > 0$, show that for every $\xi \neq 0$, $\left|\hat{f}(\xi)\right| < \hat{f}(0)$.

Solution By definition,

$$\hat{f}(0) = (2\pi)^{-d/2} \int\limits_{\mathbb{R}^d} f(x)\, dx.$$

The weak inequality follows by direct computation,

$$\begin{aligned}
\left|\hat{f}(\xi)\right| &= \left|(2\pi)^{-d/2} \int\limits_{\mathbb{R}^d} f(x)\, e^{-ix\cdot\xi} dx\right| \\
&\leqslant (2\pi)^{-d/2} \int\limits_{\mathbb{R}^d} |f(x)|\, dx \\
&\leqslant (2\pi)^{-d/2} \int\limits_{\mathbb{R}^d} f(x)\, dx \\
&= \hat{f}(0)
\end{aligned}$$

Problem 6.4:

If $f \in L^1\left(\mathbb{R}^d\right)$ and $f(x) = g(|x|)$ for some g, show that $\hat{f}(\xi) = h(|\xi|)$ for some h. Can you relate g and h?

Solution

Problem 6.6:

Show that the Fourier transform $\mathcal{F} : L^1\left(\mathbb{R}^d\right) \to C_v\left(\mathbb{R}^d\right)$ is not onto. Show that $\mathcal{F}\left(L^1\left(\mathbb{R}^d\right)\right)$ is dense in $C_v\left(\mathbb{R}^d\right)$.

Solution Some like Fourier Inverse Theorem, Open Mapping Theorem, conclude bounded inverse.