Homework 3

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Problem 6.18:

For $f \in \mathcal{S}(\mathbb{R})$, define the Hilbert transform of f by $Hf = PV\left(\frac{1}{\pi x}\right) * f$.

1. Show that $PV(1/x) \in \mathcal{S}'$.
2. Show that $\mathcal{F}\left(PV\left(1/x\right)\right) = -i\sqrt{\pi/2}sgn\left(\xi\right)$.
3. Show that $\|Hf\|_{L^2} = \|f\|_{L^2}$ and $HHf = -f$ for $f \in \mathcal{S}\left(\mathbb{R}\right)$.
4. Extend H to $L^2\left(\mathbb{R}\right)$.

Solution

1. For $\phi \in \mathcal{S}$,

$$\begin{aligned}
\left\langle PV\left(1/x\right), \phi\right\rangle &= \lim_{\varepsilon \to 0^+} \int\limits_{|x|>\varepsilon} \frac{\phi\left(x\right)}{x} dx \\
&= \lim_{\varepsilon \to 0^+, M, M' \to \infty} \left\{ \int\limits_{\varepsilon}^{M} \frac{\phi\left(x\right)}{x} dx + \int\limits_{-M'}^{-\varepsilon} \frac{\phi\left(x\right)}{x} dx \right\} \\
&= \lim_{\varepsilon \to 0^+, M, M' \to \infty} \left\{ \int\limits_{\varepsilon}^{1} \frac{\phi\left(x\right)}{x} dx + \int\limits_{1}^{M} \frac{\phi\left(x\right)}{x} dx + \int\limits_{-M'}^{-1} \frac{\phi\left(x\right)}{x} dx + \int\limits_{-1}^{-\varepsilon} \frac{\phi\left(x\right)}{x} dx \right\} \\
&= \lim_{\varepsilon \to 0^+, M, M' \to \infty} \left\{ \int\limits_{\varepsilon}^{1} \frac{\phi\left(x\right) - \phi\left(-x\right)}{x} dx + \int\limits_{1}^{M} \frac{\phi\left(x\right)}{x} dx + \int\limits_{-M'}^{-1} \frac{\phi\left(x\right)}{x} dx \right\}
\end{aligned}$$

The first integral converges because by the mean value theorem, we have that

$$\left|\frac{\phi\left(x\right) - \phi\left(-x\right)}{x}\right| = \left|\frac{\phi'\left(\xi\right) 2x}{x}\right| \leqslant 2\|\phi\|_{0,1}$$

and for the second and third integrals, for $|x| > 1$,

$$\left|\frac{\phi\left(x\right)}{x}\right| \leqslant \frac{1}{x^2}\left|x\phi\left(x\right)\right| \leqslant \frac{1}{x^2}\|\phi\|_{1,0}$$

Hence,

$$\left|\left\langle PV\left(1/x\right), \phi\right\rangle\right| \leqslant 2\left(\|\phi\|_{0,1} + \|\phi\|_{1,0}\right)$$

which proves continuity. Linearity is easy to see by linearity of integrals.

2. Recall $xPV\left(1/x\right) = 1$. Taking Fourier transform on both sides yields

$$2\pi\delta_0 = \mathcal{F}\left(xPV\left(1/x\right)\right) = i\frac{d}{d\xi}\mathcal{F}\left(PV\left(1/x\right)\right),$$

so

$$\mathcal{F}\left(PV\left(1/x\right)\right) = -(2\pi)^{-1/2} iH\left(\xi\right) + C = -i\sqrt{\pi/2}\,\mathrm{sgn}\left(\xi\right)$$

where we have used the fact that the Dirac delta is the derivative of the Heaviside function. Notice we choose the constant C and rescale in order to make the Heaviside function look like the sgn function.

3. We have for $f \in \mathcal{S}$,

$$\|Hf\| = \left\|(Hf)^{\wedge}\right\| = \left\|-i\,\mathrm{sgn}\left(x\right)\hat{f}\right\| = \left\|\hat{f}\right\| = \|f\|$$

and

$$\begin{aligned} HHf &= H\left(Hf\right) \\ &= H\left(\left(-i\,\mathrm{sgn}\left(x\right)\hat{f}\right)^{\vee}\right) \\ &= \left(-i\,\mathrm{sgn}\left(x\right)\left(\left(-i\,\mathrm{sgn}\left(x\right)\hat{f}\right)^{\vee}\right)^{\wedge}\right)^{\vee} \\ &= \left(-i\,\mathrm{sgn}\left(x\right)\left(-i\,\mathrm{sgn}\left(x\right)\hat{f}\right)\right)^{\vee} \\ &= -f \end{aligned}$$

4. By the density of $\mathcal{S}$ in L^2, H can be extended through limits. For $f \in L^2$, pick a sequence $\mathcal{S} \ni f_n \to f \in L^2$. Then define $Hf = \lim_{n\to\infty} Hf_n$. We can apply the result we had in class for a bounded linear map on a dense subset.

Problem 6.20:

Compute the Fourier transforms of the following functions considered as tempered distributions.

1. x^n for $x \in \mathbb{R}$ and integer $n \geq 0$
2. $e^{-|x|}$ for $x \in \mathbb{R}$
3. $e^{i|x|^2}$ for $x \in \mathbb{R}^d$
4. $\sin x$ and $\cos x$ for $x \in \mathbb{R}$.

Solution

1.

$$\begin{aligned}
\langle \hat{x}^n, \phi \rangle &= \left\langle x^n, \hat{\phi} \right\rangle \\
&= \left\langle 1, x^n \hat{\phi} \right\rangle \\
&= (i)^{-n} \left\langle 1, (ix)^n \hat{\phi} \right\rangle \\
&= (i)^{-n} \left\langle 1, (D^n \phi)^\wedge \right\rangle \\
&= (i)^{-n} \left\langle \hat{1}, (D^n \phi) \right\rangle \\
&= (i)^{-n} \left\langle (2\pi)^{1/2} \delta_0, (D^n \phi) \right\rangle \\
&= (2\pi)^{1/2} (i)^{-n} \left\langle (-1)^n D^n \delta_0, \phi \right\rangle \\
&= \left\langle (2\pi)^{1/2} (i)^n D^n \delta_0, \phi \right\rangle
\end{aligned}$$

2. In this case, $e^{-|x|}$ is actually an L^1 function. Therefore, we can equivalently just compute its Fourier transform in the usual way.

$$\begin{aligned}
\mathcal{F}\left(e^{-|x|}\right) &= (2\pi)^{-1/2} \int\limits_{\mathbb{R}} e^{-|x| - ix\xi} dx \\
&= (2\pi)^{-1/2} \left[\int\limits_{-\infty}^{0} e^{x - ix\xi} dx + \int\limits_{0}^{\infty} e^{-x - ix\xi} dx \right] \\
&= (2\pi)^{-1/2} \left[\frac{1}{1 - i\xi} + \frac{1}{1 + i\xi} \right] \\
&= (2\pi)^{-1/2} \frac{2}{1 + \xi^2}
\end{aligned}$$

so

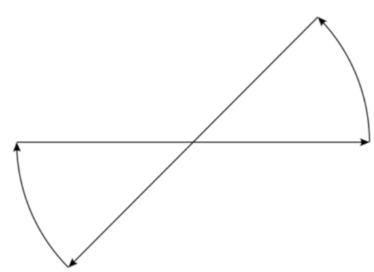
$$\left\langle \left(e^{-|x|}\right)^\wedge, \phi \right\rangle = \left\langle (2\pi)^{-1/2} \frac{2}{1 + \xi^2}, \phi \right\rangle.$$

3. We will attempt to evaluate the integral using contour integration. First of all, notice that even

though we are in $\mathbb{R}^d$, $|x|^2 \in \mathbb{R}$, so it is sufficient to consider only the 1D case. The steps are

$$\begin{aligned}
\left(e^{i|x|^2}\right)^\wedge &= (2\pi)^{-1/2} \int_{\mathbb{R}} e^{i|x|^2 - ix\xi} dx \\
&= (2\pi)^{-1/2} \sqrt{1/2} \int_{\mathbb{R}} e^{i|x|^2/2 - ix\xi\sqrt{1/2}} dx \\
&= (2\pi)^{-1/2} \sqrt{1/2} e^{-i\xi^2/4} \int e^{i\frac{\left(x - \xi\sqrt{1/2}\right)^2}{2}} dx \\
&= (2\pi)^{-1/2} \sqrt{1/2} e^{-i\xi^2/4} \int e^{i\frac{(x)^2}{2}} dx \\
&= (2\pi)^{-1/2} \sqrt{1/2} e^{-i\xi^2/4} e^{i\pi/4} \int e^{-\frac{(x)^2}{2}} dx \\
&= (2\pi)^{-1/2} \sqrt{1/2} e^{-i\xi^2/4} e^{i\pi/4} \left(\sqrt{2\pi}\right) \\
&= \sqrt{1/2} e^{i\left(-\xi^2 + \pi\right)/4}.
\end{aligned}$$

We use the substitutions $t \mapsto t\sqrt{a}$, completing the square, $t \mapsto t + x\sqrt{a}$. The contour is



Over this contour, the integral is zero. The integrals along the arc vanish as $R \to \infty$. The integral alon the horizontal line tends to $\int \exp(it^2/2) = \sqrt{2\pi}$, and the integral alon the diagonal is the negative of $\exp(i\pi/4) \int \exp(it^2/2)$.

4. Follows from the Fourier transform of x^n by linearity.

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \Rightarrow (\cos x)^\wedge = \sum_{n=0}^{\infty} \frac{(-1)^n \left(x^{2n}\right)^\wedge}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} (2\pi)^{1/2} i^{2n} D^{2n} \delta_0$$

and

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \Rightarrow (\sin x)^\wedge \sum_{n=0}^{\infty} \frac{(-1)^n \left(x^{2n+1}\right)^\wedge}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (2\pi)^{1/2} i^{2n+1} D^{2n+1} \delta_0$$

Problem 6.23:

Define the space H, endowed with inner product $(f, g)_H$.

1. Show that H is complete.

2. Prove that H is continuously imbedded in $L^2(\mathbb{R})$.

3. If $m(\xi) \geq \alpha|\xi|^2$ for some $\alpha > 0$, prove that for $f \in H$, the tempered distributional derivative $f' \in L^2(\mathbb{R})$.

Solution Notice first that the norm on H is merely a weighted L^2 norm on Fourier space.

1. Let $f_n \in H$ be a Cauchy sequence. Since the Fourier transform preserves norm, then $\hat{f}_n$ is also a Cauchy sequence in $L^2_m(\mathbb{R})$. Since m is a non-negative measure, L^2_m is complete, so the sequence posesses a limit $\hat{f}_n \to g \in L^2_m$. Now, g, as the limit of measurable functions $\hat{f}_n$, is also measurable. Furthermore, $g \in L^2$ since $||g||_{L^2_m} \geq m^*\|g\|_{L^2}$. If we define now $f := \mathcal{F}^{-1}(g)$ using the L^2 Fourier inverse, we uncover the remaining properties we need. As the limit of measurable functions f_n, f is measurable. Additionally, $f \in L^2$ and so it immediately generates a tempered distribution, so $f \in \mathcal{S}'(\mathbb{R})$. Finally, $\|f\|_H = \|\mathcal{F}^{-1}(g)\|_H = \|g\|_{L^2} < \infty$.

2. The inclusion map
$$H \ni f \to i(f) = f \in L^2$$
is bounded, since
$$\|f\|_H = \left\|\hat{f}\right\|_{L^2_m} \geqslant m^*\left\|\hat{f}\right\|_{L^2} = m^*\|f\|_{L^2}$$
i.e. the continuity constant is $1/m^*$.

3. We have
$$\begin{aligned}\|f\|_H^2 &= \int \left|\hat{f}(\xi)\right|^2 m(\xi)\, d\xi \\ &\geqslant \alpha \int \left|\xi \hat{f}(\xi)\right|^2 d\xi \\ &= \alpha \int \left|(Df)^\wedge(\xi)\right|^2 d\xi \\ &= \alpha\left\|(Df)^\wedge\right\|^2 \Rightarrow f' \in L^2.\end{aligned}$$

Problem 6.25:

Use the Fourier transform to find a solution to
$$u - \frac{\partial^2 u}{\partial x_1^2} - \frac{\partial^2 u}{\partial x_2^2} = e^{-x_1^2 - x_2^2}$$
Can you find a fundamental solution?

Solution If we can find a fundamental solution, then we have solved the given right hand side. Note $d = 2$. Let us attempt to solve
$$u_0 - \frac{\partial^2 u_0}{\partial x_1^2} - \frac{\partial^2 u_0}{\partial x_2^2} = \delta_0.$$
Taking Fourier transform,
$$\hat{u}_0 - (i\xi_1)^2\hat{u}_0 - (i\xi_2)^2\hat{u}_0 = \hat{u}_0\left(1 + |\xi_1|^2 + |\xi_2|^2\right) = (2\pi)^{-1}$$

so, in the Fourier domain, the fundamental solution is

$$\hat{u}_0 = (2\pi)^{-1} \frac{1}{1 + |\xi_1|^2 + |\xi_2|^2} \Rightarrow u_0 = \left[(2\pi)^{-1} \frac{1}{1 + |\xi_1|^2 + |\xi_2|^2} \right]^\vee.$$

Now for any right hand side f, the solution is given by $u = u_0 * f$. In this case,

$$u = \left[(2\pi)^{-1} \frac{1}{1 + |\xi_1|^2 + |\xi_2|^2} \right]^\vee * e^{-x_1^2 - x_2^2}$$

Problem 6.27:

Telegrapher's equation

$$u_{tt} + 2u_t + u = c^2 u_{xx}$$

for $x \in \mathbb{R}$ and $t > 0$, and $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$ are given in L^2.

1. Use Fourier transform in x and its inverse to find an explicit representation of the solution.
2. Justify your representation is a solution.
3. Show that the solution can be viewed as the sum of two wave packets, one moving to the right with a constant speed, and one moving to the left with the same speed.

Solution Proceed formally. Take the Fourier transform in x:

$$\hat{u}_{tt} + 2\hat{u}_t + \hat{u} \left(1 + c^2\xi^2\right) = 0$$

Using the ansatz $\hat{u}(\xi, t) = F(\xi)e^{rt}$, we arrive at the values for r,

$$r^2 + 2r + \left(1 + c^2\xi^2\right) = 0 \Rightarrow r = \frac{-2 \pm 2ic\xi}{2} = -1 \pm ic\xi$$

$$\hat{u}(\xi, t) = F(\xi) e^{(-1+ic\xi)t} + G(\xi) e^{(-1-ic\xi)t} = e^{-t} \left[F(\xi) e^{ic\xi t} + G(\xi) e^{-ic\xi t} \right]$$

The initial conditions can also be transformed, and building those in, we find formulas for the unknown coefficients F and G.

$$\begin{aligned} \hat{u}(\xi, 0) &= \hat{f}(\xi) = F(\xi) + G(\xi) \\ \hat{u}_t(\xi, 0) &= \hat{g}(\xi) = F(\xi)(-1 + ic\xi) + G(\xi)(-1 - ic\xi) \end{aligned}$$

which yield

$$F(\xi) = \frac{1}{2ic\xi} \left[(1 + ic\xi) \hat{f}(\xi) + \hat{g}(\xi) \right], \qquad G(\xi) = -\frac{1}{2ic\xi} \left[(1 - ic\xi) \hat{f}(\xi) + \hat{g}(\xi) \right].$$

The final solution is then the inverse Fourier transform

$$u(x, t) = (2\pi)^{-1/2} e^{-t} \int \left(F(\xi) e^{ic\xi t} + G(\xi) e^{-ic\xi t} \right) e^{i\xi x} d\xi.$$

Furthermore, the representation is indeed a solution. Notice that (almost by construction), the solution satisfies the initial conditions. Formally, without justifying interchange of integral and time derivative, it also satisfies the original PDE. (Note this is much easier to check in the Fourier domain, but can still be verified in the physical domain).

We can write the solution as

$$\begin{aligned} u(x,t) &\propto e^{-t} \int \left(F(\xi) e^{ic\xi t + i\xi x} + G(\xi) e^{-ic\xi t + i\xi x} \right) d\xi \\ &\propto e^{-t} \int \left(F(\xi) e^{i\xi(ct+x)} + G(\xi) e^{i\xi(-ct+x)} \right) d\xi \\ &\propto e^{-t} \left(\check{F}(x+ct) + \check{G}(x-ct) \right) \end{aligned}$$

which represent two wave packets, both moving with speed c.

Problem 6.30:

Consider

$$\Delta^2 u + u = f(x)$$

for $x \in \mathbb{R}^d$ and $\Delta^2 = \Delta\Delta$.

1. Suppose that $f \in \mathcal{S}'\left(\mathbb{R}^d\right)$. Use the Fourier transform to find a solution $u \in \mathcal{S}'$. Leave answer as a multiplier operator.
2. Is the solution unique?
3. Suppose that $f \in \mathcal{S}\left(\mathbb{R}^d\right)$. Write the solution as a convolution operator.
4. Show that $\check{m} \in L^2\left(\mathbb{R}^d\right)$ for some range of d and use this to extend the convolution solution to $f \in L^1\left(\mathbb{R}^d\right)$. In that case, in what L^p space is u?

Solution Let us attempt to find a fundamental solution. Setting the right hand side equal to δ and taking Fourier transform, we see that

$$\left((i\xi)^4 + 1\right)\hat{u} = (2\pi)^{-d/2} \Rightarrow \hat{u} = \frac{(2\pi)^{-d/2}}{|\xi|^4 + 1}$$

and the corresponding solution in a distributional sense is

$$u = \left(\frac{1}{1+|\xi|^4}\hat{f}\right)^{\vee}$$

i.e. $m(\xi) = 1/1 + |\xi|^4$.

This solution is unique. Suppose to the contrary that there exist two different solutions, u_1 and u_2. Define $v = u_1 - v_2$. Then v satisfies the homogeneous equation, i.e.

$$v = \left(\frac{1}{1+|\xi|^4}\hat{0}\right)^{\vee} = 0^{\vee} = 0$$

but this implies that $u_1 = u_2$.

For $f \in \mathcal{S}$, the solution is given by the convolution with the fundamental solution

$$u = \left(\frac{(2\pi)^{-d/2}}{|\xi|^4 + 1}\right)^{\vee} * f$$

Finally, computing the norm of $\check{m}$ is the same as the norm of m. Since m is radial, the integral is of the form

$$\|m\|^2 = \int \frac{1}{\left(1+|\xi|^4\right)^2} d\xi \propto \int_0^\infty \frac{r^{d-1}dr}{(1+r^4)^2}$$

We are concerned just with when the integral converges, so

$$\int_0^\infty \frac{r^{d-1}dr}{(1+r^4)^2} \sim \int_0^\infty \frac{dr}{r^{8+1-d}}$$

which converges only when $9-d>1$, i.e. $d<8$. In the case when $f \in L^1$, we claim $u \in L^2$, since $\|u\| = \|\hat{u}\|$ which is bounded by the Cauchy Schwartz inequality.