Homework 6

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Exercise (7.10). *Suppose $\Omega \subset \mathbb{R}^d$ is bounded with Lipschitz boundary and $f_j \rightharpoonup f$ and $g_j \rightharpoonup g$ weakly in $H^1(\Omega)$. Show that, for a subsequence, $\nabla(f_j g_j) \to \nabla(fg)$ as a distribution. Find all p in $[1,\infty]$ such that the convergence can be taken weakly in $L^p(\Omega)$.*

Weak convergence of f_j in H^1 implies strong convergence of f_j in L^2. To see this, consider any subsequence f_{j_k}. By Rellich-Kondrachov, then every subsequence of f_{j_k} converges strongly to f in L^2. But if that is the case, then $f_j \to f$ in L^2. Same with g. Holder's implies then that the product of $f_j g_j$ converges strongly in L^1 to fg.

Now, as a distribution,

$$\langle \nabla(f_j g_j) - \nabla(fg), \phi \rangle = \langle \nabla(f_j g_j - fg), \phi \rangle = \langle f_j g_j - fg, \nabla \phi \rangle \leq \|f_j g_j - fg\|_1 \|\nabla \phi\|_\infty.$$

We claim that weak convergence in L^p works for any $p \in [1,\infty]$. Notice first that on bounded domains, L^p spaces are nested. Additionally, $\nabla(f_j g_j) = f_j \nabla g_j + g_j \nabla f_j$. Each term is a product of a weakly convergent and strongly convergent sequence, so each term converges weakly in L^1. We have just shown that both the sequence $f_j g_j$ and the sequence $\nabla(f_j g_j)$ converge weakly in L^1, so $f_j g_j$ converges in $W^{1,1}$. Now, Rellich implies that for some subsequence, $f_j g_j$ converges strongly in $L^1(\Omega)$, and thus every L^p, $p > 1$ as well.

Exercise (7.15). *Suppose that $w \in L^1(\mathbb{R})$, $w(x) > 0$, and w is even. Moreover, for $x > 0$, $w \in C^2[0,\infty)$, $w'(x) < 0$, and $w''(x) > 0$. Consider the equation*

$$w * u - u'' = f \in L^2(\mathbb{R}).$$

1. *Show that $\hat{w}(\xi) > 0$.*
2. *Fund a fundamental solution. Leave answer in terms of inverse Fourier transform.*
3. *Find the solution operator as a convolution operator.*
4. *Show that $u \in H^2(\mathbb{R})$.*

Proof. 1. First, use some properties of w:

$$\begin{aligned}
\hat{w}(\xi) &= (2\pi)^{-1/2} \int_{\mathbb{R}} w(x) e^{-ix\xi} dx \\
&= (2\pi)^{-1/2} \int_{\mathbb{R}} w(x) (\cos(x\xi) - i \sin(x\xi)) dx \\
&= (2\pi)^{-1/2} \int_{\mathbb{R}} w(x) \cos(x\xi) dx \\
&= 2(2\pi)^{-1/2} \int_0^\infty w(x) \cos(x\xi) dx
\end{aligned}$$

Since the product $w(x)\sin(x\xi)$ is odd, its integral over the real line is zero. IBP tells us

$$\int_0^\infty w(x)\cos(x\xi)\,dx = \left[w(x)\sin(x\xi)\frac{1}{\xi}\right]_0^\infty - \int_0^\infty \frac{1}{\xi}w'(x)\sin(x\xi)\,dx$$
$$= \lim_{x\to\infty} w(x)\sin(x\xi)\frac{1}{\xi} - \int_0^\infty \frac{1}{\xi}w'(x)\sin(x\xi)\,dx.$$

Let us examine the integral

$$-\int_0^\infty \frac{1}{\xi}w'(x)\sin(x\xi)\,dx.$$

and consider it first on an interval $(0, 2\pi/|\xi|)$. For the moment, let us restrict ourselves to $\xi > 0$. We will demonstrate that over a single interval, the integral is positive. Then, we will extend the same reasoning to all the inervals, and conclude that the original integral is positive.

Let us decompose the integral as

$$\int_0^{2\pi/|\xi|} w'(x)\sin(x\xi)\,dx = \int_0^{\pi/|\xi|} w'(x)\sin(x\xi)\,dx + \int_{\pi/|\xi|}^{2\pi/|\xi|} w'(x)\sin(x\xi)\,dx.$$

For the second integral, change variables to $u = x - \pi/|\xi|$:

$$\int_0^{2\pi/|\xi|} w'(x)\sin(x\xi)\,dx = \int_0^{\pi/|\xi|} w'(x)\sin(x\xi)\,dx + \int_0^{\pi/|\xi|} w'(u+\pi/|\xi|)\sin((u+\pi/|\xi|)\xi)\,dx$$
$$= \int_0^{\pi/|\xi|} w'(x)\sin(x\xi)\,dx - \int_0^{\pi/|\xi|} w'(u+\pi/|\xi|)\sin(u\xi)du$$
$$= \int_0^{\pi/|\xi|} \big(w'(u) - w'(u+\pi/|\xi|)\big)\sin(u\xi)du.$$

Now by Taylor expansion,

$$w'(u+\pi/|\xi|) = w'(u) + w''(\tilde{u})\pi/|\xi| \quad \text{for some } \tilde{u} \in [u, u+\pi/|\xi|].$$

So the integral simplifies to

$$\int_0^{\pi/|\xi|} -\pi/|\xi|w''(\tilde{u})\sin(u\xi)du.$$

The sign of this integral is negative by inspection. The same reasoning applies for the integral over the interval $(n\pi/|\xi|, (n+1)\pi/|\xi|)$. That means the original integral is positive since it is premultiplied by $-1/\xi$. For negative ξ, we remark that the same holds by the oddity of sine.

Lastly, we argue that $\lim_{x\to\infty} w(x) = 0$. We know that w is monotone decreasing, but at a decreasing rate. We also know that w is strictly positive. At best case, $w(x) \to 0$ as $x \to \infty$. At worst, $w(x)$ approaches some constant nonzero, but this cannot be, since w is integrable. The final result is that

$$\hat{w}(\xi) = -2(2\pi)^{-1/2}\frac{1}{\xi}\int_0^\infty w'(x)\sin(x\xi)dx > 0$$

2. Let $f = \delta_0$. Take the Fourier transform and solve for $\hat{G}$.

$$(2\pi)^{1/2}\hat{w}\hat{G} - (i\xi)^2\hat{G} = (2\pi)^{-1/2}$$

so

$$G = \left[\frac{(2\pi)^{-1/2}}{(2\pi)^{1/2}\hat{w} + \xi^2}\right]^\vee$$

is the fundamental solution.

3. Now for any right hand side, the solution is $u = G * f$.

4. $u \in H^2$ since

$$\|u\|_{H^2}^2 = \int\left(1+\xi^2\right)^2\hat{u}^2 = \int\left(1+\xi^2\right)^2\frac{\hat{f}^2}{((2\pi)^{1/2}\hat{w}+\xi^2)^2}$$

is finite, since by Holders, it is bounded by $\|f\|_2^2\left\|\frac{(1+\xi^2)^2}{((2\pi)^{1/2}\hat{w}+\xi^2)^2}\right\|_\infty$.

□

Exercise (7.16). *Let H be a Hilbert space and Z a closed linear subspace. Prove that H/Z has the inner product*

$$(\hat{x},\hat{y})_{H/Z} = \left(P_Z^\perp x, P_Z^\perp y\right)_H$$

and that H/Z is isomorphic to $Z^\perp$.

We want to show the following:

1. The proposed inner product is well defined.
2. The inner product in H/Z is in fact an inner product.
3. There is a natural isomorphism between H/Z and $Z^\perp$.

Proof. 1. Let $x_1, x_2 \in \hat{x}$ be two elements in the equivalence class. Then,

$$(\hat{x},\hat{y})_{H/Z} = \left(P_Z^\perp x_1, P_Z^\perp y\right) = \left(P_Z^\perp x_2, P_Z^\perp y\right)$$

if and only if

$$\left(P_Z^\perp(x_1 - x_2), P_Z^\perp y\right) = 0.$$

But, since x_1 and x_2 belong to the same equivalence class, their difference is a member of Z. Therefore, the projection $P_Z^\perp(x_1 - x_2) = 0$, so the value of $(\hat{x},\hat{y})_{H/Z}$ is independent of the choice of representative.

2. Symmetry:
$$(\hat{x}, \hat{y})_{H/Z} = \left(P_Z^{\perp} x_1, P_Z^{\perp} y\right)_H = \left(P_Z^{\perp} y, P_Z^{\perp} x_1\right)_H = (\hat{y}, \hat{x})_{H/Z}.$$
Positivity:
$$(\hat{x}, \hat{x})_{H/Z} = \left(P_Z^{\perp} x_1, P_Z^{\perp} x_1\right)_H = \left\|P_Z^{\perp} x_1\right\|^2 \geqslant 0,$$
and we see clearly that $(\hat{x}, \hat{x})_{H/Z} = 0$ only when $x \in Z$, or equivalently, $x \in \hat{0}$. Linearity: Projections are linear, and the inner product in H is linear.

3. The two spaces are connected by the natural isomorphism
$$i : H/Z \ni \hat{x} \to P_Z^{\perp} x_1 \in Z^{\perp}; \quad i^{-1} : Z^{\perp} \ni x \to \hat{x} \in H/Z.$$
Map i is a well defined map, since we just showed the projection is independent of the choice of representative.

□

Exercise (7.17). *Elliptic regularity theorey shows that if $\Omega \subset \mathbb{R}^d$ has a smooth boundary and $f \in H^s(\Omega)$, then $-\Delta u = f$ in Ω, $u = 0$ on $\partial\Omega$ has a unique solution $u \in H^{s+2}$. For what values of s will u be continuous? Can you be sure that a fundamental solution is continuous? Answer depends on d.*

Use the Sobolev embedding theorem with $m = s + 2$, $p = 2$, and $j = 0$. This tells us that $H^{s+2} \hookrightarrow C_B^0$ (i.e. u is continuous) only if $2(s + 2) > d$, that is, $s > (d - 1)/2$.

For the fundamental solution, recall that $\delta \in H^s$ for $s < -d/2$. The two conditions tell us that $d = 1$ is the only possible value, so the fundamental solution is continuous only if $d = 1$ and $s \in [-3/2, -1/2]$.

Exercise (7.19). *Let $\Omega \subset \mathbb{R}^d$ be a domain with Lipschitz boundary. Let $w \in L^\infty(\Omega)$. Define*
$$H_w(\Omega) = \left\{f \in L^2(\Omega) : \nabla(wf) \in \left(L^2(\Omega)\right)^d\right\}$$

1. *Give reasonable conditions on w so that $H_w(\Omega) = H^1(\Omega)$.*
2. *Prove that $H_w(\Omega)$ is a Hilbert space. What is the inner product?*
3. *Can you define the trace of f on $\partial\Omega$? First consider trace of wf.*
4. *Characterize $H_w(\mathbb{R})$ if $w(x)$ is the Heaviside function.*

1. First, we remark that an easy condition is that $w \in \mathbb{R}$ and $w \neq 0$. However, we can admit a larger class of w. By the product rule, $\nabla(wf) = f\nabla w + w\nabla f$, and if we want the result to be in L^2, then a reasonable condition is that $\nabla w \in L^\infty$. Finally, we need that $|w| \geq a > 0$.

2. It is clear that H_w is a vector space. We need to show now that it is complete. Let f_j be a Cauchy sequence in $H_w(\Omega)$. Then f_j converges in L^2 to f. At the same time, $\nabla(wf_j)$ is Cauchy in $\left(L^2\right)^d$ and converges to $h \in \left(L^2\right)^d$. Our claim is that $h = \nabla(wf)$. Indeed,
$$\langle h, \varphi\rangle \leftarrow \langle\nabla(wf_j), \varphi\rangle = \langle wf_j, \nabla\varphi\rangle \to \langle wf, \nabla\varphi\rangle = \langle\nabla(wf), \varphi\rangle$$
and so by the Lebesgue lemma, we have $h = \nabla(wf)$. The inner product on H_w is defined by
$$(f, g)_{H_w} = (f, g)_{L^2} + (\nabla(wf), \nabla(wg))_{\boldsymbol{L}^2}$$