Homework 1

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Problem 6.2:

Compute the Fourier transform of $\exp\left(-a|x|^2\right)$, $a > 0$, directly for $x \in \mathbb{R}$.

Solution First of all, notice that the Gaussian is an L^1 function. Computing directly,

Problem 6.3:

If $f \in L^1\left(\mathbb{R}^d\right)$ and $f > 0$, show that for every $\xi \neq 0$, $\left|\hat{f}\left(\xi\right)\right| < \hat{f}\left(0\right)$.

Solution By definition,

$$\hat{f}\left(0\right) = \left(2\pi\right)^{-d/2} \int\limits_{\mathbb{R}^d} f\left(x\right) dx.$$

The weak inequality follows by direct computation,

$$\begin{aligned}
\left|\hat{f}\left(\xi\right)\right| &= \left|\left(2\pi\right)^{-d/2} \int\limits_{\mathbb{R}^d} f\left(x\right) e^{-ix\cdot\xi} dx\right| \\
&\leqslant \left(2\pi\right)^{-d/2} \int\limits_{\mathbb{R}^d} \left|f\left(x\right)\right| dx \\
&\leqslant \left(2\pi\right)^{-d/2} \int\limits_{\mathbb{R}^d} f\left(x\right) dx \\
&= \hat{f}\left(0\right).
\end{aligned}$$

We argue now that $\xi = 0$ is the only point where equality is attained. Assume to the contrary that there exists a $\xi^* \neq 0$ such that $|\hat{f}\left(\xi^*\right)| = \hat{f}\left(0\right)$. However, this means that

$$\left|\int\limits_{\mathbb{R}^d} f\left(x\right) e^{-ix\cdot\xi^*} dx\right| = \int\limits_{\mathbb{R}^d} f\left(x\right) dx$$

from which we see that

$$e^{-ix\cdot\xi^*} = \pm 1$$

is a necessary condition. But, to hold for every x, ξ^* must equal zero, a contradiction.

Problem 6.4:

If $f \in L^1\left(\mathbb{R}^d\right)$ and $f(x) = g\left(|x|\right)$ for some g, show that $\hat{f}\left(\xi\right) = h\left(|\xi|\right)$ for some h. Can you relate g and h?

Solution

Problem 6.6:

Show that the Fourier transform $\mathcal{F} : L^1\left(\mathbb{R}^d\right) \rightarrow C_v\left(\mathbb{R}^d\right)$ is not onto. Show that $\mathcal{F}\left(L^1\left(\mathbb{R}^d\right)\right)$ is dense in $C_v\left(\mathbb{R}^d\right)$.

Solution Some like Fourier Inverse Theorem, Open Mapping Theorem, conclude bounded inverse, and then a contradiction with indicator function after that