Homework 9

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Unless otherwise stated, denote by $(\cdot,\cdot)$ as the L^2 inner product, and $\|\cdot\|$ as $\|\cdot\|_{L^2(\Omega)}$.

Exercise (19). *Let $\Omega \subset \mathbb{R}^4$ be a bounded domain with smooth boundary, $f \in L^2(\Omega)$. Consider the BVP*

$$-\nabla^2 u + u^3 = f$$

with $u = 0$ on $\partial\Omega$. We attempt to solve iteratively from $u_0 = 0$ by computing for each $n = 1, 2, \ldots$ the solutions to the linear BVP

$$-\nabla^2 u_n + u_{n-1}^2 u_n = f, \qquad u_n = 0 \text{ on } \partial\Omega.$$

1. *Find appropraite variational problems for the linear BVP's and show that they are well defined in $H_0^1\left(\mathbb{R}^4\right)$ provided that $u_{n-1} \in H_0^1\left(\mathbb{R}^4\right)$. (Hint: Sobolev Imbedding Theorem.)*
2. *Show that there is a unique solution $u_n \in H_0^1\left(\mathbb{R}^4\right)$ assuming that $u_{n-1} \in H_0^1(\mathbb{R}^4)$. Moreover, find a bound for the norm of u_n.*
3. *Show that the nonlinear BVP has a weak solution. Extract a subsequence of u_n that converges weakly to some u and show that u satisfies the weak form of the nonlinear BVP.*

Proof. 1. Assume that $u_{n-1} \in H_0^1\left(\mathbb{R}^4\right)$ is given. Multiply by a test function and integrate by parts the first term,

$$\begin{aligned} -\left(\nabla^2 u_n, v\right) + \left(u_{n-1}^2 u_n, v\right) &= (f, v) \\ (\nabla u_n, \nabla v) - \langle \nabla u_n \cdot n, v\rangle + \left(u_{n-1}^2 u_n, v\right) &= (f, v) \end{aligned}$$

For $v \in H_0^1(\Omega)$, we have

$$B(u_n, v) := (\nabla u_n, \nabla v) + \left(u_{n-1}^2 u_n, v\right) = (f, v) =: F(v).$$

The VP reads

$$\begin{cases} u_n \in H_0^1(\Omega), \\ (\nabla u_n, \nabla v) + \left(u_{n-1}^2 u_n v\right) = (f, v) \\ \quad \text{for every } v \in H_0^1(\Omega). \end{cases}$$

We need to check that the second integral is well-defined. We recall that for the zero spaces, we simply extend by zero to the whole space with the equality $\|f\|_{H_0^1(\Omega)} = \|f\|_{H_0^1(\mathbb{R}^4)}$.

Let $p = 2$, $j = 0$, $m = 1$, and $d = 4$. The Sobolev Embedding Theorem tells us that $H_0^1(\Omega) \equiv W_0^{1,2}(\Omega) \hookrightarrow W^{0,q}(\Omega) \equiv L^q(\Omega)$ for all finite $q \le dp/(d - mp) = 4$. We will use exactly $q = 4$ for the estimate

$$\begin{aligned} \left|\left(u_{n-1}^2 u_n, v\right)\right| &\le \|u_{n-1}\|_{L^4(\Omega)}^2 \|u_n\|_{L^4(\Omega)} \|v\|_{L^4(\Omega)} && \text{Hölder's} \\ &\le C\|u_{n-1}\|_{H_0^1(\mathbb{R}^4)}^2 \|u_n\|_{H_0^1(\Omega)} \|v\|_{H_0^1(\Omega)} && \text{Sobolev Emb.} \\ &= C\|u_{n-1}\|_{H_0^1(\mathbb{R}^4)}^2 \|u_n\|_{H_0^1(\mathbb{R}^4)} \|v\|_{H_0^1(\mathbb{R}^4)} && \text{Bd. Ext. Op.} \end{aligned}$$

So, the problem is well-defined in the space $H^1(\mathbb{R}^4)$ in the sense that all terms are well-defined, and the solution obtained from solving the VP which lives presently in $H_0^1(\Omega)$ can be easily extended to $H^1(\mathbb{R}^4)$.

2. We have a symmetric functional setting, so Lax-Milgram is the tool. The right hand side is trivially continuous by Hölder's Inequality given $f \in L^2(\Omega)$. The left hand side is also continuous,

$$\begin{aligned} B(u_n, v) &= (\nabla u_n, \nabla v) + \left(u_{n-1}^2 u_n, v\right) \\ &\leq \|\nabla u_n\|\|\nabla v\| + C\|u_{n-1}\|^2_{H_0^1(\mathbb{R}^4)}\|u_n\|_{H_0^1(\mathbb{R}^4)}\|v\|_{H_0^1(\mathbb{R}^4)} \\ &\leq \left(1 + C\|u_{n-1}\|^2_{H_0^1(\mathbb{R}^4)}\right)\|u_n\|_{H_0^1(\Omega)}\|v\|_{H_0^1(\Omega)}. \end{aligned}$$

B is also coercive,

$$\begin{aligned} B(v,v) &= \|\nabla v\|^2 + \int_\Omega u_{n-1}^2 v^2 \\ &\geq \|\nabla v\|^2 \\ &\geq C_P^{-2}\|v\|^2_{H_0^1(\Omega)}. \end{aligned}$$

By Lax-Milgram, there exists unique $u_n \in H_0^1(\Omega)$ which we can extend to $u_n \in H_0^1(\mathbb{R}^4)$. Moreover,

$$\|u_n\|_{H_0^1(\Omega)} = \|u_n\|_{H_0^1(\mathbb{R}^4)} \leq C_P^2\|f\|.$$

3. Notice that u_n has just been shown to be a bounded sequence in the Hilbert space $H_0^1(\Omega)$. Therefore, there is a subsequence $u_{n_k} \rightharpoonup u$ in $H_0^1(\Omega)$, furthermore, $u_{n_k} \to u$ in $L^2(\Omega)$ and $\nabla u_{n_k} \rightharpoonup \nabla u$ in $L^2(\Omega)^4$. We argue that u is indeed a weak solution to the nonlinear PDE by arguing that $B(u_n - u, v) \to 0$. For every test function v, $(\cdot, \nabla v)$ is a linear functional on $L^2(\Omega)^4$, so $(\nabla u_{n_k}, \nabla v) \to (\nabla u, \nabla v)$. Additionally, sequence $u_{n_k-1}^2$ is bounded, and $u_{n_k} - u \to 0$, so we conclude $u_{n_k-1}^2(u_{n_k} - u) \to 0$. (Product of a bounded sequence and a sequence converging to zero converges to zero.) That is, $B(u_n - u, v) \to 0$, so u satisfies the weak form of the nonlinear PDE.

$\square$

Exercise (21). *$\Omega \subset \mathbb{R}^d$ bounded domain with Lipschitz $\partial\Omega$, define*

$$H\left(div, \Omega\right) = \left\{ v \in (L^2(\Omega))^d : \nabla \cdot v \in L^2(\Omega) \right\}.$$

1. *Show that $H\left(div, \Omega\right)$ is a Hilbert space with inner product*

$$(u, v)_{H(div,\Omega)} = (u, v) + (\nabla \cdot u, \nabla \cdot v)\,.$$

2. *The trace Theorem does not imply that $\partial_\nu v = v \cdot \nu$ exists on $\partial\Omega$. Nevertheless, show that $\partial_\nu : H\left(div, \Omega\right) \to H^{-1/2}\left(\partial\Omega\right) = \left(H^{1/2}(\partial\Omega)\right)^*$ is a well defined bounded linear operator in the sense that*

$$\int_{\partial\Omega} v \cdot \nu \phi d\sigma(x) = \int_\Omega \nabla \cdot v \phi dx + \int_\Omega v \cdot \nabla \phi dx.$$

3. *Prove the following inf-sup condition: there exists $\gamma > 0$ such that*

$$\inf_{w \in L^2} \sup_{v \in H(div,\Omega)} \frac{(w, \nabla \cdot v)}{\|w\| \|v\|_{H(div,\Omega)}} \geq \gamma > 0.$$

Hint solve $\Delta\varphi = w$ in $H_0^1\left(\Omega\right)$ and consider $v = \nabla\varphi$.

Proof. 1. $H\left(\text{div}, \Omega\right) \subset L^2\left(\Omega\right)^d$ is clearly a linear subspace. What remains to show is that it is closed. Let u_n be a Cauchy sequence in $H\left(\text{div}, \Omega\right)$ with $u_n \to u$ in $L^2\left(\Omega\right)^d$. Since $L^2\left(\Omega\right)$ is complete, then $u \in L^2\left(\Omega\right)^d$. Furthermore, for any $\phi \in \mathcal{D}$,

$$\left(\nabla \cdot u_n, \phi\right) = -\left(u_n, \nabla\phi\right) \to -\left(u, \nabla\phi\right) = \left(\nabla \cdot u, \phi\right).$$

So, $\nabla \cdot u_n \to \nabla \cdot u$ in the distributional sense. Since u_n is Cauchy in $H\left(\text{div}, \Omega\right)$, then $\nabla \cdot u_n$ is Cauchy in $L^2\left(\Omega\right)$, and therefore $\nabla \cdot u_n \to g \in L^2\left(\Omega\right)$. Then by comparison to what we have showed, $\nabla \cdot u = g \in L^2\left(\Omega\right)$, so $u \in H\left(\text{div}, \Omega\right)$.

2. The operator is clearly linear since all the involved operations are linear. Furthermore, we have the estimate

$$\begin{aligned}\left|\int_{\partial\Omega} v \cdot \nu \phi d\sigma\left(x\right)\right| &\leqslant \left|\int_\Omega \nabla \cdot v \phi dx\right| + \left|\int_\Omega v \cdot \nabla\phi dx\right| \\ &\leqslant \left\|\nabla \cdot v\right\| \left\|\phi\right\| + \left\|v\right\| \left\|\nabla\phi\right\| \\ &\leqslant 2 \left\|v\right\|_{H(\text{div},\Omega)} \left\|\phi\right\|_{H^1(\Omega)}\end{aligned}$$

for $v \in H\left(\text{div}, \Omega\right)$ and $\phi \in H^1\left(\Omega\right)$. We remark that integration by parts holds for both spaces by density of test functions. A better approach is to start with test functions and then extend to the entire space, which is implicit here. The trace operator $\gamma_0 : H^1\left(\Omega\right) \to H^{1/2}\left(\partial\Omega\right)$ is bounded and surjective, and so for every $g \in H^{1/2}(\partial\Omega)$, there exists a $\phi \in H^1\left(\Omega\right)$ such that $\gamma_0\phi = g$ and $\|\phi\|_{H^1(\Omega)} \leq \|g\|_{H^{1/2}(\partial\Omega)}$. Thus, we write

$$\left|\int_{\partial\Omega} v \cdot \nu g d\sigma\left(x\right)\right| \leqslant 2 \left\|v\right\|_{H(\text{div},\Omega)} \left\|g\right\|_{H^{1/2}(\partial\Omega)}$$

for every $v \in H(\text{div}, \Omega)$, $g \in H^{1/2}(\partial\Omega)$. We have just demonstrated that the proposed operator ∂_ν is bounded, which implies also that it is well-defined between the spaces $H(\text{div}, \Omega)$ and $H^{-1/2}(\partial\Omega)$.

3. First consider $\nabla^2 \varphi = w$. The VP is

$$\begin{cases} \varphi \in H_0^1(\Omega) \\ (\nabla\varphi, \nabla v) = -(w, v) \\ \text{for every } v \in H_0^1(\Omega). \end{cases}$$

Since $w \in L^2(\Omega)$, the RHS is continuous. The bilinear form is continuous with constant one and coercive by the Poincaré inequality. Therefore, the problem admits a unique solution in $H_0^1(\Omega)$. Moreover, by elliptic regularity, the solution $\varphi \in H^2(\Omega)$ and $\|\varphi\|_{H^2(\Omega)} \leq C\|w\|$. Now, let us consider $v = \nabla\varphi \implies \nabla \cdot v = \nabla \cdot \nabla\varphi = \Delta\varphi = w$. Then

$$\begin{aligned} \frac{(w, \nabla \cdot (\nabla\varphi))}{\|w\|\|\nabla\varphi\|_{H(\text{div},\Omega)}} &= \frac{\|w\|^2}{\|w\|\|\nabla\varphi\|_{H(\text{div},\Omega)}} \\ &= \frac{\|w\|}{\|\nabla\varphi\|_{H(\text{div},\Omega)}} \\ &= \frac{\|w\|}{\sqrt{\|\nabla\varphi\|^2 + \|\nabla \cdot \nabla\varphi\|^2}} \\ &\geq \frac{\|w\|}{\|\varphi\|_{H^2(\Omega)}} \\ &\geq \frac{1}{C} > 0, \end{aligned}$$

which implies the result.

$\square$

Exercise (25). *Consider the finite element method.*

1. *Modify the method to account for nonhomogeneous Neumann conditions.*
2. *Modify the method to account for nonhomogeneous Dirichlet conditions.*

Proof. 1. We wish to solve

$$\begin{cases} -u'' = f & x \in (0,1) \\ u'(0) = a, \ u'(1) = b \end{cases}$$

The VP is

$$\begin{cases} u \in \tilde{H}^1(0,1) \\ (u', v') = (f, v) - av(0) + bv(1) \\ \forall v \in H^1(0,1) \end{cases}$$

which is accompanied by the compatibility condition

$$\int_0^1 f = a - b.$$

We have the same mesh setup, $h = 1/N$, $N \in \mathbb{N}$. $x_j = jh$ for $j = 0, \ldots, N$. We define the finite element space as

$$H_n = \left\{ v \in C^0(0,1) : v|_{[x_{j-1}, x_j]} \in \mathcal{P}^1(x_{j-1}, x_j) \right\}.$$

The space is dense in $H^1(\Omega)$, and when we seek the Galerkin approximation, we look in the trial space $H^1(\Omega)$ as opposed to $\tilde{H}^1$ with the understanding that we can shift the numerical solution by any constant. The discrete problem is

$$\begin{cases} u_n \in H_n \\ ({u'}_n, {v'}_n) = (f, v_n) - av_n(0) + bv_n(1) \\ \forall v \in H_n \end{cases}$$

It is sufficient to test simply with the hat functions $\phi_{n,j}$ which form a basis for H_n. We represent the solution as

$$u_n = \sum_{j=0}^{N} \alpha_j \phi_{n,j}.$$

The resulting linear system is

$$\underbrace{\sum_{j=0}^{N} \alpha_j \left({\phi_{n,j}}', {\phi_{n,i}}' \right)}_{M\alpha} = \underbrace{(f, \phi_{n,i}) - a\phi_{n,i}(0) + b\phi_{n,i}(1)}_{b}.$$

Matrix M is invertible and tridiagonal, so the problem $M\alpha = b$ has a unique solution. Once the coefficients are obtained, then any solution $u_n + c$, $c \in \mathbb{R}$ is also a solution.

2. Consider the nonhomogeneous Dirichlet boundary conditions

$$\begin{cases} -u'' = f & x \in (0,1) \\ u(0) = u_0 \\ u(1) = u_1. \end{cases}$$

Define the lift $u_D(x) = u_0 + (u_1 - u_0)x$, and write $u = w + u_D$. Remark: one may use any admissible lift that lives in the finite element space. We proved already that the end solution is independent of the choice of lift. Then, we solve essentially the same system, just with a modified right hand side. Notice now that function w satisfies the homogeneous boundary conditions. The problem is now

$$\begin{cases} w_n \in H_{n,0} \\ ({w_n}', {v_n}') = (f, v_n) - ({u_{D,n}}', {v_n}'), \quad \forall v_n \in H_{n,0} \end{cases}$$

This problem results in a similar linear system (same left hand side, actually) as the original problem for the weights α_j of the numerical solution $w_n = \sum \alpha_j \phi_{n,j}$. The ultimate solution is $u = w + u_D$.

□

Exercise (27). *Suppose $u \in H^1(\Omega)$ where $\Omega \subset \mathbb{R}^d$ is bounded connected. Recall the H^1 seminorm is $|u|_{H^1} = \left\{\sum_{|\alpha|=1} \|D^\alpha u\|^2\right\}^{1/2}$.*

1. *Show that there is a constant C_Ω such that*
$$\inf_{c\in\mathbb{R}} \|u - c\| \le C_\Omega |u|_{H^1}.$$

2. *Let $\Omega = (0,h)^d$ for $h > 0$. Show that there is a constant C independent of h and u such that*
$$\inf_{c\in\mathbb{R}} \|u - c\| \le Ch|u|_{H^1}.$$
Change var. to integrate over $(0,1)^d$ and then use previous.

3. *Let $\Omega = (0,1)^d$ and let P be the set of piecewise discontinuous constants over the grid of spacing $h = 1/N$ for some positive integer N. Show that there is a constant C independent of h, u, such that*
$$\inf_{p\in P} \|u - p\| \le Ch|u|_{H^1}.$$

Proof. 1. We always have
$$\inf_{c\in\mathbb{R}} \|u - c\| \le \|u - d\|,$$
so it is sufficient to demonstrate the result for a particular $d \in \mathbb{R}$. Let $d = |\Omega|^{-1}\int_\Omega u$. Then, $u - d$ has average zero, and the Poincare Inequality implies
$$\exists C_P > 0 : \|u - d\| \le C_P\|\nabla u\| = C_P|u|_{H^1(\Omega)}.$$

2. Use scaling arguments. Let $\xi \in \hat{\Omega} = (0,1)^d$ be the master element coordinates. Notice $x_i = h\xi_i$, $dx = h^d d\xi$. Then
$$\begin{aligned}
\|u - c\|^2_{L^2(\Omega)} &= \int_\Omega (u(x) - c)^2 dx \\
&= \int_{\hat{\Omega}} ((u\circ h)(\xi) - c)^2 h^d d\xi && (x = h\xi) \\
&= h^d \|u\circ h - c\|^2_{L^2(\hat{\Omega})} \\
&\le h^d C^2_{\hat{\Omega}} |u\circ h|^2_{H^1(\hat{\Omega})} && \text{(by previous)} \\
&= h^d C^2_{\hat{\Omega}} \|\hat{\nabla}(u\circ h)\|^2_{L^2(\hat{\Omega})^d} \\
&= h^d C^2_{\hat{\Omega}} \int_{\hat{\Omega}} (\hat{\nabla}(u\circ h)(\xi))^2 d\xi \\
&= C^2_{\hat{\Omega}} \int_\Omega h^2 (\nabla u)^2 dx \\
&= h^2 C^2_{\hat{\Omega}} |u|^2_{H^1(\Omega)}.
\end{aligned}$$
Finally, take infimum on both sides with respect to c.

3. Idea: consider each element separately, and apply the previous result. In 1D, for example, we have

$$\|u - p\|^2 = \int_0^1 (u(x) - p)^2 dx = \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} (u(x) - p)^2 dx.$$

On each interval, we can apply the previous result. Summing a finite number of times, we just increase the constant. In d dimensions, with each dimension split into N_i elements, we have $N_1 \times N_2 \times \cdots \times N_d$ small domains. Then we have

$$\begin{aligned}
\inf_{p \in P} \|u - p\|^2 &= \inf_{p \in P} \int_{\Omega} (u(x) - p)^2 dx \\
&= \sum_{i=1}^{N_1 \times \cdots \times N_d} \inf_{p_i \in \mathbb{R}} \int_{\Omega_i} (u(x) - p_i)^2 dx \\
&\leq \sum_{i=1}^{N_1 \times \cdots \times N_d} C^2 h^2 |u|^2_{H^1(\Omega_i)} && \text{(previous result on } \Omega_i) \\
&= C^2 h^2 |u|^2_{H^1(\Omega)}.
\end{aligned}$$

We remark that we have slightly glanced over the translation transformation required to shift an arbitrary element Ω_i to align with $(0, h)^d$. However, in the case where $\Omega = (0, 1)^d$, this is merely a translation which the integral does not care about.

$\square$

Exercise (29). *Consider the problem* $\Omega = (0,1) \subset \mathbb{R}$, $f \in L^2(0,1)$,

$$\begin{cases} -u'' = f & x \in \Omega \\ u(0) = u(1) = 0. \end{cases}$$

1. *Find the Green's Function.*
2. *Instead impose Neumann BC's, and find the Green's function.*
   Recall we now require $-\partial^2/\partial x^2 G(x,y) = \delta_y(x) - 1$.

Proof. 1. We want to solve

$$-\partial_x^2 G = \delta_y(x)$$

on the interval $(0,1)$ and subject to $G(0,y) = G(1,y) = 0$. We split the problem into two domains and equivalently seek the solution to

$$\begin{cases} -\partial_x^2 G_1 = 0 & x \in (0,y) \\ G_1(0,y) = 0 \end{cases}$$
$$\begin{cases} -\partial_x^2 G_2 = 0 & x \in (y,1) \\ G_2(1,y) = 0 \end{cases}$$

We see immediately that $G_1 = c_1(y)x$ and $G_2 = c_2(y)(x-1)$. The continuity and jump conditions imply

$$G_1(y,y) = G_2(y,y)$$
$$\partial_x (G_1(x,y) - G_2(x,y))|_{x=y} = 1$$

which implies that $c_1(y) = 1 - y$ and $c_2(y) = -y$. The final Green's function is

$$G(x,y) = \begin{cases} x(1-y) & x \in (0,y) \\ (1-x)y & x \in (y,1) \end{cases}$$

Finally, we verify that G satisfies the BC and is a solution to the equivalent variational problem for $v \in H_0^1(0,1)$:

$$\begin{aligned} \int_0^1 -\partial_x^2 G v dx &= \int_0^1 \partial_x G v' dx - \partial_x G v|_0^1 \\ &= \int_0^1 \partial_x G v' dx \\ &= \int_0^y (1-y) v' dx + \int_y^1 -y v' dx \\ &= v(y) \\ &= \int_0^1 \delta_y(x) v dx \end{aligned}$$

2. Essentially the same procedure. We are solving

$$-\partial_x^2 G = \delta_y(x) - 1$$

on $(0,1)$ subject to the conditions $\partial_x G(0,y) = \partial_x G(1,y) = 0$. Decompose the domain into

$$\begin{cases} -\partial_x^2 G_1 = 1 & x \in (0,y) \\ \partial_x G_1(0,y) = 0 \end{cases}$$
$$\begin{cases} -\partial_x^2 G_2 = 1 & x \in (y,1) \\ \partial_x G_2(1,y) = 0 \end{cases}$$

It is easy to solve the equations, we see that

$$G_i(x,y) = \frac{1}{2}x^2 + C_{i,1}(y)\,x + C_{i,2}(y)$$

for $i = 1, 2$. Applying the BC's,

$$\begin{aligned} G_1 &= \frac{1}{2}x^2 + C_1(y) \\ G_2 &= \frac{1}{2}x^2 - x + C_2(y)\,. \end{aligned}$$

Continuity and jump conditions imply that

$$C_1(y) = -y + C_2(y) \qquad y - (y-1) = 1.$$

We satisfy automatically the jump condition, so we choose a $C_1(y) = c(y)$ sufficiently regular as a free variable, arriving at the result

$$G(x,y) = \begin{cases} \frac{1}{2}x^2 + c(y) & x \in (0,y) \\ \frac{1}{2}x^2 - x + y + c(y) & x \in (y,1) \end{cases}$$

Finally, for $v \in H^1(\Omega)$, we have

$$\begin{aligned} \int_0^1 -\partial_x^2 G v &= \int_0^1 \partial_x G v' - \partial_x G v\Big|_0^1 \\ &= \int_0^y x v' + \int_y^1 (x-1)\,v' \\ &= \int_0^1 x v' - v(1) + v(y) \\ &= v(y) - \int_0^1 v \\ &= \int_0^1 (\delta_y - 1)\,v \end{aligned}$$

$P_m \underline{\downarrow} H_m$

□