Homework 1

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Problem 6.2:

Compute the Fourier transform of $\exp\left(-a|x|^2\right)$, $a > 0$, directly for $x \in \mathbb{R}$.

Solution First of all, notice that the Gaussian is an L^1 function, so this integral makes sense. Computing directly,

$$\hat{f}\left(\xi\right) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{-ax^2 - ix\xi} dx$$

By completing the square,

$$\hat{f}\left(\xi\right) = (2\pi)^{-1/2}\, e^{-\xi^2/4a} \int_{-\infty}^{\infty} e^{-a(x+i\xi/2a)^2} dx.$$

We attempt to evaluate the integral by considering the corresponding complex integral over a rectangle. Disregarding the details for a moment to conclude the result, the original integral is equal to

$$\hat{f}\left(\xi\right) = (2\pi)^{-1/2}\, e^{-\xi^2/4a} \sqrt{\frac{\pi}{a}}.$$

To compute the integral, we consider the counter-clockwise countour in $\mathbb{C}$ given by the vertices $R, -R, R - i\xi/2a, -R - i\xi/2a$. Since the integrand is holomorphic, Cauchy's Theorem tells us the integral over this contour must be zero. Further, the integrals on the sides of the rectangles vanish as $R \to \infty$ as they are each bounded by e^{-aR^2}. (This can be seen by letting $z(t) = -R - i\xi t/2a$ for $t \in [0, 1]$ on the left edge, say. Then plug z into the integral, expand the exponent, and see that the integral in magnitude is dominated by e^{-aR^2}.) Denote the upper edge contour as Γ_1 and the lower as Γ_3. Cauchy's Theorem tells us

$$\int_{\Gamma_1} + \int_{\Gamma_3} = 0$$

In particular, since Γ_1 is oriented from right to left, we are actually interested in $-\int_{\Gamma_1} = \int_{\Gamma_3}$. On Γ_3, $z = t - i\xi/2a$ and so the integral is

$$\int_{\Gamma_3} e^{-a(z+i\xi/2a)^2} dz = \int_{-\infty}^{\infty} e^{-at^2} dt = \sqrt{\frac{\pi}{a}}.$$

Problem 6.3:

If $f \in L^1\left(\mathbb{R}^d\right)$ and $f > 0$, show that for every $\xi \neq 0$, $\left|\hat{f}\left(\xi\right)\right| < \hat{f}\left(0\right)$.

Solution By definition,

$$\hat{f}\left(0\right) = \left(2\pi\right)^{-d/2} \int\limits_{\mathbb{R}^d} f\left(x\right) dx.$$

The weak inequality follows by direct computation,

$$\begin{aligned}\left|\hat{f}\left(\xi\right)\right| &= \left|\left(2\pi\right)^{-d/2} \int\limits_{\mathbb{R}^d} f\left(x\right) e^{-ix\cdot\xi} dx\right| \\ &\leqslant \left(2\pi\right)^{-d/2} \int\limits_{\mathbb{R}^d} \left|f\left(x\right)\right| dx \\ &\leqslant \left(2\pi\right)^{-d/2} \int\limits_{\mathbb{R}^d} f\left(x\right) dx \\ &= \hat{f}\left(0\right).\end{aligned}$$

What remains is to show that the inequality is strict. We can decompose the real and imaginary parts of the fourier transform as

$$\hat{f}\left(\xi\right) = \left(2\pi\right)^{-d/2} \int f(x) \cos\left(x\cdot\xi\right) dx - \left(2\pi\right)^{-d/2} i \int f(x) \sin\left(x\cdot\xi\right) dx$$

Now, notice that since $f > 0$, $f(x)\cos(x\cdot\xi) \leq f(x)$, and so $\Re\left(\hat{f}\right)\left(\xi\right) \leq \int f(x)dx$. If $\xi \neq 0$, then $\cos(x\cdot\xi) < 1$ apart from a set of measure zero, so equality is reached only if $\|f\|_1 = 0$, but that would imply that $f = 0$ almost everywhere, a contradiction to the assumption that $f > 0$. So, this inequality is strict whenever $\xi \neq 0$.

Problem 6.4:

If $f \in L^1\left(\mathbb{R}^d\right)$ and $f(x) = g\left(|x|\right)$ for some g, show that $\hat{f}\left(\xi\right) = h\left(|\xi|\right)$ for some h. Can you relate g and h?

Solution Essentially, we are arguing that the Fourier transform of an even function is again another even function. Its Fourier transform is

$$\hat{f}\left(\xi\right) = \left(2\pi\right)^{-d/2} \int\limits_{\mathbb{R}^d} g\left(|x|\right) e^{-ix\cdot\xi} dx$$

since f is real valued, $f = \overline{f}$ and we have

$$\overline{\hat{f}\left(\xi\right)} = \hat{f}\left(-\xi\right) = \left(2\pi\right)^{-d/2} \int\limits_{\mathbb{R}^d} g\left(|x|\right) e^{ix\cdot\xi} dx$$

Letting $u = -x$, $du = -dx$,

$$\overline{\hat{f}(\xi)} = \hat{f}(-\xi) = (2\pi)^{-d/2} \int\limits_{\mathbb{R}^d} g(|u|)\, e^{-iu\cdot\xi} du = \hat{f}(\xi).$$

That is, $\hat{f}$ is a real-valued, even function, so we conclude $\hat{f}(\xi) = h(|\xi|)$. In particular, we see that

$$\hat{f}(\xi) = (\mathcal{F}f(x))(\xi) = (\mathcal{F}g(|x|))(\xi) = h(|\xi|).$$

That is, g and h are related indirectly, since they are each being composed with the absolute value, so $h \circ |\cdot| = \mathcal{F}(g \circ |\cdot|)$.

Problem 6.6:

Show that the Fourier transform $\mathcal{F} : L^1(\mathbb{R}^d) \to C_v(\mathbb{R}^d)$ is not onto. Show that $\mathcal{F}(L^1(\mathbb{R}^d))$ is dense in $C_v(\mathbb{R}^d)$.

Solution The general strategy is to go by contradiction. We will assume that $\mathcal{F}$ is onto, and thus, by the Fourier Inversion Theorem, $\mathcal{F}$ is one to one. Then by open mapping theorem, we conclude that there must exist a bounded inverse operator from $C_v \to L^1$. Without loss of generality, let us proceed in 1-D.

Let $f_n \in C_v$ be defined as

$$f_n(x) = \begin{cases} 1 & x \in [-n, n] \\ 0 & \text{otherwise} \end{cases}$$

and consider the convolution $f_n * f_1$:

$$(f_n * f_1)(x) = \int\limits_{-\infty}^{\infty} f_n(x-y) f_1(y) dy = \int\limits_{-1}^{1} f_n(x-y) dy = \int\limits_{-1}^{1} f_n(y-x) dy$$

which is zero in the interval $(-\infty, -n-1)$, linear with slope 1 in $(-n-1, -n+1)$, unity in $(-n+1, n-1)$, linear with slope -1 in $(n-1, n+1)$, and zero in $(n+1, \infty)$. (i.e. a trapezoid). It can be shown that the sequence $f_n * f_1$ is precisely the Fourier transform of the sequence

$$g_n(x) = \frac{\sin(2\pi x)\sin(2\pi n x)}{\pi^2 x^2}.$$

Moreover,

$$\begin{aligned}
\|g_1\|_1 &= \int \frac{|\sin(2\pi x)\sin(2\pi n x)|}{\pi^2 x^2} dx \\
&= \frac{2n}{\pi} \int \frac{|\sin(x)\sin(x/n)|}{x^2} dx \\
&= \frac{4n}{\pi} \int_0^\infty \frac{|\sin(x)\sin(x/n)|}{x^2} \\
&\geq \frac{4n}{\pi} \int_0^n \frac{|\sin(x)\sin(x/n)|}{x^2}
\end{aligned}$$

To estimate, $\sin t \geq t - t^3/6 \geq 5/6t$ for $t \in [0,1]$, and $\sin(x/n) \geq 5x/6n$ on $[0,n]$. Therefore we may estimate that $\|g_n\|_1 \to \infty$ as $n \to \infty$. However, this contradicts the statement that there exists a bounded inverse operator from C_v to L^1. Therefore, the Fourier transform is not onto.

To show that the image of L^1 is dense in C_v, we first show that if f is continuous with compact support and twice differentiable, then $\hat{f} \in L^1$. Indeed, for $\xi \neq 0$,

$$\hat{f}(\xi) = \int f(x)e^{-ix\xi}dx = \frac{1}{i\xi}\int f'(x)e^{-ix\xi}dx = \frac{1}{-\xi^2}\int f''(x)e^{-ix\xi}dx$$

with each integration by parts, the boundary terms disappear due to compact support. We see that the last expression is a C_v function times $1/\xi^2$ which shows that $\hat{f}$ is bounded by a constant times ξ^{-2}. Furthermore, around zero, we have already shown that $\hat{f}$ is bounded. Hence, we conclude that the range of $\mathcal{F}$ on L^1 is dense in C_v.