

# Homework 9

Jonathan Zhang      EID: jdz357

Unless otherwise stated, denote by  $(\cdot, \cdot)$  as the  $L^2$  inner product, and  $\|\cdot\|$  as  $\|\cdot\|_{L^2(\Omega)}$ .

**Exercise (19).** Let  $\Omega \subset \mathbb{R}^4$  be a bounded domain with smooth boundary,  $f \in L^2(\Omega)$ . Consider the BVP

$$-\nabla^2 u + u^3 = f$$

with  $u = 0$  on  $\partial\Omega$ . We attempt to solve iteratively from  $u_0 = 0$  by computing for each  $n = 1, 2, \dots$  the solutions to the linear BVP

$$-\nabla^2 u_n + u_{n-1}^2 u_n = f, \quad u_n = 0 \text{ on } \partial\Omega.$$

1. Find appropriate variational problems for the linear BVP's and show that they are well defined in  $H_0^1(\mathbb{R}^4)$  provided that  $u_{n-1} \in H_0^1(\mathbb{R}^4)$ . (Hint: Sobolev Imbedding Theorem.)
2. Show that there is a unique solution  $u_n \in H_0^1(\mathbb{R}^4)$  assuming that  $u_{n-1} \in H_0^1(\mathbb{R}^4)$ . Moreover, find a bound for the norm of  $u_n$ .
3. Show that the nonlinear BVP has a weak solution. Extract a subsequence of  $u_n$  that converges weakly to some  $u$  and show that  $u$  satisfies the weak form of the nonlinear BVP.

*Proof.* 1. Assume that  $u_{n-1} \in H_0^1(\mathbb{R}^4)$  is given. Multiply by a test function and integrate by parts the first term,

$$\begin{aligned} -(\nabla^2 u_n, v) + (u_{n-1}^2 u_n, v) &= (f, v) \\ (\nabla u_n, \nabla v) - \langle \nabla u_n \cdot n, v \rangle + (u_{n-1}^2 u_n, v) &= (f, v) \end{aligned}$$

For  $v \in H_0^1(\Omega)$ , we have

$$B(u_n, v) := (\nabla u_n, \nabla v) + (u_{n-1}^2 u_n, v) = (f, v) =: F(v).$$

The VP reads

$$\begin{cases} u_n \in H_0^1(\Omega), \\ (\nabla u_n, \nabla v) + (u_{n-1}^2 u_n, v) = (f, v) \\ \text{for every } v \in H_0^1(\Omega). \end{cases}$$

We need to check that the second integral is well-defined. We recall that for the zero spaces, we simply extend by zero to the whole space with the equality  $\|f\|_{H_0^1(\Omega)} = \|f\|_{H_0^1(\mathbb{R}^4)}$ .

Let  $p = 2$ ,  $j = 0$ ,  $m = 1$ , and  $d = 4$ . The Sobolev Embedding Theorem tells us that  $H_0^1(\Omega) \equiv W_0^{1,2}(\Omega) \hookrightarrow W^{0,q}(\Omega) \equiv L^q(\Omega)$  for all finite  $q \leq dp/(d - mp) = 4$ . We will use exactly  $q = 4$  for the estimate

$$\begin{aligned} |(u_{n-1}^2 u_n, v)| &\leq \|u_{n-1}\|_{L^4(\Omega)}^2 \|u_n\|_{L^4(\Omega)} \|v\|_{L^4(\Omega)} && \text{Hölder's} \\ &\leq C \|u_{n-1}\|_{H_0^1(\mathbb{R}^4)}^2 \|u_n\|_{H_0^1(\Omega)} \|v\|_{H_0^1(\Omega)} && \text{Sobolev Emb.} \\ &= C \|u_{n-1}\|_{H_0^1(\mathbb{R}^4)}^2 \|u_n\|_{H_0^1(\mathbb{R}^4)} \|v\|_{H_0^1(\mathbb{R}^4)} && \text{Bd. Ext. Op.} \end{aligned}$$

So, the problem is well-defined in the space  $H^1(\mathbb{R}^4)$  in the sense that all terms are well-defined, and the solution obtained from solving the VP which lives presently in  $H_0^1(\Omega)$  can be easily extended to  $H^1(\mathbb{R}^4)$ .

2. We have a symmetric functional setting, so Lax-Milgram is the tool. The right hand side is trivially continuous by Hölder's Inequality given  $f \in L^2(\Omega)$ . The left hand side is also continuous,

$$\begin{aligned} B(u_n, v) &= (\nabla u_n, \nabla v) + (u_{n-1}^2 u_n, v) \\ &\leq \|\nabla u_n\| \|\nabla v\| + C \|u_{n-1}\|_{H_0^1(\mathbb{R}^4)}^2 \|u_n\|_{H_0^1(\mathbb{R}^4)} \|v\|_{H_0^1(\mathbb{R}^4)} \\ &\leq \left(1 + C \|u_{n-1}\|_{H_0^1(\mathbb{R}^4)}^2\right) \|u_n\|_{H_0^1(\Omega)} \|v\|_{H_0^1(\Omega)}. \end{aligned}$$

$B$  is also coercive,

$$\begin{aligned} B(v, v) &= \|\nabla v\|^2 + \int_{\Omega} u_{n-1}^2 v^2 \\ &\geq \|\nabla v\|^2 \\ &\geq C_P^{-2} \|v\|_{H_0^1(\Omega)}^2. \end{aligned}$$

By Lax-Milgram, there exists unique  $u_n \in H_0^1(\Omega)$  which we can extend to  $u_n \in H_0^1(\mathbb{R}^4)$ . Moreover,

$$\|u_n\|_{H_0^1(\Omega)} = \|u_n\|_{H_0^1(\mathbb{R}^4)} \leq C_P^2 \|f\|.$$

3. Notice that  $u_n$  has just been shown to be a bounded sequence in the Hilbert space  $H_0^1(\Omega)$ . Therefore, there is a subsequence  $u_{n_k} \rightharpoonup u$  in  $H_0^1(\Omega)$ , furthermore,  $u_{n_k} \rightarrow u$  in  $L^2(\Omega)$  and  $\nabla u_{n_k} \rightharpoonup \nabla u$  in  $L^2(\Omega)^4$ . We argue that  $u$  is indeed a weak solution to the nonlinear PDE. For every test function  $v$ ,  $(\cdot, \nabla v)$  is a linear functional on  $L^2(\Omega)^4$ , so  $(\nabla u_{n_k}, \nabla v) \rightarrow (\nabla u, \nabla v)$ . Additionally,  $u_{n_k-1}^2 u_{n_k} \rightarrow u^3$ , so  $(u_{n_k-1}^2 u_{n_k}, v) \rightarrow (u^3, v)$ . This implies

$$B(u_{n_k}, v) \rightarrow (\nabla u, \nabla v) + (u^3, v) = (f, v)$$

which is exactly the weak form of the original nonlinear problem.

□

**Exercise (21).**  $\Omega \subset \mathbb{R}^d$  bounded domain with Lipschitz  $\partial\Omega$ , define

$$H(\operatorname{div}, \Omega) = \left\{ v \in (L^2(\Omega))^d : \nabla \cdot v \in L^2(\Omega) \right\}.$$

1. Show that  $H(\operatorname{div}, \Omega)$  is a Hilbert space with inner product

$$(u, v)_{H(\operatorname{div}, \Omega)} = (u, v) + (\nabla \cdot u, \nabla \cdot v).$$

2. The trace Theorem does not imply that  $\partial_\nu v = v \cdot \nu$  exists on  $\partial\Omega$ . Nevertheless, show that  $\partial_\nu : H(\operatorname{div}, \Omega) \rightarrow H^{-1/2}(\partial\Omega) = (H^{1/2}(\partial\Omega))^*$  is a well defined bounded linear operator in the sense that

$$\int_{\partial\Omega} v \cdot \nu \phi d\sigma(x) = \int_{\Omega} \nabla \cdot v \phi dx + \int_{\Omega} v \cdot \nabla \phi dx.$$

3. Prove the following inf-sup condition: there exists  $\gamma > 0$  such that

$$\inf_{w \in L^2} \sup_{v \in H(\operatorname{div}, \Omega)} \frac{(w, \nabla \cdot v)}{\|w\| \|v\|_{H(\operatorname{div}, \Omega)}} \geq \gamma > 0.$$

Hint solve  $\Delta\varphi = w$  in  $H_0^1(\Omega)$  and consider  $v = \nabla\varphi$ .

*Proof.* 1.  $H(\operatorname{div}, \Omega) \subset L^2(\Omega)^d$  is clearly a linear subspace. What remains to show is that  $H(\operatorname{div}, \Omega)$  is closed with respect to  $\|\cdot\|_{H(\operatorname{div}, \Omega)}$ . Let  $u_n$  be a Cauchy sequence in  $H(\operatorname{div}, \Omega)$  with  $u_n \rightarrow u$  in  $L^2(\Omega)^d$ . **Convergence in norm? div is continuous in this norm?**

2.

3. First consider  $\nabla^2\varphi = w$ . The VP is

$$\begin{cases} \varphi \in H_0^1(\Omega) \\ (\nabla\varphi, \nabla v) = -(w, v) \\ \text{for every } v \in H_0^1(\Omega). \end{cases}$$

Since  $w \in L^2(\Omega)$ , the RHS is continuous. The bilinear form is continuous with constant one and coercive by the Poincaré inequality. Therefore, the problem admits a unique solution in  $H_0^1(\Omega)$ . Moreover, by elliptic regularity, the solution  $\varphi \in H^2(\Omega)$  and  $\|\varphi\|_{H^2(\Omega)} \leq C\|w\|$ . Now, let us consider  $v = \nabla\varphi \implies \nabla \cdot v = \nabla \cdot \nabla\varphi = \Delta\varphi = w$ . Then

$$\begin{aligned} \frac{(w, \nabla \cdot (\nabla\varphi))}{\|w\| \|\nabla\varphi\|_{H(\operatorname{div}, \Omega)}} &= \frac{\|w\|^2}{\|w\| \|\nabla\varphi\|_{H(\operatorname{div}, \Omega)}} \\ &= \frac{\|w\|}{\|\nabla\varphi\|_{H(\operatorname{div}, \Omega)}} \\ &= \frac{\|w\|}{\sqrt{\|\nabla\varphi\|^2 + \|\nabla \cdot \nabla\varphi\|^2}} \\ &\geq \frac{\|w\|}{\|\varphi\|_{H^2(\Omega)}} \\ &\geq \frac{1}{C} > 0, \end{aligned}$$

which implies the result.



**Exercise (25).** Consider the finite element method.

1. Modify the method to account for nonhomogeneous Neumann conditions.
2. Modify the method to account for nonhomogeneous Dirichlet conditions.

*Proof.* 1. Compatibility condition:

2. Consider the nonhomogeneous Dirichlet boundary conditions

$$\begin{cases} -u'' = f & x \in (0, 1) \\ u(0) = u_0 \\ u(1) = u_1. \end{cases}$$

Define the lift  $u_D(x) = u_0 + (u_1 - u_0)x$ , and write  $u = w + u_D$ .

□

**Exercise (27).** Suppose  $u \in H^1(\Omega)$  where  $\Omega \subset \mathbb{R}^d$  is bounded connected. Recall the  $H^1$  seminorm is  $|u|_{H^1} = \left\{ \sum_{|\alpha|=1} \|D^\alpha u\|^2 \right\}^{1/2}$ .

1. Show that there is a constant  $C_\Omega$  such that

$$\inf_{c \in \mathbb{R}} \|u - c\| \leq C_\Omega |u|_{H^1}.$$

2. Let  $\Omega = (0, h)^d$  for  $h > 0$ . Show that there is a constant  $C$  independent of  $h$  and  $u$  such that

$$\inf_{c \in \mathbb{R}} \|u - c\| \leq Ch |u|_{H^1}.$$

Change var. to integrate over  $(0, 1)^d$  and then use previous.

3. Let  $\Omega = (0, 1)^d$  and let  $P$  be the set of piecewise discontinuous constants over the grid of spacing  $h = 1/N$  for some positive integer  $N$ . Show that there is a constant  $C$  independent of  $h, u$ , such that

$$\inf_{p \in P} \|u - p\| \leq Ch |u|_{H^1}.$$

*Proof.* 1. We always have

$$\inf_{c \in \mathbb{R}} \|u - c\| \leq \|u - d\|,$$

so it is sufficient to demonstrate the result for a particular  $d \in \mathbb{R}$ . Let  $d = |\Omega|^{-1} \int_\Omega u$ . Then,  $u - d$  has average zero, and the Poincare Inequality implies

$$\exists C_P > 0 : \|u - d\| \leq C_P \|\nabla u\| = C_P |u|_{H^1(\Omega)}.$$

2. Use scaling arguments. Let  $\xi \in \hat{\Omega} = (0, 1)^d$  be the master element coordinates. Notice  $x_i = h\xi_i, dx = h^d d\xi$ . Then

$$\begin{aligned} \|u - c\|_{L^2(\Omega)}^2 &= \int_\Omega (u(x) - c)^2 dx \\ &= \int_{\hat{\Omega}} ((u \circ h)(\xi) - c)^2 h^d d\xi && (x = h\xi) \\ &= h^d \|u \circ h - c\|_{L^2(\hat{\Omega})}^2 \\ &\leq h^d C_\Omega^2 |u \circ h|_{H^1(\hat{\Omega})}^2 && (\text{by previous}) \\ &= h^d C_\Omega^2 \|\hat{\nabla}(u \circ h)\|_{L^2(\hat{\Omega})^d}^2 \\ &= h^d C_\Omega^2 \int_{\hat{\Omega}} (\hat{\nabla}(u \circ h)(\xi))^2 d\xi \\ &= C_\Omega^2 \int_\Omega h^2 (\nabla u)^2 dx \\ &= h^2 C_\Omega^2 |u|_{H^1(\Omega)}^2. \end{aligned}$$

Finally, take infimum on both sides with respect to  $c$ .

3. Idea: consider each element separately, and apply the previous result. In 1D, for example, we have

$$\|u - p\|^2 = \int_0^1 (u(x) - p)^2 dx = \sum_{i=1}^N \int_{x_{i-1}}^{x_i} (u(x) - p)^2 dx.$$

On each interval, we can apply the previous result. Summing a finite number of times, we just increase the constant. In  $d$  dimensions, with each dimension split into  $N_i$  elements, we have  $N_1 \times N_2 \times \cdots \times N_d$  small domains. Then we have

$$\begin{aligned} \|u - p\|^2 &= \int_{\Omega} (u(x) - p)^2 dx \\ &= \sum_{i=1}^{N_1 \times \cdots \times N_d} \int_{\Omega_i} (u(x) - p)^2 dx \\ &\leq \sum_{i=1}^{N_1 \times \cdots \times N_d} C^2 h^2 |u|_{H^1(\Omega_i)}^2 \quad (\text{previous result on } \Omega_i) \\ &= C^2 h^2 |u|_{H^1(\Omega)}^2. \end{aligned}$$

We remark that we have slightly glanced over the translation transformation required to shift an arbitrary element  $\Omega_i$  to align with  $(0, h)^d$ . However, in the case where  $\Omega = (0, 1)^d$ , this is merely a translation which the integral does not care about.

□

**Exercise (29).** Consider the problem  $\Omega = (0, 1) \subset \mathbb{R}$ ,  $f \in L^2(0, 1)$ ,

$$\begin{cases} -u'' = f & x \in \Omega \\ u(0) = u(1) = 0. \end{cases}$$

1. Find the Green's Function.

2. Instead impose Neumann BC's, and find the Green's function.

Recall we now require  $-\partial^2/\partial x^2 G(x, y) = \delta_y(x) - 1$ .

*Proof.* 1.

2.

□