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ORIGINAL PAPER

Estimation of the largest Lyapunov exponent from the perturbation vector and its derivative dot product

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Abstract Nowadays, the largest Lyapunov exponent (LLE) is employed in many different areas of the scientific research. Thus, there is still need to elaborate fast and simple methods of LLE calculation. The new method of the LLE estimation is presented in this paper. The method (LLEDP) applies the perturbation vector and its derivative dot product to calculate Lyapunov exponent in the direction of disturbance. Value of this exponent is the LLE. The theoretical improvement was introduced. Results of the numerical simulations were shown and compared with the Stefański method (Stefański in Chaos Solitons Fractals 11(15):2443–2451, 2000; Stefański and Kapitaniak in Chaos Solitons Fractals 15:233–244, 2003; Stefański et al. in Chaos Solitons Fractals 23:1651–1659, 2005; Stefański in J. Theor. Appl. Mech. 46:665–678, 2008). Investigations of the Duffing oscillators with external excitation and three Duffing oscillators coupled in ring scheme without external excitation were made with use of the presented method. Fast time LLE estimation results convergence was shown.

Keywords Lyapunov exponent · Stability · Nonlinear dynamics

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1 Introduction

Depending on the dynamical system type and kind of the information that is useful in its investigations, there are applied different types of invariants characterizing the system dynamics. One can use, for instance, Kolmogorov entropy [1] or correlation dimension [2, 3], to determine chaotic level or complexity of the system dynamics [4]. But when there is a need to predict behavior of the real system with possibility of different disturbances existence, Lyapunov exponents are one of the most often applied tools. That is because these exponents determine the exponential convergence or divergence of trajectories that start close to each other. The existence of such numbers has been proved by Oseledec theorem [5], but the first numerical study of the system behavior using Lyapunov exponents has been done by Henon and Heiles [6], before the Oseledec theorem publication. The most important algorithms for calculating Lyapunov exponents for a continuous systems have been developed by Benettin et al. [7] and Shimada and Nagashima [8], later improved by Benettin et al. [9, 10] and Wolf [11]. For the system with discontinuities or time delay, one possible approach is the estimation of Lyapunov exponents from the scalar time series basing on Takens procedure [12]. Numerical algorithms for such estimation have been developed by Wolf et al. [13], Sano and Sawada [14], and later improved by Eckmann et al. [15], Rosenstein et al. [16], and Parlitz [17].

The set of Lyapunov exponents contains much physical information characterizing the considered dynamical system, but calculation of the full spectrum demands much time and labor. Hence, only the largest Lyapunov exponent (LLE), which determines the predictability of the dynamical system, is frequently referred. That is because the presence of at least one positive Lyapunov exponent, by definition, is the most important evidence for chaos [18]. The algorithm for calculating the largest Lyapunov exponent was independently presented by Rosentein et al. [16] and Kantz [19]. These methods make use of the statistical properties of the local divergence rates of nearby trajectories. An improved algorithm based on Rosentein and Kantz was recently presented by Kim and Choe [20]. The next method of the LLE calculation was introduced by Stefański [21–24]. This method based on the synchronization phenomena allows the LLE estimation for both, continuous and not continuous systems, and thus can be applied for system with flow and maps dynamics representation.

Nowadays, LLE is employed in many different areas of the scientific research [25–34]. Thus, there is still need to elaborate fast and simple methods of LLE calculation. The new method of the LLE estimation is presented in this paper. The method applies the perturbation vector and its derivative dot product to calculate Lyapunov exponent in the direction of disturbance, which is the LLE.

2 The method

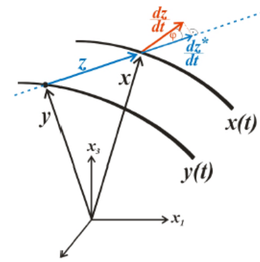
Let the dynamical system be described by the following set of the differential equations written in the vector form:

$$\frac{d\mathbf{x}}{dt} = \mathbf{F}(\mathbf{x}, t) \tag{1}$$

The method for such a system's LLE determination employs a concept of the vectors dot product. Generally, the essence of this approach is based on the analysis of the disturbance vector $\mathbf{z}(t)$ (Fig. 1). Vector $\mathbf{z}(t)$ determines difference between solution $\mathbf{y}(t)$ and disturbed solution $\mathbf{x}(t)$:

$$\mathbf{z}(t) = \mathbf{x}(t) - \mathbf{y}(t) \tag{2}$$

Fig. 1 Graphic illustration of the method



After linearization one can obtain variational equation describing approximate disturbance behavior.

$$\frac{d\mathbf{z}}{dt} = U\big(\mathbf{y}(t)\big)\mathbf{z}, \tag{3}$$

where $\mathbf{U}(\mathbf{y}(t))$ is the Jacobi matrix in the point $\mathbf{y}(t)$. Analyzing transformation of the vector $\mathbf{z}$ in the direction of the chosen eigenvector $\mathbf{w}^*$ of the $\mathbf{U}$ transformation it can be written:

$$\frac{d\mathbf{z}^*}{dt} = \lambda^* \mathbf{z}^* \tag{4}$$

After transformation,

$$\frac{d\mathbf{z}^*}{\mathbf{z}^*} = \lambda^* \, dt \tag{5}$$

and integration for $\mathbf{z}^*(0) = \mathbf{z}_0^*$ one obtains

$$\mathbf{z}^* = \mathbf{z}_0^* e^{\lambda^* t} \tag{6}$$

where $\mathbf{z}^*$—component of the vector $\mathbf{z}$ in direction of the eigenvector $\mathbf{w}^*$. λ^*—eigenvalue in direction of the eigenvector $\mathbf{w}^*$.

Equation (6) describes mean transformation of the vector $\mathbf{z}^*$ after time t. For $t \to \infty$, λ^* is a value of Lyapunov exponent in direction of $\mathbf{w}^*$. For n-dimensional eigenvectors set $(\mathbf{w}_1^*, \mathbf{w}_2^*, \dots, \mathbf{w}_n^*)$ and respective eigenvalues $(\lambda_1^*, \lambda_2^*, \dots, \lambda_n^*)$, one can obtain the whole disturbance $\mathbf{z}$ transformation in the following form:

$$\mathbf{z} = a_1 \mathbf{w}_1^* e^{\lambda_1^* t} + a_2 \mathbf{w}_2^* e^{\lambda_2^* t} + \dots + a_2 \mathbf{w}_2^* e^{\lambda_n^* t} \tag{7}$$

where $a_1, a_2, \dots, a_n$ depend on the initial conditions. In the case $\lambda^* > \lambda_1^*, \lambda_2^*, \dots, \lambda_n^*$ as $t \to \infty$ terms $e^{\lambda_1^* t}, e^{\lambda_2^* t}, \dots, e^{\lambda_n^* t}$ become negligible and perturbation evolves as $e^{\lambda^* t}$ what is of the most importance for presented method of Largest Lyapunov Exponent (LLE) calculation. LLE is of course value of λ^*.

In the presented method, (4) is used for the system stability determination. Using vectors dot product properties (Fig. 1):

$$\mathbf{z} \circ \frac{d\mathbf{z}}{dt} = |\mathbf{z}| \left| \frac{d\mathbf{z}}{dt} \right| \cos(\varphi) \tag{8}$$

After transformations

$$\frac{\frac{d\mathbf{z}^*}{dt}}{\mathbf{z}} = \frac{\mathbf{z} \circ \frac{d\mathbf{z}}{dt}}{|\mathbf{z}|^2} = \lambda^* \tag{9}$$

The average value $\hat{\lambda}^*$ of the λ^* is the parameter allowing to determine stability of the system in the direction of the disturbance $\mathbf{z}$. Following presented explanations it is the value of the LLE.

3 Numerical simulations

Program code was written in Delphi. The numerical simulations were based on integration of the differential equations (1), (3) with use of the Runge–Kutta (RK4) method of the fourth order. The step of integration was fit to the system type with use of the procedure of integration with control of the integration error. Then it was applied into the procedure RK4 with constant integration step. Both the equations sets, the system variables $\mathbf{x}$ and the perturbation variables $\mathbf{z}$, were integrated in each step of RK4. For the linear system analysis of the second of them would be enough, but for the considered in the paper nonlinear cases $\mathbf{z}$ derivative is dependent on $\mathbf{x}$. After each integration step considered in (9) the value λ^* was calculated.

$$\frac{\mathbf{z} \circ \frac{d\mathbf{z}}{dt}}{|\mathbf{z}|^2} = \lambda^* \tag{10}$$

An average value of the $\hat{\lambda}^*$ was calculated in time of all the computations. The attained value of the LLE was the $\hat{\lambda}^*$ value for the $\hat{\lambda}^*$ increment smaller than the matched ε value.

Program code was written in Delphi. The numerical simulations were based on integration of the differential equations (1), (3) with use of the Runge–Kutta (RK4) method of the fourth order. The step of integration was fit to the system type by comparing results of the ones obtained with use of the procedure of integration with control of the integration error (Runge–Kutta4system [35]) with results obtained from standard procedure (RK4). The step value was referred to the average oscillations period, and the value of the required steps for the average period T was obtained. Then it was applied into (1), (3) integration with constant integration step, and fit to each system parameters set with use of the required steps per average period T. Both the equations sets, the system variables $\mathbf{x}$ and the $\mathbf{z}$, were integrated in each step of RK4.

To obtain the value λ^* at first dot product of the vectors $\mathbf{z}$ and $\frac{d\mathbf{z}}{dt}$ were calculated. The vector $\mathbf{z}$ coordinates were estimated from (4) with use of the procedure (RK4). $\frac{d\mathbf{z}}{dt}$ vector was obtained with use (4); also, but it was got from inside of the RK4 procedure during estimation of $\mathbf{z}$ coordinates. Vector $\mathbf{z}$ was normalized before integration in the RK4 procedure, what preserved it from the fast growth.

Then Standard dot product in form

$$\mathbf{z} \circ \frac{d\mathbf{z}}{dt} = \sum_{i=1}^{m} \left(z_i \frac{dz_i}{dt} \right) \tag{11}$$

m—the dimension of the node system space was applied according (10) to calculate actual value λ^*:

$$\lambda^* = \frac{\mathbf{z} \circ \frac{d\mathbf{z}}{dt}}{|\mathbf{z}|^2} \tag{12}$$

The value λ^* was calculated at each integration step. To obtain its average value $\hat{\lambda}^*$ actual values λ^* were summed up in time of all the computations, and after fixed number of the average periods T an average value $\hat{\lambda}^*$ of the λ^* was calculated. Computations were continued till the increment $\delta\hat{\lambda}^*$ of the $\hat{\lambda}^*$ in the number of the consecutive $\hat{\lambda}^*$ calculations periods was smaller than the matched ε value.

$$|\delta\hat{\lambda}^*| < \varepsilon \tag{13}$$

The latest attained value $\hat{\lambda}^*$ was accepted for TLE value.

Results of the numerical simulations were compared with the Stefański method of LLE estimation [21–24] and comparison was presented in following paragraphs.

The basics of the Stefański method result from the connection between the value of the LLE and coupling coefficient between two synchronized identical systems. To estimate LLE of a dynamical system with use of this method one has to take the second identical system, then make a coupling between them and increase coupling coefficient till systems synchronize.

Fig. 2 Time dependence of the λ^* and $\hat{\lambda}^*$ (λ^* —— and $\hat{\lambda}^*$ ▬)

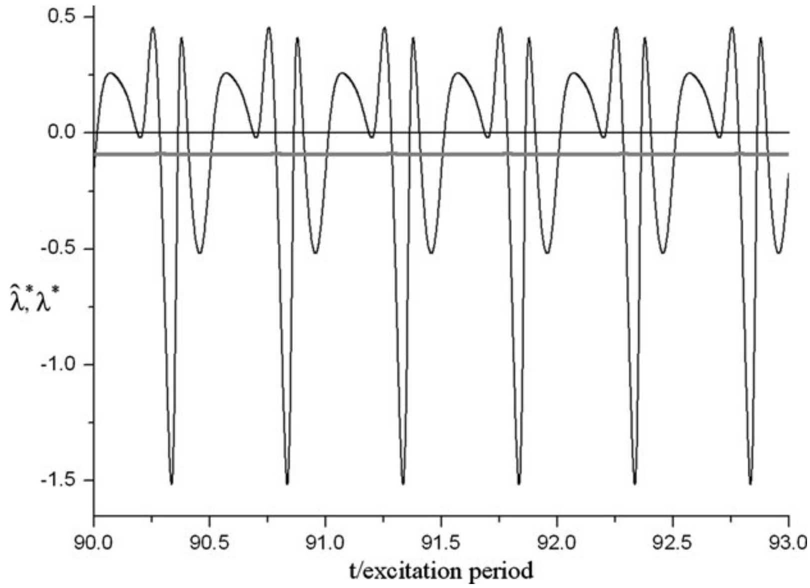
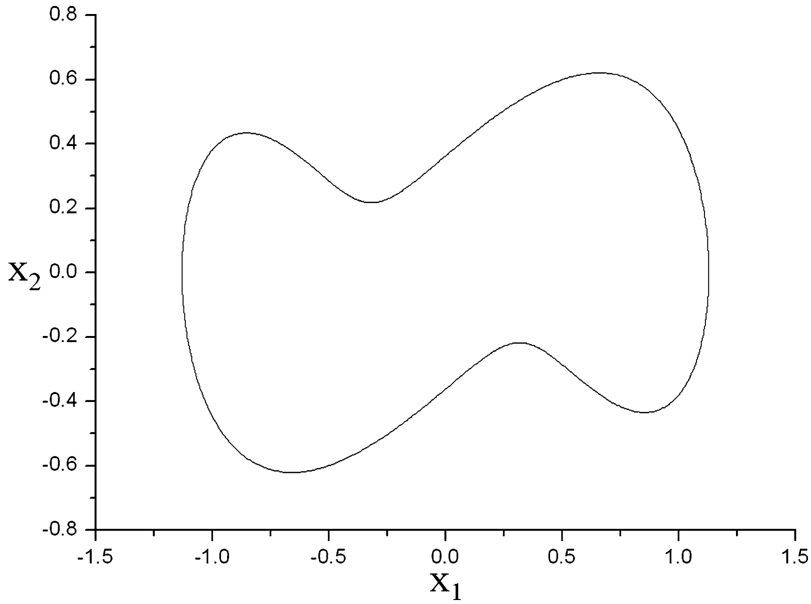


Fig. 3 The phase plane of x_1 and x_2



Value of this coefficient for the state of the complete synchronization is the value of the LLE. Theoretical background for the method is given in [21–24]. The strength of Stefański method is its effectiveness in very wide spectrum of dynamical systems, continuous, not continuous, with time delay, discrete ones and so on. On the second hand, estimation of LLE with use of it requires for each system's parameters set testing the wide spectrum of the coupling coefficients to find the value of the synchronization coefficient. Such a process requires a lot of computations, what is a weakness of the method.

3.1 Duffing oscillator with external excitation

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\beta x_2 - \alpha x_1^3 + q\sin(eta * t) \\
\dot{z}_1 &= z_2 \\
\dot{z}_2 &= -\beta z_2 - 3\alpha x_1^2 z_1
\end{aligned}
\tag{14}
$$

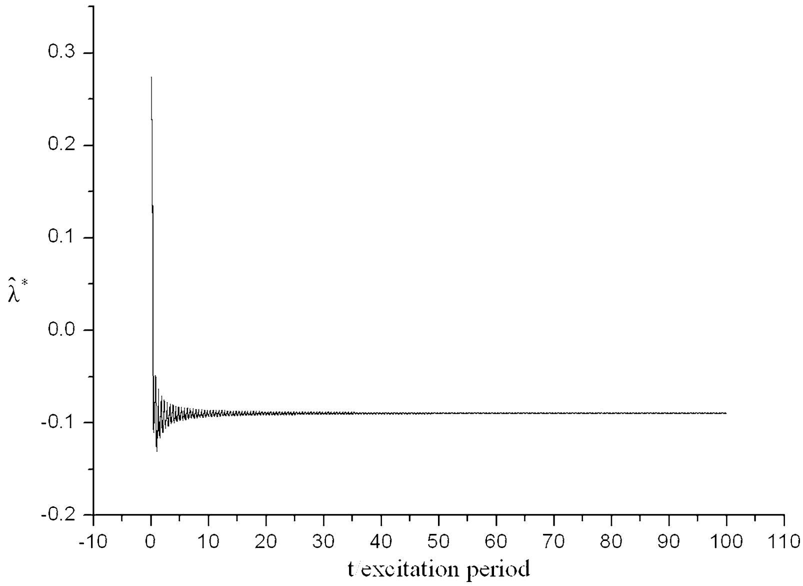


Fig. 4 Time dependence of the $\hat{\lambda}^*$

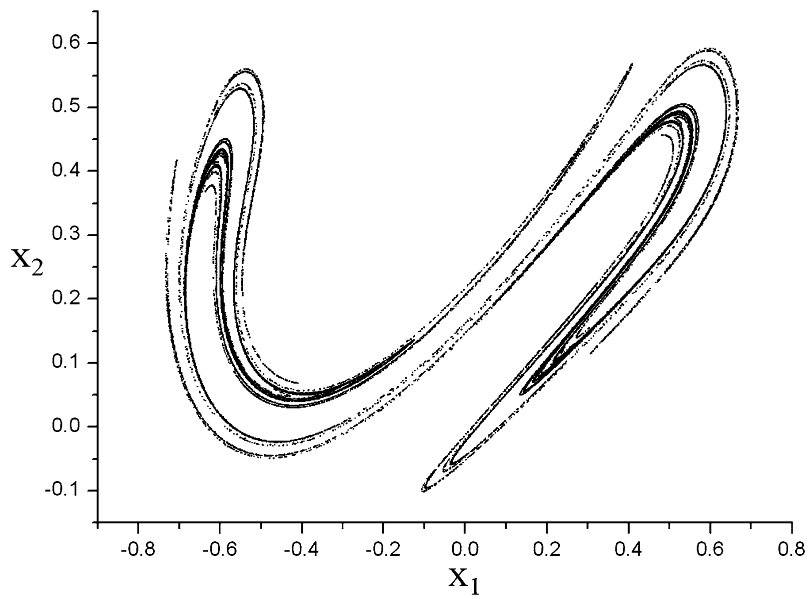


Fig. 5 The Poincaré map

The time dependent results of λ^* and $\hat{\lambda}^*$ estimation for the periodically excited Duffing system described by (14) is shown in Fig. 2.

One can see large oscillations of the λ^* value and very stable $\hat{\lambda}^*$ value. The phase portrait for considered case is shown in Fig. 3.

In Fig. 4, one can see that in the considered case value of the $\hat{\lambda}^*$ stabilizes in almost 20 excitation periods.

Estimation of the LLE in the chaotic state can be observed in Fig. 6. Poincaré map for that range of parameters is shown in Fig. 5. In that case, $\hat{\lambda}^*$ stabilization requires about three times more time than in the periodic state.

In the bifurcation diagram (Fig. 7), it can be seen different types of the system dynamics and values of LLE confirming chaotic, periodic dynamics existence. Zero LLE values that determine period doubling bifurcation points are visible as well. The time of the LLE estimation for each bifurcation parameter value depends on the actual $\hat{\lambda}^*$ increment. The attained value of the LLE was the $\hat{\lambda}^*$ value for the $\hat{\lambda}^*$ increment

Fig. 6 Time dependence of the $\hat{\lambda}^*$

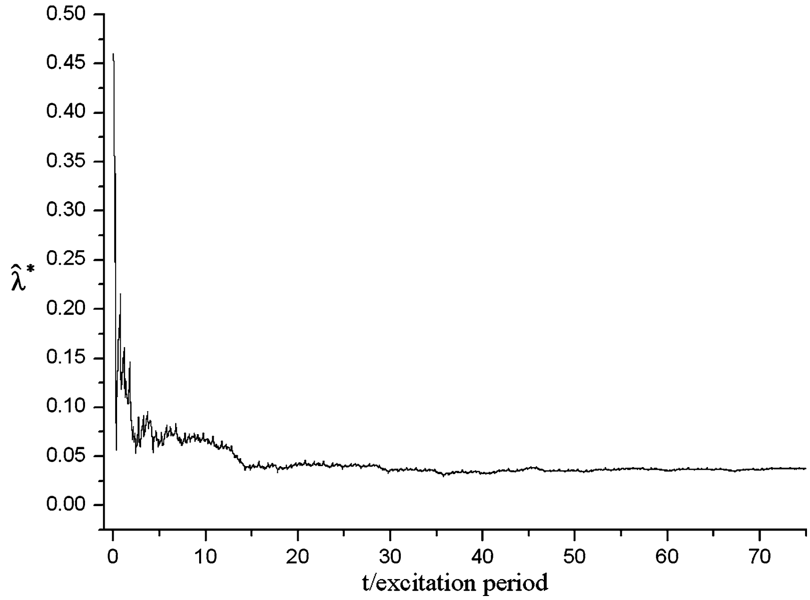
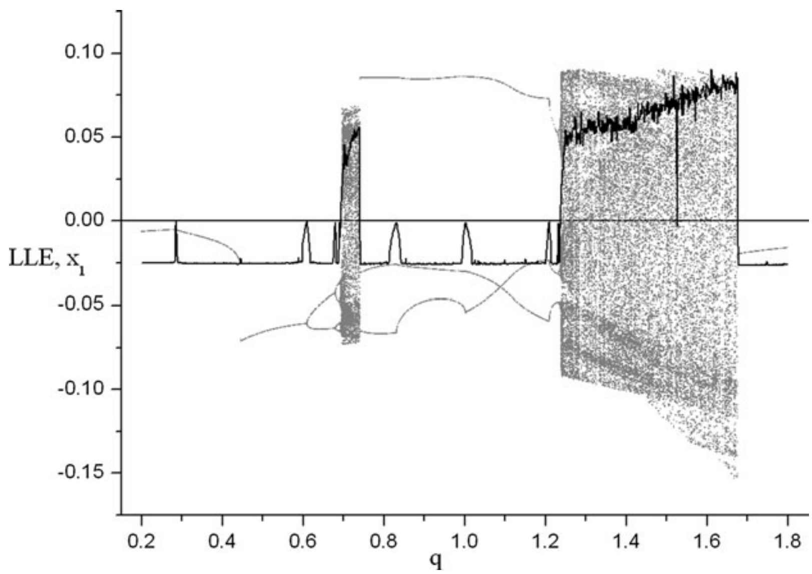


Fig. 7 Bifurcation diagram of the LLE and x_1



smaller than the matched ε value. To determine the exact bifurcation parameter values maintaining the fast algorithm operation, ε was changed depending on the actual LLE value.

Comparison of the LLE estimated with use of the presented method and Stefański method [20] based on the systems synchronization phenomena is shown in Fig. 8. While the results are very similar, times of simulations differed more than one magnitude order.

3.2 Three duffing oscillators coupled directionally in a ring scheme

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\beta x_2 - \alpha x_1^3 - \alpha_2(x_1 - x_5) \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= -\beta x_4 - \alpha x_3^3 - \alpha_2(x_3 - x_1) \\
\dot{x}_5 &= x_6 \\
\dot{x}_6 &= -\beta x_6 - \alpha x_5^3 - \alpha_2(x_5 - x_3)
\end{aligned} \tag{15}
$$

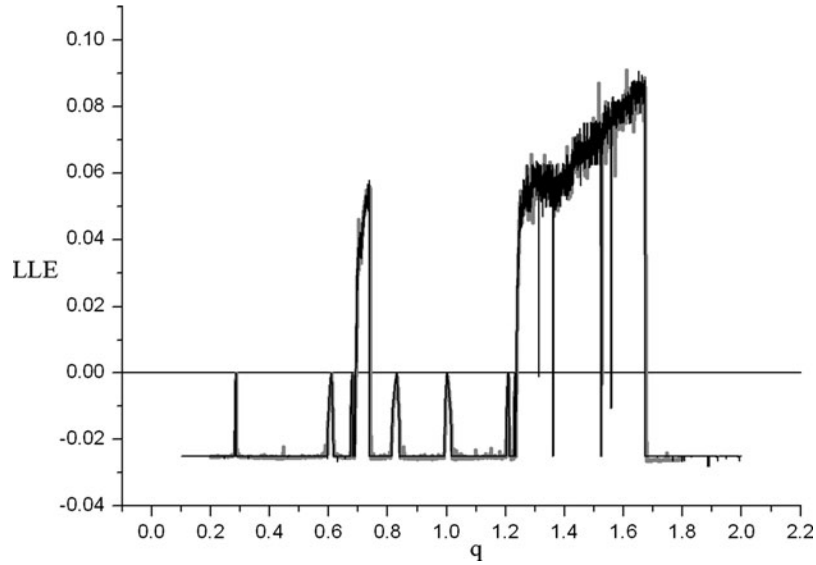


Fig. 8 Comparison of the LLE bifurcation diagrams estimated with use if the presented method and Stefański method

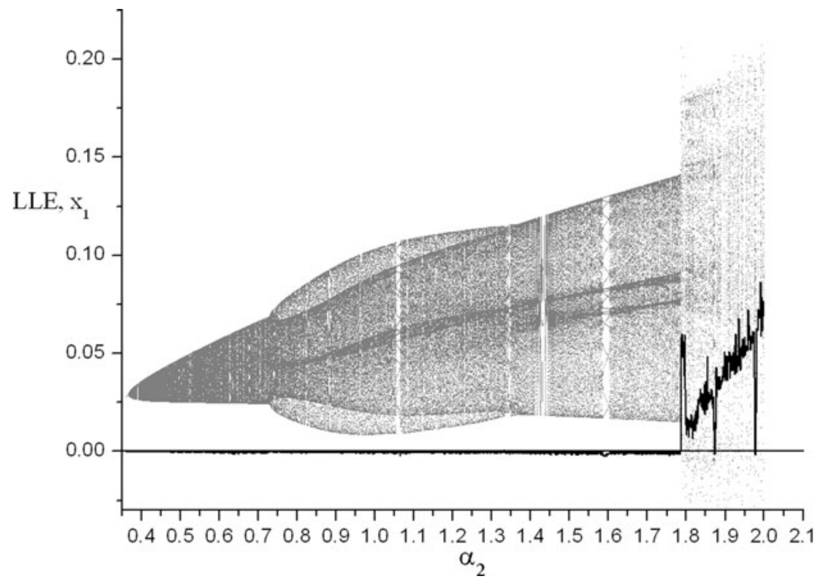


Fig. 9 Bifurcation diagram of the LLE and x_1

$$
\begin{aligned}
\dot{z}_1 &= z_2 \\
\dot{z}_2 &= -\beta z_2 - 3\alpha x_1^2 z_1 - \alpha_2(z_1 - z_5) \\
\dot{z}_3 &= z_4 \\
\dot{z}_4 &= -\beta z_4 - 3\alpha x_3^2 z_3 - \alpha_2(z_3 - z_1) \\
\dot{z}_5 &= z_6 \\
\dot{z}_6 &= -\beta z_6 - 3\alpha x_5^2 z_5 - \alpha_2(z_5 - z_3)
\end{aligned}
$$

Different dynamics can be observed for the three Duffing oscillators coupled one directionally in the ring scheme. Dynamics of the system is described by (15). One can see in Fig. 9 chaotic and quasiperiodic regimes of the system. Poincaré map in the quasiperiodic state can be seen in Fig. 10. Estimation of the LLE in the quasiperiodic state can be observed in the Fig. 11.

For the range of quasiperiodic dynamics, obtaining the zero LLE value requires much more time of the simulations and accuracy of the numerical integration (see Fig. 9). Comparing time of the LLE stabilization in Fig. 11 with Fig. 4 or Fig. 6, one can see that LLE stabilization requires much more time of computations.

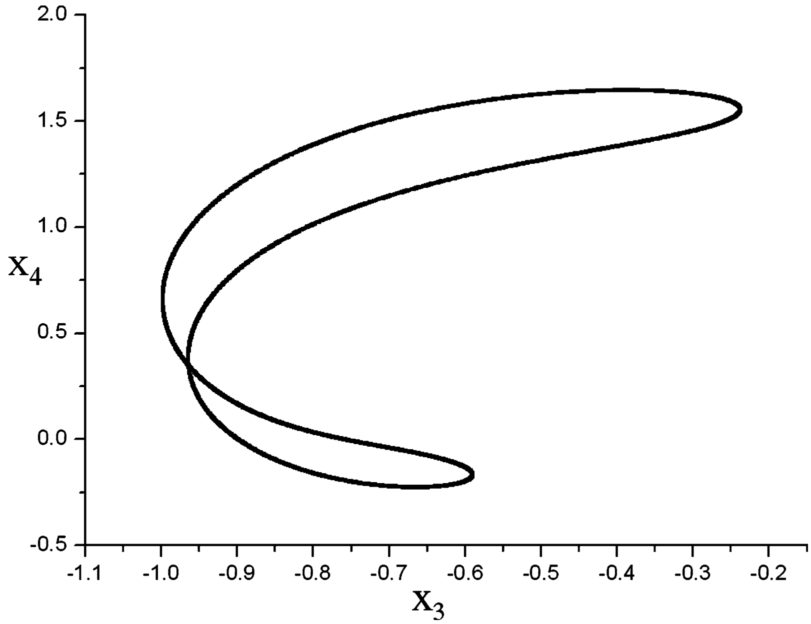


Fig. 10 The Poincaré map

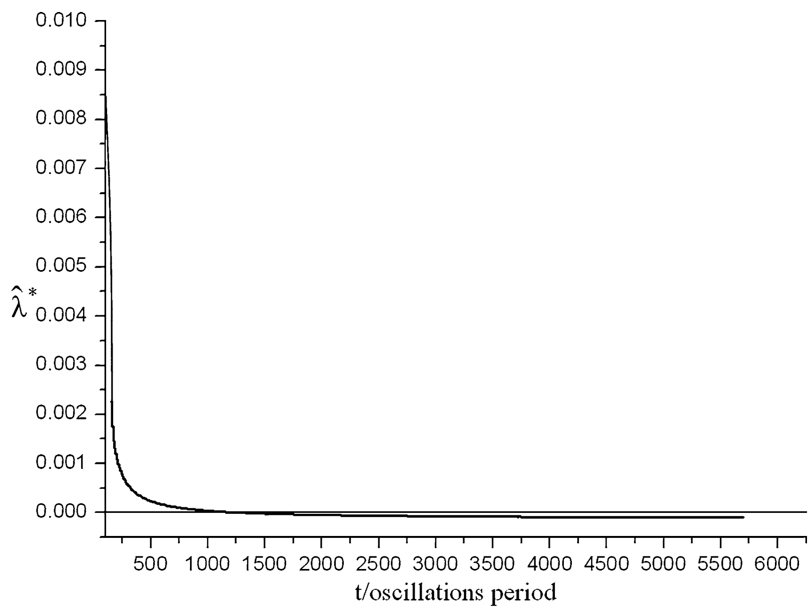


Fig. 11 Time dependence of the $\hat{\lambda}^*$

4 Conclusions

The new method of the LLE estimation is presented in this paper. The method applies the perturbation vector and its derivative dot product to calculate Lyapunov exponent in the direction of disturbance. Value of this exponent is the LLE. The theoretical improvement was introduced. It was shown that basics of the method are very simple. Results of the numerical simulations were shown and compared with the Stefański method [21–24]. Investigations of the Duffing oscillators with external excitation and three Duffing oscillators coupled in ring scheme without external excitation were made with use of the presented method. Periodic, quasiperiodic, and chaotic regimes of the system were shown. Times of the LLE stabilization were compared for these cases. Fast time LLE results convergence was proved.

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