Chapter 3, section 3.2: **Nonlinear** equations

Newton's method can be used to find a root of an equation with single variable

$$f(x) = 0.$$

It can also be applied to solve a system of equations.

In Newton's method, the updating formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \big(f'(x_n)\big)^{-1} f(x_n).$$

Derivation: We use the first two terms in Taylor series to approximate f at x_{n+1}, and set it equals zero:

$$f(x_{n+1}) = f(x_n + h) \approx f(x_n) + hf'(x_n) = 0$$

$$\Rightarrow \qquad h = -\frac{f(x_n)}{f'(x_n)}.$$

3 nonlinear equations with 3 variables

Now we solve a system $\boldsymbol{F}(\boldsymbol{X}) = 0$.

The system can be displayed in the form

$$\begin{aligned} f_1\left(x_1, x_2, x_3\right) &= 0 \\ f_2\left(x_1, x_2, x_3\right) &= 0 \\ f_3\left(x_1, x_2, x_3\right) &= 0 \end{aligned}$$

We start with the initial guess (x_1^0, x_2^0, x_3^0), which is close to the solution of the nonlinear system.

Recall the Taylor expansion at $\boldsymbol{X}^0 = (x_1^0, x_2^0, x_3^0)$ with step size $\boldsymbol{H} = (h_1, h_2, h_3) \rightarrow$ next slide.

3 nonlinear equations with 3 variables

Linearization / truncated Taylor series gives

$$
\begin{aligned}
0 &= f_1\left(x_1^0 + h_1, x_2^0 + h_2, x_3^0 + h_3\right) \\
&\approx f_1\left(x_1^0, x_2^0, x_3^0\right) + \frac{\partial f_1}{\partial x_1} h_1 + \frac{\partial f_1}{\partial x_2} h_2 + \frac{\partial f_1}{\partial x_3} h_3 \\
0 &= f_2\left(x_1^0 + h_1, x_2^0 + h_2, x_3^0 + h_3\right) \\
&\approx f_2\left(x_1^0, x_2^0, x_3^0\right) + \frac{\partial f_2}{\partial x_1} h_1 + \frac{\partial f_2}{\partial x_2} h_2 + \frac{\partial f_2}{\partial x_3} h_3 \\
0 &= f_3\left(x_1^0 + h_1, x_2^0 + h_2, x_3^0 + h_3\right) \\
&\approx f_3\left(x_1^0, x_2^0, x_3^0\right) + \frac{\partial f_3}{\partial x_1} h_1 + \frac{\partial f_3}{\partial x_2} h_2 + \frac{\partial f_3}{\partial x_3} h_3
\end{aligned}
$$

Newton's method

The blue equations are linear in (h_1, h_2, h_3):

$$
\begin{aligned}
0 &= f_1\left(x_1^0 + h_1, x_2^0 + h_2, x_3^0 + h_3\right) \\
&\approx f_1\left(x_1^0, x_2^0, x_3^0\right) + \frac{\partial f_1}{\partial x_1} h_1 + \frac{\partial f_1}{\partial x_2} h_2 + \frac{\partial f_1}{\partial x_3} h_3 = 0 \\
0 &= f_2\left(x_1^0 + h_1, x_2^0 + h_2, x_3^0 + h_3\right) \\
&\approx f_2\left(x_1^0, x_2^0, x_3^0\right) + \frac{\partial f_2}{\partial x_1} h_1 + \frac{\partial f_2}{\partial x_2} h_2 + \frac{\partial f_2}{\partial x_3} h_3 = 0 \\
0 &= f_1\left(x_1^0 + h_1, x_2^0 + h_2, x_3^0 + h_3\right) \\
&\approx f_3\left(x_1^0, x_2^0, x_3^0\right) + \frac{\partial f_3}{\partial x_1} h_1 + \frac{\partial f_3}{\partial x_2} h_2 + \frac{\partial f_3}{\partial x_3} h_3 = 0
\end{aligned}
$$

We use the first two terms in Taylor series to approximate $\boldsymbol{F}$ at $\boldsymbol{X}^0 + H$, and set it equals zero.

Newton's method

The blue system can be rewritten as:

$$\frac{\partial f_1}{\partial x_1}h_1 + \frac{\partial f_1}{\partial x_2}h_2 + \frac{\partial f_1}{\partial x_3}h_3 = -f_1\left(x_1^0, x_2^0, x_3^0\right)$$
$$\frac{\partial f_2}{\partial x_1}h_1 + \frac{\partial f_2}{\partial x_2}h_2 + \frac{\partial f_2}{\partial x_3}h_3 = -f_2\left(x_1^0, x_2^0, x_3^0\right)$$
$$\frac{\partial f_3}{\partial x_1}h_1 + \frac{\partial f_3}{\partial x_2}h_2 + \frac{\partial f_3}{\partial x_3}h_3 = -f_3\left(x_1^0, x_2^0, x_3^0\right)$$

Or:

$$\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} & \frac{\partial f_3}{\partial x_3} \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix} = - \begin{bmatrix} f_1\left(x_1^0, x_2^0, x_3^0\right) \\ f_2\left(x_1^0, x_2^0, x_3^0\right) \\ f_3\left(x_1^0, x_2^0, x_3^0\right) \end{bmatrix}$$

Newton's method for nonlinear equations

The System from the former slide:

$$\underbrace{\boldsymbol{F'}(\boldsymbol{X}^0)}_{\text{matrix}}\underbrace{\boldsymbol{H}}_{\text{vector}} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} & \frac{\partial f_3}{\partial x_3} \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix} = - \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = -\underbrace{\boldsymbol{F}(\boldsymbol{X}^0)}_{\text{vector}},$$

where $\boldsymbol{F'}$ is called as **Jacobian matrix** for the vector function $\boldsymbol{F}$.

A Newton iteration from $\mathrm{X}^{(k)}$ is: Solve the linear system

$$\boldsymbol{F'}(\boldsymbol{X}^{(k)})\boldsymbol{H}^{(k)} = -\boldsymbol{F}(\boldsymbol{X}^{(k)})$$

(we assume that the matrix $\boldsymbol{F'}(\boldsymbol{X}^{(k)})$ is nonsingular) and update

$$\boldsymbol{X}^{(k+1)} = \boldsymbol{X}^{(k)} + \boldsymbol{H}^{(k)}.$$

Remark: For a system of equations, $(f'(x_k))^{-1}f(x_k)$ becomes $\left(\boldsymbol{F'}(\boldsymbol{X}^{(k)})\right)^{-1}\boldsymbol{F}(\boldsymbol{X}^{(k)})$.

Example: Newton's method for solving 2 nonlinear equations

The complex equation $z^3 = 1 \Leftrightarrow z^3 - 1 = 0$ has the simple roots

$$1 + 0i \quad \text{and} \quad -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}, \qquad i^2 = -1.$$

With $z = x + iy$ we have the complex equation

$$0 = z^3 - 1 = (x + iy)^3 - 1 = x^3 + 3ix^2y - 3xy^2 - iy^3 - 1 = 0 + i0$$

which corresponds to two real equations (for real- and imaginary-part, respectively):

$$\boldsymbol{F}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x^3 - 3xy^2 - 1 \\ 3x^2y - y^3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The Jacobian matrix of $\boldsymbol{F}$ is

$$\boldsymbol{F}' = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix} = \begin{bmatrix} 3x^2 - 3y^2 & -6xy \\ 6xy & 3x^2 - 3y^2 \end{bmatrix}.$$

Example: Newton's method for solving 2 nonlinear equations

With the initial guess $\boldsymbol{X}^{(0)} = \begin{bmatrix} 1 \\ 0.1 \end{bmatrix}$, $\boldsymbol{F}\left(\begin{bmatrix} 1 \\ 0.1 \end{bmatrix}\right) = \begin{bmatrix} -0.03 \\ 0.299 \end{bmatrix}$. The Jacobian matrix is

$$\boldsymbol{F'}\left(\boldsymbol{X}^{(0)}\right) = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix} = \begin{bmatrix} 2.97 & -0.6 \\ 0.6 & 2.97 \end{bmatrix}.$$

Now, we solve the linear system

$$\boldsymbol{F'}\left(\boldsymbol{X}^{(0)}\right) \boldsymbol{H}^{(0)} = -\boldsymbol{F}\left(\boldsymbol{X}^{(0)}\right),$$

for $\boldsymbol{H}^{(0)}$ and update $\boldsymbol{X}^{(1)} = \boldsymbol{X}^{(0)} + \boldsymbol{H}^{(0)}$. The new $\boldsymbol{X}^{(1)} = \begin{bmatrix} 0.9902 \\ 0.0013 \end{bmatrix}$ is obviously closer to the nearest root $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

After $0, 1, 2, 3, 4$ iterations, the errors to this root is

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0.10, 1.0e-2, 1.0e-4, 1.0e-8, 8.7e-17.
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How fast does Newton's method converge?

Example: Which root does Newton's method find?

In the figure, each point is assigned to the same color as the root

$1 + 0i$ (blue)

$-\frac{1}{2} + i\frac{\sqrt{3}}{2}$ (green)

$-\frac{1}{2} - i\frac{\sqrt{3}}{2}$ (red)

that Newton's method converges to.