02601 Introduction to numerical algorithms

Homework assignment 4

Hand in on DTU Learn before 5 December 10pm

1 Multiple choice (40%)

Each question has only ONE correct answer. In the pdf file with your answers, you just need type the number of the answer. You get 10% for a correct answer, 0% for no answer, and -5% for a wrong answer.

A – Permutation matrix. When we multiply a $n \times n$ matrix $\boldsymbol{A}$ from the left side with a permutation matrix $\boldsymbol{P}$, the following result applies to the product $\boldsymbol{P}\boldsymbol{A}$:

1. We calculate the upper triangular $\boldsymbol{U}$ in LU factorization.
2. We interchange rows in the matrix $\boldsymbol{A}$.
3. We interchange columns in the matrix $\boldsymbol{A}$.
4. We reduce the number of “Long Operations” (LOps) from $\frac{1}{3}n^3$ to $\frac{1}{2}n^2$.

B – LU factorization with partial pivoting. Apply LU factorization with partial pivoting on the matrix

$$\boldsymbol{A} = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 1 & 1 \\ 3 & 2 & 1 \end{bmatrix}$$

and obtain $\boldsymbol{P}\boldsymbol{A} = \boldsymbol{L}\boldsymbol{U}$, where $\boldsymbol{P}$ is a permutation matrix, $\boldsymbol{L}$ is a unit lower-triangular matrix, and $\boldsymbol{U}$ is an upper triangular matrix. Concerning the pivoting $\boldsymbol{P}$, which of the following statement is true?

1. It’s not necessary to use pivoting.
2. We need partial pivoting. We need interchange row 1 and 3.
3. We need partial pivoting. We need interchange row 1 and 2.
4. We need partial pivoting. We need interchange row 1 and 3, and then interchange row 2 and 3.

C – Sensitivity analysis. We consider two systems of linear equations

$$\boldsymbol{A}\,\boldsymbol{x} = \boldsymbol{b}, \qquad \boldsymbol{A}\,\tilde{\boldsymbol{x}} = \tilde{\boldsymbol{b}}, \qquad \tilde{\boldsymbol{b}} = \boldsymbol{b} + \delta\boldsymbol{b}. \tag{1}$$

where $\boldsymbol{A}$ is a 100×100 matrix that is invertible. We set $\delta\boldsymbol{x} = \tilde{\boldsymbol{x}} - \boldsymbol{x}$, the condition number of $\boldsymbol{A}$ is denoted by $\kappa(\boldsymbol{A})$, and we have

$$\kappa(\boldsymbol{A}) = 5.3, \qquad \|\boldsymbol{b}\|_2 = 1.4, \qquad \boldsymbol{x} = [1, -1, 1, -1, \cdots].$$

If the absolute error $\delta\boldsymbol{b}$ satisfies:

$$\|\delta\boldsymbol{b}\|_2 \leq 0.07, \tag{2}$$

we obtain the *upper bound* for the relative error $\|\delta\boldsymbol{x}\|_2 / \|\boldsymbol{x}\|_2$ in the solution as

1. 0.371.
2. 0.265.
3. 2.65.
4. 7.42.

D – Solving linear system. For a given 1000-by-1000 matrix A, assume that it takes about 10 seconds to find the inverse of A by the use of the LU factorization, that is, calculating LU factorization once, and then doing forward substitution and back substitution 1000 times using the 1000 columns of the identity matrix as the right hand side vector. The approximate time, in seconds, that it will take to find the inverse of A by repeated use of Naive Gaussian Elimination method, that is, doing Gaussian elimination and back substitution 1000 times by using the 1000 columns of the identity matrix is

1. 500.
2. 2500.
3. 7500.
4. 10000.

2 Newton's method for systems of nonlinear equations (40%)

Consider a system of nonlinear equations $\boldsymbol{F}(\boldsymbol{X}) = \boldsymbol{0}$ with

$$\boldsymbol{X} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \text{and} \qquad \boldsymbol{F}(\boldsymbol{X}) = \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix} = \begin{bmatrix} x_1^2 + 2x_2 - 2 \\ x_1 + 4x_2^2 - 4 \end{bmatrix}.$$

We use Newton's method to find a solution with a given starting point.

1. (10%) Derive the Jacobian matrix $\boldsymbol{F}'$ for the vector function $\boldsymbol{F}$, and write a Python function `funFdF` to return the vector $\boldsymbol{F}(\boldsymbol{X})$ and the Jacobian matrix $\boldsymbol{F}'(\boldsymbol{X})$ with a given $\boldsymbol{X}$. Include your Python function in the pdf file with your answers.

2. (10%) Write the formula of Newton iteration for solving the nonlinear equations $\boldsymbol{F}(\boldsymbol{X}) = \mathbf{0}$.

3. (10%) Run 8 iterations of Newton's method with the starting point $\boldsymbol{X}^{(0)} = [1, 2]^T$, and state your final solution $\boldsymbol{X}^{(8)}$. The implementation of Newton's method was used for Exercise 20 in Block 4.

4. (10%) Save the last iterate as $\boldsymbol{X}^*$, and for each iteration calculate the 2-norm of the absolute error, which is defined as:

$$e_k = \|\boldsymbol{X}^{(k)} - \boldsymbol{X}^*\|_2, \quad \text{for } k = 1, 2, \cdots, 7.$$

   Plot e_k with $k = 1, 2, \cdots, 7$ by using `semilogy`. Include the figure in your answer.

3 Matrix factorization (20%)

Consider the symmetric matrix

$$\boldsymbol{A} = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 3 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

This question does not need use computer. The derivations and explanations should be included in your answers.

1. (10%) Calculate a LU factorization of $\boldsymbol{A}$ and state the matrices $\boldsymbol{L}$ and $\boldsymbol{U}$.

2. (5%) Check if $\boldsymbol{A}$ has a $\boldsymbol{L}\,\boldsymbol{D}\,\boldsymbol{L}^T$ factorization. If applicable, state the matrices $\boldsymbol{L}$ and $\boldsymbol{D}$.

3. (5%) Check if $\boldsymbol{A}$ has a Cholesky factorization. If applicable, state the matrix $\boldsymbol{L}$ which gives $A = \boldsymbol{L}\,\boldsymbol{L}^T$.