Homework set 4

shsp
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Solution suggestion

a) V is a real vector space $\mathbb{R}^{3\times 3}$. A 6-dimensional subspace of V can be spanned by 6 basis vectors, so 6 linearly independent elements from V. We note that $\dim(V) = 9 < 6$, so a 6-dimensional subspace of V exists according to Lemma 9.31.

We can choose the 6 linearly independent vectors from V:

$$W = \text{Span}\left(\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&1&0\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\1&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&1\\0&0&0\end{bmatrix}\right)$$

which span a subset from V. According to the remark after Definition 9.36, a spanning is a subspace, hence W is a 6-dimensional subspace within V.

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b) In $\mathbb{R}^3$ we are given a vector space W as:

$$W = \text{Span}\left(\underset{\underline{v}_1}{\begin{bmatrix}1\\-1\\0\end{bmatrix}}, \underset{\underline{v}_2}{\begin{bmatrix}5\\4\\3\end{bmatrix}}, \underset{\underline{v}_3}{\begin{bmatrix}7\\11\\6\end{bmatrix}}\right).$$

A basis for W must according to Definition 9.14 span W and its vectors must be linearly independent. The spanning is given, and thus only lin. independence must be investigated. For this, the vectors are merged as columns in a matrix:

$$\begin{bmatrix}\underline{v}_1 & \underline{v}_2 & \underline{v}_3\end{bmatrix} = \begin{bmatrix}1&5&7\\-1&4&11\\0&3&6\end{bmatrix} \xrightarrow{R_2: R_2+R_1} \begin{bmatrix}1&5&7\\0&9&18\\0&3&6\end{bmatrix} \xrightarrow{R_2: \frac{1}{9}R_2} \begin{bmatrix}1&5&7\\0&1&2\\0&3&6\end{bmatrix} \xrightarrow{R_3: R_3-3R_2} \begin{bmatrix}1&5&7\\0&1&2\\0&0&0\end{bmatrix}$$

$$\xrightarrow{R_1: R_1-5R_2} \begin{bmatrix}1&0&-3\\0&1&2\\0&0&0\end{bmatrix} = \text{rref}\left(\begin{bmatrix}\underline{v}_1 & \underline{v}_2 & \underline{v}_3\end{bmatrix}\right)$$

$$\uparrow \quad \uparrow$$

The three vectors are according to Theorem 7.8 not linearly independent, since the rank is not 3. Without $\underline{v}_3$ a matrix $[\underline{v}_1\ \underline{v}_2]$ would have rank 2, and $\underline{v}_1$ and $\underline{v}$

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are thus linearly Independent. A basis for W is hence $(\underline{v}_1, \underline{v}_2) = \left(\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 4 \\ 3 \end{bmatrix} \right)$.

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c)

A map $L: C_\infty \to C_\infty$ between real vector spaces is given by:

$$L(f) = f' + f + 1.$$

L is linear, according to Definition 10.1, if $L(u + c \cdot v) = L(u) + c \cdot L(v)$ for any two functions $u, v \in C_\infty$ and any scalar $c \in \mathbb{R}$.
We check if that's the case:

$$L(u + c \cdot v) = (u + c \cdot v)' + (u + c \cdot v) + 1 = u' + c \cdot v' + u + c \cdot v + 1$$
$$= u' + u + c(v' + v) + 1$$
$$L(u) + c \cdot L(v) = u' + u + 1 + c(v' + v + 1) + 1 = u' + u + c(v' + v) + c + 2$$

We see that $L(u + cv) \neq L(u) + c \cdot L(v)$ and L is thus not linear.

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d) A map $L: \mathbb{C}^3 \to \mathbb{C}^2$ is given as:

$$L\left(\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \right) = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -4 & 5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \quad , \quad v_1, v_2, v_3 \in \mathbb{C}$$

Two ordered bases are given:
$\beta = \left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right)$ for $\mathbb{C}^2$ and $\gamma = \left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)$ for $\mathbb{C}^3$.

The shown matrix $\underline{\underline{A}} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -4 & 5 \end{bmatrix}$ can be considered a mapping matrix of $L = L_{\underline{\underline{A}}}$ with respect to the standard basis e. Hence $\underline{\underline{A}} = {}_e[L]_e$. We now wish to change bases as follows, according to Theorem 10.33:

according to Definition 10.30

$${}_\beta[\mathrm{id}]_e \, {}_e[L]_e \, {}_e[\mathrm{id}]_\gamma = {}_\beta[L]_\gamma$$

to a mapping matrix with respect to the γ- and β-bases, respectively.

The Change-of-basis matrices from the standard basis to β or γ basis, respectively, are needed for this.

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. We can from the given basis vectors create:

$$_e[id]_\beta = \begin{bmatrix}1 & -2\\ 2 & 1\end{bmatrix} \quad \text{and} \quad _e[id]_\gamma = \begin{bmatrix}1 & 0 & 0\\ 1 & 1 & 0\\ 1 & 1 & 1\end{bmatrix},$$

and $_\beta[id]_e = (_e[id]_\beta)^{-1}$ according to Lemma 10.37, bullet 3. Finding this inverse:

$$[\,_e[id]_\beta \mid \underline{\underline{I}}_2\,] = \left[\begin{array}{cc|cc}1 & -2 & 1 & 0\\ 2 & 1 & 0 & 1\end{array}\right] \xrightarrow{R_2:R_2-2R_1} \left[\begin{array}{cc|cc}1 & -2 & 1 & 0\\ 0 & 5 & -2 & 1\end{array}\right]$$

$$\xrightarrow{R_2:\frac{1}{5}R_2} \left[\begin{array}{cc|cc}1 & -2 & 1 & 0\\ 0 & 1 & -2/5 & 1/5\end{array}\right] \xrightarrow{R_1:R_1+2R_2} \left[\begin{array}{cc|cc}1 & 0 & 1-4/5 & 2/5\\ 0 & 1 & -2/5 & 1/5\end{array}\right]$$

$$= \left[\begin{array}{cc|cc}1 & 0 & 1/5 & 2/5\\ 0 & 1 & -2/5 & 1/5\end{array}\right]$$

$$\underbrace{\begin{matrix}1/5 & 2/5\\ -2/5 & 1/5\end{matrix}}_{_\beta[id]_e}$$

so, $_\beta[id]_e = \begin{bmatrix}1/5 & 2/5\\ -2/5 & 1/5\end{bmatrix}$.

Computing the new mapping matrix:

$$_\beta[L]_\gamma = {}_\beta[id]_e\; {}_e[L]_e\; {}_e[id]_\gamma = \begin{bmatrix}1/5 & 2/5\\ -2/5 & 1/5\end{bmatrix} \cdot \begin{bmatrix}1 & 1 & 1\\ 2 & -4 & 5\end{bmatrix} \cdot \begin{bmatrix}1 & 0 & 0\\ 1 & 1 & 0\\ 1 & 1 & 1\end{bmatrix}$$

$$= \begin{bmatrix}1 & -7/5 & 11/5\\ 0 & -6/5 & 3/5\end{bmatrix} \cdot \begin{bmatrix}1 & 0 & 0\\ 1 & 1 & 0\\ 1 & 1 & 1\end{bmatrix} = \begin{bmatrix}9/5 & 4/5 & 11/5\\ -3/5 & -3/5 & 3/5\end{bmatrix}$$

$$= \frac{1}{5}\begin{bmatrix}9 & 4 & 11\\ -3 & -3 & 3\end{bmatrix}$$

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e) For a real $\mathbb{R}^{2\times 2}$ vector space of 2×2 matrices with real coefficients V, a basis is chosen as: $\beta = \left(\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\right)$

A linear map $M: V \to V$ is given by

$$M(\underline{\underline{A}}) = \begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix} \cdot \underline{\underline{A}}, \quad \underline{\underline{A}} \in V.$$

A mapping matrix $_\beta[M]_\beta$ will, according to Theorem 10.31, consist of images of the β-basis vectors as columns.

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We find the image by M of each β-basis vector:

$$M\left(\begin{bmatrix}1&0\\0&0\end{bmatrix}\right)=\begin{bmatrix}1&2\\3&4\end{bmatrix}\begin{bmatrix}1&0\\0&0\end{bmatrix}=\begin{bmatrix}1&0\\3&0\end{bmatrix}=1\cdot\begin{bmatrix}1&0\\0&0\end{bmatrix}+3\begin{bmatrix}0&0\\1&0\end{bmatrix}$$

$$M\left(\begin{bmatrix}0&1\\0&0\end{bmatrix}\right)=\begin{bmatrix}1&2\\3&4\end{bmatrix}\begin{bmatrix}0&1\\0&0\end{bmatrix}=\begin{bmatrix}0&1\\0&3\end{bmatrix}=1\cdot\begin{bmatrix}0&1\\0&0\end{bmatrix}+3\begin{bmatrix}0&0\\0&1\end{bmatrix}$$

$$M\left(\begin{bmatrix}0&0\\1&0\end{bmatrix}\right)=\begin{bmatrix}1&2\\3&4\end{bmatrix}\begin{bmatrix}0&0\\1&0\end{bmatrix}=\begin{bmatrix}2&0\\4&0\end{bmatrix}=2\cdot\begin{bmatrix}1&0\\0&0\end{bmatrix}+4\begin{bmatrix}0&0\\1&0\end{bmatrix}$$

$$M\left(\begin{bmatrix}0&0\\0&1\end{bmatrix}\right)=\begin{bmatrix}1&2\\3&4\end{bmatrix}\begin{bmatrix}0&0\\0&1\end{bmatrix}=\begin{bmatrix}0&2\\0&4\end{bmatrix}=2\begin{bmatrix}0&1\\0&0\end{bmatrix}+4\begin{bmatrix}0&0\\0&1\end{bmatrix}$$

Expressed in β-basis we have

$$_\beta[M]_\beta=\begin{bmatrix}1&0&2&0\\0&1&0&2\\3&0&4&0\\0&3&0&4\end{bmatrix}$$

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f) Given: $\underline{\underline{A}}=\begin{bmatrix}1&0&0\\2&0&-1\\2&-1&0\end{bmatrix}\in\mathbb{R}^{3\times3}$

Eigenvalues are found by solving the characteristic equation, according to Theorem 11.9:

$$P_{\underline{\underline{A}}}(\lambda)=\det([\underline{\underline{A}}-\lambda\underline{\underline{I}}_3])=\det\left(\begin{bmatrix}1-\lambda&0&0\\2&-\lambda&-1\\2&-1&-\lambda\end{bmatrix}\right)=0$$

Finding the characteristic polynomial can be done using the expansion method for the determinant by choosing to expand along row $i=1$:

according to Theorem 8.20

$$P_{\underline{\underline{A}}}(\lambda)=\det\left(\begin{bmatrix}1-\lambda&0&0\\2&-\lambda&-1\\2&-1&-\lambda\end{bmatrix}\right)=\sum_{j=1}^{3}(-1)^{1+j}a_{1j}\det(\underline{\underline{A}}(1;j))$$

$$=(-1)^2(1-\lambda)\det\left(\begin{bmatrix}-\lambda&-1\\-1&-\lambda\end{bmatrix}\right)+(-1)^3\cdot\cancel{0\cdot\det(\underline{\underline{A}}(1;2)}+(-1)^4\cdot\cancel{0\cdot\det(A(1}$$

$$=(1-\lambda)((-\lambda)^2-(-1)^2)=(1-\lambda)(\lambda^2-1)=\lambda^2-1-\lambda^3+\lambda=-\lambda^3+\lambda^2+\lambda-1$$

Solving the characteristic equation:

$-\lambda^3+\lambda^2+\lambda-1=0$ can be done by first guessing a root $\lambda_1=1$, which holds since

$$-1^3+1^2+1-1=0$$
$$\Updownarrow\quad -1+1+1-1=0$$
$$\Updownarrow\quad 0=0$$

Then we perform polynomial division folling Example 4.18:

$$\begin{array}{l|l|l}
\lambda-1 & -\lambda^3+\lambda^2+\lambda-1 & -\lambda^2+1 \\
 & -\lambda^3+\lambda^2 & \\
\hline
 & \qquad\quad\ \lambda-1 & \\
 & \qquad\quad\ \lambda-1 & \\
\hline
 & \qquad\qquad 0 &
\end{array}$$

$\nwarrow$ $Q(\lambda)$

In factorized form we now have $P_{\underline{\underline{A}}}(\lambda)=(\lambda-1)\,Q(\lambda)$

$$=(\lambda-1)(-\lambda^2+1)$$

$\downarrow$ continues...

. Finding roots of $Q(\lambda)$: $Q(\lambda)=0 \Leftrightarrow -\lambda^2+1=0$

$\Leftrightarrow \lambda^2=1 \Leftrightarrow \lambda=\pm 1$

All roots, that is all eigenvalues of $\underline{\underline{A}}$, are thus: $\lambda_1=1$ and $\lambda_2=-1$, with

$gm(1)=2$ and $gm(-1)=1$.

Finding corresponding eigenspaces by solving $(\underline{\underline{A}}-\lambda_i \underline{\underline{I}}_3)\underline{v}_i=\underline{0}$ for each eigenvalue λ_i:

For $\lambda_1=1$:
$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 2 & -1 & -1 & 0 \\ 2 & -1 & -1 & 0 \end{array}\right] \xrightarrow{R_3 \leftrightarrow R_1} \left[\begin{array}{ccc|c} 2 & -1 & -1 & 0 \\ 2 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_2 : R_2 - R_1} \left[\begin{array}{ccc|c} 2 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_1 : \frac{1}{2}R_1} \left[\begin{array}{ccc|c} 1 & -1/2 & -1/2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Leftrightarrow v_1 - \tfrac{1}{2}v_2 - 1/2 = 0$$

$\Updownarrow$

$v_1 = \frac{1}{2}v_2 + \frac{1}{2}v_3 = \frac{1}{2}t + \frac{1}{2}s$

$\Updownarrow$

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} t + \begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \end{bmatrix} s \quad , t,s \in \mathbb{R}$$

For $\lambda_2=-1$:
$$\left[\begin{array}{ccc|c} 2 & 0 & 0 & 0 \\ 2 & 1 & -1 & 0 \\ 2 & -1 & 1 & 0 \end{array}\right] \xrightarrow{R_3 : R_3 - R_1} \left[\begin{array}{ccc|c} 2 & 0 & 0 & 0 \\ 2 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right] \xrightarrow{R_2 : R_2 - R_1} \left[\begin{array}{ccc|c} 2 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{R_1 : \frac{1}{2}R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right] \xrightarrow{R_3 : R_3 + R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Leftrightarrow \begin{cases} v_1 = 0 \\ v_2 - v_3 = 0 \end{cases}$$

$\Updownarrow$

$$\begin{cases} v_1 = 0 \\ v_2 = v_3 = t \end{cases}$$

$\Updownarrow$

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} t \quad , t \in \mathbb{R}$$

We have denoted the coordinates of eigenvectors by $\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$.

We see eigenspace E_1 having ordered basis $\left(\begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1/2 \\ 0 \\ 1 \end{bmatrix} \right)$ and $gm(1)=2$ and eigenspace E_{-1} having (ordered) basis $\left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right)$ and $gm(-1)=1$

According to Corollary 11.28, $\underline{\underline{A}}$ is diagonalizable since $am(1)=gm(1)=2$ and $am(-1)=gm(-1)=1$.

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$\therefore$ Hence a matrix V must exist such that

$\underline{\underline{V}}^{-1}\underline{\underline{A}}\,\underline{\underline{V}} = \underline{\underline{\Lambda}}$ where $\underline{\underline{\Lambda}}$ is a diagonal matrix.

Following Example 11.31, such a diagonal matrix will consist of eigenvalues in the diagonal, and $\underline{\underline{V}}$ of corresponding linearly independent eigenvectors is corresponding columns, $\underline{\underline{V}} = [\underline{v}_1\ \underline{v}_2\ \underline{v}_3]$, $\underline{\underline{\Lambda}} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$

Hence $\underline{\underline{\Lambda}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$. For $\lambda = 1$ where $am(1) = 2$ we choose two linearly independent eigenvectors, $\begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1/2 \\ 0 \\ 1 \end{bmatrix}$, and have $\underline{\underline{V}} = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$.

The inverse $\underline{\underline{V}}^{-1}$ of $\underline{\underline{V}}$ is found:

$$\left[\ \underline{\underline{V}}\ \middle|\ \underline{\underline{I}}_3\ \right] = \left[\begin{array}{ccc|ccc} 1/2 & 1/2 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 1/2 & 1/2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 : R_2 - \frac{1}{2}R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1/2 & -1/2 & 1 & -1/2 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2 : 2 \cdot R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 2 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 : R_3 - R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 2 & -1 & 0 \\ 0 & 0 & 2 & -2 & 1 & 1 \end{array}\right] \xrightarrow{R_3 : \frac{1}{2}R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 2 & -1 & 0 \\ 0 & 0 & 1 & -1 & 1/2 & 1/2 \end{array}\right]$$

$$\xrightarrow{R_2 : R_2 + R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & -1/2 & 1/2 \\ 0 & 0 & 1 & -1 & 1/2 & 1/2 \end{array}\right] \xrightarrow{R_1 : R_1 - R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & \underbrace{\begin{matrix} 1 & 1/2 & -1/2 \\ 1 & -1/2 & 1/2 \\ -1 & 1/2 & 1/2 \end{matrix}}_{\underline{\underline{V}}^{-1}} \\ 0 & 0 & 1 & \end{array}\right]$$

so, $\underline{\underline{V}}^{-1} = \begin{bmatrix} 1 & 1/2 & -1/2 \\ 1 & -1/2 & 1/2 \\ -1 & 1/2 & 1/2 \end{bmatrix}$

In the equation $\underline{\underline{Q}}\,\underline{\underline{A}}\,\underline{\underline{Q}}^{-1} = \underline{\underline{\Lambda}}$ we see $\underline{\underline{Q}} = \underline{\underline{V}}^{-1}$.

(note, there are several correct $\underline{\underline{Q}}$'s)
