01004 Mathematics 1b Project

Permanent Halbach Magnets

Group 09

Máni Nielsen (s225112)

Silas Søbakke Hinsch Rojas (s222192)

Isabella Blinkenberg Madsen (s236178)

Matyas Steiner (s234512)

Elias Haynie-Gay (s234535)

Fedir Vaslyliev (s234542)

April 2024

1 Introduction

Permanent Halbach Magnets are permanent magnets held in a Halbach array, that strengthens the magnetic field on one side and almost nullifies the magnetic field on the other side. They are used in many things in our daily lives, from refrigerators to maglev trains.

2 Cylindrical coordinates

Exercise 1

A vector field in the (x, y, z) coordinates is given by:

$$V = (V_x, V_y, V_z)$$

However, if we wish to represent this in cylindrical coordinates, we can use an orthonormal basis (e_r, e_θ, e_z), such that e_r is parallel to the point $(x, y, 0)$; e_θ is orthogonal to e_r, and parallel to $(-y, x, 0)$; and e_z is $(0, 0, 1)$.

We can rewrite these new basis vectors as a function of rectangular coordinates. To do so, we must orthonormalize the vectors. Nonetheless, since the vectors are already orthogonal one to another (which can also be proven by checking that their dot product is equal to zero), we must only normalize them. This can be done by taking the vectors

e_r, e_θ and e_z and dividing them by their length $|r|$:

$$e_r = \frac{(x, y, 0)}{|r_{e_r}|} = \frac{(x, y, 0)}{\sqrt{(x^2 + y^2)}}$$
$$e_\theta = \frac{(-y, x, 0)}{|r_{e_\theta}|} = \frac{(-y, x, 0)}{\sqrt{(x^2 + y^2)}}$$
$$e_r = \frac{(0, 0, 1)}{|r_z|} = \frac{(0, 0, 1)}{1}$$

This can easily be rewritten to:

$$e_r = (x^2 + y^2)^{\frac{1}{2}} \cdot (x, y, 0)$$

$$e_\theta = (x^2 + y^2)^{\frac{1}{2}} \cdot (x, y, 0)$$

$$e_z = (0, 0, 1)$$

Exercise 2

To show that the basis vectors (e_r, e_θ, e_z), expressed in cylindrical coordinates, are the following:

$$e_r = (cos(\theta), sin(\theta), 0)$$

$$e_\theta = (-sin(\theta), cos(\theta), 0)$$

$$e_z = (0, 0, 1)$$

We will begin with e_r and divide the basis vector by the length of r, as done in exercise 1.

$$e_r = \frac{(0, 0, 1)}{\sqrt{x^2 + y^2}}$$

We are given the information that:

$$x = rcos(\theta)$$

$$y = rsin(\theta)$$

$$z = z$$

Which we can use to substitute x and y:

$$\sqrt{x^2 + y^2} \rightarrow \sqrt{r^2cos(\theta)^2 + r^2 + sin(\theta)^2} = r$$

Therefore, $e_r = \dfrac{rcos(\theta), rsin(\theta), 0}{r}$ which becomes $(cos(\theta), sin(\theta), 0)$ when divided by r.

For e_θ and e_z, the working out is similar:

$$e_\theta = \frac{(-y, x, 0)}{||-y, x||} \rightarrow \frac{-rsin(\theta, rcos(\theta), 0)}{r} = (-sin(\theta), cos(\theta), 0)$$

$$e_z = \frac{(0,0,1)}{1} = 1$$

Exercise 3

We are trying to prove that $\dfrac{\partial f}{\partial r} = \nabla f \cdot e_r$ To do this we can expand on the equation. First we can look at the gradient $\nabla f \cdot e_r$:

$$\nabla f \cdot e_r = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}) \cdot (cos(\theta), sin(\theta), 0) = \frac{\partial f}{\partial x} cos(\theta) + \frac{\partial f}{\partial y} sin(\theta) + \frac{\partial f}{\partial z} \cdot 0$$

Now, we have the following values for x, y and z, which we can derive to get the following:

$$x = rcos(\theta) \Leftrightarrow \frac{\partial x}{\partial r} = cos(\theta)$$
$$y = rsin(\theta) \Leftrightarrow \frac{\partial y}{\partial r} = sin(\theta)$$
$$z = 1 \Leftrightarrow \frac{\partial z}{\partial r} = 0$$

We can now substitute on our equation to generate a chain rule as following:

$$\frac{\partial f}{\partial x}cos(\theta) + \frac{\partial f}{\partial y}sin(\theta) + \frac{\partial f}{\partial z} \cdot 0 = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial r} = \frac{\partial f}{\partial r}$$

And thus, we end up with the formula:

$$\nabla f \cdot e_r = \frac{\partial f}{\partial r}$$

Exercise 4

Similar to what we did in the previous exercise, we will prove that

$$\frac{\partial f}{\partial \theta} = r\nabla f \cdot e_\theta$$

We know that

$$\nabla f \cdot e_\theta = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}) \cdot (-sin(\theta), cos(\theta), 0) = \frac{\partial f}{\partial x}(-sin(\theta)) + \frac{\partial f}{\partial y}cos(\theta) + \frac{\partial f}{\partial z} \cdot 0$$

Like the last exercise, we can formulate an expression for the partial derivatives with respect to θ by evaluating the values for x, y and z:

$$x = rcos(\theta) \Leftrightarrow \frac{\partial x}{\partial \theta} = -rsin(\theta) \Leftrightarrow \frac{1}{r}\frac{\partial x}{\partial \theta} = -sin(\theta)$$
$$y = rsin(\theta) \Leftrightarrow \frac{\partial y}{\partial \theta} = rcos(\theta) \Leftrightarrow \frac{1}{r}\frac{\partial y}{\partial \theta} = cos(\theta)$$
$$z = 1 \Leftrightarrow \frac{\partial z}{\partial \theta} = 0$$

If we now substitute these values in our original equation we get:

$$\frac{\partial f}{\partial x}(-sin(\theta)) + \frac{\partial f}{\partial y}cos(\theta) + \frac{\partial f}{\partial z} \cdot 0 = \frac{\partial f}{\partial x}\frac{1}{r}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{1}{r}\frac{\partial y}{\partial \theta} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial \theta}$$
$$= \frac{1}{r}(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}) = \frac{1}{r}\frac{\partial f}{\partial \theta}$$

This implies that

$$\nabla f \cdot e_\theta = \frac{1}{r}\frac{\partial f}{\partial \theta} \Leftrightarrow r\nabla f \cdot e_\theta = \frac{\partial f}{\partial \theta}$$

We can see that they are equal and therefore

$$r\nabla f \cdot e_\theta = \frac{\partial f}{\partial \theta}$$

Exercise 5

In this exercise we are showing the gradient is given by the equation:
$\frac{\partial f}{\partial r} \cdot e_r + \frac{1}{r} \cdot \frac{\partial f}{\partial \theta} \cdot e_\theta + \frac{\partial f}{\partial \theta} \cdot e_z$
From script equation 13 we have that:

$$\nabla f(r,\theta,z) = \nabla f_r \cdot e_r + \nabla f_\theta \cdot e_\theta + \nabla f_z \cdot e_z$$

And since script equation 14 gives equations:

$$\nabla f_r = \nabla f \cdot e_r, \nabla f_\theta = \nabla f \cdot e_\theta, \nabla f_z \cdot = \nabla f \cdot e_z$$

We can now use our answers from the last exercise to substitute, as follows:

$$\nabla f(r,\theta,z) = \nabla f_r \cdot e_r + \nabla f_\theta \cdot e_\theta + \nabla f_z \cdot e_z = \nabla f \cdot e_r \cdot e_r + \nabla f_\theta \cdot e_\theta \cdot e_\theta + \nabla f_z \cdot e_z \cdot e_z$$
$$= \frac{\partial f}{\partial r} \cdot e_r + \frac{1}{r} \cdot \frac{\partial f}{\partial \theta} \cdot e_\theta + \frac{\partial f}{\partial \theta} \cdot e_z$$

Exercise 6

In this exercise, we have the function $f(x, y, z)$ and vector field $V(x, y, z)$. We want to show that the following equation is true:

$$Div(fV) = \nabla f \cdot V + fDiv(V)$$

To do this we will expand on the term of the left using the equations we have already derived, and using the following definition for the divergence of a vector field:

$$Div(fV) = \frac{\partial fV_x}{\partial x} + \frac{\partial fV_y}{\partial y} + \frac{\partial fV_z}{\partial z}$$

We can now use the product rule, to get the following:

$$Div(fV) = \frac{\partial fV_x}{\partial x} + \frac{\partial fV_y}{\partial y} + \frac{\partial fV_z}{\partial z}$$
$$= (\frac{\partial f}{\partial x}V_x + \frac{\partial V_x}{\partial x}f) + (\frac{\partial f}{\partial y}V_y + \frac{\partial V_y}{\partial y}f) + (\frac{\partial f}{\partial z}V_z + \frac{\partial V_z}{\partial z}f)$$

We can now group the terms and factorize:

$$(\frac{\partial f}{\partial x}V_x + \frac{\partial V_x}{\partial x}f) + (\frac{\partial f}{\partial y}V_y + \frac{\partial V_y}{\partial y}f) + (\frac{\partial f}{\partial z}V_z + \frac{\partial V_z}{\partial z}f)$$
$$= (\frac{\partial f}{\partial x}V_x + \frac{\partial f}{\partial y}V_y + \frac{\partial f}{\partial z}V_z) + f(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z})$$

Since we know ∇f from script equation 12, we can substitute this and vector V in as follows:

$$\nabla f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z})$$

$$V = (V_x, V_y, V_z)$$

$$\nabla f \cdot V = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}) \cdot (V_x, V_y, V_z) = (\frac{\partial f}{\partial x}V_x, \frac{\partial f}{\partial y}V_y, \frac{\partial f}{\partial z}V_z)$$

$$Div(V) = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$$

We can now substitute in our equation and get the following:

$$(\frac{\partial f}{\partial x}V_x + \frac{\partial f}{\partial y}V_y + \frac{\partial f}{\partial z}V_z) + f(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z})$$

$$= \nabla f \cdot V + f \cdot Div(V)$$

We see that the original equation $Div(fV) = \nabla fV + fDiv(V)$, thus holds true.

Exercise 7

We must find the divergence of e_r, e_θ and e_z: We are told that only one of these is not equal to zero.

$$Div(e_r) = \nabla \cdot e_r = \nabla \cdot [\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}, 0] = \frac{\partial}{\partial x}\frac{x}{\sqrt{x^2+y^2}} + \frac{\partial}{\partial y}\frac{y}{\sqrt{x^2+y^2}} + \frac{\partial}{\partial z}0$$

We can now use the quotient rule for solving each of these individual derivatives:

$$\frac{\partial}{\partial x}\frac{x}{\sqrt{x^2+y^2}}+\frac{\partial}{\partial y}\frac{y}{\sqrt{x^2+y^2}}+\frac{\partial}{\partial z}0$$

$$=\frac{\sqrt{x^2+y^2}-x\frac{1}{2}(2x)\frac{1}{\sqrt{x^2+y^2}}}{x^2+y^2}+\frac{\sqrt{x^2+y^2}-y\frac{1}{2}(2y)\frac{1}{\sqrt{x^2+y^2}}}{x^2+y^2}+0$$

$$=\frac{\frac{x^2+y^2-x^2}{\sqrt{x^2+y^2}}}{x^2+y^2}+\frac{\frac{x^2+y^2-y^2}{\sqrt{x^2+y^2}}}{x^2+y^2}=\frac{y^2}{(x^2+y^2)^{\frac{3}{2}}}+\frac{x^2}{(x^2+y^2)^{\frac{3}{2}}}$$

$$=\frac{x^2+y^2}{(x^2+y^2)^{\frac{3}{2}}}$$

$$=\frac{1}{\sqrt{x^2+y^2}}$$

Thus, we can see that $Div(e_r)$ is the non-zero component. We can now calculate for e_θ and e_z:

$$Div(e_z)=\frac{\partial e_z x}{\partial x}+\frac{\partial e_z y}{\partial y}+\frac{\partial e_z z}{\partial z}$$

$$=0+0+0$$

$$=0$$

$$Div(e_\theta)=\frac{\partial l_{\theta_x}}{\partial x}+\frac{\partial l_{\theta_y}}{\partial y}+\frac{\partial l_{\theta_z}}{\partial z}$$

$$=0+0+0$$

$$=0$$

So, now we have:

$$Div(e_r)=\frac{1}{\sqrt{x^2+y^2}}$$

$$Div(e_z)=0$$

$$Div(e_\theta)=0$$

Which thus implies that $Div(e_r)$ can be rewritten in cylindrical coordinates (r, θ, z) as:

$$Div(e_r) = \frac{1}{\sqrt{x^2 + y^2}} = \frac{1}{r}$$

Exercise 8

We want to show the divergence of a vector field expressed in cylindrical coordinates is given by equation: $Div(V) = \frac{\partial V_r}{\partial r} + \frac{V_r}{r} + \frac{1}{r}\frac{\partial V_\theta}{\partial \theta} + \frac{\partial V_z}{\partial z}$ We are given the projections along the vector field in line 2 of the script:

$$V_r = V \cdot e_r$$

$$V_\theta = V \cdot e_\theta$$

$$V_z = V \cdot e_z$$

Because the vector field is made of these three projections, we can rewrite $Div(V)$ as:

$$Div(V) = Div(V_r \cdot e_r) + Div(V_\theta \cdot e_\theta) + Div(V_z \cdot e_z)$$

Using the formula we proved to be true in exercise 6, the components are:

$$Div(V_r \cdot e_r) = \nabla V_r \cdot e_r + V_r \cdot Div(e_r)$$

$$Div(V_\theta \cdot e_\theta) = \nabla V_\theta \cdot e_\theta + V_\theta \cdot Div(e_\theta)$$

$$Div(V_z \cdot e_z) = \nabla V_z \cdot e_z + V_z \cdot Div(e_z)$$

Following the equation proven in 5:

$$\nabla V_r = \frac{\partial V_r}{\partial r}e_r + \frac{1}{r}\frac{\partial V_r}{\partial \theta}e_\theta + \frac{\partial V_r}{\partial z}e_z$$
$$\nabla V_\theta = \frac{\partial V_\theta}{\partial r}e_r + \frac{1}{r}\frac{\partial V_\theta}{\partial \theta}e_\theta + \frac{\partial V_\theta}{\partial z}e_\theta$$
$$\nabla V_z = \frac{\partial V_z}{\partial r}e_z + \frac{1}{r}\frac{\partial V_z}{\partial \theta}e_\theta + \frac{\partial V_z}{\partial z}e_z$$

Since we know $Div(e_r)$, $Div(e_\theta)$, and $Div(e_z)$ from exercise 7, we can plug them in along with the gradients calculated above:

$$Div(V_r \cdot e_r) = (\frac{\partial V_r}{\partial r}e_r + \frac{1}{r}\frac{\partial V_r}{\partial \theta}e_\theta + \frac{\partial V_r}{\partial z}e_z)e_r + V_r \cdot \frac{1}{r}$$
$$= \frac{\partial V_r}{\partial r}e_r \cdot e_r + \frac{1}{r}\frac{\partial V_r}{\partial \theta}e_\theta \cdot e_r + \frac{\partial V_r}{\partial z}e_z \cdot e_r + V_r \cdot \frac{1}{r}$$

$$Div(V_\theta \cdot e_\theta) = (\frac{\partial V_\theta}{\partial r}e_r + \frac{1}{r}\frac{\partial V_\theta}{\partial \theta}e_\theta + \frac{\partial V_\theta}{\partial z}e_z)e_\theta + V_\theta \cdot 0$$
$$= \frac{\partial V_\theta}{\partial r}e_r \cdot e_\theta + \frac{1}{r}\frac{\partial V_\theta}{\partial \theta}e_\theta \cdot e_\theta + \frac{\partial V_\theta}{\partial z}e_z \cdot e_\theta + V_\theta \cdot 0$$

$$Div(V_z \cdot e_z) = (\frac{\partial V_z}{\partial r}e_r + \frac{1}{r}\frac{\partial V_z}{\partial \theta}e_\theta + \frac{\partial V_z}{\partial z}e_z)e_z + V_z \cdot 0 = \nabla V_z \cdot e_z$$
$$= \frac{\partial V_z}{\partial r}e_r \cdot e_z + \frac{1}{r}\frac{\partial V_z}{\partial \theta}e_z \cdot e_\theta + \frac{\partial V_z}{\partial z}e_z \cdot e_z + V_z \cdot 0 = \nabla V_z \cdot e_z$$

Since the dot product of a normal vector with itself is 1, and the dot product of a normal vector with another normal vector that is orthogonal to itself is 0, we can reduce the terms as $e_r \cdot e_r = 1$, $e_r \cdot e_\theta = 0$ and $e_z \cdot e_r = 0$. We repeat the same for the vectors e_θ and e_z. Thus, we can reduce any terms that have the normal vectors as dot products:

$$\frac{\partial V_r}{\partial r}e_r \cdot e_r + \frac{1}{r}\frac{\partial V_r}{\partial \theta}e_\theta \cdot e_r + \frac{\partial V_r}{\partial z}e_z \cdot e_r + V_r \cdot \frac{1}{r}$$
$$= \frac{\partial V_r}{\partial r} + \frac{V_r}{r}$$

$$\frac{\partial V_\theta}{\partial r}e_r \cdot e_\theta + \frac{1}{r}\frac{\partial V_\theta}{\partial \theta}e_\theta \cdot e_\theta + \frac{\partial V_\theta}{\partial z}e_z \cdot e_\theta + V_\theta \cdot 0$$
$$= \frac{1}{r}\frac{V_\theta}{\partial \theta}$$

$$\frac{\partial V_z}{\partial r}e_r \cdot e_\theta + \frac{1}{r}\frac{\partial V_z}{\partial \theta}e_\theta \cdot e_\theta + \frac{\partial V_z}{\partial z}e_z \cdot e_z + V_z \cdot 0 = \nabla V_z \cdot e_z$$
$$= \frac{\partial V_z}{\partial z}$$

We can now simplify:

$$Div(\mathbf{V}) = \frac{\partial V_r}{\partial r} + \frac{V_r}{r} + \frac{1}{r}\frac{\partial V_\theta}{\partial \theta} + \frac{\partial V_z}{\partial z} \tag{1}$$

We also get the definition for the curl of this vector in cylindrical coordinates:

$$\mathbf{Rot}(\mathbf{V}) = (\frac{1}{r}\frac{\partial V_z}{\partial \theta} - \frac{\partial V_\theta}{\partial z})\mathbf{e_r} + (\frac{\partial V_r}{\partial z} - \frac{\partial V_z}{\partial r})\mathbf{e}_\theta + (\frac{\partial V_\theta}{\partial r} + \frac{V_\theta}{r} - \frac{1}{r}\frac{\partial V_r}{\partial \theta})\mathbf{e_z} \tag{2}$$

3 Magnetic field around a conductor

Exercise 9

We are told the current density **J** within a circle centered on the infinitely long wire that is expressed in the unit A/m^2. **A** is amperes, a measurement of current (**I**), m^2 is a cross-section of area, which in the case of a cable takes the form πr^2. This takes the form:

$$J = \frac{I}{\pi r^2} \tag{3}$$

We also must give it a direction, which we do by multiplying by unit vector e_z:

$$J = \frac{I}{\pi r^2} \cdot e_z \tag{4}$$

Exercise 10

We want to show:

$$\frac{\partial B_z}{\partial \theta} = 0$$
$$\frac{\partial B_z}{\partial r} = 0$$

We can achieve this by using the curl equation in cylindrical coordinates 4.1, the 4^{th} Maxwell's equation 8, and the fact that the current passing through the conductor has only component in z direction, e.g. the current density $\mathbf{J} = (0, 0, J_z)$. Because the current doesn't change over time e.g. $\frac{\partial \mathbf{E}}{\partial t} = 0$, the Maxwell equation will be:

$$\mathbf{Rot}(\mathbf{B}) = \mu_0 \begin{bmatrix} 0 \\ 0 \\ J_z \end{bmatrix}$$

We also know the following:

$$\frac{\partial B_r}{\partial z} = 0, \frac{\partial B_\theta}{\partial z} = 0, \frac{\partial B_z}{\partial z} = 0$$

So, the curl is now:

$$\mathbf{Rot}(\mathbf{B}) = \left(\frac{1}{r}\frac{\partial B_z}{\partial \theta} - 0\right) e_r + \left(0 - \frac{\partial B_z}{\partial r}\right) e_\theta + \left(\frac{\partial B_\theta}{\partial r} + \frac{B_\theta}{r} - \frac{1}{r}\frac{\partial B_r}{\partial \theta}\right) e_z$$

As we only can have z component for the $\mathbf{Rot}(\mathbf{B})$ and the e_r, $e_\theta \neq 0$ - the only possible scenario is

$$\frac{\partial B_z}{\partial \theta} = 0$$
$$\frac{\partial B_z}{\partial r} = 0$$

So, the

$$\mathbf{Rot}(\mathbf{B}) = (\frac{\partial B_\theta}{\partial r} + \frac{B_\theta}{r} - \frac{1}{r}\frac{\partial B_r}{\partial}\theta)e_z \tag{5}$$

We now can conclude that B_z is a constant, $B_z = 0$. Due to the symmetry $\dfrac{\partial B_r}{\partial \theta} = 0$.

Exercise 11

In this exercise, we want to write and solve a differential equation for $B_r(r)$

We will start by using the fact that $Div(B) = 0$:

$$Div(B) = \frac{\partial B_r}{\partial r} + \frac{B_r}{r} + \frac{1}{r}\frac{\partial B_\theta}{\partial \theta} + \frac{\partial B_z}{\partial z} = 0 \tag{6}$$

We will use the fact that $\dfrac{\partial B_\theta}{\partial \theta} = 0$ for the same reason as $\dfrac{\partial B_r}{\partial \theta} = 0$, as was concluded in the previous exercise.

$$\frac{\partial B_r}{\partial r} = -\frac{B_r}{r}$$

$$B_r(r) = e^{A(r)} \int e^{-A(r)} \cdot 0\, dr, \text{ where } A(r) = \int -\frac{1}{r}\, dr = -ln(r)$$

$$B_r(r) = e^{-ln(r)} \cdot c$$

$$B_r(r) = \frac{c}{r},\ c = constant$$

Utilizing the symmetry of the wire by mirroring a cross-section of the conductor we can argue that $c = 0$.

Exercise 12

As the only non-zero component left is B_θ it is of interest to actually find the expression for it. This can be done by utilizing Stroke's equation (7) and Maxwell's 4^{th} equation (8), assuming the constant current through the wire to be I

$$\int_{F_r} \mathbf{Rot}(\mathbf{B}) \cdot \mathbf{n_F}\, d\mu = \int_{\partial F} \mathbf{B}_\theta \cdot \mathbf{e_F}\, d\mu \tag{7}$$

The Stroke's theorem states that the surface integral over the flux of the curl of a vector field is equivalent to the tangential curve integral of the vector field along the edge of the surface.

And the 4^{th} Maxwell's equation:

$$\mathbf{Rot}(\mathbf{B}) = \mu_0 \cdot \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} = \mu_0 \cdot \mathbf{J} \tag{8}$$

Combining these two equations one can get the following equality

$$\int \mu_0 J\, d\vec{s} = \int B_\theta\, d\vec{l}$$

as we have seen in Exercise 9: current density $J = \frac{I}{A}$, when we integrate J over an area we get the current I. That is what we see on the left side. On the right side: B_θ is a constant across the line, so the integral $\displaystyle\int dl$ becomes a length of the path $2\pi r$.

$$\mu_0 I = B_\theta 2\pi r$$

Solve for B_θ:

$$B_\theta = \frac{\mu_0 I}{2\pi r} \tag{9}$$

Exercise 13

Since we know from question 12 that $B_\theta = \frac{\mu_0 I}{2\pi r}$, we can use this to plot a vector field showing the cross-sectional magnetic field around an infinitely long conductor centered at 0,0.

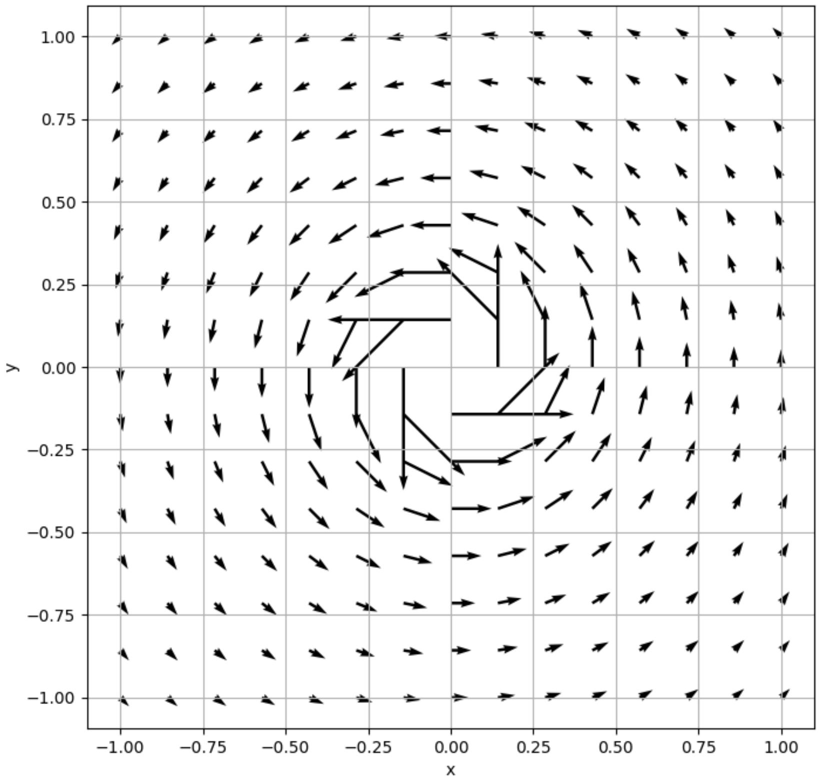

Figure 1: Vector field around infinitely long conductor

Here we can see that the vectors get shorter, and thus weaker as they get further from the center, due to B_θ being inversely proportional to the distance from the center (r). This shows that the magnetic field is strongest in the middle of the conductor.

4 A Halbach magnet

4.1 The magnetic field exterior to the magnetic material

The local magnetisation in a Halbach magnet varies with r and θ as:

$$\mathbf{B_{rem}} = \begin{pmatrix} B_{rem,r} \\ B_{rem,\theta} \end{pmatrix} = B_{rem} \begin{pmatrix} cos(\rho\theta) \\ sin(\rho\theta) \end{pmatrix} \tag{10}$$

Exercise 14

We can plot the vector field described by the remanence of the magnetic field B_{rem}. We need to plot first with positive ρ values, and then with negative values to better understand the role of ρ in the magnetic field.

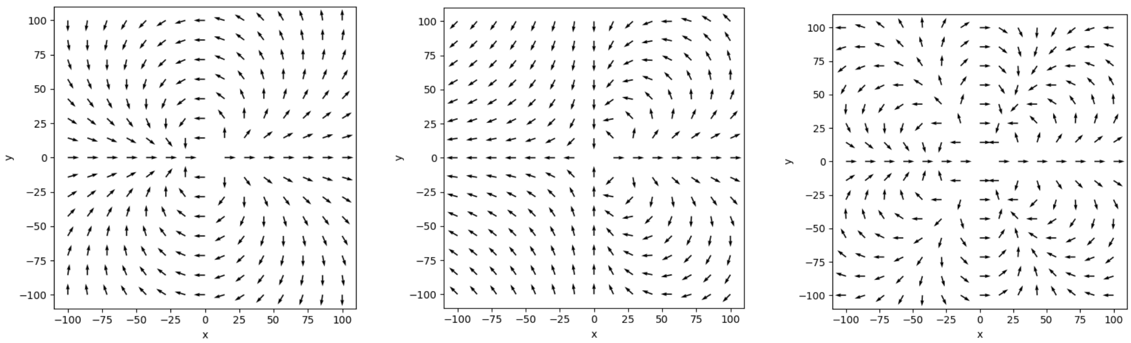

Figure 2: Vector field for a permanent Halbach magnet with positive ρ values.

We can now compare with the negative counterpart for these plots:

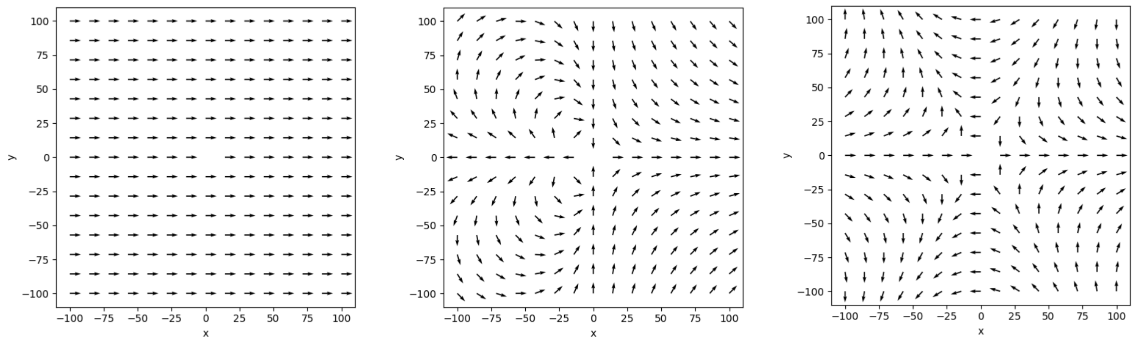

Figure 3: Vector field for a permanent Halbach magnet with negative ρ values.

It can be noted from these graphs that the role the parameter ρ plays in the shape of these magnetic fields is important, since its sign defines the direction of the vector field's flow, while its integer value defines its symmetry, or more precisely, the number of axis of symmetry that compose the magnetic field. This behavior can be seen by comparing the different plots. For example, plot (a) shows one axis of symmetry while plot (b) shows two, whereas their negative counterparts (d) and (f) show the same symmetry but in opposite direction. The relevant code for these plots can be found in the appendix.

Exercise 15

We have to show that $Rot(\nabla f) = \mathbf{0}$. We know that:

$$\nabla f = (\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3})$$

We can now use the definition of the curl and the cross product to calculate $Rot(\nabla f)$.

$$Rot(\nabla f) = \nabla \times \nabla f = \begin{bmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} \end{bmatrix} \times \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \frac{\partial f}{\partial x_3} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\partial^2 f}{\partial x_2 \partial x_3} - \frac{\partial^2 f}{\partial x_2 \partial x_3} \\ \frac{\partial^2 f}{\partial x_1 \partial x_3} - \frac{\partial^2 f}{\partial x_3 \partial x_1} \\ \frac{\partial^2 f}{\partial x_1 \partial x_2} - \frac{\partial^2 f}{\partial x_2 \partial x_1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This returns a zero vector, and thus $Rot(\nabla f) = 0$.

Technically, when working with magnetic fields, it is an advantage to use the so-called magnetic vector potential, A, defined in the following way:

$$\mathbf{B} = \mathbf{Rot}(\mathbf{A}) \tag{11}$$

Exercise 16

In this exercise, we want to show that we can freely choose A so that its divergence is zero. We know that

$$Div(A) = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} \frac{\partial}{\partial x_3}) \cdot \underline{A}$$

$$A = A_0 + \nabla f$$

Thus we can try to expand these terms:

$$Div(A) = Div(A_0) + Div(\nabla_f) = 0$$

$$\Leftrightarrow Div(\nabla f) = -Div(A)$$

And thus, we can choose any f such that $Div(\nabla f) = -Div(A_0)$. Therefore, the curl would be calculated as:

$$Rot(A) = Rot(A_0) + Rot(\nabla f)$$

And since $Rot(\nabla f) = 0$, $Rot(A) = Rot(A_0)$

Exercise 17

We want to show now that the following expression is true:

$$-\nabla^2 \mathbf{A} = \mathbf{Rot}(\mathbf{B_{rem}}) \tag{12}$$

Since $B = Rot(A)$, and $Rot(B) = Rot(B_{rem})$, we can substitute and get the following equation:

$$Rot(B_{rem}) = Rot(Rot(A))$$

We expand this equation by considering the properties of the cross product and the definition of the curl of a vector, as shown:

$$Rot(Rot(A)) = \nabla \times \nabla \times A = (\nabla \cdot A)\nabla - (\nabla \cdot \nabla)A$$

However, we know that $\nabla \cdot A = Div(A)$, and the divergence of A is defined to be zero $Div(A) = 0$. Then we can substitute in the following way:

$$(\nabla \cdot A)\nabla - (\nabla \cdot \nabla)A = \nabla(Div(A)) - \nabla^2 A = 0 - \nabla^2 A$$

Hence, it is shown that the expression $-\nabla^2 \mathbf{A} = \mathbf{Rot}(\mathbf{B_{rem}})$ holds true.

Exercise 18

We must now calculate the components of $Rot(B_{rem})$. We can use the definition for B_{rem} and the equation for calculating the curl of the vector in cylindrical coordinates for this, by substituting the x, y and z components into the curl equation:

$$\mathbf{Rot}(\mathbf{V}) = (\frac{1}{r}\frac{\partial V_z}{\partial \theta} - \frac{\partial V_\theta}{\partial z})\mathbf{e_r} + (\frac{\partial V_r}{\partial z} - \frac{\partial V_z}{\partial r})\mathbf{e}_\theta + (\frac{\partial V_\theta}{\partial r} + \frac{V_\theta}{r} - \frac{1}{r}\frac{\partial V_r}{\partial \theta})\mathbf{e_z}$$

$$B_{rem} = \begin{bmatrix} cos\rho\theta \\ sin\rho\theta \\ 0 \end{bmatrix}$$

$$\begin{aligned}
\Leftrightarrow Rot(B_{rem}) &= (\frac{1}{r}(0-0)e_r + (0-0)e_\theta + (0 + \frac{sin(\rho\theta)}{r} + \frac{1}{r}\rho sin(\rho\theta))e_z \\
&= (\frac{1}{r}sin(\rho\theta) + \frac{1}{r}\rho sin(\rho\theta))e_z \\
&= \frac{1}{r}sin(\rho\theta)(1+\rho)e_z(B_{rem}) \\
&= \frac{1}{r}B_{rem}sin(\rho\theta)e_z + \frac{\rho}{r}B_{rem}sin(\rho\theta)e_z
\end{aligned}$$

Since ρ is an integer such that $\rho \neq 0$. We can evaluate for a simple scenario $\rho = 1$, which gives the following:

$$\frac{1}{r}B_{rem}sin(\rho\theta)e_z + \frac{\rho}{r}B_{rem}sin(\rho\theta)e_z = \frac{2}{r}B_{rem}sin(\theta)e_z$$

It can be noted however, that no matter the value for ρ used, the only non-zero component of the curl of B_{rem} is $Rot(B_{rem}) \cdot e_z = \frac{1}{r}B_{rem}sin(\rho\theta)e_z + \frac{\rho}{r}B_{rem}sin(\rho\theta)e_z$.

Exercise 19

We now have the differential equation:

$$-\nabla^2 A_z(r,\theta) = \mathbf{Rot}(\mathbf{B_{rem}}) \cdot \mathbf{e_z} \tag{13}$$

We wish to show that for $\rho = 1$, the linear combination of the solution to the homogeneous and particular solutions is:

$$A_z(r,\theta) = (Cr + Dr^{-1})sin(\theta) - B_{rem}rln(r)sin(\theta) \tag{14}$$

The first step in proving this is expanding the left-hand side $-\nabla^2 A_z(r,\theta)$. To do this we use the following equation, known as the Laplace operator for cylindrical coordinates, which is equivalent to ∇^2. The derivation for this equation is also shown in the appendix:

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial}{\partial r}) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial}{\partial z}$$
$$\Leftrightarrow -\nabla^2 A_z(r,\theta) = -\frac{1}{r}\frac{\partial A_z}{\partial r}(r\frac{\partial A_z}{\partial r}) - \frac{1}{r^2}\frac{\partial^2 A_z}{\partial \theta^2} - \frac{\partial A_z}{\partial z}$$

For the following part, it is easier to divide each of the different derivatives making up the equation in the following way:

$$r\frac{\partial A_z}{\partial r} = r(Csin(\theta) - \frac{D}{r^2}sin\theta - B_{rem}sin\theta ln(r) - B_{rem}sin(\theta))$$
$$= rCsin(\theta) - \frac{D}{r} - B_{rem}sin(\theta)rln(r) - rB_{rem}sin(\theta)$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial A_z}{\partial r}) = \frac{1}{r}\frac{\partial}{\partial r}(rCsin(\theta) - \frac{D}{r} - B_{rem}sin(\theta)rln(r) - rB_{rem}sin(\theta))$$
$$= \frac{1}{r}(Csin(\theta) + \frac{D}{r^2} - B_{rem}sin(\theta)ln(r) - B_{rem}sin(\theta) - B_{rem}sin(\theta))$$
$$= \frac{1}{r}Csin(\theta) + \frac{D}{r^3} - \frac{1}{r}B_{rem}sin(\theta)ln(r) - \frac{2}{r}B_{rem}sin(\theta)$$

$$\frac{1}{r^2}\frac{\partial^2 A_z}{\partial \theta^2} = \frac{1}{r^2}\frac{\partial}{\partial \theta}(rCcos(\theta) + \frac{D}{r}cos(\theta) - B_{rem}rln(r)cos(\theta))$$
$$= \frac{1}{r^2}(-rCsin(\theta) - \frac{D}{r}sin(\theta) + B_{rem}rln(r)sin(\theta))$$
$$= -\frac{1}{r}Csin(\theta) - \frac{D}{r^3}sin(\theta) + \frac{1}{r}B_{rem}ln(r)sin(\theta)$$

$$\frac{\partial A_z}{\partial z} = 0$$

We can now substitute these values in our original equation and cancel terms:

$$-\nabla^2 A_z(r,\theta) = -\frac{1}{r}\frac{\partial A_z}{\partial r}(r\frac{\partial A_z}{\partial r}) - \frac{1}{r^2}\frac{\partial^2 A_z}{\partial \theta^2} - \frac{\partial A_z}{\partial z}$$
$$= -(\frac{1}{r}Csin(\theta) + \frac{D}{r^3} - \frac{1}{r}B_{rem}sin(\theta)ln(r) - \frac{2}{r}B_{rem}sin(\theta)) -$$
$$(-\frac{1}{r}Csin(\theta) - \frac{D}{r^3}sin(\theta) + \frac{1}{r}B_{rem}ln(r)sin(\theta)) - 0$$
$$= -\frac{1}{r}Csin(\theta) - \frac{D}{r^3} + \frac{1}{r}B_{rem}sin(\theta)ln(r) + \frac{2}{r}B_{rem}sin(\theta)$$
$$+\frac{1}{r}Csin(\theta) + \frac{D}{r^3}sin(\theta) - \frac{1}{r}B_{rem}ln(r)sin(\theta)$$
$$= \frac{2}{r}B_{rem}sin(\theta) = Rot(B_{rem}) \cdot e_z|_{\rho=1}$$

We can see that $\frac{2}{r}B_{rem}sin(\theta)$ is the same result we got from the last exercise when evaluating the curl of B_{rem} when evaluated at $\rho = 1$. Therefore, A_z satisfies the differential equation.

4.2 Solution of the partial differential equation

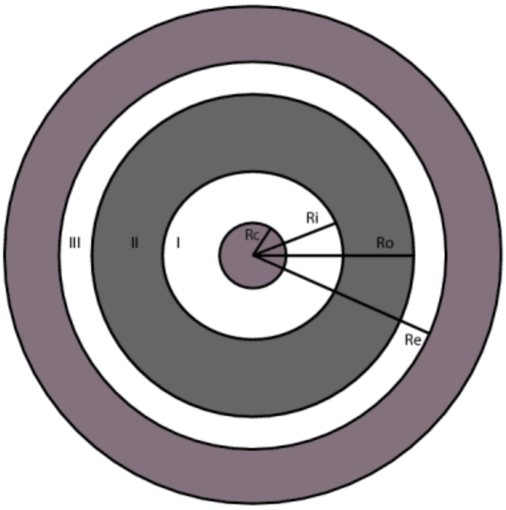

Figure 4: Drawing of the relevant domain for the solution. The Halbach magnet is placed in the grey area between r = Ri and r = Ro .

Magnetic vector field **H** through:

$$\mathbf{B} = \mu_o \mu_r \mathbf{H} + \mathbf{B_{rem}} \tag{15}$$

Using electromagnetic theory it is possible to derive boundary conditions that are generally valid for **B** and **H**. These may be written in the following way:

$$B_r^I = B_r^{II} | r = R_i \tag{16}$$

$$B_r^{II} = B_r^{III} | r = R_o \tag{17}$$

$$H_\theta^I = 0 | r = R_c \tag{18}$$

$$H_\theta^I = H_\theta^{II} | r = R_i \tag{19}$$

$$H_\theta^{II} = H_\theta^{III} | r = R_o \tag{20}$$

$$H_\theta^{III} = 0 | r = R_e \tag{21}$$

Boundary conditions show if any changes occur if we change the domain. If the equation has a $0|r$ term this means the magnetic field in 2 neighboring domains is different.

Exercise 20

To find the components of the **B** and **H** we first find the curl of **A**:

$$\mathbf{Rot}(\mathbf{A}) = (\frac{1}{r}\frac{\partial A_z}{\partial \theta} - \frac{\partial A_\theta}{\partial z})\mathbf{e_r} + (\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r})\mathbf{e}_\theta + (\frac{\partial A_\theta}{\partial r} + \frac{A_\theta}{r} - \frac{1}{r}\frac{\partial A_r}{\partial \theta})\mathbf{e_z}$$

As we have seen in section 4.1, **A** has only z component, so, all te terms besides the ones with A_z are zeros.

$$\begin{aligned}\mathbf{Rot}(\mathbf{A}) &= (\frac{1}{r}\frac{\partial A_z}{\partial \theta} - 0)\mathbf{e_r} + (0 - \frac{\partial A_z}{\partial r})\mathbf{e}_\theta + (0 + 0 - 0)\mathbf{e_z} = \\ &= (\frac{1}{r}\frac{\partial A_z}{\partial \theta})\mathbf{e_r} - (\frac{\partial A_z}{\partial r})\mathbf{e}_\theta + (0)\mathbf{e_z}\end{aligned}$$

This means according to the equations 11 and 14 that

$$\mathbf{B} = \begin{bmatrix} B_r \\ B_\theta \end{bmatrix} = \begin{bmatrix} \frac{1}{r}\frac{\partial A_z}{\partial \theta} \\ -\frac{\partial A_z}{\partial r} \end{bmatrix} = \begin{bmatrix} cos(\theta)((C + Dr^{-2}) - B_{rem}ln(r)) \\ sin(\theta)(B_{rem}(ln(r) + 1) + (Dr^{-2} - C)) \end{bmatrix} \tag{22}$$

From now on z component will not be written as it is always 0 and we are working with a 2D model.

H can be derived from the equations 22 and 15 by solving them for **H**:

$$\begin{bmatrix} B_r \\ B_\theta \end{bmatrix} = \mu_0\mu_r \begin{bmatrix} H_r \\ H_\theta \end{bmatrix} + B_{rem} \begin{bmatrix} cos(\theta) \\ sin(\theta) \end{bmatrix} \tag{23}$$

$$\begin{bmatrix} H_r \\ H_\theta \end{bmatrix} = \frac{1}{\mu_0\mu_r} \begin{bmatrix} B_r - B_{rem}cos(\rho\theta) \\ B_\theta - B_{rem}sin(\rho\theta) \end{bmatrix} \tag{24}$$

After substituting B_r and B_θ one should end up with:

$$\begin{bmatrix} H_r \\ \\ H_\theta \end{bmatrix} = \frac{1}{\mu_0\mu_r}\begin{bmatrix} -B_{rem}cos(\rho\theta) + cos(\theta)((C + Dr^{-2}) - B_{rem}ln(r)) \\ \\ -B_{rem}sin(\rho\theta) + sin(\theta)(B_{rem}(ln(r) + 1) + (Dr^{-2} - C)) \end{bmatrix} \tag{25}$$

To find the constants C and D one can use boundary conditions to construct a system of equation that holds C and D for each of 3 domains introduced at the beginning of this section.

Given the equation for the first boundary condition 16, the remaining 5 are found in the following 3 exercises.

$$C^I + D^I R_i^{-2} - C^{II} - D^{II} R_i^{-2} = -B_{rem}ln(R_i) \tag{26}$$

Exercise 21

It is stated that remanence (B_{rem}) does not exist in the domains I and III, so we don't have to worry about it. We can write B_r for the domains II and III as follows:

$$B_r^{II} = cos(\theta)(C^{II} + D^{II} R_o^{-2} - B_{rem}ln(R_o)) \tag{27}$$

$$B_r^{III} = cos(\theta)(C^{III} + D^{III} R_o^{-2} - B_{rem}ln(R_o)) = cos(\theta)(C^{III} + D^{III} R_o^{-2}) \tag{28}$$

Now we equate those two. The $cos(\theta)$ cancels out, and after re-arranging we end up with

$$C^{II} - C^{III} + (D^{II} - D^{III})R_o^{-2} = B_{rem}ln(R_o) \tag{29}$$

Exercises 22 and 23

We know that $\mu_r = \infty$ on the intervals $0 \leq r \leq R_c$ and $R_e \leq r$, this is evident from the equation 25 that **H** is 0 on these intervals. Also we are informed that $\mu_r = 1$ in I and III domains.

Following the logic from the previous exercise one can construct missing 4 equations.

Boundary condition 18

$$H_\theta^I = \frac{1}{\mu_0 \mu_r} sin(\theta)(D^I R_c^{-2} - C^I) = 0 \tag{30}$$

We set it equal to 0 because $\mu_r = \infty$ on the intervals $0 \leq r \leq R_c$.

$$D^I R_c^{-2} - C^I = 0 \tag{31}$$

Boundary condition 19

$$H_\theta^I = \frac{1}{\mu_0} sin(\theta)(D^I R_i^{-2} - C^I) \tag{32}$$

$$H_\theta^{II} = \frac{sin(\theta)}{\mu_0 \mu_r}(B_{rem}(ln(R_i) + 1) - B_{rem} + D^{II} R_i^{-2} - C^{II}) \tag{33}$$

$$H_\theta^I = H_\theta^{II} \tag{34}$$

After rearrangement:

$$\mu_r R_i^{-2} D^I - R_i^{-2} D^{II} + C^{II} - \mu_r C^I = B_{rem} ln(R_i) \tag{35}$$

Boundary condition 20

$$H_\theta^{II} = \frac{sin(\theta)}{\mu_0 \mu_r}(B_{rem}(ln(R_o) + 1) - B_{rem} + D^{II} R_o^{-2} - C^{II}) \tag{36}$$

$$H_\theta^{III} = \frac{sin(\theta)}{\mu_0}(D^{III} R_o^{-2} - C^{III}) \tag{37}$$

$$H_\theta^{II} = H_\theta^{III} \tag{38}$$

$$C^{II} - \mu_r C^{III} + \mu_r R_o^{-2} D^{III} - R_o^{-2} D^{II} = B_{rem} ln(R_o) \tag{39}$$

Boundary condition 21

$$H_\theta^{III} = \frac{1}{\mu_0} sin(\theta)(D^{III} R_e^{-2} - C^{III}) = 0$$

Again, we set it equal to 0 as the $\mu_r = \infty$ on the interval $R_e \leq r$.

$$D^{III} R_e^{-2} - C^{III} = 0 \tag{40}$$

Exercise 24

We now must find the 6 constants in the equations above. To do this we can write these 6 equations into a matrix and use Python to solve it.

$$\begin{bmatrix} 1 & -1 & 0 & R_i^{-2} & -R_i^{-2} & 0 \\ 0 & 1 & -1 & 0 & R_o^{-2} & -R_o^{-2} \\ -1 & 0 & 0 & R_c^{-2} & 0 & 0 \\ -\mu_r & 1 & 0 & \mu_r R_i^{-2} & -R_i^{-2} & 0 \\ 0 & 1 & -\mu_r & 0 & -R_o^{-2} & \mu_r R_o^{-2} \\ 0 & 0 & -1 & 0 & 0 & R_e^{-2} \end{bmatrix} \begin{bmatrix} C^I \\ C^{II} \\ C^{III} \\ D^I \\ D^{II} \\ D^{III} \end{bmatrix} = \begin{bmatrix} -B_{rem} ln(R_i) \\ B_{rem} ln(R_o) \\ 0 \\ B_{rem} ln(R_i) \\ B_{rem} ln(R_o) \\ 0 \end{bmatrix} \tag{41}$$

We get a long unsimplified output which would be difficult to show here, but can be reproduced by running the code in Appendix 7.2.

Exercise 25

In this exercise, we want to show the solutions we got in exercise 24 match the given solutions. To do this we will construct the given solutions in Python, and check that they are equal to our found solutions. The code to do this can be seen in Appendix 7.4.

4.3 A single Halbach cylinder in air

Exercise 26

To show that $\lim_{R_e \to \infty} a = 1$, we first find the equation a, which we know is as following:

$$a = \frac{R_e^2 - R_0^2}{R_e^2 + R_0^2}$$

We will be using L'Hopital's Rule to show that $\lim_{R_e \to \infty} a = 1$: $\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$

$$\lim_{R_e \to \infty} \frac{R_e^2 - R_0^2}{R_e^2 + R_0^2} = \lim_{R_e \to \infty} \frac{\frac{d}{dR_e}(R_e^2 - R_0^2)}{\frac{d}{dR_e}(R_e^2 + R_0^2)}$$
$$\lim_{R_e \to \infty} \frac{\frac{d}{dR_e}(R_e^2 - R_0^2)}{\frac{d}{dR_e}(R_e^2 + R_0^2)} = \lim_{R_e \to \infty} \frac{2R_e}{2R_e} = 1$$

To show that $\lim_{R_c \to 0} b = -1$, we first find the equation b, which we know is:

$$b = -\frac{R_i^2 - R_c^2}{R_i^2 + R_c^2}$$

Since $R_c \to 0$, R_c^2 can be considered zero, which means we can cancel them out, leaving $-\frac{R_i^2}{R_i^2}$.

$$-\frac{R_i^2}{R_i^2} = -1$$

Exercise 27

Using the following equation for D^{II}, we will find the limit as μ_r approaches 1.

$$D^{II} = -\left(\frac{(a\mu_r - 1)}{(a\mu_r + 1)} \cdot R_0^{-2} + \frac{(b\mu_r - 1)}{(b\mu_r + 1)} \cdot R_i^{-2}\right)^{-1} \cdot B_{rem} \ln\left(\frac{R_i}{R_0}\right)$$

Since μ_r goes towards 1, we can remove it from the equation, leaving it as:

$$D^{II} = -\left(\frac{(a-1)}{(a+1)} \cdot R_0^{-2} + \frac{(b-1)}{(b+1)} \cdot R_i^{-2}\right)^{-1} \cdot B_{rem} \ln\left(\frac{R_i}{R_0}\right)$$

To simplify further, we can substitute in a and b, but first, we can simply them a little to make them easier to work with. The equations for a and b are the same as used to solve the previous exercise and have been written below as a refresher:

$$a = \frac{R_e^2 - R_0^2}{R_e^2 + R_0^2} \text{ and } b = -\frac{R_i^2 - R_c^2}{R_i^2 + R_c^2} \tag{42}$$

For a, we can rewrite it as $\dfrac{R_e^2 - R_0^2}{R_e^4 - R_0^4}$ because we can multiply with the conjugate to make the denominator a subtraction instead of an addition:

$$a = \frac{R_e^2 - R_0^2}{R_e^2 + R_0^2} \rightarrow \frac{R_e^2 - R_0^2}{R_e^2 + R_0^2} \cdot \frac{R_e^2 - R_e^2}{R_e^2 - R_e^2} \rightarrow \frac{(R_e^2 - R_0^2)(R_e^2 - R_e^2)}{(R_e^2 + R_0^2)(R_e^2 - R_e^2)} \rightarrow \frac{R_e^2 - R_0^2}{R_e^4 - R_0^4}$$

We can do the same for b:

$$b = -\frac{R_i^2 - R_c^2}{R_i^2 + R_c^2} \rightarrow -\frac{R_i^2 - R_c^2}{R_i^2 + R_c^2} \cdot \frac{R_i^2 - R_i^2}{R_i^2 - R_i^2} \rightarrow -\frac{(R_i^2 - R_c^2)(R_i^2 - R_i^2)}{(R_i^2 + R_c^2)(R_i^2 - R_i^2)} \rightarrow -\frac{R_i^2 - R_c^2}{R_i^4 - R_c^4}$$

This leaves us with the following equation:

$$D^{II} = -\left(\frac{\left(\frac{R_e^2 - R_0^2}{R_e^4 - R_0^4}\right) - 1}{\left(\frac{R_e^2 - R_0^2}{R_e^4 - R_0^4}\right) + 1} \cdot R_0^{-2} + \frac{\left(-\frac{R_i^2 - R_c^2}{R_i^4 - R_c^4}\right) - 1}{\left(-\frac{R_i^2 - R_c^2}{R_i^4 - R_c^4}\right) + 1} \cdot R_i^{-2}\right)^{-1} \cdot B_{rem} \ln\left(\frac{R_i}{R_0}\right)$$

We can simplify the first fraction by adding and subtracting the numerator and the denominator by $R_e^4 - R_0^4$, respectively:

$$\begin{aligned} &\frac{(R_e^2 - R_0^2) - (R_e^4 - R_0^4)}{(R_e^2 - R_0^2) + (R_e^4 - R_0^4)} \\ &= \frac{R_e^2 - R_0^2 - R_e^4 + R_0^4}{R_e^2 - R_0^2 + R_e^4 - R_0^4} \\ &= \frac{-R_e^4 + R_0^4 - R_e^2 + R_0^2}{R_e^4 - R_0^4 + R_e^2 - R_0^2} \end{aligned}$$

By factorising -1, we can effectively cancel out the rest, leaving only -1.

$$\frac{-1(R_e^4 - R_0^4 + R_e^2 - R_0^2)}{R_e^4 - R_0^4 + R_e^2 - R_0^2} = -1$$

We can do the same with the second fraction and we also get -1.

Now, when we substitute them back into D^{II}, we get:

$$D^{II} = -\left(-1 \cdot R_0^{-2} + -1 \cdot R_i^{-2}\right)^{-1} \cdot B_{rem} \ln\left(\frac{R_i}{R_0}\right)$$

$$D^{II} = -\left(-R_0^{-2} - R_i^{-2}\right)^{-1} \cdot B_{rem} \ln\left(\frac{R_i}{R_0}\right)$$

This way, we get the fully simplified answer:

$$D^{II} = \left(R_0^{-2} + R_i^{-2}\right)^{-1} \cdot B_{rem} \ln\left(\frac{R_i}{R_0}\right)$$

Exercise 28

We know that:

$$C^{II} = -D^{II}\frac{\mu_r a - 1}{\mu_r a + 1}R_0^{-2} + B_{rem}\ln(R_0)$$

$$D^{III} = C^{III}R_e^2$$

$$C^{III} = \frac{D^{II}}{R_0^2 + R_e^2}\left(1 - \frac{\mu_r a - 1}{\mu_r a + 1}\right)$$

Firstly, to find the limit of C^{III}, we can cancel out μ_r:

$$\lim_{\mu_r \to 1} C^{II} = -\left(\lim_{\mu_r \to 1} D^{II}\right)\left(\frac{a-1}{a+1} \cdot R_0^{-2}\right) + B_{rem}\ln(R_0)$$

Since we know what $\lim_{\mu_r \to 1} D^{II}$ is, we can substitute that in to get the final answer:

$$\lim_{\mu_r \to 1} C^{II} = -\left(R_0^{-2} + R_i^{-2}\right)^{-1} \cdot B_{rem} \ln\left(\frac{R_i}{R_0}\right) \times -R_0^{-2} + B_{rem} \ln(R_0)$$

We get $-R_0^{-2}$ because after μ_r is cancelled out we are left with $\frac{a-1}{a+1} R_0^{-2}$ and we know that $\frac{a-1}{a+1}$ can be simplified to -1.

When we do the same, we get the following for C^{III}:

$$\lim_{\mu_r \to 1} C^{III} = 2 \cdot \frac{\left(R_0^{-2} + R_i^{-2}\right)^{-1} \cdot B_{rem} \ln\left(\frac{R_i}{R_0}\right)}{R_0^2 + R_e^2}$$

Here, we get 2 because after μ_r is cancelled out we are left with $1 - \frac{a-1}{a+1}$ and we know that $\frac{a-1}{a+1}$ can be simplified to -1, which means that we get $1 - (-1) = 2$.

For D^{III}, it is simply a matter of taking the entirety of C^{III} and multiplying that with R_e^2 and we get:

$$\lim_{\mu_r \to 1} D^{III} = 2 \cdot \frac{\left(R_0^{-2} + R_i^{-2}\right)^{-1} \cdot B_{rem} \ln\left(\frac{R_i}{R_0}\right)}{R_0^2 + R_e^2} \cdot R_e^2$$

Exercise 29

Now we can find the limits of C^I and D^I when $\mu_r \to 1$.

We know the following:

$$D^{II} = -\left(\frac{(a\mu_r - 1)}{(a\mu_r + 1)} \cdot R_0^{-2} + \frac{(b\mu_r - 1)}{(b\mu_r + 1)} \cdot R_i^{-2}\right)^{-1} \cdot B_{rem} \ln\left(\frac{R_i}{R_0}\right) \tag{43}$$

$$C^I = \frac{D^{II}}{R_i^2 + R_c^2}\left(1 - \frac{\mu_r b - 1}{\mu_r b + 1}\right)$$

$$D^I = C^I \cdot R_c^2$$

$$R_e \to \infty$$

$$R_c \to 0$$

We use equations 42 and 43 to substitute D^{II} and start simplifying from there.

$$D^{II} = -R_0^{-2} - R_i^{-2}$$

$$-(-R_0^{-2} - R_i^{-2}) \cdot B_{rem} ln(\frac{R_i}{R_0})$$

$$\frac{D^{II}}{R_1^2 + R_c^2}(1 + \frac{\mu_r + 1}{-\mu_r + 1})$$

$$= \lim_{\mu_r \to 1} \frac{1}{R_i^2} \cdot \left[-\frac{\mu_r - 1}{\mu_r + 1} \cdot R_o^{-2} + \frac{\mu_r - 1}{-\mu_r + 1} R_i^{-2} \right]^{-1} \cdot B_{rem} ln(\frac{R_i}{R_0}) \cdot (1 + \frac{\mu_r + 1}{1 - \mu_r})$$

$$= B_{rem} ln(\frac{R_i}{R_0}) \lim_{\mu_r \to 1} \frac{1}{R_i^2}[(-\frac{(\mu - 1) \cdot (-\mu + 1)}{(\mu_r + 1) \cdot (-\mu_r + 1)} \cdot \frac{R_0^{-2}(\mu + 1)(-\mu + 1)}{(\mu_r + 1)(-\mu_r + 1)}$$

$$+ \frac{(\mu_r - 1)(\mu + 1)}{(\mu_r + 1) \cdot (-\mu + 1)} \cdot \frac{R_i^{-2}(\mu + 1)(-\mu + 1)}{(\mu_r + 1) \cdot (-\mu_r + 1)})]^{-1} \cdot 1 + (\frac{\mu_r + 1}{1 - \mu_r})$$

Here we can see that all of the terms cancel out to 0, so we are left with

$$C^I = 0$$

$$D^I = 0$$

We can then use these values to calculate the components of the magnetic field, matching the equations given to us in the document:

$$B_r^I = B_{rem} ln(\frac{R_o}{R_i}) cos(\theta) \tag{44}$$

$$B_\theta^I = -B_{rem} ln(\frac{R_o}{R_i}) sin(\theta) \tag{45}$$

$$B_r^{II} = B_{rem} ln(\frac{R_o}{r}) cos(\theta) \tag{46}$$

$$B_\theta^{II} = -B_{rem} ln(\frac{R_o}{r} - 1) sin(\theta) \tag{47}$$

Exercise 30

In this exercise we will show graphically how the norm $B = \sqrt{B_r^2 + B_\theta^2}$ behaves, as plotted in python using sympy. Since we are looking for the magnetic field inside the magnet, we use the values:

$$B_r^{II} = B_{rem} ln(\frac{R_o}{r} cos(\theta))$$
$$B_\theta^{II} = -B_r em(ln(\frac{R_o}{r}) - 1) sin(\theta)$$

Thus, the magnetic field is given by:

$$B = \sqrt{(B_r^{II})^2 + (B_\theta^{II})^2}$$

We can now generate a plot of the magnetic field by choosing arbitrary values for R_o and R_i. For our plot we have $B_{rem} = 1.4T$, $R_o = 8$ and $R_i = 1.4$, since these parameters draw a nice graph:

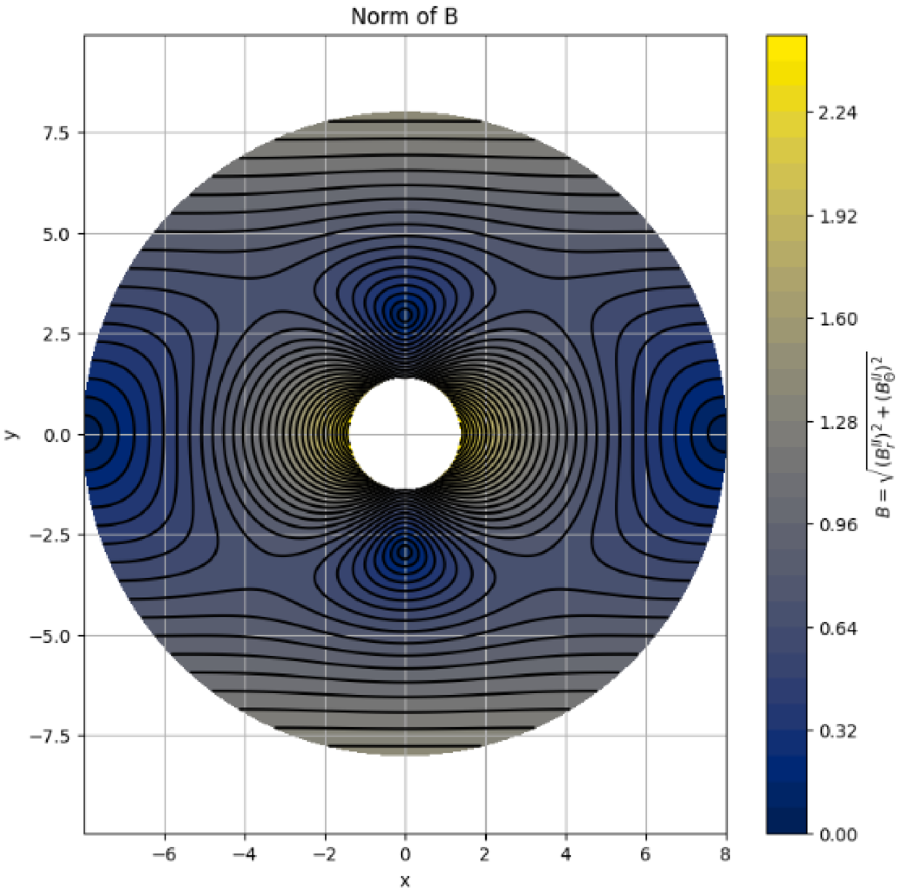

Figure 5: Graph for the norm of B with $B_{rem} = 1.4, R_o = 8$ and $R_i = 1.4$

The plot shows the strength of the magnetic field in different areas within the magnet, showing the areas with strongest magnetic field in yellow, while the weakest areas are in blue. The contour lines on the other hand show the change in magnitude of the strength of the magnetic field, or its slope. Thus, closer lines represent a faster rate change whereas more separated lines signify a slower rate of change.

Exercise 31

We want to find the norm of B using the following equation:

$$B = \sqrt{B_r^2 + B_\theta^2} \tag{48}$$

To do this we insert B_r (44) and B_θ (45) from domain 1 into the equation above, and then simplify.

$$B = \sqrt{(B_{rem} ln(\frac{R_o}{R_i}) cos(\theta))^2 + (B_{rem} ln(\frac{R_o}{R_i}) sin(\theta))^2} = \sqrt{(B_{rem}^2 ln(\frac{R_o}{R_i}))^2} \tag{49}$$

We expect a positive result and for R_o and R_i not to be zero, so can further simplify to $Norm(B) = B_{rem} ln(\frac{R_o}{R_i})$.

5 Halbach magnets and their applications

5.1 Halbach magnet for MRI

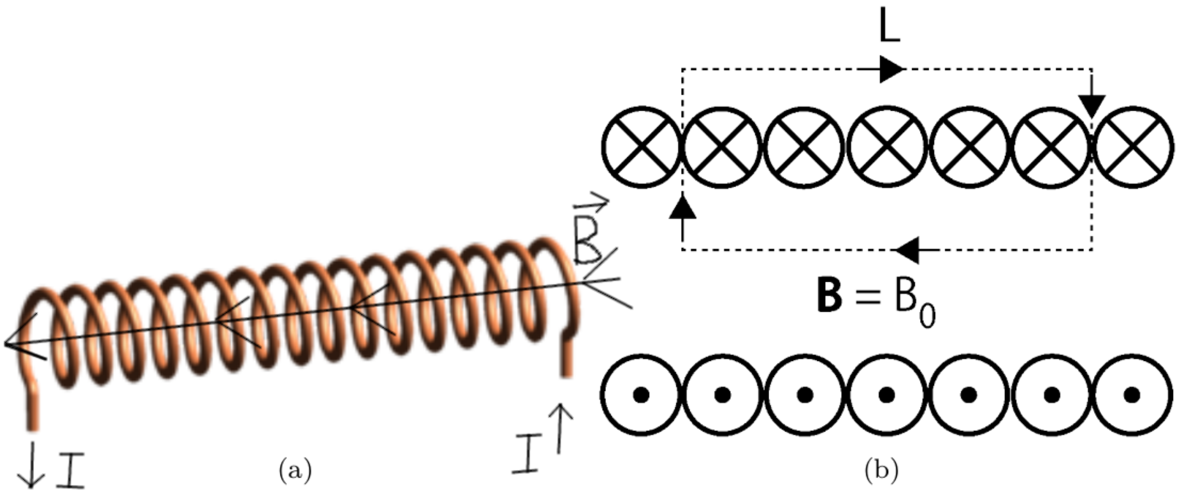

Figure 6: A coil. The schematic of the coil is seen from the side, The cross indicates that the current runs into the page while a dot indicates that it runs out of the page.

Exercise 32

To determine the direction of the magnetic field induced by a solenoid one can use the right-hand rule. In Figure 6 we chose the direction of electrical current, shown by the arrows I, and applying the right-hand rule we got the direction of the magnetic field pointing to the left, $\vec{B}$.

$$\oint_C \mathbf{B}\, d\mathbf{l} = \mu_0 I_{enc} \tag{50}$$

Exercise 33

Now let us find an expression for the magnetic field produced by the coil with N windings and the length of L, the current through the coil is I.

The easiest way to find this expression will be using Ampere's law 50. What it says is a magnetic field that goes across the closed(starts and ends at the same point) line equals the current enclosed by this line times permeability.

So, if we take a look at Figure 6.b, the dotted line you can see is the closed line we need to integrate across. The magnetic field goes from the right to the left across the bottom part of the contour(dotted line). As the vertical lines of the contour are perpendicular to the magnetic field the dot product will become zero, and the current outside the coil may be considered to be zero, so the only non-zero component of the integral is the magnetic field times the length of line L. The left part of the Ampere's law can be written as:

$$\oint_C \mathbf{B}\, d\mathbf{l} = BL \tag{51}$$

We are only left with the right side. I_{enc} can be thought of as all currents enclosed inside the contour:

$$I_{enc} = NI$$

Now we can contract the final equation:

$$BL = \mu_0 NI$$

Solve for B, and get:

$$B = \frac{\mu_0 NI}{L} \tag{52}$$

Exercise 34

Hole diameter $= 60cm$, a field $B_0 = 1.5T$, and a length of $50cm$. The remanence of a permanent magnet is $Brem = 1.4T$.

To find the outer diameter for the Halbach magnet we can use equations 44, 45, and the actual norm 48.

We know the inner diameter $R_i = 0.6m$ the magnitudes of the magnetic field and the remanence are respectively $1.5T$ and $1.4T$, so the only unknown is R_o - the outer diameter. (the whole solution is in the appendix). What we get: the outer radius is $1.75m$. The thickness can be calculated as $1.75 - 0.6 = 1.05m$.

Exercise 35

Now we would love to see if it is possible to have a solenoid with 5000 windings instead of the permanent magnet. Using the formula 52 and the values for the magnetic fields from the previous exercise we can find the current:

$$I = \frac{BL}{\mu_0 N} = \frac{1.5 \cdot 0.5}{4\pi \cdot 10^{-7} \cdot 5000} \approx 119.4\,A$$

Based on a simple calculation of the resistance of a copper wire of $1mm$ in diameter with resistance under the temperature of 20^o, which is also should be taken into consideration, as under that large current the wire will become blazing hot, which will increase the resistivity. And, so, the voltage should then be something of the magnitude of 12000 volts. Taking this into account having a coil instead of a permanent magnet introduces too many technical complications.

Exercise 36

To find the length of the wire needed we can simply multiply the number of windings be the circumference of the coil.

$$N \cdot 2\pi r = 5000 \cdot 2 \cdot \pi \cdot 0.6 \approx 18850m \tag{53}$$

Exercise 37

The power(heating) generated by a current is calculated by

$$P = I^2 R \approx 1.4MW \tag{54}$$

To get a sense if it is a lot or not: FuelCell Energy's 1500 power plant generates 1.4 MW of reliable and efficient power using carbonate fuel cells. According to this page[1]. So, it is quite a substantial amount of energy that cannot be neglected.

Exercise 38

For the supper conducting wire, the maximum current can be expressed with the formula:

$$I_C(B) = \frac{I_0}{1 + B/B_0} \tag{55}$$

We now can substitute the formula 52 for B derived in Exercise 33.

$$I_C(B) = \frac{I_0}{1 + \frac{I_C \mu_0 N}{L B_0}} = \\ = \frac{I_0 L B_0}{L B_0 + I_C \mu_0 N}$$

[1] https://www.fuelcellenergy.com/platform/fuel-cell-power-plants

Solve for I_C

$$I_C L B_0 + I_C^2 \mu_0 N = I_0 L B_0 \tag{56}$$

Disregarding negative solution we get

$$I_C = \frac{-LB_0 + \sqrt{L^2 B_0^2 + 4\mu_0 N L B_0 I_0}}{2\mu_0 N} \tag{57}$$

The expression for N can be derived from the equation 56.

$$N = LB_0 \frac{I_0 - I_C}{I_C^2 \mu_0} \tag{58}$$

Substituting $I_C = 1.2kA$, $B = 1.5T$, $B_0 = 3T$, $L = 0.5m$ and $I_0 = 1.8kA$ we get:

$$N = 0.5 \cdot 3 \frac{(1.8 - 1.2) \cdot 10^3}{(1.2 \cdot 10^3)^2 \cdot 4\pi 10^{-7}} = 497$$

6 References

Marchio, Cathy. "Everything You Need to Know about Halbach Arrays." Stanford Magnets, 9 Apr. 2024, www.stanfordmagnets.com/everything-you-need-to-know-about-halbach-arrays.html.

Marchio, Cathy. "What Is a Permanent Magnet?" Stanford Magnets, 16 Apr. 2024, www.stanfordmagnets.com/what-is-a-permanent-magnet.html.

7 Appendix

7.1 Code for Exercise 13

Python code to generate Figure 1:

```
import numpy as np
import matplotlib.pyplot as plt


#Defining the grid to span between -1, and 1 in both x and y coordinates
x = np.linspace(-1, 1, 15)
y = np.linspace(-1, 1, 15)
X, Y = np.meshgrid(x, y)


#Defining the vector field function from question 12
def B_field(x, y):
    mu_0 = 4*np.pi*10**-7
```

```
    I = 1
    R = np.sqrt(x**2 + y**2)
    B_theta = mu_0 * I / (2 * np.pi * R)
    B_r = 0
    return B_theta, B_r
B_theta, B_r = B_field(X, Y)


#Converting to cartesian coordinates
Bx = -B_theta * Y / np.sqrt(X**2 + Y**2)
By = B_theta * X / np.sqrt(X**2 + Y**2)


plt.figure(figsize=(8, 8))
plt.quiver(X, Y, Bx, By)
plt.xlabel('x')
plt.ylabel('y')
plt.axis('equal')
plt.grid(True)
plt.show()
```

7.2 Section 4.2 appendix

7.3 Finding solution for the domain constants

```
from sympy import symbols, ln, sin, simplify, Matrix
```

```
# Define symbols
R_i, R_o, R_c, R_e, B_rem, p, th, mu_r = symbols('R_i R_o R_c R_e B_rem p th mu_r', re

lhs_matrix = Matrix([
    [1,-1, 0, R_i**(-2), -R_i**(-2), 0],
    [0, -1, 1, 0, -R_o**(-2), R_o**(-2)],
    [mu_r, -1, 0, -R_i**(-2) * mu_r, R_i**(-2), 0],
    [0, -1, mu_r, 0, R_o**(-2), -R_o**(-2) * mu_r],
    [1, 0, 0, -R_c**(-2), 0, 0],
    [0, 0, 1, 0, 0, -R_e**(-2)]
])

# Define right-hand side expressions
rhs_matrix = Matrix([
    -B_rem * ln(R_i),
    -B_rem * ln(R_o),
    -B_rem * (1 + ln(R_i) - (sin(th)*p/sin(th))),
    -B_rem * (1 + ln(R_o) - (sin(th)*p/sin(th))),
    0,
    0
])

solution = simplify(lhs_matrix**(-1)*rhs_matrix)
```

7.4 Comparison of the matrix results

:

```
a, b, c1, c2, c3, d1, d2, d3 = symbols('a bc c1 c2 c3 d1 d2 d3', real=True, positive=T

R_i, R_o, R_c, R_e, B_rem, p, th, mu_r = symbols('R_i R_o R_c R_e B_rem p th mu_r', re

a_eq = (R_e**2 - R_o**2)/(R_e**2 + R_o**2)

b_eq = -(R_i**2 - R_c**2)/(R_i**2 + R_c**2)


d2_eq = -((mu_r*a-1)/(mu_r*a+1)*R_o**(-2) - (mu_r*b-1)/(mu_r*b+1)*R_i**(-2))**(-1) * B

c1_eq = (d2/(R_i**2+R_c**2))*(1-((mu_r*b-1)/(mu_r*b+1)))

d1_eq = c1*R_c**2

c2_eq = -d2*((mu_r*a-1)/(mu_r*a+1))*R_o**(-2) + B_rem*ln(R_o)

c3_eq = (d2/(R_o**2+R_e**2))*(1-((mu_r*a-1)/(mu_r*a+1)))

d3_eq = c3*R_e**2


d2_eq = d2_eq.subs([(a, a_eq), (b, b_eq), (c1, c1_eq), (c2, c2_eq),  (c3, c3_eq), (d1

c1_eq = c1_eq.subs([(a, a_eq), (b, b_eq), (c1, c1_eq), (c2, c2_eq), (c3, c3_eq), (d1,

d1_eq = d1_eq.subs([(a, a_eq), (b, b_eq), (c1, c1_eq), (c2, c2_eq), (c3, c3_eq), (d1,

c2_eq = c2_eq.subs([(a, a_eq), (b, b_eq), (c1, c1_eq), (c2, c2_eq), (c3, c3_eq), (d1,

c3_eq = c3_eq.subs([(a, a_eq), (b, b_eq), (c1, c1_eq), (c2, c2_eq), (c3, c3_eq), (d1,

d3_eq = d3_eq.subs([(a, a_eq), (b, b_eq), (c1, c1_eq), (c2, c2_eq), (c3, c3_eq), (d1,


solution_p_1 = solution.subs(p,1)
```

```
display(solution_p_1)


display(simplify(solution_p_1[0]-c1_eq))
display(simplify(solution_p_1[1]-c2_eq))
display(simplify(solution_p_1[2]-c3_eq))
display(simplify(solution_p_1[3]-d1_eq))
display(simplify(solution_p_1[4]-d2_eq))
display(simplify(solution_p_1[5]-d3_eq))
```

7.5 Plot for the magnetic field inside the magnet

here is the code used for generating the plot for exercise 30:

```
#libraries
import numpy as np
import matplotlib.pyplot as plt


#values
B_rem = 1.4
R_o = 8
R_i = 1.4


#r and theta variables
r = np.linspace(R_i, R_o, 1000)
theta = np.linspace(0, 2*np.pi, 1000)
```

```
r_grid, theta_grid = np.meshgrid(r, theta)

#equations
def B_r(r, theta):
    return B_rem*np.log(R_o/r)*np.cos(theta)

def B_theta(r, theta):
    return -B_rem*(np.log(R_o/r) - 1)*np.sin(theta)

def B_norm(r, theta):
    return np.sqrt((B_r(r, theta)**2) + (B_theta(r, theta)**2))

#compute
B = B_norm(r_grid, theta_grid)

#we convert to cartesian
x = r_grid * np.cos(theta_grid)
y = r_grid * np.sin(theta_grid)

#plot
plt.figure(figsize=(8, 8))
contourf = plt.contourf(x, y, B, 30, cmap='cividis')
plt.colorbar(contourf, label='$B = \sqrt{(B^{II}_r)^2 + (B^{II}_{\Theta})^2}$')
```

```
contour = plt.contour(x, y, B, 30, colors='black')


plt.xlabel('x')

plt.ylabel('y')

plt.grid(True)

plt.title('Norm of B')

plt.axis('equal')

plt.show()
```

7.6 Section 5 appendix

```
Ri, Ro, Rc, Re, Brem, mu_r = symbols("Ri, Ro, Rc, Re, Brem, mu_r")

B_r, B_th = symbols("B_r, B_th")

B_r = Brem*ln(Ro/Ri)*cos(th)

B_th = -Brem*ln(Ro/Ri)*sin(th)

B = sqrt(B_r**2 + B_th**2)

mag_d = solve(B.subs({Ri:0.6, Brem:1.4})-1.5,Ro)[1]

I_sol = 0.75/(4*pi*10**(-7)*5000)

length = 2*pi*0.6*5000

A = pi*0.001**2

rho = 8960

resistivity_copper = 1.68 * 10**(-8)

L = 0.5
```

```
B = 1.5

B_0 = 3

I_0 = 1.8 * 10**3

res_copper = resistivity_copper * length / A

mass = length * A * rho

power = I_sol**2 * res_copper

N = 0.5 * 3 *(1.8 - 1.2) * 10**3 / (1.2 * 10**3)**2 / (4 * pi * 10**(-7))


print("Outer radius =", mag_d.evalf())

print("Current through the wire =", I_sol.evalf())

print("length =", length.evalf())

print("resistivity =", res_copper)

print("mass =", mass.evalf())

print("power =", power.evalf())

print("N =", N.evalf())


Outer radius = 1.75172838247696

Current through the wire = 119.366207318921

length = 18849.5559215388

resistivity = 100.800000000000

mass = 530.589932602564

power = 1436227.77813014

N = 497.359197162173
```

7.7 $\nabla^2 f$ in cylindrical coordinates

Parametrization of (x, y, z) in cylindrical coordinates: $(rcos(\theta), rsin(\theta), z)$.

$$\frac{\partial}{\partial r} = \frac{\partial}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial}{\partial z}\frac{\partial z}{\partial r} = cos(\theta)\frac{\partial}{\partial x} + sin(\theta)\frac{\partial}{\partial y}$$
$$\frac{\partial}{\partial \theta} = \frac{\partial}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial}{\partial y}\frac{\partial y}{\partial \theta} + \frac{\partial}{\partial z}\frac{\partial z}{\partial \theta} = rcos(\theta)\frac{\partial}{\partial y} - rsin(\theta)\frac{\partial}{\partial x}$$
$$\frac{\partial}{\partial z} = \frac{\partial}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial}{\partial y}\frac{\partial y}{\partial z} + \frac{\partial}{\partial z}\frac{\partial z}{\partial z} = \frac{\partial}{\partial z}$$

Now we solve for $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$. The result is

$$\frac{\partial}{\partial x} = cos(\theta)\frac{\partial}{\partial r} - \frac{sin(\theta)}{r}\frac{\partial}{\partial \theta} \tag{59}$$
$$\frac{\partial}{\partial y} = sin(\theta)\frac{\partial}{\partial r} + \frac{cos(\theta)}{r}\frac{\partial}{\partial \theta} \tag{60}$$
$$\frac{\partial}{\partial z} = \frac{\partial}{\partial z} \tag{61}$$

Now we can write ∇ as follows:

$$\nabla = cos(\theta)\frac{\partial}{\partial r} - \frac{sin(\theta)}{r}\frac{\partial}{\partial \theta})\hat{\mathbf{i}} + (sin(\theta)\frac{\partial}{\partial r} + \frac{cos(\theta)}{r}\frac{\partial}{\partial \theta})\hat{\mathbf{j}} + (\frac{\partial}{\partial z})\hat{\mathbf{k}} \tag{62}$$

$$\nabla \cdot \nabla \cdot f = (cos(\theta)\frac{\partial}{\partial r} - \frac{sin(\theta)}{r}\frac{\partial}{\partial \theta})\hat{\mathbf{i}} + (sin(\theta)\frac{\partial}{\partial r} + \frac{cos(\theta)}{r}\frac{\partial}{\partial \theta})\hat{\mathbf{j}} + (\frac{\partial}{\partial z})\hat{\mathbf{k}}) \cdot (cos(\theta)\frac{\partial f}{\partial r} - \frac{sin(\theta)}{r}\frac{\partial f}{\partial \theta})\hat{\mathbf{i}} +$$
$$+ (sin(\theta)\frac{\partial f}{\partial r} + \frac{cos(\theta)}{r}\frac{\partial f}{\partial \theta})\hat{\mathbf{j}} + (\frac{\partial f}{\partial z})\hat{\mathbf{k}})$$

$$\nabla^2 f = (cos(\theta)\frac{\partial}{\partial r} - \frac{sin(\theta)}{r}\frac{\partial}{\partial \theta})(cos(\theta)\frac{\partial f}{\partial r} - \frac{sin(\theta)}{r}\frac{\partial f}{\partial \theta}) +$$
$$+ (sin(\theta)\frac{\partial}{\partial r} + \frac{cos(\theta)}{r}\frac{\partial}{\partial \theta})(sin(\theta)\frac{\partial f}{\partial r} + \frac{cos(\theta)}{r}\frac{\partial f}{\partial \theta}) + (\frac{\partial}{\partial z})(\frac{\partial f}{\partial z})$$

$$\nabla^2 f = cos^2(\theta)\frac{\partial^2 f}{\partial r^2} - cos(\theta)sin(\theta)\frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial f}{\partial \theta}) - \frac{sin(\theta)}{r}\frac{\partial}{\partial \theta}(cos(\theta)\frac{\partial f}{\partial r}) + \frac{sin(\theta)}{r^2}\frac{\partial}{\partial \theta}(sin(\theta)\frac{\partial f}{\partial \theta})+$$
$$+sin^2(\theta)\frac{\partial^2 f}{\partial r^2} + +cos(\theta)sin(\theta)\frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial f}{\partial \theta}) + \frac{cos(\theta)}{r}\frac{\partial}{\partial \theta}(sin(\theta)\frac{\partial f}{\partial r}) + \frac{cos(\theta)}{r^2}\frac{\partial}{\partial \theta}(cos(\theta)\frac{\partial f}{\partial \theta} + \frac{\partial^2 f}{\partial z^2} =$$
$$= \frac{\partial^2 f}{\partial r^2} + \frac{sin^2(\theta)}{r}\frac{\partial f}{\partial r} - \frac{cos(\theta)sin(\theta)}{r}\frac{\partial^2 f}{\partial r \partial \theta} + \frac{cos(\theta)sin(\theta)}{r^2}\frac{\partial f}{\partial \theta} + \frac{sin^2(\theta)}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{cos^2(\theta)}{r}\frac{\partial f}{\partial r}+$$
$$+\frac{cos(\theta)sin(\theta)}{r}\frac{\partial^2 f}{\partial \theta \partial r} - \frac{cos(\theta)sin(\theta)}{r^2}\frac{\partial f}{\partial \theta} + \frac{cos^2(\theta)}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2} =$$
$$= \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}$$

Usually written as

$$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial f}{\partial r}) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2} \tag{63}$$