02601 Introduction to numerical algorithms

Homework assignment 2

Hand in on DTU Learn before 24 October 10pm

1 Multiple choice (40%)

Each question has only ONE correct answer. In the pdf file with your answers, you just need type the number of the answer. You get 10% for a correct answer, 0% for no answer, and -5% for a wrong answer.

A – Interpolation. Consider the table

x	0	3	3	5
y	-2	-2	0	9

1. The table **does not satisfy** the condition to be an interpolation table because the values of x can not be 0 (zero).
2. The table **does not satisfy** the condition to be an interpolation table because the values of x must be equally spaced.
3. The table **does not satisfy** the condition to be an interpolation table because there cannot be two points with the same y values.
4. The table **does not satisfy** the condition to be an interpolation table because there cannot be two points with the same x values.
5. The table **satisfies** the condition to be an interpolation table.

B – Interpolation. We approximate the function $f(x) = \cos(2x)$ by an interpolation polynomial of degree 3 on $3+1$ equally spaced nodes in the interval $[0, \frac{\pi}{2}]$, and we name this polynomial as p_3. Based on the second interpolation error theorem, what is the theoretical upper bound of the interpolation error for p_3, i.e., the upper bound of $|f(x) - p_3(x)|$?

1. 7.516×10^{-2}.
2. 3.758×10^{-2}.
3. 1.235×10^{-2}.
4. 3.906×10^{-3}.

C – Numerical integration. The function f has the function values:

$$f(1) = 2, \qquad f(2) = 2, \qquad f(3) = 3. \tag{1}$$

With three function values, the integral $\int_1^3 f(x)\,dx$ can be approximated by using the composite trapezoid rule and Simpson rule. The integral also can be approximated by using the trapezoid rule with only two function values $f(1)$ and $f(3)$. The three approximations $\tilde{I}_{\text{composite trapezoid}}$, $\tilde{I}_{\text{Simpson}}$ and $\tilde{I}_{\text{trapezoid}}$ satisfy

1. $\tilde{I}_{\text{trapezoid}} < \tilde{I}_{\text{composite trapezoid}} < \tilde{I}_{\text{Simpson}}$
2. $\tilde{I}_{\text{composite trapezoid}} < \tilde{I}_{\text{Simpson}} < \tilde{I}_{\text{trapezoid}}$
3. $\tilde{I}_{\text{Simpson}} < \tilde{I}_{\text{composite trapezoid}} < \tilde{I}_{\text{trapezoid}}$
4. $\tilde{I}_{\text{Simpson}} < \tilde{I}_{\text{trapezoid}} < \tilde{I}_{\text{composite trapezoid}}$

D – Numerical integration. Consider the function $f(x) = xe^x$ on the interval $I = [0, 4]$. Apply the composite Simpson's rule on n equally spaced subintervals to approximate the integral $\int_0^4 f(x)\,dx$. We know that the mth derivative of f is $f^{(m)}(x) = (m + x)e^x$. What is the smallest value n such that the largest approximation error is guaranteed to not exceed 10^{-4}?

1. $n = 71$
2. $n = 72$
3. $n = 2090$
4. $n = 84$

2 Approximation of a function (25%)

In this question, we will approximate the function $f(x) = \ln x$ on the interval $I = [1, 2]$ and study the approximation error. We know that the absolute value of the nth derivative of f is $|f^{(n)}(x)| = \frac{(n-1)!}{x^n}$ for $n > 1$.

2.1) (10%) Approximate the function $f(x)$ by an interpolation polynomial of degree 9, named p_9, on $9 + 1$ equally spaced nodes in the interval I with the first point at $x = 1$ and the last point at $x = 2$. Compute and state the maximum absolute error $\max_{x \in I} |f(x) - p_9(x)|$ at 400 equally spaced points in the interval I. For this question, you will need the Python function `InterpolerLagrangeForm.py` from Exercise 7.2.

2.2) (5%) Based on the second interpolation error theorem, find a theoretical upper bound of the interpolation error for p_9. State this upper bound. (Hint: The factorial $n!$ can be calculated by using the Matlab function `factorial(n)`.)

Now we approximate f by a polynomial of degree 25, p_{25}, on $25 + 1$ equally spaced nodes in the interval I.

2.3) (5%) Compute and state the maximum absolute error $\max_{x \in I} |f(x) - p_{25}(x)|$ in these 400 points as in Question **2.1**.

2.4) (5%) Note that the result in Question **2.3** is larger than the theoretical upper bound given in the second interpolation error theorem. Is there an error in the theoretical upper bound or is there another explanation?

3 The composite Simpson's rule (35%)

Modify your function for computing the composite trapezoid method slightly to carry out the composite Simpson's rule, and call the new Python function as `MySimpson`. Same as for the composite trapezoid method, the inputs of the new function should be `( f, a, b, n)`, where `f` is the integrand, `[a, b]` is the interval of integration, and `n` is the number of the subintervals. Include your code in the report.

- (10%) Your Python function must check if `n` is an even number before continuing the calculation. If n is not even, then it should display an error message and return `NaN`.
- (10%) Now, let's use the composite Simpson's rule to approximate $\int_0^\pi \sin(x)dx$. By increasing n we should obtain better approximation. Investigate if the error reduces as fast as you expected. State and comment on the results.
- (10%) Theory: Note that for the integral $\int_0^\pi \sin(x)dx$, we know the upper bound of the fourth derivative of the function $\sin(x)$. Use this and the error term (8) in page 231 in the textbook to derive the n-dependent upper bound of the absolute error for the composite Simpson's rule.
- (5%) Theory: Find a n_0 such that the absolute error is less than 10^{-4}.