02601 Introduction to numerical algorithms

Homework assignment 3

Hand in on DTU Learn before 14 November 10pm

1 Multiple choice (40%)

Each question has only ONE correct answer. In the pdf file with your answers, you just need type the number of the answer. You get 10% for a correct answer, 0% for no answer, and -5% for a wrong answer.

A – Systems of ordinary differential equations. A system of ordinary differential equations

$$\begin{aligned} x''' &= (y')^2 - x + t \\ y'' &= \sin(t) + x' - x'' \end{aligned}$$

can be rewritten as a system of first-order differential equations:

1. $$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} x_2^2 - x_1 + t \\ \sin(t) + x_1 - x_2 \end{pmatrix}$$

2. $$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \\ x_5' \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ x_5^2 - x_1 + t \\ x_5 \\ \sin(t) + x_2 - x_3 \end{pmatrix}$$

3. $$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{pmatrix} = \begin{pmatrix} x_2 \\ x_4^2 - x_1 + t \\ x_4 \\ \sin(t) + x_1 - x_2 \end{pmatrix}$$

4. $$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \\ x_5' \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ t \\ 0 \\ \sin(t) \end{pmatrix}$$

B – Numerical methods for solving ordinary differential equations. When we use numerical methods to solve an ordinary differential equation, we try to approximate the curve of the solution. In which of the following methods, this curve is approximated by the tangent in each interval?

1. Trapezoid method.
2. Newton's method.
3. Euler's method.
4. Runge-Kutta method.

C – Runge-Kutta method.. Using the Runge-Kutta method of order 4 with the step size $h = 0.25$ to solve an initial values problem, we obtain the global truncation error of order 10^{-2}. If we want to have the error of order 10^{-6}, which size of h should we choose:

1. $h \approx 0.05$
2. $h \approx 0.025$
3. $h \approx 0.005$
4. $h \approx 0.0396$

D – Initial-value problems. We wish to solve an initial value problem numerically using a large number of iterations with step size h. We are considering to use the Taylor series method of order 4 or the Runge-Kutta method of order 4 described on page 314 in the textbook. The truncation error after all the iterations, i.e., the global error, can be expected to be

1. of order $O(h^5)$ for both methods
2. of order $O(h^5)$ for the Taylor series method and $O(h^4)$ for the Runge-Kutta method
3. of order $O(h^4)$ for both methods
4. of order $O(h^4)$ for the Taylor series method and $O(h^5)$ for the Runge-Kutta method

2 Runge-Kutta methods (20%)

This question is about the numerical methods for solving initial-value problems. The method considered here is called as `RKvariant` and is a Runge-Kutta method of order 2, which was introduced in the lecture and the book.

$$K_1 = f(t, x) \tag{1}$$
$$K_2 = f(t + \alpha h, x + \beta h K_1) \tag{2}$$
$$x(t + h) = x(t) + h\big(\omega_1 K_1 + \omega_2 K_2\big) \tag{3}$$

with

$$\omega_1 = \omega_2 = \alpha = \frac{1}{2}, \qquad \beta = 1. \tag{4}$$

Note that this method is different from the Heun's method and midpoint method, which had been practiced in the course.

1. (10%) Use the derivation in the textbook or the lecture to show that if the differential equation has the form

$$\frac{dx}{dt} = f(x),$$

where the right-hand side is only a function on x, then the `RKvariant`'s approximation agrees with the Taylor expansion of $x(t+h)$ up to and including the term in h^2.

2. (10%) Use the derivation in the textbook or the lecture to show that if the differential equation has the form

$$\frac{dx}{dt} = g(t),$$

where the right-hand side is only a function on t, then the `RKvariant`'s approximation agrees with the Taylor expansion of $x(t+h)$ only up to and including the term in h.

3 Boundary-value problem (40%)

Objective: Solving a boundary-value problem by using the shooting method on a nonlinear differential equation.

In this question, we will solve a boundary-value problem

$$\begin{aligned} &x''(t) = x'(t) + x(t) - (2t-1)e^t, \qquad 1 < t < 2 \\ &x(1) = 3e, \quad x(2) = 5e^2 \end{aligned} \tag{5}$$

- (10%) Rewrite the equation (5) as a system of first-order differential equation, and implement a Python function to return the right-hand side of the system. Include the Python code in your report.

- (10%) Apply Runge-Kutta method of order 4 with $n = 50$ to perform two "shootings" for the ordinary differential equation (5) with the initial conditions

$$x(1) = 3e, \quad x'(1) = z = 10 \qquad \text{and} \qquad x(1) = 3e, \quad x'(1) = z = 25,$$

  respectively. State the values of $x(2)$ in these two shootings. (You will need the implementation of Runge-Kutta method of order 4 from Exercise 15.1 in Block 3.)

- (10%) In the shooting method, we consider the value of x at the end point, i.e., $x(2)$, as a function $\varphi(z)$ of z and $z = x'(1)$. Based on the answers to the previous question, you have two points on the graph $(z, \varphi(z))$. Use one iteration of the secant method on the equation $\varphi(z) = x(2) = 5e^2$ to calculate a new iterate $\hat{z}$. State the value of the new $\hat{z}$.

- (10%) Perform a new shooting for the ordinary differential equation (5) with the initial condition

$$x(1) = 3e, \quad x'(1) = \hat{z}.$$

Include a plot of the new shooting $(t, x(t))$ for $t \in [1, 2]$ in your answer.