Advanced Engineering Mathematics

Mini Project 1

Fedir Vasyliev

29/09/2024

$$\frac{d^2x}{dt^2} + \alpha\frac{dx}{dt} = j \tag{1}$$

$$\frac{dj}{dt} = -\beta j - \frac{dx}{dt} + u(t) \tag{2}$$

1

1.1

We will first differentiate equation (1).

$$\frac{d^3x}{dt^3} + \alpha\frac{d^2x}{dt^2} = \frac{dj}{dt} \tag{3}$$

Then equate what we got to the equation (2).

$$\frac{d^3x}{dt^3} + \alpha\frac{d^2x}{dt^2} = -\beta j - \frac{dx}{dt} + u(t)$$

We now want to isolate j and substitute what we will get into (1).

$$j = -\frac{1}{\beta}(\frac{d^3x}{dt^3} + \alpha\frac{d^2x}{dt^2} + \frac{dx}{dt} - u(t))$$

$$-\beta\frac{d^2x}{dt^2} - \alpha\beta\frac{dx}{dt} = \frac{d^3x}{dt^3} + \alpha\frac{d^2x}{dt^2} + \frac{dx}{dt} - u(t)$$

$$\frac{d^3x}{dt^3} + (\alpha+\beta)\frac{d^2x}{dt^2} + (1+\alpha\beta)\frac{dx}{dt} = u(t) \tag{4}$$

1.2

Based on the equation (4) we will construct the transfer function, assuming that $u(t) = e^{st}$

$$H(s) = \frac{1}{s^3 + (\alpha+\beta)s^2 + (1+\alpha\beta)s} \tag{5}$$

$s^3 + (\alpha+\beta)s^2 + (1+\alpha\beta)s \neq 0.$

1.3

Now, as we know the actual function $u(t) = sin(2t)$, we can find general real solution to (4). We will start by finding the particular solution.

$$u(t) = sin(2t) = Im(e^{i2t})$$

$$y_{par}(t) = Im(H(i2)e^{i2t})$$

$$H(i2) = \frac{1}{(2i)^3 + (\alpha+\beta)(2i)^2 + (1+\beta\alpha)2i} = \frac{1}{-8i - 4(\alpha+\beta) + 2i(1+\beta\alpha)} =$$

$$= \frac{1}{-4(\alpha+\beta) + i(-8 + 2(1+\beta\alpha))} = \frac{-4(\alpha+\beta) - i(-6+2\beta\alpha))}{16(\alpha+\beta)^2 + (-6+2\beta\alpha)^2}$$

$$y_{par}(t) = Im(\frac{-4(\alpha+\beta) - i(-6+2\beta\alpha))}{16(\alpha+\beta)^2 + (-6+2\beta\alpha)^2}(cos(2t) + isin(2t))) =$$

$$= cos(2t)(-\frac{-6+2\beta\alpha}{16(\alpha+\beta)^2 + (-6+2\beta\alpha)^2}) - sin(2t)(\frac{-4(\alpha+\beta)}{16(\alpha+\beta)^2 + (-6+2\beta\alpha)^2})$$

We now need to find general real solution to the homogeneous part.

$$\lambda^3 + (\alpha + \beta)\lambda^2 + \lambda(1 + \alpha\beta) = 0$$

$$\lambda^2 + (\alpha + \beta)\lambda + (1 + \alpha\beta) = 0$$

$$\lambda = \frac{-(\alpha + \beta) \pm \sqrt{\alpha^2 + \beta^2 - 4\beta\alpha - 4}}{2}$$

So we have 3 different λ as the solution to the characteristic polynomial, $\lambda = 0/\frac{-(\alpha+\beta)\pm\sqrt{\alpha^2+\beta^2-4\beta\alpha-4}}{2}$. As we know that alpha and beta are very small positive numbers, we can state that $\alpha^2+\beta^2-4\beta\alpha-4 < 0$, therefore, we have two complex solutions of the form $a+wt$. Thus, a general complex solution is

$$y_{hom}(t) = c_1e^{0\cdot t} + c_2e^{a+wt} + c_3e^{a-wt},\ \ c_1, c_2, c_3 \in \mathbb{C}$$

The general complex solution is then

$$y_{hom}(t) = c_1e^{0\cdot t} + c_2Re(e^{a+wt}) + c_3Im(e^{a+wt})$$

$$y_{hom}(t) = c_1e^{0\cdot t} + c_2e^{at}cos(wt) + c_3e^{at}sin(wt)\ c_1, c_2, c_3 \in \mathbb{R}$$

We can now combine two solutions to get a general real solution to (4).

$$y(t) = y_{hom}(t)+y_{par}(t) = c_1e^{0\cdot t}+c_2e^{at}cos(wt)+c_3e^{at}sin(wt)+Acos(2t)+Bsin(2t) \tag{6}$$

Where,

$A = \frac{6-2\beta\alpha}{16(\alpha+\beta)^2+(-6+2\beta\alpha)^2}$

$B = \frac{4(\alpha+\beta)}{16(\alpha+\beta)^2+(-6+2\beta\alpha)^2}$

$a = -\frac{\alpha+\beta}{2}$

$w = \frac{\sqrt{\alpha^2+\beta^2-4\beta\alpha-4}}{2}$

1.4

$$v(t) = \frac{dx}{dt}$$
$$\frac{dv(t)}{dt} = j(t) - \alpha v(t)$$
$$\frac{dj}{dt} = -\beta j - v(t) + u(t)$$

Now we can write it in the matrix form with the solution for $x(t)$.

$$\begin{cases} \begin{bmatrix} \frac{dx}{dt} \\ \frac{dv}{dt} \\ \frac{dj}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & \alpha & 1 \\ 0 & -1 & -\beta \end{bmatrix} \begin{bmatrix} x \\ v \\ j \end{bmatrix} + u(t) \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\ y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ v \\ j \end{bmatrix} \end{cases}$$

2

2.1

$$f(t) = z(t) - x(t)$$

Substitute this equation into the given one and simplify.

$$z^{'''}(t) - x^{'''}(t) + (\alpha + \beta)(z^{''}(t) - x^{''}(t)) +$$

$$+(1 + \alpha\beta + g_2)(z^{'}(t) - x^{'}(t)) + g_1(z(t) - x(t)) =$$

$$= z^{'''}(t) + (\alpha + \beta)z^{''}(t) + (1 + \alpha\beta)z^{'}(t)$$

$$-x^{'''}(t) - (\alpha + \beta)x^{'''}(t) - (1 + \alpha\beta)x^{'}(t) + g_2(z^{'}(t) - x^{'}(t)) + g_1(z(t) - x(t)) = 0$$

It is not hard to see $g_2(z^{'}(t) - x^{'}(t)) + g_1(z(t) - x(t)) = g_1 f + g_2 f^{'} = u(t)$.

$$x^{'''}(t) + (\alpha + \beta)x^{'''}(t) + (1 + \alpha\beta)x^{'}(t) = u(t)$$

We arrived to the same equation (4).

2.2

For this task we will use Theorem 2.47, which implies that if the inhomogeneous system of first-order differential equations is asymptotically stable, then the homogeneous part is also stable.

So, from Routh-Hurwitz Method we can construct the matrix

$$\begin{bmatrix} \alpha+\beta & g_1 \\ 1 & 1+\alpha\beta+g_2 \end{bmatrix},$$

All entries and determinant of this matrix must be greater than 0.

$$det = \alpha + \beta + \alpha^2\beta + \beta^2\alpha + (\alpha+\beta)g_2 - g_1 > 0$$

$$g_2 > -(1+\alpha\beta)$$

These two conditions can be represented on (g_1, g_2) plane as follows