Differential equations and infinite series

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Preface

This book is written for the course Mathematics 2, offered at DTU Compute. The most important topics are linear nth order differential equations, systems of first order differential equations, infinite series, and Fourier series. In order to understand the presentation it is important to have a good knowledge of function theory, simple differential equations, and linear algrebra.

The book is based on previous teaching material developed at the former Department of Mathematics. A lot of the material goes back to the books *Mathematical Analysis I-IV*, written by Helge Elbrønd Jensen in 1974, and later revised together with Tom Høholdt and Frank Nielsen. Other parts of the book are revised versions of notes written by Martin Bendsøe, Wolfhard Kliem, resp. Poul G. Hjorth. The chapters on infinite series are revised versions of material from [3].

I would like to thank the colleagues who have contributed to the material over the years, especially G. Hjorth who wrote the first version of Chapter 3, and Preben Alsholm, Per Goltermann, Ernst E. Scheufens, Ole Jørsboe, Carsten Thomassen, Mads Peter Sørensen, Magnus Ögren. I also thank Jens Gravesen for the section about solving a differential equation using the Fourier transform, and David Brander for corrections to the English version and for providing translations of several exercises.

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1

Differential equations of af nth order

Differential equations play an important role for mathematical models in science and technology. In this chapter we study nth order linear differential equations. We expect the reader to have a good understanding of second order differential equations, and we will see that many results are parallel for higher order differential equations. Vi also introduce the transfer function, which can be used to find solutions on a simple form for certain classes of differential equations.

1.1 Introduction

Let us consider a *linear differential equation of nth order with constant coefficients,* i.e., an equation of the form

$$a_0\frac{d^ny}{dt^n}+a_1\frac{d^{n-1}y}{dt^{n-1}}+\cdots+a_{n-1}\frac{dy}{dt}+a_ny=u(t),\quad t\in I. \tag{1.1}$$

Here $a_1,\dots,a_n\in\mathbb{R}$, and we assume that $a_0\neq 0$; furthermore u is a given function, defined on the interval I. We want to find the solutions y. We will often write the equation using alternative notation for the derivatives, i.e.,

$$\dot{y}=y^{(1)}=\frac{dy}{dt},\ \ddot{y}=y^{(2)}=\frac{d^2y}{dt^2},\dots,\ y^{(n)}=\frac{d^ny}{dt^n}.$$

The equation (1.1) is said to be *inhomogeneous.* The *homogeneous equation* corresponding to (1.1) is

$$a_0\frac{d^ny}{dt^n}+a_1\frac{d^{n-1}y}{dt^{n-1}}+\cdots+a_{n-1}\frac{dy}{dt}+a_ny=0. \tag{1.2}$$

A solution to the homogeneous equation (1.2) is called a *homogeneous solution.*

Denoting the left-hand side in (1.1) by $D_n(y)$, we can alternatively write the equation as

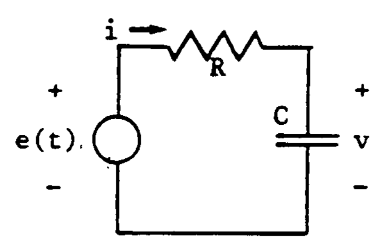
$$D_n(y) = u. \tag{1.3}$$

The homogeneous equation corresponding to (1.3) is then

$$D_n(y) = 0. \tag{1.4}$$

Many physical systems (e.g., linear circuits) can be modelled by systems of the form (1.3).

Example 1.1 The circuit on the figure consists of a power supply connected with a resistant R and a capacitor C. The voltage delivered by the power supply at time t is denoted by $e(t)$, and the current in the circuit is denoted by $i = i(t)$. We will now derive a differential equation that can be used to find the voltage v over the capacitor as a function of t.



Using Kirchhoff's law,

$$Ri + v = e(t);$$

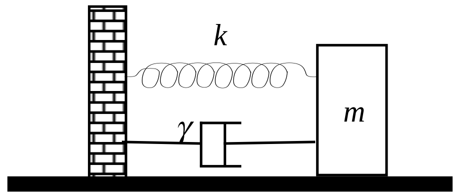
since $i = C\dot{v}$ it follows that

$$RC\dot{v} + v = e(t).$$

This is a differential equation that can be used to determine v. □

Example 1.2 Consider a mass m, connected to a wall via a spring. We assume that the mass moves on the ground with a certain friction (illustrated by the damper and denoted by γ on the figure).

Let us first assume that there is no friction. This corresponds to the situation on the figure, but without the damper, corresponding to $\gamma = 0$. Assume that the spring is moved away form its equilibrium position and released. Denote its position related to the equilibrium by $y(t)$, where t denotes the time.



The acceleration of the mass is given by $\frac{d^2y}{dt^2}$, so according to Newton's second law the force F acting on the mass is proportional with $y(t)$ but with the opposite sign, i.e,

$$F = -ky,$$

where the constant $k > 0$ is called the spring constant. Thus

$$-ky = m\frac{d^2y}{dt^2},$$

which leads to the differential equation

$$\frac{d^2y}{dt^2} + \frac{k}{m}y = 0.$$

In the more realistic case where we include the possibility of friction, we obtain a damping term in the differential equation:

$$\frac{d^2y}{dt^2} + \frac{\gamma}{m}\frac{dy}{dt} + \frac{k}{m}y = 0.$$

The number $\gamma > 0$ is called the damping constant. As we can see, the equation is a homogeneous second order linear differential equation. If the mass is under influence of an external force F_0, its displacement will be described by the equation

$$\frac{d^2y}{dt^2} + \frac{\gamma}{m}\frac{dy}{dt} + \frac{k}{m}y = \frac{F_0}{m}.$$

This is a linear inhomogeneous second order differential equation. □

1.2 The homogeneous equation

In this section we describe the structure of the solutions to the homogeneous equation

$$a_0\frac{d^ny}{dt^n} + a_1\frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_{n-1}\frac{dy}{dt} + a_ny = 0,$$

as well as how to find the solutions. We will write the equation on the short form

$$D_n(y) = 0. \tag{1.5}$$

The following Theorem 1.3 expresses a fundamental linearity property of the solutions to (1.5):

Theorem 1.3 *If the functions y_1 and y_2 are solutions to* (1.5) *and $c_1, c_2 \in \mathbb{R}$, then the function $y = c_1y_1 + c_2y_2$ is also a solution to* (1.5).

The result shows that the solutions to (1.5) form a vector space. The dimension of the space is given by the order of the differential equation:

Theorem 1.4 *The solution to* (1.5) *form an n-dimensional vector space. Thus, all solutions to* (1.5) *can be written in the form*

$$y = c_1y_1 + c_2y_2 + \cdots + c_ny_n,$$

where $c_1, c_2, \ldots, c_n$ denote certain constants and $y_1, y_2, \ldots, y_n$ form a set of linearly independent solutions.

The purpose with this section is to describe how we can determine n linearly independent solutions to the homogeneous equation $D_n(y) = 0$. Let us consider the *characteristic equation*, given by

$$a_0\lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n = 0. \tag{1.6}$$

The polynomial on the left-hand side is called the *characteristic polynomial,* and is denoted by

$$P(\lambda) := a_0\lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n. \tag{1.7}$$

Theorem 1.5 *If λ is a real or complex root in the characteristic equation, then* (1.5) *has the solution*

$$y(t) = e^{\lambda t}. \tag{1.8}$$

Solutions corresponding to different values of λ are linearly independent.

Proof: We will prove the first part, namely, that the function $y(t) = e^{\lambda t}$ is a solution to the differential equation if λ is a root in the characteristic equation. First observe that

$$y'(t) = \lambda\, e^{\lambda t}, \quad y''(t) = \lambda^2\, e^{\lambda t};$$

more generally, for $k = 1, 2, \ldots,$

$$y^{(k)}(t) = \lambda^k e^{\lambda t}.$$

Thus

$$\begin{aligned} D_n(y) &= a_0\frac{d^ny}{dt^n} + a_1\frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_{n-1}\frac{dy}{dt} + a_ny \\ &= a_0\lambda^n e^{\lambda t} + a_1\lambda^{n-1}e^{\lambda t} + \cdots + a_{n-1}\lambda e^{\lambda t} + a_ne^{\lambda t} \\ &= \left[a_0\lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n\right] e^{\lambda t}. \end{aligned}$$

The term in the bracket is exactly the characteristic polynomial, and thus vanishes when λ is a root in the characteristic equation. Thus $D_n(y) = 0$, and we have proved that $y(t) = e^{\lambda t}$ is a solution to the differential equation. $\square$

In the case where the characteristic equation has n pairwise different roots (real or complex), Theorem 1.5 now tells us how to find the complete set of solutions to the homogeneous eqations: we simply have to form all possible linear combinations of the solutions in (1.8).

Example 1.6 Consider the differential equation

$$\dddot{y} + \ddot{y} + 4\dot{y} + 4y = 0. \tag{1.9}$$

The characteristic equation is

$$\lambda^3 + \lambda^2 + 4\lambda + 4 = 0;$$

it has three roots, namely, $\lambda = -1$ and $\lambda = \pm 2i$. Via Theorem 1.5 we conclude that (1.9) has the linearly independent solutions

$$y(t) = e^{-t}, \quad y(t) = e^{2it}, \quad y(t) = e^{-2it}.$$

The complete solution to (1.9) is therefore

$$y(t) = c_1 e^{-t} + c_2 e^{2it} + c_3 e^{-2it}, \;\; c_1, c_2, c_3 \in \mathbb{C}. \qquad \square$$

In modelling of physical problems or problems in engineering we are often interested in the space of solutions taking real values. In this context we are facing two challenges:

- The characteristic equation might have complex roots, leading to complex solutions to (1.5). We will describe how to find corresponding real solutions.
- Theorem 1.5 only provide us with a basis for the solution space if the characteristic equation has n roots that are *pairwise different.* Thus, we still need to describe how to find the general solution in case of roots appearing with multiplicity (the exact definition will be given in Definition 1.12).

In the rest of the introduction we will discuss these problems in detail.

We have assumed that all the coefficients in the differential equation are real, so all the coefficients in the characteristic equation

$$a_0\lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n = 0$$

are real. It follows that all complex roots in the characteristic equation appear in *complex conjugated pairs:* that is, if $\lambda = a + i\omega$ is a root in the characteristic equation, then also $\overline{\lambda} = a - i\omega$ is a root.

If λ is a complex root in the characteristic equation, then the solution given in Theorem 1.5 is also complex. Writing $\lambda = a + i\omega$, we can write the solution as

$$\begin{aligned} y(t) = e^{\lambda t} = e^{(a+i\omega)t} = e^{at}e^{i\omega t} &= e^{at}(\cos\omega t + i\sin\omega t) \\ &= e^{at}\cos\omega t + ie^{at}\sin\omega t. \end{aligned} \tag{1.10}$$

In order to extract real solutions we will use the following result:

Lemma 1.7 *If the function y is a solution to* (1.5)*, then the real functions* $\mathrm{Re}\, y$ *and* $\mathrm{Im}\, y$ *are also solutions to* (1.5)*.*

Proof: Assume that $y = \mathrm{Re}\, y + i\,\mathrm{Im}\, y$ is a complex solution to (1.5), i.e., that $D_n(y) = 0$. Since

$$D_n(y) = D_n(\mathrm{Re}\, y + i\,\mathrm{Im}\, y) = D_n(\mathrm{Re}\, y) + iD_n(\mathrm{Im}\, y) = 0$$

we can conclude that $D_n(\mathrm{Re}\, y) = 0$ and $D_n(\mathrm{Im}\, y) = 0$. Thus $\mathrm{Re}\, y$ and $\mathrm{Im}\, y$ are also solutions to the differential equation. □

Via (1.10) and Lemma 1.7 it follows that if $\lambda = a + i\omega$ is a complex root in the characteristic equation associated with (1.5), then the two functions

$$y(t) = \mathrm{Re}\, e^{\lambda t} = e^{at}\cos\omega t \quad \text{and} \quad y(t) = \mathrm{Im}\, e^{\lambda t} = e^{at}\sin\omega t$$

are real solutions to the differential equations (1.5). One can prove that the functions are linearly independent. This leads to a method to determine a set of linearly independent solutions:

Theorem 1.8 *Consider the homogeneous equation* $D_n(y) = 0$*. We can determine a set of real and linearly independent solutions as follows:*

(i) For every real root λ in the characteristic equations we have the real solution $y(t) = e^{\lambda t}$.

(ii) For any pair of complex conjugated complex roots $\lambda = a + i\omega$ and $\overline{\lambda} = a - i\omega$, we have the real solutions

$$y(t) = Re(e^{\lambda t}) = e^{at}\cos\omega t \quad \text{and} \quad y(t) = \mathrm{Im}\,(e^{\lambda t}) = e^{at}\sin\omega t. \tag{1.11}$$

The solutions obtained via (i) and (ii) are linearly independent.

If the characteristic equation has n pairwise different roots, then Theorem 1.8 yields n linearly independent homogeneous solutions. We then find the general solution by building linear combinations, see Theorem 1.4.

Note that the solutions in (1.11) correspond to the *pair* of roots $a \pm i\omega$: thus, we only have to calculate these solutions for either the root $a + i\omega$ *or* the root $a - i\omega$. This is illustrated in the next example.

Example 1.9 Consider again the differential equation in Example 1.6,

$$\dddot{y} + \ddot{y} + 4\dot{y} + 4y = 0.$$

In Example 1.6 we saw that the characteristic equation has the complex roots $\lambda = \pm 2i$. Thus $a = 0$ and $\omega = 2$, and according to Theorem 1.8 we have the real solutions

$$y(t) = \operatorname{Re} e^{2it} = \cos 2t \quad \text{and} \quad y(t) = \operatorname{Im} e^{2it} = \sin 2t.$$

Remember that we in Example 1.6 also found the real solution $y(t) = e^{-t}$. Thus, the general real solution to the differential equation is

$$y(t) = c_1 e^{-t} + c_2 \cos 2t + c_3 \sin 2t, \quad \text{where} \quad c_1, c_2, c_3 \in \mathbb{R}. \qquad \square$$

Example 1.10 The differential equation

$$\ddot{y} + \omega^2 y = 0, \quad \omega > 0,$$

has the characteristic equation $\lambda^2 + \omega^2 = 0$. The roots are $\lambda = \pm i\omega$. Thus, according to Theorem 1.8(ii) the general real solution to the differential equation is

$$y(t) = c_1 \cos \omega t + c_2 \sin \omega t, \text{ where } c_1, c_2 \in \mathbb{R}. \qquad \square$$

We will now consider the issue of roots appearing with multiplicity. First, note that since the characteristic polynomial has degree n, it always has n roots. However, there might not be n pairwise different roots, and in this case we speak about roots with multiplicity. Let us consider an illustrating example before we give the formal definition.

Example 1.11 Consider the differential equation

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = 0.$$

The characteristic equation is $\lambda^2 + 2\lambda + 1 = 0$, which we can also write as $(\lambda + 1)^2 = 0$. Thus, even though it is a second order equation, there is only one solution, namely, $\lambda = -1$. We say that $\lambda = -1$ is a *double root,* or a *multiple root.* $\square$

The formal definition of multiple roots and multiplicity is based on the fundamental theorem of algebra, which tells us that the characteristic polynomial can be written as a product of first order polynomials:

Definition 1.12 (Algebraic multiplicity) *Write the characteristic polynomial as a product of polynomials of first order,*

$$P(\lambda) = a_0(\lambda - \lambda_1)^{p_1}(\lambda - \lambda_2)^{p_2} \cdots (\lambda - \lambda_k)^{p_k},$$

where $\lambda_i \neq \lambda_j$ whenever $i \neq j$. The number p_i is called the algebraic multiplicity of the root λ_i.

Thus, in Example 1.11 the root $\lambda = -1$ has algebraic multiplicity $p = 2$.

If one or more roots in the characteristic equation has algebraic multiplicity $p > 1$, the method in Theorem 1.8 does not give us all solutions. In this case we need the following result:

Theorem 1.13 *Assume that λ is root in the characteristic equation, with algebraic multiplicity p. Then the differential equation $D_n(y) = 0$ has the linearly independent solutions*

$$y_1(t) = e^{\lambda t},\ y_2(t) = te^{\lambda t},\ \cdots, y_p(t) = t^{p-1}e^{\lambda t}. \tag{1.12}$$

If a pair of complex conjugated roots $a \pm i\omega$ appear with multiplicity $p > 1$, then we get (similarly as in Theorem 1.8) $2p$ linearly independent real solutions, given by

$$y_1(t) = e^{at}\cos\omega t,\ y_2(t) = te^{at}\cos\omega t, \cdots, y_p(t) = t^{p-1}e^{at}\cos\omega t$$

and

$$y_{p+1}(t) = e^{at}\sin\omega t,\ y_{p+2}(t) = te^{at}\sin\omega t, \cdots, y_{2p}(t) = t^{p-1}e^{at}\sin\omega t.$$

Let us collect the obtained results in the following theorem, which describes how to find the general solution to (1.5):

Theorem 1.14 (General solution to (1.5)**)** *The general solution to the homogeneous differential equation $D_n(y) = 0$ is determnied as follows:*

(i) (complex solutions) For any root λ in the characteristic equation, consider the solution $y(t) = e^{\lambda t}$, and, if λ has algebraic multiplicity $p > 1$, the solutions

$$y(t) = te^{\lambda t}, \cdots, y(t) = t^{p-1}e^{\lambda t}.$$

The general complex solution is obtained by forming linear combinations of these n solutions, with complex coefficients.

(ii) (real solutions) For any real root λ in the characteristic equation, consider the solutions found in (i). For any pair of complex conjugated roots $a \pm i\omega$, consider furthermore the solutions

$$y(t) = e^{at}\cos\omega t \quad \textit{and} \quad y(t) = e^{at}\sin\omega t,$$

and, if $a \pm i\omega$ has algebraic multiplicity $p > 1$, the solutions

$$y(t) = te^{at}\cos\omega t, \cdots, y(t) = t^{p-1}e^{at}\cos\omega t$$

and

$$y(t) = te^{at}\sin\omega t, \cdots, y(t) = t^{p-1}e^{at}\sin\omega t.$$

The general real solution is obtained by forming linear combinations of these n solutions, with real coefficients.

Example 1.15 The characteristic polynomial associated with the differential equation

$$\dddot{y} - 3\ddot{y} + 4y = 0$$

is given by

$$P(\lambda) = \lambda^3 - 3\lambda^2 + 4.$$

Factorising the polynomial (see, e.g., the Maple Appendix) shows that

$$P(\lambda) = (\lambda - 2)^2(\lambda + 1).$$

Thus the roots are $\lambda = -1$ and $\lambda = 2$, the last mentioned one with algebraic multiplicity $p = 2$. According to Theorem 1.14 we obtain the following three linearly independent solutions:

$$y(t) = e^{-t}, \quad y(t) = e^{2t}, \quad \text{and} \quad y(t) = te^{2t}.$$

The general real solution to the differential equation is therefore

$$y(t) = c_1e^{-t} + c_2e^{2t} + c_3te^{2t},$$

where $c_1, c_2, c_3 \in \mathbb{R}$. □

Example 1.16 The characteristic polynomial associated with the differential equation

$$y^{(5)} + y^{(4)} + 8y^{(3)} + 8y^{(2)} + 16y' + 16y = 0$$

is given by

$$P(\lambda) = \lambda^5 + \lambda^4 + 8\lambda^3 + 8\lambda^2 + 16\lambda + 16.$$

Factorising the polynomial shows that

$$P(\lambda) = (\lambda^2 + 4)^2(\lambda + 1).$$

Thus the roots are $\lambda = -1$ and $\lambda = \pm 2i$, the last ones with algebraic multiplicity $p = 2$. According to Theorem 1.14 we obtain 5 linearly independent solutions

$$y(t) = e^{-t},\ y(t) = e^{2it},\ y(t) = te^{2it},\ y(t) = e^{-2it},\ y(t) = te^{-2it}.$$

The general complex solution is therefore

$$y(t) = c_1e^{-t} + c_2e^{2it} + c_3te^{2it} + c_4e^{-2it} + c_5te^{-2it},\ c_1, c_2, c_3, c_4, c_5 \in \mathbb{C}.$$

According to Theorem 1.14 the general real solution is

$$y(t) = c_1e^{-t} + c_2\cos 2t + c_3t\cos 2t + c_4\sin 2t + c_5t\sin 2t,$$

where $c_1, c_2, c_3, c_4, c_5 \in \mathbb{R}$. □

1.3 The inhomogeneous equation

We will now turn to the inhomogeneous equation

$$a_0\frac{d^ny}{dt^n}+a_1\frac{d^{n-1}y}{dt^{n-1}}+\cdots+a_{n-1}\frac{dy}{dt}+a_ny=u(t),$$

where u is a conntinuous function taking real (or complex) values. The function u is called the *forcing function* or the *input function;* we will use the name forcing function because it often correspond to its physical significance.

We assume that the function u is defined on an interval I, and we seek a solution y that is defined on the same interval. We will write the equation on the short form

$$D_n(y)=u. \tag{1.13}$$

If u is continuous, the equation (1.13) always has infinitely many solutions:

Theorem 1.17 (Existence and uniqueness) *Assume that the function $u(t), t\in I$, is continuous. For any $t_0\in I$ and any vector $(v_1,\dots,v_n)\in\mathbb{R}^n$ the equation (1.13) has exactly one solution $y(t), t\in I$, for which*

$$(y(t_0),y'(t_0),\dots,y^{(n-1)}(t_0))=(v_1,\dots,v_n)$$

The given values for the coordinates of the vector

$$(y(t_0),y'(t_0),\dots,y^{(n-1)}(t_0))$$

are called *initial conditions.*

Example 1.18 The differential equation

$$\frac{d^2y}{dt^2}+2\frac{dy}{dt}+y=\cos(2t)$$

is an inhomogeneous equation of second order. For any set of real numbers v_1,v_2 and any given t_0, Theorem 1.17 shows that the differential equation has exactly one solution $y(t)$ for which

$$y(t_0)=v_1,\quad y'(t_0)=v_2.$$

We will return to this differential equation in Example 1.20, where we find the solution associated to a concrete initial condition. □

Let us now state a result, expressing a fundamental relationship between the homogeneous solutions and the inhomogeneous solutions.

Theorem 1.19 *Let y_0 denote a solution to the equation* (1.13), $D_n(y)=u$, *and let y_{HOM} denote the general solution to the associated homogeneous equation. Then the general solution to* (1.13) *is given as*

$$y=y_0+y_{HOM}.$$

Proof: Assume first that y_0 is a solution to the inhomogeneous equation $D_n(y) = u$ and that y_{HOM} is a solution to the corresponding homogeneous equation. Letting $y = y_0 + y_{HOM}$ we then have that

$$D_n(y) = D_n(y_0 + y_{HOM}) = D_n(y_0) + D_n(y_{HOM}) = D_n(y_0) = u.$$

Thus y is a solution to the inhomogeneous equation. On the other hand, if the functions y and y_0 are solutions to the inhomogeneous equation (1.13), then

$$D_n(y - y_0) = D_n(y) - D_n(y_0) = u - u = 0.$$

Thus the function $y - y_0$ is a solution to the homogeneous equation. This shows that we can write $y = y_0 + (y - y_0)$, i.e., that y is a sum of the function y_0 and a solution to the homogeneous equation. □

The solution y_0 is called a *particular* solution. A particular solution can frequently be found using a "qualified guess," i.e., by searching for a solution of a similar form as u. In the literature the method is called *the method of undetermined coefficients* or the *guess and test method.* The most important cases are

- Assume that $u(t)$ is an exponential function
$$u(t) = be^{st}.$$
We will then search for a solution $y(t) = ce^{st}$, where c is a (possibly complex) constant, which is determined by inserting the functions u and y in the differential equation.

- Assume that $u(t) = \sin(kt)$. Then we will search for a solution of the form
$$y(t) = A\cos(kt) + B\sin(kt).$$
The same method is applied when $u(t) = \cos(kt)$.

- Assume that $u(t)$ is a polynomial of degree k,
$$u(t) = c_0t^k + c_1t^{k-1} + \cdots + c_{k-1}t + c_k.$$
We will then search for a solution which is a polynomial of degree $n + k$,
$$y(t) = d_0t^{n+k} + d_1t^{n+k-1} + \cdots + d_{n+k-1}t + d_{n+k}.$$

Example 1.20 We want to find the general solution to the differential equation

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = \cos(2t), \tag{1.14}$$

and find the particular solution for which $y(0) = y'(0) = 1$. We will search for a solution of the form

$$y(t) = A\cos(2t) + B\sin(2t).$$

Then

$$y'(t) = -2A\sin(2t) + 2B\cos(2t)$$

and

$$y''(t) = -4A\cos(2t) - 4B\sin(2t).$$

Inserting this in the differential equation leads to

$$\begin{aligned}-4A\cos(2t) - 4B\sin(2t) + 2(-2A\sin(2t) + 2B\cos(2t)) + A\cos(2t) + B\sin(2t)\\ = \cos(2t).\end{aligned}$$

The equation can be reformulated as

$$(-4A + 4B + A)\cos(2t) + (-4B - 4A + B)\sin(2t) = \cos(2t),$$

or

$$(-3A + 4B)\cos(2t) + (-4A - 3B)\sin(2t) = \cos(2t).$$

Comparing the coefficients for $\cos(2t)$ and $\sin(2t)$ we see that the equation is satisfied if

$$-3A + 4B = 1 \text{ og } -4A - 3B = 0.$$

The solution to this set of equations is $A = \frac{-3}{25},\ B = \frac{4}{25}$, so the differential equation has the particular solution

$$y(t) = A\cos(2t) + B\sin(2t) = \frac{-3}{25}\cos(2t) + \frac{4}{25}\sin(2t).$$

In order to find the general solution we now consider the corresponding homogeneous equation,

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = 0.$$

The characteristic equation is $\lambda^2 + 2\lambda + 1 = 0$, which can also be written as

$$(\lambda + 1)^2 = 0.$$

Thus the root is $\lambda = -1$, with algebraic multiplicity 2. According to Theorem 1.14 the homogeneous equation thus has the general solution

$$y(t) = c_1 e^{-t} + c_2 t e^{-t},$$

where $c_1, c_2 \in \mathbb{R}$. By Theorem 1.19 the equation (1.14) therefore has the general real solution

$$\begin{aligned} y(t) &= y_0(t) + y_{HOM}(t) \\ &= \frac{-3}{25}\cos(2t) + \frac{4}{25}\sin(2t) + c_1e^{-t} + c_2te^{-t}, \end{aligned}$$

where $c_1, c_2 \in \mathbb{R}$. Now Theorem 1.17 tells us that among all these solutions there is exactly one solution such that $y(0) = y'(0) = 1$. In order to find that solution we note that

$$y'(t) = \frac{6}{25}\sin(2t) + \frac{8}{25}\cos(2t) - c_1e^{-t} + c_2(e^{-t} - te^{-t}).$$

Inserting $t = 0$ shows that the conditions $y(0) = y'(0) = 1$ lead to

$$\frac{-3}{25} + c_1 = 0, \text{ and } \frac{8}{25} - c_1 + c_2 = 0,$$

which has the solution $c_1 = \frac{3}{25}$, $c_2 = \frac{-5}{25} = -\frac{1}{5}$. The particular solution satisfying the initial conditions is therefore

$$y(t) = \frac{-3}{25}\cos(2t) + \frac{4}{25}\sin(2t) + \frac{3}{25}e^{-t} - \frac{1}{5}te^{-t}.$$

□

The following theorem shows how a differential equation with a right-hand side consisting of several terms can be related to differential equations with only a single term on the right-hand side.

Theorem 1.21 (The principle of superposition) *Let u_1 and u_2 denote two given functions. Let y_1 denote a solution to the equation $D_n(y) = u_1$, and y_2 a solution to the equation $D_n(y) = u_2$. Then the function $y = c_1y_1 + c_2y_2$ is a solution to the equation $D_n(y) = c_1u_1 + c_2u_2$.*

Proof: Inserting the function $y = c_1y_1 + c_2y_2$ in the expression for $D_n(y)$ shows that

$$D_n(c_1y_1 + c_2y_2) = c_1D_n(y_1) + c_2D_n(y_2) = c_1u_1 + c_2u_2.$$

Thus the function $y = c_1y_1 + c_2y_2$ is a solution to the equation $D_n(y) = c_1u_1 + c_2u_2$. □

1.4 Transfer functions

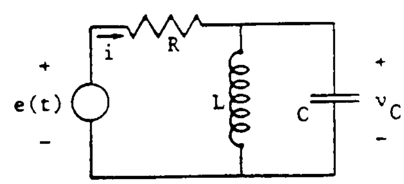
In this section we give an introduction to the very important transfer function associated with a differential equation of nth order. We will consider differential equations on the special form

$$a_0y^{(n)} + a_1y^{(n-1)} + \cdots + a_ny = b_0u^{(m)} + b_1u^{(m-1)} + \cdots + b_mu; \quad (1.15)$$

as before, u is a given function, and we are searching for a solution y. We assume that all the coefficients $a_0, \ldots, a_n, b_0, \ldots, b_m$ are real. A differential equation on this form often appears by modelling of a physical system. In such cases the function $u(t)$ is typical an outer force acting on a system. For example, u can be a force acting on a mechanical system, and y can denote the corresponding position of one of the components.

A differential equation in the form (1.15) can be analyzed exactly like the equation (1.13), simply by considering the right-hand side of (1.15) as the forcing function. However, it turns out to be convenient to consider the form (1.15). In fact, we will introduce the transfer function and show that it can be used to describe the behavior of the system, independently of the chosen function u. If for example u describes a force acting on a system, we will then be able to derive general conclusions about the behavior of the system and then choose u accordingly.

Example 1.22 We want to derive a differential equation describing the current I on the figure, where a power supply delivers the voltage $e(t)$ to a circuit consisting of a resistant R, a capacitor C, and a inductor L.



By the physical laws we have the following three equations:

$$\begin{cases} v_C = L\dot{I}_L, \\ I = I_L + C\dot{v}_C, \\ e = RI + v_C. \end{cases}$$

In order to determine a differential equation involving the current I we would like to eliminate v_C and I_L from these equations. From the last equation we derive that $v_C = e - RI$. Inserting this in the two first equations leads to $e - RI = L\dot{I}_L$ and $I = I_L + C(\dot{e} - R\dot{I})$. From the last equation we derive that $I_L = I - C\dot{e} + RC\dot{I}$. Inserting this in the first equation then gives that

$$e - RI = L(\dot{I} - C\ddot{e} + RC\ddot{I}),$$

or

$$RCL\ddot{I} + L\dot{I} + RI = LC\ddot{e} + e. \quad (1.16)$$

This equation has the form (1.15). □

Often it is useful to consider the differential equation (1.15) as a map, which maps a forcing function $u(t)$ (and a set of initial conditions) onto the solution $y(t)$. Let us now consider the case where

$$u(t) = e^{st}, \tag{1.17}$$

for a given $s \in \mathbb{C}$. We will search for a solution on the same form, i.e.,

$$y(t) = H(s)e^{st}, \tag{1.18}$$

where $H(s)$ is a constant that does not depend on t. We will now insert the functions u and y in the differential equation (1.15). Note that $u'(t) = se^{st}$, and more generally, for $k = 1, 2, \ldots,$

$$u^{(k)}(t) = s^k e^{st}.$$

Similarly $y'(t) = H(s)se^{st}$, and more generally, for $k = 1, 2, \ldots,$

$$y^{(k)}(t) = H(s)s^k e^{st}.$$

Thus (1.15) takes the form

$$\begin{aligned} & a_0 H(s)s^n e^{st} + a_1 H(s)s^{n-1}e^{st} + \cdots + a_n H(s)e^{st} \\ = \; & b_0 s^m e^{st} + b_1 s^{m-1} e^{st} + \cdots + b_m e^{st}. \end{aligned}$$

We can also write this as

$$\begin{aligned} H(s)(a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n)e^{st} = \\ (b_0 s^m + b_1 s^{m-1} + \cdots + b_{m-1}s + b_m)e^{st}. \end{aligned}$$

Note that the factor

$$P(s) = a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n$$

is the characteristic polynomial for (1.15).Thus, if s is not a root in the characteristic equation associated with (1.15), i.e., whenever $P(s) \neq 0$, then (1.18) is a solution to (1.15) if and only if

$$H(s) = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_{m-1}s + b_m}{a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n}. \tag{1.19}$$

The function H of the complex variable s is called the *transfer function* associated with (1.15). We note that $H(s)$ only is defined for the values of $s \in \mathbb{C}$ for which $P(s) \neq 0$. Whenever we want to determine the transfer function we must remember to determine as well the expression for the function as the domain where it is defined; see Example 1.24.

Let us formulate the above result formally:

Theorem 1.23 (Stationary solution) *Consider the equation* (1.15) *and let*

$$u(t) = e^{st}.$$

Then, if $P(s) \neq 0$, *the equation* (1.15) *has exactly one solution on the form*

$$y(t) = H(s)e^{st}. \tag{1.20}$$

This particular solution is obtained by letting $H(s)$ *denote the transfer function associated with* (1.15). *The solution* (1.20) *is called the stationary solution associated with* $u(t) = e^{st}$.

Note that we can find the formal expression for the transfer function directly based on the given differential equation, without repeating the above calculations:

Example 1.24 Consider the differential equation

$$\ddot{y} + y = \dot{u}. \tag{1.21}$$

Via (1.19) we see directly that the transfer function is

$$H(s) = \frac{s}{s^2+1}.$$

The transfer function is defined whenever $s^2 + 1 \neq 0$, i.e., for $s \neq \pm i$. If $u(t) = e^{2t}$ we have $s = 2$ and we see immediately from Theorem 1.23 that the equation (1.21) has the solution

$$y(t) = H(2)e^{2t} = \frac{2}{5}e^{2t}. \qquad \square$$

1.5 Frequency characteristics

The most important application of the transfer function appears when $s = i\omega$, where ω is a real positive constant. Then the function

$$u(t) = e^{i\omega t} \tag{1.22}$$

is periodic with period $T = \frac{2\pi}{\omega}$. Thus the function u describes an oscillation with frequency $\frac{\omega}{2\pi}$. Theorem 1.23 shows that if $P(i\omega) \neq 0$, then the differential equation has a periodic solution y with period $T = \frac{2\pi}{\omega}$, namely,

$$y(t) = H(i\omega)e^{i\omega t}. \tag{1.23}$$

We will now apply this result to derive the expression for a solution to the equation (1.15) when $u(t) = \cos \omega t$. For this purpose we need the following result.

Theorem 1.25 *Suppose that y_0 is a complex solution to the equation $D_n(y) = u$. Then the following hold:*

(i) The function $\mathrm{Re}\, y_0$ *is a solution to the equation* $D_n(y) = \mathrm{Re}\, u$*;*

(ii) The function $\mathrm{Im}\, y_0$ *is a solution to the equation* $D_n(y) = \mathrm{Im}\, u$*.*

Proof: From the equation $D_n(y_0) = u$ it follows by complex conjugation that $D_n(\bar{y}_0) = \bar{u}$. Adding the two equations leads to $D_n(\mathrm{Re}\, y_0) = \mathrm{Re}\, u$. Thus the function $y = \mathrm{Re}\, y_0$ is a solution to the equation $D_n(y) = \mathrm{Re}\, u$. The second result is derived in a similar fashion by subtraction. □

Theorem 1.25 and (1.23) imply that if

$$u(t) = \mathrm{Re}\, e^{i\omega t} = \cos \omega t$$

and $P(i\omega) \neq 0$, then the function

$$y(t) = \mathrm{Re}\, (H(i\omega)e^{i\omega t})$$

is a solution to the differential equation. We will now derive an alternative expression for this. First, recall that any complex number $a + ib$ can be written on polar form as

$$a + ib = |a + ib|\, e^{iArg(a+ib)};$$

here $Arg(a + ib)$ denotes the *argument*, i.e., the angle in the complex plane (belonging to the interval $]-\pi, \pi]$) between the real axis and the line through the points 0 and $a + ib$. We can write $H(i\omega)$ on the form

$$H(i\omega) = |H(i\omega)|\, e^{iArgH(i\omega)};$$

then

$$\begin{aligned} Re\left(H(i\omega)e^{i\omega t}\right) &= Re\left(|H(i\omega)|\, e^{iArgH(i\omega)}e^{i\omega t}\right) = Re\left(|H(i\omega)|\, e^{i(\omega t + ArgH(i\omega))}\right) \\ &= |H(i\omega)|\, \cos(\omega t + \mathrm{Arg}\, H(i\omega)). \end{aligned}$$

We have now proved (i) in the following theorem:

Theorem 1.26 (Stationary solution) *Assume that $\omega \in \mathbb{R}$ and that $i\omega$ is not a root in the characteristic equation associated with* (1.15)*. Then the following hold:*

(i) If $u(t) = \cos \omega t$, then (1.15) *has the solution*

$$y(t) = \mathrm{Re}\, (H(i\omega)e^{i\omega t}) = |H(i\omega)| \cos(\omega t + \mathrm{Arg}\, H(i\omega)). \tag{1.24}$$

(ii) If $u(t) = \sin \omega t$, then (1.15) *has the solution*

$$y(t) = \mathrm{Im}\, (H(i\omega)e^{i\omega t}) = |H(i\omega)| \sin(\omega t + \mathrm{Arg}\, H(i\omega)). \tag{1.25}$$

The solution (1.24) *is called the* stationary solution *associated with $u(t) = \cos \omega t$, and* (1.25) *is the stationary solution associated with $u(t) = \sin \omega t$.*

Theorem 1.26 shows that we can determine the stationary solutions directly by calculating the functions $|H(i\omega)|$ and $\operatorname{Arg} H(i\omega)$. These functions have been given names in the literature:

Definition 1.27 (Frequency characteristics) *The function*

$$A(\omega) := |H(i\omega)|, \ \omega > 0, \tag{1.26}$$

is called the amplitude characteristic for (1.15) *and*

$$\varphi(\omega) := \operatorname{Arg} H(i\omega), \ \omega > 0, \tag{1.27}$$

is called the phase characteristic for (1.15). *Together, the functions are called frequency characteristics.*

In order to determine the phase characteristic, recall that for a complex number $a + ib, \ a, b \in \mathbb{R}$, we have that

$$\operatorname{Arg}(a+ib) = \begin{cases} \operatorname{Arctan} \frac{b}{a}, & \text{if } a+ib \text{ is in the 1. quadrant,} \\ \operatorname{Arctan} \frac{b}{a} + \pi, & \text{if } a+ib \text{ is in the 2. quadrant,} \\ \operatorname{Arctan} \frac{b}{a} - \pi, & \text{if } a+ib \text{ is in the 3. quadrant,} \\ \operatorname{Arctan} \frac{b}{a}, & \text{if } a+ib \text{ is in the 4. quadrant.} \end{cases} \tag{1.28}$$

Inserting $s = i\omega$ in the expression for $H(s)$ leads to

$$H(i\omega) = \frac{a(\omega) + ib(\omega)}{c(\omega) + id(\omega)}, \tag{1.29}$$

where the functions $a(\omega), b(\omega), c(\omega)$ and $d(\omega)$ are real. Thus

$$\operatorname{Arg} H(i\omega) = \operatorname{Arg}(a(\omega) + ib(\omega)) - \operatorname{Arg}(c(\omega) + id(\omega)) \ \ (\pm 2\pi), \tag{1.30}$$

where the adjustment $\pm 2\pi$ is used in order to obtain an angle belonging to the interval $-\pi < \operatorname{Arg} H(i\omega) \leq \pi$. The amplitude characteristic can be found using the formula

$$A(\omega) = \frac{\sqrt{a(\omega)^2 + b(\omega)^2}}{\sqrt{c(\omega)^2 + d(\omega)^2}}. \tag{1.31}$$

A good feeling for the functions $A(\omega)$ and $\varphi(\omega)$ can be obtained by drawing a plot, see the commands in the Maple Appendix in Section D.3.3. Frequently the information in the phase characteristics are collected in a single curve, the so-called *Nyquist plot:* this is the curve in the complex plane obtained by plotting the points $H(i\omega)$ for ω running through the interval $]0, \infty[$ (or a subinterval hereof). The Nyquist plot contains the information from both of the plots of $A(\omega)$ and $\varphi(\omega)$ in a single curve: for a given value of ω the Nyquist plot is showing the *point* $H(i\omega)$, while $A(\omega)$ and $\varphi(\omega)$ only show the distance from the point $H(i\omega)$ to the point 0 and the associated angle. On the other hand the Nyquist plot does not show the value of ω giving rise to a certain point on the curve.

Example 1.28 Taking $L = R = C = 1$ the differential equation (1.16) in Example 1.22 takes the form

$$\ddot{I} + \dot{I} + I = \ddot{e} + e.$$

Let us calculate the associated frequency characteristics. The transfer function is

$$H(s) = \frac{s^2+1}{s^2+s+1};$$

it is defined whenever $s^2 + s + 1 \neq 0$, i.e., for $s \neq \frac{-1\pm i\sqrt{3}}{2}$. Therefore

$$H(i\omega) = \frac{1-\omega^2}{1-\omega^2+i\omega},\ \omega > 0.$$

The amplitude characteristic is thus

$$A(\omega) = \frac{|1-\omega^2|}{\sqrt{(1-\omega^2)^2+\omega^2}}, \tag{1.32}$$

and the phase characteristic is

$$\varphi(\omega) = \mathrm{Arg}\,(1-\omega^2) - \mathrm{Arg}\,(1-\omega^2+i\omega) \quad (\pm 2\pi).$$

Let us derive a shorter expression for the function φ. For $0 < \omega < 1$ the complex number $1 - \omega^2 + i\omega$ belongs to the first quadrant, so

$$\begin{aligned} &\mathrm{Arg}\,(1-\omega^2) = 0, \\ &\mathrm{Arg}\,(1-\omega^2+i\omega) = \mathrm{Arctan}\,\frac{\omega}{1-\omega^2}, \end{aligned}$$

Thus

$$\varphi(\omega) = -\mathrm{Arctan}\,\frac{\omega}{1-\omega^2},\ \omega \in]0,1[. \tag{1.33}$$

Note that we do not need the correction with $\pm 2\pi$ as the function Arctan takes values in the interval $]-\pi/2, \pi/2[$. For $\omega > 1$ the complex number $1 - \omega^2 + i\omega$ belongs to the second quadrant, so

$$\begin{aligned} &\mathrm{Arg}\,(1-\omega^2) = \pi, \\ &\mathrm{Arg}\,(1-\omega^2+i\omega) = \pi + \mathrm{Arctan}\,\frac{\omega}{1-\omega^2}. \end{aligned}$$

Thus we obtain the same expression for $\varphi(\omega)$ as we did for $\omega \in]0,1[$, i.e.,

$$\varphi(\omega) = -\mathrm{Arctan}\,\frac{\omega}{1-\omega^2}, \quad \omega \neq 1. \tag{1.34}$$

For $\omega = 1$ we have that

$$H(i\omega) = H(i) = 0.$$

Thus the phase characteristic $\varphi(\omega)$ is not defined for $\omega = 1$. See Figure 1.29. Furthermore we see that $\omega = 1$ corresponds to the point $(0, 0)$ in the Nyquist plot. Note also that (1.34) shows that

$$\varphi(\omega) \to -\frac{\pi}{2} \text{ as } \omega \to 1^-,$$

while

$$\varphi(\omega) \to \frac{\pi}{2} \text{ as } \omega \to 1^+.$$

This information can also be seen from the Nyquist plot: for $\omega \to 1^-$ the point $H(i\omega)$ approaches $(0, 0)$ via the piece of the curve in the fourth quadrant, while it for $\omega \to 1^+$ approaches the point via the piece of the curve in the first quadrant.

Note that $A(1) = 0$; the stationary solution corresponding to $u(t) = e^{it}$ is therefore

$$y(t) = H(i)e^{it} = 0. \qquad \square$$

It is useful to think about a differential equation of the form (1.15) as a "black box", which gives the output $y(t)$ corresponding to a given input $u(t)$. When the input has the form $u(t) = \cos \omega t$, (1.24) shows that the output also is a harmonic oscillation: the input is just amplified with the factor $A(\omega) = H(i\omega)$ (if $A(\omega) < 1$ it is more appropriate to speak about a damping) and the phase is shifted by $\varphi(\omega) = \operatorname{Arg} H(i\omega)$. For example, in Example 1.28 we see via (1.32) and (1.34) that for $\omega = 2$ the signal is getting amplified by $A(2) \approx 0.83$ while the phase shift is $\varphi(2) \approx 0.59$. Whenever $\omega \approx 1$ we have $A(\omega) << 1$; thus the "black box" leads to a strong damping of the signal. Concretely, the "black box" could be an amplifier used to music signals. In this case it is desirable that the "amplifying factor" $A(\omega)$ is approximately constant for all frequencies that can be heard by the human ear.

Let us finally notice that our requirement that the argument $\varphi(\omega)$ belongs to the interval $]-\pi, \pi]$ frequently leads to a solution that is not continuous. This sometimes contradicts the "physical reality," and in this context the requirement is often omitted.

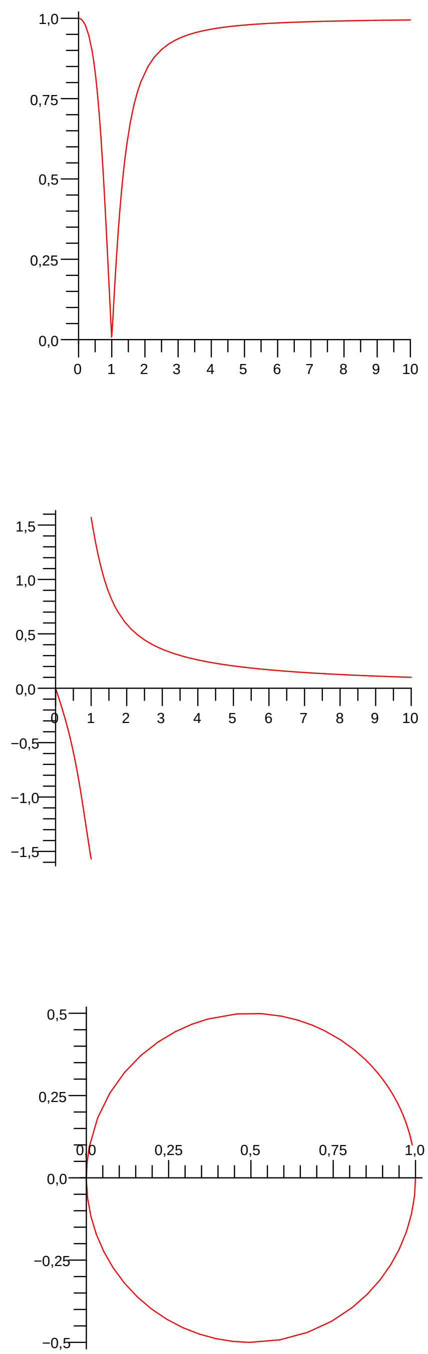


Figure 1.29 *The amplitude characteristic, the phase characteristic, and a Nyquist plot for the differential equation in Example 1.28. The Nyquist plot is drawn for* $\omega \in [0, 10]$*; the curve begings at* $H(i0) = 1$*, and ends at* $H(10i) = \frac{-99}{-99+10i} = \frac{-99(-99-10i)}{99^2+10^2} = \frac{9801}{9901} + i\frac{990}{9901} \approx 1 + 0.1i$*.*

1.6 Equations with variable coefficients

So far we have only considered differential equations with constant coefficients. However, applications often lead to differential equations where the coefficients depend on a variable t, typically the time. For the second order case, consider an equation of the form

$$\frac{d^2y}{dt^2} + a_1(t)\frac{dy}{dt} + a_2(t)y = u(t), \quad t \in I, \tag{1.35}$$

where a_1, a_2 and u are given functions. As usual the corresponding homogeneous differential equation is given by

$$\frac{d^2y}{dt^2} + a_1(t)\frac{dy}{dt} + a_2(t)y = 0, \quad t \in I. \tag{1.36}$$

The treatment of such equations is complicated, and the methods we have derived for differential equations with constant coefficients are not applicable. Often it is even impossible to derive an explicit solutions in terms of the elementary functions. But *if* we can find an explicit expression for a solution to the homogeneous equation (1.36), the following result can often be used to solve the inhomogeneous equation (1.35).

Theorem 1.30 *Assume that the function y_1 is a solution to the homogeneous equation* (1.36) *and that $y_1(t) \neq 0$ for all $t \in I$. Let*

$$A_1(t) = \int a_1(t)dt \quad \text{and} \quad \Omega(t) = e^{A_1(t)}. \tag{1.37}$$

Then the following hold

(i) The homogeneous equation (1.36) *has the solution*

$$y_2(t) = y_1(t)\left(\int \frac{1}{y_1(t)^2\Omega(t)}\,dt\right), \tag{1.38}$$

and the functions y_1, y_2 are linearly independent;

(ii) The general solution to the homogeneous equation (1.36) *is*

$$y(t) = c_1y_1(t) + c_2y_2(t),$$

where c_1 and c_2 denote arbitrary constants.

(iii) The inhomogeneous equation (1.35) *has the particular solution*

$$y_{par}(t) = y_1(t)\left(\int \frac{1}{y_1(t)^2\Omega(t)}\left[\int y_1(t)\Omega(t)u(t)dt\right]dt\right). \tag{1.39}$$

(iv) The general solution to the inhomogeneous equation (1.35) *is*

$$y(t) = y_{par}(t) + c_1y_1(t) + c_2y_2(t),$$

where c_1 and c_2 are arbitrary constants.

In the expression (1.37) for the function $A_1(t)$ we do not need to include an arbitrary constant. The main difficulty in order to apply the result is to find the solution y_1 to the homogeneous equation. Frequently the solution y_1 can be found using the method of power series, see Section 5.2, or using Fourier series, see Section 6.1.

Example 1.31 Let us determine the general solution to the differential equation

$$\frac{d^2y}{dt^2} - \frac{2}{t}\frac{dy}{dt} + \frac{2}{t^2}y = \frac{1}{t}, \ t \in]0, \infty[. \tag{1.40}$$

It is seen directly that the function $y_1(t) = t$ is a solution to the homogeneous equation corresponding to (1.40). Thus we can determine the general solution to (1.40) using Theorem 1.30. Let

$$A_1(t) = \int \frac{-2}{t}\,dt = -2\ln t$$

and

$$\Omega(t) = e^{-2\ln t} = \frac{1}{t^2}.$$

Via (1.38) we see that the homogeneous equation has the solution

$$\begin{aligned} y_2(t) = y_1(t)\left(\int \frac{1}{y_1(t)^2\Omega(t)}\,dt\right) &= t\int \frac{1}{t^2}\,t^2\,dt \\ &= t\int dt \\ &= t^2. \end{aligned}$$

Similarly (1.39) shows that the inhomogeneous equation has the solution

$$\begin{aligned} y_{\mathrm{par}}(t) &= y_1(t)\left(\int \frac{1}{y_1(t)^2\Omega(t)}\left[\int y_1(t)\Omega(t)u(t)dt\right]dt\right) \\ &= t\left(\int\left[\int \frac{1}{t^2}dt\right]dt\right) \\ &= t\left(\int\left[-\frac{1}{t}\right]dt\right) \\ &= -t\ln t. \end{aligned}$$

Thus the general solution to (1.40) is given by

$$y(t) = -t\ln t + c_1t^2 + c_2t,$$

where $c_1, c_2 \in \mathbb{R}$. □

1.7 Partial differential equations

In this chapter we have exclusively considered differential equations depending on a single variable. However, several differential equations appearing in physics and engineering involve several variable, e.g., variables corresponding to the physical coordinates and the time, and their partial derivatives. For example a vibrating string is modelled by an equation of the form

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2},\ 0 < x < L. \tag{1.41}$$

The equation (1.41) is called the *one dimensional wave equation.* Solution of such equations involve more advanced methods than discussed in the current chapter, e.g., methods based on Fourier series. We refer the interested reader to the course 01418 *Partial differential equations* and the book [11].

2
Systems of first-order differential equations

Mathematical modelling of physical systems often involve several time-dependent functions. In the context of an electric circuit these functions could for instance describe the time-dependence of the current through a number of components; or, in a mechanical context, the position or speed of a collection of masses on which certain forces act. Typically the modelling is based on the physical laws and consists of the derivation of a system of differential equations.

In this chapter we will discuss the case of a system of linear differential equations, meaning that the unknown functions and their derivatives only appear in a linear combination. We will furthermore restrict ourself to the case of first-order differential equations. We will follow the same plan as we did for the nth order differential equation, and indeed many results turn out to be parallel. At the end of the chapter we will discuss stability issues for systems of differential equations, an issue of extreme importance for applications.

There exist many physical systems that can not be modelled using a system of linear differential equations. In such cases the exact model is often linearized using a Taylor expansion of the non-linear terms (for example, by substituting $\sin x$ with x for small values of x). We will give a short introduction to nonlinear differential equations in Chapter 3.

2.1 Introduction

We will now consider a system consisting of n first-order differential equations of the form

$$\begin{cases} \dot{x}_1(t) = a_{11}x_1(t) + a_{12}x_2(t) + \cdots + a_{1n}x_n(t) + u_1(t), \\ \dot{x}_2(t) = a_{21}x_1(t) + a_{22}x_2(t) + \cdots + a_{2n}x_n(t) + u_2(t), \\ \quad\cdot \\ \quad\cdot \\ \dot{x}_n(t) = a_{n,1}x_1(t) + a_{n,2}x_2(t) + \cdots + a_{n,n}x_n(t) + u_n(t). \end{cases} \tag{2.1}$$

Here $a_{rs} \in \mathbb{R}$, $r, s = 1, \dots, n$ and $u_1, \dots, u_n$ are given functions, defined on an interval I; $x_1, \dots, x_n$ are the unknown functions which we would like to find. The equations are said to form a *linear system of first-order differential equations with n unknowns*. The system can be represented on matrix form as

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \cdot \\ \cdot \\ \dot{x}_n(t) \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdot & \cdot & a_{1n} \\ a_{21} & a_{22} & \cdot & \cdot & a_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdot & \cdot & a_{nn} \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ \cdot \\ \cdot \\ x_n(t) \end{pmatrix} + \begin{pmatrix} u_1(t) \\ u_2(t) \\ \cdot \\ \cdot \\ u_n(t) \end{pmatrix}. \tag{2.2}$$

Usually we will write the system on a shorter form. Define the *system matrix* $\mathbf{A}$ by

$$\mathbf{A} := \begin{pmatrix} a_{11} & a_{12} & \cdot & \cdot & a_{1n} \\ a_{21} & a_{22} & \cdot & \cdot & a_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdot & \cdot & a_{nn} \end{pmatrix},$$

and let

$$\mathbf{x}(t) := \begin{pmatrix} x_1(t) \\ x_2(t) \\ \cdot \\ \cdot \\ x_n(t) \end{pmatrix}, \qquad \mathbf{u}(t) := \begin{pmatrix} u_1(t) \\ u_2(t) \\ \cdot \\ \cdot \\ u_n(t) \end{pmatrix}.$$

Using this notation we can write the system as

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{u}(t).$$

Often we will omit to mention the dependence on the variable t, and simply write the system as

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{u}. \tag{2.3}$$

A *solution* is a vector $\mathbf{x} = \mathbf{x}(t)$, which satisfies the system. The system (2.3) is said to be *inhomogeneous* whenever $\mathbf{u} \neq 0$, i.e., if there exists $t \in I$ such that $\mathbf{u}(t) \neq \mathbf{0}$. The function $\mathbf{u}$ is called the *forcing function* or *input function.* The corresponding *homogeneous system* is

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}. \tag{2.4}$$

Example 2.1 As an example of a linear inhomogeneous system of first-order differential equations, consider

$$\begin{cases} \dot{x}_1(t) = -\dfrac{2}{3}x_1(t) - \dfrac{4}{3}x_2(t) + \cos t \\ \dot{x}_2(t) = \dfrac{1}{3}x_1(t) - \dfrac{1}{3}x_2(t) + \sin t. \end{cases} \tag{2.5}$$

We can write the system on matrix form as

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{pmatrix} -\frac{2}{3} & -\frac{4}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}.$$

We will now show that the vector function

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} 2\cos t + \sin t \\ \sin t - \cos t \end{pmatrix} \tag{2.6}$$

is a solution. Differentiating leads to

$$\dot{\mathbf{x}}(t) = \begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{pmatrix} -2\sin t + \cos t \\ \cos t + \sin t \end{pmatrix}.$$

Inserting (2.6) in (2.5) leads to

$$\begin{aligned} -\frac{2}{3}x_1(t) - \frac{4}{3}x_2(t) + \cos t &= -\frac{2}{3}(2\cos t + \sin t) - \frac{4}{3}(\sin t - \cos t) + \cos t \\ &= -2\sin t + \cos t = \dot{x}_1(t), \end{aligned}$$

and

$$\begin{aligned} \frac{1}{3}x_1(t) - \frac{1}{3}x_2(t) + \sin t &= \frac{1}{3}(2\cos t + \sin t) - \frac{1}{3}(\sin t - \cos t) + \sin t \\ &= \cos t + \sin t = \dot{x}_2(t). \end{aligned}$$

Thus the function in (2.6) satisfies (2.5), as desired. Later in this chapter we will consider general methods to solve systems of the mentioned type.
□

In this course we will have more emphasis on solution of first order systems of differential equations than the treatment of the nth order differential equation. One of the reasons for this is that an nth order linear differential equation can be rephrased via a system of first-order equations. Indeed, consider an nth order differential equation

$$y^{(n)}(t) + b_1 y^{(n-1)}(t) + \cdots + b_{n-1} y'(t) + b_n y(t) = u(t), \tag{2.7}$$

and introduce new variable $x_1, x_2, \ldots, x_n$ by

$$x_1(t) = y(t),\ x_2(t) = y'(t), \ldots, x_n(t) = y^{(n-1)}(t). \tag{2.8}$$

Then the definition of the functions x_1 and x_2 shows that $\dot{x}_1(t) = x_2(t)$, and similarly,

$$\dot{x}_2(t) = x_3(t), \cdots, \dot{x}_{n-1}(t) = x_n(t).$$

Furthermore (2.7) shows that

$$y^{(n)}(t) = -b_n y(t) - b_{n-1} y'(t) - \cdots - b_2 y^{(n-2)}(t) - b_1 y^{(n-1)}(t) + u(t),$$

which, in terms of the new variables, implies that

$$\dot{x}_n(t) \ = -b_n x_1(t) - b_{n-1} x_2(t) - \cdots - b_2 x_{n-1}(t) - b_1 x_n(t) + u(t).$$

Thus the new variables satisfy the system

$$\begin{cases} \dot{x}_1(t) = x_2(t), \\ \dot{x}_2(t) = x_3(t), \\ \quad\cdot \\ \quad\cdot \\ \dot{x}_{n-1}(t) = x_n(t), \\ \dot{x}_n(t) = -b_n x_1(t) - b_{n-1} x_2(t) - \cdots - b_1 x_n(t) + u(t). \end{cases} \tag{2.9}$$

This is a system of n first-order differential equations, which can be written on matrix form as

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \cdot \\ \cdot \\ \dot{x}_{n-1}(t) \\ \dot{x}_n(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & \cdot & 0 & 0 \\ 0 & 0 & 1 & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & 0 & 1 \\ -b_n & -b_{n-1} & -b_{n-2} & \cdot & -b_2 & -b_1 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ \cdot \\ \cdot \\ x_{n-1}(t) \\ x_n(t) \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ 0 \\ 1 \end{pmatrix} u(t).$$

Any solution $y(t)$ to (2.7) yields via (2.8) a solution $\mathbf{x}$ to the system of differential equations. On the other hand, every solution to the system has the property that the first coordinate function $x_1(t)$ is a solution to (2.7). In this sense the nth order differential equation is a special case of a system of n first-order differential equations. Furthermore one can "translate" all the results for nth order differential equations from the previous chapter to more general results that are valid for systems (2.3) with arbitrary real matrices $\mathbf{A}$ and forcing functions $u(t)$. In particular we will introduce a characteristic polynomial for general systems of the form (2.4); for systems corresponding to the nth order equation (2.7) this polynomial is identical with the characteristic polynomial for (2.7) as introduced in Section 1.2.

2.2 The homogeneous equation

In this section we describe how to determine n linearly independent solutions to the homogeneous equation

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}. \tag{2.10}$$

Let $\mathbf{I}$ denote the identity matrix. Recall that a (real or complex) number λ is an *eigenvalue* for the matrix $\mathbf{A}$ if λ is a root in the *characteristic polynomial*

$$P(\lambda) = \det(\mathbf{A} - \lambda\mathbf{I}).$$

The corresponding *eigenvectors* $\mathbf{v}$ are the solutions to the equation

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0}.$$

We will find the general solution to the homogeneous equation (2.10) based on the following result.

Theorem 2.2 *If λ is a (real or complex) eigenvalue for $\mathbf{A}$ and $\mathbf{v}$ is an eigenvector, then* (2.10) *has the solution*

$$\mathbf{x}(t) = e^{\lambda t}\mathbf{v}.$$

Solutions corresponding to linearly independent eigenvectors (for example, nonzero eigenvectors corresponding to different eigenvalues) are linearly independent.

Proof of the first claim: Suppose that λ is an eigenvalue for $\mathbf{A}$, with corresponding eigenvector $\mathbf{v}$. Let $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$; then

$$\dot{\mathbf{x}}(t) = e^{\lambda t}\lambda\mathbf{v} = e^{\lambda t}\mathbf{A}\mathbf{v} = \mathbf{A}\left(e^{\lambda t}\mathbf{v}\right) = \mathbf{A}\mathbf{x}(t).$$

This proves the result. $\square$

As for the nth order differential equation it holds that if $\lambda = a + i\omega$ is a root in the characteristic equation, then also $\overline{\lambda} = a - i\omega$ is a root. If we have found an eigenvector corresponding to the eigenvalue $\lambda = a + i\omega$ we can use the following result to find an eigenvector corresponding to the eigenvalue $\lambda = a - i\omega$.

Lemma 2.3 *If* $\mathbf{v} = \mathrm{Re}\,\mathbf{v} + i\,\mathrm{Im}\,\mathbf{v}$ *is an eigenvector for the matrix* $\mathbf{A}$ *corresponding to the eigenvalue* $\lambda = a + i\omega$*, then* $\overline{\mathbf{v}} = \mathrm{Re}\,\mathbf{v} - i\,\mathrm{Im}\,\mathbf{v}$ *is an eigenvector corresponding to the eigenvalue* $\lambda = a - i\omega$.

Let us apply Theorem 2.2 and Lemma 2.3 on a concrete example:

Example 2.4 Consider the system

$$\begin{cases} \dot{x}_1(t) = x_1(t) + x_2(t) \\ \dot{x}_2(t) = -x_1(t) + x_2(t), \end{cases}$$

or, on matrix form,

$$\dot{\mathbf{x}} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \mathbf{x}.$$

Thus, consider the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix},$$

which has the characteristic polynomial

$$P(\lambda) = \begin{vmatrix} 1-\lambda & 1 \\ -1 & 1-\lambda \end{vmatrix} = \lambda^2 - 2\lambda + 2.$$

The roots are $\lambda_1 = 1 + i$ and $\lambda_2 = 1 - i$. The eigenvectors corresponding to the eigenvalue $\lambda_1 = 1 + i$ are determined by the equation

$$\begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix} \mathbf{v} = \mathbf{0}.$$

Thus the vector $\mathbf{v} = \begin{pmatrix} 1 \\ i \end{pmatrix}$ is an eigenvector corresponding to the eigenvalue λ_1. According to Lemma 2.3 the vector $\begin{pmatrix} 1 \\ -i \end{pmatrix}$ is therefore an eigenvector corresponding to the eigenvalue λ_2. By Theorem 2.2 the system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ therefore has the two linearly independent solutions

$$\mathbf{x}(t) = e^{(1+i)t} \begin{pmatrix} 1 \\ i \end{pmatrix}$$

and

$$\mathbf{x}(t) = e^{(1-i)t} \begin{pmatrix} 1 \\ -i \end{pmatrix}.$$

The general solution is therefore

$$\mathbf{x}(t) = c_1 e^{(1+i)t} \begin{pmatrix} 1 \\ i \end{pmatrix} + c_2 e^{(1-i)t} \begin{pmatrix} 1 \\ -i \end{pmatrix},$$

where $c_1, c_2 \in \mathbb{C}$. □

Analogous to Theorem 1.4, the solutions to the homogeneous first-order system with n unknown form an n-dimensional vector space. Often we are only interested in the real solutions (and we want to find all of them), so we need to cake care of the following issues:

- The characteristic polynomial might have complex roots. How can we then find the real-valued solutions to the system of differential equations?
- Theorem 2.2 only yields the general solution if we can find n linearly independent eigenvectors. We must discuss how the general solution can be found in the case of multiple roots.

We will now discuss these issues in detail. Analogous to Theorem 1.8 we first formulate a result showing how to determine real-valued solutions in case of complex eigenvalues. If $\mathbf{A}$ has n pairwise distinct eigenvalues we will then be able to find n linearly independent solutions to (2.10).

Theorem 2.5 *A collection of real-valued solutions to the homogeneous equation $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, can be found as follows:*

(a) For any real eigenvalue with corresponding real and nonzero eigenvector $\mathbf{v}$, we have the solution

$$\mathbf{x}(t) = e^{\lambda t}\mathbf{v}.$$

(b) Complex eigenvalues appear in pairs $a \pm i\omega$. Choose a nonzero eigenvector $\mathbf{v}$ corresponding to the eigenvalue $\lambda = a + i\omega$; then we have the solutions

$$\mathbf{x}(t) = \mathrm{Re}\,(e^{\lambda t}\mathbf{v}) = e^{at}(\cos\omega t\ \mathrm{Re}\,\mathbf{v} - \sin\omega t\ \mathrm{Im}\,\mathbf{v}) \tag{2.11}$$

and

$$\mathbf{x}(t) = \mathrm{Im}\,(e^{\lambda t}\mathbf{v}) = e^{at}(\sin\omega t\ \mathrm{Re}\,\mathbf{v} + \cos\omega t\ \mathrm{Im}\,\mathbf{v}) \tag{2.12}$$

Proof: We will only derive the explicit expression for the solutions in (b). Write the eigenvector $\mathbf{v}$ as a sum of its real and imaginary part, $\mathbf{v} = \mathrm{Re}\,\mathbf{v} + i\,\mathrm{Im}\,\mathbf{v}$; then (1.10) implies that

$$\begin{aligned} e^{\lambda t}\mathbf{v} &= e^{at}(\cos\omega t + i\sin\omega t)\,(\mathrm{Re}\,\mathbf{v} + i\,\mathrm{Im}\,\mathbf{v}) \\ &= e^{at}(\cos\omega t\ \mathrm{Re}\,\mathbf{v} - \sin\omega t\ \mathrm{Im}\,\mathbf{v}) + ie^{at}(\sin\omega t\ \mathrm{Re}\,\mathbf{v} + \cos\omega t\ \mathrm{Im}\,\mathbf{v}). \end{aligned}$$

Extracting the real part and the imaginary part yields precisely the solutions stated in (b). □

Note that in (b) it is enough to find an eigenvector $\mathbf{v}$ corresponding to the eigenvalue $\lambda = a + i\omega$: indeed, the two solutions corresponding to the pair $a \pm i\omega$ will be substituted by the two solutions in (2.11) and (2.12), similarly to the situation for the nth order differential equation.

Example 2.6 We want to find the general real solution to the system in Example 2.4,

$$\dot{\mathbf{x}} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \mathbf{x}.$$

Corresponding to the eigenvalue $\lambda = 1 + i$ we found the eigenvector

$$\mathbf{v} = \begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + i \begin{pmatrix} 0 \\ 1 \end{pmatrix};$$

via Theorem 2.5, i.e., the formulas (2.11) and (2.12), we therefore obtain the solutions

$$\mathbf{x}(t) = e^t \left(\cos t \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \sin t \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) = e^t \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}$$

and

$$\mathbf{x}(t) = e^t \left(\sin t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \cos t \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) = e^t \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}.$$

The general real solution is therefore

$$\mathbf{x}(t) = c_1 e^t \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + c_2 e^t \begin{pmatrix} \sin t \\ \cos t \end{pmatrix},$$

where $c_1, c_2 \in \mathbb{R}$. □

Example 2.7 Assume that $\omega \neq 0$. Then one can prove (do it!) that the system

$$\dot{\mathbf{x}} = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix} \mathbf{x}$$

has the general real solution

$$\mathbf{x}(t) = c_1 \begin{pmatrix} \sin \omega t \\ \cos \omega t \end{pmatrix} + c_2 \begin{pmatrix} -\cos \omega t \\ \sin \omega t \end{pmatrix},$$

where $c_1, c_2 \in \mathbb{R}$. □

The characteristic polynomial

$$P(\lambda) = \det(\mathbf{A} - \lambda \mathbf{I})$$

is a polynomial of nth degree, where the coefficient corresponding to λ^n is $(-1)^n$. It is therefore a product of n polynomials of the form $\lambda_n - \lambda$, where the scalars λ_n are the eigenvalues. The number of appearances of a given eigenvalue is called the *algebraic multiplicity of the eigenvalue,* see Definition 1.12. If we for each eigenvalue λ with algebraic multiplicity p can find p linearly independent eigenvectors, then the solutions in Theorem

2.2 lead to n linearly independent solutions to (2.10): we can then find the general solution to (2.10) by taking linear combinations of these solutions.

However, there are also cases where it is more complicated to find the general solution, namely, if an eigenvalue λ with algebraic multiplicity p only leads to $q < p$ linearly independent eigenvectors. In this context we need the concept of *geometric multiplicity:*

Definition 2.8 (Geometric multiplicity) *The geometric multiplicity q of an eigenvalue λ is the dimension of the null space for the matrix $\mathbf{A}-\lambda\mathbf{I}$.*

Note that the geometric multiplicity q of an eigenvalue λ equals the number of linearly independent eigenvectors. In example 2.11 we demonstrate how to calculate the geometric multiplicity.

If $q < p$ for certain eigenvalues, we can use the following result to find linearly independent solutions:

Theorem 2.9 (Linearly independent solutions to $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$)

(a) Assume that λ is a real eigenvalie for A, with algebraic multiplicity $p \geq 2$ and geometric multiplicity $q < p$. Then there exist vectors $\mathbf{b}_{jk} \in \mathbb{R}^n$ such that the functions

$$\begin{aligned}
\mathbf{x_1}(t) &= \mathbf{b}_{11}e^{\lambda t} \\
\mathbf{x_2}(t) &= \mathbf{b}_{21}e^{\lambda t} + \mathbf{b}_{22}te^{\lambda t} \\
&\cdot \\
&\cdot \\
\mathbf{x_p}(t) &= \mathbf{b}_{p1}e^{\lambda t} + \mathbf{b}_{p2}te^{\lambda t} + \cdots + \mathbf{b}_{pp}t^{p-1}e^{\lambda t}
\end{aligned}$$

are linearly independent real solutions to the system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$.

(b) Assume that $a \pm i\omega$ is a pair of complex eigenvalues with algebraic multiplicity $p \geq 2$ and geometric multiplicity $q < p$. Let $\lambda := a + i\omega$. Then there exist vectors $\mathbf{b}_{jk} \in \mathbb{C}^n$ such that the functions

$$\begin{aligned}
\mathbf{x_1}(t) &= \mathrm{Re}\,(\mathbf{b}_{11}e^{\lambda t}) \\
\mathbf{x_2}(t) &= \mathrm{Re}\,(\mathbf{b}_{21}e^{\lambda t} + \mathbf{b}_{22}te^{\lambda t}) \\
&\cdot \\
&\cdot \\
\mathbf{x_p}(t) &= \mathrm{Re}\,(\mathbf{b}_{p1}e^{\lambda t} + \mathbf{b}_{p2}te^{\lambda t} + \cdots + \mathbf{b}_{pp}t^{p-1}e^{\lambda t}) \\
\mathbf{x_{p+1}}(t) &= \mathrm{Im}\,(\mathbf{b}_{11}e^{\lambda t}) \\
\mathbf{x_{p+2}}(t) &= \mathrm{Im}\,(\mathbf{b}_{21}e^{\lambda t} + \mathbf{b}_{22}te^{\lambda t}) \\
&\cdot \\
&\cdot \\
\mathbf{x_{2p}}(t) &= \mathrm{Im}\,(\mathbf{b}_{p1}e^{\lambda t} + \mathbf{b}_{p2}te^{\lambda t} + \cdots + \mathbf{b}_{pp}t^{p-1}e^{\lambda t})
\end{aligned}$$

are linearly independent real solutions to the system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$.

For as well (a) as (b) it holds that $\mathbf{b}_{11}$ is chosen as a nonzero eigenvector corresponding to the eigenvalue λ; one can show that we can also choose the vector $\mathbf{b}_{22}$ as an eigenvector corresponding to the eigenvalue λ, see Example 2.11; the same example illustrates how to find the remaining vectors $\mathbf{b}_{jk}$.

Theorem 2.9 yields a method to find the general real solution to the system (2.10):

Theorem 2.10 (General real solution to (2.10)**)** *The general real solution to the homogeneous system* $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ *can be found as follows:*

(i) For each real eigenvalue λ*, consider the solutions obtained in Theorem 2.9 (a);*

(ii) For each pair of complex eigenvalues $a \pm i\omega$*, consider the solutions in Theorem 2.9 (b).*

The general real solution to the system is obtained by forming linear combinations of the mentioned solutions, with real coefficients.

In Example 2.30 we return to the solutions in Theorem 2.10 and illustrate their behavior, as well for real eigenvalues as complex eigenvalues.

Example 2.11 Consider the system

$$\begin{cases} \dot{x}_1(t) = 3x_1(t) - x_2(t) \\ \dot{x}_2(t) = x_1(t) + x_2(t), \end{cases}$$

or, on matrix form,

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

Thus, consider the matrix

$$\mathbf{A} = \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix}.$$

The characteristic polynomial is

$$P(\lambda) = \det(\mathbf{A} - \lambda\mathbf{I}) = (3-\lambda)(1-\lambda) + 1 = \lambda^2 - 4\lambda + 4 = (\lambda - 2)^2.$$

The only eigenvalue is $\lambda = 2$, having algebraic multiplicity $p = 2$. In order to find the eigenvectors, consider the equation $\mathbf{A}\mathbf{v} = 2\mathbf{v}$, i.e.,

$$\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \mathbf{v} = \mathbf{0};$$

the equation has the solutions

$$\mathbf{v} = c \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \ c \in \mathbb{R}.$$

Thus the eigenvalue $\lambda = 2$ has geometric multiplicity $q = 1$. Via Theorem 2.2 we obtain the solution

$$\mathbf{x}_1(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}. \tag{2.13}$$

According to Theorem 2.9 we now seek a solution of the form

$$\mathbf{x}_2(t) = \mathbf{b}_1 e^{2t} + \mathbf{b}_2 t e^{2t}.$$

By differentiating we see that then

$$\dot{\mathbf{x}}_2(t) = 2\mathbf{b}_1 e^{2t} + \mathbf{b}_2 e^{2t} + 2\mathbf{b}_2 t e^{2t}.$$

Inserting this in the system we see that $\mathbf{x}_2$ is a solution if

$$2\mathbf{b}_1 e^{2t} + \mathbf{b}_2 e^{2t} + 2\mathbf{b}_2 t e^{2t} = \mathbf{A}\mathbf{b}_1 e^{2t} + \mathbf{A}\mathbf{b}_2 t e^{2t} \text{ for all } t \in \mathbb{R},$$

i.e., if

$$2\mathbf{b}_1 + \mathbf{b}_2 + 2\mathbf{b}_2 t = \mathbf{A}\mathbf{b}_1 + \mathbf{A}\mathbf{b}_2 t \text{ for all } t \in \mathbb{R}. \tag{2.14}$$

Taking $t = 0$ in (2.14) leads to

$$\mathbf{A}\mathbf{b}_1 = 2\mathbf{b}_1 + \mathbf{b}_2; \tag{2.15}$$

taking now $t = 1$ in (2.14) leads to

$$\mathbf{A}\mathbf{b}_2 = 2\mathbf{b}_2. \tag{2.16}$$

The equation (2.16) shows that $\mathbf{b}_2$ must be an eigenvector corresponding to the eigenvalue $\lambda = 2$, so we can take

$$\mathbf{b}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The equation (2.15) can be written as $(\mathbf{A} - 2\mathbf{I})\mathbf{b}_1 = \mathbf{b}_2$, where we just found $\mathbf{b}_2$: thus the equation takes the form

$$\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \mathbf{b}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

having, e.g., the solution

$$\mathbf{b}_1 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

Thus we arrive at

$$\mathbf{x}_2(t) = \mathbf{b}_1 e^{2t} + \mathbf{b}_2 t e^{2t} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} e^{2t} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} t e^{2t}.$$

The solutions $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ are linearly independent. Thus the system of differential equations has the general real solution

$$\begin{aligned}\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) &= c_1 \begin{pmatrix}1\\1\end{pmatrix} e^{2t} + c_2 \left(\begin{pmatrix}0\\-1\end{pmatrix} e^{2t} + \begin{pmatrix}1\\1\end{pmatrix} te^{2t} \right) \\ &= c_1 \begin{pmatrix}1\\1\end{pmatrix} e^{2t} + c_2 \begin{pmatrix}0\\-1\end{pmatrix} e^{2t} + c_2 \begin{pmatrix}1\\1\end{pmatrix} te^{2t},\end{aligned}$$

where $c_1, c_2 \in \mathbb{R}$.

Note that there does not exists a nonzero solution of the form $\mathbf{x}_2(t) = \mathbf{b}_2 te^{2t}$: such a solution would correspond to $\mathbf{b}_1 = 0$, but then also $\mathbf{b}_2 = 0$ according to (2.16). □

2.3 The fundamental matrix

In this section we will formulate our results concerning the solutions to the homogeneous system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ using the so-called *fundamental matrix*, introduced as follows:

Definition 2.12 (Fundamental matrix) *Corresponding to any set of n linearly independent solutions $\mathbf{x}_1(t), \mathbf{x}_2(t), \dots, \mathbf{x}_n(t)$ to the homogeneous system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, the associated fundamental matrix is the $n \times n$ matrix given by*

$$\mathbf{\Phi}(t) := (\mathbf{x}_1(t), \mathbf{x}_2(t), \dots, \mathbf{x}_n(t)),\ t \in I. \tag{2.17}$$

Thus, the columns in $\Phi(t)$ consist of the vectors $\mathbf{x}_i(t),\ i = 1, \dots, n$.

We note that many different sets of n linearly independent solutions to the system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ can be found; thus the fundamental matrix is not unique.

Example 2.13 Assume that $\omega \neq 0$. According to Example 2.7 the system

$$\dot{\mathbf{x}} = \begin{pmatrix}0 & \omega\\ -\omega & 0\end{pmatrix} \mathbf{x}$$

has the linearly independent solutions

$$\mathbf{x}(t) = \begin{pmatrix}\sin \omega t\\ \cos \omega t\end{pmatrix} \quad \text{and} \quad \mathbf{x}(t) = \begin{pmatrix}-\cos \omega t\\ \sin \omega t\end{pmatrix}.$$

Thus

$$\mathbf{\Phi}(t) = \begin{pmatrix}\sin \omega t & -\cos \omega t\\ \cos \omega t & \sin \omega t\end{pmatrix}$$

is a fundamental matrix. Another fundamental matrix is

$$\begin{pmatrix} -\cos\omega t & \sin\omega t \\ \sin\omega t & \cos\omega t \end{pmatrix} \qquad \square$$

The following result will play an important role in connection with inhomogeneous systems of differential equations

Theorem 2.14 *Let* $\mathbf{x}_1(t), \mathbf{x}_2(t), \ldots, \mathbf{x}_n(t)$ *denote a set of* n *linearly independent solutions to the homogeneous system* $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$. *Then the corresponding fundamental matrix* $\mathbf{\Phi}(t)$ *is regular (i.e., invertible) for all* $t \in I$.

Proof: It is well-known from linear algebra that an $n \times n$ matrix consisting of n linearly independent columns is invertible. $\square$

The fundamental matrix allows us to write the general solution on a convenient form, and also gives a concrete formula for a certain particular solution:

Theorem 2.15

(i) Every fundamental matrix $\mathbf{\Phi}(t)$ *associated with the homogeneous system* $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ *is a differentiable matrix function, which satisfies the matrix equation*

$$\dot{\mathbf{\Phi}}(t) = \mathbf{A}\mathbf{\Phi}(t). \tag{2.18}$$

(ii) Given an arbitrary real fundamental matrix $\mathbf{\Phi}(t)$, *the general real solution to* $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ *is*

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{c}, \;\; \mathbf{c} = (c_1 \; c_2 \; \ldots \; c_n)^T \in \mathbb{R}^n, \tag{2.19}$$

and the particular solution, for which $\mathbf{x}(t_0) = \mathbf{x}_0$, *is given by*

$$\mathbf{x}(t) = \mathbf{\Phi}(t)[\mathbf{\Phi}(t_0)]^{-1}\mathbf{x}_0. \tag{2.20}$$

2.4 The inhomogeneous equation

We will now discuss the inhomogeneous equation

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{u}, \tag{2.21}$$

where $\mathbf{u}$ is a vector consisting of n continuous functions, defined on the interval I. According to the following theorem such an equation always has infinitely many solutions:

Theorem 2.16 (The existence and uniqueness theorem) *For every $t_0 \in I$ and every vector $\mathbf{v} = (v_1, v_2, \ldots, v_n)^\top \in \mathbb{R}^n$ there exist precisely one solution to* (2.21) *for which*

$$\mathbf{x}(t_0) = \mathbf{v}.$$

The given values of the vector $\mathbf{x}(t_0)$ are called *initial conditions.*

In the modelling of a linear physical system one can choose between a description in terms of an nth order differential equation as in (2.7), or a system of n first-order differential equations as in (2.9). In the first case the initial condition appearing in the existence and uniqueness theorem is $(y(t_0), y'(t_0), \ldots, y^{(n-1)}(t_0))$; in the second case the necessary initial conditions are given by $\mathbf{x}(t_0)$. Considering the case of modelling of an electric circuit, the values of $\mathbf{x}(t_0)$ usually reflect physical quantities, while the values of the higher-order derivatives in $(y(t_0), y'(t_0), \ldots, y^{(n-1)}(t_0))$ might not have physical interpretations. For this reason it is often preferred to model the system using systems of first-order equations.

The structural properties for the solutions to the inhomogeneous equation (2.21) are similar to what we know for the nth order differential equation. In particular the principle of superposition and Theorem 1.19 have similar versions; we leave the precise formulation to the reader. Often a particular solution can be found using the *guess and test method* as illustrated by the following example. The principle is the same as what we described for the nth order differential equation, namely, that we search for a solution of a similar form as the forcing function $\mathbf{u}(t)$:

Example 2.17 We want to find the general real solution to the system

$$\begin{cases} \dot{x}_1(t) = x_1(t) + x_2(t) + t \\ \dot{x}_2(t) = -x_1(t) + x_2(t) + t + 1. \end{cases}$$

On matrix form the system is

$$\dot{\mathbf{x}} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} t \\ t+1 \end{pmatrix}. \tag{2.22}$$

We will therefore search for a solution of the form

$$\mathbf{x}(t) = \mathbf{a} + \mathbf{b}t.$$

Then $\dot{\mathbf{x}}(t) = \mathbf{b}$. Inserting this in (2.22) shows that $\mathbf{x}$ is a solution if and only if

$$\mathbf{b} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} (\mathbf{a} + \mathbf{b}t) + \begin{pmatrix} t \\ t+1 \end{pmatrix}.$$

Since

$$\begin{pmatrix} t \\ t+1 \end{pmatrix} = t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

we can write the condition for having a solution as

$$\mathbf{b} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \mathbf{a} + t \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \mathbf{b} + t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Collecting the terms involving t shows that this is equivalent with

$$\mathbf{b} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \mathbf{a} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} + t \left[\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \mathbf{b} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right]. \tag{2.23}$$

This must hold for all $t \in \mathbb{R}$. Taking $t = 0$ shows that therefore

$$\mathbf{b} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \mathbf{a} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{2.24}$$

Inserting this in (2.23) shows that then

$$\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \mathbf{b} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This equation has the solution

$$\mathbf{b} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix};$$

inserting this in (2.24) leads to the equation

$$\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \mathbf{a} = \begin{pmatrix} 0 \\ -2 \end{pmatrix}$$

with the solution

$$\mathbf{a} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Thus we have proved that the given system has the solution

$$\mathbf{x}(t) = \mathbf{a} + \mathbf{b}t = \begin{pmatrix} 1 \\ -1 \end{pmatrix} + t \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

In Example 2.6 we found the general real solution to the corresponding homogeneous system. Thus the general real solution to the inhomogeneous equation is

$$\mathbf{x}(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} + t \begin{pmatrix} 0 \\ -1 \end{pmatrix} + c_1 e^t \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + c_2 e^t \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}, \quad c_1, c_2 \in \mathbb{R}. \quad \square$$

Let us finally apply the fundamental matrix to express the general solution to the inhomogeneous system.

Theorem 2.18 *Let $\mathbf{\Phi}(t)$ denote an arbitrary real fundamental matrix associated with the homogeneous system* (2.4) *with a real forcing function* $\mathbf{u}$, *and choose $t_0 \in I$ arbitrarily. Then the following holds:*

(i) Any real solution to the linear inhomogeneous system (2.21) *has the form*

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{c} + \mathbf{\Phi}(t)\int_{t_0}^{t} [\mathbf{\Phi}(\tau)]^{-1}\mathbf{u}(\tau)d\tau, \quad t \in I, \tag{2.25}$$

with $\mathbf{c} = (c_1\ c_2\ \dots\ c_n)^T \in \mathbb{R}^n$.

(ii) The particular solution to (2.21) *for which $\mathbf{x}(t_0) = \mathbf{x}_0$ is given by*

$$\mathbf{x}(t) = \mathbf{\Phi}(t)[\mathbf{\Phi}(t_0)]^{-1}\mathbf{x}_0 + \mathbf{\Phi}(t)\int_{t_0}^{t} [\mathbf{\Phi}(\tau)]^{-1}\mathbf{u}(\tau)d\tau, \quad t \in I. \tag{2.26}$$

Based on knowledge of two linearly independent solutions to a homogeneous second-order differential equation one can find the solutions to the corresponding inhomogeneous differential equation:

Example 2.19 We want to solve the differential equation

$$\ddot{y}(t) + a_1\dot{y}(t) + a_2 y(t) = q(t), \text{ where } a_1, a_2 \in \mathbb{R}. \tag{2.27}$$

Assume that we have found two linearly independent solutions $y_1(t), y_2(t)$ to the homogeneous equation

$$\ddot{y}(t) + a_1\dot{y}(t) + a_2 y(t) = 0;$$

we can do this as described in Section 1.2. Let us now reformulate (2.27) as a system of first-order differential equations as described on page 30. Thus, we let $x_1(t) = y(t)$, $x_2(t) = \dot{y}(t)$ and arrive at the system

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -a_2 & -a_1 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} 0 \\ q(t) \end{pmatrix}. \tag{2.28}$$

Note that $[y_1(t) \quad \dot{y}_1(t)]^T$ and $[y_2(t) \quad \dot{y}_2(t)]^T$ are linearly independent solutions to the homogeneous system

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -a_2 & -a_1 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}.$$

Thus we obtain the fundamental matrix

$$\mathbf{\Phi}(t) = \begin{pmatrix} y_1(t) & y_2(t) \\ \dot{y}_1(t) & \dot{y}_2(t) \end{pmatrix}.$$

Letting $W(t) = \det(\mathbf{\Phi}(t))$ (the so-called Wronski-determinant), we see that

$$[\mathbf{\Phi}(t)]^{-1} = \frac{1}{W(t)} \begin{pmatrix} \dot{y}_2(t) & -y_2(t) \\ -\dot{y}_1(t) & y_1(t) \end{pmatrix}$$

and

$$[\mathbf{\Phi}(t)]^{-1}\begin{pmatrix}0\\ q(t)\end{pmatrix}=\frac{q(t)}{W(t)}\begin{pmatrix}-y_2(t)\\ y_1(t)\end{pmatrix}.$$

Thus the general solution of (2.28) is

$$\mathbf{x}(t)=\begin{pmatrix}x_1(t)\\ x_2(t)\end{pmatrix}=\begin{pmatrix}y_1(t) & y_2(t)\\ \dot{y}_1(t) & \dot{y}_2(t)\end{pmatrix}\left(\begin{pmatrix}c_1\\ c_2\end{pmatrix}+\int_{t_0}^{t}\frac{q(\tau)}{W(\tau)}\begin{pmatrix}-y_2(\tau)\\ y_1(\tau)\end{pmatrix}d\tau\right) \tag{2.29}$$

The general solution of (2.27) is given by the first coordinate function of the solution of (2.29). That is,

$$\begin{aligned} y(t) &= x_1(t)\\ &= c_1y_1(t)+c_2y_2(t)+y_1(t)\int_{t_0}^{t}\frac{-y_2(\tau)q(\tau)}{W(\tau)}d\tau+y_2(t)\int_{t_0}^{t}\frac{y_1(\tau)q(\tau)}{W(\tau)}d\tau, \end{aligned}$$

where $c_1, c_2 \in \mathbb{C}$. □

2.5 Systems with variable coefficients

So far we have only treated systems of differential equations with constant coefficients. However, applications often lead to systems where the coefficients are continuous functions of the variable t; that is systems of the form

$$\begin{cases} \dot{x}_1(t)=a_{11}(t)x_1(t)+a_{12}(t)x_2(t)+\cdots+a_{1n}(t)x_n(t)+u_1(t),\\ \dot{x}_2(t)=a_{21}(t)x_1(t)+a_{22}(t)x_2(t)+\cdots+a_{2n}(t)x_n(t)+u_2(t),\\ \quad\quad .\\ \quad\quad .\\ \dot{x}_n(t)=a_{n1}(t)x_1(t)+a_{n2}(t)x_2(t)+\cdots+a_{nn}(t)x_n(t)+u_n(t), \end{cases} \tag{2.30}$$

where a_{rs} and $u_1, \dots, u_n$ are given functioner. On matrix form the system is

$$\dot{\mathbf{x}}(t)=\mathbf{A}(t)\mathbf{x}(t)+\mathbf{u}(t), \tag{2.31}$$

where the matrix $\mathbf{A}(t)$ depends on the variable t.

Such systems are considerably more complicated than the systems with constant coefficients; in particular the method of finding the solutions via the eigenvalues and eigenvectors does not apply anymore. Often it is not possible to find explicit solutions in terms of the classical standard functions – and in case it can be done, mathematical programs like Maple or Mathematica can find them more efficiently than we can do with theoretical methods. For this reason we will not go into a detailed discussion of how to find explicit solutions. On the other hand it is important to know the structure of the solution space for (2.30). The good news is that the structural results we have for the systems with constant coefficients also hold for the more general case of variable coefficients. In particular,

- the existence and uniqueness theorem holds;
- the solution space of (2.30) is n-dimensional;
- one can introduce the fundamental matrix in the same manner as for systems with constant coefficients; and using the fundamental matrix the solutions to the homogeneous equation can be described as in Theorem 2.15.

2.6 The transfer function for systems

Even when considering systems of differential equations involving n variables $x_1, x_2, \ldots, x_n$, there are cases where one is only interested in a single variable (or maybe a linear combination of the variables). Another issue is that the forcing function $\mathbf{u}$ often is given in terms of a certain "direction" (i.e., a vector), multiplied by a scalar-valued function. For these reasons we will now consider a first-order system of differential equations with n unknowns, of the form

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u, \\ y = \mathbf{d}^\top \mathbf{x}, \end{cases} \tag{2.32}$$

where $u : I \to \mathbb{R}$ is a continuous function, and $\mathbf{d}$ and $\mathbf{b}$ are vectors in $\mathbb{R}^n$. For the system (2.32) the solution is y. Note that we can find y simply by solving the differential equation $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u$, and then inserting $\mathbf{x}$ in the expression $y = \mathbf{d}^\top \mathbf{x}$. For special choices of the function u we will now introduce a method that directly leads to a solution y. The method is parallel with our method for the nth order differential equation in Section 1.4.

The corresponding homogeneous system is obtained by letting $u(t) = 0$. Then, as usual, if $y_0(t)$ is a solution to the inhomogeneous equation and y_{H} denotes all the homogeneous solutions, the general solution to the inhomogeneous equation is given by $y = y_0 + y_{\mathrm{H}}$.

Note that a system of the form (2.32) appears when an nth order differential equation is reformulated via a system of differential equations: more precisely, as seen in Section 2.1, when an equation of the form af formen

$$y^{(n)} + b_1 y^{(n-1)} + \cdots + b_{n-1} y' + b_n y = u,$$

is rephrased as a system, the function y is precisely the first coordinate of the solution, see (2.8). This means that

$$y = x_1 = 1 \cdot x_1 + 0 \cdot x_2 + \cdots + 0 \cdot x_n = \mathbf{d}^T \mathbf{x},$$

where

$$\mathbf{d} = \begin{pmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ 0 \end{pmatrix}.$$

Frequently it is convenient to consider (2.32) as a *mapping* that maps the forcing function u (and a set of initial conditions) to a solution y. We will now introduce the transfer function for systems of the form (2.32). Considering $u(t) = e^{st}$ we can show that if

$$\det(\mathbf{A} - s\mathbf{I}) \neq 0,$$

then the system has precisely one solution of the form

$$\mathbf{x}(t) = \mathbf{x}_0 e^{st}; \tag{2.33}$$

the solution appears for

$$\mathbf{x}_0 = -(\mathbf{A} - s\mathbf{I})^{-1}\mathbf{b}. \tag{2.34}$$

Inserting $\mathbf{x}$ in the equation $y = \mathbf{d}^\top \mathbf{x}$ leads to

$$y(t) = H(s)e^{st}, \tag{2.35}$$

where

$$H(s) := -\mathbf{d}^\top (\mathbf{A} - s\mathbf{I})^{-1}\mathbf{b}. \tag{2.36}$$

This result is parallel with the result we found for the nth order differential equation in Section 1.4 – we just need to observe that the function $H(s)$ is defined via different expressions in the two cases. We will again call the solution (2.35) for the *stationary solution* associated with $u(t) = e^{st}$. The function in (2.36) is called the *transfer function* for (2.32). As for the case of the nth order differential equation we must remember to find its domain: the transfer function (2.36) is defined for the $s \in \mathbb{C}$ for which $\det(\mathbf{A} - s\mathbf{I}) \neq 0$.

The above considerations prove the existence part of the following theorem:

Theorem 2.20 (Stationary solution) *Consider* (2.32)*, and let* $u(t) = e^{st}$*, where* s *is not an eigenvalue for the matrix* $\mathbf{A}$*, i.e.,* $\det(\mathbf{A} - s\mathbf{I}) \neq 0$*. Then* (2.32) *has precisely one solution of the form*

$$y(t) = H(s)e^{st}. \tag{2.37}$$

The solution is obtained when

$$H(s) = -\mathbf{d}^\top (\mathbf{A} - s\mathbf{I})^{-1}\mathbf{b}.$$

The uniqueness of a solution of the form (2.37) follows from the fact that all solutions to the inhomogeneous equation appear by adding a homogeneous solution to a particular solution, see Theorem 1.19: indeed, none of the solutions to the homogeneous equation have the form Ce^{st} for the same value of s.

We remind the reader that if

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and $\det(\mathbf{A}) = ad - bc \neq 0$, then $\mathbf{A}$ is invertible, and

$$\mathbf{A}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}. \tag{2.38}$$

Example 2.21 Let us find the transfer function for the system

$$\begin{cases} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -4 & 0 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 4 \\ 0 \end{pmatrix} u(t), \\ \quad y = \begin{pmatrix} 1 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \end{cases}$$

Now,

$$\mathbf{A} - s\mathbf{I} = \begin{pmatrix} -4 - s & 0 \\ 1 & -1 - s \end{pmatrix}$$

so

$$\det(\mathbf{A} - s\mathbf{I}) = (-4 - s)(-1 - s) = (s + 1)(s + 4).$$

Using (2.38) we see that if $s \neq -1$ and $s \neq -4$, then

$$(\mathbf{A} - s\mathbf{I})^{-1} = \frac{1}{(s + 1)(s + 4)} \begin{pmatrix} -1 - s & 0 \\ -1 & -4 - s \end{pmatrix}.$$

Thus, according to (2.36) the transfer function is

$$\begin{aligned} H(s) &= -\begin{pmatrix} 1 & 3 \end{pmatrix} \frac{1}{(s + 1)(s + 4)} \begin{pmatrix} -1 - s & 0 \\ -1 & -4 - s \end{pmatrix} \begin{pmatrix} 4 \\ 0 \end{pmatrix} \\ &= \frac{-1}{(s + 1)(s + 4)} \begin{pmatrix} 1 & 3 \end{pmatrix} \begin{pmatrix} -4 - 4s \\ -4 \end{pmatrix} = \frac{4s + 16}{(s + 1)(s + 4)} \\ &= \frac{4(s + 4)}{(s + 1)(s + 4)} = \frac{4}{s + 1}, \quad s \notin \{-1, -4\}. \end{aligned}$$

It is very important that we remember to state the domain for the transfer function: indeed, the final expression for $H(s)$ does not reveal that the transfer function is not defined for $s = -4$. □

The transfer function play an important role for solution of systems of differential equations with periodic forcing functions u. In order to show this in detail we will need Fourier series; we return to this in Section 7.2. The following example shows how far we can go at the moment.

Example 2.22 Consider a system of the form (2.32), with transfer function $H(s)$. Consider $u(t) = e^{int}$ for some $n \in \mathbb{Z}$ such that $\det(\mathbf{A} - in\mathbf{I}) \neq 0$; then according to Theorem 2.20 we have a solution of the form

$$y(t) = H(in)e^{int}.$$

Using the principle of superposition we can extend this result. In fact, consider a function u, which is a finite linear combination of exponential functions e^{int}, where $n \in \mathbb{Z}$; in other words,

$$u(t) = \sum_{n=-N}^{N} c_n e^{int} \tag{2.39}$$

for some $N \in \mathbb{N}$ and some scalar coefficients c_n. The principle of superposition now tells us that if we can solve the system (2.32) for each of the exponential functions appearing in (2.39), i.e., if $\det(\mathbf{A} - in\mathbf{I}) \neq 0$ for all $n = -N, -N+1, \dots, N$, then considering (2.32) with $u(t)$ in (2.39) we obtain the solution

$$y(t) = \sum_{n=-N}^{N} c_n H(in)e^{int}.$$

This leads to a method to solve (2.32) for a large class of periodic functions u. In the theory for Fourier series we will show that "any" function u with period 2π can be considered as a limit of a sequence of functions of the form (2.39): that is, for suitable coefficients $c_n,\ n \in \mathbb{Z}$, we have

$$u(t) = \lim_{N\to\infty} \sum_{n=-N}^{N} c_n e^{int}.$$

This is written on short form as

$$u(t) = \sum_{n=-\infty}^{\infty} c_n e^{int}.$$

Now it turns out that a generalized version of the principle of superposition holds: if the system (2.32) satisfies a stability condition that will be introduced in the next section, then (2.32) has a solution given by

$$y(t) = \lim_{N\to\infty} \sum_{n=-N}^{N} c_n H(in)e^{int} = \sum_{n=-\infty}^{\infty} c_n H(in)e^{int}.$$

We return to a detailed description of this method in Section 7.2. □

2.7 Frequency characteristics

The transfer function $H(s)$ in (2.36) plays exactly the same role for systems of differential equations as the transfer function (1.19) play for the $n$the order differential equation. The following result is parallel to Theorem 1.26:

Theorem 2.23 (Stationary solution) *If*

$$u(t) = \cos \omega t,$$

where $\det(\mathbf{A} - i\omega \mathbf{I}) \neq 0$*, then* (2.32) *has the solution*

$$y(t) = \mathrm{Re}\,(H(i\omega)e^{i\omega t}) = |H(i\omega)| \cos(\omega t + \mathrm{Arg}\, H(i\omega)). \tag{2.40}$$

Under the same assumption, if $u(t) = \sin \omega t$ *then* (2.32) *has the solution*

$$y(t) = \mathrm{Im}\,(H(i\omega)e^{i\omega t}) = |H(i\omega)| \sin(\omega t + \mathrm{Arg}\, H(i\omega)).$$

The function

$$A(\omega) = |H(i\omega)|, \ \omega > 0,$$

is called the *amplitude characteristic* for (2.32). The function

$$\varphi(\omega) = \mathrm{Arg}\, H(i\omega), \ \omega > 0,$$

is called the *phase characteristic* for (2.32). Put together, the two functions are called *frequency characteristics*. Using these definitions, all results about frequency characteristics in Section 1.5 have a similar version for the systems considered here.

Example 2.24 In Example 2.21 we considered a system of differential equations having the transfer function

$$H(s) = \frac{4}{s+1}, \ s \notin \{-1, -4\}.$$

Thus, for any $\omega > 0$,

$$H(i\omega) = \frac{4}{1 + i\omega};$$

therefore the amplitude characteristic is

$$A(\omega) = |H(i\omega)| = \left| \frac{4}{1+i\omega} \right| = \frac{4}{\sqrt{1+\omega^2}}.$$

Using (1.30) we see that the phase characteristic is

$$\begin{aligned} \varphi(\omega) = \mathrm{Arg}\,(H(i\omega)) &= \mathrm{Arg}\,(\frac{4}{1+i\omega}) \\ &= \mathrm{Arg}\,(4) - Arg(1+i\omega) \ \ (\pm 2\pi) \\ &= -\mathrm{Arctan}\, \omega, \ \omega > 0. \end{aligned}$$

Since the function Arctan takes values in the interval $]-\pi/2, \pi/2[$ we do not need the correction with $\pm 2\pi$ in the expression for the phase characteristic. □

2.8 Stability for homogeneous systems

So far we have derived general formulas for solutions to homogeneous and inhomogeneous systems of differential equations, and considered solutions satisfying certain initial conditions. However, often we are in a situation where neither the exact forcing function nor the exact initial conditions are known. In this case we need a qualitative analysis of the behaviour of the solutions; this leads to certain stability issues, which is the topic for the current section.

Stable physical or chemical systems are characterized by the property that small changes in the initial conditions or the "forces" acting on the system only leads to bounded changes in the behaviour of the system. On the other hand, certain systems react strongly even on small changes: in worst case a bridge can break down if the wind trigs undesirable oscillations in the construction, a chemical reaction can lead to a explosion, etc. We will now formulate precise mathematical definitions of stability and instability for a linear system of first-order differential equations with constant coefficients,

$$\dot{\mathbf{x}}(t) \quad = \quad \mathbf{A}\mathbf{x}(t), \ t \in [t_0, \infty[. \tag{2.41}$$

We will need the concept of a bounded vector function. First, recall that a function $f : \mathbb{R} \to \mathbb{C}$ is *bounded* if all the values $|f(t)|, \ t \in \mathbb{R}$, are kept under a certain threshold, i.e., if there exists a constant $K > 0$ such that $|f(t)| \leq K$ for all $t \in \mathbb{R}$. Similarly, a vector function

$$\mathbf{x} : I \to \mathbb{C}^n, I \subseteq \mathbb{R}, \ \mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \cdot \\ \cdot \\ x_n(t) \end{pmatrix}$$

is *bounded*, if each of the coordinate functions $x_1(t), \ldots, x_n(t)$ is bounded, or, equivalently, if there exists a constant $K > 0$, such that

$$||\mathbf{x}(t)|| := \sqrt{|x_1(t)|^2 + |x_2(t)|^2 + \cdots + |x_n(t)|^2} \leq K \text{ for all } t \in I. \tag{2.42}$$

Observe that $||\mathbf{x}(t)||$ is the length of the vector $\mathbf{x}(t)$. We will consider two types of stability:

Definition 2.25 (Stability and asymptotically stability)

(i) The homogeneous system (2.41) *is stable if every solution* $\mathbf{x}(t), t \in [t_0, \infty[$ *is bounded. In the literature this type of stability is also known under the name "marginal stability."*

(ii) The system (2.41) *is asymptotically stable if every solution* $\mathbf{x}(t), t \in [t_0, \infty[$, *satisfies that*

$$\mathbf{x}(t) \to \mathbf{0} \text{ as } t \to \infty.$$

In the literature this type of stability is also known under the name "internal stability".

A system that is not stable, is called instable.

Note that an asymptotically stabile system automatically is stable.

If a physical system is modelled by a stable linear system of the form (2.41), then all choices of the initial conditions lead to bounded solutions. Figure 2.26 illustrates the various types of stability.

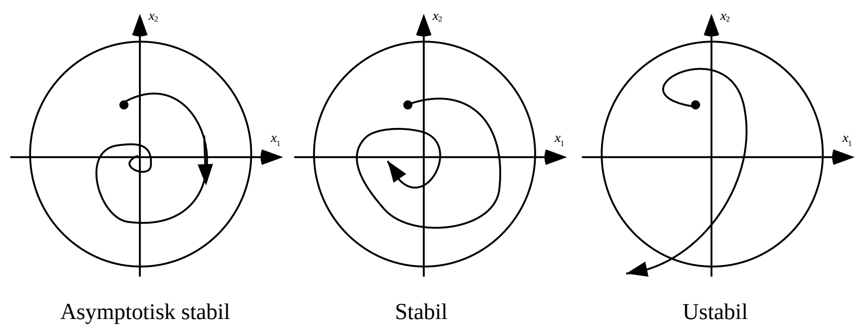


Figure 2.26 *The figure illustrates the behaviour of the solutions* $\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$ *associated with a system of the type* (2.41) *for an* 2×2 *matrix. If the system is asymptotically stable then* $\mathbf{x}(t)$ *converges towards* $\mathbf{0}$ *as* $t \to \infty$; *on the other hand, if we only know that the system is stable we can only say that* $\mathbf{x}(t)$ *stays inside a ball in* $\mathbb{R}^2$ *with radius* K, *see* (2.42). *For an instable system the vectors* $\mathbf{x}(t)$ *might not stay inside a ball.*

According to the definition we can check the stability properties for a system by studying the behavior of the solutions as $t \to \infty$. We will illustrate this with some examples; after that we will show that there exist more simple methods that allow us to characterize the stability properties directly from the matrix describing the system, without having to find the solutions to the system of differential equations.

Example 2.27 Consider the homogeneous system

$$\begin{cases} \dot{x}_1(t) = -x_2(t) \\ \dot{x}_2(t) = -x_1(t) \end{cases}$$

for $t \geq 0$. On matrix form the system has the form

$$\dot{\mathbf{x}}(t) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{x}(t), \ t \geq 0.$$

According to Example 2.7 (with $\omega = 1$) the system has the solutions

$$\mathbf{x}(t) = c_1 \begin{pmatrix} \sin t \\ \cos t \end{pmatrix} + c_2 \begin{pmatrix} -\cos t \\ \sin t \end{pmatrix} = \begin{pmatrix} c_1 \sin t - c_2 \cos t \\ c_1 \cos t + c_2 \sin t \end{pmatrix}.$$

Thus the coordinate functions are

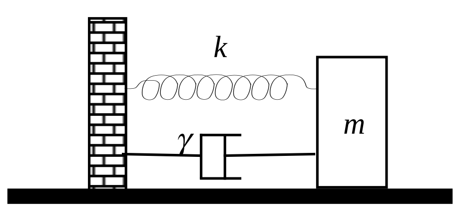
$$x_1(t) = c_1 \sin t - c_2 \cos t \text{ and } x_2(t) = c_1 \cos t + c_2 \sin t$$

In order to show that the system is stable we must verify that the coordinate functions are bounded. We see that

$$\begin{aligned} |x_1(t)| = |c_1 \sin t - c_2 \cos t| &\leq |c_1 \sin t| + |c_2 \cos t| \\ &= |c_1|\,|\sin t| + |c_2|\,|\cos t| \\ &\leq |c_1| + |c_2|. \end{aligned}$$

This shows that $|x_1(t)|$ is bounded; similarly we can show that $x_2(t)$ is bounded. Thus the system is stable. On the other hand, since the coordinate functions do not tend to zero as $t \to \infty$ (except for the special solution with $c_1 = c_2 = 0$), the system is not asymptotically stable. $\square$

Example 2.28 Consider a mass-spring-damper system as shown on the figure.



In Example 1.2 we have modelled the oscillations of the system via the differential equation

$$\ddot{y}(t) + \frac{\gamma}{m}\dot{y}(t) + \frac{k}{m}y(t) = 0, \ t \geq 0, \tag{2.43}$$

where m is the mass, $\gamma > 0$ is the damping constant, $k > 0$ is the spring constant, and $y(t)$ is the position of the mass (relatively to its equilibrium) as a function of the time t. As discussed in Section 2.1 we can rephrase the differential equation as a system of differential equations using the substitution $x_1 = y, x_2 = \dot{y}$; the resulting system is

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{\gamma}{m} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \tag{2.44}$$

The eigenvalues for the system matrix in (2.44) are exactly the roots in the characteristic equation for the differential equation (2.43), i.e., the roots in the polynomial

$$\lambda^2 + \frac{\gamma}{m}\lambda + \frac{k}{m} = 0.$$

In the case $\frac{k}{m} > \frac{\gamma^2}{4m^2}$ we obtain two (complex) roots given by

$$\lambda_{1,2} = -d \pm i\omega, \text{ where } d = \frac{\gamma}{2m} > 0, \ \omega = \sqrt{\frac{k}{m} - \frac{\gamma^2}{4m^2}} > 0.$$

The fundamental matrix $\mathbf{\Phi}_0$ for which $\mathbf{\Phi}_0(0) = \mathbf{I}$ is

$$\begin{aligned} \mathbf{\Phi}_0(t) &= \begin{pmatrix} e^{-dt}(\cos\omega t + \frac{d}{\omega}\sin\omega t) & e^{-dt}(\frac{1}{\omega}\sin\omega t) \\ e^{-dt}(-\frac{d^2+\omega^2}{\omega}\sin\omega t) & e^{-dt}(\cos\omega t - \frac{d}{\omega}\sin\omega t) \end{pmatrix} \\ &= e^{-dt}\begin{pmatrix} \cos\omega t + \frac{d}{\omega}\sin\omega t & \frac{1}{\omega}\sin\omega t \\ -\frac{d^2+\omega^2}{\omega}\sin\omega t & \cos\omega t - \frac{d}{\omega}\sin\omega t \end{pmatrix}. \end{aligned}$$

From Theorem 2.15 we know that every solution $\mathbf{x}(t)$ has the form $\mathbf{x}(t) = \mathbf{\Phi}_0(t)\mathbf{x}_0, \mathbf{x}_0 \in \mathbb{R}^2$. Since $d > 0$ and $|\cos\omega t| \leq 1, |\sin\omega t| \leq 1$, we can use the same method as in Example 2.27 and show that all these solutions are bounded and tend to $\mathbf{0}$ as $t \to \infty$. Thus the system is asymptotically stable (which is not surprising, as we are modelling a damped oscillation). □

Example 2.29 The homogeneous system

$$\begin{cases} \dot{x}_1(t) = -\frac{1}{3}x_1(t) + \frac{2}{3}x_2(t) \\ \dot{x}_2(t) = \frac{4}{3}x_1(t) + \frac{1}{3}x_2(t), \end{cases}$$

considered for $t \geq 0$, has the matrix form

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) = \begin{pmatrix} -\frac{1}{3} & \frac{2}{3} \\ \frac{4}{3} & \frac{1}{3} \end{pmatrix} \mathbf{x}(t), \ t \geq 0. \tag{2.45}$$

Using the methods in Section 2.2 one can show that the system has the solution

$$\mathbf{x}(t) = \begin{pmatrix} e^t \\ 2e^t \end{pmatrix}.$$

Since $e^t \to \infty$ as $t \to \infty$ the solution is unbounded. Thus the system is instable. $\square$

In practice it is inconvenient to be forced to find all solutions to a system of differential equations in order to decide whether it is stable: if a modelling leads to an instable system one will usually change the modelling (or various free parameters) in order to obtain a stable system, and in this sense the work of finding the solutions is superfluous. We will now prove some very important results, telling us that we can characterize the stability properties directly based on the eigenvalues of the system matrix. For this purpose we will first consider the structure of the solutions to the homogeneous system (2.41).

Example 2.30 Let us consider a system of the form (2.41) consisting of two equations. Thus the solutions $\mathbf{x}(t)$ have two coordinates,

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}.$$

We will analyze the form of the coordinate functions $x_1(t)$ and $x_2(t)$. We consider various cases:

(i) Assume that λ is a real eigenvalue for the matrix $\mathbf{A}$, with an associated nonzero eigenvector $\mathbf{v}$. According the Theorem 2.2 the system (2.41) has the solution

$$\mathbf{x}(t) = e^{\lambda t}\mathbf{v}.$$

Thus the coordinate functions $x_1(t)$ and $x_2(t)$ have the same form, so let us just consider $x_1(t)$. Denoting the first coordinate of the vector $\mathbf{v}$ by v_1 we can write $x_1(t) = v_1 e^{\lambda t}$. Let us assume that v_1 is real and positive. If $\lambda > 0$ it follows that x_1 is an increasing exponential function; if $\lambda < 0$ we obtain a decreasing exponential function. If $\lambda = 0$ we obtain a constant function. The behaviour of these functions is shown on the left part of Figure 2.33.
(ii) If λ is complex, $\lambda = a + i\omega$, then similar considerations lead to

$$x_1(t) = v_1 e^{\lambda t} = v_1 e^{(a+i\omega)t} = v_1 e^{at}(\cos\omega t + i\sin\omega t).$$

If v_1 is real, the real part of x_1 is given by

$$Re(x_1(t)) = v_1 e^{at} cos\omega t.$$

The real part of x_1 is shown on the right part of Figure 2.33, for the cases $a > 0, a < 0, a = 0$.

(iii) If the eigenvalue λ appears with a geometric multiplicity equal to its algebraic multiplicity, all solutions have the described form. But if the geometric multiplicity of λ is smaller than the algebraic multiplicity, Theorem 2.9 shows that we also obtain solutions of the form

$$\mathbf{x}(t) = \mathbf{b}_{21} e^{\lambda t} + \mathbf{b}_{22} t e^{\lambda t}. \tag{2.46}$$

Here $\mathbf{b}_{21}$ and $\mathbf{b}_{22}$ are vectors; let us denote their first coordinates by resp. u_1 and v_1. Using this notation the first coordinate of the solution $\mathbf{x}(t)$ in (2.46) has the form

$$x_1(t) = u_1 e^{\lambda t} + v_1 t e^{\lambda t}.$$

The behavior of this solution (resp. its real part in case it is complex) is shown on Figure 2.34. □

The following results show how the stability properties for a system can be analyzed using knowledge of the eigenvalues.

Theorem 2.31 (Stability) *The linear homogeneous system* (2.41) *is stable if and only if the following two conditions are satisfied:*

(i) no eigenvalue for the system matrix $\mathbf{A}$ *has a positive real part, and*

(ii) every eigenvalue for the system matrix $\mathbf{A}$ *having real part equal to zero, has the same algebraic and geometric multiplicity.*

Note that if $\mathbf{A}$ has n pairwise different eigenvalues, then $p = q = 1$ for all eigenvalues. Thus, in this case we only need to check that no eigenvalue has a positive real part.

Corollary 2.32 *If* $\mathbf{A}$ *has* n *pairwise different eigenvalues, then the system* (2.41) *is stable if and only if no eigenvalue for* $\mathbf{A}$ *has a positive real part.*

Concerning asymptotically stability we have the following important characterization:

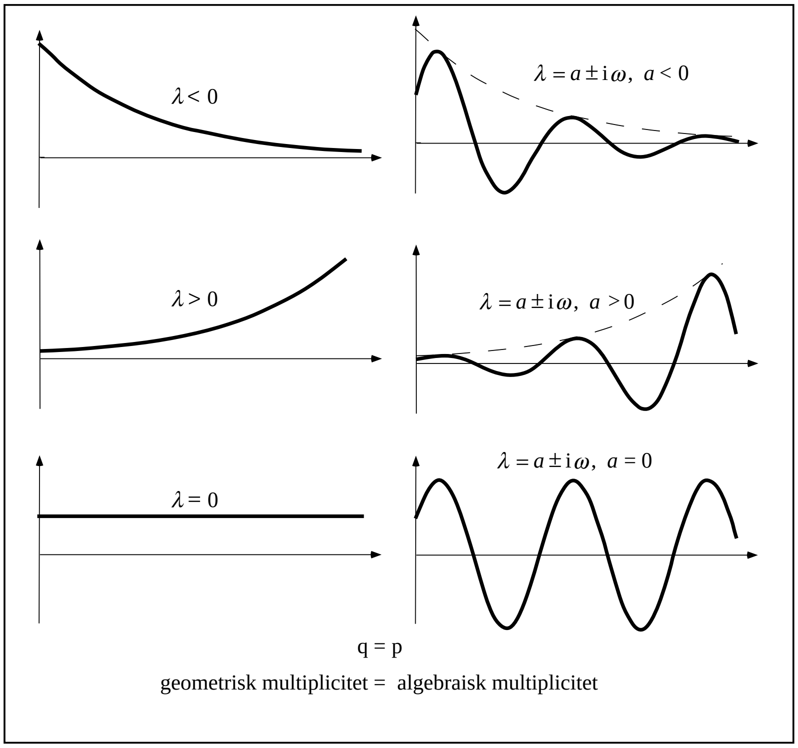


Figure 2.33

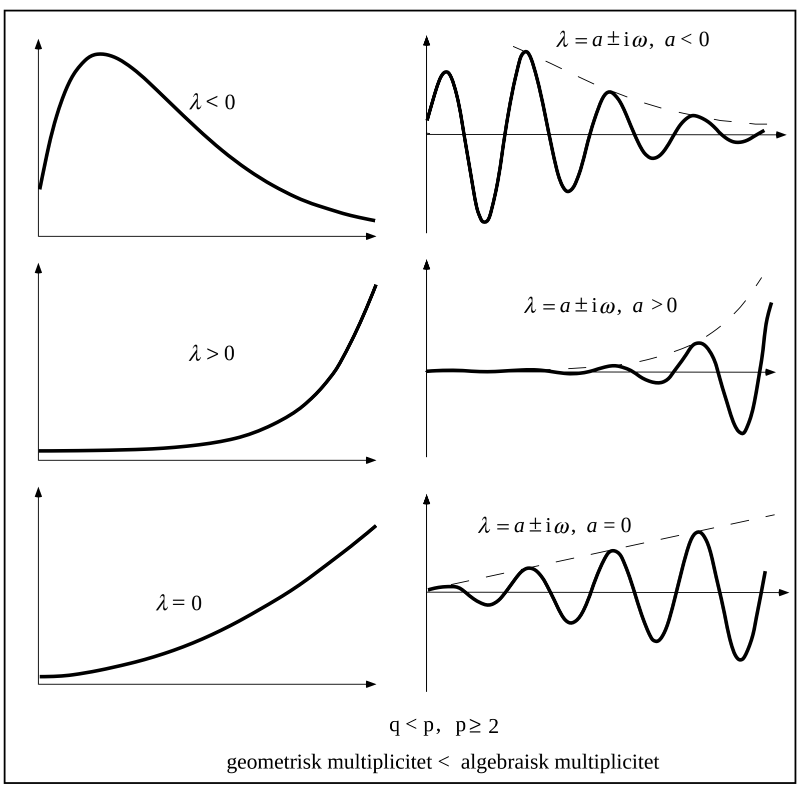


Figure 2.34

Theorem 2.35 (Asymptotically stability) *The system* (2.41) *is asymptotically stable if and only if all eigenvalues for the system matrix* $\mathbf{A}$ *have a negative real part.*

Proof: Theorem 2.9 shows how to find a basis for the vector space of all solutions to the system (2.41). All these solutions tend to $\mathbf{0}$ as $t \to \infty$, if and only if all eigenvalues have a negative real part. $\square$

Example 2.36 Consider the system $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t),\ t \geq 0$, where

$$\mathbf{A} = \begin{pmatrix} 2 & -2 \\ 2 & -2 \end{pmatrix}.$$

The characteristic polynomial is

$$P(\lambda) = (2-\lambda)(-2-\lambda) + 4 = \lambda^2.$$

Therefor the only eigenvalue is $\lambda = 0$ and its algebraic multiplicity is $p = 2$. All solutions to the system $\mathbf{A}\mathbf{v} = \mathbf{0}$ have the form

$$\mathbf{v} = c \begin{pmatrix} 1 \\ 1 \end{pmatrix};$$

thus the geometric multiplicity is $q = 1$. According to Theorem 2.31 the system is instable. $\square$

Example 2.37

(i) The matrix

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

considered in Example 2.27 has the eigenvalues $\pm i$, both with algebraic multiplicity and geometric multiplicity $p = q = 1$. Thus the system $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t), t \geq 0$ is stable, but not asymptotically stable.

(ii) The matrix

$$\mathbf{A} = \begin{pmatrix} -1/3 & 2/3 \\ 4/3 & 1/3 \end{pmatrix}$$

considered in Example 2.29 has the eigenvalues ± 1. Thus the corresponding homogeneous system is instable.

$\square$

According to Theorem 2.35 we can check whether a system is asymptotically stable by calculating all eigenvalues. However, the characteristic polynomial has degree n, and analytic expressions for the roots are only known whenever $n \leq 4$. Furthermore the modelling of concrete system often involve certain parameters that will also occur in the characteristic polynomials. Thus it is very important to have tools to characterize the stability properties without the need of calculating the eigenvalues explicitly. One of the most important results in this direction is the so-called *Routh-Hurwitz' criterion*. We will formulate the result for polynomials of degree n with the additional property that the coefficient corresponding to the term λ^n is 1:

Theorem 2.38 (Routh-Hurwitz' criterion) *All roots in a polynomial with real coefficients of the form*

$$P(\lambda) = \lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n \tag{2.47}$$

have negative real parts, if and only if the following two conditions are satisfied:

(i) all the coefficients are positive, i.e., $a_k > 0$ *for* $k = 1, \ldots, n$*;*

(ii) all $k \times k$ *determinants,* $k = 2, \ldots, n-1$*, of the form*

$$D_k = det \begin{pmatrix} a_1 & a_3 & a_5 & a_7 & \cdot & \cdot & a_{2k-1} \\ 1 & a_2 & a_4 & \cdot & \cdot & \cdot & a_{2k-2} \\ 0 & a_1 & a_3 & a_5 & \cdot & \cdot & a_{2k-3} \\ 0 & 1 & a_2 & \cdot & \cdot & \cdot & a_{2k-4} \\ 0 & 0 & a_1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & a_k \end{pmatrix} \tag{2.48}$$

are positive, i.e., $D_k > 0$ *(in the expression for* D_k *we put* $a_p = 0$ *for* $p > n$*).*

We will now consider the explicit forms of the Routh-Hurwitz' criterion for polynomials of degree up to four. First, for a polynomial of degree two,

$$P(\lambda) = \lambda^2 + a_1\lambda + a_2$$

the condition concerning D_k is void. We therefore obtain the following result:

Corollary 2.39 *All the roots in a polynomial of degree two with real coefficients,*

$$P(\lambda) = \lambda^2 + a_1\lambda + a_2,$$

have a negative real part if and only if

$$a_1 > 0, \quad a_2 > 0. \tag{2.49}$$

Considering now a polynomial of degree three,

$$P(\lambda) = \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3$$

we also need to consider the determinant D_2 for an 2×2 matrix:

Corollary 2.40 *All the roots in a polynomial of degree three with real coefficients,*

$$P(\lambda) = \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3$$

have a negative real part if and only if

$$a_1 > 0, \quad a_2 > 0, \quad a_3 > 0, \quad det \begin{pmatrix} a_1 & a_3 \\ 1 & a_2 \end{pmatrix} > 0. \tag{2.50}$$

Let us finally consider a polynomial of degree four,

$$P(\lambda) = \lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4.$$

We have already found the determinant D_2, and in order to calculate D_3 we take $a_5 = 0$. We then obtain the following result:

Corollary 2.41 *All the roots in a polynomial of degree four with real coefficients,*

$$P(\lambda) = \lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4$$

have negative real parts, if and only if

$$a_1 > 0,\ a_2 > 0,\ a_3 > 0,\ a_4 > 0,\ det \begin{pmatrix} a_1 & a_3 \\ 1 & a_2 \end{pmatrix} > 0,\ det \begin{pmatrix} a_1 & a_3 & 0 \\ 1 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{pmatrix} > 0.$$

Example 2.42 We would like to determine the values of the parameter a for which the system

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) \text{ with } \mathbf{A} = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ a & -1 & -1 \end{pmatrix}$$

is asymptotically stable. The characteristic polynomial is

$$P(\lambda) = \det \begin{pmatrix} -\lambda & -1 & 0 \\ 0 & -\lambda & 1 \\ a & -1 & -1-\lambda \end{pmatrix} = -(\lambda^3 + \lambda^2 + \lambda + a).$$

In order to apply Corollary 2.40 the coefficient corresponding to the term λ^3 must be 1. Thus we will use that

$$P(\lambda) = 0 \Leftrightarrow \lambda^3 + \lambda^2 + \lambda + a = 0.$$

By Corollary 2.40 the system is asymptotically stable if and only if

$$a > 0 \text{ og } \det \begin{pmatrix} 1 & a \\ 1 & 1 \end{pmatrix} = 1 - a > 0,$$

i.e., if and only if $0 < a < 1$. □

2.9 Stability for inhomogeneous systems

We will now consider the linear inhomogeneous system

$$\dot{\mathbf{x}}(t) \quad = \quad \mathbf{A}\mathbf{x}(t) + \mathbf{u}(t), \ t \in [t_0, \infty[. \tag{2.51}$$

A natural question is how a small change in the initial conditions change the behaviour of the solutions. In order to describe this we will introduce the following definition:

Definition 2.43 (Asymptotically stability) *The system* (2.51) *is asymptotically stable if it for every forcing function* $\mathbf{u}$ *holds that two arbitrary solutions* $\mathbf{x}_1(t)$ *and* $\mathbf{x}_2(t)$ *to the system satisfy that*

$$\mathbf{x}_1(t) - \mathbf{x}_2(t) \to \mathbf{0} \textit{ for } t \to \infty.$$

Thus, asymptotically stability of the system (2.51) means that the difference between any two solutions will tend to zero when the time tends to infinity.

It turns out that asymptotically stability of the inhomogeneous system (2.51) is equivalent with asymptotically stability of the associated homogeneous system:

Theorem 2.44 (Asymptotically stability) *The inhomogeneous system* (2.51) *is asymptotically stable if and only if the associated homogeneous system* $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)$ *is asymptotically stable.*

Proof: Assume first that the homogeneous system $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)$ is asymptotically stable. Let $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ denote two arbitrary solutions to the inhomogeneous system, i.e.,

$$\dot{\mathbf{x}}_1(t) = \mathbf{A}\mathbf{x}_1(t) + \mathbf{u}(t), \quad \dot{\mathbf{x}}_2(t) = \mathbf{A}\mathbf{x}_2(t) + \mathbf{u}(t).$$

Then

$$\dot{\mathbf{x}}_1(t) - \dot{\mathbf{x}}_2(t) = \mathbf{A}(\mathbf{x}_1(t) - \mathbf{x}_2(t)),$$

meaning that $\mathbf{x}_1(t) - \mathbf{x}_2(t)$ is a solution to the homogeneous system. Since this system is assumed to be asymptotically stable it follows that

$$\mathbf{x}_1(t) - \mathbf{x}_2(t) \to \mathbf{0} \text{ as } t \to \infty.$$

Thus the inhomogeneous system is asymptotically stable.

On the other hand, assume that the inhomogeneous system is asymptotically stable. Let $\mathbf{x}(t)$ denote an arbitrary solution to the homogeneous system $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)$. Note that also $\mathbf{0}$ is a solution to the homogeneous system. Since the homogeneous system appear from the inhomogeneous system by the choice $\mathbf{u} = \mathbf{0}$, it now follows from the asymptotically stability of the inhomogeneous system, that

$$\mathbf{x}(t) = \mathbf{x}(t) - \mathbf{0} \to \mathbf{0} \text{ for } t \to \infty. \qquad \square$$

Another important issue in connection with the system (2.51) is whether bounded forcing functions $\mathbf{u}$ also lead to bounded solutions $\mathbf{x}$:

Definition 2.45 (BIBO-stability) *The inhomogeneous system* (2.51) *is said to be BIBO-stable if the solutions* $\mathbf{x}(t), t \in [t_0, \infty[$ *associated to any bounded choice of the forcing function* $\mathbf{u}$ *are bounded.*

The short name "BIBO" comes from the expression "Bounded Input – Bounded Output". It turns out that this type of stability is equivalent with asymptotically stability:

Theorem 2.46 (BIBO-stability) *The inhomogeneous system* (2.51) *is BIBO-stable if and only if the associated homogeneous system* $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)$ *is asymptotically stable.*

Example 2.47 Consider the system

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{u}(t) \text{ med } \mathbf{A} = \begin{pmatrix} -1 & -1 & 0 \\ 0 & -1 & -1 \\ -1 & 0 & -1 \end{pmatrix}.$$

The characteristic polynomial is

$$P(\lambda) = \det(\mathbf{A} - \lambda\mathbf{I}) = (-1-\lambda)^3 - 1 = -(\lambda^3 + 3\lambda^2 + 3\lambda + 2).$$

Thus

$$P(\lambda) = 0 \Leftrightarrow \lambda^3 + 3\lambda^2 + 3\lambda + 2 = 0.$$

We now apply 2.40 with $a_1 = 3, a_2 = 3, a_3 = 2$: since

$$a_1 > 0, \quad a_2 > 0, \quad a_3 > 0, \quad \det\begin{pmatrix} a_1 & a_3 \\ 1 & a_2 \end{pmatrix} = 7 > 0,$$

we conclude that all three eigenvalues have a negative real part. Thus the system is asymptotically stable. □

Let us finally return to the system (2.32) and relate the asymptotical stability and the stationary solutions:

Example 2.48 Assume that (2.32) is asymptotiscally stable, and consider and arbitrary $\omega > 0$. According to Theorem 2.35 the complex number $i\omega$ is not an eigenvalue for the matrix $\mathbf{A}$, i.e., $i\omega$ is not a root in the characteristic equation. Letting $u(t) = e^{i\omega t}$, the associated stationary solution is according to Theorem 2.20 given by

$$y(t) = H(i\omega)e^{i\omega t}.$$

The general solution to (2.32) is therefore

$$y(t) = H(i\omega)e^{i\omega t} + y_h(t),$$

where y_h denotes the general solution to the associated homogeneous equation. But according to the definition of asymptotical stability we know that

$$y_h(t) \to 0 \text{ for } t \to \infty.$$

Thus, for an arbitrary solution y to (2.32),

$$y(t) \approx H(i\omega)e^{i\omega t}$$

for large values of t. Thus, every solution to (2.32) will behave like the stationary solution after sufficiently long time. $\square$

3

Nonlinear differential equations

In this chapter we will give a very short introduction to nonlinear differential equations. One of the problems with such differential equations is that it often is impossible to find the solutions explicitly; thus, we either have to use methods describing the qualitative behaviour of the differential equation, various methods to find approximate solutions, or numerical solutions. We will present a few methods to analyze nonlinear differential equations using stability techniques.

3.1 Introduction

Laet $f_1, f_2, \dots, f_n$ denote functions defined on $\mathbb{R}^n$ and taking values in $\mathbb{R}$. We will consider a system of differential equations of the form

$$\begin{cases} \dot{x}_1(t) = f_1(x_1(t), x_2(t), \dots, x_n(t)), \\ \dot{x}_2(t) = f_2(x_1(t), x_2(t), \dots, x_n(t)), \\ \qquad \cdot \\ \qquad \cdot \\ \dot{x}_n(t) = f_n(x_1(t), x_2(t), \dots, x_n(t)). \end{cases} \tag{3.1}$$

Using the vector notation

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \cdot \\ \cdot \\ x_n(t) \end{pmatrix}, \quad \mathbf{f}(\mathbf{x}(t)) = \begin{pmatrix} f_1(x_1(t), x_2(t), \dots, x_n(t)) \\ f_2(x_1(t), x_2(t), \dots, x_n(t)) \\ \cdot \\ \cdot \\ f_n(x_1(t), x_2(t), \dots, x_n(t)) \end{pmatrix}$$

we can write the system (3.1) as

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(\mathbf{t})). \tag{3.2}$$

Note that the homogeneous linear system of differential equations analyzed in Section 2,

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t), \tag{3.3}$$

is a special case of (3.1), corresponding to the case where the function $\mathbf{f}$ is linear,

$$\mathbf{f}(\mathbf{x}(\mathbf{t})) = \mathbf{A}\mathbf{x}(\mathbf{t}). \tag{3.4}$$

In the general case a system of differential equations of the form (3.2) is called a *nonlinear autonomous system.* The word autonomous is used to differentiate the class of systems of differential equations from the more general class of systems having the form $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}(t), t)$. Note that the system (3.3) is autonomous, while the inhomogeneous system

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{u}(t),$$

is not autonomous, unless the function $\mathbf{u}$ is constant.

Example 3.1 The system

$$\begin{cases} \dot{x}_1(t) = & x_1(t) + 3x_1(t)x_2(t), \\ \dot{x}_2(t) = & x_1(t) + x_1(t)x_2(t) \end{cases}$$

is an autonomous nonlinear system. Usually we suppress the dependence on the variable t in the notation and simply write

$$\begin{cases} \dot{x}_1 = & x_1 + 3x_1x_2, \\ \dot{x}_2 = & x_1 + x_1x_2 \end{cases}$$

This shows that the function $\mathbf{f}$ in (3.2) is

$$\mathbf{f}(x_1, x_2) = \begin{pmatrix} x_1 + 3x_1x_2 \\ x_1 + x_1x_2 \end{pmatrix}.$$

The system

$$\begin{cases} \dot{x}_1(t) = & x_1(t) + 3x_1(t)x_2(t) + t^2, \\ \dot{x}_2(t) = & x_1(t) + x_1(t)x_2(t) \end{cases}$$

is a nonlinear system, but it is not autonomous; indeed, the left-hand side not only depends on the functions $x_1(t)$ and $x_2(t)$, but also directly on t.
□

The existence and uniqueness theorem is also valid for autonomous nonlinear systems of differential equations:

Theorem 3.2 (Existence and uniqueness theorem) *Assume that the function* $\mathbf{f}$ *has continuous partial derivatives. Then for each* $t_0 \in \mathbb{R}$ *and each* $\mathbf{v} \in \mathbb{R}^n$ *the system* (3.2) *has a unique solution* $\mathbf{x}(t)$ *satisfying the initial condition* $\mathbf{x}(t_0) = \mathbf{v}$.

Unfortunately it is often impossible to find an explicit solution to (3.2) in terms of elementary functions. Thus, an important question is how to analyze the system and the behavior of its solutions. We will now give an introduction to some of the standard tools.

In order to get an overview of the solutions to (3.2) we will use the so-called phase portraits. A *phase portrait* consists of a sketch of selected solutions, corresponding to a choice of initial conditions. Thus, for $n = 2$ we obtain curves in $\mathbb{R}^2$, consisting of points of the form $(x_1(t), x_2(t)),\ t \in \mathbb{R}$. We note that these curves to not reveal information about the value of t corresponding to the point $(x_1(t), x_2(t))$ on the curve.

In case we do not know the solutions to a two-dimensional system of differential equations we can not use the definition of the phase portrait directly. Instead, we will use that a solution $\mathbf{x}$ at a given time t_0 pass the point $\mathbf{x}(t_0)$, and that the tangent is given by

$$\varphi(t) = \dot{\mathbf{x}}(t_0)(t - t_0) + \mathbf{x}(t_0).$$

Thus, at the point $\mathbf{x}(t_0)$ the solution has a direction given by the vector

$$\dot{\mathbf{x}}(t_0) = \mathbf{f}(\mathbf{x}(t_0)).$$

At any given point (x_1, x_2) we can now consider the solution $\mathbf{x}(t)$, which at time t_0 satisfy that $\mathbf{x}(t_0) = (x_1, x_2)$; according to the above discussion this solution has at the point (x_1, x_2) a direction determined by the vector $\mathbf{f}(\mathbf{x}(t_0)) = \mathbf{f}(x_1, x_2)$. The phase portrait can now be sketched by marking the direction of the vector $\mathbf{f}(x_1, x_2)$ for a collection of points (x_1, x_2).

It follows that for the considered autonomous systems the phase portrait is independent of the choice of t_0. For this reason we will often simply consider $t_0 = 0$.

Example 3.3 Consider the system of differential equations

$$\begin{cases} \dot{x}_1 = & x_2, \\ \dot{x}_2 = & x_1 + x_1 x_2. \end{cases}$$

At the point $(1, 0)$ we have

$$\begin{pmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{pmatrix} = \begin{pmatrix} x_2 \\ x_1 + x_1 x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix};$$

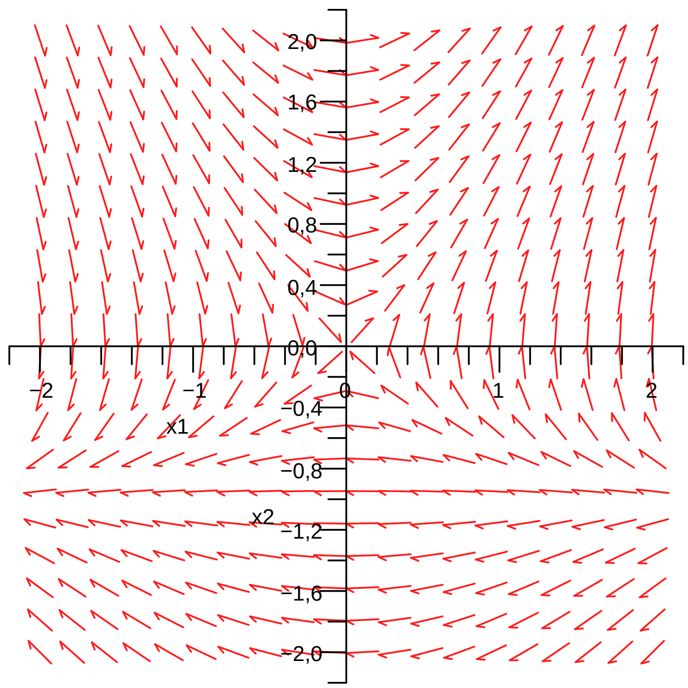


Figure 3.4 *A phase portrait for the system in Example 3.3.*

thus, the phase portrait marks en vector in that direction at the point $(1, 0)$. Following this procedure for a large number of points leads to a phase portrait as shown on Figure 3.4. In practice a phase portrait is usually made using Maple. □

Frequently there exist solution curves for (3.2) consisting of a single point. They play an important role in the theory, so we need the following definition.

Definition 3.5 (Stationary point) *Consider an autonomous system* (3.2)*. A point* $\mathbf{x}_s \in \mathbb{R}^n$ *for which* $\mathbf{f}(\mathbf{x_s}) = \mathbf{0}$ *is called a stationary point.*

Note that if $\mathbf{x}_s$ is a stationary point, the constant function

$$\mathbf{x}(t) = \mathbf{x}_s$$

is a solution to (3.2).

As already mentioned the choice of the "starting time" t_0 does not play any role for the behaviour of the solution curves, so in the sequel we will choose $t_0 = 0$. Let us now consider an initial condition $\mathbf{x}(0) = \mathbf{v}$, where $\mathbf{v}$ is "close" to a stationary point $\mathbf{x_s}$. According to the existence and uniqueness theorem, exactly one solution passes the point $\mathbf{v}$ at the time $t = 0$. Assume for a moment that the solution curve will be *attracted* to the the stationary point, i.e., that

$$\mathbf{x}(t) \to \mathbf{x_s} \text{ for } t \to \infty. \tag{3.5}$$

If (3.5) happens for all initial conditions sufficiently close to $\mathbf{x}_s$, we say that the stationary point is *asymptotically stable.* Using the "distance" in (2.42) we will now give a precise definition, followed by an explanation in words.

Definition 3.6 (Stability for nonlinear system)

(i) A stationary point $\mathbf{x}_s \in \mathbb{R}^n$ is stable if there for every $\epsilon > 0$ exists a $\delta > 0$ such that if $\mathbf{x}(t)$ is a solution with $||\mathbf{x}(0) - \mathbf{x}_s|| < \delta$, then $||\mathbf{x}(t) - \mathbf{x}_s|| < \epsilon$ for all $t > 0$.

(ii) A stationary point $\mathbf{x}_s \in \mathbb{R}^n$ is asymptotically stable if it is stable and there exists a $\delta > 0$ such that if $\mathbf{x}(t)$ is a solution and

$$||\mathbf{x}(0) - \mathbf{x_s}|| < \delta,$$

then

$$\mathbf{x}(t) \to \mathbf{x_s} \textit{ as } t \to \infty.$$

A stationary point that is not stable, is said to be instable.

In words Definition 3.6 expresses the following:

- A stationary point $\mathbf{x}_s$ is stable if every solution that starts sufficiently close to $\mathbf{x}_s$ will stay close to $\mathbf{x}_s$ for all $t \geq t_0$;
- A stationary point is asymptotically stable if it is stable and every solution starting sufficiently close to $\mathbf{x}_s$ will converge to $\mathbf{x}_s$ as $t \to \infty$.

Note that this wording correspond precisely with the geometric intuition gained by examining Figure 2.26 in our discussion of stability for linear systems.

We will now state a result which shows that the type of stability occuring at a stationary point can be determined via methods that are parallel to the methods used in Section 2.8. Given the function $\mathbf{f} : \mathbb{R}^\mathbf{n} \to \mathbb{R}^\mathbf{n}$, the *functional matrix* or *Jacobi matrix* at the point $\mathbf{x} \in \mathbb{R}^n$ is defined by

$$D\mathbf{f}(\mathbf{x}) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(\mathbf{x}) & \frac{\partial f_1}{\partial x_2}(\mathbf{x}) & \cdot & \cdot & \frac{\partial f_1}{\partial x_n}(\mathbf{x}) \\ \frac{\partial f_2}{\partial x_1}(\mathbf{x}) & \frac{\partial f_2}{\partial x_2}(\mathbf{x}) & \cdot & \cdot & \frac{\partial f_2}{\partial x_n}(\mathbf{x}) \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{\partial f_n}{\partial x_1}(\mathbf{x}) & \frac{\partial f_n}{\partial x_2}(\mathbf{x}) & \cdot & \cdot & \frac{\partial f_n}{\partial x_n}(\mathbf{x}) \end{pmatrix}$$

Theorem 3.7 (Stability for nonlinear system)

(i) A stationary point $\mathbf{x}_s$ is instable if $D\mathbf{f}(\mathbf{x}_s)$ has an eigenvalue with positive real part.

(ii) A stationary point $\mathbf{x}_s$ is asymptotically stable if all eigenvalues for $D\mathbf{f}(\mathbf{x}_s)$ have negative real part.

Thus, to check whether a stationary point is asymptotically stable is done in a similar way as for the linear system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, see Theorem 2.35: we just have to replace the system matrix $\mathbf{A}$ by the functional matrix evaluated at the stationary point. However, we also note that if there exists an eigenvalue with real part equal to zero, the analysis can be complicated: the above theorem does not always provide a solution in that case.

Example 3.8 The system in Example 3.3,

$$\begin{cases} \dot{x}_1 = & x_2, \\ \dot{x}_2 = & x_1 + x_1 x_2 \end{cases}$$

is of the type (3.2), with

$$\mathbf{f}(x_1, x_2) = \begin{pmatrix} x_2 \\ x_1 + x_1 x_2 \end{pmatrix}.$$

Since the function $\mathbf{f}$ is nonlinear, the system is a nonlinear autonomous system. In order to find the stationary points we consider the equation

$$\mathbf{f}(x_1, x_2) = \mathbf{0}.$$

The only solution is $\mathbf{x}_s = (x_1, x_2) = (0, 0)$, which is thus the only stationary point.

The functional matrix is

$$D\mathbf{f}(\mathbf{x}) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 + x_2 & x_1 \end{pmatrix};$$

thus

$$D\mathbf{f}(\mathbf{x}_s) = D\mathbf{f}(0, 0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The eigenvalues for $D\mathbf{f}(0, 0)$ are $\lambda = \pm 1$; thus the stationary point $(0, 0)$ is instable. This corresponds to the intuition gained by considering Figure 3.4. □

4

Infinite series with constant terms

In this chapter we introduce the theory for infinite series. Later, in Chapter 7, we will show how infinite series can be used as a tool to solve differential equations.

The analysis of infinite series requires certain tools, which we introduce first. In Section 4.1 we discuss integrals over infinite intervals. Section 4.2 introduces sequences of numbers, and Section 4.3 deals with Taylor's theorem. Section 4.4 states the definition of an infinite series; standard tests to decide whether a given infinite series is convergent or not are presented in Section 4.5. In Section 4.6 we discuss various methods to approximate the sum of an infinite series. Alternating series are introduced in Section 4.7.

4.1 Integrals over infinite intervals

In connection with the theory for infinite series we need to be able to integrate functions f over infinite intervals. A typical example is an integral of the form

$$\int_a^\infty f(x)\,dx, \tag{4.1}$$

where $a \in \mathbb{R}$ is a given scalar. Integrals of the type (4.1) are introduced as follows:

Definition 4.1 *Let $a \in \mathbb{R}$, and consider a piecewise continuous function $f : [a, \infty[\to \mathbb{C}$.*

(i) Assume that the integral $\int_a^t f(x)\,dx$ has a limit as $t \to \infty$. Then we say that the integral $\int_a^\infty f(x)\,dx$ is convergent, and we write

$$\int_a^\infty f(x)\,dx := \lim_{t\to\infty} \int_a^t f(x)\,dx. \tag{4.2}$$

(ii) If the integral $\int_a^t f(x)\,dx$ does not have a limit as $t \to \infty$, we say that the integral $\int_a^\infty f(x)\,dx$ is divergent.

Example 4.2 We will check whether the integral

$$\int_1^\infty \frac{1}{x^2}\,dx$$

is convergent or divergent. Thus, we first calculate the integral

$$\int_1^t \frac{1}{x^2}\,dx = \int_1^t x^{-2}\,dx = \left[-\frac{1}{x}\right]_{x=1}^t = 1 - \frac{1}{t}.$$

Since $1 - \frac{1}{t} \to 1$ as $t \to \infty$, we conclude that $\int_1^\infty \frac{1}{x^2}\,dx$ is convergent, and that

$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^2}\,dx = 1. \qquad \square$$

In many case we will consider integrals that depend on a certain parameter:

Example 4.3 Let $b \in \mathbb{R}$. We want to find the values of b for which the integral

$$\int_1^\infty e^{bx}\,dx$$

is convergent.

We first observe that the integral is divergent for $b = 0$ (check it!). For $b \neq 0$ we get that

$$\int_1^t e^{bx}\,dx = \frac{1}{b}\left[e^{bx}\right]_{x=1}^t = \frac{1}{b}\left(e^{bt} - e^b\right). \tag{4.3}$$

For $b < 0$ we have that $e^{bt} \to 0$ for $t \to \infty$. Therefore $\int_1^\infty e^{bx}\,dx$ is convergent for $b < 0$, and

$$\int_1^\infty e^{bx}\,dx = \lim_{t\to\infty} \int_1^t e^{bx}\,dx = \lim_{t\to\infty} \frac{1}{b}\left(e^{bt} - e^b\right) = \frac{1}{b}\left(0 - e^b\right) = -\frac{e^b}{b}.$$

For $b > 0$ we see that $e^{bt} \to \infty$ for $t \to \infty$. Therefore (4.3) implies that

$$\int_1^t e^{bx}\,dx = \frac{1}{b}\left(e^{bt} - e^b\right) \to \infty \text{ as } t \to \infty.$$

Thus the integral $\int_1^\infty e^{bx}\,dx$ is divergent for $b > 0$. $\square$

4.2 Sequences

In this section we describe an important ingredient in the theory for infinite series, namely, sequences consisting of scalars. Let us first consider finite sequences. A *finite sequence* of scalars is a collection or numbers, which are ordered and written as a list, i.e.,

$$\{x_1, x_2, x_3, \dots, x_N\} \tag{4.4}$$

for some $N \in \mathbb{N}$. Sequences of scalars naturally appear in all types of mathematical programs like Maple, Mathematica, and Matlab:

Example 4.4 All the computer programs that are used to do calculations are working with finite sequences of numbers. Looking, e.g., at the function

$$f(x) = x^2,\ x \in [0, 10],$$

it is natural to consider the finite sequence of numbers obtained by calculating $f(x)$ for $x = 0, 1, \dots, 10$. Writing $x_n := f(n)$, we obtain the sequence

$$\{x_n\}_{n=0}^{10} = \{f(0), f(1), \dots, f(10)\}.$$

We say that the sequence $\{x_n\}_{n=0}^{10}$ appears by *sampling* of the function f, with *sampling points* $0, 1, \dots, 10$. If we need more exact information about the function f we will normally consider more dense sampling points, and consider $x_n = f(an)$ for a small number $a > 0$. Taking, e.g., $a = 10^{-4}$, we obtain the sequence

$$\{x_n\}_{n=0}^{10000} = \{f(0), f(\frac{1}{10000}), f(\frac{2}{10000}), \dots, f(1), \dots, f(10)\}.$$

Note that the number of terms in the sequence increases considerably when we consider denser sampling points. $\square$

In the theory for infinite series we need to consider sequences consisting of infinitely many scalars. Assume that we for each $n \in \mathbb{N}$ have defined a scalar $x_n \in \mathbb{C}$. We can again write the scalars x_n on a list, i.e., of the form

$$\{x_1, x_2, x_3, \dots, x_n, \dots\}. \tag{4.5}$$

Formally we say that $\{x_1, x_2, \ldots\ldots, x_n, \ldots\}$ forms an *infinite sequence.* As a short notation we will write

$$\{x_n\}_{n=1}^{\infty} := \{x_1, x_2, x_3, \ldots, x_n, \ldots\}. \tag{4.6}$$

Frequently we will be less formal, and simply speak about the sequence $x_n,\ n \in \mathbb{N}$.

The following definition is central in the theory for sequences and its applications.

Definition 4.5 *A sequence $\{x_n\}_{n=1}^{\infty}$ is convergent if there exists some $s \in \mathbb{C}$ such that*

$$x_n \to s \text{ as } n \to \infty. \tag{4.7}$$

In this case we call s the limit of the sequence $\{x_n\}_{n=1}^{\infty}$, and we write

$$\lim_{n\to\infty} x_n = s.$$

A sequence that is not convergent, is said to be divergent.

Intuitively the condition (4.7) means that the scalars x_n are approaching s when n tends to infinity. Mathematically this is expressed in an exact way by saying that "for each $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$|s - x_n| \leq \epsilon$$

whenever $n \geq N$." In mathematical symbols we write this as

$$\forall \epsilon > 0\, \exists N \in \mathbb{N} :\ n \geq N \Rightarrow |s - x_n| \leq \epsilon. \tag{4.8}$$

Example 4.6 Consider the sequence

$$x_n = \frac{n+1}{n},\ n \in \mathbb{N}.$$

As well the counter as the denominator tend to infinity as $n \to \infty$, so it is not immediately clear whether the sequence is convergent or not. But

$$x_n = \frac{n+1}{n} = 1 + \frac{1}{n} \to 1 \text{ as } n \to \infty.$$

Thus the sequence x_n is convergent with limit 1. □

Example 4.7 Consider the sequence

$$x_n = \frac{2n+1}{n+1},\ n \in \mathbb{N}.$$

We are again in the situation that the counter and the denominator tend to infinity as $n \to \infty$. Dividing by n we see that

$$x_n = \frac{2n+1}{n+1} = \frac{2+1/n}{1+1/n} \to \frac{2}{1} = 2 \text{ for } n \to \infty.$$

Thus the sequence $\{x_n\}_{n=1}^{\infty}$ is convergent with limit $s = 2$. Let us check that this agrees with the formal definition in (4.8). First, we notice that with $s = 2$,

$$\begin{aligned} |s - x_n| = \left|2 - \frac{2n+1}{n+1}\right| = \left|\frac{2(n+1)}{n+1} - \frac{2n+1}{n+1}\right| &= \left|\frac{2(n+1) - (2n+1)}{n+1}\right| \\ &= \frac{1}{n+1}. \end{aligned}$$

Given $\epsilon > 0$ we now see that the inequality $|s - x_n| \leq \epsilon$ is satisfied whenever $\frac{1}{n+1} \leq \epsilon$, i.e., for

$$n \geq \frac{1}{\epsilon} - 1.$$

Thus the condition (4.8) is satisfied for any choice of $N \in \mathbb{N}$ for which $N \geq \frac{1}{\epsilon} - 1$; we conclude again that the sequence $\{x_n\}_{n=1}^{\infty}$ converges to $s = 2$. $\square$

Very often we can calculate the limit for a given sequence using knowledge of the behavior of elementary functions:

Example 4.8 Consider the following sequences:

(i) $x_n = \cos(n\pi/2),\ n \in \mathbb{N}$;

(ii) $x_n = 2^{-n},\ n \in \mathbb{N}$;

(iii) $x_n = \frac{n^2}{2^n},\ n \in \mathbb{N}$.

(iv) $x_n = \frac{2^n}{n^2},\ n \in \mathbb{N}$.

The sequence in (i) consists of the scalars

$$\begin{aligned} \{x_n\}_{n=1}^{\infty} = \{\cos(n\pi/2)\}_{n=1}^{\infty} &= \{\cos\frac{\pi}{2},\ \cos\pi, \cos\frac{3\pi}{2}, \dots\} \\ &= \{0, -1, 0, 1, 0, -1, 0, 1, \dots\}, \end{aligned}$$

i.e., a periodic repetition of the numbers $0, -1, 0, 1$. Thus the sequence is divergent.

For the sequence in (ii) we see that

$$x_n = 2^{-n} = \frac{1}{2^n}.$$

Since $2^n \to \infty$ as $n \to \infty$ we have that $x_n \to 0$ for $n \to \infty$. Thus the sequence $\{x_n\}_{n=1}^{\infty}$ is convergent with limit $s = 0$.

The sequence in (iii) is more complicated. As well the counter as the denominator tend to infinity as $n \to \infty$. But the function in the counter, i.e.,

n^2, is a polynomial, while the function in the denominator is an exponential function. Thus

$$\frac{n^2}{2^n} \to 0 \text{ for } n \to \infty.$$

We conclude that the sequence in (iii) is convergent with limit $s = 0$.

Comparing the sequences in (iii) and (iv) we now see that

$$x_n = \frac{2^n}{n^2} = \frac{1}{\frac{n^2}{2^n}}. \tag{4.9}$$

We know from (iii) that $\frac{n^2}{2^n} \to 0$ as $n \to \infty$. Thus (4.9) implies that $x_n \to \infty$ as $n \to \infty$. We conclude that the sequence in (iv) is divergent. □

Often we will consider sequences, where the terms depend on a variable x. In this case the question is typically how we can find the values of the variable for which the sequence is convergent.

Example 4.9 For $x \in \mathbb{R}$, consider the sequences

(i) $x_n = x^n$, $n \in \mathbb{N}$;

(ii) $x_n = \frac{\cos(nx)}{2^n}$, $n \in \mathbb{N}$.

For the sequence in (i) we see that if $x \in]-1, 1[$, then

$$x_n = x^n \to 0 \text{ as } n \to \infty;$$

thus the sequence is convergent with limit 0 in this case. For $x = 1$ we have $x_n = 1^n = 1$ for all $n \in \mathbb{N}$, and therefore the sequence is convergent with limit $s = 1$. On the other hand, for $x = -1$ we have $x_n = (-1)^n$, which takes the value 1 when n is even, and the value -1 when n is odd. Therefore the sequence is divergent.

For $x \notin [-1, 1]$, i.e., if $|x| > 1$, we see that $|x_n| \to \infty$ as $n \to \infty$. Therefore the sequence $x_n = x^n$ is divergent for $x \notin [-1, 1]$.

Now consider the sequence in (ii). We immediately see that the counter, i.e., $\cos(nx)$, is convergent for some values of x, and divergent for other values of x. But in this concrete case it turns out that the behavior of the counter is irrelevant! Indeed, the denominator, i.e., 2^n, goes toward infinity as $n \to \infty$, and the counter only takes values in the bounded interval $[-1, 1]$. That is, regardless whether $\cos(nx)$ is convergent or not, we can conclude that

$$x_n = \frac{\cos(nx)}{2^n} \to 0 \text{ as } n \to \infty.$$

Thus the sequence $\{x_n\}_{n=1}^{\infty}$ is convergent with limit $s = 0$ for all $x \in \mathbb{R}$. □

Often it is an advantage to split complicated sequences in simpler parts. For this purpose we will need the following result.

Lemma 4.10 *Assume that the sequence $\{x_n\}_{n=1}^{\infty}$ is convergent with limit x, and that the sequence $\{y_n\}_{n=1}^{\infty}$ is convergent with limit y. Then, for any $\alpha, \beta \in \mathbb{C}$ the sequence $\{\alpha x_n + \beta y_n\}_{n=1}^{\infty}$ is convergent with limit $\alpha x + \beta y$.*

Example 4.11 Consider the sequence

$$x_n = \frac{n+1}{n} + 2\cos(2^{-n}),\ n \in \mathbb{N}.$$

We know that

$$\frac{n+1}{n} = 1 + \frac{1}{n} \to 1 \text{ as } n \to \infty.$$

Furthermore $2^{-n} \to 0$ as $n \to \infty$, so

$$\cos(2^{-n}) \to \cos(0) = 1 \text{ as } n \to \infty.$$

Thus, by Lemma 4.10,

$$x_n = \frac{n+1}{n} + 2\cos(2^{-n}) \to 1 + 2 \cdot 1 = 3 \text{ as } n \to \infty.$$

The sequence $x_n,\ n \in \mathbb{N}$, is therefore convergent with limit $s = 3$. □

4.3 Taylor's theorem

In Taylor's theorem we will approximate functions f via polynomials $P_N,\ N \in \mathbb{N}$, of the form

$$\begin{aligned} &P_N(x) \\ &= f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^{(N)}(x_0)}{N!}(x-x_0)^N \\ &= \sum_{n=0}^{N} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n. \end{aligned}$$

The polynomial P_N is called the *Taylor polynomial of degree N associated to f at the point x_0* or the *Nth Taylor polynomial at x_0*. Observe that for $N = 1$,

$$P_1(x) = f(x_0) + f'(x_0)(x - x_0),$$

which is the equation for the tangent line of f at the point x_0. For higher values of N we get polynomials of higher degree, which usually approximate the function f even better for x close to x_0. Before we see a more precise version of this statement in Theorem 4.13 we will look at an example.

Example 4.12 Let us consider the function

$$f(x) = e^x, \; x \in]-1, 2[.$$

This function is arbitrarily often differentiable in $]-1,2[$, and

$$f'(x) = f''(x) = \cdots = f^{(n)}(x) = \cdots = e^x.$$

For $N \in \mathbb{N}$ the Nth Taylor polynomial at $x_0 = 0$ is given by

$$\begin{aligned} P_N(x) &= f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(N)}(0)}{N!}x^N \\ &= 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{1}{N!}x^N \\ &= \sum_{n=0}^{N} \frac{1}{n!}x^n. \end{aligned} \tag{4.10}$$

Similarly, the Nth Taylor polynomial at $x_0 = 1$ is given by

$$\begin{aligned} P_N(x) &= f(1) + \frac{f'(1)}{1!}(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \cdots + \frac{f^{(N)}(1)}{N!}(x-1)^N \\ &= e + e(x-1) + \frac{e}{2}(x-1)^2 + \cdots + \frac{e}{N!}(x-1)^N. \end{aligned}$$

We do not unfold the terms in P_N, since this would lead to a less transparent expression. But for $N = 1$ we see that

$$P_1(x) = e + e(x-1) = ex,$$

and for $N = 2$,

$$P_2(x) = e + e(x-1) + \frac{e}{2}(x-1)^2 = \frac{e}{2}x^2 + \frac{1}{2}e.$$

The second Taylor polynomial gives a better approximation of f around $x = 1$ than the first Taylor polynomial. One can reach an even better approximation by choosing a Taylor polynomial of higher degree, which however requires more calculations. In the present example it is easy to find the Taylor polynomials of higher degree; in real applications however, there will always be a tradeoff between how good an approximation we want to obtain and how much calculation power we have at our disposal. □

We are now ready to formulate Taylor's theorem. It shows that under certain conditions, we can find a polynomial which is as close to f in a bounded interval containing x_0 as we wish:

Theorem 4.13 *Let $I \subset \mathbb{R}$ be an interval. Assume that the function $f : I \to \mathbb{R}$ is arbitrarily often differentiable and that there exists a constant $C > 0$ such that*

$$\left|f^{(n)}(x)\right| \leq C, \quad \textit{for all } n \in \mathbb{N} \textit{ and all } x \in I. \tag{4.11}$$

Let $x_0 \in I$. Then for all $N \in \mathbb{N}$ and all $x \in I$,

$$\left| f(x) - \sum_{n=0}^{N} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n \right| \leq \frac{C}{(N+1)!} \left|x - x_0\right|^{N+1}. \tag{4.12}$$

In particular, if I is a bounded interval, there exists for arbitrary $\epsilon > 0$ an $N_0 \in \mathbb{N}$ such that

$$\left| f(x) - \sum_{n=0}^{N} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n \right| \leq \epsilon \quad \textit{for all } x \in I \textit{ and all } N \geq N_0. \tag{4.13}$$

The proof of Theorem 4.13 is given in Section A.2.

For convenience we will usually assume that we can choose $x_0 = 0$; under the assumptions in Theorem 4.13 we can therefore approximate f as well as we wish with a polynomial

$$P(x) = \sum_{n=0}^{N} \frac{f^{(n)}(0)}{n!} x^n,$$

by choosing the degree N sufficiently high.

Formula (4.12) can be used to decide how many terms we should include in the Taylor polynomial in order to reach a certain approximation of a given function on a fixed interval. The following example illustrates this for the exponential function, and also shows how the number of terms depend on the considered interval and on how well we want the function to be approximated.

Example 4.14 We consider again the function $f(x) = e^x$. We aim at finding three values of $N \in \mathbb{N}$, namely

(i) $N \in \mathbb{N}$ such that

$$\left| e^x - \sum_{n=0}^{N} \frac{x^n}{n!} \right| \leq 0.015 \quad \text{for all } x \in [-1, 2]. \tag{4.14}$$

(ii) $N \in \mathbb{N}$ such that

$$\left| e^x - \sum_{n=0}^{N} \frac{x^n}{n!} \right| \leq 0.0015 \quad \text{for all } x \in [-1, 2]. \tag{4.15}$$

(iii) $N \in \mathbb{N}$ such that

$$\left| e^x - \sum_{n=0}^{N} \frac{x^n}{n!} \right| \leq 0.015 \quad \text{for all } x \in [-5, 5]. \tag{4.16}$$

In order to do so, we first observe that the estimate (4.12) involves the constant C, which we can choose as the maximum of all derivatives of f on the considered interval. For the function f we have that $f^{(n)}(x) = e^x$, i.e., all derivatives equal the function itself.

For (i) we can thus take $C = e^2$. Now (4.12) shows that (4.14) is obtained if we choose N such that

$$\frac{e^2}{(N+1)!} 2^{N+1} \leq 0.015; \tag{4.17}$$

this is satisfied for all $N \geq 8$.

In (ii), we obtain an appropriate value for $N \in \mathbb{N}$ by replacing the number 0.015 on the right-hand side of (4.17) by 0.0015; the obtained inequality is satisfied for all $N \geq 10$.

Note than in (iii), we enlarge the interval on which we want to approximate f. In order to find an appropriate N-value in (iii), we need to replace the previous value for C with $C = e^5$; we obtain the inequality

$$\frac{e^5}{(N+1)!} 5^{N+1} \leq 0.015,$$

which is satisfied for $N \geq 19$. $\square$

When dealing with mathematical theorems, it is essential to understand the role of the various conditions; in fact, "mathematics" can almost be defined as the science of finding the minimal condition leading to a certain conclusion. For a statement like Theorem 4.13, this means that we should examine the conditions once more and find out whether they are really needed. Formulated more concretely in this particular case: is it really necessary that f is infinitely often differentiable on I and that (4.11) holds, in order to reach the conclusion in (4.12)?

This discussion would take us too far at the moment, but we can indicate the answer. In order for the conclusion in (4.12) to make sense for all $n \in \mathbb{N}$, we at least need f to be arbitrarily often differentiable at x_0. Furthermore, in Section 5.2 we will see that (4.12) leads to a so-called power series representation of f in a neighborhood of x_0 (i.e., for x sufficiently close to x_0); and Theorem 5.17 will show us that this forces f to be arbitrarily often differentiable in that neighborhood. What remains is a discussion concerning the necessity of (4.11). For this, we refer to our later Example 5.23. This shows that there exists a nonconstant function $f : \mathbb{R} \to \mathbb{R}$ for which

$$f \text{ is arbitrarily often differentiable and } f^{(n)}(0) = 0 \text{ for all } n \in \mathbb{N}.$$

Taking $x_0 = 0$, for such a function we have

$$f(x) - \sum_{n=0}^{N} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n = f(x) - f(0) \text{ for all } \ x \in \mathbb{R}.$$

This shows that (4.12) can not hold for all $N \in \mathbb{N}$ (otherwise we would have $f(x) = f(0)$ for all $x \in \mathbb{R}$). The reason that this does not contradict Theorem 4.13 is that the function in Example 5.23 does not satisfy (4.11). This shows that the assumption (4.11) actually plays a role in the theorem; it is not just added for convenience of the proof, and the conclusion might not hold if it is removed.

Example 4.14 shows that the better approximation we need, the higher the degree of the approximating polynomial P has to be. There is nothing that is called a "polynomial of infinite degree", but one can ask if one can obtain that $f(x) = P(x)$ if one has the right to include "infinitely many terms" in $P(x)$. So far we have not defined a sum of infinitely many numbers, but our next purpose is to show that certain functions actually can be represented in the form

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots = \sum_{n=0}^{\infty} a_n x^n. \tag{4.18}$$

At the moment it is far from being clear how such an infinite sum should be interpreted. In the next chapter we give the formal definition, and consider functions f which can be represented this way.

4.4 Infinite series of numbers

We are now ready to introduce infinite series, a topic that will follow us throughout the rest of the book. We begin by defining an infinite sum of real (or complex) numbers $a_1, a_2, a_3, \ldots, a_n, \ldots$.

Formally, an *infinite series* is an expression of the form

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots = \sum_{n=1}^{\infty} a_n.$$

The numbers a_n are called the *terms* in the infinite series. In order to explain how an infinite series should be interpreted, we first define the Nth *partial sum* associated to the infinite series as

$$S_N := a_1 + a_2 + a_3 + \cdots + a_N = \sum_{n=1}^{N} a_n. \tag{4.19}$$

Definition 4.15 (Convergence of infinite series) *Let* $\sum_{n=1}^{\infty} a_n$ *be an infinite series with partial sums* S_N. *If* S_N *is convergent, i.e., if there exists a number* S *such that*

$$S_N \to S \text{ as } N \to \infty, \tag{4.20}$$

we say that the infinite series $\sum_{n=1}^{\infty} a_n$ *is convergent with sum* S. *We write this as*

$$\sum_{n=1}^{\infty} a_n = S.$$

In case S_N *is divergent,* $\sum_{n=1}^{\infty} a_n$ *is said to be divergent. No sum is attributed to a divergent series.*

Note that $\sum_{n=1}^{\infty} a_n$ is convergent with sum S precisely if

$$\sum_{n=1}^{N} a_n \to S \text{ as } N \to \infty;$$

thus, to consider an infinite sum really corresponds to the intuitive idea of just "adding more and more terms".

To avoid confusion, we mention that sometimes we will consider infinite series starting with a_0 instead of a_1. The difference between $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=0}^{\infty} a_n$ is that in the second case we have added the number a_0; for the question whether the series is convergent or not, it does not matter whether we start with $n = 0$ or $n = 1$. In case of convergence,

$$\sum_{n=0}^{\infty} a_n = a_0 + \sum_{n=1}^{\infty} a_n.$$

When a given infinite series begins with a term a_N we will often use the following result to change the summation index.

Lemma 4.16 (Change of index) *Assume that the infinite series* $\sum_{n=0}^{\infty} a_n$ *is convergent. Then, for any* $N \in \mathbb{N}$ *it holds that*

$$\sum_{n=N}^{\infty} a_n = \sum_{n=0}^{\infty} a_{n+N}. \tag{4.21}$$

Proof: By looking at the terms on the left-hand side of (4.21) we see that

$$\sum_{n=N}^{\infty} a_n = a_N + a_{N+1} + \cdots.$$

Looking now at the terms on the right-hand side, we see that the terms in the partial sums of the considered infinite series are exactly the same. $\square$

The next result deals with infinite series that appear by taking linear combinations of the terms in two given infinite series that are known to be convergent.

Lemma 4.17 (Linear combinations of infinite series) *Assume that the infinite series* $\sum_{n=1}^{\infty} a_n$ *and* $\sum_{n=1}^{\infty} b_n$ *are convergent. Then, for any* $\alpha, \beta \in \mathbb{C}$ *the infinite series* $\sum_{n=1}^{\infty}(\alpha a_n + \beta b_n)$ *is convergent, and*

$$\sum_{n=1}^{\infty}(\alpha a_n + \beta b_n) = \alpha \sum_{n=1}^{\infty} a_n + \beta \sum_{n=1}^{\infty} b_n.$$

Proof: The Nth partial sum of the infinite series $\sum_{n=1}^{\infty}(\alpha a_n + \beta b_n)$ is

$$S_N = \sum_{n=1}^{N}(\alpha a_n + \beta b_n) = \alpha \sum_{n=1}^{N} a_n + \beta \sum_{n=1}^{N} b_n. \tag{4.22}$$

According to Definition 4.15 we must prove that

$$S_N \to \alpha \sum_{n=1}^{\infty} a_n + \beta \sum_{n=1}^{\infty} b_n \text{ as } n \to \infty.$$

By assumption we know that

$$\sum_{n=1}^{N} a_n \to \sum_{n=1}^{\infty} a_n \text{ as } N \to \infty$$

and

$$\sum_{n=1}^{N} b_n \to \sum_{n=1}^{\infty} b_n \text{ as } N \to \infty.$$

Using Lemma 4.10 we can now apply (4.22) to conclude that

$$S_N = \alpha \sum_{n=1}^{N} a_n + \beta \sum_{n=1}^{N} b_n \to \alpha \sum_{n=1}^{\infty} a_n + \beta \sum_{n=1}^{\infty} b_n \text{ as } n \to \infty,$$

as desired. □

To determine whether a series is convergent or not, one usually applies the so-called convergence tests, to which we return later in this section. In some cases we can apply the definition directly:

Example 4.18 In this example we consider the infinite series

(i) $\sum_{n=1}^{\infty} n = 1 + 2 + \cdots + n + \cdots;$

(ii) $\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n} + \cdots;$

(iii) $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)} + \cdots.$

For the infinite series in (i), the Nth partial sum is

$$S_N = \sum_{n=1}^{N} n = 1 + 2 + \cdots + N.$$

It is clear that

$$S_N \to \infty \text{ for } N \to \infty$$

(in fact, $S_N = \frac{N(N+1)}{2}$ for all $N \in \mathbb{N}$). Thus

$$\sum_{n=1}^{\infty} n \text{ is divergent.}$$

Now we consider the infinite series in (ii). The Nth partial sum is

$$S_N = \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^N} = \sum_{n=1}^{N} \frac{1}{2^n}.$$

We will give a geometric argument showing that

$$\sum_{n=1}^{N} \frac{1}{2^n} = 1 - \frac{1}{2^N}. \tag{4.23}$$

In fact, $S_1 = \frac{1}{2}$, which is exactly half of the distance from 0 to 1. That is,

$$S_1 = 1 - \frac{1}{2}.$$

Further, $S_2 = \frac{1}{2} + \frac{1}{2^2}$; thus, the extra contribution compared to S_1 is $\frac{1}{2^2}$, which is half the distance from S_1 to 1. Going to the next step, i.e., looking at S_3, again cuts the distance to 1 by a factor of 2:

$$S_3 = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} = 1 - \frac{1}{2^3}.$$

The same happens in all the following steps: when looking at S_N, i.e., after N steps, the distance to 1 will be exactly $1/2^N$, which shows (4.23). It follows that

$$\sum_{n=1}^{N} \frac{1}{2^n} \to 1 \text{ for } N \to \infty;$$

we conclude that

$$\sum_{n=1}^{\infty} \frac{1}{2^n} \text{ is convergent with sum 1.}$$

Concerning the series in (iii), we first observe that for any $n \in \mathbb{N}$,

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1};$$

thus, the Nth partial sum is

$$\begin{aligned} S_N &= \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{N(N+1)} \\ &= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{N} - \frac{1}{N+1}\right) \\ &= 1 - \frac{1}{N+1}. \end{aligned}$$

Since $S_N \to 1$ as $N \to \infty$, we conclude that the series is convergent with sum 1. □

4.5 Convergence tests for infinite series

In this section we will introduce a number of tests that are useful to check whether a given infinite series is convergent or not.

Let us first return to the divergent series

$$\sum_{n=1}^{\infty} n$$

considered in Example 4.18. Intuitively, the "problem" with this series is that the terms $a_n = n$ are too large: considering partial sums S_N for larger and larger values of N means that we continue to add numbers to S_N, and when they grow as in this example, it prevents S_N from having a finite limit as $N \to \infty$. A refinement of this intuitive statement leads to a useful criterion, which states that an infinite series only has a chance to be convergent if $a_n \to 0$ as $n \to \infty$; this result is called the *nth term test.*

Theorem 4.19 *If $a_n \nrightarrow 0$ as $n \to \infty$, then $\sum_{n=1}^{\infty} a_n$ is divergent.*

Proof: Assume that $\sum_{n=1}^{\infty} a_n$ is convergent with sum S. By definition of the partial sums, this means that

$$S_N = a_0 + a_1 + \cdots + a_N \to S \text{ as } N \to \infty;$$

this implies that

$$a_N = S_N - S_{N-1} \to S - S = 0 \text{ as } N \to \infty. \qquad \square$$

An alternative (but equivalent) formulation of Theorem 4.19 says that

$$\text{if } \sum_{n=1}^{\infty} a_n \text{ is convergent, then } a_n \to 0 \text{ for } n \to \infty.$$

It is important to notice that Theorem 4.19 (at most) can give us a negative conclusion: we can *never* use the nth term test to conclude that a series is convergent. In fact, even if $a_n \to 0$ as $n \to \infty$, it can happen that $\sum_{n=1}^{\infty} a_n$ is divergent; see Example 4.34.

Intuitively, it is very reasonable that if a certain infinite series with positive terms is convergent, then any series with smaller (positive) terms is also convergent; in fact, the partial sums of the series with the smaller terms are forced to stay bounded. The formal statement of this result is called the *comparison test*:

Theorem 4.20 (The comparison test) *Assume that* $0 \leq a_n \leq b_n$ *for all* $n \in \mathbb{N}$. *Then the following holds:*

(i) If $\sum_{n=1}^{\infty} b_n$ *is convergent, then* $\sum_{n=1}^{\infty} a_n$ *is convergent as well.*

(ii) If $\sum_{n=1}^{\infty} a_n$ *is divergent, then* $\sum_{n=1}^{\infty} b_n$ *is divergent as well.*

We note that the comparison test also applies if the inequalities $0 \leq a_n \leq b_n$ are just satisfied from a "certain step," i.e., for $n \geq K$ for some $K \in \mathbb{N}$. The following example illustrates this.

Example 4.21 Consider the infinite series $\sum_{n=1}^{\infty} a_n$, where

$$a_n = \begin{cases} n^4 & \text{for } n = 1, \dots, 1000, \\ \frac{2}{2^n+1} & \text{for } n = 1001, \dots . \end{cases}$$

The first 1000 term in the infinite series play no role for the convergence. Thus we will ignore the first terms and only consider a_n for $n > 1000$; for these values of n we have that

$$0 \leq a_n = \frac{2}{2^n + 1} \leq \frac{2}{2^n}.$$

Since the infinite series

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$$

is convergent according to Example 4.18, the series $\sum_{n=1}^{\infty} \frac{2}{2^n}$ is also convergent. It now follows fom the comparison test that the infinite series $\sum_{n=1}^{\infty} a_n$ is convergent. □

Example 4.22 Consider the infinite series

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)^2} = \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{(n+1)^2} + \cdots .$$

First, observe that for all $n \in \mathbb{N}$,

$$\frac{1}{(n+1)^2} \leq \frac{1}{(n+1)n}.$$

By Example 4.18(iii) we know that $\sum_{n=1}^{\infty} \frac{1}{(n+1)n}$ is convergent, so the comparison test shows that $\sum_{n=1}^{\infty} \frac{1}{(n+1)^2}$ is convergent. This coincides with a result we will obtain much later in Example 6.26, with a very different argument which even allows us to find the sum of this particular infinite series. $\square$

For the question whether a given series $\sum_{n=1}^{\infty} a_n$ is convergent or not, the first terms in the series do not matter: the question is whether a_n tends sufficiently fast to zero as $n \to \infty$. It turns out that if two infinite series with positive terms "behave similarly" for large values of n, they are either both convergent, or both divergent. In order to state this result formally, we need a definition:

Definition 4.23 (Equivalent infinite series) *Two infinite series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ with positive terms are equivalent if there exists a constant $C > 0$ such that*

$$\frac{a_n}{b_n} \to C \text{ for } n \to \infty.$$

Proposition 4.24 (The equivalence test) *Assume that the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ have positive terms and are equivalent. Then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are either both convergent, or both divergent.*

Example 4.25 Proposition 4.24 gives an alternative argument for the result in Example 4.22: the reader can check that the series $\sum_{n=1}^{\infty} \frac{1}{(n+1)n}$ and $\sum_{n=1}^{\infty} \frac{1}{(n+1)^2}$ are equivalent. $\square$

Given a sequence of numbers, we note that our definition of the associated infinite series depends on how the terms are ordered: different orderings will lead to different partial sums, and (potentially) different answers to the question whether the series is convergent or not. This is not merely a theoretical concern, but a problem that appears in practice: in fact, there exist sequences where certain orderings lead to convergent series, and other lead to divergent series. The following definition introduces a condition on the terms in an infinite series, which implies that the series is convergent with respect to any ordering.

Definition 4.26 (Absolute convergence) *An infinite series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent if $\sum_{n=1}^{\infty} |a_n|$ is convergent.*

Theorem 4.27 (Absolute convergence) *If $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent; furthermore,*

$$\left| \sum_{n=1}^{\infty} a_n \right| \le \sum_{n=1}^{\infty} |a_n|.$$

There exist series which are convergent, but not absolutely convergent; they are said to be *conditionally convergent.*

Definition 4.28 (Conditionally convergence infinite series) *An infinite series* $\sum_{n=1}^{\infty} a_n$ *is conditionally convergent if* $\sum_{n=1}^{\infty} a_n$ *is convergent and* $\sum_{n=1}^{\infty} |a_n|$ *is divergent.*

For conditionally convergent series with real terms, something very strange happens. In fact, the terms can be reordered such that we obtain a divergent series; and for *any* given real number $\tilde{S}$ we can find an ordering such that the corresponding infinite series is convergent with sum equal to $\tilde{S}$. Example 4.39 exhibits an infinite series which is convergent, but not absolutely convergent.

Example 4.29 We will check whether the infinite series

$$\sum_{n=1}^{\infty} \frac{\sin n}{2^n} \tag{4.24}$$

is absolutely convergent, conditionally convergent, or divergent.

Intuitively it is reasonable to expect that the infinite series is absolutely convergent: indeed, we know from Example 4.18 that the infinite series $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is convergent, and the term $\sin n$ in the counter in (4.24) is only taking values in the bounded interval $[-1, 1]$. In order to show that the intuition is correct, let

$$a_n = \frac{\sin n}{2^n};$$

then

$$0 \le |a_n| = \left| \frac{\sin n}{2^n} \right| = \frac{|\sin n|}{2^n} \le \frac{1}{2^n} = \left(\frac{1}{2} \right)^n.$$

We will now apply the comparison test: since $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$ is convergent according to Example 4.18, we can now conclude that $\sum_{n=1}^{\infty} |a_n|$ er konvergent. The conclusion is that

$$\sum_{n=1}^{\infty} \frac{\sin n}{2^n} \quad \text{is absolutely convergent.} \qquad \square$$

A sufficient condition for (absolute) convergence of an infinite series is given by the *quotient test*:

Theorem 4.30 (The quotient test) *Assume that* $a_n \neq 0$ *for all* $n \in \mathbb{N}$ *and that there exists a number* $C \geq 0$ *such that*

$$\left|\frac{a_{n+1}}{a_n}\right| \to C \quad \textit{for} \quad n \to \infty. \tag{4.25}$$

Then the following hold:

(i) If $C < 1$*, then* $\sum a_n$ *is absolutely convergent.*

(ii) If $C > 1$*, then* $\sum a_n$ *is divergent.*

We prove Theorem 4.30 in Section A.3. Note that the case where (4.25) is satisfied with $C = 1$ has a very special status: Theorem 4.30 does not give any conclusion in that case. Example 4.34 will show us that there exist convergent as well as divergent series which lead to $C = 1$; if we end up with this particular value, we have to use other methods to find out whether the given series is convergent or not.

Example 4.31 Let $\alpha > 0$ be given, and consider the infinite series

$$\sum_{n=1}^{\infty} \frac{1}{n^\alpha} = 1 + \frac{1}{2^\alpha} + \frac{1}{3^\alpha} + \cdots + \frac{1}{n^\alpha} + \cdots.$$

Let $a_n = \frac{1}{n^\alpha}$; then

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\frac{1}{(n+1)^\alpha}}{\frac{1}{n^\alpha}} = \frac{n^\alpha}{(n+1)^\alpha} = \left(\frac{n}{n+1}\right)^\alpha.$$

By division with n in the counter and denominator we see that

$$\left|\frac{a_{n+1}}{a_n}\right| = \left(\frac{1}{1+1/n}\right)^\alpha \to 1 \text{ for } n \to \infty.$$

Thus the quotient test does not lead to any conclusion. We return to the given infinite series in Example 4.34. □

Example 4.32 Consider the infinite series

$$\sum_{n=1}^{\infty} \frac{n^2}{2^n} = \frac{1}{2} + 1 + \frac{9}{8} + \cdots + \frac{n^2}{2^n} + \cdots.$$

Letting $a_n = \frac{n^2}{2^n}$, we see that

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^2}{2^{n+1}} \cdot \frac{2^n}{n^2} = \left(1 + \frac{1}{n}\right)^2 \frac{1}{2} \to \frac{1}{2} \quad \text{for } n \to \infty.$$

Thus the quotient test shows that the series is convergent. □

4.6 Estimating the sum of an infinite series

Only in relatively rare cases is it possible to find the exact sum of an infinite series. This raises the question how one can find approximative values for the sum. Using computers, we are able to calculate the partial sums S_N for large values of $N \in \mathbb{N}$; but in order for this information to be useful, we need to be able to estimate the difference

$$\left|\sum_{n=1}^{\infty} a_n - S_N\right| = \left|\sum_{n=1}^{\infty} a_n - \sum_{n=1}^{N} a_n\right| = \left|\sum_{n=N+1}^{\infty} a_n\right|.$$

One important result which can frequently be used in order to do so can be derived from the following result, called the *integral test:*

Theorem 4.33 (The integral test) *Assume that the function $f : [1, \infty[\to [0, \infty[$ is continuous and decreasing. Then the following holds:*

(i) If $\int_1^t f(x)dx$ has a finite limit as $t \to \infty$, then $\sum_{n=1}^{\infty} f(n)$ is convergent, and

$$\int_1^{\infty} f(x)dx < \sum_{n=1}^{\infty} f(n) < \int_1^{\infty} f(x)dx + f(1). \tag{4.26}$$

(ii) If $\int_1^t f(x)dx \to \infty$ as $t \to \infty$, then $\sum_{n=1}^{\infty} f(n)$ is divergent.

The theorem is proved in Section A.3. As an application of Theorem 4.33 we now return to the infinite series in Example 4.31.

Example 4.34 Let $\alpha > 0$, and consider again the infinite series in Example 4.31,

$$\sum_{n=1}^{\infty} \frac{1}{n^\alpha} = 1 + \frac{1}{2^\alpha} + \frac{1}{3^\alpha} + \cdots + \frac{1}{n^\alpha} + \cdots. \tag{4.27}$$

In order to apply the integral test, consider now the function

$$f(x) = \frac{1}{x^\alpha}, \ x \in [1, \infty[.$$

This function is decreasing and continuous. If $\alpha > 1$, then

$$\int_1^t \frac{1}{x^\alpha}dx = \frac{1}{-\alpha+1}\left[x^{-\alpha+1}\right]_1^t = \frac{1}{-\alpha+1}(t^{-\alpha+1} - 1).$$

It follows that

$$\int_1^t \frac{1}{x^\alpha}dx \to \frac{1}{\alpha - 1} \quad \text{for } t \to \infty.$$

Via Theorem 4.33 we conclude that

$$\sum_{n=1}^{\infty} \frac{1}{n^{\alpha}} \text{ is convergent if } \alpha > 1.$$

Furthermore (4.26) shows that

$$\frac{1}{\alpha-1} \leq \sum_{n=1}^{\infty} \frac{1}{n^{\alpha}} \leq \frac{1}{\alpha-1} + 1. \tag{4.28}$$

For $\alpha = 1$,

$$\int_1^t \frac{1}{x} dx = [\ln x]_1^t = \ln t \to \infty \quad \text{for } t \to \infty.$$

Again via Theorem 4.33 we conclude that

$$\sum_{n=1}^{\infty} \frac{1}{n} \text{ is divergent.}$$

A similar argument (which is left to the reader) shows that the infinite series (4.27) is divergent for any $0 < \alpha < 1$. □

The inequalities (4.26) give some information about the value of the sum $\sum_{n=1}^{\infty} f(n)$: indeed, it belongs to the interval

$$\left[\int_1^{\infty} f(x)\, dx, \int_1^{\infty} f(x)\, dx + f(1) \right].$$

Note that the length of the interval is given by the number $f(1)$. Thus, if $f(1)$ is a large number we do not get a precise estimate on the value of the sum. For example, using (4.28) with $\alpha = 2$ shows only that

$$1 \leq \sum_{n=1}^{\infty} \frac{1}{n^2} \leq 2.$$

By a small modification of Theorem 4.33 we will now derive a method that can deliver estimates for the infinite sum $\sum_{n=1}^{\infty} f(n)$ with arbitrary precision. In order to discuss this method we refer to the proof of Theorem 4.33 in Appendix A.3. The proof shows that it does not matter whether the summation in the integral test starts with $n = 1$ or any other value of $n \in \mathbb{N}$. In other words, if the conditions in Theorem 4.33 are satisfied, then

$$\int_{N+1}^{\infty} f(x)\, dx \leq \sum_{n=N+1}^{\infty} f(n) \leq \int_{N+1}^{\infty} f(x)\, dx + f(N+1) \tag{4.29}$$

for any $N \in \mathbb{N}$. This announced methods to find approximate values of $\sum_{n=1}^{\infty} f(n)$ are based on the following result:

Corollary 4.35 *Assume that the function $f : [1, \infty[\to [0, \infty[$ is continuous and decreasing, at least for $x \geq N$ for some $N \in \mathbb{N}$. Assume furthermore that $\int_1^\infty f(x)\,dx$ is convergent. Then the following hold:*

(i) For any N as above,

$$\sum_{n=N+1}^{\infty} f(n) \leq \int_{N+1}^{\infty} f(x)\,dx + f(N+1). \tag{4.30}$$

(ii) For any N as above,

$$\begin{aligned} \sum_{n=1}^{N} f(n) \quad + & \quad \int_{N+1}^{\infty} f(x)\,dx \leq \sum_{n=1}^{\infty} f(n) \\ \leq & \quad \sum_{n=1}^{N} f(n) + \int_{N+1}^{\infty} f(x)\,dx + f(N+1). \end{aligned} \tag{4.31}$$

Proof: The result in (i) follows directly from (4.29). In order to prove (ii) we note that

$$\sum_{n=N+1}^{\infty} f(n) = \sum_{n=1}^{\infty} f(n) - \sum_{n=1}^{N} f(n).$$

Inserting this in (4.29) shows that

$$\int_{N+1}^{\infty} f(x)\,dx \leq \sum_{n=1}^{\infty} f(n) - \sum_{n=1}^{N} f(n) \leq \int_{N+1}^{\infty} f(x)\,dx + f(N+1);$$

now (4.31) follows by adding the finite sum $\sum_{n=1}^{N} f(n)$ in the inequalities.
$\square$

We will now explain how these results can be used to find approximate values of $\sum_{n=1}^{\infty} f(n)$. Assume that we want to find this sum within a tolerance of some $\epsilon > 0$. We will discuss two methods:

Method (i): Choose $N \in \mathbb{N}$ such that

$$\int_{N+1}^{\infty} f(x)\,dx + f(N+1) \leq \epsilon.$$

Using (4.30) it follows that for this value of N,

$$\sum_{n=N+1}^{\infty} f(n) < \epsilon.$$

Since

$$\sum_{n=1}^{\infty} f(n) = \sum_{n=1}^{N} f(n) + \sum_{n=N+1}^{\infty} f(n)$$

we conclude that the *finite sum*

$$\sum_{n=1}^{N} f(n)$$

at most deviate with ϵ from the sum $\sum_{n=1}^{\infty} f(n)$. We can now calculate the finite sum $\sum_{n=1}^{N} f(n)$ using Maple or any other program.

Method (ii): Choose $N \in \mathbb{N}$ such that

$$f(N+1) \leq \epsilon.$$

Then (4.31) shows that

$$\sum_{n=1}^{N} f(n) + \int_{N+1}^{\infty} f(x)\,dx \leq \sum_{n=1}^{\infty} f(n) \leq \sum_{n=1}^{N} f(n) + \int_{N+1}^{\infty} f(x)\,dx + \epsilon;$$

but this means that the number

$$\sum_{n=1}^{N} f(n) + \int_{N+1}^{\infty} f(x)\,dx$$

at most deviate with ϵ from the sum $\sum_{n=1}^{\infty} f(n)$.

Let us apply these methods to the case $\alpha = 2$ in Example 4.34.

Example 4.36 We want to calculate the sum

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

within a tolerance of $\epsilon = 0.01$. We first apply the metod in (i) with $f(x) := \frac{1}{x^2}$. Precisely as in Example 4.2 we can prove that (do it!) for any $N \in \mathbb{N}$,

$$\int_{N+1}^{\infty} \frac{1}{x^2}\,dx = \frac{1}{N+1}.$$

Thus

$$\begin{aligned}
\int_{N+1}^{\infty} f(x)\,dx + f(N+1) &= \int_{N+1}^{\infty} \frac{1}{x^2}\,dx + \frac{1}{(N+1)^2} \\
&= \frac{1}{N+1} + \frac{1}{(N+1)^2} \\
&= \frac{N+1+1}{(N+1)^2} \\
&= \frac{N+2}{N^2+2N+1} \\
&= \frac{N+2}{N(N+2)+1}.
\end{aligned}$$

Since $N(N+2)+1 \geq N(N+2)$, we conclude that

$$\int_{N+1}^{\infty} f(x)\,dx + f(N+1) \leq \frac{N+2}{N(N+2)} \leq \frac{1}{N}.$$

Via (4.30) it follows that

$$\sum_{n=N+1}^{\infty} \frac{1}{n^2} \;\leq\; \frac{1}{N}. \tag{4.32}$$

Now, $1/N \leq 0.01$ for $N = 100$; this shows that within a tolerance of 0.01,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \approx \sum_{n=1}^{100} \frac{1}{n^2} = 1.635. \tag{4.33}$$

The finite sum in (4.33) is calculated using Maple.

Let us now apply the method in (ii). For the function $f(x) = \frac{1}{x^2}$, the inequality $f(N+1) \leq 0.01$ is satisfied for $N = 9$; thus, within a tolerance of 0.01,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \approx \sum_{n=1}^{9} \frac{1}{n^2} + \int_{10}^{\infty} \frac{1}{x^2}\,dx = \sum_{n=1}^{9} \frac{1}{n^2} + \frac{1}{10} = 1.640.$$

In Example 6.26 we will show that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \approx 1.6449. \qquad \square$$

The method in (i) usually lead to larger values for N than the method in (ii). Consider, e.g., Example 4.36 again: if we substitute $\epsilon = 0.01$ with $\epsilon = 10^{-4}$, then the method in (i) leads to $N = 10^4$ terms, while the method in (ii) leads to $N = 99$. On the other hand the method in (ii) also requires that we calculate the exact value of the integral $\int_{N+1}^{\infty} f(x)\,dx$.

In cases where we just want to find an approximate value for the sum $\sum_{n=1}^{\infty} f(n)$, the method in (ii) is usually preferable. However, in connection withe Fourier series we often need to find a value of $N \in \mathbb{N}$ such that

$$\sum_{n=N+1}^{\infty} f(n) \leq \epsilon;$$

in this case we are forced to apply method (i).

4.7 Alternating series

Corollary 4.35 only applies for infinite series with positive terms. However, there exists another important class of infinite series for which we can obtain estimates of the sum with arbitrary precision, namely the so-called *alternating series.* Hereby we understand infinite series, where the terms change between beeing positive and negative in each step:

Definition 4.37 (Alternating infinite series) *An alternating series is an infinite series of the form*

$$\sum_{n=1}^{\infty}(-1)^{n-1}b_n = b_1 - b_2 + b_3 - b_4 + \cdots + (-1)^{n-1}b_n + \cdots , \tag{4.34}$$

or

$$\sum_{n=1}^{\infty}(-1)^{n}b_n = -b_1 + b_2 - b_3 + \cdots + (-1)^{n}b_n + \cdots . \tag{4.35}$$

where $b_n > 0$ for all $n \in \mathbb{N}$.

The following criterion, which is know under the name *Leibniz' test,* is useful in order to check convergence for an alternating series. It can also be used to find an estimate for how many terms we need to include in the partial sum in order to reach a certain approximation of the sum:

Theorem 4.38 (Leibniz' test) *Consider an infinite series of the form* (4.34) *or* (4.35)*, and assume the following:*

(i) The numbers b_n are positive, i.e., $b_n > 0$ for all $n \in \mathbb{N}$;

(ii) The numbers b_n decrease monotonically, i.e.,

$$b_1 \geq b_2 \geq b_3 \geq \cdots .$$

(iii) $b_n \to 0$ as $n \to \infty$.

Then the infinite series (4.34) *and* (4.35) *are convergent. Furthermore, for all $N \in \mathbb{N}$,*

$$\left|\sum_{n=1}^{\infty}(-1)^{n-1}b_n - \sum_{n=1}^{N}(-1)^{n-1}b_n\right| \leq b_{N+1} \tag{4.36}$$

and

$$\left|\sum_{n=1}^{\infty}(-1)^{n}b_n - \sum_{n=1}^{N}(-1)^{n}b_n\right| \leq b_{N+1}. \tag{4.37}$$

Note that a similar version of Theorem 4.38 holds for for alternating infinite series that do not start with the summation index $n = 1$. Considering an alternating infinite series $\sum_{n=p}^{\infty}(-1)^{n-1}b_n$ we have the estimate

$$\left|\sum_{n=p}^{\infty}(-1)^{n-1}b_n - \sum_{n=p}^{N}(-1)^{n-1}b_n\right| \leq b_{N+1} \tag{4.38}$$

whenever $N \geq p$.

Example 4.39 By Theorem 4.38, the infinite series

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}$$

is convergent. According to Example 4.34, the series is not absolutely convergent; thus it is conditionally convergent. □

Example 4.40 Consider the infinite series

$$\sum_{n=2}^{\infty}(-1)^{n-1}\frac{1}{1+\ln n} = -\frac{1}{1+\ln 2}+\frac{1}{1+\ln 3}-\frac{1}{1+\ln 4}+\cdots. \quad (4.39)$$

We note that the natural logarithm ln is an increasing function; thus the sequence

$$b_n = \frac{1}{1+\ln n}$$

is decreasing. We also see that

$$b_n \to 0 \text{ as } n \to \infty.$$

Thus the given infinite series is convergent according to Theorem 4.38. In order to find an approximate value for the sum within a tolerance of $\epsilon = 0.2$, choose $N \in \mathbb{N}$ such that $b_{N+1} \leq 0.2$, i.e., such that

$$\frac{1}{1+\ln(N+1)} \leq 0.2.$$

We can rewrite this inequality as

$$\frac{1}{0.2} \leq 1+\ln(N+1),$$

or

$$4 \leq \ln(N+1).$$

This inequality is satisfied if $N+1 \geq e^4$, i.e., whenever $N \geq e^4 - 1 \approx 53.6$. According to (4.36) we can now conclude that within a tolerance of $\epsilon = 0.2$,

$$\sum_{n=2}^{\infty}(-1)^{n-1}\frac{1}{1+\ln n} \approx \sum_{n=2}^{54}(-1)^{n-1}\frac{1}{1+\ln n} = -0.43. \quad (4.40)$$

The finite sum in (4.40) is calculated using Maple.

We note that the infinite series (4.39) converges very slow. If we want to find the sum within a tolerance of at most $\epsilon = 0.05$, Leibniz' test shows that we should take $N \geq e^{19} - 1 \approx 1.78 \cdot 10^8$ terms! □

5

Infinite series of functions

So far we have only discussed infinite series $\sum_{n=1}^{\infty} a_n$ where the terms a_n are numbers. The next step is to consider infinite series where the terms depend on a variable. In Section 5.1 we consider geometric series, i.e., infinite series of the type

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots + x^n + \cdots, \; x \in \mathbb{R}. \tag{5.1}$$

In this case we can characterize the $x \in \mathbb{R}$ for which the series is convergent, and we can calculate the sum. In Section 5.2 we consider "similar" infinite series, where we allow the terms in (5.1) to be multiplied with certain constants depending on n, i.e., infinite series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots$$

for certain constants c_n; infinite series of this type are called *power series*. Section 5.3 takes an important further step: here we consider infinite series of general functions $f_0, f_1, \ldots$, i.e., series of the form

$$\sum_{n=0}^{\infty} f_n(x) = f_0(x) + f_1(x) + f_2(x) + \cdots .$$

An understanding of this type of infinite series is important for our treatment of Fourier series in Chapter 6. Finally Section 5.4 introduces the concept "uniform convergence," and discusses its role and importance. An application of power series is sketched in Section 5.5.

5.1 Geometric series

The next step is to consider series where the terms depend on a variable. One of the simplest examples is a *geometric series:*

Definition 5.1 *A geometric series is an infinite series of the form*

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots + x^n + \cdots , \tag{5.2}$$

where $x \in \mathbb{R}$ (or more generally, $x \in \mathbb{C}$). When dealing with a specific value of x, this number is called the quotient.

We now show that a geometric series is convergent if and only if $|x| < 1$; furthermore, for these values of x we calculate the sum of the infinite series.

Theorem 5.2 *A geometric series with quotient x is convergent if and only if $|x| < 1$; and for $|x| < 1$ the sum is*

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}. \tag{5.3}$$

Proof: First we find the values of x for which the series is convergent. For this purpose we notice that for any $N \in \mathbb{N}$,

$$\begin{aligned} (1-x)(1 + x + x^2 + \cdots + x^N) &= 1 - x + x - x^2 + \cdots + x^N - x^{N+1} \\ &= 1 - x^{N+1}; \end{aligned}$$

thus, for $x \neq 1$, the Nth partial sum of (5.2) is given by

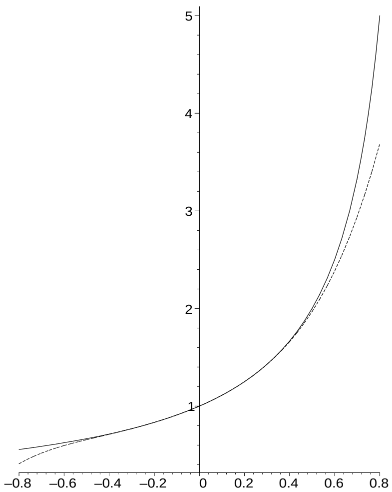
$$S_N = 1 + x + \cdots + x^N = \frac{1 - x^{N+1}}{1-x}.$$

When $|x| < 1$, we see that $x^{N+1} \to 0$ as $N \to \infty$; therefore S_N converges for $N \to \infty$ and the limit is $\frac{1}{1-x}$. When $|x| \geq 1$, S_N does not have a limit for $N \to \infty$. □

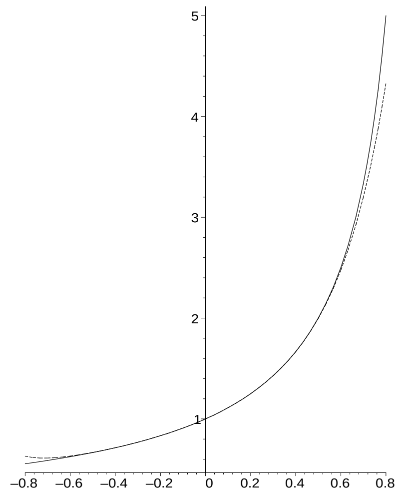
Example 5.3 We consider Example 4.18(ii) again. The series

$$\sum_{n=1}^{\infty} \frac{1}{2^n}$$

is a geometric series with $x = \frac{1}{2}$; thus, according to Theorem 5.2, the series is convergent with sum $\frac{1}{1-\frac{1}{2}} - 1 = 1$. □



(a)



(b)

Figure 5.4 *(a) The function* $f(x) = \frac{1}{1-x}$ *and the partial sum* $S_5(x) = \sum_{n=0}^{5} x^n$ *of the series representation in* (5.3) *(dotted). (b) The function* $f(x) = \frac{1}{1-x}$ *and the partial sum* $S_8(x) = \sum_{n=0}^{8} x^n$ *(dotted).*

Note that we in Example 5.3 found the sum of a geometric series starting by $n = 1$ by subtraction of the first term in (5.3). Alternatively, we could have used the following consequence of Theorem 5.2:

Corollary 5.5 *For all* $N \in \mathbb{N}$ *and all* $x \in]-1,1[$,

$$\sum_{n=N}^{\infty} x^n = \sum_{n=0}^{\infty} x^{n+N} = \frac{x^N}{1-x}. \tag{5.4}$$

Example 5.6 Using (5.4) we see that

$$\sum_{n=3}^{\infty} \frac{5}{3^{n+1}} = \sum_{n=3}^{\infty} \frac{5}{3 \cdot 3^n} = \frac{5}{3} \sum_{n=3}^{\infty} \frac{1}{3^n} = \frac{5}{3} \sum_{n=3}^{\infty} \left(\frac{1}{3}\right)^n = \frac{5}{3} \frac{\left(\frac{1}{3}\right)^3}{1-\frac{1}{3}} = \frac{5}{54}. \quad \square$$

For reasons that will be clear in Section 5.4, we emphasize that (5.3) is a *pointwise* identity: it states, that when we *fix* a certain $x \in]-1,1[$, then the series in (5.3) converges to the stated limit. This *does not* imply that we can find a partial sum S_N of (5.2) which is close to $\frac{1}{1-x}$ *simultaneously for all* $x \in]-1,1[$. Using a wording that will be introduced in Section 5.4, we say that the partial sums do not converge to $\frac{1}{1-x}$ uniformly. See Figure 5.4.

5.2 Power series

We are now ready to connect the infinite series with the approximation theory discussed in Section 4.3. We begin with a reformulation of Theorem 4.13 using infinite series:

Theorem 5.7 *Let I be an interval and f an arbitrarily often differentiable function defined on I. Let $x_0 \in I$, and assume that there exists a constant $C > 0$ such that*

$$\left|f^{(n)}(x)\right| \leq C \quad \textit{for all } n \in \mathbb{N} \quad \textit{and all } x \in I.$$

Then

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n \quad \textit{for all } x \in I. \tag{5.5}$$

Proof: The statement in Theorem 4.13 shows that the Nth partial sum of the infinite series in (5.5) converges to $f(x)$ when $N \to \infty$; thus, the result follows by the definition of the sum of an infinite series. □

In the remainder of this chapter we will concentrate on the case where $0 \in I$ and we can choose $x_0 = 0$.

Example 5.8 Let us go back to Example 4.12, where we considered the function

$$f(x) = e^x, \; x \in]-1, 2[.$$

Since

$$f'(x) = f''(x) = \cdots = e^x,$$

we see that

$$|f^{(n)}(x)| \leq e^2 \text{ for all } n \in \mathbb{N}, \; x \in]-1, 2[. \tag{5.6}$$

Thus, according to Theorem 5.7,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n, \; x \in]-1, 2[. \tag{5.7}$$

The particular interval $]-1, 2[$ does not play any role in (5.7): the arguments leading to this identity can be performed for any bounded interval containing 0. This implies that the identity holds pointwise for *any* $x \in \mathbb{R}$. Using that $f^{(n)}(0) = 1$ for all $n \in \mathbb{N}$, we arrive at

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \; x \in \mathbb{R}. \tag{5.8}$$

Formula (5.8) shows that calculation of e^x can be reduced to repeated use of the operations addition, multiplication, and division. All calculators (and mathematical calculating programs like Maple, Mathematica or Matlab) are based only on these operations (together with subtraction) and all other computations are done by reduction to these. In fact, calculators evaluate expressions involving e^x using series expansions like the one in (5.8)! However, a mathematical program can not add infinitely many terms in finite time, so in practice it calculates

$$\sum_{n=0}^{N} \frac{x^n}{n!} \approx e^x$$

for a large value of N; see Example 4.14, where we saw that $N = 8$ was sufficient to approximate e^x with an error of at most 0.015 on the interval $[-1, 2]$. The commercial mathematical programs use way more terms, and the magnitude of the round-off error is considerably smaller; under "normal" conditions one will not notice the round-off error, but sometimes with long calculations the errors become noticeable. $\square$

Example 5.9 Consider the function $f(x) = \sin x,\ x \in \mathbb{R}$. Then

$$\begin{aligned}
f'(x) &= \cos x = f^{(1+4n)}(x), \ \text{ and } f^{(1+4n)}(0) = 1, \ n \in \mathbb{N}, \\
f''(x) &= -\sin x = f^{(2+4n)}(x), \ \text{ and } f^{(2+4n)}(0) = 0, \ n \in \mathbb{N}, \\
f^{(3)}(x) &= -\cos x = f^{(3+4n)}(x), \ \text{ and } f^{(3+4n)}(0) = -1, \ n \in \mathbb{N}, \\
f^{(4)}(x) &= \sin x = f^{(4n)}(x), \ \text{ and } f^{(4n)}(0) = 0, \ n \in \mathbb{N}.
\end{aligned}$$

We see that $|f^{(n)}(x)| \leq 1$ for all $n \in \mathbb{N}$ and all $x \in \mathbb{R}$. By Theorem 5.7,

$$\begin{aligned}
\sin x &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \\
&= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}. \qquad (5.9)
\end{aligned}$$

Our calculators find $\sin x$ using this expression. The partial sums of (5.9) converge very fast to the sine function, mainly because of the denominator $(2n+1)!$ in (5.9), which makes the coefficients in the power series decay quickly. Another reason for the fast convergence is the oscillatory behavior of the terms in the partial sums: the series (5.9) is an alternating series, and according to Proposition 4.38,

$$\left| \sin x - \sum_{n=0}^{N} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \right| \leq \frac{|x|^{2N+3}}{(2N+3)!}. \qquad (5.10)$$

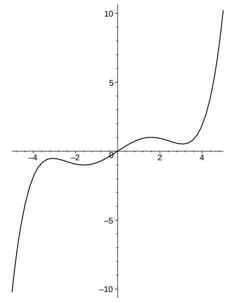


Figure 5.10 *The partial sum $S_5(x)$ for $f(x) = \sin x$.*

If we want the difference in (5.10) to be smaller than 0.01 for all $x \in [-5, 5]$, it is enough to take $N \geq 7$; for $N = 7$, the corresponding partial sum is a polynomial of degree 15.

It is actually surprising that the series representation (5.9) of the sine function is valid for all $x \in \mathbb{R}$. None of the functions that appear in the series representation are periodic, but nevertheless the infinite sum, that is, the sine function, is periodic with period 2π! See Figure 5.10, which shows the partial sum

$$S_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!};$$

this is clearly not a 2π-periodic function. This illustrates that very contra-intuitive things might happen for infinite series; we will meet several other such instances later. □

Let us collect the derived results:

Theorem 5.11 (Series representations for $e^x, \cos x, \sin x$) *For every $x \in \mathbb{R}$ we have the following series representations:*

(i) $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

(ii) $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$

(iii) $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$

The proof for (iii) is analogous to the proof of (ii).

The geometric series, as well as the representations (5.8) and (5.9) derived for the exponential function and the sine function, all have the form $\sum_{n=0}^{\infty} c_n x^n$ for some coefficients c_n. We will now introduce such series formally.

Definition 5.12 (Power series) *An infinite series of the form*

$$\sum_{n=0}^{\infty} c_n x^n \tag{5.11}$$

is called a power series, and a function which can be written in the form

$$f(x) = \sum_{n=0}^{\infty} c_n x^n$$

for some coefficients c_n, is said to have a power series representation.

Power series appear in many contexts, where one needs to approximate complicated functions. Unfortunately not all functions can be represented with help of a power series; we will come back to this issue on page 102.

Concerning convergence of power series, a very interesting result holds: basically, it says that a power series always is convergent on a certain symmetric interval around $x = 0$ and divergent outside the interval.

Theorem 5.13 *For every power series $\sum_{n=0}^{\infty} c_n x^n$ one of the following options holds:*

(i) The series only converges for $x = 0$.

(ii) The series is absolutely convergent for all $x \in \mathbb{R}$.

(iii) There exists a number $\rho > 0$ such that the series is absolutely convergent for $|x| < \rho$, and divergent for $|x| > \rho$.

We give a proof of this result (under an extra assumption) in Section A.3. If the case (iii) in Theorem 5.13 occur, the number ρ is called the *radius of convergence* of the power series. In the case (i) we put $\rho = 0$, and in the case (ii), $\rho = \infty$. Theorem 5.13 also holds for complex values of x, which explains the word "radius of convergence"; see Figure 5.15, which corresponds to the case $\rho = 1$.

Example 5.14 In order to find the radius of convergence for the power series

$$\sum_{n=0}^{\infty} \frac{x^n}{n!},$$

we consider $x \neq 0$ and put $a_n = \frac{x^n}{n!}$. Then

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{(n+1)!}\,\frac{n!}{x^n}\right| = \frac{|x|}{n+1} \to 0 \quad \text{for } n \to \infty.$$

The quotient test shows that the series is convergent for any value of x, so $\rho = \infty$. This is in accordance with our previous observation that (5.8) holds for all x. $\square$

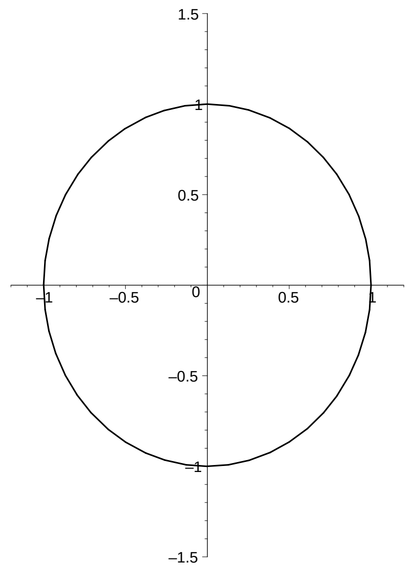


Figure 5.15 *Circle of convergence for the series* $\sum_{n=0}^{\infty}(-1)^n x^{2n}$ *in Example 5.16, with radius of convergence* $\rho = 1$*. The series is convergent for* x*-values on the part of the real axis which is inside the circle, and divergent outside the circle. Thinking about* x *as a complex variable and the plane* $\mathbb{R}^2$ *as* $\mathbb{C}$*, the series is convergent for all* x *inside the circle.*

Example 5.16 The power series $\sum_{n=0}^{\infty}(-1)^n x^{2n}$ is convergent for $|x| < 1$ and divergent for $|x| > 1$ (check this!). Thus $\rho = 1$. □

In the interval given by $|x| < \rho$, it turns out that a power series defines an infinitely often differentiable function:

Theorem 5.17 *Assume that the power series* $\sum_{n=0}^{\infty} c_n x^n$ *has radius of convergence* $\rho > 0$*, and define the function* f *by*

$$f :]-\rho, \rho[\to \mathbb{C}, \quad f(x) = \sum_{n=0}^{\infty} c_n x^n. \tag{5.12}$$

Then f *is infinitely often differentiable. Moreover*

$$f'(x) = \sum_{n=1}^{\infty} c_n n x^{n-1}, \ |x| < \rho, \tag{5.13}$$

and more generally, for every $k \in \mathbb{N}$*,*

$$f^{(k)}(x) = \sum_{n=k}^{\infty} c_n n(n-1)\cdots(n-k+1)x^{n-k}, \ |x| < \rho. \tag{5.14}$$

Observe the changes in the summation index that appear in Theorem 5.17. They are explained in the following example.

Example 5.18 The terms in the infinite series (5.12) are

$$f(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots .$$

Here the first led (corresponding to the index $n = 0$) is constant, and thus does not contribute to $f'(x)$. Therefore the infinite series in (5.13) starts with $n = 1$ instead of $n = 0$. Similarly, the expression for the kth derivative $f^{(k)}(x)$ starts with $n = k$, see (5.14).

Note that we need to be very careful when we apply the rule for differentiating a power series: indeed, we shall *only* change the summation index if the first term in the power series is a constant. Consider, e.g., the infinite series

$$f(x) = \sum_{n=1}^{\infty} x^n = x + x^2 + x^3 + \cdots, \ x \in]-1,1[;$$

here the first term is x, which indeed contribute to the derivative. Thus

$$f'(x) = \sum_{n=1}^{\infty} n x^{n-1} = 1 + 2x + 3x^2 + \cdots, \ x \in]-1,1[.$$

That is, in this case no change in the summation index appears. □

Example 5.19

(i) The power series $\sum_{n=1}^{\infty}(n+1)x^{2n}$ has radius of convergence $\rho = 1$. Thus the function

$$f(x) = \sum_{n=1}^{\infty}(n+1)x^{2n}, \ x \in]-1,1[$$

is differentiable, and

$$f'(x) = \sum_{n=1}^{\infty} 2n(n+1)x^{2n-1}, \ x \in]-1,1[.$$

(ii) The power series $\sum_{n=0}^{\infty}(n+1)(2x)^n$ has radius of convergence $\rho = 1/2$. Thus the function

$$f(x) = \sum_{n=0}^{\infty}(n+1)(2x)^n = \sum_{n=0}^{\infty}(n+1)2^n x^n, \ x \in]-1/2,1/2[$$

is differentiable, and

$$f'(x) = \sum_{n=1}^{\infty} n(n+1)2^n x^{n-1}, \ x \in]-1/2,1/2[. \qquad \square$$

Theorem 5.17 gives a part of the answer to our question about which functions have a power series expansion: in order for a function f to have such a representation, the function f must be arbitrarily often differentiable: this is a *necessary* condition. Example 5.23 will show that the condition is not sufficient to guarantee the desired representation. Before we present that example, we connect Theorem 5.17 with Taylor's theorem.

Theorem 5.20 *Consider a power series* $\sum_{n=0}^{\infty} c_n x^n$ *with radius of convergence* $\rho > 0$, *and let*

$$f(x) = \sum_{n=0}^{\infty} c_n x^n, \; x \in]-\rho, \rho[.$$

Then

$$c_n = \frac{f^{(n)}(0)}{n!}, \; n = 0, 1, \ldots, \tag{5.15}$$

i.e.,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n, \; x \in]-\rho, \rho[.$$

Proof: First, $f(0) = c_0$. Letting $x = 0$ in Theorem 5.17 we get that

$$f'(0) = c_1, \; f''(0) = 2 \cdot 1 \cdot c_2,$$

and more generally, via (5.14),

$$f^{(n)}(0) = c_n \cdot n!, \; n \in \mathbb{N}.$$

□

Proposition 5.20 shows that if a function f has a power series representation with some arbitrary coefficients c_n, then it is *necessarily* the representation we saw in Theorem 5.7. In particular, the power series representation is *unique*, i.e., only one choice of the coefficients c_n is possible:

Corollary 5.21 *Assume that the coefficients* c_n *satisfies that*

$$\sum_{n=0}^{\infty} c_n x^n = 0, \; x \in]-\rho, \rho[$$

for some $\rho > 0$, *then* $c_n = 0$ *for all* n.

As promised, we now show that not every arbitrarily often differentiable function $f : \mathbb{R} \to \mathbb{R}$ has a power series representation. We do so by example, i.e., we exhibit a concrete function without such a representation:

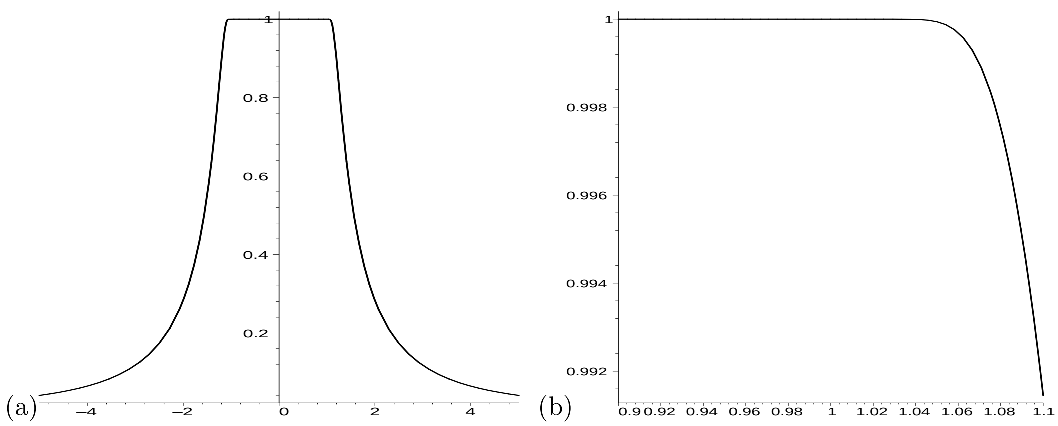


Figure 5.22 *(a): The function in* (5.17)*, shown on the interval* $[-5,5]$*. (b): The function in* (5.17)*, shown on the interval* $[0.9,1.1]$*. Notice the unit on the second axis!*

Example 5.23 Consider a function $f:\mathbb{R}\to\mathbb{R}$ with the following properties:

(i) $f(x)=1$ for all $x\in[-1,1]$.

(ii) f is arbitrarily often differentiable.

Assume that f has a power series representation,

$$f(x)=\sum_{n=0}^{\infty}c_nx^n \text{ for all } x\in\mathbb{R}. \tag{5.16}$$

Proposition 5.20 shows us that then

$$c_n=\frac{f^{(n)}(0)}{n!}=\begin{cases}1 \text{ if } n=0,\\ 0 \text{ otherwise.}\end{cases}$$

Inserting this in (5.16) shows that

$$f(x)=\sum_{n=0}^{\infty}c_nx^n=1,\ \forall x\in\mathbb{R}.$$

It follows that unless f is a constant function, f does not have a power series representation that holds for all $x\in\mathbb{R}$. So all we have to do is to exhibit a concrete function $f:\mathbb{R}\to\mathbb{R}$, which is not constant, but satisfies (i) and (ii). One choice is the function

$$f(x)=\begin{cases}1 \text{ for } |x|\le 1,\\ 1-e^{\frac{-1}{x^2-1}} \text{ for } |x|>1.\end{cases} \tag{5.17}$$

This function is shown in Figure 5.22 (a). At a first glance, the function does not appear to be differentiable at $x=1$! However, that this is indeed the case seems more reasonable from Figure 5.22 (b), which shows the function in a small neighborhood of $x = 1$. A mathematical proof of the fact that the function is arbitrarily often differentiable falls outside the scope of this book. □

The result in Example 5.23 can be formulated differently. In fact the example exhibits a function f for which the Taylor series $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$ converges for all $x \in \mathbb{R}$, but with a sum which does not equal $f(x)$ for all $x \in \mathbb{R}$.

We notice that it is known exactly which functions can be represented via a power series: it is the class of analytic functions (i.e., the functions that are complex differentiable). More details on this can be found in any standard textbook on complex analysis, e.g. [4].

5.3 General infinite sums of functions

As we have seen, a power series consists of infinite sums of terms $c_n x^n$, $n \in \mathbb{N}$. Since not all functions have a representation of this type, it is natural also to consider infinite sums of other types of "simple" functions. More generally, we will consider a family of functions $f_0, f_1, f_2, \ldots$ with the *same* domain of definition I, and attempt to define the function

$$f(x) = f_0(x) + f_1(x) + f_2(x) + \cdots + f_n(x) + \cdots = \sum_{n=0}^{\infty} f_n(x). \quad (5.18)$$

Here it is of course important to know for which values of x the expression for $f(x)$ makes sense: this is the case exactly for the $x \in I$ for which *the series of numbers* $\sum_{n=0}^{\infty} f_n(x)$ is convergent. The function f is called *the sum of the infinite series* associated to $f_0, f_1, \ldots$. As before we will call the expression

$$S_N(x) = f_0(x) + f_1(x) + \cdots + f_N(x)$$

the Nth partial sum of $\sum_{n=0}^{\infty} f_n(x)$; it is now a function depending on x.

In practice, one can only deal with a representation of the type (5.18) if the functions f_n have some kind of common structure. In the next chapter we will consider infinite series where the functions f_n are trigonometric functions.

We emphasize once more that care is needed while working with infinite sums: many of the results we use for finite sums can not be generalized to infinite sums. For example, we know that any finite sum of continuous functions is continuous; the next example shows that the sum of infinitely many continuous functions does *not* need to be continuous. This demonstrates clearly that we can not just take the known rules for finite sums and use them on infinite sums.

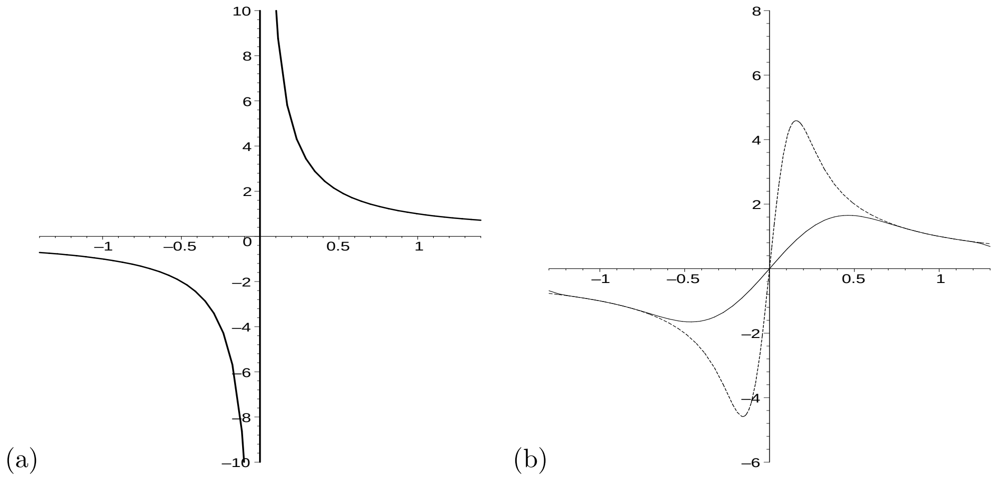


Figure 5.24 *(a): The function* $f(x) = \sum_{n=0}^{\infty} x(1-x^2)^n, \ x \in]-\sqrt{2}, \sqrt{2}[$. *(b): The partial sums* $S_5(x) = \sum_{n=0}^{5} x(1-x^2)^n$ *(unbroken line) and* $S_{50}(x) = \sum_{n=0}^{50} x(1-x^2)^n$ *(dotted). The more terms we include in the partial sum, the more the graph looks like the graph of the sum of the infinite series, shown in Figure 5.24 (a). However, all partial sums "turn off" near the point* $(0,0)$*, where all the graphs go through.*

Example 5.25 Consider the series

$$\sum_{n=0}^{\infty} x(1-x^2)^n.$$

We want to find the values of $x \in \mathbb{R}$ for which the series is convergent, and to determine the sum of the infinite series.

For $x = 0$ the series is convergent with sum 0. For a fixed $x \neq 0$ we can view the series as a geometric series with quotient $1 - x^2$, which is multiplied with the number x; by Theorem 5.2, the series is convergent for $|1 - x^2| < 1$, i.e., for $0 < |x| < \sqrt{2}$, and divergent for $|x| \geq \sqrt{2}$. For $0 < |x| < \sqrt{2}$ we obtain the sum

$$\sum_{n=0}^{\infty} x(1-x^2)^n = x\frac{1}{1-(1-x^2)} = x\frac{1}{x^2} = \frac{1}{x}.$$

Thus the sum of the infinite series is

$$f(x) = \sum_{n=0}^{\infty} x(1-x^2)^n = \begin{cases} \frac{1}{x} \text{ for } 0 < |x| < \sqrt{2}, \\ 0 \text{ for } x = 0. \end{cases}$$

We see that the sum of the infinite series has a discontinuity at $x = 0$ even though all functions $x(1-x^2)^n$ are continuous! Figure 5.24 (a) shows the sum of the infinite series as a function of x; compare with the partial sums for $N = 5$ and for $N = 50$, shown on Figure 5.24 (b). □

The difference between finite and infinite sums can be used to make surprising and non-intuitive constructions. For example, one can construct a function $f : \mathbb{R} \to \mathbb{R}$, which is continuous, but nowhere differentiable:

Example 5.26 Given constants $A > 1, B \in]0,1[$, for which $AB \geq 1$, we attempt to consider the function

$$f(x) = \sum_{n=1}^{\infty} B^n \cos(A^n x), \;\; x \in \mathbb{R}.$$

Since $|B^n \cos(A^n x)| \leq B^n$ and $B \in]0,1[$, Theorem 4.20 combined with Theorem 5.2 show that the series defining f is actually convergent. A result stated formally in the next section, Theorem 5.33, tells us that f is continuous. Assuming that A is an odd integer and that the product AB is sufficiently large, Weierstrass proved in 1887 that the function f is nowhere differentiable. We will not go into details here. □

5.4 Uniform convergence

So far, our definition of convergence of an infinite series of functions has been *pointwise:* in fact, (5.18), i.e.,

$$f(x) = \sum_{n=1}^{\infty} f_n(x), \;\; x \in I,$$

merely means that if we fix an arbitrary x in the domain of f, the partial sums of the series converge to $f(x)$. In mathematical terms this means that for every fixed $x \in I$ and any given $\epsilon > 0$ we can find $N_0 \in \mathbb{N}$ such that

$$\left| f(x) - \sum_{n=1}^{N} f_n(x) \right| \leq \epsilon \tag{5.19}$$

for all $N \geq N_0$.

However, as we will explain now, this type of convergence is not sufficient in all situations. As already mentioned, the motivation behind (5.18) is to write a complicated signal f as an infinite sum of simpler functions. However, since a computer can only deal with finite sums, for applications of this result in practice we always have to replace the infinite sum in (5.18) by a *finite sum*, i.e., by a certain partial sum. In other words, we want to be able to find $N \in \mathbb{N}$ such that

$$|f(x) - S_N(x)|$$

is sufficiently small, *simultaneously for all x in the considered interval.* As the following example demonstrates, this is not always possible.

Example 5.27 Consider the function

$$f(x) = \frac{1}{1-x}, \;\; x \in]-1,1[.$$

Then, as we saw in Theorem 5.2,

$$f(x) = \sum_{n=0}^{\infty} x^n, \tag{5.20}$$

understood in the sense that the partial sums $S_N(x)$ of the right-hand side of (5.20) converges to $f(x)$ for any $x \in]-1,1[$. However, if we fix any $N \in \mathbb{N}$, we have

$$\begin{aligned} \left| f(x) - \sum_{n=0}^{N} x^n \right| &= \left| \sum_{n=N+1}^{\infty} x^n \right| \\ &= \frac{|x|^{N+1}}{1-x}. \end{aligned}$$

No matter how we choose $N \in \mathbb{N}$, there exists an $x \in]-1,1[$ for which this difference is large (see Figure 5.4). Thus, we are not able to obtain a good approximation simultaneously for all $x \in]-1,1[$.

Note that the above problem disappears if we only consider the function f on any interval of the type $[-r,r]$ for some $r \in]0,1[$; in this case, for any $x \in [-r,r]$ we have

$$\begin{aligned} \left| f(x) - \sum_{n=0}^{N} x^n \right| &= \frac{|x|^{N+1}}{1-x} \\ &\leq \frac{r^{N+1}}{1-r}. \end{aligned}$$

Since

$$\frac{r^{N+1}}{1-r} \to 0 \text{ as } N \to \infty,$$

we can thus make $\left| f(x) - \sum_{n=0}^{N} x^n \right|$ as small as we want, simultaneously for all $x \in [-r,r]$, by choosing $N \in \mathbb{N}$ sufficiently large. □

Example 5.27 suggests that we introduce the following type of convergence.

Definition 5.28 *Given functions $f_1, f_2, \dots, f_n, \dots$ defined on an interval I, assume that the function*

$$f(x) = \sum_{n=1}^{\infty} f_n(x), \ \ x \in I,$$

is well defined. Then we say that $\sum_{n=1}^{\infty} f_n$ converges uniformly to f on the interval I if for each $\epsilon > 0$ we can find an $N_0 \in \mathbb{N}$ such that

$$\left| f(x) - \sum_{n=1}^{N} f_n(x) \right| \leq \epsilon \text{ for all } \ x \in I \text{ and all } N \geq N_0.$$

Let us return to Example 5.27:

Example 5.29 Formulated in terms of uniform convergence, our result in Example 5.27 shows that

- $\sum_{n=0}^{\infty} x^n$ does not converge uniformly to $f(x) = \frac{1}{1-x}$ on $I =]-1, 1[$;
- $\sum_{n=0}^{\infty} x^n$ converges uniformly to $f(x) = \frac{1}{1-x}$ on any interval $I = [-r, r], \ r \in]0, 1[$.

It turns out that Example 5.29 is typical for power series: in general, nothing guarantees that a power series with radius of convergence ρ converges uniformly on $]-\rho, \rho[$. On the other hand we always obtain uniform convergence if we shrink the interval slightly:

Proposition 5.30 *Let $\sum_{n=0}^{\infty} c_n x^n$ be a power series with radius of convergence $\rho > 0$. Then $\sum_{n=0}^{\infty} c_n x^n$ is uniformly convergent on any interval of the form $[-r, r]$, where $r \in]0, \rho[$.*

Proof: Given any $x \in [-r, r]$,

$$\begin{aligned} \left| \sum_{n=0}^{\infty} c_n x^n - \sum_{n=0}^{N} c_n x^n \right| &= \left| \sum_{n=N+1}^{\infty} c_n x^n \right| \\ &\leq \sum_{n=N+1}^{\infty} |c_n x^n| \\ &\leq \sum_{n=N+1}^{\infty} |c_n r^n|. \end{aligned}$$

By the fact that $\sum_{n=1}^{\infty} c_n r^n$ is absolutely convergent, we can make $\sum_{n=N+1}^{\infty} |c_n r^n|$ as small as we want, simply by choosing N sufficiently large. This proves the result. □

We will now introduce the so-called *majorant series*, associated with a given infinite series with terms depending on a variable. As we will see soon, the existence of such series turn out to be important for the continuity and differentiability of the given infinite series.

Definition 5.31 (Majorant series) *Assume that the functions $f_1, f_2, \ldots, f_n, \ldots$ are defined on an interval I and consider the infinite series*

$$\sum_{n=1}^{\infty} f_n(x), \ \ x \in I. \tag{5.21}$$

(i) An infinite series $\sum_{n=1}^{\infty} k_n$ with constant and positive terms k_n is a majorant series for the infinite series (5.21) *on the interval I if for any $n \in \mathbb{N}$*

$$|f_n(x)| \leq k_n, \ \ \forall\, x \in I. \tag{5.22}$$

(ii) An infinite series $\sum_{n=1}^{\infty} k_n$ that is convergent and satisfies (5.22)*, is called a convergent majorant series for the infinite series* (5.21)*.*

Example 5.32 We will check the existence of a convergent majorant series for the following infinite series:

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(nx)}{n^2}, \ \ x \in [-\pi, \pi];$

(ii) $\displaystyle\sum_{n=1}^{\infty} \frac{e^{-nx}}{n}, \ \ x \in [0, \infty[;$

(iii) $\displaystyle\sum_{n=1}^{\infty} \frac{e^{-nx}}{n}, \ \ x \in [1, \infty[.$

For the infinite series in (i) we have $f_n(x) = \frac{\sin(nx)}{n^2}$; thus

$$|f_n(x)| = \left|\frac{\sin(nx)}{n^2}\right| = \frac{|\sin(nx)|}{n^2}.$$

Since the sine-function exclusively takes values in the interval $[-1, 1]$ it follows that

$$|f_n(x)| \leq \frac{1}{n^2}, \ \ \forall x \in [-\pi, \pi].$$

Thus $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a majorant series for the infinite series in (i). Using Example 4.34 we see that it is indeed a convergent majorant series.

We now consider the infinite series in (ii): here we have $f_n(x) = \frac{e^{-nx}}{n}$. Since the exponential function only assumes positive values, we have that

$$|f_n(x)| = \left|\frac{e^{-nx}}{n}\right| = \frac{e^{-nx}}{n}.$$

For $n \in \mathbb{N}$ the function $x \mapsto e^{-nx}$ is decreasing. Thus, for $x \in [0, \infty[$ we obtain the largest value precisely for $x = 0$. We conclude that

$$|f_n(x)| = \frac{e^{-nx}}{n} \leq \frac{e^0}{n} = \frac{1}{n}, \ \forall x \in [0, \infty[.$$

Thus the series $\sum_{n=1}^{\infty} \frac{1}{n}$ is a majorant series for the infinite series in (ii). According to Example 4.34 the majorant series is divergent. Now, since the above calculation shows that the infinite series $\sum_{n=1}^{\infty} \frac{1}{n}$ is the majorant series with the *smallest possible terms,* we conclude that

$$\sum_{n=1}^{\infty} \frac{e^{-nx}}{n} \text{ does not have a conv. majorant series on the interval } [0, \infty[. \quad (5.23)$$

For the infinite series in(iii) we do precisely as in (ii), but we apply that $x \in [1, \infty[$. Therefore the maximal value for the function $x \mapsto e^{-nx}$ is assumed for $x = 1$, and we conclude that

$$|f_n(x)| = \frac{e^{-nx}}{n} \leq \frac{e^{-n}}{n} = \frac{1}{ne^n}.$$

Thus, the infinite series $\sum_{n=1}^{\infty} \frac{1}{e^n n}$ is a majorant series for the infinite series (iii). The majorant series is convergent (check it, e.g., using the quotient test). The conclusion is that

$$\sum_{n=1}^{\infty} \frac{e^{-nx}}{n} \text{ has a convergent majorant series on the interval } [1, \infty[. \quad (5.24)$$

□

Note the two different conclusions in (5.23) and (5.24): they show that when we claim that an infinite series has a convergent majorant series we must carefully state the interval we are working with.

We end this section with a few important results concerning continuity and differentiability of sums of infinite series of functions.

Theorem 5.33 (Continuity of the sum of infinite series) *Assume that the functions $f_1, f_2, \ldots$ are defined and continuous on an interval I, and that there exist positive constants $k_1, k_2, \ldots$ such that*

(i) $|f_n(x)| \leq k_n, \ \forall x \in I, \ n \in \mathbb{N};$

(ii) $\sum_{n=1}^{\infty} k_n$ is convergent.

Then the infinite series $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly; and the function

$$f(x) = \sum_{n=1}^{\infty} f_n(x), \ x \in I$$

is continuous.

Theorem 5.33 is known in the literature under the name *Weierstrass' M-test*.

Finally, we mention that uniform convergence of an infinite series consisting of integrable functions allows us to integrate term-wise:

Theorem 5.34 *Assume that the functions $f_1, f_2, \dots$ are continuous on the interval I and that $\sum_{n=1}^{\infty} f_n(x)$ is uniformly convergent. Then, the function $x \mapsto \sum_{n=1}^{\infty} f_n(x)$ is continuous on I; furthermore, for any $a, b \in I$,*

$$\int_a^b \sum_{n=1}^{\infty} f_n(x)dx = \sum_{n=1}^{\infty} \int_a^b f_n(x)dx.$$

If we want the sum of an infinite series to be differentiable, we need slightly different conditions, stated below; on the theoretical level, the result can be used to prove Theorem 5.17.

Theorem 5.35 *Assume that the functions $f_1, f_2, \dots$ are defined and differentiable with a continuous derivative on the interval I, and that the function $f(x) = \sum_{n=1}^{\infty} f_n(x)$ is well defined on I. Assume also that there exist positive constants $k_1, k_2, \dots$ such that*

(i) $|f_n'(x)| \le k_n, \ \forall x \in I, \ n \in \mathbb{N}$;

(ii) $\sum_{n=1}^{\infty} k_n$ *is convergent.*

Then f is differentiable on I, and

$$f'(x) = \sum_{n=1}^{\infty} f_n'(x).$$

Theorem 5.34 has an important consequence for integration of power series:

Corollary 5.36 *Let $\sum_{n=0}^{\infty} c_n x^n$ be a power series with radius of convergence $\rho > 0$. Then, for any $b \in]-\rho, \rho[$,*

$$\int_0^b \sum_{n=0}^{\infty} c_n x^n dx = \sum_{n=0}^{\infty} \frac{c_n}{n+1} b^{n+1}.$$

Example 5.37 The infinite series

$$\sum_{n=0}^{\infty} \frac{1}{3^n} \cos(nx), \ x \in \mathbb{R},$$

has the form $\sum_{n=0}^{\infty} f_n(x)$ with

$$f_n(x) = \frac{1}{3^n} \cos(nx).$$

We see that

$$|f_n(x)| = \left|\frac{1}{3^n}\cos(nx)\right| \leq \frac{1}{3^n} = \left(\frac{1}{3}\right)^n, \ \forall x \in \mathbb{R}.$$

Thus $\sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^n$ is a convergent majorant series for $\sum_{n=0}^{\infty} f_n(x)$. It follows now from Theorem 5.33 that the function

$$f(x) := \sum_{n=0}^{\infty}\frac{1}{3^n}\cos(nx), \ x \in \mathbb{R},$$

is continuous. Since $f_n'(x) = -\frac{n}{3^n}\sin(nx)$, we furthermore see that

$$\sum_{n=0}^{\infty} f_n'(x) = -\sum_{n=0}^{\infty}\frac{n}{3^n}\sin(nx), \ x \in \mathbb{R}. \tag{5.25}$$

The infinite series (5.25) has the convergent majorant series $\sum_{n=0}^{\infty}\frac{n}{3^n}$ (check this, e.g., using the quotient test). Via Theorem 5.34 we now conclude that the function f is differentiable, and that

$$f'(x) = \sum_{n=0}^{\infty} f_n'(x) = -\sum_{n=0}^{\infty}\frac{n}{3^n}\sin(nx), \ x \in \mathbb{R}. \qquad \square$$

We will now present an application of power series. It is know that the integral $\int e^{-x^2}\,dx$ can not be expressed as a simple sum of the classical standard functions; nevertheless the following example shows how we can use results about infinite series to approximate the integral $\int_0^1 e^{-x^2}\,dx$ within any desired tolerance.

Example 5.38 We will estimate the integral

$$\int_0^1 e^{-x^2}dx.$$

Now, by (5.8),

$$e^{-x^2} = \sum_{n=0}^{\infty}\frac{(-x^2)^n}{n!}, \ x \in \mathbb{R}.$$

It is allowed to integrate a power series over a bounded interval term-wise, see Corollary 5.36; in the present case, this shows that

$$\begin{aligned}
\int_0^1 e^{-x^2} dx &= \int_0^1 \left(\sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} \right) dx \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^1 x^{2n} \, dx \\
&= \left[\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)n!} \right]_{x=0}^{1} \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)n!}. \qquad (5.26)
\end{aligned}$$

Thus, we have now obtained an exact expression for the integral in terms of an infinite series. We can obtain an arbitrarily precise value for the integral by considering a partial sum of (5.26) with sufficiently many terms. We can even estimate how many terms we need to include in order to obtain a certain precision: indeed, according to Theorem 4.38,

$$\left| \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)n!} - \sum_{n=0}^{N} (-1)^n \frac{1}{(2n+1)n!} \right| \leq \frac{1}{(2N+3)(N+1)!}.$$

If we want to find the sum of the infinite series in (5.26) within a tolerance of 10^{-3}, we can choose $N \in \mathbb{N}$ such that

$$\frac{1}{(2N+3)(N+1)!} \leq 10^{-3}.$$

This inequality is satisfied already for $N = 4$, so with an error of maximally 10^{-3} we have

$$\sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1) \cdot n!} \approx \sum_{n=0}^{4} (-1)^n \frac{1}{(2n+1) \cdot n!} = 0.747. \qquad \square$$

.

5.5 An application: signal transmission

Modern technology often requires that information can be sent from one place to another; one speaks about *signal transmission*. It occurs, e.g., in connection with wireless communication, the internet, computer graphics, or transfer of data from CD-ROM to computer.

All types of signal transmission are based on transmission of a series of numbers. The first step is to convert the given information (called the *signal*) to a series of numbers, and this is where the question of having a series representation comes in: if we know that a signal is given as a function f which has a power series representation

$$f(x) = \sum_{n=0}^{\infty} c_n x^n, \tag{5.27}$$

then all information about the function f is stored in the coefficients $\{c_n\}_{n=0}^{\infty}$. In other words: if we know the coefficients $\{c_n\}_{n=0}^{\infty}$, then we can immediately use (5.27) to find out which signal we are dealing with.

Let us explain how this can be used in signal transmission. Assume for example that a sender $\mathcal{S}$ wishes to send a picture to a receiver $\mathcal{R}$. For simplification, we assume that the picture is given as a graph of a function f, and that f can be represented as a power series. Then the transmission can be done in the following way:

- $\mathcal{S}$ finds the coefficients $c_0, c_1, \dots$ such that $f(x) = \sum_{n=0}^{\infty} c_n x^n$;
- $\mathcal{S}$ sends the coefficients $a_0, a_1, \dots$;
- $\mathcal{R}$ receives the coefficients $c_0, c_1, \dots$;
- $\mathcal{R}$ reconstructs the signal by multiplying the coefficients c_n by x^n and forming the infinite series $f(x) = \sum_{n=0}^{\infty} c_n x^n$.

In practice there are a few more steps not described here. For example, $\mathcal{S}$ can not send an infinite sequence of numbers $c_0, c_1, \dots$: it is possible to send only a finite sequence of numbers $c_0, c_1, \dots, c_N$. The choice of the numbers to send needs to be done with care: we want that the signal we actually transmit, i.e.,

$$f^\sharp(x) = c_0 + c_1 x + \cdots + c_N x^N = \sum_{n=0}^{N} c_n x^n,$$

looks and behaves like the original signal

$$f(x) = c_0 + c_1 x + \cdots + c_N x^N + \cdots = \sum_{n=0}^{\infty} c_n x^n.$$

This usually forces N to be large. Another issue is that $\mathcal{S}$ can only send the numbers $c_0, \dots, c_N$ with a certain precision, for example, 50 digits. So in reality $\mathcal{S}$ will send some numbers

$$\tilde{c}_0 \simeq c_0, \;\; \tilde{c}_1 \simeq c_1, \; \tilde{c}_2 \simeq c_2, \; \dots, \tilde{c}_N \simeq c_N,$$

and $\mathcal{R}$ receives coefficients corresponding to the signal

$$\tilde{f}(x) = \sum_{n=0}^{N} \tilde{c}_n x^n.$$

The fact that the numbers c_n can only be sent with a certain precision is well known in engineering; one speaks about *quantification*.

The process above needs to be made such that

$$\tilde{f}(x) \simeq f(x)$$

— it does not help if $\mathcal{S}$ sends a picture of an elephant and $\mathcal{R}$ receives something that looks like a giraffe! Sometimes it is unsuitable just to send the *first* $N+1$ coefficients $c_0, \ldots, c_N$: it will be much more reasonable to send the $N+1$ most important coefficients, which usually means the (numerically) *largest* coefficients. In this case one also has to send information about which powers x^n the coefficients that are sent belong to; we speak about *coding* of the information.

The idea of sending the $N+1$ largest coefficients instead of just the first $N+1$ coefficients can be seen as the first step towards *best N-term approximation,* a subject to which we return in Section 6.9. Observe that in the technical sense, to send the largest coefficients first usually corresponds to *reordering* the terms in the given infinite series; as we have seen in Section 4.4 this can lead to further complications if the series does not converge unconditionally, so it has to be done with care.

We also note that to send only a finite number of numbers $c_0, a_1, \ldots, c_N$, rather than the full sequence, corresponds to a *compression,* in the sense that a part of the information determining the signal is thrown away. In the situation discussed here, this type of compression can not be avoided: we need to have a finite sequence in order to be able to transmit. Usually the word "compression" appears in a slightly different sense, namely, that we want the information given in a finite sequence to be represented efficiently using a shorter sequence.

Example 5.39 Assume that $\mathcal{S}$ wishes to send the function

$$f(x) = \sin x, \;\; x \in [0, 3]$$

to $\mathcal{R}$. In Example 5.9 we showed that

$$\sin x = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n+1}}{(2n+1)!} = 0 + x + 0 \cdot x^2 - \frac{1}{6}x^3 + 0 \cdot x^4 + \frac{1}{120}x^5 + \cdots .$$

Since $\frac{1}{6} \approx 0.1666, \; \frac{1}{120} \approx 0.0083$, $\mathcal{S}$ can choose to send the numbers

$$0; 1; \;\; 0; \;\; -0.1666; \;\; 0; \;\; 0.0083,$$

according to which $\mathcal{R}$ will reconstruct the function as

$$\tilde{f}(x) = x - 0.1666x^3 + 0.0083x^5 \approx f(x).$$

Figure 5.40 shows that the reconstructed signal is close to the original signal when $x \in [0, 2]$, but then the approximation starts to deviate from

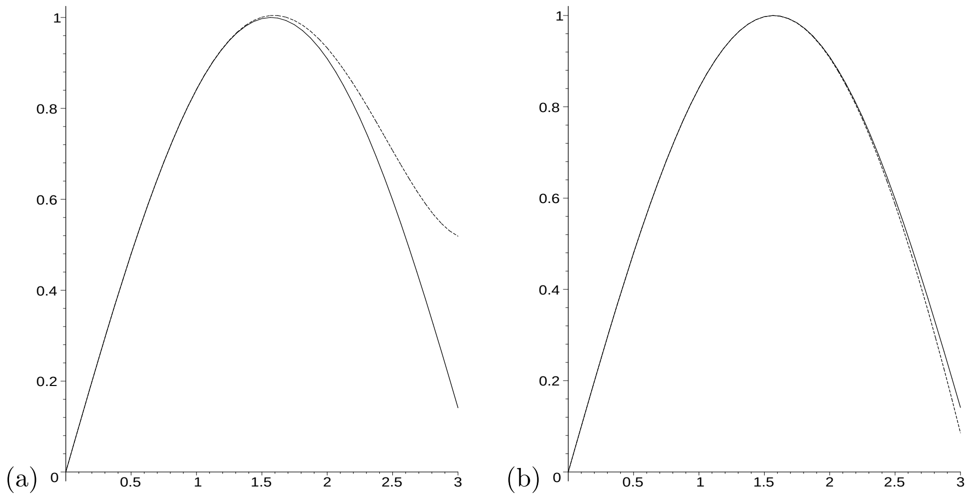


Figure 5.40 *(a): The functions* $f(x) = \sin x$ *(unbroken line) and* $\tilde{f}(x) = x - 0.1666x^3 + 0.0083x^5$. *(b): The functions* $f(x) = \sin x$ *(unbroken line) and* $\tilde{f}(x) = x - 0.1666x^3 + 0.0083x^5 - 0.000198x^7$.

the signal. A better reconstruction can be reached if we also send the coefficient belonging to x^7, namely $-\frac{1}{7!} \simeq -0.000198$; then

$$\tilde{f}(x) = x - 0.1666x^3 + 0.0083x^5 - 0.000198x^7,$$

see Figure 5.40. In practice, transmission of pictures is done in a different way than described here. One of the problems with the principle above is that not all functions have a power series representation. Thus, in many situations we are forced to work with other types of series representations, a topic to which we return in the next chapters.

6
Fourier Analysis

Although we already saw the definition of an infinite series of general functions, the main topic in the previous chapter was power series. Unfortunately, only a relatively limited class of functions has a power series expansion, so often we need to seek other tools to represent functions.

Fourier series and the Fourier transform are such tools. Fourier analysis is a large and classical area of mathematics, dealing with representations of functions on $\mathbb{R}$ via trigonometric functions; as we will see, periodic functions have series expansions in terms of cosine functions and sine functions, while aperiodic functions f have expansions in terms of integrals involving trigonometric functions.

The sections below are of increasing complexity. Section 6.1 and Section 6.2, presenting Fourier series for periodic functions and their approximation properties, are understandable with the background obtained by reading Chapter 4. Section 6.3 introduces Fourier series on complex form, which is a very convenient rewriting of the series in terms of complex exponential functions; we will stick to this format in the rest of the chapter. Section 6.4 presents Parseval's theorem and some of its applications. Section 6.5 relates Fourier series with expansions in terms of orthonormal bases in vector spaces, and can be understood with prior knowledge of vector spaces with inner products. Section 6.6 gives a short introduction to the Fourier transform, which is used to obtain integral representations of aperiodic functions. Section 6.7 motivates Fourier series from the perspective of signal analysis. Section 6.8 exhibits the relationship between the regularity of a function (i.e., how many derivatives it has) with the decay of the Fourier coefficients (i.e., the question of how fast the coefficients in the Fourier series tend to zero). Section 6.9 deals with best N-term approximation, which is a method to obtain efficient approximations of infinite series using as few terms as possible.

6.1 Fourier series

We now introduce a type of series representation, which is well suited for analysis of periodic functions. Consider a function f defined on $\mathbb{R}$, and with period 2π; that is, we assume that

$$f(x+2\pi) = f(x), \;\; x \in \mathbb{R}.$$

We further assume that f belongs to the vector space

$$L^2(-\pi,\pi) := \left\{ f : \quad \int_{-\pi}^{\pi} |f(x)|^2 dx < \infty \right\}. \tag{6.1}$$

Definition 6.1 *To any 2π-periodic function $f \in L^2(-\pi,\pi)$, we associate formally (which we write "$\sim$") the infinite series*

$$f \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx)), \tag{6.2}$$

where

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) \; dx, \;\; n = 0, 1, 2, \ldots$$

and

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) \; dx, \;\; n = 1, 2, \ldots \; .$$

The series (6.2) is called the Fourier series associated to f, and a_n, b_n are called Fourier coefficients.

The assumption (6.1) implies that the integrals defining a_n and b_n are well defined.

Note that in (6.2) and in the expressions for the Fourier coefficients we have written parentheses around nx. Traditionally, these parentheses are skipped, and we will only include them in a few cases; compare, e.g., (6.2) with (6.3) below.

As before, the Nth partial sum of the Fourier series is

$$S_N(x) = \frac{1}{2}a_0 + \sum_{n=1}^{N} (a_n \cos nx + b_n \sin nx). \tag{6.3}$$

The expression (6.2) only introduces Fourier series on the formal level: so far we have not discussed whether the series is convergent or not. Convergence issues are actually quite delicate for Fourier series; we come back to this issue in Section 6.2. However, already now it is clear why we restrict attention to 2π-periodic functions: all the trigonometric functions appearing in the Fourier series are 2π-periodic, so if we want any pointwise relationship between f and the Fourier series, this assumption is necessary.

Calculation of the Fourier coefficients can often be simplified using the rules stated below:

Lemma 6.2 *Let $f \in L^2(-\pi, \pi)$. Then the following hold:*

(R1) *If f is 2π-periodic, then*

$$\int_0^{2\pi} f(x)dx = \int_a^{a+2\pi} f(x)dx, \ \forall a \in \mathbb{R}.$$

(R2) *If f is even, i.e., $f(x) = f(-x)$ for all x, then*

$$\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx, \ \forall a > 0.$$

(R3) *If f is odd, i.e., $f(x) = -f(-x)$ for all x, then*

$$\int_{-a}^{a} f(x)dx = 0, \ \forall a > 0.$$

If f is an even function, then $x \mapsto f(x)\cos nx$ is even and $x \mapsto f(x)\sin nx$ is odd; if f is odd, then $x \mapsto f(x)\cos nx$ is odd and $x \mapsto f(x)\sin nx$ is even. If we combine these observations with the rules above, we obtain the following result:

Theorem 6.3 (Fourier coefficients for even and odd functions)

(i) If f is an even function, then $b_n = 0$ for all n, and

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx \ dx, \ n = 0, 1, 2, \dots \ .$$

(ii) If f is odd, then $a_n = 0$ for all n, and

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx \ dx, \ n = 1, 2, \dots \ .$$

Theorem 6.3 also holds if the function f is not even (resp. odd) in a strict sense, i.e., if there exists at most a finite number of x-values such that $f(x) \neq f(-x)$ (resp. $f(x) \neq -f(-x)$). Let us formulate Theorem 6.3 in words:

Theorem 6.4 (Fourier coefficients for even and odd functions)

(i) The Fourier series for an even function is an infinite series consisting only of a constant term and cosine-functions,

$$f \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx; \tag{6.4}$$

(ii) The Fourier series for an odd function is an infinite series consisting only of sine-functions,

$$f \sim \sum_{n=1}^{\infty} b_n \sin nx. \tag{6.5}$$

Let us find the Fourier series for a few 2π-periodic functions. Note that for such functions, it is sufficient to specify its function values on an interval of length 2π, e.g., the interval $[-\pi, \pi[$.

Example 6.5 Consider the *step function*

$$f(x) = \begin{cases} -1 & \text{if } x \in]-\pi, 0[, \\ 0 & \text{if } x = 0, \pi \\ 1 & \text{if } x \in]0, \pi[, \end{cases} \tag{6.6}$$

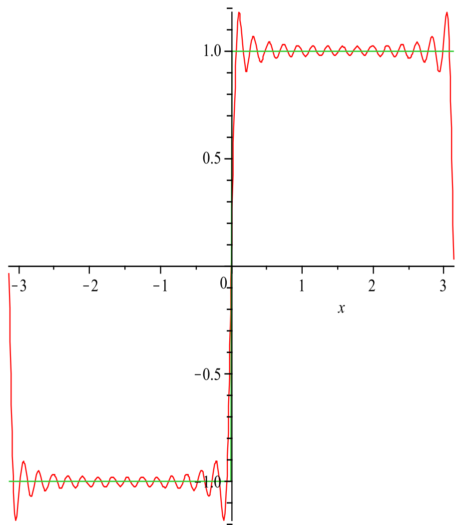
extended to a 2π-periodic function. The function f is odd, so via Theorem 6.3 we see that $a_n = 0$ for all n and that

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \; dx \\ &= \frac{2}{\pi}\int_0^{\pi} \sin nx \; dx \\ &= \begin{cases} 0 & \text{if } n \text{ is even,} \\ \frac{4}{n\pi} & \text{if } n \text{ is odd.} \end{cases} \end{aligned}$$

Thus, the Fourier series is

$$\begin{aligned} f &\sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \\ &= \sum_{n \ odd} \frac{4}{n\pi} \sin nx \\ &= \frac{4}{\pi}(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots) \\ &= \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{2n-1} \sin((2n-1)x). \end{aligned} \tag{6.7}$$

Figure 6.6 shows that the partial sums approximate f well at points where f is continuous. The figure also shows that there is a problem in the



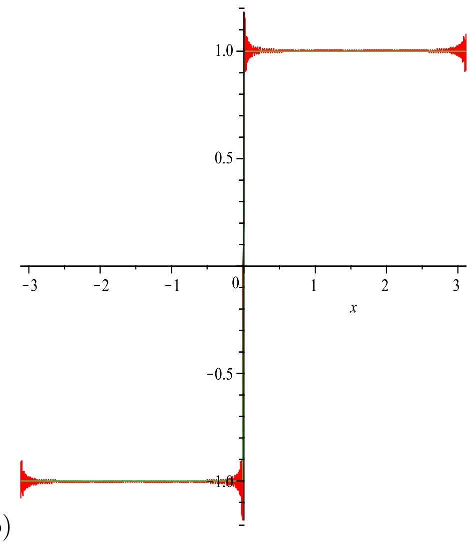


Figure 6.6 *(a): The function f given by* (6.6) *and the partial sum S_{15}. (b): The function f given by* (6.6) *and the partial sum S_{100}. Observe the overshooting of the partial sum around the points where f is discontinuous.*

neighborhood of points where f is discontinuous: all the considered partial sums are "shooting over the correct value". This turns out to be a general phenomena for Fourier series; it is called *Gibb's phenomena,* and appears for arbitrary Fourier series of functions with a jump. One can show that the overshooting is about 9% of the size of the jump, regardless of the considered function.

It is interesting to notice that the Fourier series (6.7) actually converges pointwise for all $x \in \mathbb{R}$; this follows from Theorem 6.12 stated in the next section, but is nontrivial to prove directly based on the results we have presented in the tutorial on infinite series. In order to illustrate the difficulties, we mention that the series $\sum_{n=1}^{\infty} \frac{1}{2n-1}$ is divergent; this can be derived from Example 4.34. So the convergence of (6.7) is a consequence of the terms $\frac{1}{2n-1}$ being multiplied with $\sin((2n-1)x)$. We also note that (6.7) does not converge absolutely: in fact,

$$\sum_{n=1}^{\infty} \left| \frac{1}{2n-1} \sin((2n-1)x) \right|$$

is clearly divergent for $x = \pi/2$. The conclusion of these observations is that the convergence of (6.7) is a consequence of the sign-changes in the term $\sin((2n-1)x)$.

For later use we note that the bad convergence properties of the Fourier series in (6.7) comes from the fact that the coefficients $\frac{1}{2n-1}$ only tend quite slowly to zero as $n \to \infty$. In Section 6.8 we will see that this is related to the function f being nondifferentiable at $x = \pi$. $\square$

Example 6.7 Consider the function

$$f(x) = x, \ x \in]-\pi, \pi[,$$

again extended to a 2π-periodic function. Via Theorem 6.3 we see that $a_n = 0$ for all n and that

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx \ dx \\ &= \frac{2}{\pi}\int_0^{\pi} x\sin nx \ dx. \end{aligned}$$

Using partial integration,

$$\begin{aligned} \int_0^{\pi} x\sin nx \ dx &= [-\frac{x}{n}\cos nx]_0^{\pi} + \frac{1}{n}\int_0^{\pi}\cos nx \ dx \\ &= -\frac{\pi}{n}\cos n\pi + \frac{1}{n^2}[\sin nx]_0^{\pi} \\ &= -\frac{\pi}{n}(-1)^n + 0 \\ &= \frac{\pi}{n}(-1)^{n+1}. \end{aligned}$$

Thus,

$$b_n = \frac{2}{n}(-1)^{n+1},$$

and the Fourier series is

$$\begin{aligned} f &\sim \sum_{n=1}^{\infty}\frac{2}{n}(-1)^{n+1}\sin nx \qquad (6.8) \\ &= 2\left(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x + \cdots\right). \end{aligned}$$

We encourage the reader to plot the function f as well as the first few partial sums of the Fourier series: comparison of the graphs again shows that the partial sums approximate f well, except around points where f is nondifferentiable. Theorem 6.12 in the next section will give a formal explanation of this. □

Section D.4.5 contains several commands that can be used to calculate the Fourier coefficients. Note that we often need to consider certain special cases separately, especially when dealing with the Fourier coefficient a_0:

Example 6.8 Consider the 2π-periodic function given by

$$f(x) = x^2, \ x \in [-\pi, \pi[.$$

Using Maple, one would immediately get that

$$\int x^2\cos nx dx = \frac{n^2x^2\sin nx - 2\sin nx + 2nx\cos nx}{n^3}. \qquad (6.9)$$

However, it is clear that this expression can not be valid for $n = 0$, so the Fourier coefficient a_0 must be calculated separately:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,dx = \frac{1}{\pi}\left[\frac{1}{3}x^3\right]_{x=-\pi}^{\pi} = \frac{2\pi^2}{3}.$$

The other Fourier coefficients can be calculated using (6.9), and the Fourier series is

$$f \sim \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}(-1)^n \frac{1}{n^2}\cos nx. \qquad \square$$

6.2 Fourier's theorem and approximation

We now turn to a discussion of pointwise convergence of Fourier series. Having our experience with power series in mind, it is natural to ask for the Fourier series for a function f to converge pointwise toward $f(x)$ for each $x \in \mathbb{R}$. However, without extra knowledge about the function, this is too optimistic:

Example 6.9 We make a small modification of the function f in Example 6.5, and consider the 2π-periodic function given by

$$g(x) = \begin{cases} -1 & \text{if } x \in]-\pi, 0[, \\ 0 & \text{if } x = 0, \\ 1 & \text{if } x \in]0, \pi]. \end{cases}$$

Note that the only difference between the function f in Example 6.5 and the function g is that the value in $x = \pi$ (and $x = \pi + p2\pi, p \in \mathbb{Z}$) is changed from $f(\pi) = 0$ to $g(\pi) = 1$. The integral of a function does not change if the values of the function are changed in a finite number of points; thus the functions f and g have the same Fourier coefficients and therefore the same Fourier series. In the points $x \in \mathbb{R}$ for which $f(x) \neq g(x)$, the Fourier series can not converge to both values $f(x)$ and $g(x)$. Thus, in general we can not expect that the Fourier series for a function f evaluated in a given point x converges to $f(x)$. $\square$

This example shows that certain conditions are necessary if we want any pointwise relationship between a function and its Fourier series. It turns out that conditions on the smoothness of f will be sufficient in order to obtain such relationships.

Definition 6.10 (Piecewise differentiable function) *A 2π-periodic function f on $\mathbb{R}$ is said to be piecewise differentiable if there exist finitely many points $x_1, x_2, \ldots, x_n$,*

$$-\pi = x_1 < x_2 < \cdots <_{n-1} < x_n = \pi$$

and differentiable functions $f_i : [x_i, x_{i+1}] \to \mathbb{C}$ such that f_i' is continuous and

$$f(x) = f_i(x), \; x \in]x_i, x_{i+1}[.$$

The conditions in Definition 6.10 are technical and needs some explanation. As we have seen in Example 6.5 we will often consider 2π-periodic functions that are given by different expressions in different intervals. Let us assume that we are only dealing with two intervals. In generel notation this means that we consider a function

$$f(x) = \begin{cases} f_1(x), & \text{for } x \in]-\pi, x_2[, \\ f_2(x), & \text{for } x \in]x_2, \pi[, \end{cases} \tag{6.10}$$

for some $x_2 \in]-\pi, \pi[$. For a function of this type Definition 6.10 is satisfied if we can verify that the function f_1 can be considered as a differentiable function on the closed interval $[-\pi, x_2]$, and the function f_2 can be considered as a differentiable function on the closed interval $[x_2, \pi]$. Let us illustrate this by an example.

Example 6.11 Let us verify that the step function f in Example 6.5,

$$f(x) = \begin{cases} -1 & \text{hvis } x \in]-\pi, 0[, \\ 0 & \text{hvis } x = 0, \\ 1 & \text{hvis } x \in]0, \pi[, \\ 0 & \text{hvis } x = \pi. \end{cases},$$

is piecewise differentiable. The function has the form (6.10) with

$$f_1(x) = -1, \; x \in]-\pi, 0[, \quad f_2(x) = -1, \; x \in]0, \pi[.$$

Both functions f_1 and f_2 can be extended to differentiable functions, respectively on $[-\pi, 0]$ and on $[0, \pi]$: indeed, letting

$$f_1(x) = -1, \; x \in [-\pi, 0]$$

we have a differentiable function, and similarly for f_2. Furthermore f_1' and f_2' are continuous. We conclude that the step function is piecewise differentiable, as desired. □

We are now ready to state the main result concerning convergence of Fourier series. We will use the following short notation for the limits of the function values from right, resp. from left at a point x:

$$f(x^+) := \lim_{y \to x^+} f(y), \qquad f(x^-) := \lim_{y \to x^-} f(y).$$

Theorem 6.12 *Assume that the function f is piecewise differentiable and 2π-periodic. Then the Fourier series converges pointwise for all $x \in \mathbb{R}$. For the sum of the Fourier series we have the following:*

(i) If f is continuous in x, then

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) = f(x); \tag{6.11}$$

(ii) If x_j is a point of discontinuity for f, then

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx_j + b_n \sin nx_j) = \frac{1}{2}\left(\lim_{x \to x_j^+} f(x) + \lim_{x \to x_j^-} f(x)\right).$$

The reader is encouraged to be very careful with the exact formulation of Theorem 6.12: under the given assumptions we know that the Fourier series converges for all x, but the sum of the Fourier series does not always equal the value $f(x)$! If the Fourier series converges to $f(x)$ for all $x \in \mathbb{R}$ we will substitute the symbol "$\sim$" in the definition of the Fourier series by "$=$". Theorem 6.12 shows that this always happens for continuous and piecewise differentiable functions:

Corollary 6.13 *Assume that the function f is continuous, piecewise differentiable, and 2π-periodic. Then the following hold:*

(i) For any $x \in \mathbb{R}$,

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx).$$

(ii) The Fourier series converges uniformly to f, and the maximal deviation between $f(x)$ and the partial sum $S_N(x)$ can be estimated by

$$|f(x) - S_N(x)| \leq \frac{1}{\sqrt{N}}\frac{1}{\sqrt{\pi}}\sqrt{\int_{-\pi}^{\pi}|f'(t)|^2 dt}. \tag{6.12}$$

Note that there also exist functions that are not continuous, but for which we can still replace the symbol "$\sim$" by "$=$":

Example 6.14 For the step function in (6.6) we have

$$f(x) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin((2n-1)x), \ \forall x \in \mathbb{R}. \tag{6.13}$$

We can prove (6.13) by splitting the argument into the cases $x \notin \pi\mathbb{Z} := \{0, \pm\pi, \pm 2\pi, \dots\}$ and $x \in \pi\mathbb{Z}$. For $x \notin \pi\mathbb{Z}$ the result follows from Theorem 6.12(i). For $x \in \pi\mathbb{Z}$ it follows from (ii) in the same theorem and the special definition of $f(0)$ and $f(\pi)$: a different choice of these values would not change the Fourier series, but (6.13) would no longer be satisfied for $x \in \pi\mathbb{Z}$. If we, e.g., consider the function g in Example 6.9 we can not substitute the symbol "$\sim$" by "$=$." □

The first part of Theorem 6.12, that is, the statement about the convergence of Fourier series, is called *Fourier's theorem.* Fourier published the result in 1822 in the article [5], actually with the claim that the Fourier series converges without any assumption on the function. He gave quite intuitive reasons for his claim, and the article does not reach the level of precision required for mathematical publications nowadays. Already around the time of its publication, many researchers protested against the result, and later it was proved that the result does not hold as generally as Fourier believed. In fact, things can go very bad if we do not impose the conditions in Theorem 6.12: there exist functions for which all the Fourier coefficients are well defined, but for which the Fourier series does not converge at any point! These functions, however, do not belong to $L^2(-\pi, \pi)$. For functions in $L^2(-\pi, \pi)$ the Fourier series converges pointwise almost everywhere; the exact meaning of "almost everywhere" can be found in textbooks on measure theory.

When this is said, we should also add that one has to admire Fourier for his intuition. Basically he was right, and his claim had a strong influence on development of mathematics: a large part of the mathematics from the nineteenth and twentieth century was invented in the process of finding the conditions for convergence of Fourier series and applying the results to, e.g., solution of differential equations.

The importance of Theorem 6.12 lies in the fact that it shows how a large class of functions can be decomposed into a sum of elementary sine and cosine functions. As a more curious consequence we mention that this frequently allows us to determine the exact sum of certain infinite series:

Example 6.15 Applying (6.13) with $x = \pi/2$ shows that

$$1 = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} \sin((2n-1)\frac{\pi}{2}) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} (-1)^{n-1},$$

or,

$$\sum_{n=1}^{\infty} \frac{1}{2n-1} (-1)^{n-1} = \frac{\pi}{4}.$$ □

We stress the fact that the calculations in Example 6.15 only are valid because we know that we have equality in (6.13): we can not perform this kind of manipulation before we have proved that the symbol "$\sim$" can be replaced by "=."

For a continuous function f, the assumptions in Theorem 6.12 imply that the Fourier series converges uniformly to f. Equation (6.12) can be used to estimate how many terms we need to keep in the Fourier series in order to guarantee a certain approximation of the function f:

Corollary 6.16 *Assume that the function f is continuous, piecewise differentiable, and 2π-periodic. Given any $\epsilon > 0$, the inequality*

$$|f(x) - S_N(x)| \leq \epsilon$$

is satisfied for all $x \in \mathbb{R}$ if we choose N such that

$$N \geq \frac{\int_{-\pi}^{\pi} |f'(t)|^2 dt}{\pi \epsilon^2}. \tag{6.14}$$

Proof: According to Corollary 6.13 we have that

$$|f(x) - S_N(x)| \leq \frac{1}{\sqrt{N}} \frac{1}{\sqrt{\pi}} \sqrt{\int_{-\pi}^{\pi} |f'(t)|^2 dt};$$

thus the condition $|f(x) - S_N(x)| \leq \epsilon$ is satisfied for all $x \in \mathbb{R}$ if we choose $N \in \mathbb{N}$ such that

$$\frac{1}{\sqrt{N}} \frac{1}{\sqrt{\pi}} \sqrt{\int_{-\pi}^{\pi} |f'(t)|^2 dt} \leq \epsilon.$$

This leads to the stated result. □

Note that (6.14) is a "worst case estimate": it gives a value of $N \in \mathbb{N}$ which can be used for all functions satisfying the conditions in Theorem 6.12. In concrete cases where the Fourier coefficients are known explicitly, the next result can often be used to prove that smaller values of N than suggested in (6.14) are sufficient.

Theorem 6.17 *Assume that f is continuous, piecewise differentiable and 2π-periodic, with Fourier coefficients a_n, b_n. Then, for any $N \in \mathbb{N}$,*

$$|f(x) - S_N(x)| \leq \sum_{n=N+1}^{\infty} (|a_n| + |b_n|), \ \forall x \in \mathbb{R}. \tag{6.15}$$

Proof: By Theorem 6.12, the assumptions imply that the Fourier series converges to $f(x)$ for all $x \in \mathbb{R}$. Via (6.11) and (6.3),

$$\begin{aligned} |f(x) - S_N(x)| &= \left| \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \right. \\ &\quad \left. - \left(\frac{1}{2}a_0 + \sum_{n=1}^{N}(a_n \cos nx + b_n \sin nx) \right) \right| \\ &= \left| \sum_{n=N+1}^{\infty} (a_n \cos nx + b_n \sin nx) \right| \\ &\leq \sum_{n=N+1}^{\infty} |a_n \cos nx + b_n \sin nx| \\ &\leq \sum_{n=N+1}^{\infty} (|a_n| + |b_n|). \end{aligned}$$

□

In the next example we compare Corollary 6.16 and Theorem 6.17.

Example 6.18 Consider the 2π-periodic function given by

$$f(x) = x^2, \ x \in [-\pi, \pi[.$$

Our purpose is to find estimates for $N \in \mathbb{N}$ such that

$$|f(x) - S_N(x)| \leq 0.1 \text{ for all } x \in \mathbb{R}. \tag{6.16}$$

By Example 6.8 the Fourier series of f is given by

$$f \sim \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}(-1)^n \frac{1}{n^2} \cos nx.$$

According to Theorem 6.12 we can replace the sign "$\sim$" by "$=$". We first apply (6.14) in Corollary 6.16: it shows that (6.16) is satisfied if

$$N \geq \frac{\int_{-\pi}^{\pi}(2t)^2 dt}{\pi \cdot 0.1^2} = \frac{8\pi^3}{3\pi \cdot 0.1^2} = \frac{800\pi^2}{3} \approx 2632.$$

Let us now apply Theorem 6.17. Via (6.15), followed by an application of (4.32) in Example 4.34,

$$|f(x) - S_N(x)| = 4 \sum_{n=N+1}^{\infty} \frac{1}{n^2} \leq \frac{4}{N}.$$

Thus, (6.16) is satisfied if

$$\frac{4}{N} \leq 0.1,$$

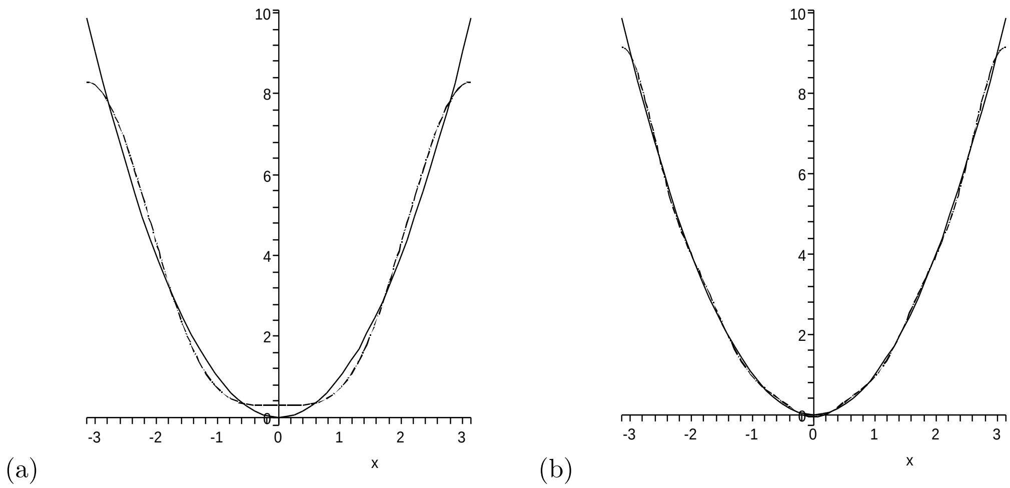


Figure 6.19 *(a): The function $f(x) = x^2$ and the partial sum $S_2(x)$ (dotted), shown on the interval $[-\pi, \pi]$. (b): The function $f(x) = x^2$ and the partial sum $S_5(x)$ (dotted), shown on the interval $[-\pi, \pi]$.*

i.e., if $N \geq 40$.

For the function considered here we see that Theorem 6.17 leads to a much smaller value of N than Corollary 6.16. The difference between the estimates obtained via these two results is getting larger when we ask for better precision: if we decrease the error-tolerance by a factor of 10,

- Corollary 6.16 will increase the value of N by a factor of 100;

- Theorem 6.17 will increase the value of N by a factor of approximately 10.

We note that this result is based on the choice of the considered function: the difference between the use of Corollary 6.16 or Theorem 6.17 depends on the given function.

Figure 6.19 shows that already the partial sum S_5 gives a very good approximation to f, except for values of x which are close to $\pm\pi$. In order also to obtain good approximations close to $x = \pm\pi$, much larger values of N are needed. For example, S_{20} gives a very good approximation of f on the interval $[-3.1, 3.1]$, with a deviation not exceeding 0.005; but on the interval $[3.1, \pi]$, the error increases, and the maximal deviation is approximately 0.2. □

6.3 Fourier series in complex form

Complex exponential functions give a convenient way to rewrite Fourier series. In fact, with our definition of Fourier series in Section 6.1 we have to keep track of the numbers a_n and b_n for $n \in \mathbb{N}$, as well as the special definition of a_0; as we will see, considering Fourier series in complex form will allow us to work with one set of coefficients $\{c_n\}_{n=-\infty}^{\infty}$, where all coefficients are given by a single formula.

Let us denote the imaginary unit number by i. In order to arrive at the announced expression for the Fourier series, we first recall that the complex exponential function is defined by

$$e^{i\theta} = \cos\theta + i\sin\theta,\ \ \theta \in \mathbb{R}.$$

Replacing θ by $-\theta$ in this formula, a few manipulations lead to expressions for the basic trigonometric functions via complex exponentials:

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \quad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}. \tag{6.17}$$

Lemma 6.20 (Partial sum of Fourier series on complex form) *The Nth partial sum of the Fourier series of a function f can be written*

$$S_N(x) = \frac{1}{2}a_0 + \sum_{n=1}^{N}(a_n\cos nx + b_n \sin nx) = \sum_{n=-N}^{N} c_n e^{inx},$$

where

$$c_n := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx, \quad n \in \mathbb{Z}. \tag{6.18}$$

Proof: Let us insert the expressions (6.17) in the formula for the nth term, $n \neq 0$, in the Fourier series for a function $f \in L^2(-\pi,\pi)$:

$$\begin{aligned}
a_n\cos nx + b_n\sin nx &= \frac{1}{\pi}\left(\int_{-\pi}^{\pi} f(x)\cos nx\ dx\right)\frac{e^{inx}+e^{-inx}}{2} \\
&\quad + \frac{1}{\pi}\left(\int_{-\pi}^{\pi} f(x)\sin nx\ dx\right)\frac{e^{inx}-e^{-inx}}{2i} \\
&= \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)(\cos nx - i\sin nx)dx\right)e^{inx} \\
&\quad + \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)(\cos nx + i\sin nx)dx\right)e^{-inx} \\
&= \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx\right)e^{inx} \\
&\quad + \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{inx}dx\right)e^{-inx}.
\end{aligned}$$

Also,

$$\frac{1}{2}a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx.$$

Let us now define the numbers c_n as in (6.18). With this definition, the Nth partial sum of the Fourier series can be written as

$$S_N(x) \quad = \quad \frac{1}{2}a_0 + \sum_{n=1}^{N}(a_n \cos nx + b_n \sin nx) = \sum_{n=-N}^{N} c_n e^{inx},$$

as desired. □

The calculation in Lemma 6.20 leads to the following definition.

Definition 6.21 *The Fourier series of a 2π-periodic function $f \in L^2(-\pi,\pi)$ in complex form is the infinite series given by*

$$f \sim \sum_{n=-\infty}^{\infty} c_n e^{inx}, \tag{6.19}$$

with the coefficients c_n defined as in (6.18).

Mathematically, nothing has changed going from Fourier series in the original form to the complex form: an infinite series is defined via the finite partial sums, and for Fourier series in complex form, the partial sums are exactly the same as before, just written in a different way.

Note also that there are simple relationships between the Fourier coefficients with respect to sine functions and cosine functions, and the Fourier coefficients in complex form:

Lemma 6.22 (From real form to complex form and vice versa) *Let $f \in L^2(-\pi,\pi)$ denote a 2π-periodic function, with Fourier coefficients a_n, b_n on real form and c_n on complex form. Then the following hold:*

(i) The coefficients c_n, $n \in \mathbb{Z}$, are given by

$$c_0 = \frac{1}{2}a_0, \text{ and for } n \in \mathbb{N}, \; c_n = \frac{a_n - ib_n}{2}, \; c_{-n} = \frac{a_n + ib_n}{2}. \tag{6.20}$$

(ii) Alternatively,

$$c_n = \begin{cases} \frac{1}{2}a_0, & n = 0, \\ \frac{1}{2}(a_n - ib_n), & n = 1, 2, \dots, \\ \frac{1}{2}(a_{-n} + ib_{-n}), & n = -1, -2, \dots. \end{cases} \tag{6.21}$$

(iii) The coefficients a_n, b_n are given by

$$a_0 = 2c_0, \; a_n = c_n + c_{-n}, \; b_n = i(c_n - c_{-n}). \tag{6.22}$$

Proof: In order to prove (i) we will use (6.18),

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx, \quad n \in \mathbb{Z}.$$

For $n = 0$ it follows directly that

$$c_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{0} dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{2} a_0.$$

For $n > 0$, using that cosine is an even function and sine is an odd function,

$$\begin{aligned} c_n &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \left(\cos(-nx) + i \sin(-nx)\right) \, dx \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) \, dx - \frac{i}{2\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) \, dx \\ &= \frac{1}{2}(a_n - ib_n). \end{aligned}$$

The proof for the case $n < 0$ is similar. The result in (ii) is a rewriting of the result in (i), and (iii) follows by solving the equations in (i) with respect to a_n and b_n. $\square$

For functions f that only assume real values there is a useful relation between the Fourier coefficients c_n and c_{-n}:

Lemma 6.23 (Fourier coefficients for real functions)
Assume that the function $f \in L^2(-\pi, \pi)$ only assumes real values. Then

$$c_{-n} = \overline{c_n}, \ n \in \mathbb{Z}.$$

Proof: By assumption we have $f(x) = \overline{f(x)}$ for all x. By direct calculation it follows that

$$\begin{aligned} c_{-n} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{inx} dx &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \overline{f(x) e^{-inx}} dx \\ &= \overline{\frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx} \\ &= \overline{c_n}. \end{aligned}$$

$\square$

Let us find the Fourier series on complex form for the step function:

Example 6.24 For the step function in Example 6.5,

$$f(x) = \begin{cases} -1 & \text{if } x \in]-\pi, 0[, \\ 0 & \text{if } x = 0, \\ 1 & \text{if } x \in]0, \pi[, \\ 0 & \text{if } x = \pi \end{cases} \tag{6.23}$$

we saw that the Fourier coefficients are

$$a_n = 0 \text{ for all } n, \text{ and } b_n = \begin{cases} 0 & \text{if } n \text{ is even,} \\ \frac{4}{n\pi} & \text{if } n \text{ is odd.} \end{cases} \tag{6.24}$$

It follows immediately from (6.20) that

$$c_n = 0 \text{ if } n \text{ even.}$$

For odd values of n we split into positive and negative n: for $n = 1, 3, \ldots,$ we have

$$c_n = \frac{1}{2}(a_n - ib_n) = \frac{1}{2}(0 - i\frac{4}{n\pi}) = \frac{-2i}{n\pi},$$

and for $n = -1, -3, \ldots,$

$$c_n = \frac{1}{2}(a_{-n} + ib_{-n}) = \frac{1}{2}(0 + i\frac{4}{-n\pi}) = \frac{-2i}{n\pi}.$$

In this case we obtain the same expression for positive and negative n, so the result can be expressed as

$$c_n = \begin{cases} 0, & n \text{ even,} \\ -\frac{2i}{\pi n}, & n \text{ odd.} \end{cases}$$

Thus, the Fourier series on complex form is

$$f \sim -\frac{2i}{\pi} \sum_{n \in \mathbb{Z}, n \text{ odd}} \frac{1}{n} e^{inx}. \qquad \square$$

Note finally that when we speak about the Fourier series on real contra complex form, it only deals with the question whether we express the Fourier series in terms of cosine/sine functions or in terms of complex exponential functions. It has nothing to do with the question whether the coefficients a_n and b_n are real or complex. Indeed, the Fourier coefficients on real form, a_n and b_n, can in general be complex numbers.

6.4 Parseval's theorem

So far, we have used Fourier series as a tool for approximation of functions. We now mention another type of result, which shows that we can use

Fourier series to determine certain infinite sums. The key to this type of result is *Parseval's theorem*:

Theorem 6.25 *Assume that the function $f \in L^2(-\pi,\pi)$ has the Fourier coefficients $\{a_n\}_{n=0}^{\infty}, \{b_n\}_{n=1}^{\infty}$, or, in complex form, $\{c_n\}_{n=-\infty}^{\infty}$. Then*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)|^2dx = \frac{1}{4}|a_0|^2+\frac{1}{2}\sum_{n=1}^{\infty}(|a_n|^2+|b_n|^2) = \sum_{n=-\infty}^{\infty}|c_n|^2.$$

As a consequence of Parseval's theorem, we note that if $f \in L^2(-\pi,\pi)$, then

$$\sum_{n=-\infty}^{\infty}|c_n|^2 < \infty. \tag{6.25}$$

One can prove that the opposite holds as well: if (6.25) holds, then

$$f(x) := \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

defines a function in $L^2(-\pi,\pi)$. Thus, we have a one-to-one correspondence between functions in $L^2(-\pi,\pi)$ and sequences satisfying (6.25). We will not go into a discussion of all the important theoretical implications of this result (see, e.g., [9]), but restrict ourselves to a few computational consequences of Theorem 6.25:

Example 6.26
(i) In Example 6.5, we saw that the Fourier series for the function

$$f(x) = \begin{cases} -1 & \text{if } x \in [-\pi,0[, \\ 0 & \text{if } x = 0, \\ 1 & \text{if } x \in]0,\pi[\end{cases}$$

is

$$f \;\sim\; \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin((2n-1)x).$$

Now,

$$\int_{-\pi}^{\pi}|f(x)|^2dx = \int_{-\pi}^{\pi}dx = 2\pi,$$

and

$$\frac{1}{4}|a_0|^2+\frac{1}{2}\sum_{n=1}^{\infty}(|a_n|^2+|b_n|^2) = \frac{1}{2}\sum_{n\ odd}(\frac{4}{n\pi})^2 = \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}.$$

By Parseval's theorem this implies that

$$\frac{1}{2\pi}\,2\pi = \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2},$$

i.e.,

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}.$$

(ii) By Example 6.7, the Fourier series for the function

$$f(x) = x,\ \ x \in [-\pi, \pi[$$

is given by

$$f \sim \sum_{n=1}^{\infty}\frac{2}{n}(-1)^{n+1}\sin nx.$$

Furthermore,

$$\int_{-\pi}^{\pi}|f(x)|^2dx = \int_{-\pi}^{\pi}x^2dx = \frac{1}{3}[x^3]_{-\pi}^{\pi} = \frac{2\pi^3}{3}.$$

Now, by Parseval's theorem,

$$\begin{aligned}\frac{1}{2\pi}\,\frac{2\pi^3}{3} &= \frac{1}{2}\sum_{n=1}^{\infty}\left|\frac{2}{n}(-1)^{n+1}\right|^2\\ &= 2\sum_{n=1}^{\infty}\frac{1}{n^2}.\end{aligned}$$

Therefore

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}. \tag{6.26}$$

This result is very surprising: think about how the number π is introduced as the fraction between the circumference and the diameter in a circle. The result in (6.26) has been known for a long time, but an *explanation* of *why* π appears in this context was only given in the second half of the twentieth century by the Austrian mathematician Hlawka. □

Let us for a moment concentrate on Fourier series in complex form. Given a function $f \in L^2(-\pi, \pi)$ with Fourier coefficients $\{c_n\}_{n=-\infty}^{\infty}$, an interesting consequence of Parseval's theorem is that

$$\sum_{n=-\infty}^{\infty}|c_n|^2 < \infty.$$

In particular, this implies that

$$c_n \to 0 \text{ as } n \to \pm\infty.$$

However, c_n might tend quite slowly to zero, and nothing guarantees that $\sum_{n=-\infty}^{\infty} c_n$ is convergent. For example, one can show that there is a function $f \in L^2(-\pi, \pi)$ for which $c_0 = 0$ and $c_n = 1/|n|^{2/3}$ for $n \neq 0$; in this case Example 4.34 shows that $\sum_{n=-\infty}^{\infty} c_n$ is divergent. Another example for which $\sum_{n=-\infty}^{\infty} c_n$ is divergent will appear in Example 6.34. In applications of Fourier series one will always have to work with certain partial sums (computers can not handle infinite sums); and here it is a complication if the coefficients c_n tend slowly to zero, because it forces the engineers to include many terms in the partial sums in order to obtain a reasonable approximation.

6.5 Fourier series and Hilbert spaces

From our description so far, it is not clear that there is a link between Fourier series and (infinite-dimensional) linear algebra; the purpose of this section is to show that this is indeed the case.

Before we relate Fourier series and linear algebra, let us consider a finite-dimensional vector space V equipped with an inner product $\langle \cdot, \cdot \rangle$; in case the vector space is complex, we choose the inner product to be linear in the first entry. Recall that a family of vectors $\{e_k\}_{k=1}^n$ belonging to V is an *orthonormal basis* for V if

(i) $\text{span}\{e_k\}_{k=1}^n = V$, i.e., each $f \in V$ has a representation as a linear combination of the vectors e_k; and

(ii)

$$\langle e_k, e_j \rangle = \begin{cases} 1 \text{ if } k = j, \\ 0 \text{ if } k \neq j. \end{cases}$$

If $\{e_k\}_{k=1}^n$ is an orthonormal basis for V, then each element $f \in V$ has a representation

$$f = \sum_{k=1}^{n} \langle f, e_k \rangle e_k. \tag{6.27}$$

A large part of the theory for linear algebra has an extension to a class of infinite-dimensional vector spaces called *Hilbert spaces.* A Hilbert space $\mathcal{H}$ is defined exactly as a vector space with an inner product $\langle \cdot, \cdot \rangle$ (and associated norm $|| \cdot || = \sqrt{\langle \cdot, \cdot \rangle}$), with an extra condition added to ensure that "things go nicely" in the infinite-dimensional case.

Before we give the formal definition, we mention that the definition of convergence for a sequence of numbers has an analogue in any vector space $\mathcal{H}$ with an inner product, or, more generally, in any vector space equipped with a norm $||\cdot||$. In fact, we say that a sequence $\{x_k\}_{k=1}^{\infty} \subset \mathcal{H}$ converges to $x \in \mathcal{H}$ if

$$||x - x_k|| \to 0 \text{ as } k \to \infty.$$

Also, in analogy with sequences in $\mathbb{R}$ or $\mathbb{C}$, we say that a sequence $\{x_k\}_{k=1}^{\infty}$ belonging to a normed vector space $\mathcal{H}$ is a *Cauchy sequence* if for each given $\epsilon > 0$ we can find a number $N \in \mathbb{N}$ such that

$$||x_k - x_\ell|| \leq \epsilon \text{ whenever } k, \ell \geq N. \tag{6.28}$$

In the case of the normed vector spaces $\mathbb{R}$ or $\mathbb{C}$, we know that a sequence is convergent if and only if it is a Cauchy sequence; that is, we can use the criterion (6.28) to check whether a sequence is convergent or not, without involving a potential limit. In general infinite-dimensional normed vector spaces, the situation is different: there might exist Cauchy sequences which are not convergent. Hilbert spaces are introduced in order to avoid this type of complication:

Definition 6.27 *A Hilbert space is a vector space equipped with an inner product $\langle \cdot, \cdot \rangle$, with the property that each Cauchy sequence is convergent.*

Every finite-dimensional vector space with an inner product is a Hilbert space; and one can view Hilbert spaces as an extension of this framework, where we now allow certain infinite-dimensional vector spaces. In Hilbert spaces, several of the main results from linear algebra still hold; however, in many cases, extra care is needed because of the infinitely many dimensions. The exact meaning of this statement will be clear soon.

In Hilbert spaces, one can also define orthonormal bases, which now become infinite sequences $\{e_k\}_{k=1}^{\infty}$; one characterization is that $\{e_k\}_{k=1}^{\infty}$ is an orthonormal basis for $\mathcal{H}$ if

$$||f||^2 = \sum_{k=1}^{\infty} |\langle f, e_k \rangle|^2, \ \forall f \in \mathcal{H} \ \text{ and } \ \langle e_k, e_j \rangle = \begin{cases} 1 \text{ if } k = j, \\ 0 \text{ if } k \neq j. \end{cases} \tag{6.29}$$

If $\{e_k\}_{k=1}^{\infty}$ is an orthonormal basis for $\mathcal{H}$, then one can prove that each $f \in \mathcal{H}$ has the representation

$$f = \sum_{k=1}^{\infty} \langle f, e_k \rangle e_k. \tag{6.30}$$

At a first glance, this representation looks similar to (6.27), but some words of explanation are needed: in fact, the vectors $\{e_k\}_{k=1}^{\infty}$ belong to an abstract vector space, so we have not yet defined what an infinite sum like (6.30) should mean! The exact meaning appears by a slight modification of our definition of convergence for a series of numbers, namely that we measure the difference between f and the partial sums of the series in (6.30) in the norm $||\cdot||$ associated to our Hilbert space. That is, (6.30) means by definition that

$$\left\| f - \sum_{k=1}^{N} \langle f, e_k \rangle e_k \right\| \to 0 \text{ as } N \to \infty.$$

In our context, the important fact is that (if we identify functions which are equal almost everywhere) $L^2(-\pi, \pi)$ is a Hilbert space when equipped with the inner product

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(x)\overline{g(x)}dx, \;\; f, g \in L^2(-\pi, \pi). \tag{6.31}$$

Furthermore, one can check that the functions

$$\left\{ \frac{1}{\sqrt{2\pi}} \right\} \bigcup \left\{ \frac{1}{\sqrt{\pi}} \cos nx \right\}_{n=1}^{\infty} \bigcup \left\{ \frac{1}{\sqrt{\pi}} \sin nx \right\}_{n=1}^{\infty} \tag{6.32}$$

form an orthonormal basis for $L^2(-\pi, \pi)$. Using (6.30) on this orthonormal basis leads to

$$f \;\; = \;\; \langle f, \frac{1}{\sqrt{2\pi}} \rangle \frac{1}{\sqrt{2\pi}} + \sum_{n=1}^{\infty} \langle f, \frac{1}{\sqrt{\pi}} \cos n\cdot \rangle \frac{1}{\sqrt{\pi}} \cos nx \tag{6.33}$$

$$+ \sum_{n=1}^{\infty} \langle f, \frac{1}{\sqrt{\pi}} \sin n\cdot \rangle \frac{1}{\sqrt{\pi}} \sin nx. \tag{6.34}$$

Calculation of the coefficients in this expansion leads to

$$f \;\; = \;\; \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx), \tag{6.35}$$

where a_n, b_n are the Fourier coefficients. But this is just the expression we used as definition of the Fourier series! This shows that the Fourier series for a function f actually is the expansion of f in terms of the orthonormal basis (6.32). This coincidence explains the troubles we have with pointwise convergence of Fourier series: viewed as a Hilbert space-identity, the exact meaning of the identity (6.35) is that

$$\left\| f - \frac{1}{2} a_0 - \sum_{n=1}^{N} (a_n \cos(n\cdot) + b_n \sin(n\cdot)) \right\| \to 0 \text{ as } N \to \infty,$$

i.e., that

$$\int_{-\pi}^{\pi} \left| f(x) - \frac{1}{2} a_0 - \sum_{n=1}^{N} (a_n \cos nx + b_n \sin nx) \right|^2 dx \to 0 \text{ as } N \to \infty; \tag{6.36}$$

this is different from requiring pointwise convergence of the Fourier series to $f(x)$.

The reader might wonder why Fourier series are introduced in such a way that the correct interpretation of convergence of Fourier series is the one stated in (6.36). We will explain this in two steps. First, since we are looking at functions f defined on an interval, it is not sufficient to have pointwise convergence: even if a series converges to $f(x)$ for all x in the interval, it might be that no partial sum approximates f well *over the entire interval;* see our discussion in Section 5.4. So we need a concept of convergence that takes all points in the interval into account simultaneously.

The next natural idea would be to require that

$$\int_{-\pi}^{\pi} \left| f(x) - \frac{1}{2}a_0 - \sum_{n=1}^{N}(a_n \cos nx + b_n \sin nx) \right| dx \to 0 \text{ as } N \to \infty; \quad (6.37)$$

this condition involves all x in the interval. Furthermore, geometrically it has the nice interpretation that the area of the set limited by the graphs of f, the partial sum S_N, and the line segments $x = \pm\pi$, tend to zero as $N \to \infty$. We note that the condition (6.37) is similar to (6.36); the difference is that (6.36) involves a *square* on the integrand. This square destroys the above geometric interpretation of the condition; but it has the advantage that Fourier series now are determined in terms of convergence in the space $L^2(-\pi, \pi)$, which is a Hilbert space. A reader with a deeper knowledge of Hilbert spaces will recognize the importance of this fact; one important feature is that the Hilbert space structure involves having an inner product; and this, in turn, guarantees that certain important minimization problems always have a unique solution; see, e.g., [9].

As a final remark, we notice that (6.27) and (6.30) connect linear algebra with our description of signal transmission in Section 5.5: for example, (6.27) shows that the vector $f \in V$ is completely determined by the sequence of numbers $\{\langle f, e_k \rangle\}_{k=1}^{n}$. That is, one can transmit information about the vector f simply by sending this sequence.

Let us finally relate the complex exponential functions to the topic of Section 6.5:

Theorem 6.28 *The functions* $\{\frac{1}{\sqrt{2\pi}} e^{imx}\}_{m \in \mathbb{Z}}$ *form an orthonormal basis for* $L^2(-\pi, \pi)$.

6.6 The Fourier transform

As we have seen, Fourier series is a useful tool for representation and approximation of periodic functions via trigonometric functions. For aperiodic functions we need to search for other methods. The classical tool is the Fourier transform, which we introduce here.

Fourier series expand 2π-periodic functions in terms of the trigonometric functions $\cos nx, \sin nx$; all of these have period 2π. Trigonometric functions $\cos \lambda x, \sin \lambda x$, where λ is not an integer, do not have period 2π; this makes it very reasonable that they do not appear in the Fourier series for a 2π-periodic function. We have also seen that it is natural to think of a Fourier series as a decomposition of the given function f into harmonic oscillations with frequencies $\frac{n}{2\pi}, n = 0, 1, ...$; the size of the contributions at these frequencies are given by the Fourier coefficients.

If we want to expand aperiodic functions, the situation is different: all frequencies can appear in the signal, and the discrete sum over the special frequencies $\frac{n}{2\pi}$ must be replaced with an integral over all frequencies. We will consider functions in the vector space

$$L^1(\mathbb{R}) = \left\{ f : \mathbb{R} \to \mathbb{C} \mid \int_{-\infty}^{\infty} |f(x)| \, dx < \infty \right\}.$$

For functions $f \in L^1(\mathbb{R})$, the *Fourier transform* is defined by

$$\hat{f}(\gamma) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \gamma} dx.$$

Frequently it is also convenient to use the notation

$$(\mathcal{F}f)(\gamma) = \hat{f}(\gamma);$$

this notation indicates that we can look at the Fourier transform as an *operator*, which maps the function f to the function $\hat{f}$.

The Fourier transform indeed contains information about the frequency content of the function f: as we will see in Example 6.30, the Fourier transform of a function oscillating like a cosine function with frequency ν over a bounded interval has a peak at $\gamma = \nu$, i.e., the Fourier transform tells us that the signal contains an oscillation with this frequency. We shall derive this result as a consequence of a chain of rules for calculations with the Fourier transform, which we present below.

Before we proceed to these results, we mention that the Fourier transform can be defined on several function spaces other than $L^1(\mathbb{R})$. For $f \in L^1(\mathbb{R})$, one can prove that $\hat{f}$ is a continuous function which tends to zero as $\gamma \to \pm\infty$.

Below we state some of the important rules for calculations with the Fourier transform:

Theorem 6.29 *Given $f \in L^1(\mathbb{R})$, the following holds:*

(i) If f is even, then

$$\hat{f}(\gamma) = 2\int_0^\infty f(x)\cos(2\pi x\gamma)dx.$$

(ii) If f is odd, then

$$\hat{f}(\gamma) = -2i\int_0^\infty f(x)\sin(2\pi x\gamma)dx.$$

(iii) For $a \in \mathbb{R}$, let $(T_a f)(x) = f(x-a)$. Then the Fourier transform of the function $T_a f$ is

$$(\mathcal{F}T_a f)(\gamma) = \hat{f}(\gamma)e^{-2\pi i a\gamma}.$$

(iv) Let $\theta \in \mathbb{R}$. Then the Fourier transform of the function $g(x) = f(x)e^{2\pi i\theta x}$ is

$$(\mathcal{F}g)(\gamma) = \hat{f}(\gamma-\theta).$$

(v) If f is differentiable and $f' \in L^1(\mathbb{R})$, then

$$(\mathcal{F}f')(\gamma) = 2\pi i\gamma\hat{f}(\gamma).$$

Theorem 6.29 is proved in almost all textbooks dealing with the Fourier transform. Note in particular the rules (iii) and (iv); in words rather than symbols, they say that

- taking the Fourier transform of a translated version of f is done by multiplying $\hat{f}$ with a complex exponential function;
- taking the Fourier transform of a function f which is multiplied with a complex exponential function, corresponds to a translation of $\hat{f}$.

Let us show how we can use some of these rules to find the Fourier transform of a function that equals the cosine function on a certain interval. interval and vanishes outside the interval. In order to describe such a function, we define the *characteristic function* associated to a given interval $I \subset \mathbb{R}$ by

$$\chi_I(x) = \begin{cases} 1 & \text{if } x \in I, \\ 0 & \text{if } x \notin I. \end{cases}$$

Example 6.30 Given constants $a, \omega > 0$, we want to calculate the Fourier transformation of the function

$$f(x) = \cos(\omega x)\chi_{[-\frac{a}{2},\frac{a}{2}]}(x). \tag{6.38}$$

This signal corresponds to an oscillation which starts at the time $x = -a/2$ and last till $x = a/2$. If x is measured in seconds, we have $\frac{\omega}{2\pi}$ oscillations per second, i.e., the frequency is $\nu = \frac{\omega}{2\pi}$. In order to find the Fourier transform, we first look at the function $\chi_{[-\frac{a}{2},\frac{a}{2}]}$ separately. Since this is an even function, Theorem 6.29(i) shows that for $\gamma \neq 0$,

$$\begin{aligned} \mathcal{F}\chi_{[-\frac{a}{2},\frac{a}{2}]}(\gamma) &= 2\int_0^\infty \chi_{[-\frac{a}{2},\frac{a}{2}]}(x)\cos(2\pi x\gamma)dx \\ &= 2\int_0^{\frac{a}{2}} \cos(2\pi x\gamma)dx \\ &= \frac{2}{2\pi\gamma}\left[\sin(2\pi x\gamma)\right]_{x=0}^{x=\frac{a}{2}} \\ &= \frac{\sin \pi a\gamma}{\pi\gamma}. \end{aligned}$$

Returning to the function f, we can use (6.17) to write

$$f(x) = \frac{1}{2}e^{i\omega x}\chi_{[-\frac{a}{2},\frac{a}{2}]}(x) + \frac{1}{2}e^{-i\omega x}\chi_{[-\frac{a}{2},\frac{a}{2}]}(x).$$

Via Theorem 6.29(iii) this shows that

$$\begin{aligned} \hat{f}(\gamma) &= \frac{1}{2}\mathcal{F}\chi_{[-\frac{a}{2},\frac{a}{2}]}(\gamma - \frac{\omega}{2\pi}) + \frac{1}{2}\mathcal{F}\chi_{[-\frac{a}{2},\frac{a}{2}]}(\gamma + \frac{\omega}{2\pi}) \\ &= \frac{1}{2}\left(\frac{\sin \pi a(\gamma - \omega/2\pi)}{\pi(\gamma - \omega/2\pi)} + \frac{\sin \pi a(\gamma + \omega/2\pi)}{\pi(\gamma + \omega/2\pi)}\right). \end{aligned}$$

Figure 6.31 shows $\hat{f}$ for $\omega = 20\pi$ and different values of a in (6.38). A larger value of a corresponds to the oscillation $\cos(\omega x)$ being present in the signal over a larger time interval; we see that this increases the peak of $\hat{f}$ at the frequency $\gamma = \nu = 10$. □

A very important fact about the Fourier transform is its invertibility: if we happen to know the Fourier transform $\hat{f}$ of an, in principle, unknown function f, we are able to come back to the function f via

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\gamma)e^{2\pi i x\gamma}d\gamma. \tag{6.39}$$

An exact statement of this result will need some more care; for us, it is enough to know that this formula holds at least if we know that f is a continuous function in $L^1(\mathbb{R})$ which vanishes at $\pm\infty$.

The most widespread use in Engineering for the Fourier transform is within signal processing; but the Fourier transform can also be used to solve linear differential equations, as briefly described here.

Consider a linear differential equation with constant coefficients:

$$\frac{\mathrm{d}^n y}{\mathrm{d}t^n} + a_1\frac{\mathrm{d}^{n-1}y}{\mathrm{d}t^{n-1}} + \cdots + a_{n-1}\frac{\mathrm{d}y}{\mathrm{d}t} + a_n\, y = u(t)\,, \qquad t \in \mathbb{R}\,, \tag{6.40}$$

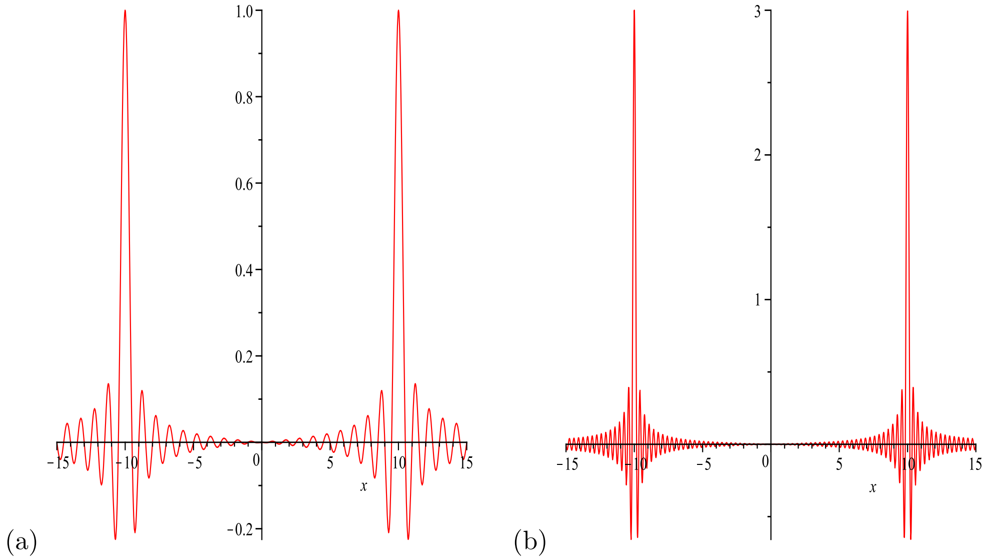


Figure 6.31 *(a): The Fourier transform of the function f in (6.38) for $\omega = 20\pi, a = 2$; this corresponds to a signal with frequency $\nu = 10$ which is present during a time interval of length 2. (b): The Fourier transform of the function f in (6.38) for $\omega = 20\pi, a = 6$; this corresponds to a signal with frequency $\nu = 10$ which is present during a time interval of length 6.*

where $a_1, a_2, \ldots, a_n \in \mathbb{R}$, and u is an L^1-function. If we assume that a solution y exists such that $y, \frac{\mathrm{d}y}{\mathrm{d}t}, \ldots, \frac{\mathrm{d}^n y}{\mathrm{d}t}$ are all L^1-functions, then we can take the Fourier transform of both sides of the equation and get:

$$(2\pi\nu i)^n \widehat{y}(\nu) + a_1(2\pi\nu i)^{n-1}\widehat{y}(\nu) + \cdots + a_{n-1}2\pi\nu i\widehat{y}(\nu) + a_n\widehat{y}(\nu) = \widehat{u}(\nu)\,.$$

Letting $P(s) = s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n$ be the characteristic polynomial, the above can be rewritten

$$P(2\pi\nu i)\widehat{y}(\nu) = \widehat{u}(\nu)\,, \tag{6.41}$$

and from this we obtain

$$\widehat{y}(\nu) = \frac{\widehat{u}(\nu)}{P(2\pi\nu i)}\,. \tag{6.42}$$

We can now write down a particular solution:

$$y(t) = \int_{-\infty}^{\infty} \frac{\widehat{u}(\nu)}{P(2\pi\nu i)} e^{2\pi i t\nu}\,\mathrm{d}\nu\,. \tag{6.43}$$

It is far from always the case that this method works. There might not be solutions that are L^1-functions. For example it is a requirement that P does not have any pure imaginary roots, because in such a case $\widehat{y}$ has singularities, and the integral at (6.43) will not converge.

Example 6.32 Consider the odd function f given by

$$f(t) = \begin{cases} 0 & \text{for } t \leq -2, \\ -2-t & \text{for } -2 < t \leq -1, \\ t & \text{for } -1 < t \leq 1, \\ 2-t & \text{for } 1 < t \leq 2, \\ 0 & \text{for } t > 2, \end{cases} \tag{6.44}$$

The Fourier transform is, by the formula for odd functions:

$$\begin{aligned}
\widehat{f}(\nu) = -2i \int_0^{\infty} f(t)\sin(2\pi\nu t)\,\mathrm{d}t &= -2i \int_0^1 f(t)\sin(2\pi\nu t)\,\mathrm{d}t \\
&- 2i \int_1^2 f(t)\sin(2\pi\nu t)\,\mathrm{d}t - 2i \int_2^{\infty} f(t)\sin(2\pi\nu t)\,\mathrm{d}t \\
&= -2i \int_0^1 t\sin(2\pi\nu t)\,\mathrm{d}t - 2i \int_1^2 (2-t)\sin(2\pi\nu t)\,\mathrm{d}t \\
&= -2i \left[t\frac{-\cos(2\pi\nu t)}{2\pi\nu} \right]_0^1 + 2i \int_0^1 \frac{-\cos(2\pi\nu t)}{2\pi\nu}\,\mathrm{d}t \\
- 2i \left[2\frac{-\cos(2\pi\nu t)}{2\pi\nu} \right]_1^2 &+ 2i \left[t\frac{-\cos(2\pi\nu t)}{2\pi\nu} \right]_1^2 - 2i \int_1^2 \frac{-\cos(2\pi\nu t)}{2\pi\nu}\,\mathrm{d}t \\
&= i\frac{\cos(2\pi\nu)}{\pi\nu} - 2i \left[\frac{\sin(2\pi\nu t)}{(2\pi\nu)^2} \right]_0^1 + 2i\frac{\cos(4\pi\nu)}{\pi\nu} - 2i\frac{\cos(2\pi\nu)}{\pi\nu} \\
&- 2i\frac{\cos(4\pi\nu)}{\pi\nu} + 2i\frac{\cos(2\pi\nu)}{2\pi\nu} + 2i \left[\frac{\sin(2\pi\nu t)}{(2\pi\nu)^2} \right]_1^2 \\
&= -2i\frac{\sin(2\pi\nu)}{4\pi^2\nu^2} + 2i\frac{\sin(4\pi\nu)}{4\pi^2\nu^2} - 2i\frac{\sin(2\pi\nu)}{4\pi^2\nu^2} \\
&= 2i\frac{2\cos(2\pi\nu)\sin(2\pi\nu)}{4\pi^2\nu^2} - 4i\frac{\sin(2\pi\nu)}{4\pi^2\nu^2} \\
&= i\frac{(\cos(2\pi\nu)-1)\sin(2\pi\nu)}{\pi^2\nu^2} = -4i\frac{\cos(\pi\nu)\sin^3(\pi\nu)}{\pi^2\nu^2}\,.
\end{aligned}$$

Let us now consider the differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} - 4y = f\,.$$

The Fourier transform gives:

$$(2\pi i\nu)^2\widehat{y} - 4\widehat{y} = \widehat{f} \iff (-4\pi^2\nu^2 - 4)\widehat{y} = \widehat{f}\,,$$

which is to say:

$$\widehat{y} = \frac{-\widehat{f}}{4\pi^2\nu^2 + 4} = i\frac{\cos(\pi\nu)\sin^3(\pi\nu)}{(\pi^2\nu^2+1)\pi^2\nu^2}\,.$$

By the inverse Fourier transform we obtain

$$\begin{aligned}
y(t) &= \int_{-\infty}^{\infty} i\frac{\cos(\pi\nu)\sin^3(\pi\nu)}{(\pi^2\nu^2+1)\pi^2\nu^2} e^{2\pi i\nu t}\,\mathrm{d}\nu \\
&= 2i\int_0^{\infty} i\frac{\cos(\pi\nu)\sin^3(\pi\nu)}{(\pi^2\nu^2+1)\pi^2\nu^2}\sin(2\pi\nu t)\,\mathrm{d}\nu \\
&= -2\int_0^{\infty} \frac{\cos(\pi\nu)\sin^3(\pi\nu)}{(\pi^2\nu^2+1)\pi^2\nu^2}\sin(2\pi\nu t)\,\mathrm{d}\nu\,.
\end{aligned}$$

Maple has problems with this integral – it gives the wrong result. We can instead approximate the integral by

$$y(t) \approx -2\int_0^{C} \frac{\cos(\pi\nu)\sin^3(\pi\nu)}{(\pi^2\nu^2+1)\pi^2\nu^2}\sin(2\pi\nu t)\,\mathrm{d}\nu\,.$$

Maple can manage this correctly, but remember to take the real part when plotting the result. The error in this approximation can be estimated by

$$\begin{aligned}
&\left| y(t) - \left(-2\int_0^{C} \frac{\cos(\pi\nu)\sin^3(\pi\nu)}{(\pi^2\nu^2+1)\pi^2\nu^2}\sin(2\pi\nu t)\,\mathrm{d}\nu\right)\right| \\
&\qquad = \left| 2\int_C^{\infty} \frac{\cos(\pi\nu)\sin^3(\pi\nu)}{(\pi^2\nu^2+1)\pi^2\nu^2}\sin(2\pi\nu t)\,\mathrm{d}\nu\right| \\
&\qquad \le 2\int_C^{\infty} \left|\frac{\cos(\pi\nu)\sin^3(\pi\nu)}{(\pi^2\nu^2+1)\pi^2\nu^2}\sin(2\pi\nu t)\right|\,\mathrm{d}\nu \\
&\qquad \le 2\int_C^{\infty} \frac{1}{(\pi^2\nu^2+1)\pi^2\nu^2}\,\mathrm{d}\nu \le 2\int_C^{\infty}\frac{1}{\pi^4\nu^4}\,\mathrm{d}\nu = \frac{2}{3\pi^4C^3}\,.
\end{aligned}$$

If we want an error less than ϵ, we can choose C, such that

$$\frac{2}{3\pi^4C^3} \le \epsilon \iff C^3 \ge \frac{2}{3\pi^4\epsilon} \iff C \ge \sqrt[3]{\frac{2}{3\pi^4\epsilon}}\,.$$

For example, if we choose $\epsilon = 0.001$, we just need to choose $C \ge 1.8986\ldots$.

6.7 Fourier series and signal analysis

Fourier's theorem tells us that the Fourier series for a continuous piecewise differentiable and 2π-periodic function converges pointwise toward the function; that is, we can write

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx),\;\; x\in\mathbb{R}. \tag{6.45}$$

Our aim is now to describe how to interpret this identity.

If we think about the variable x as time, the terms in the Fourier series correspond to oscillations with varying frequencies: for a given value of

$n \in \mathbb{N}$, the functions $\cos nx$ and $\sin nx$ run through $\frac{n}{2\pi}$ periods in a time interval of unit length, i.e., they correspond to oscillations with frequency $\nu = \frac{n}{2\pi}$. Now, given f, the identity (6.45) represents f as a superposition of harmonic oscillations with various frequencies; the coefficients in (6.45) clarify which oscillations are present in f, and what the amplitudes are (i.e., the "magnitude" of the oscillations). Let us make this clearer by considering an example.

Example 6.33 A harmonic oscillation with frequency $\frac{100}{\pi}$ can be written in the form

$$f(x) = A\cos(200x + \varphi),$$

where A denotes the amplitude and φ is the phase (describing "when the oscillation starts"). We can think of this signal as the tone produced by a musical instrument. Using standard trigonometric formulas, we can rewrite the expression for f as

$$f(x) = A\cos\varphi\cos(200x) - A\sin\varphi\sin(200x);$$

this expression equals the Fourier series for f, i.e., we have

$$a_{200} = A\cos\phi,\ b_{200} = -A\sin\phi,\ \text{and } a_n = b_n = 0 \text{ for } n \neq 200.$$

The Fourier series tells us that only the frequency $\nu = \frac{100}{\pi}$ is present in the signal. More generally, if we consider a signal made up of several oscillations with varying amplitudes and phases, the Fourier series will again reveal exactly which frequencies are present; for example, if the signal consists of several tones played with different forces and with varying phases, the Fourier series will express this. Note, however, that we are speaking about quite abstract musical instruments: real instruments of course produce tones which begin and end at a certain time, i.e., the signals are not periodic considered as functions on $\mathbb{R}$. They furthermore produce higher harmonics, i.e., oscillations with frequencies which are integer-multiples of the tone itself. Real musical signals should rather be analyzed via the Fourier transform, as discussed in Section 6.6. □

Fourier analysis plays a central role in signal analysis. Very often, it is used to extract certain information from a given signal. As an imaginary (but realistic) example, we can think about a signal which measures the vibrations of a certain device such as, e.g., an engine. Due to mechanical limitations, it can be that we know in advance that appearance of certain frequencies with large amplitudes in the vibrations will make the engine break down; this is a case where it is important to be able to analyze the frequency content of the signal.

6.8 Regularity and decay of the Fourier coefficients

Motivated by the end of the last section, we now analyze how fast the Fourier coefficients c_n tend to zero as $n \to \pm\infty$. As already mentioned, we wish the coefficients to converge fast to zero because this, at least on the intuitive level and for reasonable functions, implies that the partial sums $S_N(x)$ of the Fourier series of f will look and behave similar to f already for small values of $N \in \mathbb{N}$.

One way of measuring how fast c_n converges to zero is to ask if we can find constants $C > 0$ and $p \in \mathbb{N}$ such that

$$|c_n| \leq \frac{C}{|n|^p} \text{ for all } n \neq 0; \tag{6.46}$$

if this is satisfied, we say that c_n *decays polynomially.*

We note that from a practical point of view, we want (6.46) to hold with as small a value of $C > 0$ as possible, and as large a value of $p \in \mathbb{N}$ as possible.

It turns out that the decay rate of the Fourier coefficients is strongly related with the smoothness of the given function. Before we state this result formally in Theorem 6.35, we consider a small example which lends strong support to this fact.

Example 6.34 In this example we consider the Fourier coefficients for the following two functions:

(i) $f(x) = 1, \; x \in]-\pi, \pi[$;

(ii)

$$f(x) = \begin{cases} -1 \text{ if } x \in [-\pi, 0[, \\ 0 \text{ if } x = 0, \\ 1 \text{ if } x \in]0, \pi[. \end{cases} \tag{6.47}$$

Note that the function in (ii) is a seemingly very innocent modification of the function in (i): instead of having f constant on the interval $]-\pi, \pi[$, we now allow a jump in the middle of the interval, i.e., the function is piecewise constant. As we will see, this modification has a drastic influence on the Fourier coefficients.

First, the function in (i) has the Fourier coefficients

$$c_0 = 1, \text{ and for } n \neq 0, \; c_n = 0;$$

that is, just by working with one Fourier coefficient, we can recover the function completely. The decay condition (6.46) is satisfied for any $C > 0, p \in \mathbb{N}$.

For the function in (ii), we calculated the Fourier coefficients in real form in Example 6.5; via (6.20),

$$c_0 = 0, \text{ and for } n \in \mathbb{N}, \; c_n = \begin{cases} 0 \text{ if } n \text{ is even,} \\ \frac{-2i}{n\pi} \text{ if } n \text{ is odd,} \end{cases} \qquad c_{-n} = \begin{cases} 0 \text{ if } n \text{ is even,} \\ \frac{2i}{n\pi} \text{ if } n \text{ is odd.} \end{cases}$$

Thus, the decay condition (6.46) is only satisfied if we take $p = 1$ and $C \geq 2/\pi$, i.e., the Fourier coefficients decay quite slowly. Note also that $\sum_{n=-\infty}^{\infty} |c_n|$ is divergent; however, $\sum_{n=-\infty}^{\infty} |c_n|^2$ is convergent, in accordance with Parseval's theorem.

The example (ii) illustrates once more that it is problematic to approximate discontinuous functions with Fourier series: in an attempt to resolve the discontinuity, the Fourier series introduces oscillations with frequencies which do not appear in the analyzed function, and with quite large coefficients. □

In the following analysis we will use the Fourier series in complex form,

$$f \sim \sum_{n \in \mathbb{Z}} c_n e^{inx}, \text{ where } c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx.$$

Similar results can be stated for Fourier series in real form, and proved via (6.22). We will only consider continuous, piecewise differentiable and 2π-periodic functions, for which Fourier's theorem says that the Fourier series converges pointwise to the function; that is, we can actually write

$$f(x) = \sum_{n \in \mathbb{Z}} c_n e^{inx}, \; x \in [-\pi, \pi[.$$

Theorem 6.35 *Assume that f is 2π-periodic, continuous, and piecewise differentiable. Then the following holds:*

(i) Assume that f is p times differentiable for some $p \in \mathbb{N}$ and that $f^{(p)}$ is bounded, i.e., there is a constant $D > 0$ such that $|f^{(p)}(x)| \leq D$ for all x. Then there exists a constant $C > 0$ such that

$$|c_n| \leq \frac{C}{|n|^p} \text{ for all } n \in \mathbb{Z} \setminus \{0\}.$$

(ii) Assume that for some $p \in \mathbb{N}$ there exists a constant $C > 0$ such that

$$|c_n| \leq \frac{C}{|n|^{p+2}} \text{ for all } n \in \mathbb{Z} \setminus \{0\}. \tag{6.48}$$

Then f is p times differentiable, and

$$f^{(p)}(x) = \sum_{n \in \mathbb{Z}} c_n (in)^p e^{inx}.$$

We prove Theorem 6.35 in Section A.4. Theorem 6.35 indicates that there is a relationship between the smoothness of a function and its content of oscillations with high frequencies (measured by the size of the Fourier coefficients for large values of n): the smoother the function is, the faster these coefficients tend to zero, i.e., the smaller the content of high-frequency oscillations in f is.

6.9 Best N-term approximation

As mentioned before, infinite series is a mathematical tool, aiming at exact representation of certain functions. When working with series representations in practice, we are only able to deal with finite sums; that is, even if a function f has an exact representation via, e.g., a Fourier series, we will have to let the computers work with the finite partial sums S_N for a certain value of $N \in \mathbb{N}$.

For Fourier series, the inequality (6.14) shows how to choose N such that the partial sum S_N approximates f sufficiently well (under the conditions in Theorem 6.12). Unfortunately, this expression shows that N becomes very large if we need a good approximation, i.e., if ϵ is small. We also see that if we reduce ϵ by a factor of ten, then N might increase by a factor of a hundred! In practice, this type of calculations is performed using computers, but it is actually a problem that the number of needed terms increases so fast with the desired precision. Many calculations hereby get complicated to perform, simply because they take too much time. One way of obtaining an improvement is to replace the partial sum S_N by another finite sum which approximates f equally well, but which can be expressed using fewer coefficients. This is the idea behind the subject *nonlinear approximation.*

We will not give a general presentation of nonlinear approximation, but focus on a special type, the so-called *best N-term approximation.*

The basic idea is that, given a function f, not all terms in S_N are "equally important". To be more exact, to find S_N amounts to calculating the numbers $a_0, \dots, a_N, b_1, \dots, b_N$ (or $c_{-N}, \dots, c_N$ in complex form); however, some of these numbers might be very small and might not contribute in an essential way to S_N, or, in other words, to the approximation of f.

If our capacity limits us to calculate and store only N numbers, it is much more reasonable to pick the N coefficients which represent f "as well as possible". This usually corresponds to picking the N (numerically) largest coefficients in the Fourier series.

Best N-term approximation is a delicate issue for Fourier series when we speak about obtaining approximations in the pointwise sense. The reason is that picking the largest coefficients first amounts to a *reordering* of the Fourier series; since Fourier series do not always converge unconditionally, this might have an influence on the convergence properties. However, if

$$\sum_{n=1}^{\infty}(|a_n|+|b_n|)<\infty, \text{ or, in complex form, } \sum_{n=-\infty}^{\infty}|c_n|<\infty, \qquad (6.49)$$

then Proposition 4.27 shows that the Fourier series converges unconditionally (in the pointwise sense), and this complication does not appear. Note that Theorem 6.35(i) specifies some classes of functions for which (6.49) is satisfied, e.g., by taking $p \geq 2$.

Best N-term approximation often takes place in some norms rather than in the pointwise sense. That is, given a certain norm on $L^2(-\pi,\pi)$, we want to find an index set $J \subset \mathbb{Z}$ containing N elements, such that $x \mapsto f(x) - \sum_{n\in J} c_n e^{inx}$ is as small as possible, measured in that norm. If we use the norm $||\cdot||$ associated to the inner product $\langle\cdot,\cdot\rangle$ in (6.31), then for any choice $J \subset \mathbb{Z}$,

$$\begin{aligned}\left|\left|f-\sum_{n\in J}c_n e^{in\cdot}\right|\right| &= \left|\left|\sum_{n\notin J}c_n e^{in\cdot}\right|\right| = \left(2\pi\sum_{n\notin J}|c_n|^2\right)^{1/2}\\ &= \left(||f||^2-2\pi\sum_{n\in J}|c_n|^2\right)^{1/2};\end{aligned}$$

this shows that the optimal approximation (measured in that particular norm) indeed is obtained when $\{c_n\}_{n\in J}$ consists of the N numerically largest coefficients.

If a central part of the given signal contains high frequencies with large amplitudes, then best N-term approximation is an important issue. Consider for example the signal

$$f(x) = \cos(10^7 x).$$

For this signal the Fourier coefficients are

$$a_{10^7} = 1, \quad a_n = b_n = 0 \text{ otherwise};$$

that is, a truncation of the Fourier series will treat the signal as zero if it does not include sufficiently many terms. On the other hand, a best N-term approximation will immediately realize the existence of the component with the large frequency, and recover f completely already at the first step.

Let us finally explain why the name nonlinear approximation is associated with the method. In order to do so, let us first consider the traditional approximation, simply via the Nth partial sum in the Fourier series. Assume that two functions f, g are given, and that the Nth partial sums in their Fourier series are

$$\frac{1}{2}a_0 + \sum_{n=1}^{N}(a_n \cos nx + b_n \sin nx),$$

respectively,

$$\frac{1}{2}\alpha_0 + \sum_{n=1}^{N}(\alpha_n \cos nx + \beta_n \sin nx).$$

Then, the Nth partial sum in the Fourier series for the function $f+g$ is

$$\frac{1}{2}(a_0+\alpha_0) + \sum_{n=1}^{N}((a_n+\alpha_n) \cos nx + (b_n+\beta_n) \sin nx),$$

i.e., we obtain the coefficients in the Fourier expansion simply by adding the coefficients for f and g. With best N-term approximation, the situation is different: only in rare cases, the coefficients leading to the best approximation of $f+g$ appear by adding the coefficients giving the best N-term approximation of f and g: that is, there is no linear relationship between the given function and the coefficients. Let us illustrate this with an example:

Example 6.36 Let

$$f(x) = \cos x + \frac{1}{5}\sin x, \quad g(x) = -\cos x + \frac{1}{10}\sin x.$$

Then the best approximation of f, respectively, g, using one term of the Fourier series is

$$\tilde{f}(x) = \cos x, \text{ respectively, } \tilde{g}(x) = -\cos x.$$

Thus,

$$\tilde{f}(x) + \tilde{g}(x) = 0.$$

On the other hand,

$$(f+g)(x) = \frac{1}{5}\sin x + \frac{1}{10}\sin x;$$

the best approximation of this function via a single trigonometric function is

$$\frac{1}{5}\sin x. \qquad \square$$

7

Differential equations and infinite series

We will now combine the topics discussed in the book and show how power series and Fourier series can be used as a tool to solve certain differential equations.

7.1 Power series and differential equations

In Chapter 1 we considered differential equation of nth order. We only considered differential equations with constant coefficients, for the simple reason that there does not exist a general solution formula for the more complicated equations with varying coefficients. Thus, for example, we can not find a general formula for the solution of an equation of the form

$$a_0(t)\frac{d^n y}{dt^n} + a_1(t)\frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_{n-1}(t)\frac{dy}{dt} + a_n(t)y = 0, \ t \in I, \tag{7.1}$$

where $n \geq 2$, and the coefficients are functions depending on the variable t. In particular, we can not use the characteristic polynomial, as we did for equations with constant coefficients.

On the other hand, if all the functions $a_0(t), \ldots, a_n(t)$ in (F.12) are polynomials or power series, we can often find solutions to (F.12) having the form

$$y(t) = \sum_{n=0}^{\infty} c_n t^n. \tag{7.2}$$

Indeed, we can insert (7.2) in the equation (F.12). Using our rules for manipulation with power series we can then collect the expression on the left-hand side of (F.12) on the form of a power series $\sum_{n=0}^{\infty} b_n t^n$. Now, if the given power series in (7.2) is a solution to (F.12), we conclude that

$$\sum_{n=0}^{\infty} b_n t^n = 0$$

for all t. According to the identity theorem for power series (Corollary 5.21) this implies that $b_n = 0$ for all n. The final step is then to use the relationship between the coefficients b_n and c_n to find the coefficients c_n in (7.2).

The sketched method often leads to solutions that can be expressed in terms of the classical functions (typically exponential functions and trigonometric functions). In other cases it leads to "new functions," i.e., functions that do not have a simple description in terms of classical functions. Some of the functions that have been derived this way have been proved to be useful in several contexts, and have therefore been given names (Bessel functions, Weber functions, Hankel function). Thus, in such cases we have defined functions exclusively in terms of their power series expansions.

We will now demonstrate the method on a concrete example.

Example 7.1 We want to find a solution to the differential equation

$$t\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + ty = 0. \tag{7.3}$$

As described above we will search for a solution of the form

$$y(t) = \sum_{n=0}^{\infty} c_n t^n, \ t \in]-\rho, \rho[. \tag{7.4}$$

Using the rules for differentiation of a power series (Theorem 5.17) we see that

$$\frac{dy}{dt} = \sum_{n=1}^{\infty} c_n n t^{n-1}$$

and

$$\frac{d^2y}{dt^2} = \sum_{n=2}^{\infty} c_n (n-1) n t^{n-2}.$$

Inserting this in the differential equation, we see that the function in (7.4) is a solution within its radius of convergence if and only if

$$t\sum_{n=2}^{\infty} c_n n(n-1)t^{n-2} + 2\sum_{n=1}^{\infty} c_n n t^{n-1} + t\sum_{n=0}^{\infty} c_n t^n = 0, \ t \in]-\rho, \rho[,$$

i.e., if and only if

$$\sum_{n=2}^{\infty} c_n n(n-1)t^{n-1} + \sum_{n=1}^{\infty} 2c_n n t^{n-1} + \sum_{n=0}^{\infty} c_n t^{n+1} = 0,\ t \in]-\rho,\rho[. \tag{7.5}$$

Using the calculations

$$\begin{aligned}
\sum_{n=2}^{\infty} c_n n(n-1)t^{n-1} &= \sum_{n=0}^{\infty} c_{n+2}(n+2)(n+1)t^{n+1}, \\
\sum_{n=1}^{\infty} 2c_n n t^{n-1} &= 2c_1 + \sum_{n=2}^{\infty} 2c_n n t^{n-1} \\
&= 2c_1 + \sum_{n=0}^{\infty} 2c_{n+2}(n+2)t^{n+1},
\end{aligned}$$

by collecting the terms we see that the equation (7.5) is equivalent with

$$2c_1 + \sum_{n=0}^{\infty} \left[(n+2)(n+1)c_{n+2} + 2(n+2)c_{n+2} + c_n\right] t^{n+1} = 0,\ t \in]-\rho,\rho[.$$

According to the identity theorem for power series (Corollary 5.21) this is satisfied if and only if

$$2c_1 = 0,\ \text{and } (n+2)(n+1)c_{n+2} + 2(n+2)c_{n+2} + c_n = 0 \text{ for all } n \geq 0.$$

The conclusion is that every power series solution to the differential equation satisfies that

$$c_1 = 0,\ c_{n+2} = -\frac{1}{(n+2)(n+3)}c_n,\ n \geq 0. \tag{7.6}$$

Thus the coefficients $c_0, c_1, \ldots, c_n, \ldots$ form a recursively given sequence. We see directly that

$$c_1 = 0,\ c_3 = 0,\ c_5 = 0, \ldots,$$

i.e., that

$$c_{2n+1} = 0 \text{ for all } n \geq 0.$$

Concrete values for all the coefficients c_n can now be obtained from (7.6) by selecting an arbitrary value for the coefficient c_0. In order to verify that we indeed obtain a solution to the differential equation, we finally calculate the radius of convergence ρ for the obtained power series and check that $\rho > 0$. We will do this in Example 7.2. □

Often we search solutions y that satisfy certain conditions, e.g., concerning the function values $y(0)$ and $y'(0)$. Such conditions are very convenient when y is given by a power series as in (7.2): indeed, Theorem 5.17 shows that then

$$y(0) = c_0, \ y'(0) = c_1, \ \cdots, y^{(n)} = n!c_n. \tag{7.7}$$

Example 7.2 We now continue Example 7.1 and search for a solution y to (7.3) which satisfies the initial conditions

$$y(0) = 1, y'(0) = 0.$$

According to (7.7), for the power series (7.4) the condition $y(0) = 1$ implies that $c_0 = 1$, while the condition $y'(0) = 0$ implies that $c_1 = 0$. As we saw in (7.6) we know that every power series solution to (7.3) satisfies that $c_1 = 0$. Thus we will now focus on the condition $c_0 = 1$; via the recursion formula (7.6) it implies that

$$c_0 = 1, \quad c_2 = \frac{-1}{2 \cdot 3} = \frac{-1}{3!}, \quad c_4 = \frac{-c_2}{4 \cdot 5} = \frac{1}{5!}, \quad c_6 = \frac{-c_4}{6 \cdot 7} = \frac{-1}{7!}.$$

In order to obtain a general expression for the coefficients we now write

$$\begin{aligned} c_{2n} &= \frac{-1}{(2n+1)2n} c_{2n-2} \\ &= \frac{-1}{(2n+1)2n} \frac{-1}{(2n-1)2n-2} c_{2n-4} \\ &= \frac{-1}{(2n+1)2n} \frac{-1}{(2n-1)2n-2} \frac{-1}{(2n-3)2n-4} c_{2n-6} \\ &= \cdots\cdots \\ &= \frac{(-1)^n}{(2n+1)!}. \end{aligned}$$

We have now shown that the function

$$y(t) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} t^{2n}, \ t \in]-\rho, \rho[, \tag{7.8}$$

is a solution to the differential equation (7.3), and that it satisfies the initial conditions $y(0) = 1, y'(0) = 0$. Using the quotient test, we see that the radius of convergence is $\rho = \infty$. Thus the function is a solution to the differential equation for all $t \in \mathbb{R}$.

In the concrete case we can find the sum of the power series (7.8) directly. Indeed, using the power series for $\sin t$ we see that for $t \neq 0$,

$$y(t) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} t^{2n} = \frac{1}{t} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} t^{2n+1} = \frac{\sin t}{t},$$

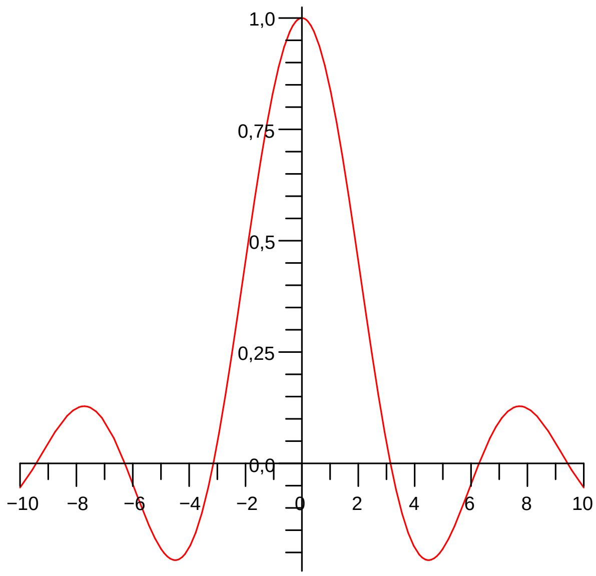


Figure 7.3 *The function y in (7.9). In the literature the function is known as the sinc-function.*

and for $t = 0$ we have $y = 1$. Thus the function

$$y(t) = \begin{cases} \frac{\sin t}{t}, & t \neq 0 \\ 1, & t = 0 \end{cases} \tag{7.9}$$

is a C^∞-function, which solves the initial condition problem. See Figure 7.3. □

Sometimes there does not exist a power series solution that satisfies given initial conditions:

Example 7.4 The initial value problem

$$t\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + ty = 0, \;\; y(0) = 1, y'(0) = 1$$

does not have a power series solution: indeed, as we saw in (7.6) in Example 7.1, every power solution to the differential equation satisfies that $c_1 = 0$; on the other hand, the initial condition $y'(0) = 1$ implies that $c_1 = 1$. Thus, no power series solution will satisfy the given initial conditions. □

The described method also applies for the inhomogeneous differential equation

$$a_0(t)\frac{d^ny}{dt^n} + a_1(t)\frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_{n-1}(t)\frac{dy}{dt} + a_0(t)y = u(t), \;\; t \in I, \tag{7.10}$$

where $u(t)$ has a known power series expansion,

$$u(t) = \sum_{n=0}^{\infty} d_n t^n.$$

Searching for a solution of the form

$$y(t) = \sum_{n=0}^{\infty} c_n t^n,$$

we can again collect all the terms on the left-hand side of (7.10) in the form $\sum_{n=0}^{\infty} b_n t^n$, and we thus arrive at the equation

$$\sum_{n=0}^{\infty} b_n t^n = \sum_{n=0}^{\infty} d_n t^n.$$

Using Corollary 5.21 this implies that $b_n = d_n$ for all n. Since the coefficients b_n are given in terms of the coefficients c_n we can now try to derive concrete expressions for the coefficients c_n such that the function y is a solution to the differential equation.

For second-order equations we will often first try to find a solution to the corresponding homogeneous differential equation: Indeed, if we can do so, Theorem 1.30 yields a method to find the solutions to the given inhomogeneous differential equation based on the solution to the homogeneous equation.

Example 7.5 Let us return to Example 1.31, where the goal was to solve the differential equation

$$\frac{d^2y}{dt^2} - \frac{2}{t}\frac{dy}{dt} + \frac{2}{t^2}y = \frac{1}{t},\ t \in]0, \infty[. \tag{7.11}$$

In Example 1.31 we did so by applying Theorem 1.30 and using that $y(t) = t$ is a solution to the corresponding homogeneous equation

$$\frac{d^2y}{dt^2} - \frac{2}{t}\frac{dy}{dt} + \frac{2}{t^2}y = 0,\ t \in]0, \infty[. \tag{7.12}$$

We will now explain how the solution $y(t) = t$ can be found using the power series method. First we note that (7.12) is equivalent with

$$t^2\frac{d^2y}{dt^2} - 2t\frac{dy}{dt} + 2y = 0,\ t \in]0, \infty[. \tag{7.13}$$

We now search a solution of the form

$$y(t) = \sum_{n=0}^{\infty} c_n t^n.$$

Inserting this in (7.12), a concrete calculation shows that y is a solution to (7.12) if and only if

$$c_0 = 0, \text{ and } c_n(n^2 - 3n + 2) = 0, \ n = 1, 2, 3, \ldots.$$

Since $n^2 - 3n + 2 \neq 0$ for $n \notin \{1, 2\}$ this means that

$$c_0 = 0, \text{ and } c_n = 0, \ n = 3, 4, \ldots.$$

Therefore (7.13) has the solutions

$$y(t) = c_1 t + c_2 t^2,$$

where $c_1, c_2 \in \mathbb{R}$. In particular, the function $y(t) = t$ is a solution, as claimed. □

So far we did not discuss the question about when the described method is applicable. For second-order equations we have the following result:

Theorem 7.6 *Consider a differential equation of the form*

$$\frac{d^2y}{dt^2} + a_1(t)\frac{dy}{dt} + a_2(t)y = 0, \tag{7.14}$$

and assume that the functions $a_1(t)$ and $a_2(t)$ are analytic, i.e., that the Taylor series for $a_1(t)$ converges to $a_1(t)$ for $t \in]-\rho_1, \rho_1[$, and that the Taylor series for $a_2(t)$ converges to $a_2(t)$ for $t \in]-\rho_2, \rho_2[$. Then (7.14) has a solution

$$y(t) = \sum_{n=0}^{\infty} c_n t^n$$

for $t \in]-\min(\rho_1, \rho_2), \min(\rho_1, \rho_2)[$.

Note that the differential equation (7.14) is *normalized* compared with (F.12), i.e., that the term $\frac{d^2y}{dt^2}$ appears with the coefficient 1 (when $a_0(t) \neq 0$ we can achieve this in (F.12) by division sith $a_0(t)$). The condition that the function $a_1(t)$ is analytic is satisfied if $a_1(t)$ is a fraction, where the counter and denominator consist of polynomials and the polynomial in the denominator is nonzero for $t = 0$. In case either $a_1(t)$ or $a_2(t)$ are not analytic, we can frequently use the same principle to find a solution of the form

$$y(t) = t^{\alpha} \sum_{n=0}^{\infty} c_n t^n = \sum_{n=0}^{\infty} c_n t^{n+\alpha},$$

where α is a real constant. The method is known under the name *the method of Frobenius.* We refer to, e.g., [1] for a more detailed description.

7.2 Fourier's method

We now return to the analysis of a system consisting of n differential equations of first order, of the form (2.32),

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u, \\ y = \mathbf{d}^\top \mathbf{x}, \end{cases} \tag{7.15}$$

where $\mathbf{d}$ is a real vector with n coordinates. We will assume that the forcing function u is 2π-periodic, and we will apply the Fourier series of u as a tool to find a solution to the system.

Let $H(s)$ denote the transfer function for (7.15); as we saw in (2.36) it is given by

$$H(s) = -\mathbf{d}^\top (\mathbf{A} - s\mathbf{I})^{-1}\mathbf{b}.$$

Recall that the transfer function is defined whenever $s \in \mathbb{C}$ satisfies that $\det(A - s\mathbf{I}) \neq 0$. Let us first check how far we can go without involving Fourier series:

Lemma 7.7 *Assume that the system* (7.15) *is asymptotically stable. For any forcing function of the form*

$$u = \sum_{n=-N}^{N} c_n e^{int}. \tag{7.16}$$

we obtain the solution

$$y = \sum_{n=-N}^{N} c_n H(in) e^{int}. \tag{7.17}$$

Proof: Let $n \in \mathbb{Z}$. Since the system is easymptotically stable, all roots in the characteristic equation have negative real part, according to Theorem 2.35. Thus $s = in$ is not a root in the characteristic polynomial. According to Theorem 2.20 it follows that if the forcing function is

$$u(t) = e^{int}, \tag{7.18}$$

then (7.15) has the solution

$$y(t) = H(in)e^{int}. \tag{7.19}$$

The function in (7.16) is a linear combination of the functions in (7.18) for $n = -N, \ldots, N$, so via the principle of superposition we obtain a solution corresponding to the forcing function in (7.16) as the corresponding linear combination of the solutions corresponding to the functions in (7.18). This yields exactly the expression in (7.17). □

In the following we will assume that (7.15) is an asymptotically stable system. Let the forcing function u be an arbitrary 2π-periodic, piecewise differentiable and continuous function in $L^2(-\pi,\pi)$. We write the Fourier series for u on complex form,

$$u(t) = \sum_{n=-\infty}^{\infty} c_n e^{int}, \quad t \in \mathbb{R}, \tag{7.20}$$

It is now a very natural question whether the function

$$y(t) \sim \sum_{n=-\infty}^{\infty} c_n H(in) e^{int}, \quad t \in \mathbb{R} \tag{7.21}$$

is a solution corresponding to the forcing function u. The answer turns out to be yes:

Theorem 7.8 (Method of Fourier series) *Let $H(s)$ denote the transfer function associated with the asymptotically stable system* (7.15)*. Consider a 2π-periodic piecewise differentiable and continuous forcing function $u(t)$, given by the Fourier series*

$$u(t) = \sum_{n=-\infty}^{\infty} c_n e^{int}, \quad t \in \mathbb{R}.$$

Then (7.15) *has a solution given by the Fourier series*

$$y(t) = \sum_{n=-\infty}^{\infty} c_n H(in) e^{int}, \quad t \in \mathbb{R}. \tag{7.22}$$

Observe that in Theorem 7.8 we do not obtain the solution to the system of differential equations in terms of the usual standard functions, but via the infinite series (7.22). In practice we can only work with finite sums, so it is necessary to substitute the infinite series (7.22) by a finite partial sum. This is one of the reasons that we have developed methods to decide how many terms we need to include in the partial sum in order to obtain a given precision (see Corollary 6.16 and Theorem 6.17).

Example 7.9 Consider the "sawtooth function", i.e., the 2π-periodic function given by

$$u(t) = |t|, \ t \in [-\pi, \pi[.$$

By direct calculation we see that the Fourier coefficients are

$$b_n = 0, \text{ and } a_n = \begin{cases} \pi & \text{for } n = 0, \\ 0 & \text{for n even, } n > 0, \\ -\frac{4}{\pi}\frac{1}{n^2} & \text{for n odd.} \end{cases}$$

Thus the Fourier series is

$$u(t) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos((2n-1)t).$$

According to (6.21) the Fourier series on complex form is therefore given by

$$u(t) = \sum_{n=-\infty}^{\infty} c_n e^{int},$$

where

$$c_n = \begin{cases} \frac{\pi}{2} & \text{for } n = 0, \\ 0 & \text{for n even, } n \neq 0, \\ -\frac{2}{\pi}\frac{1}{n^2} & \text{for n odd.} \end{cases} \tag{7.23}$$

Note that since $b_n = 0$ for all n, we do not need to distinguish between $n > 0$ and $n < 0$ in the calculation of c_n. We can now write the Fourier series on complex form as

$$u(t) = \frac{\pi}{2} - \frac{2}{\pi} \sum_{n\in\mathbb{Z}, n \text{ odd}} \frac{1}{n^2} e^{int}.$$

Let us apply $u(t)$ as forcing function on an asymptotically stable system of differential equations with transfer function $H(s) = \frac{1}{1+s}$, $s \neq -1.$. According to Theorem 7.8 we obtain a solution with the Fourier series

$$y(t) = \sum_{n=-\infty}^{\infty} c_n H(in) e^{int} \tag{7.24}$$

$$= \sum_{n=-\infty}^{\infty} \frac{c_n}{1+in} e^{int}. \tag{7.25}$$

Using that $c_n = 0$ for n even (except when $n = 0$) we can write (7.24) as

$$y(t) = c_0 H(0) + \sum_{n\in\mathbb{Z},\ n \text{ odd}} c_n H(in) e^{int}.$$

Inserting c_n in this expression we obtain that

$$y(t) = \frac{\pi}{2} H(0) - \frac{2}{\pi} \sum_{n\in\mathbb{Z}, n \text{ odd}} \frac{1}{n^2} H(in) e^{int}$$

$$= \frac{\pi}{2} - \frac{2}{\pi} \sum_{n\in\mathbb{Z}, n \text{ odd}} \frac{1}{n^2} \frac{1}{1+in} e^{int}. \tag{7.26}$$

The function y in (7.26) is not given explicitly; thus, we will apply our methods from Fourier analysis to estimate how many terms we need to include in the partial sum in order to obtain a reasonable approximation. It turns out to be most convenient to start with the expression (7.25). Via calculations exactly like in the proof of Theorem 6.17 we see that

$$\begin{aligned}
\left| y(x) - \sum_{n=-N}^{N} \frac{c_n}{1+in} e^{inx} \right| &= \left| \sum_{|n|>N} \frac{c_n}{1+in} e^{inx} \right| \\
&\le \sum_{|n|>N} \left| \frac{c_n}{1+in} \right| \\
&\le \sum_{|n|>N} \frac{|c_n|}{\sqrt{1+n^2}} \\
&\le \frac{4}{\pi} \sum_{n=N+1}^{\infty} \frac{1}{n^2} \\
&\le \frac{4}{\pi} \frac{1}{N} \qquad (7.27)
\end{aligned}$$

(see (4.32)). For $N \ge 128$, the expression in (7.27) is smaller than 0.01; we conclude that for $N \ge 128$,

$$\left| y(x) - \sum_{n=-N}^{N} \frac{c_n}{1+in} e^{inx} \right| \le 0.01, \ \forall x \in \mathbb{R}.$$

Note that some of the estimates above are quite coarse. Thus, the found value for N is larger than strictly necessary. □

Appendix A

Appendix A

A.1 Definitions and notation

The absolute value of a real number x is defined by:

$$|x| = \begin{cases} x & \text{if } x \geq 0; \\ -x & \text{if } x < 0 \ . \end{cases}$$

For a complex number $a + ib$ where $a, b \in \mathbb{R}$,

$$|a + ib| = \sqrt{a^2 + b^2}.$$

We will often use the *triangular inequality*, which says that

$$|x + y| \leq |x| + |y| \text{ for all } x, y \in \mathbb{C}.$$

A subset E of $\mathbb{R}$ is *bounded above* if there exists $\beta \in \mathbb{R}$ such that

$$x \leq \beta, \ \forall x \in E; \tag{A.1}$$

the smallest number β satisfying (A.1) is called the *supremum of* E, and is written $\sup E$, or, $\sup_{x \in E} x$. Similarly, the set E is bounded below if there exists $\alpha \in \mathbb{R}$ such that

$$\alpha \leq x, \ \forall x \in E;$$

and the largest number with this property is called the *infimum of* E, $\inf E$. For example,

$$\sup]0,4] = 4, \quad \sup[-2,5[= 5, \quad \inf]-2,1] = -2, \quad \inf(\mathbb{Q} \cap [\pi, 7]) = \pi.$$

Let $I \subset \mathbb{R}$ be an interval. A function $f : I \to \mathbb{R}$ is *continuous* at a point $x_0 \in I$ if for any given $\epsilon > 0$ we can find $\delta > 0$ such that

$$|f(x) - f(x_0)| \leq \epsilon \text{ for all } x \in I \text{ for which } |x - x_0| \leq \delta. \tag{A.2}$$

We say that f is continuous if f is continuous at every point in I.

A continuous function on a bounded and closed interval $[a,b]$ is *uniformly continuous*; that is, for each $\epsilon > 0$ we can find $\delta > 0$ such that

$$|f(x) - f(x_0)| \leq \epsilon \text{ for all } x, x_0 \in [a,b] \text{ for which } |x - x_0| \leq \delta. \tag{A.3}$$

Observe the difference between (A.2) and (A.3): in the first case x_0 is a *fixed* point in the interval $[a,b]$, while in the second case we also allow x_0 to vary. Therefore (A.3) is in general more restrictive than (A.2).

A continuous real-valued function defined on a bounded and closed interval $[a,b]$ assumes its supremum, i.e., there exists $x_0 \in [a,b]$ such that

$$f(x_0) = \sup \{ f(x) \mid x \in [a,b] \}.$$

Similarly, the function assumes its infimum. If $x_0 \in]a,b[$ is a point in which a differentiable function $f : [a,b] \to \mathbb{R}$ assumes its maximal or minimal value, then $f'(x_0) = 0$.

A.2 Proof of Taylor's theorem

Before we prove Taylors theorem we state some results which will be used in the proof. We begin with *Rolle's theorem:*

Theorem A.1 *Assume that the function $f : [a,b] \to \mathbb{R}$ is differentiable and that $f(a) = f(b) = 0$. Then there exists a point $\xi \in]a,b[$, for which $f'(\xi) = 0$.*

Proof: The function f has a maximal value as well as a minimal value in the interval $[a,b]$. If the maximal value equals the minimal value, then f is constant, and $f'(x) = 0$ for all $x \in]a,b[$. On the other hand, if the minimal value is different from the maximal value, then there exists $\xi \in]a,b[$ where f assumes an extremal value, because of the assumption $f(a) = f(b)$; but this implies that $f'(\xi) = 0$. □

Lemma A.2 *Assume that $f : [a,b] \to \mathbb{R}$ is arbitrarily often differentiable, and let x, x_0 be arbitrary points in $[a,b]$, with $x > x_0$. Assume that for some $N \in \mathbb{N}$,*

$$f(x) = 0 \text{ and } f^{(j)}(x_0) = 0 \text{ for all } j \in \{0, 1, \dots, N\}. \tag{A.4}$$

Then there exists $\xi \in]x_0, x[$ such that $f^{(N+1)}(\xi) = 0$.

Proof: The proof consists of a continued application of Rolle's theorem. We first prove the result for $N = 0$, and then move on with $N = 1$, etc. Observe that when we want to prove the theorem for one value of N, the assumption (A.4) is available for all $j \in \{0, 1, \dots, N\}$; this will play a key role in the proof.

For $N = 0$, our assumption is that $f(x) = f(x_0) = 0$. We shall prove the existence of a point $\xi_1 \in]x_0, x[$ for which$f'(\xi_1) = 0$; but this is exactly the conclusion in Rolle's theorem.

Now consider $N = 1$; in this case, the assumption (A.4) is available for $j = 0$ and $j = 1$. Via the assumption for $j = 0$ we just proved the existence of $\xi_1 \in]x_0, x[$, for which $f'(\xi_1) = 0$. Now, if we use the assumption for $j = 1$ we also have that $f'(x_0) = 0$. Applying Rolle's theorem again, this time to the function f' and the interval $[x_0, \xi_1]$, we conclude that there exists $\xi_2 \in]x_0, \xi_1[$ such that $f''(\xi_2) = 0$. This is exactly the conclusion we wanted to obtain for $N = 1$.

We now consider the general case, where $N > 1$. Thus, the assumption (A.4) means that

$$f(x) = f(x_0) = f'(x_0) = f''(x_0) = \cdots = f^{(N)}(x_0) = 0.$$

Applying Rolle's theorem to the function f and the interval $[x_0, x]$, we can find $\xi_1 \in]x_0, x[$ for which $f'(\xi_1) = 0$. Applying Rolle's theorem to f' and the interval $[x_0, \xi_1]$ leads to a point $\xi_2 \in]x_0, \xi_1[$ for which $f''(\xi_2) = 0$. Applying this procedure $N+1$ times finally leads to a point ξ_{N+1} in $]x_0, \xi_N[$ for which $f^{(N+1)}(\xi_{N+1}) = 0$. $\square$

Now we only need one step before we are ready to prove Taylor's theorem. As before, we let P_N denote the Nth Taylor polynomial for f at x_0:

Theorem A.3 *Assume that $f : [a, b] \to \mathbb{R}$ is arbitrarily often differentiable, and let $x_0 \in]a, b[,\ N \in \mathbb{N}$. Then, for each $x \in [a, b]$ we can find ξ between x and x_0 such that*

$$f(x) = P_N(x_0) + \frac{f^{(N+1)}(\xi)}{(N+1)!}(x - x_0)^{N+1}.$$

Proof: Fix $x \in [a, b]$. The statement certainly holds if $x = x_0$, so we assume that $x_0 < x$ (the proof is similar if $x_0 > x$). Consider the function ϕ defined via

$$\phi(t) = f(t) - P_N(t) - K(t - x_0)^{N+1}; \tag{A.5}$$

here K is a constant, which we choose such that $\phi(x) = 0$, i.e.,

$$K = \frac{f(x) - P_N(x)}{(x - x_0)^{N+1}}.$$

Differentiating leads to

$$\phi'(t) = f'(t) - P_N'(t) - (N+1)K(t - x_0)^N,$$

and more generally, for each integer $j \in \{1, 2, 3, \ldots, N+1\}$,

$$\begin{aligned}\phi^{(j)}(t) &= f^{(j)}(t) - P_N^{(j)}(t) \\ &\quad -(N+1)N(N-1)\cdots(N-j+2)K(t-x_0)^{N-j+1}.\end{aligned}$$

In particular,

$$\phi^{(N+1)}(t) = f^{(N+1)}(t) - P_N^{(N+1)}(t) - (N+1)!K. \tag{A.6}$$

By definition,

$$\begin{aligned}P_N(t) &= f(x_0) + \frac{f'(x_0)}{1!}(t-x_0) + \frac{f''(x_0)}{2!}(t-x_0)^2 \\ &\quad + \frac{f^{(3)}(x_0)}{3!}(t-x_0)^3 + \cdots + \frac{f^{(N)}(x_0)}{N!}(t-x_0)^N.\end{aligned}$$

In this expression the first term is a constant, so

$$\begin{aligned}P_N'(t) &= f'(x_0) + \frac{f''(x_0)}{2!}2(t-x_0) \\ &\quad + \frac{f^{(3)}(x_0)}{3!}3(t-x_0)^2 + \cdots + \frac{f^{(N)}(x_0)}{N!}N(t-x_0)^{N-1} \\ &= f'(x_0) + f''(x_0)(t-x_0) \\ &\quad + \frac{f^{(3)}(x_0)}{2!}(t-x_0)^2 + \cdots + \frac{f^{(N)}(x_0)}{(N-1)!}(t-x_0)^{N-1}.\end{aligned}$$

Differentiating once more,

$$\begin{aligned}P_N''(t) &= f''(x_0) + \frac{2}{2!}f^{(3)}(x_0)(t-x_0) \\ &\quad + \cdots + \frac{(N-1)}{(N-1)!}f^{(N)}(x_0)(t-x_0)^{N-2} \\ &= f''(x_0) + f^{(3)}(x_0)(t-x_0) + \cdots + \frac{f^{(N)}(x_0)}{(N-2)!}(t-x_0)^{N-2}.\end{aligned}$$

More generally, for $j \in \{1, 2, 3, \ldots, N\}$,

$$\begin{aligned}P_N^{(j)}(t) &= f^{(j)}(x_0) + f^{(j+1)}(x_0)(t-x_0) \\ &\quad + \frac{f^{(j+2)}(x_0)}{2!}(t-x_0)^2 + \cdots + \frac{f^{(N)}(x_0)}{(N-j)!}(t-x_0)^{N-j}.\end{aligned}$$

Note that

$$P_N^{(N+1)}(t) = 0, \ \forall t \in \mathbb{R}.$$

For $j \in \{0, 1, \ldots, N\}$, we have $P_n^{(j)}(x_0) = f^{(j)}(x_0)$, and therefore $\phi^{(j)}(x_0) = 0$. This means that the function ϕ satisfies the conditions in Lemma A.2. Thus, there exists $\xi \in]x_0, x[$ for which $\phi^{(N+1)}(\xi) = 0$. But according to

(A.6),

$$\phi^{(N+1)}(\xi) = f^{(N+1)}(\xi) - (N+1)!K,$$

so

$$f^{(N+1)}(\xi) = (N+1)!K \text{ and } K = \frac{f^{(N+1)}(\xi)}{(N+1)!}.$$

Using the expression (A.5) for ϕ and that $\phi(x) = 0$, we see that

$$f(x) = P_N(x) + \frac{f^{(N+1)}(\xi)}{(N+1)!}(x-x_0)^{N+1}. \qquad \square$$

Here is finally the proof for Taylor's theorem, in the case $I = [a, b]$:

Proof of Theorem 4.13: Let $N \in \mathbb{N}$ and $x \in]a, b[$ be given. According to Theorem A.3 we can find $\xi \in]a, b[$ such that

$$f(x) = \sum_{n=0}^{N} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + \frac{f^{(N+1)}(\xi)}{(N+1)!}(x-x_0)^{N+1}.$$

Thus,

$$f(x) - \sum_{n=0}^{N} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n = \frac{f^{(N+1)}(\xi)}{(N+1)!}(x-x_0)^{N+1}.$$

By assumption we have that $|f^{(N+1)}(x)| \leq C$ for all $x \in]a, b[$, so this implies that

$$\begin{aligned} \left| f(x) - \sum_{n=0}^{N} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n \right| &= \left| \frac{f^{(N+1)}(\xi)}{(N+1)!}(x-x_0)^{N+1} \right| \\ &\leq \frac{C}{(N+1)!} \left| x - x_0 \right|^{N+1}. \end{aligned}$$

$\square$

A.3 Infinite series

Proof of Theorem 4.30 First we assume that (4.25) is satisfied with $C < 1$. Then we can find $d \in]0, 1[$ such that $\left|\frac{a_{n+1}}{a_n}\right| \leq d$ for all sufficiently large values of n, say, for all $n \geq N$. Now, considering $n > N$,

$$\begin{aligned} |a_n| &= \left| \frac{a_n}{a_{n-1}} \cdot \frac{a_{n-1}}{a_{n-2}} \cdots \frac{a_{N+1}}{a_N} \cdot a_N \right| \\ &= \left| \frac{a_n}{a_{n-1}} \right| \cdot \left| \frac{a_{n-1}}{a_{n-2}} \right| \cdots \left| \frac{a_{N+1}}{a_N} \right| \cdot |a_N| \\ &\leq d \cdot d \cdots d \cdot |a_N| \\ &= d^{n-N} \, |a_N| \\ &= d^n \, \frac{|a_N|}{d^N}. \end{aligned}$$

The series $\sum d^n$ is convergent according to Theorem 5.2; since $\frac{|a_N|}{d^N}$ is a constant, this implies that the series $\sum_{n=N}^{\infty} d^n \, \frac{|a_N|}{d^N}$ is convergent as well. Via Theorem 4.20(i) we see that $\sum_{n=N}^{\infty} |a_n|$ is convergent; finally, Theorem 4.20(ii) shows that $\sum_{n=1}^{\infty} a_n$ is convergent.

Now consider the case where $C > 1$. Then $|a_{n+1}| > |a_n|$ for sufficiently large values of n; but this means that the sequence $|a_1|, |a_2|, \ldots$ is increasing from a certain term, and therefore $a_n \nrightarrow 0$ as $n \to \infty$. According to the nth term test this implies that $\sum_{n=1}^{\infty} a_n$ is divergent. $\square$

Proof of Theorem 4.33: Consider Figure A.4. For each $N \in \mathbb{N}$,

$$\sum_{n=2}^{N} f(n) = f(2) + \cdots + f(N) < \int_1^N f(x)dx.$$

If $\int_1^t f(x)dx$ has a finite limit as $t \to \infty$, then the sequence $\sum_{n=2}^{N} f(n),\ n \geq 2$, is increasing and bounded, and therefore convergent. It also follows that

$$\sum_{n=2}^{\infty} f(n) < \int_1^{\infty} f(x)dx$$

(it needs an extra argument to see that we have sharp inequality in the limit). So

$$\sum_{n=1}^{\infty} f(n) = f(1) + \sum_{n=2}^{\infty} f(n) < f(1) + \int_1^{\infty} f(x)dx.$$

Similarly,

$$\sum_{n=1}^{N} f(n) = f(1) + f(2) + \cdots + f(N) > \int_1^{N+1} f(x)dx,$$

which implies that

$$\sum_{n=1}^{\infty} f(n) > \int_1^{\infty} f(x)dx.$$

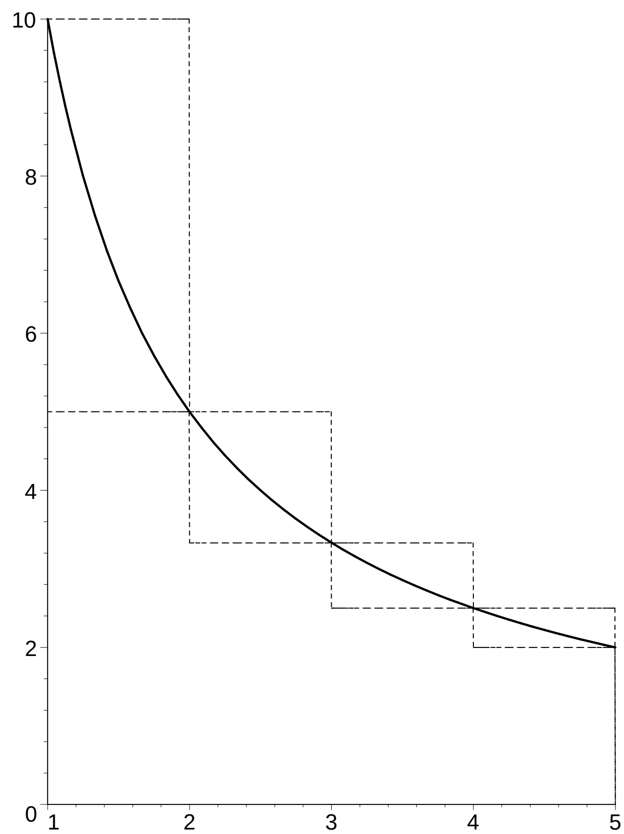


Figure A.4 *Graph for a decreasing function $f(x), x \in [1,5]$. We see that $\sum_{n=1}^{4} f(n)$ is an upper sum for the Riemann integral $\int_1^5 f(x)dx$, and that $\sum_{n=2}^{5} f(n)$ is a lower sum.*

This concludes the proof of (i), and proves at the same time (ii). $\square$

Proof of Theorem 5.13: Let us prove the theorem under the (superfluous) assumption that

$$\left|\frac{a_{n+1}}{a_n}\right| \to C \in [0,\infty] \text{ for } n \to \infty. \tag{A.7}$$

First we observe that the series $\sum_{n=0}^{\infty} a_n x^n$ is convergent for $x = 0$; we will therefore assume that $x \neq 0$. Applying the quotient test to the series $\sum_{n=0}^{\infty} a_n x^n$ leads to

$$\left|\frac{a_{n+1}x^{n+1}}{a_n x^n}\right| \to C\ |x| \text{ for } n \to \infty. \tag{A.8}$$

We now split the rest of the proof into three cases. If $C = 0$, we see that

$$\left|\frac{a_{n+1}x^{n+1}}{a_n x^n}\right| \to 0 \text{ for } n \to \infty.$$

According to the quotient test this implies that $\sum_{n=1}^{\infty} a_n x^n$ is convergent; since x was arbitrary, we conclude that the series is convergent for all $x \in \mathbb{R}$.

If $C = \infty$, (A.8) shows that

$$\left| \frac{a_{n+1}x^{n+1}}{a_n x^n} \right| \to \infty \text{ for } n \to \infty;$$

in this case the terms in the series $\sum_{n=1}^{\infty} |a_n x^n|$ are increasing, so

$$a_n x^n \nrightarrow 0 \text{ for } n \to \infty.$$

Thus, $\sum_{n=0}^{\infty} a_n x^n$ is divergent for $x \neq 0$ according to the nth term test.

Finally, if $C \in]0, \infty[$, then (A.8) together with the quotient test shows that $\sum_{n=1}^{\infty} a_n x^n$ is convergent if $C|x| < 1$, i.e., for $|x| < \frac{1}{C}$, and divergent if $|x| > \frac{1}{C}$. $\square$

A.4 Proof of Theorem 6.35

Proof: Let us consider $p = 1$. Using partial integration, for $n \neq 0$,

$$\begin{aligned} c_n &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx \\ &= \frac{1}{2\pi} \left(\left[\frac{1}{-in} e^{-inx} f(x) \right]_{x=-\pi}^{x=\pi} - \frac{1}{-in} \int_{-\pi}^{\pi} f'(x) e^{-inx} dx \right) \\ &= \frac{1}{2\pi i n} \int_{-\pi}^{\pi} f'(x) e^{-inx} dx. \end{aligned} \tag{A.9}$$

Thus,

$$\begin{aligned} |c_n| &\leq \frac{1}{2\pi |n|} \left| \int_{-\pi}^{\pi} f'(x) e^{-inx} dx \right| \\ &\leq \frac{1}{2\pi |n|} \int_{-\pi}^{\pi} \left| f'(x) e^{-inx} \right| dx \\ &\leq \sup_x |f'(x)| \frac{1}{|n|}. \end{aligned}$$

This proves (i) for $p = 1$. Continuing using partial integration on (A.9) shows that for any value of $p \in \mathbb{N}$ we have

$$c_n = \frac{1}{(2\pi i n)^p} \int_{-\pi}^{\pi} f^{(p)}(x) e^{-inx} dx;$$

repeating the above argument now leads to

$$|c_n| \leq \sup_x |f^{(p)}(x)| \frac{1}{(2\pi)^{p-1}} \frac{1}{|n|^p}.$$

(ii) Regardless of the considered value of $p \in \mathbb{N}$, the assumption (6.48) implies that

$$|c_n| \le \frac{C}{|n|^3} \text{ for all } n \in \mathbb{Z} \setminus \{0\}.$$

Thus, Theorem 5.35 implies that f is differentiable and that

$$f'(x) = \sum_{n \in \mathbb{Z}} c_n i n e^{inx}.$$

Iterating the argument shows that f in fact is p times differentiable, and that the announced expression for $f^{(p)}$ holds. □

Appendix B

Appendix B

In this appendix we list a number of important power series and Fourier series. For the series appearing in the book we state the reference as well.

B.1 Power series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \; |x| < 1 \tag{5.3}$$

$$\frac{x}{(1-x)^2} = \sum_{n=0}^{\infty} nx^n, \; |x| < 1$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \; x \in \mathbb{R} \tag{5.8}$$

$$\ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1}, \; |x| < 1$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}, \; x \in \mathbb{R} \tag{5.9}$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}, \; x \in \mathbb{R}$$

$$
\begin{aligned}
\arctan x &= \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1},\ |x| < 1 \\
\sinh x &= \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} x^{2n+1},\ x \in \mathbb{R} \\
\cosh x &= \sum_{n=0}^{\infty} \frac{1}{(2n)!} x^{2n},\ x \in \mathbb{R}
\end{aligned}
$$

B.2 Fourier series for 2π-periodic functions

$$
f(x) = \begin{cases} -1 \text{ if } x \in]-\pi, 0[, \\ 0 \text{ if } x = 0, x = \pi, \\ 1 \text{ if } x \in]0, \pi[, \end{cases}, \quad f \sim \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} \sin((2n-1)x) \tag{6.7}
$$

$$
f(x) = x,\ x \in]-\pi, \pi[: \quad f \sim \sum_{n=1}^{\infty} \frac{2}{n} (-1)^{n+1} \sin nx \tag{6.8}
$$

$$
\begin{aligned}
f(x) &= |x|,\ x \in]-\pi, \pi[: \ f \sim \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos((2n-1)x) \\
f(x) &= x^2,\ x \in]-\pi, \pi[: \quad f \sim \frac{\pi^2}{3} - 4 \sum_{n=1}^{\infty} \frac{1}{n^2} (-1)^{n+1} \cos nx \\
f(x) &= |\sin x|,\ x \in]-\pi, \pi[: \quad f \sim \frac{2}{\pi} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos(2nx)}{(2n-1)(2n+1)}
\end{aligned}
$$

Appendix C

Appendix C

In this appendix we collect the formulas for Fourier series for functions with period T for some $T > 0$.

C.1 Fourier series for T-periodic functions

Given $T > 0$, let $\omega = 2\pi/T$. For a function $f : \mathbb{R} \to \mathbb{C}$ with period T, which belongs to the vector space

$$L^2(-\frac{T}{2}, \frac{T}{2}) := \left\{ f : \quad \int_{-T/2}^{T/2} |f(x)|^2 dx < \infty \right\},$$

the Fourier series is

$$f \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega x + b_n \sin n\omega x),$$

where

$$a_n = \frac{2}{T} \int_0^T f(x) \cos n\omega x \; dx, \;\; n = 0, 1, 2, \ldots$$

and

$$b_n = \frac{2}{T} \int_0^T f(x) \sin n\omega x \; dx, \;\; n = 1, 2, \ldots \;\; .$$

The Nth partial sum of the Fourier series is

$$S_N(x) = \frac{1}{2}a_0 + \sum_{n=1}^{N} (a_n \cos n\omega x + b_n \sin n\omega x).$$

Similar to Theorem 6.3, we have that

(i) If f is an even function, then $b_n = 0$ for all n, and

$$a_n = \frac{4}{T}\int_0^{\frac{T}{2}} f(x)\cos n\omega x \; dx, \;\; n = 0, 1, 2, \dots \;\; .$$

(ii) If f is odd, then $a_n = 0$ for all n, and

$$b_n = \frac{4}{T}\int_0^{\frac{T}{2}} f(x)\sin n\omega x \; dx, \;\; n = 1, 2, \dots \;\; .$$

The Fourier series of f in complex form is

$$f \sim \sum_{n=-\infty}^{\infty} c_n e^{in\omega x},$$

where

$$c_n = \frac{1}{T}\int_0^T f(x)e^{-in\omega x}dx, \quad n \in \mathbb{Z}.$$

Under assumptions similar to the one used in Theorem 6.12, the maximal deviation between $f(x)$ and the partial sum $S_N(x)$ can be estimated by

$$|f(x) - S_N(x)| \le \frac{1}{\pi}\sqrt{\frac{T}{2N}\int_0^T |f'(t)|^2 dt}.$$

Alternatively, as in Proposition 6.17,

$$|f(x) - S_N(x)| \le \sum_{n=N+1}^{\infty} (|a_n| + |b_n|).$$

Parseval's Theorem takes the form

$$\frac{1}{T}\int_0^T |f(x)|^2 dx = \frac{1}{4}|a_0|^2 + \frac{1}{2}\sum_{n=1}^{\infty}(|a_n|^2 + |b_n|^2) = \sum_{n=-\infty}^{\infty} |c_n|^2.$$

Appendix D
Maple

In this appendix we collect the most important Maple commands related to the topics in this book.

D.1 Generel commands

D.1.1 Decomposition

A fraction, where the counter and the denominator are polynomials, can be decomposed according to the roots of the polynomial in the counter. Consider, e.g.,

$$\frac{x^2-11}{x^4-4x^3+7x^2-12x+12}$$

```
> convert((x^2-11)/(x^4-4*x^3 +7*x^2-12*x+12),parfrac,x);
```

$$\frac{8}{7}\frac{1}{x-2}-\frac{1}{(x-2)^2}-\frac{2}{7}\frac{1+4x}{x^2+3}$$

D.1.2 Factorization of nth order polynomial

Every nth order polynomial

$$P(x)=a_0x^n+a_1x^{n-1}+\cdots+a_n$$

can be factorized as a product of n first-order polynomials:

$$P(\lambda)=a_0(x-x_1)^{p_1}(x-x_2)^{p_2}\cdots(x-x_k)^{p_k}.$$

Consider, e.g.,

$$P(x)=x^2+x.$$

In Maple, factorization is done using the command

> factor(x^2+x);

$$x(x+1)$$

If the polynomial has complex roots this command does not give the full decomposition in terms of first-order polynomials. Consider, e.g.,

> factor(x^2+1);

$$x^2+1$$

In this case we can use the following addition to the command:

> factor(x^2+1, complex);

$$(x+I)(x-I)$$

D.1.3 Egenvalues, eigenvectors, and the exponential matrix

Consider the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

In order to determine the eigenvalues and the associated eigenvectors we can do the following:

> with(LinearAlgebra):

> A:=Matrix(3,3,[1,0,0,0,0,1,0,1,0]);

$$\mathbf{A} := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

> Eigenvalues(A);

$$\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

The result says that the eigenvalues are er -1 (with algebraic multiplicity 1) and 1 (with algebraic multiplicity 2). The following command yields as well the eigenvalues as the associated eigenvectors:

> Eigenvectors(A);

$$\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

The result says that -1 is an eigenvalue with associated eigenvector

$$\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix};$$

similarly, 1 is an eigenvalue with associated linearly independent eigenvectors

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

D.2 Differential equations

D.2.1 Solution of differential equations

We want to solve

$$y'' + 5y' + 6y = 0.$$

First we find the general solution:

```
> L := diff(y(t),t,t)+5*diff(y(t),t)+6*y(t)=0;
```

$$L := \frac{d^2}{dt^2}y(t) + 5\frac{d}{dt}y(t) + 6y(t) = 0$$

```
> dsolve(L);
```

$$y(t) = \_C1e^{(-3t)} + \_C2e^{(-2t)}$$

Let us now determine the solution for which $y(0) = 1, y'(0) = 1$:

> `dsolve({L,` $y(0) = 1, D(y)(0) = 1$`});`

$$y(t) = -3e^{(-3t)} + 4e^{(-2t)}$$

D.3 Systems of differential equations

D.3.1 Solution of systems of differential equations

We want to slove the system

$$\begin{cases} \dot{x}_1 = & x_1 + 2x_2 \\ \dot{x}_2 = & -2x_1 - x_2 \end{cases}$$

First we find the general solution:

```
> eq1 := diff(x1(t),t)=x1(t)+2*x2(t);
```

$$eq1 := \frac{d}{dt}x1(t) = x1(t) + 2x2(t)$$

```
> eq2 := diff(x2(t),t)=-2*x1(t)-x2(t);
```

$$eq2 := \frac{d}{dt}x2(t) = -2x1(t) - x2(t)$$

```
> dsolve({eq1,eq2});
```

$$\begin{aligned} \{x1(t) &= \_C1\sin(\sqrt{3}t) + \_C2\cos(\sqrt{3}t), \\ x2(t) &= \frac{1}{2}\_C1\cos(\sqrt{3}t)\sqrt{3} - \frac{1}{2}\_C2\sin(\sqrt{3}t)\sqrt{3} - \frac{1}{2}\_C1\sin(\sqrt{3}t) - \frac{1}{2}\_C2\cos(\sqrt{3}t)\} \end{aligned}$$

Let us now determine the solution for which $x_1(0) = 1, x_2(0) = 1$:

> dsolve({eq1,eq2, $x1(0) = 1, x2(0) = 1$});

$$\{x1(t) = \sin(\sqrt{3}t)\sqrt{3} + \cos(\sqrt{3}t),\ x2(t) = \cos(\sqrt{3}t) - \sin(\sqrt{3}t)\sqrt{3}\}$$

D.3.2 Phase portraits

Considering the system

$$\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \end{pmatrix} = \begin{pmatrix} -2x_1 & -x_2 \\ -3x_1 & x_2 \end{pmatrix}$$

the MAPLE commands

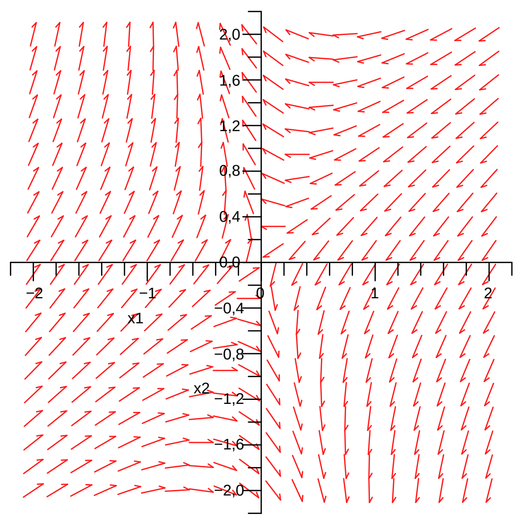
```
> with(DEtools):
> eqn1 := diff(x1(t),t)=-2*x1(t)-x2(t);eqn2 := diff(x2(t),t)=-3*x1(t)+x2(t);
> ic1 := x1(0)=-0.5, x2(0)=1;ic2 := x1(0)=0.5, x2(0)=-0.5;
> DEplot([eqn1,eqn2],[x1(t),x2(t)], t=-1..1,[[ic1],[ic2]],stepsize=.05);
```

will yield a plot of a phase portrait around the point $(0, 0)$.

We used “ic” in the Maple command as a short name for “initial condition.” Using this command, Maple plots the solution curves in yellow. We should choose the initial conditions such that we get a good feeling for the behavior of the solutions; often it is a good idea to choose points close to the initial conditions.

Using the command “DEplot” Maple can make a plot of a part of the plane: if we are interested in the behavior around the point $(1, 2)$, we can write $x1 = 0..2, x2 = 1..3$ in the Maple command, e.g., right after the stepsize. Using the following command Maple can make a phase portrait without using any initial conditions:

```
> with(DEtools):
> eq1 := diff(x1(t),t)=-2*x1(t)-x2(t);eq2 := diff(x2(t),t)=-3*x1(t)+x2(t);
> DEplot({eq1, eq2}, [x1, x2], t = 0..10, x1 = -2..2, x2 = -2..2, stepsize = .1);
```



D.3.3 Frequency characteristics

Consider the transfer function

$$H(s) = \frac{s+3}{s^3+4s^2+3s}.$$

```
>  H:=s− >(s+3)/(s^3+4*s^2+3*s);
```

$$H := s- > \frac{s+3}{s^3+4s^2+3s}.$$

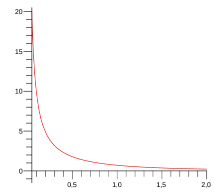
```
>  A:=omega− >abs(H(I*omega));
```

$$A := \omega- > |H(I\omega)|$$

```
>  A(omega) assuming omega::positive;
```

$$\frac{\sqrt{9+\omega^2}}{\sqrt{10\omega^4+\omega^6+9\omega^2}}$$

> plot(A, 0.05..2,-1.5..20);



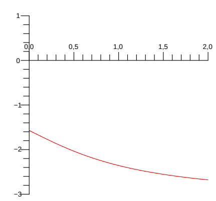
> Phi:=omega− >argument(H(I*omega));

$$\Phi := \omega - > argument(H(I\omega));$$

> Phi(omega) assuming omega::positive;

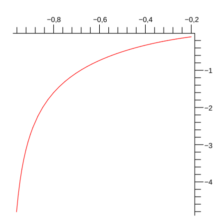
$$\arctan(\frac{1}{\omega}) - \pi.$$

> plot(Phi, 0..2, -3..1);



The following command provides a Nyquist plot:

> with(plots);
> complexplot(H(I*omega), omega=0.2..2);



D.4 Infinite series

D.4.1 Geometric series

Consider the infinite series

$$\sum_{n=1}^{\infty} \frac{(3n)!3^n}{n^{3n}2^{2n}}.$$

```
> a[n]:= (3*n)!*3^n/(n^(3*n)*2^(2*n));
```

$$a_n := \frac{(3n)!3^n}{n^{(3n)}2^{(2n)}}$$

```
> sum(a[n], n=1..infinity);
```

$$\sum_{n=1}^{\infty} \frac{(3n)!3^n}{n^{(3n)}2^{(2n)}}$$

Determination of $\frac{a_{n+1}}{a_n}$:

```
> subs(n=n+1,a[n]);
```

$$\frac{(3n+3)!3^{(n+1)}}{(n+1)^{(3n+3)}2^{(2n+2)}}$$

Calculation of $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$:

```
> limit((%)/a[n],n=infinity);
```

$$\frac{81}{4}e^{-3}$$

```
> evalf(%);
```

$$1.008188134$$

Since $1.008188134 > 1$ the series is divergent.

D.4.2 The quotient test, varying terms

Consider the power series

$$\sum_{n=0} \frac{(2n)!x^n}{(n!)^2}$$

```
> a[n]:=abs((2*n)!*x^n/(n!)^2);
```

$$a_n := \left|\frac{(2n)!x^n}{(n!)^2}\right|$$

> subs(n=n+1,a[n]);

$$\left|\frac{(2n+2)!x^{n+1}}{(n+1)!^2}\right|$$

Calculation of $\lim_{n\to\infty}\frac{a_{n+1}}{a_n}$:

> limit((%)/a[n],n=infinity);

$$\lim_{n\to\infty}\frac{\left|\frac{(2n+2)!x^{n+1}}{((n+1)!)^2}\right|}{\left|\frac{(2n)!x^n}{(n!)^2}\right|}$$

> simplify(%);

$$4\,|x|$$

According to the quotient test the infinite series is convergent if $4\,|x| < 1$, i.e., for $|x| < 1/4$, and divergent for $|x| > 1/4$ (the case $|x| = 1/4$ must be examined using different methods). Thus the radius of convergence is $\rho = 1/4$.

D.4.3 The sum of an infinite series

In certain cases Maple can find the sum of an infinite series, even if it contains a variable. Consider, e.g., the power series

$$\sum_{n=1}\frac{(2n)!x^n}{(n!)^2}$$

Note the difference between use of "sum" and "Sum" below:

> a[n]:=(2*n)!*x^n/(n!)^2;

$$a_n := \frac{(2n)!x^n}{n!^2}$$

> Sum(a[n], n=1..infinity);

$$\sum_{n=1}^{\infty}\frac{(2n)!x^n}{n!^2}$$

> sum(a[n], n=1..infinity);

$$\frac{4x}{\sqrt{1-4x}\left(1+\sqrt{1-4x}\right)}$$

Of course there are many cases where Maple (or any other program) will not be able to determine a concrete expression for an infinite sum.

D.4.4 The integral test and Theorem 4.38

We want to check whether the infinite series

$$\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{(n+1)(n+2)(n+3)} \tag{D.1}$$

is absolutely convergent, conditionally convergent, or divergent.

Consider first the infinite series

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n n^2}{(n+1)(n+2)(n+3)} \right| = \sum_{n=1}^{\infty} \frac{n^2}{(n+1)(n+2)(n+3)} \tag{D.2}$$

We first want to apply the integral test, so let us check that the assumptions are satisfied:

> f:=x− > x^2/((x+1)*(x+2)*(x+3));

$$f := x \to \frac{x^2}{(x+1)(x+2)(x+3)}$$

> simplify(diff(f(x),x));

$$-\frac{x(x^3 - 11x - 12)}{(x+1)^2(x+2)^2(x+3)^2}$$

This expressions shows that $f'(x)$ is negative for sufficiently large values of x; thus the function f is decreasing on $[0, \infty[$ for N sufficiently large. Let us check how large x must be in order to ensure that $f'(x) < 0$. We first solve the equation $f'(x) = 0$. Let us try the command

> `fsolve`((%) = 0);

$$0$$

Thus the command only gives one of the roots. The problem is that Maple does not see that we just need to find out when the numerator in the fraction determining $f(x)$ equals zero. We can force Maple to use this via the command

> `fsolve`(numer(%) = 0);

$$-2.483611621, -1.282823863, 0, 3.766435484$$

("numer" is the short name for "numerator"). We now conclude that f is decreasing for $x \geq 3.8$; thus, the terms in the infinite series in (D.2) decay for $n \geq 4$.

```
> Int(f(x), x=4..infinity)= int(f(x), x=4..infinity);
```

$$\int_4^{\infty} \frac{x^2}{(x+1)(x+2)(x+3)} dx = \infty$$

According to the integral test we conclude that (D.2) is divergent.

Let us now check whether (D.1) is convergent. We will use Theorem 4.38. We already showed that the function f is decreasing for $x \geq 3.8$.

```
> limit(f(x),x=infinity);
```

$$0$$

Thus the function f converges to 0 for $x \to \infty$. According to Theorem 4.38 we conclude that the infinite series (D.1) is conditionally convergent. Let us now determine the sum of the infinite series (D.1), within a tolerance of 10^{-5}. We again apply Theorem 4.38.

```
> b:=N->N ^2/((N+1)*(N+2)*(N+3));
```

$$b := N \to \frac{N^2}{(N+1)(N+2)(N+3)}$$

```
> fsolve(b(N+1) = 1/100000, N = 4..infinity);
```

$$99992.99989$$

Since f is decreasing for $x \geq 4$, it follows that

$$b_{N+1} \leq (10)^{-5} \text{ for all } N \geq 99993.$$

We can now determine the sum of the infinite series (D.1) within the required tolerance:

```
> evalf(sum((-1)^N*N^2/((N+1)*(N+2)*(N+3)), N=1..99993));
```

$$-0.011675375$$

D.4.5 Fourier series

We would like to calculate the Fourier coefficients for the 2π-periodic function given by

$$f(t) = t, \; t \in [-\pi, \pi[.$$

Let us first write the function in Maple:

```
> f:=t- >t;
```

$$f := t \to t$$

Calculation of the Fourier coefficients:

> assume(n,integer);

> a[n]:=1/Pi*int(f(t)*cos(n*t),t=-Pi..Pi);

$$a_{n\sim} = 0$$

> b[n]:=1/Pi*int(f(t)*sin(n*t),t=-Pi..Pi);

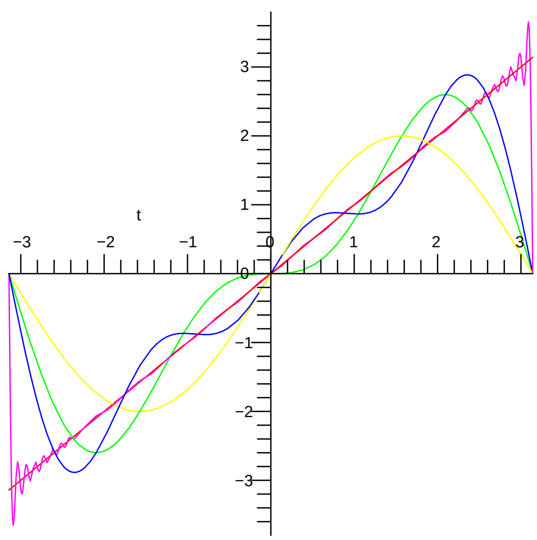
$$b_{n\sim} = -2\frac{(-1)^{n\sim}}{n\sim}$$

> S:=N− >sum(b[n]*sin(n*t),n=1..N);

The partial sum S_N is

$$S := N \to \sum_{n=1}^{N} b_n sin(nt)$$

> plot([f(t),S(1),S(2),S(3),S(60)], t=-Pi..Pi);



In connection with Fourier series on complex form we need to notice that Maple can not plot expressions that it comsider to be complex-valued. The following re-writing of the partial sum S_N is often useful:

$$\begin{aligned} S_N(t) &= \sum_{n=-N}^{N} c_n e^{int} \\ &= c_0 + \sum_{n=1}^{N} c_n e^{int} + \sum_{n=-N}^{-1} c_n e^{int}. \end{aligned}$$

> restart;

$$f(t) = t, \; t \in [-\pi, \pi[.$$

> f:=t− >t;

$$f := t \to t$$

Calculation of the Fourier coefficients on complex form:

> assume(n,integer);

> c[n]:=1/(2*Pi)*int(f(t)*exp(-I*n*t),t=-Pi..Pi);

$$c_{n\sim} = \frac{I(-1)^{n\sim}}{n \sim}.$$

This expression is obviously wrong when $n = 0$. For $n = 0$ we have

> 1/(2*Pi)*int(f(t)*exp(I*0*t),t=-Pi..Pi);

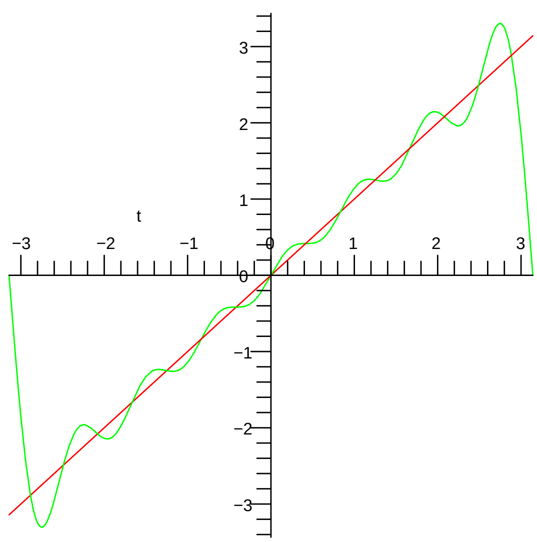
$$0$$

The partial sum S_N:

> S:=N→ sum(c[n]*exp(I*n*t),n=1..N)+sum(c[n]*exp(I*n*t), n=-N..-1);

$$S := N \to \sum_{n=1}^{N} c_n e^{Int} + \sum_{n=-N}^{-1} c_n e^{Int}.$$

> plot([f(t),S(7)], t=-Pi..Pi);



Appendix E
List of symbols

$\mathbb{R}$:	the real numbers
$\mathbb{C}$:	the complex numbers
$\mathbb{Z}$:	the integers, i.e., $0, \pm 1, \pm 2, \ldots$
$\mathbb{N}$:	the positive integers, i.e., $1, 2, \ldots$
$\mathbb{Q}$:	the fractions
$x \in A$:	x is an element in the set A
$A \cup B$:	the union of the sets A and B
$A \cap B$:	the intersection of the sets A and B
$A \subseteq B$:	the set A is a subset of the set B
$\forall$:	for all
$\exists$:	there exist
$f : A \to B$:	function f with domain A and taking values in B
$y^{(n)}$:	the nth derivative of the function y
$H(s)$:	transfer function
$L^2(-\pi, \pi)$:	the vector space of 2π-periodic functions, which are quadratically integrable over a period.

$\mathbf{A}$: matrix

$\mathbf{A}^{-1}$: inverse of matrix, defined when$(\mathbf{A}) \neq 0$.

$\mathbf{x}$: vector

$\mathbf{x}(t)$: vector function

$\dot{\mathbf{x}}(t)$: the derivative of a vector function

Some useful formulas:

$$\begin{aligned}
e^{(a+i\omega)t} &= e^{at}(\cos\omega t + i\sin\omega t),\ a,\omega \in \mathbb{R};\\
\cos\omega t &= \frac{1}{2}(e^{i\omega t} + e^{-i\omega t}),\ \omega \in \mathbb{R};\\
\sin\omega t &= \frac{1}{2i}(e^{i\omega t} - e^{-i\omega t}),\ \omega \in \mathbb{R};\\
a+ib &= re^{i\theta},\ r = |a+ib|,\ \theta = \mathrm{Arg}\,(a+ib)\\
&\ a = r\cos\theta, \quad b = r\sin\theta;\\
|a+ib| &= \sqrt{a^2+b^2},\ a,b \in \mathbb{R}\ \text{(Pythagoras)};\\
|a+b| &\leq |a| + |b|,\ a,b \in \mathbb{C}\ \text{(the triangle inequality)};\\
\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} &= ad - bc\\
\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} &= \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\ \text{when } \det(\mathbf{A}) = ad - bc \neq 0\\
\det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} &= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}\\
\sin(x+y) &= \sin x\cos y + \cos x\sin y\\
\cos(x+y) &= \cos x\cos y - \sin x\sin y
\end{aligned}$$

Relationship between growth of classical functions:

$$\begin{aligned}
\frac{x^a}{b^x} &\to 0 \text{ for } \ x \to \infty \text{ for all } a > 0, b > 1\\
\frac{\ln x}{x^a} &\to 0 \text{ for } \ x \to \infty \text{ for all } a > 0.
\end{aligned}$$

Partial integration:

$$\int f(x)g(x)\,dx = F(x)g(x) - \int F(x)g'(x)\,dx,\ \text{where } F(x) = \int f(x)\,dx.$$

Integration by substitution:

$$\int f(g(x))g'(x)\,dx = F(g(x)),\ \text{where } F(x) = \int f(x)\,dx.$$

Appendix F
Problems

F.1 Infinite series

Problem 101 For each of the following series, determine whether the sum is convergent or divergent. In case the series is convergent, determine the value of the sum.

(i) $\frac{1}{3} + \frac{1}{6} + \frac{1}{9} + \frac{1}{12} + \ldots$

(ii) $\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \ldots$

(iii) $\frac{1}{101} + \frac{2}{201} + \frac{3}{301} + \ldots$

(iv) $\frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \ldots$

(v) $\frac{\log 2}{2} + \frac{\log 3}{3} + \frac{\log 4}{4} + \ldots$

(vi) $\frac{1}{1} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt[3]{2}} + \frac{1}{\sqrt[4]{2}} + \ldots$

Problem 102 Check for each of the following infinite series whether it is absolutely convergent, conditionally convergent, or divergent.

(i) $\sum_{n=1}^{\infty} \frac{\sin n + \cos n}{n^2}$

(ii) $\sum_{n=1}^{\infty} \cos(n\pi) \frac{2}{n+5}$ (hint: $cos(n\pi) = (-1)^n$)

(iii) $\sum_{n=1}^{\infty} (-2)^n \frac{1}{n^2+7}$

Problem 103 Check for each of the following infinite series whether it is convergent of divergent; and find the sum for the convergent ones.

(a) $\sum_{n=1}^{\infty} \frac{n+3}{n+2}$

(b) $\sum_{n=1}^{\infty} \left(\frac{1}{5}\right)^n$

(c) $\sum_{n=1}^{\infty} \frac{1}{2(n+2)}$

(d) $\sum_{n=1}^{\infty} (1 + (-1)^n)$

(e) $\sum_{n=1}^{\infty} \left(\sqrt{5} - 1\right)^n$

(f) $\sum_{n=1}^{\infty} \frac{1}{(n+3)(n+4)}$ (apply the method from Example 4.18(iii))

Problem 104 Determine the current I in the infinite circuit shown in Figure F.1

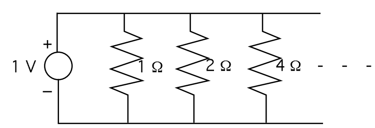


Figure F.1. A circuit with an infinite number of resistors (problem 104)

Problem 105 Prove the inequalities below via the integral test:

$$\frac{\pi}{4} < \sum_{n=1}^{\infty} \frac{1}{n^2+1} < \frac{\pi}{4} + \frac{1}{2},$$
$$\frac{1}{8} < \sum_{n=2}^{\infty} \frac{1}{n^3} < \frac{1}{4}.$$

Problem 106 A and B divide a cake according to the following rule: first A takes half of the cake, then B takes half of what remains, and then A takes half of the remainder, and so on, until there are only insignificant crumbs left. How large a portion of the original cake did A get? Solve the problem both with and without the use of infinite series.

Problem 108 Prove that the series

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{4n-3}} + \frac{1}{\sqrt{4n-1}} - \frac{1}{\sqrt{2n}} \right)$$

is divergent (hint: look at $\sqrt{n}a_n$ as $n \to \infty$).

Looking at a_n for the first few values of n, we observe that $\sum_{n=1}^{\infty} a_n$ consists of terms of the form $\pm\frac{1}{\sqrt{n}}$, with the sign plus when n is odd, and the sign minus when n is even. According to Proposition 4.38, the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$ is convergent. Explain why these results do not contradict each other.

Problem 109 Prove that the following series are convergent:

(a) $\sum_{n=1}^{\infty} \frac{1}{(2n)!}$

(b) $\sum_{n=1}^{\infty} \frac{\ln n}{n^{9/8}}$ (Hint: $\frac{\ln n}{n^{9/8}} = \frac{\ln n}{n^{1/16}} \frac{1}{n^{17/16}}$.)

(c) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln\left(\ln\left(n+\frac{1}{2}\right)\right)}$

Problem 110 Find the radius of convergence for the power series

$$\sum_{n=0}^{\infty} \frac{n+2}{n+1} x^{n+1},$$

and find an expression for the sum in terms of the standard functions.

Problem 111

(a) Show that the series $\sum_{n=1}^{\infty} \frac{\sin(nx)}{n^{5/2}}$ has a convergent majorant series.

(b) Give a convergent series for

$$\int_0^{\pi} \sum_{n=1}^{\infty} \frac{\sin(nx)}{n^{5/2}} dx \ .$$

Problem 112 We will apply Theorem 5.2.

(i) Show that

$$\sum_{n=1}^{\infty} n x^n = \frac{x}{(1-x)^2}, \ x \in]-1, 1[. \tag{F.1}$$

Hint: differentiate on each side of (5.3) and apply Theorem 5.17.

(ii) Apply the same procedure on the infinite series (F.1) and show that

$$\sum_{n=1}^{\infty} n^2 x^n = \frac{x+x^2}{(1-x)^3}, \ x \in]-1, 1[.$$

(iii) Using (i) and (ii), calculate the infnite sums

$$\sum_{n=1}^{\infty} \frac{n}{2^n} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{n^2}{2^n}.$$

Problem 113 For each of the following series, find all values of x for which the series converges. For such values of x, find the sum of the series.

(i) $\frac{x}{1} + \frac{x}{2} + \frac{x}{3} + \cdots$

(ii) $e^x + e^{2x} + e^{3x} + \cdots$

(iii) $\log x + \log 2x + \log 3x + \cdots$

Problem 114 Which of the following series have a majorant series? Which of the series have a convergent majorant series?

(a) $1 + x + x^2 + x^3 + \cdots, x \in \mathbb{R}$

(b) $1 + x + x^2 + x^3 + \cdots, |x| < 1$

(c) $1 + x + x^2 + x^3 + \cdots, |x| < 0,99$

(d) $\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots x \in \mathbb{R}$

(e) $f(x) = \frac{\sin x}{1^2} + \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} + \cdots x \in \mathbb{R}$

Is the function $f(x)$ in (e) continuous? What is the connection between $f(x)$ and the function

$$g(x) = \sum_{n=1}^{\infty} \frac{-\cos((2n-1)x)}{(2n-1)^3}?$$

Explain which results you apply to answer these questions.

Problem 115 For each of the following power series, determine the $x \in \mathbb{R}$ for which the infinite series is convergent and calculate the sum (use Theorem 5.2).

(i) $\sum_{n=0}^{\infty}(-x)^n$

(ii) $\sum_{n=0}^{\infty}(-1)^n x^{2n}$

Problem 116 Consider the power series

$$\sum_{n=0}^{\infty} \frac{(-1)^n(2n+2)}{(2n+1)!} x^{2n+1}.$$

(i) Find the radius of convergence for the series.

(ii) If $f(x)$ denotes the sum of the series, find the power series for $\int_0^x f(t)dt$, and use this to determine an explicit expression for $f(x)$.

Problem 117 Find the power series for each of the functions:

(a) $f(x) = 2\sin x + \cos x$

(b) $g(x) = 2\sin x \cos x$

Problem 118 Prove that

$$f(x) := \sum_{n=1}^{\infty} \frac{3\cos nx - \sin^2 nx}{3^n - 1}, \quad x \in \mathbb{R},$$

defines a continuous function.

Problem 119 The object of this exercise is to find e^A, where

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}.$$

(a) First observe that A has an eigenvalue 1 with corresponding eigenvector $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$, and eigenvalue 2 with eigenvector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Show that A is diagonalizable as

$$A = U \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} U^{-1},$$

where $U = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $U^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$

(b) Show that, for $k = 1, 2, ..,$

$$A^k = U \begin{pmatrix} 1 & 0 \\ 0 & 2^k \end{pmatrix} U^{-1}.$$

(c) Find e^B, where $B = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$

(d) Now find e^A.

Problem 120 Given the matrix

$$A = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$$

(i) Show that $A^2 = -\mathbf{1}$

(ii) Calculate the matrix e^A .

Problem 121 (i) Find the radius of convergence for the power series

$$\sum_{n=1}^{\infty} \frac{n+1}{n \cdot n!} x^n.$$

Let $f(x)$ denote the sum of this series.

(ii) Find a power series representation for the function

$$h(x) = \int_0^x t f'(t) dt, \ x \in \mathbb{R}.$$

(iii) Use the power series for e^x to find an expression for $h(x)$ in terms of the standard functions.

Problem 122 Investigate the convergence of the following series:

(i) $\sum_{n=1}^{\infty} e^{\frac{i\pi n}{2}} \frac{1}{n}$.

(ii) $\sum_{n=1}^{\infty} a_n$, given that the Nth partial sum is $S_N = \frac{N+1}{\ln N+2}$.

Problem 123 Is the series $\sum_{n=1}^{\infty}(-1)^n \frac{1}{\sqrt{n}}$ absolutely convergent or conditionally convergent? Find the sum with an error at most 0.1.

Problem 124

(i) Show that the series

$$\sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{x}{n}\right)$$

is absolutely convergent for all $x \in \mathbb{R}$ (Hint: Use that $|\sin x| \leq |x|$ for all $x \in \mathbb{R}$).

(ii) Show that the function

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{x}{n}\right), \quad x \in \mathbb{R},$$

is differentiable and find $f'(0)$.

Problem 125 Determine whether the series

$$\sum_{n=1}^{\infty} \cos\left(\frac{n!}{n^n}\right)$$

is convergent.

Problem 126

(a) Determine the $a \in \mathbb{R}$ for which the infinite series

$$\sum_{n=1}^{\infty} \frac{1}{(2+a)^n}$$

is convergent.

(b) For these values of $a \in \mathbb{R}$, calculate the sum of the infinite series.

Problem 127 Find a polynomial $P(x)$ for which

$$|\sin x - P(x)| \leq 0.05 \text{ for all } \; x \in [0, 3].$$

Problem 128 We use Maple to investigate the series

$$\sum_{n=1}^{\infty} \frac{n! \cdot \ln n}{n^n}.$$

(a) What information does the nth term test give here?

(b) What information does the quotient test give?

Problem 129

(i) Explain why the series

$$\frac{1}{100}+\frac{1}{101}+\frac{1}{102}+\cdots$$

is divergent.

(i) Find an integer $N > 100$ for which

$$\frac{1}{100}+\frac{1}{101}+\cdots+\frac{1}{N} > 100$$

(You can use a computer to help with this)

Problem 131 This problem deals with calculation of the sum

$$S=\sum_{n=1}^{\infty}\frac{1}{n^{11/10}}. \tag{F.2}$$

(i) Use Maple to calculate the finite sum $\sum_{n=1}^{31642}\frac{1}{n^{11/10}}$.

Intuitively one could expect that 31642 terms is enough to obtain a good approximation of the sum S. We will now check whether the intuition is correct.

(ii) Find $N \in \mathbb{N}$ such that $\left|S-\sum_{n=1}^{N}\frac{1}{n^{11/10}}\right| \le 0.1$.

(iii) Is it possible for Maple to calculate the number S within a tolerance of 0.1 using the information in (i)?

(iv) Explain in words (without any calculations) how the number of necessary terms (i.e., the value of N) changes when the infinite series (F.2) is substituted by, e.g., $\sum_{n=1}^{\infty}\frac{1}{n^{1.01}}$ eller $\sum_{n=1}^{\infty}\frac{1}{n^{5}}$.

The answer in (iii) might surprise, as Maple actually can provide the value of the infinite series in (F.2). Below we apply another method to find the value of S with a tolerance of maximally 0.1.

(v) Determine

$$\sum_{n=1}^{\infty}\frac{1}{n^{11/10}}$$

within a tolerance of 0.1 using Corollary 4.35(ii).

Problem 133 Show that the power series for $f(x)=\ln(1+x)$ does not have a convergent majorant series for $x \in \left]-1,1\right]$.

Problem 134 For which real values of x is the series

$$\frac{x}{1\cdot 2}+\frac{x^2}{2\cdot 2^2}+\frac{x^3}{3\cdot 2^3}+\frac{x^4}{4\cdot 2^4}+\cdots$$

convergent ?

Problem 135 Determine whether the following series is absolutely convergent, conditionally convergent or divergent:

$$\sum_{n=1}^{\infty}\frac{\sin n}{n^3}(-1)^n.$$

Problem 136

(i) Find the radius of convergence ρ for the infinite series

$$\sum_{n=1}^{\infty}\frac{2^n}{n}x^n.$$

Now, let

$$f(x):=\sum_{n=1}^{\infty}\frac{2^n}{n}x^n,\ \ x\in]-\rho,\rho[.$$

(ii) Find an expression for the derivative, $f'(x)$.

Problem 137

(i) Find the radius of convergence for the series:

$$\sum_{n=1}^{\infty}\frac{1}{n(n+1)}x^n.$$

(ii) Is the series convergent for $x=\pm\rho$?

Problem 138 Determine whether the following series is absolutely convergent, conditionally convergent or divergent:

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n}e^{-n}$$

Problem 139

(i) Find the radius of convergence for the series:

$$\sum_{n=1}^{\infty} \frac{x^{n+1}}{(n+1)n!}$$

Now set

$$f(x) := \sum_{n=1}^{\infty} \frac{x^{n+1}}{(n+1)n!}, \ x \in]-\rho, \rho[.$$

(ii) Using the power series for e^x, find an expression for $f'(x)$.

(iii) Now determine an expression for $f(x)$.

Problem 140 Determine whether the following series is convergent:

$$\sum_{n=1}^{\infty} \frac{e^n}{n+1}$$

Problem 142 Determine whether the following series is absolutely convergent, conditionally convergent or divergent:

(i) $\sum_{n=0}^{\infty} \frac{1}{n^2+3}$;

(ii) $\sum_{n=0}^{\infty} \frac{(-1)^n}{n+13}$;

(iii) $\sum_{n=0}^{\infty} (n!+5)^2$;

(iv) $\sum_{n=1}^{\infty} a_n$, if the Nth partial sum is given by

$$S_N = \sum_{n=1}^{N} a_n = \cos(\pi N).$$

Problem 143 Determine the radius of convergence for the series

$$\sum_{n=0}^{\infty} n^3 (3x)^n.$$

Problem 144

(i) For which values of $x \in \mathbb{R}$ is the infinite series

$$\sum_{n=0}^{\infty} (x+1)^n$$

convergent?

(ii) Find an expression for the sum of the infinite series for these $x \in \mathbb{R}$.

Problem 145 The infinite series

$$\sum_{n=1}^{\infty} \frac{n}{2^{n-1}} x^{n-1}$$

has radius of convergence $\rho = 2$ (you don't need to prove this). Let

$$f(x) = \sum_{n=1}^{\infty} \frac{n}{2^{n-1}} x^{n-1}, \ x \in]-2, 2[.$$

(i) Show that

$$\int_0^x f(t)dt = \frac{2x}{2-x}$$

for $|x| < 2$.

(ii) Is the infinite series $\sum_{n=1}^{\infty} \frac{n}{2^{n-1}} x^{n-1}$ convergent for $x = \pm 2$?

Problem 146 Check whether the following infinite series are absolutely convergent, conditionally convergent, or divergent:

(i) $\sum_{n=0}^{\infty} \frac{n}{4^{n+3}}$

(ii) $\sum_{n=0}^{\infty} (-1)^n n^2$

(iii) $\sum_{n=1}^{\infty} a_n$, where

$$a_n = \begin{cases} n^3 & \text{for} \quad n = 1, 2, \ldots, 10000 \\ \frac{(-1)^n}{n^3} & \text{for} \quad n \geq 10001 \end{cases}.$$

Problem 147 Determine the radius of convergence for the series

$$\sum_{n=0}^{\infty} \sqrt{n} x^{2n}.$$

Problem 148 Find an approximate value for $\sum_{n=1}^{\infty} \frac{1}{n^4}$, with a tolerance of maximally 0.02.

Problem 149

(i) Show that the infinite series $\sum_{n=0}^{\infty} \frac{1}{3^n} \cos(2^n x)$ is convergent for all $x \in \mathbb{R}$.

Now, let

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{3^n} \cos(2^n x), \quad x \in \mathbb{R}.$$

(ii) Show that the function f is continuous.

(iii) Show that the function f is differentiable.

(iv) Find $f'(0)$.

(v) Compare the considered infinite series and the conclusion in (iii) with Example 5.26.

Problem 150 We know that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent and that $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent. Use the comparison test and the equivalence test to check whether the following infinite series are convergent or divergent.

$$(a) \sum_{n=1}^{\infty} \frac{1}{n^3+1}, \quad (b) \sum_{n=1}^{\infty} \frac{1}{n+\sin n}, \quad (c) \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

Problem 151 Consider the infinite series for $\ln(1+x)$ in Appendix B.

(i) Show that the infinite series does not have a convergent majorant series for $x \in]-1,1]$.

(ii) Show that the function $f(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} x^n,\ x \in]-1,1[$ is continuous and differentiable for $x \in]-1,1[$.

(iii) Find an expression for $f'(x)$ in terms of the logarithm function.

Problem 152

(a) Calculate the sums

$$\sum_{n=200}^{\infty} 3(\frac{1}{4})^n, \quad \sum_{n=10}^{\infty} 3(\frac{1}{4})^{n-3}.$$

(b) Decide whether the infinite series $\sum_{n=1}^{\infty} a_n$ is convergent if

$$a_n = \begin{cases} n^2 & \text{for} \quad n = 1, 2, ..., 100 \\ \frac{1}{n^2} & \text{for} \quad n \geq 101. \end{cases}$$

(c) For a certain infinite series $\sum_{n=1}^{\infty} a_n$ it is known that the Nth partial sum is given by $S_N = \frac{1}{N}\cos(N\pi)$. Is the infinite series absolutely convergent, conditionally convergent, or divergent?

Problem 153

(i) Show that for the function $f(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} x^n$ we have

$$\int_0^{1/2} f(x)dx = \sum_{n=1}^{\infty} \frac{1}{n^2(n+1)} \frac{1}{2^{n+1}}. \tag{F.3}$$

The goal of this exercise is to find an approximate value for this sum.

(ii) Show that, for $N \in \mathbb{N}$, $\sum_{n=N+1}^{\infty} \frac{1}{2^{n+1}} = \frac{1}{2^{N+1}}$.

(iii) Let S_N denote the N'th partial sum of the series at (F.3). Determine an $N \in \mathbb{N}$ such that

$$\left| \sum_{n=1}^{\infty} \frac{1}{n^2(n+1)} \frac{1}{2^{n+1}} - S_N \right| \leq 10^{-3}.$$

Hint: First note that

$$\sum_{n=N+1}^{\infty} \frac{1}{n^2(n+1)} \frac{1}{2^{n+1}} \leq \sum_{n=N+1}^{\infty} \frac{1}{2^{n+1}}.$$

(iv) Use Maple to calculate the sum at (F.3) with an error at most 10^{-3}.

Problem 155 The following questions can be answered independently of each other:

(i) Is $\sum_{n=1}^{\infty} 2^n$ convergent?

(ii) For an infinite series $\sum_{n=1}^{\infty} a_n$ it is known that the Nth partial sum is

$$S_N = \sum_{n=1}^{N} a_n = \frac{1}{N} + 1.$$

Is the infinite series $\sum_{n=1}^{\infty} a_n$ convergent?

(iii) Find the values of $x \in \mathbb{R}$ for which

$$\sum_{n=0}^{\infty} nx^n$$

is convergent (remember to check the endpoints of the interval of convergence).

Problem 156

(i) Investigate whether the series $\sum_{n=1}^{\infty} \frac{2^n}{n}$ is convergent.

(ii) Is the series $\sum_{n=0}^{\infty}(-1)^n \frac{1}{(n+1)^2}$ absolutely convergent?

In the rest of this exercise we consider the infinite series

$$\sum_{n=0}^{\infty} \frac{1}{x^n}. \tag{F.4}$$

(iii) Determine the values of $x \in \mathbb{R}$ for which the series (F.4) is convergent.

(iv) For these values of x, determine the sum of the series (F.4).

Problem 157 It is known (you don't need to prove this) that the power series $\sum_{n=1}^{\infty} \frac{1}{n+1} x^{n+1}$ has radius of convergence $\rho = 1$. Set

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n+1} x^{n+1}, \; x \in]-1, 1[.$$

(i) Explain why f is differentiable and find an expression for $f'(x)$ via an infinite series.

(ii) Show that $f'(x) = \frac{1}{1-x} - 1$.

(iii) Using (ii), find a closed formula for $f(x)$.

Problem 158 Determine whether the following series are convergent:

(i) $\sum_{n=1}^{\infty} \ln n$

(ii) $\sum_{n=0}^{\infty} \frac{n}{4^n}$

(iii) $\sum_{n=3}^{\infty} \frac{1}{\ln n} \frac{1}{n^4}$

Problem 159 Is the series

$$\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^3+4}$$

absolutely convergent, conditionally convergent, or divergent?

Problem 160

(i) Show that the infinite series
$$\sum_{n=1}^{\infty} \frac{1}{n^3} e^{-nx}$$
is convergent for all $x > 0$.

Now, let
$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n^3} e^{-nx}, \quad x > 0.$$

(ii) Show that the function f is continuous.

(iii) Show that the infinite series
$$\sum_{n=1}^{\infty} \frac{1}{n^2} e^{-nx}$$
has a convergent majorant series on the interval $]0, \infty[$.

(iv) Is f differentiable for $x > 0$?

(v) Does the infinite series
$$\sum_{n=1}^{\infty} \frac{1}{n^2} e^{-nx}$$
have a convergent majorant series on the interval $]-1, 1[$?

Problem 161 We know that the infinite series $\sum_{n=1}^{\infty} (-1)^{n-1} \ \frac{1}{n}$ is convergent. We will now determine an approximate value for the sum.

(i) Find $N \in \mathbb{N}$ such that the difference between mellem $\sum_{n=1}^{\infty} (-1)^{n-1} \ \frac{1}{n}$ and $\sum_{n=1}^{N} (-1)^{n-1} \ \frac{1}{n}$ is at most 0.01.

(ii) Apply the value of N that was found in (i) and Maple to find an approximate value for $\sum_{n=1}^{\infty} (-1)^{n-1} \ \frac{1}{n}$, with an error of at most 0.01.

(iii) Repeat the questions (i)+(ii) with the infinite series $\sum_{n=1}^{\infty} (-1)^{n-1} \ \frac{1}{n^4}$.

(iv) For the infinite series in (iii) we had to use considerably fewer terms than in (ii). Explain this in words.

Problem 162 Show that, for all $\alpha > 1$ and $N \in \mathbb{N}$, the following hold:

(i)
$$\sum_{n=N+1}^{\infty} \frac{1}{n^\alpha} \leq \frac{1}{(N+1)^\alpha} \frac{\alpha + N}{\alpha - 1}.$$

(ii)
$$\sum_{n=N+1}^{\infty} \frac{1}{n^\alpha} \leq \frac{1}{(N+1)^{\alpha-1}} \frac{\alpha}{\alpha - 1}.$$

(Hint: for the values of α under consideration, we have $\alpha + N < \alpha + \alpha N$).

Problem 163

(i) Investigate monotonicity for the function
$$f(x) = \frac{100}{x - 10 \ln x + 30}, \qquad x \geq 1.$$

(ii) Determine whether the series
$$\sum_{n=1}^{\infty} (-1)^n \frac{100}{n - 10 \ln n + 30}$$
is absolutely convergent, conditionally convergent, or divergent.

Problem 164 Recall that the hyperbolic tangent and cotangent functions are defined respectively by
$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^{2x} - 1}{e^{2x} + 1}\ , \quad x \in \mathbb{R}$$
and
$$\coth x = \frac{\cosh x}{\sinh x} = \frac{e^{2x} + 1}{e^{2x} - 1},\ x \neq 0.$$
Determine the convergence or lack thereof for the series:
$$\sum_{n=1}^{\infty} \frac{\tanh n}{n^2} \quad \text{og} \quad \sum_{n=1}^{\infty} \frac{\coth n}{n^2}$$

F.2 Fourier series

Problem 202 Consider the 2π-periodic function f given by

$$f(x) = \frac{1}{4}x^2 - \frac{\pi}{2}x, \; x \in [0, 2\pi[.$$

(i) Sketch the graph of f.

(ii) Show that the function f is even (f is a polynomial of degree 2).

(iii) Show that the Fourier series f is given by

$$f \sim -\frac{1}{6}\pi^2 + \sum_{n=1}^{\infty} \frac{1}{n^2} \cos nx.$$

(iv) Argue that "$\sim$" can be substituted by "$=$", i.e., that for all x we have

$$f(x) = -\frac{1}{6}\pi^2 + \sum_{n=1}^{\infty} \frac{1}{n^2} \cos nx. \tag{F.5}$$

(v) Prove using (F.5) that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

(vi) Let $S_N(x)$ denote the Nth partial sum in the Fourier series. Find N such that the difference between $f(x)$ and $S_N(x)$ is at most 0.1 for all x. (*Hint:* apply Theorem 6.17 and the result in Example 4.36).

(vii) Calculate the sum $\sum_{n=1}^{\infty} \frac{1}{n^4}$.

(viii) Find for $x \in [0, 2\pi]$ the sum $\sum_{n=1}^{\infty} \frac{\sin nx}{n^3}$.

Problem 204 A harmonic forcing function can be written in any of the forms:
1) $A\cos(\omega t + \varphi)$, 2) $a\cos\omega t + b\sin\omega t$, 3) $\mathrm{Re}(c \cdot e^{i\omega t})$
where$A$, a and b are real numbers, and $c \in \mathbb{C}$. Write each of the functions below in all of the above three forms:
(1) $\sin 4t$ (2) $\cos(3t - \frac{\pi}{4})$ (3) $\mathrm{Re}((2-i)e^{i\frac{\pi}{2}t})$

Problem 205 Write each of the following functions as a sum $f_E + f_O$ of an even function $f_E(t)$ and an odd function $f_O(t)$.

(a) $f(t) = t^4 + 2t^3 - t^2 + 1$

(b) $f(t) = \cos 3t \cdot \sin t$

(c) $f(t) = |t+1|$

Problem 206 Show that the two functions $f_1(t) = 2,\; f_2(t) = t - \frac{1}{2}$, are an orthogonal system on the interval $[0, 1]$.

Problem 207 Determine the Fourier coefficients a_0, a_1, b_1 for the 2π-periodic function given by

$$f(t) = \begin{cases} \cos t & \text{for } -\pi < t \leq 0, \\ \sin t & \text{for } 0 < t \leq \pi. \end{cases}$$

Problem 209 Let f denote the 2π- periodic function, which for $t \in [-\pi, \pi[$ is given by

$$f(t) = \begin{cases} \cos t & \text{for } \frac{-\pi}{4} < t < \frac{\pi}{4}, \\ 0 & \text{ellers.} \end{cases}$$

(i) Sketch the graph of f.

(ii) Find the Fourier coefficients a_0, a_1 and $b_n,\ n = 1, 2, ..$.

(iii) Is the Fourier series convergent for all $t \in \mathbb{R}$? If yes, what does the Fourier series converges to?

(iv) Can we replace "$\sim$" in the Fourier series by "="?

(v) Does the Fourier series have a convergent majorant series?

Problem 210 Find the Fourier series in real form for the 2π-periodic function f if the complex Fourier coefficients are

$$c_n = \begin{cases} \frac{-i}{n} & \text{for} \quad n = 1, 2, ..., 10, \\ \frac{i}{n} & \text{for} \quad n = -1, -2, ..., -10, \\ 0 & \text{ellers.} \end{cases}$$

Problem 211 A function f with period 1 is given on the interval $]0, 1[$ as $f(t) = e^{-t}$. We want to determine the 2nd partial sum, $S_2(t)$, of the Fourier series four $f(t)$ in realform.

(i) Write down the formulas for the coefficients a_n and b_n - is it necessary to calculate the integrals?

(ii) Find $S_2(t)$ by first computing a suitable number of Fourier coefficients in complex form. Use the fact that f is real, so $c_{-n} = \overline{c_n}$.

Problem 213 A function $f(t)$ with period T is in the interval $[0, T]$ given by

$$f(t) = \begin{cases} 1, & 0 \le t \le \frac{T}{8},\ \frac{7T}{8} \le t \le T \\ 0, & \frac{T}{8} < t < \frac{7T}{8} \end{cases}$$

(i) Find the Fourier series for $f(t)$.

(ii) Use the Fourier series to find the sum of the infinite series

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

(hint: apply Fouriers theorem with $t = T/8$ and use that $\sin 2t = 2 \sin t \cos t$).

Problem 214 (i) Prove that

$$|\sin x| = \frac{2}{\pi} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos 2nx}{(2n-1)(2n+1)}, \ \forall x \in \mathbb{R}.$$

(ii) Calculate the number $\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)}$.

(iii) Calculate the number $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2(2n+1)^2}$.

(iv) Write the Fourier series for $|\sin(\cdot)|$ in complex form.

(v) Compare the decay of the coefficients in the Fourier series for $\sin(\cdot)$ and $|\sin(\cdot)|$; see Theorem 6.35.

(vi) Denote the Nth partial sum of the Fourier series for $|\sin(\cdot)|$ by S_N. Find N such that

$$|\,|\sin x| - S_N(x)| \le 0.1, \ \forall x \in \mathbb{R}.$$

Problem 217 Consider the odd 2π-periodic function, which for $x \in [0, \pi]$ is given by

$$f(x) = \frac{\pi}{96}(x^4 - 2\pi x^3 + \pi^3 x).$$

(i) Find $f(-\frac{\pi}{2})$.

(ii) Prove that $f(x) = \sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{(2n-1)^5}, \ x \in \mathbb{R}$.
(hint: $\int_0^{\pi} (x^4 - 2\pi x^3 + \pi^3 x) \sin nx \ dx = 24 \frac{1-(-1)^n}{n^5}, \ n \in \mathbb{N}$).

(iii) Prove that

$$|f(x) - \sin x| \le 0.01, \ \forall x \in \mathbb{R}$$

Problem 222 Consider the 2π-periodic function given by

$$f(t) = t,\ \ t \in [-\pi, \pi[.$$

(a) Show that the Fourier series (on real form) is given by

$$f(t) \sim \sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n+1} \sin nt.$$

(b) Apply a) to find the sum of the infinite series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1} \qquad \textit{Hint: use Fouriers theorem with } t = \tfrac{\pi}{2}.$$

(c) Calculate the Fourier coefficients on complex form.

(d) Does the Fourier series on real form have a convergent majorant series?

(e) Determine for each $t \in [-\pi, \pi]$ the sum of the Fourier series.

(f) The function f is differentiable for $t \in]-\pi, \pi[$. Is it allowed to "differentiate under the sum tegn" in the Fourier series, which would lead to

$$1 = 2\sum_{n=1}^{\infty}(-1)^{n+1} \cos nt,\ \ t \in] - \pi, \pi[\,?$$

Compare with Theorem 5.35.

Problem 223 Consider the 2π-periodic function f given by

$$f(x) = \begin{cases} 0 & \text{for} \quad -\pi \le x \le 0, \\ 1 & \text{for} \quad 0 < x < \pi. \end{cases}$$

(i) Prove that the complex Fourier series of f is given by

$$f \sim \frac{1}{2} + \sum_{n \text{ odd}, n=-\infty}^{\infty} \frac{1}{\pi i n} e^{inx}.$$

(ii) Use (i) to calculate

$$\textstyle\sum_{n=1}^{\infty} \dfrac{1}{(2n-1)^2}.$$

(iii) Can we replace "$\sim$" in the Fourier series by "$=$"?

Problem 224 Let f be the 2π-periodic function given by

$$f(x) = \begin{cases} 1 & \text{for } x \in [0, \frac{\pi}{4}[, \\ 0 & \text{for } x \in [\frac{\pi}{4}, 2\pi[. \end{cases}$$

(i) Find the Fourier series for f.

(ii) Does the Fourier series have a convergent majorant series?

(iii) How can you answer (ii) without looking at the Fourier series?

Problem 225 Find the Fourier coefficients for the 2π-periodic function given by $f(x) = 1$ for all $x \in \mathbb{R}$. Can we replace "$\sim$" by "$=$"?

Problem 226 Consider the 2π-periodic function f given by

$$f(x) = \begin{cases} \sin x & \text{if} \quad 0 < x \leq \pi, \\ 0 & \text{if} \quad \pi < x \leq 2\pi. \end{cases}$$

Denote the Nth partial sum of the Fourier series by S_N. Find N such that

$$|f(x) - S_N(x)| \leq 0.1, \ \forall x \in \mathbb{R}.$$

Problem 227

(i) Show that the Fourier series for the 2π-periodic function f given by

$$f(x) = |x|, \ x \in [-\pi, \pi[$$

is

$$f \sim \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos((2n-1)x)}{(2n-1)^2}.$$

(ii) Show that we can replace "$\sim$" by "$=$", i.e., that the Fourier series converges pointwise to $f(x)$ for all $x \in \mathbb{R}$.

(iii) Find N such that the difference between $f(x)$ and the Nth partial sum $S_N(x)$ is at most 0.1 for all x. Find $N \in \mathbb{N}$ such that the difference is at most 0.01.

(iv) Does the Fourier series have a convergent majorant series?

(v) Sketch the function f and the partial sum in (iii).

(vi) Calculate the Fourier coefficients on complex form.

(vii) Find the sum of the infinite series

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}.$$

Problem 228 Consider the 2π-periodic function f given by

$$f(x) = \begin{cases} \cos x & \text{for} \quad 0 < x \leq \pi, \\ 0 & \text{for} \quad \pi < x \leq 2\pi. \end{cases}$$

(i) Sketch the function f.

(ii) Show that the Fourier series for f is convergent for all $x \in \mathbb{R}$ (you do not have to calculate the Fourier coefficients to answer this).

(iii) Calculate the sum of the Fourier series for $x = 0$.

(iv) Calculate the Fourier coefficients a_0.

(v) Calculate the Fourier coefficients c_0.

Problem 229

(i) Find the transfer function for the linear system

$$\begin{cases} \dfrac{d\mathbf{x}}{dt} = \begin{pmatrix} 1 & 2 \\ -3 & -4 \end{pmatrix} \mathbf{x}(t) + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u(t) \\ y(t) = x_1(t) + x_2(t). \end{cases}$$

(ii) Show that the system is asymptotically stable.

(iii) Expressed as an infinite series, find a solution for the system when u is the 2π-periodic function given by $u(t) = t^2, \ t \in [-\pi, \pi[$ (Hint: you can use the result in Example 6.8). .

Problem 230 The calculation in Example 7.9 led to $N \geq 128$. Can you modify the calculation and get a better value for N?

Problem 231

(i) Find the transfer function for the linear system

$$\begin{cases} \dfrac{d\mathbf{x}}{dt} = \begin{pmatrix} 1 & 2 \\ -3 & -4 \end{pmatrix} \mathbf{x}(t) + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u(t) \\ y(t) = x_1(t) + x_2(t). \end{cases}$$

(ii) Show that the system is asymptotically stable.

(iii) Expressed as an infinite series, find a solution for the system when u is the 2π-periodic function given by $u(t) = \frac{1}{4}t^2 - \frac{\pi}{2}t, \ t \in [0, 2\pi[$ (Hint: Use the result in Problem 202).

Problem 232 Let f denote the function with period 2π, which in the interval $0 \le x \le 2\pi$ is given by

$$f(x) = \begin{cases} x & \text{for} \quad 0 \le x \le \pi \\ 0 & \text{for} \quad \pi < x \le 2\pi. \end{cases}$$

1. Sketch the graph for f in the interval $[-2\pi, 4\pi]$.
2. Find the Fourier series for the function f.
3. Is the Fourier series convergent for all $x \in \mathbb{R}$? If yes, determine the sum for all $x \in \mathbb{R}$.

F.3 nth order differential equations

Problem 301

(a) Find the general real solution to the differential equation

$$\ddot{y} + 5\dot{y} - 6y = 0.$$

(b) Now find the general real solution to

$$\ddot{y} + 5\dot{y} - 6y = t.$$

Problem 304 Determine the transfer function for the differential equation

$$y''' + 4y'' + 3y' = u' + 3u.$$

Problem 305 Find the transfer function for the equation

$$\dddot{y} + 3\ddot{y} - 4y = \dot{u} + 2u.$$

Find the stationary solution corresponding to the forcing functions:

$$(a)\;\; u(t) = e^{2t}, \quad (b)\;\; u(t) = e^{2it}, \quad (c)\;\; u(t) = \cos 2t.$$

Problem 306

(i) Find the general complex solution to the differential equation

$$y^{(4)} + 2y^{(2)} + y = 0.$$

(ii) Find the general real solution to the differential equation.

Problem 307 Calculate and sketch the frequency characteristics for the differential equation $\dot{y} + y = \dot{u}$. Read off the approximate values of the stationary solution corresponding to the forcing function:

$$(1) \;\; u(t) = \cos t, \qquad (2) \;\; u(t) = \cos \sqrt{3}t.$$

Problem 308 Consider the differential equation

$$y'' - y' - 6y = u' + 2u,$$

where u is the forcing function and y is the unknown.

(i) Find the transfer function for the above equation.

(ii) Find the stationary solution corresponding to the forcing function $u(t) = e^t$.

(iii) Find the general solution corresponding to the same forcing function.

Problem 309 Use the method of "guess and test" to find a solution to each of the equations

(1) $\ddot{y} + 5\dot{y} - 6y = \cos 2t$

(2) $\ddot{y} + 5\dot{y} - 6y = \sin 2t$

(3) How can you find the general solution to the equations (1) and (2)?

(4) Argue how we can find the general solution to the equation

$$\ddot{y} + 5\dot{y} - 6y = 17\cos 2t - 121\sin 2t.$$

Problem 310 Consider the differential equation

$$y'' - y = u' + u.$$

(a) Determine the transfer function $H(s)$ corresponding to the above equation.

(b) Determine the amplitude characteristic $A(\omega)$ and the phase characteristic $\phi(\omega)$.

(c) Find the stationary solution corresponding to the forcing function $u(t) = \cos\sqrt{3}t$.

Problem 311 Consider the differential equation

$$y''(t) + 3y'(t) + 2y = u'(t) + 2u(t),$$

where u is the forcing function and y the unknown.

(a) Determine the transfer function $H(s)$.

(b) Find the stationary solution corresponding to $u(t) = e^t$.

Problem 312 Does the differential equation $\frac{dy}{dt} - 2y = e^{2t}$ have a solution of the form $y(t) = Ce^{2t}$ where C is a constant?

Problem 313 Find the general real solution to the differential equation

$$\frac{\mathrm{d}^3y}{\mathrm{d}t^3} - \frac{\mathrm{d}^2y}{\mathrm{d}t^2} - \frac{\mathrm{d}y}{\mathrm{d}t} + y = \cos t.$$

Problem 314 Find the general real solution to the differential equation

$$\frac{\mathrm{d}^3y}{\mathrm{d}t^3} + 3\frac{\mathrm{d}^2y}{\mathrm{d}t^2} = \sin t.$$

Problem 316 Consider the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} + y = u.$$

(i) Find the transfer function $H(s)$.

(ii) Find the stationary solution corresponding to the forcing function $u(t) = e^{-t/2}$.

(iii) Find the general real solution with the same forcing function.

Problem 317

(i) Find the general real solution to the differential equation

$$\frac{d^4y}{dt^4} + 10\frac{d^2y}{dt^2} + 169y = 0.$$

(ii) Show that the differential equation has precisely one solution $y = f(t)$ for which

$$f(0) = 0,\ f'(0) = 1,\ \text{and}\ f(t) \to 0 \text{ for } t \to \infty.$$

Problem 318 Consider the differential equation

$$\frac{\mathrm{d}^3y}{\mathrm{d}t^3} + 2\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + \frac{\mathrm{d}y}{\mathrm{d}t} = u' + u,$$

where u is the forcing function and y is the unknown.

(i) Determine by hand an expression for the amplitude characteristic and the phase characteristic.

(ii) Use Maple to plot the frequency characteristics and the Nyquist plot.

Problem 319 For the differential equation

$$\frac{\mathrm{d}^3y}{\mathrm{d}t^3} + \frac{\mathrm{d}^2y}{\mathrm{d}t^2} - 2y = u' - u,$$

determine:

(i) The transfer function $H(s)$.

(ii) The stationary solution for the forcing function $u(t) = 1$.

(iii) The stationary solution for the forcing function $u(t) = e^{2t}$.

(iv) The stationary solution for the forcing function $u(t) = \cos t$.

Problem 320 Consider the differential equation

$$\frac{d^4y}{dt^4} + 6\frac{d^2y}{dt^2} + 25y = u' + 3u.$$

(i) Find the transfer function $H(s)$.

(ii) Calculate the stationary solution associated with the forcing function $u(t) = e^{-t}$.

(iii) Calculate the stationary solution associated with the forcing function $u(t) = e^t \cos t$ (vink: $e^t \cos t = \mathrm{Re}\,(e^{(1+i)t})$).

Problem 325 Consider the differential equation

$$\frac{\mathrm{d}^6 x}{\mathrm{d}t^6} - x = e^t \cos t\ . \tag{F.6}$$

(i) Find the general complex solution to the corresponding inhomogeneous equation.

(ii) Find the general real solution to the corresponding inhomogeneous equation.

(iii) Find the general real solution to the inhomogeneous equation (F.6) (hint: $e^t \cos t = \mathrm{Re}\,(e^{(1+i)t})$.

Problem 326 Consider the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} - \frac{1}{t}y = \sin t, \qquad t > 0.$$

Set

$$y(t) = \sum_{n=0}^{\infty} c_n t^n,$$

and determine the coefficients c_n that make $y(t)$ a solution to the differential equation. For which t-values is this solution valid?

F.4 Systems of first-order differential equations

Problem 401 Consider the system of differential equations

$$\begin{cases} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \end{pmatrix} u(t), \\ y = \begin{pmatrix} 3 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \end{cases}$$

where u is the forcing function and y is the solution.

(i) Find the transfer function $H(s)$.

(ii) Find the solution corresponding to the forcing function $u(t) = e^{5t}$.

Problem 402 Consider the system of differential equations

$$\begin{cases} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t), \\ y(t) = x_2(t), \end{cases}$$

where u is the forcing function and y is the solution.

(i) Find the transfer function $H(s)$.

(ii) Find the solution corresponding to the forcing function $u(t) = e^{-t}$.

Problem 403 Consider the differential equation

$$\frac{dy}{dt} + 2y = u(t), \tag{F.7}$$

where u is the forcing function and y the solution. We regard the equation as a system consisting of one equation.

(i) Find the transfer function $H(s)$.

(ii) Show that the system (F.7) is asymptotically stable.

(iii) Determine the stationary solution corresponding to each forcing function

$$u_n(t) = e^{int}, \quad n \in \mathbb{Z}.$$

According to Theorem 7.8, if

$$u(t) = \sum_{n=-\infty}^{\infty} \frac{1}{n^3 + \frac{1}{2}} e^{int}, \tag{F.8}$$

then (F.7) has the solution

$$y(t) = \sum_{n=-\infty}^{\infty} \frac{1}{n^3 + \frac{1}{2}} H(in) e^{int}. \tag{F.9}$$

The solution (F.9) can also be reached by direct calculation:

(iv) Show, by inserting into the differential equation (F.7) that the function y given by (F.9) is a solution when u is given by (F.8). Remember to state which results you are using.

Problem 404 Determine the transfer function $H(s)$ for the system:

$$\begin{cases} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -\frac{3}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u(t), \\ y(t) = x_1(t) + x_2(t). \end{cases}$$

Problem 408 Use the eigenvalue method to find the general solution to the equation

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Problem 409

(i) Find a fundamental matrix for the homogeneous system

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 5 & 3 \\ 9 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

(ii) Find den general real solution to the system

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 5 & 3 \\ 9 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + e^{4t} \begin{pmatrix} 3 \\ -5 \end{pmatrix}$$

Problem 410 Consider the system of differential equations

$$\begin{cases} x_1'(t) = -x_2(t) \\ x_2'(t) = -x_1(t) \end{cases}$$

(a) Find the general real solution.

(b) Find the particular solution for which $x_1(0) = -1, x_2(0) = 1$.

Problem 411 Consider the system

$$\begin{cases} \dot{x}_1 = & x_3 \\ \dot{x}_2 = & 2x_1 - x_3 \\ \dot{x}_3 = & -x_1 \end{cases} \tag{F.10}$$

(i) Find the general complex solution.

(ii) Find the general real solution.

(iii) Is the system (F.10) stable?

(iv) Is the system (F.10) asymptotically stable?

Problem 413 Let

$$A = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}.$$

Use the eigenvalue method to find the general real solution to the system of differential equations $\dot{x} = Ax$.

Problem 414 Consider the system

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 2 & 9 \\ -1 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

(1) Find the eigenvalues and the corresponding eigenvectors. Does Theorem 2.5 leads to the general solution?

(2) Find a solution of the form

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \mathbf{u}e^{-t} + \mathbf{v}te^{-t}$$

with $\mathbf{v} \neq \mathbf{0}$.

(3) Find the general real solution to the system.

Problem 417

(i) Find the transfer function for the system

$$\begin{cases} \dfrac{d\mathbf{x}}{dt} = \begin{pmatrix} 2 & -1 \\ 3 & 1 \end{pmatrix} \mathbf{x}(t) + \begin{pmatrix} 0 \\ -1 \end{pmatrix} u(t) \\ y(t) = 2x_1(t) - x_2(t). \end{cases}$$

(ii) Find the stationary solution corresponding to $u(t) = \cos t$.

Problem 418

(i) Find the transfer function for the system

$$\begin{cases} \dfrac{d\mathbf{x}}{dt} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \mathbf{x}(t) + \begin{pmatrix} 0 \\ -1 \end{pmatrix} u(t) \\ y(t) = 2x_1(t) - x_2(t). \end{cases}$$

(ii) Find the stationary solution associated with $u(t) = \sin 2t$.

Problem 421 For each of the following systems, find the general real solution:

(i) $$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}.$$

(ii) $$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -3 & 4 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 2t \\ t \end{pmatrix}.$$

Problem 422 Check whether the following systems are stable, respectively, asymptotically stable:

(i) $\dot{\mathbf{x}}(t) = \begin{pmatrix} 1 & 7 \\ 3 & -2 \end{pmatrix} \mathbf{x}(t).$

(ii) $\dot{\mathbf{x}}(t) = \begin{pmatrix} -1 & 1 & 0 \\ -5 & -1 & 1 \\ -7 & 0 & 1 \end{pmatrix} \mathbf{x}(t).$

(iii) $\dot{\mathbf{x}}(t) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & -1 & -1 & -1 \\ -1 & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \mathbf{x}(t).$

Problem 423 Consider the system

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1 \\ -4 \end{pmatrix}.$$

(i) Find the general real solution to the system.

(ii) Find the solution $\mathbf{x}$ for which $\mathbf{x}(0) = \mathbf{0}$.

Problem 424 Consider the system

$$\begin{cases} \dot{x} = \frac{3}{2}x - y - \frac{1}{2}z \\ \dot{y} = -\frac{1}{2}x + 2y + \frac{1}{2}z \\ \dot{z} = \frac{1}{2}x + y + \frac{5}{2}z. \end{cases}$$

(i) Use Maple to find the eigenvalues and the corresponding eigenvectors for the system matrix.

(ii) Determine the general real solution of the system of differential equations.

Problem 425 For each of the following systems, decide whether it is asymptotically stable:

(i) $\dot{x}(t) = \begin{pmatrix} 1 & 7 \\ 3 & -2 \end{pmatrix} x(t) + u(t).$

(ii) $\dot{x}(t) = \begin{pmatrix} -1 & 1 & 0 \\ -5 & -1 & 1 \\ -7 & 0 & 1 \end{pmatrix} x(t) + u(t).$

(iii) $\dot{x}(t) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & -1 & -1 & -1 \\ -1 & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} x(t) + u(t).$

Problem 427 Find all values of the parameter a for which the system:

$$\dot{x}(t) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & -1 & a & 1+a^2 \\ 0 & -a & 0 & a \\ -a & 1 & 0 & -1 \end{pmatrix} x(t) + u(t).$$

is asymptotically stable.

Problem 428 Determine for each of the following systems whether it is asymptotically stable:

(i) $\dot{x}(t) = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} x(t) + u(t)$,

(ii) $\dot{x}(t) = \begin{pmatrix} 1 & -1 \\ -2 & 2 \end{pmatrix} x(t)$,

(iii) $\dot{x}(t) = \begin{pmatrix} -2 & -1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} x(t) + \begin{pmatrix} \cos t \\ \cos 2t \\ \sin t \end{pmatrix}$,

(iv) $\dot{x}(t) = \begin{pmatrix} 1 & -1 & 0 \\ 0 & -1 & -1 \\ 1 & -1 & 0 \end{pmatrix} x(t) + u(t)$,

Problem 429 Check whether the following systems are stable, respectively, asymptotically stable:

(i) $\dot{\mathbf{x}}(t) = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} \mathbf{x}(t)$

(ii) $\dot{\mathbf{x}}(t) = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \mathbf{x}(t)$

(iii) $\dot{\mathbf{x}}(t) = \begin{pmatrix} 4 & 7 \\ -2 & 1 \end{pmatrix} \mathbf{x}(t)$.

Problem 430 We are given the following linear system

$$\begin{cases} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -4 & 2 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 2 \\ -1 \end{pmatrix} u, \\ y = x_1, \end{cases}$$

with forcing function $u(t)$ and solution $y(t)$.

(a) Find the transfer function $H(s)$ for the system.

(b) Find the amplitude characteristic $A(\omega)$.

(c) Plot $A(\omega)$.

(d) Find and plot the phase characteristic $\phi(\omega)$.

(e) Find and plot the stationary solution corresponding to the forcing function $u(t) = \sin t$.

Problem 431 Consider the inhomogeneous system

$$\dot{\mathbf{x}} = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

(i) Find a fundamental matrix for the corresponding homogeneous system.

(ii) Show that the metod of "guess and test" does not lead to a particular solution to the inhomogeneous system.

(iii) Find the general real solution to the inhomogeneous system using the general solution formula in Theorem 2.18.

(iv) Find the particular solution to the inhomogeneous system for which $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Problem 432 Show that the systems

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -3 & 1 \\ 4 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

and

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -3 & 1 \\ 4 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$$

are asymptotically stable.

Problem 436 Find all stationary points for the system

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x^2 + y^2 - x + 2y + 1 \\ x - y - 1 \end{pmatrix}.$$

Problem 441 Consider the system

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x^2 + 2y - 8 \\ 4x - 2y + 3 \end{pmatrix}. \tag{F.11}$$

(i) Is (F.11) a linear or nonlinear system?

(ii) Find the functional matrix.

(iii) Find the stationary points.

(iv) Decide for each stationary point whether it is asymptotically stable.

(v) Make a phase portrait, with special attention to the solution curves corresponding to initial conditions close to the stationary points.

Problem 450 We consider a constrained system of differential equations given by

$$\begin{aligned} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} &= \begin{pmatrix} 1 & -1 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u(t)\,, \\ y &= x_1 + 2x_2\,, \end{aligned} \tag{F.12}$$

where $u(t)$ is the forcing function and $y(t)$ the unknown.

(i) Show that the system (F.12) is asymptotically stable.

(ii) Determine the transfer function for the system (F.12).

(iii) Let u be the function in Problem 518 (where it is called f). Find a solution y to (F.12), expressed as a Fourier series.

(iv) Let $\widetilde{S}_N$ denote the N'th partial sum of the Fourier series you found for the solution y in (iii). One can show that $\left|\frac{5+3in}{1-n^2+in}\right| \leq 1$, for $N \geq 4$. Show that, for $N \geq 4$, one has

$$|y(t) - \widetilde{S}_N(t)| \leq \sum_{n=N+1}^{\infty} \frac{1}{n^3} \text{ for all } t \in \mathbb{R}.$$

(v) Using the result in (iv), determine an N such that

$$|y(t) - \widetilde{S}_N(t)| \leq 0.01 \text{ for all } t \in \mathbb{R}.$$

F.5 Problems from the exam

Problem 503 Show that the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{4^n}$$

is convergent and determine its sum. Next, investigate whether the series is conditionally or absolutely convergent.

Problem 504 We consider the function f which is periodic with period 2π and in the interval $]-\pi, \pi]$ given by

$$f(t) = \frac{1}{12}(\pi^2 t - t^3), -\pi < t \leq \pi.$$

(i) Sketch the graph for f on the interval $[-3\pi, 3\pi]$.

(ii) Show that f has the Fourier series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3} \sin nt.$$

Hint: you can use that for all $n \in \mathbb{N}$,

$$\begin{aligned} \int t \sin nt \, dt &= \frac{\sin nt}{n^2} - t\frac{\cos nt}{n} \\ \int t^3 \sin nt \, dt &= -6\frac{\sin nt}{n^4} + 6t\frac{\cos nt}{n^3} + 3t^2\frac{\sin nt}{n^2} - t^3\frac{\cos nt}{n}. \end{aligned}$$

(iii) Is the Fourier series convergent with sum $f(t)$ for all $t \in \mathbb{R}$?

(iv) Show that

$$\sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945}.$$

Problem 505 Consider the differential equation

$$\frac{\mathrm{d}^3 y}{\mathrm{d}t^3} + 3\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + 2\frac{\mathrm{d}y}{\mathrm{d}t} = u'' + 5u' + 6u,$$

where u is the forcing function and y the solution.

(i) Find the transfer function $H(s)$.

(ii) Find the solution for the forcing function $u(t) = \sin 3t$.

Problem 506 Determine the general real solution to the systems:

(i) $\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -8 & -5 \\ 20 & -8 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$

(ii) $\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -8 & -5 \\ 20 & -8 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 19 \\ -2 \end{pmatrix} e^t.$

Problem 507 Let $a \in \mathbb{R}$ and consider the system of differential equations

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}, \text{ where } \mathbf{A} = \begin{pmatrix} a & 2 \\ a & 1 \end{pmatrix}.$$

(i) Find the values of a for which the system is asymptotically stable.

(ii) Check whether the system is stable for $a = -1$.

(iii) Check whether the system is stable for $a \geq 0$.

Problem 508

(i) Check whether the infinite series

$$\sum_{n=1}^{\infty} (-1)^{n-1} \tan\left(\frac{1}{\sqrt{n}}\right)$$

is absolutely convergent, conditionally convergent, or divergent. (Hint: you might use that $\tan x > x$ for $x \in]0, \pi/2[$).

(ii) Show that the infinite series

$$\sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{x^n}{\sqrt{n}}\right)$$

is uniformly convergent on the interval $[-1, 1]$. (Hint: you might use that $|\sin x| \leq |x|$ for all $x \in \mathbb{R}$).

Problem 509 Consider the power series

$$\sum_{n=1}^{\infty} \frac{x^{2n}}{3^n (n^2 + 1)}.$$

(i) Determine the radius of convergence ρ.

(ii) Show that the power series is convergent for $x = \pm\rho$.

(iii) For a given $x \in [-\rho, \rho]$, let $s(x)$ denote the sum of the power series. Show that

$$\frac{\pi}{4} \leq s(\sqrt{3}) \leq \frac{\pi}{4} + \frac{1}{2}.$$

Problem 510 Consider the functionen f which is odd, periodic with period 2π, and in the interval $[0, \pi]$ given by

$$f(t) = \frac{\pi}{8}(\pi t - t^2), 0 \leq t \leq \pi.$$

(i) Sketch the graph for f on the interval $[-2\pi, 2\pi]$.

(ii) Show that f has the Fourier series

$$\sum_{n=1}^{\infty} \frac{\sin((2n-1)t)}{(2n-1)^3}.$$

Hint: you might use that for all $n \in \mathbb{N}$,

$$\int t \sin nt \, dt = \frac{\sin nt}{n^2} - t\frac{\cos nt}{n}$$

and

$$\int t^2 \sin nt \, dt = 2\frac{\cos nt}{n^3} + 2t\frac{\sin nt}{n^2} - t^2\frac{\cos nt}{n}.$$

(iii) Is the Fourier series convergent with sum $f(t)$ for all $t \in \mathbb{R}$?

(iv) Show that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^3} = \frac{\pi^3}{32}.$$

Problem 511 Consider the differential equation

$$\frac{d^3y}{dt^3} + 2\frac{d^2y}{dt^2} + \frac{dy}{dt} = u' + u; \tag{F.13}$$

Here u is the forcing function and y the solution sought.

(i) Determine the general real solution of the homogeneous system corresponding to (F.13).

(ii) Determine the transfer function corresponding to (F.13).

(iii) Determine the stationary solution corresponding to the forcing function

$$u(t) = e^{2t}.$$

(iv) Write down the general real solution to (F.13) when $u(t) = e^{2t}$.

Problem 512 Consider the system of differential equations

$$\begin{cases} \dot{x}_1 = & x_1 - 2x_2 \\ \dot{x}_2 = & 2x_1 + x_2 \end{cases} \tag{F.14}$$

(i) Determine the general complex solution.

(ii) Determine the general real solution.

(iii) Is the system (F.14) stable?

Problem 513

(i) Is the series $\sum_{n=1}^{\infty} \sin(n)$ convergent?

(ii) Consider the series $\sum_{n=1}^{\infty} a_n$, where it is known that the N'th partial sum is

$$S_N = \sum_{n=1}^{N} a_n = 1 + e^N.$$

Is the series $\sum_{n=1}^{\infty} a_n$ convergent?

(iii) Find the radius of convergence for the series $\sum_{n=1}^{\infty} (-2)^n\, n^2 x^n$.

Problem 514 Consider the system of differential equations

$$\begin{cases} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -2 & 3 \\ 0 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \end{pmatrix} u(t), \\ \qquad y = x_1 + x_2, \end{cases} \tag{F.15}$$

where $u(t)$ is the forcing function and $y(t)$ is the solution.

(i) Show that the system (F.15) is asymptotically stable.

(ii) Show that the transfer function for the system is given by

$$H(s) = \frac{2}{2+s}, \quad s \notin \{-2, -4\}.$$

Now consider the 2π-periodic forcing function

$$u(t) = t(\pi - t)(\pi + t), \; t \in [-\pi, \pi[. \tag{F.16}$$

You are given that the Fourier series for u is

$$u(t) \sim -12 \sum_{n=1}^{\infty} (-1)^n \frac{\sin(nt)}{n^3}. \tag{F.17}$$

(iii) Plot the function u on the interval $[-2\pi, 2\pi]$.

(iv) Explain why the symbol "$\sim$" in (F.17) can be replaced by "$=$".

(v) Find, expressed as a Fourier series, a solution for (F.15) when the forcing function u is given by (F.16).

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Written 2 hours exam, 17.5.2006

Course : Mathematics 2 Course no.: 01035

Allowed aids: all

Weight: Problem 515: 25%, Problem 516: 25%, Problem 517: 20%, Problem 518: 30%.

In order to get full credit it is necessary to give full arguments for the solutions.

Problem 515 Consider the differential equation

$$\frac{d^4y}{dt^4} - 16y = u' + u; \tag{F.18}$$

here u is the forcing function, and y is the solution.

(i) Find the general real solution to the homogeneous differential equation corresponding to (F.18).

(ii) Find the transfer function for (F.18).

(iii) Find the stationary solution corresponding to the forcing function

$$u(t) = e^t.$$

(iv) Find a solution to the differential equation when

$$u(t) = e^{3t} + 3e^t.$$

Problem 516 Consider the system of differential equations

$$\begin{cases} \dot{x}_1 = & -x_1^2 + 2x_2 \\ \dot{x}_2 = & -2x_1 - x_2 \end{cases} \tag{F.19}$$

(i) Is the system (F.19) linear or nonlinear?

(ii) Find the functional matrix for the system (F.19).

(iii) Determine all stationary points for the system (F.19).

(iv) For each stationary point, check whether it is asymptotically stable.

Problem 517

(i) Prove that the infinite series

$$\sum_{n=1}^{\infty}(-1)^n \frac{1}{n+\sqrt{n}} \tag{F.20}$$

is convergent.

(ii) Consider the Nth partial sum

$$S_N = \sum_{n=1}^{N}(-1)^n \frac{1}{n+\sqrt{n}}.$$

Find N such that S_N yields an approximate value for the sum of the infinite series in (F.20), within a tolerance of 0.1.

(iii) Is the infinite series (F.20) conditionally convergent or absolutely convergent?

Problem 518 A certain 2π-periodic, piecewise differentiable and continuous function $f : \mathbb{R} \to \mathbb{R}$ has the Fourier coefficients

$$a_n = 0 \text{ for all } n, \text{ and } b_n = \frac{1}{n^3}, \; n \in \mathbb{N}.$$

(i) Find the Fourier series on complex form.

In the rest of the problem you can either work with the Fourier series on real form or on complex form.

(ii) Find an Nth partial sum S_N such that

$$|f(x) - S_N(x)| \leq 0.1, \; \forall x.$$

(iii) Let $P(f)$ denote the power of the function f. Show that

$$P(f) > \frac{1}{2}.$$

(iv) Find an Nth partial sum S_N such that S_N contains at least 99.9% of the power of f.

Hint: it is enough to choose N such that

$$\sum_{n=N+1}^{\infty} \frac{1}{n^6} \leq 0.001.$$

The Problems are finished!

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Written 2 hour exam, 16.12.2006

Course: Mathematics 2 Course no.: 01035

Allowed aids: all

Weight: Problem 519: 20%, Problem 520: 25%, Problem 521: 30%, Problem 522: 25%.

In order to get full credit it is necessary to give full arguments.

Problem 519 Consider the differential equation

$$y''' - y' = u' + u; \tag{F.21}$$

here u is the forcing function, and y is the solution.

(i) Find the general real solution to the homogeneous equation corresponding to (F.21).

(ii) Find the transfer function for (F.21).

(iii) Find the stationary solution corresponding to the forcing function

$$u(t) = e^{2t}.$$

(iv) Find a solution to (F.21) whenever

$$u(t) = e^{-t}.$$

(*Hint: calculate the left-hand side in* (F.21) *for* $u(t) = e^{-t}$.)

Problem 520 Consider the system of differential equations given by

$$\begin{cases} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t), \\ \\ y = x_2, \end{cases} \tag{F.22}$$

where $u(t)$ is the forcing function and $y(t)$ is the solution.

The Problem continues!

1. Show that the transfer function for (F.22) is given by
$$H(s) = \frac{1}{1+s}, \ \ s \neq -1.$$
2. Is the system (F.22) asymptotically stable?
3. Find the amplitude characteristic for the system (F.22).
4. Find the phase characteristic for the system (F.22).

Problem 521 Consider the infinite series
$$\sum_{n=1}^{\infty} \frac{n}{n^5+1}. \tag{F.23}$$

1. Show that for every $n \in \mathbb{N}$ it holds that
$$\frac{n}{n^5+1} \leq \frac{1}{n^4}. \tag{F.24}$$
2. Show that the infinite series in (F.23) is convergent.
3. It is known that (you are not supposed to prove this) that for any $N \in \mathbb{N}$,
$$\sum_{n=N+1}^{\infty} \frac{1}{n^4} \leq \frac{4}{3}\frac{1}{(N+1)^3}. \tag{F.25}$$
Using (F.24) and (F.25), find $N \in \mathbb{N}$ such that
$$\sum_{n=N+1}^{\infty} \frac{n}{n^5+1} \leq 0.05.$$
4. Find the sum of the infinite series in (F.23) within a tolerance of 0.05, using the result in (iii).

Problem 522 Consider the 2π-periodic function u given by
$$u(t) = \frac{1}{3}t^3, \ \ t \in [-\pi, \pi[.$$

1. Sketch the function u on the interval $[-2\pi, 2\pi]$.
2. Does the Fourier series converges to $u(t)$ for all $t \in]-\pi, \pi[$?
3. What is the sum of the Fourier series for $t = \pi$?
4. Denote the Fourier coefficients on complex form by $c_n, n \in \mathbb{Z}$. Show that $c_{-n} = -c_n$ for all $n \in \mathbb{Z}$.

The Problems are finished!

Problem 523 Consider the power series

$$\sum_{n=1}^{\infty} \frac{2n}{2^n}(-1)^n x^n.$$

(i) Determine the radius of convergence ρ. Answer:

a) $\rho = 0$; b) $\rho = \infty$; c) $\rho = 2$; d) $\rho = \frac{1}{2}$; e) $\rho = 1$.

(ii) For $x = 1$, the power series is:

a) absolutely convergent; b) conditionally convergent; c) divergent;

(iii) For $x = -2$ the series is:

a) absolutely convergent; b) conditionally convergent; c) divergent;

Problem 526 A 2π-periodic, piecewise differentiable and continuous function $f : \mathbb{R} \to \mathbb{R}$ is given by the Fourier series

$$f(x) = 1 + \frac{1}{2}\sin x + \sum_{n=1}^{\infty} \frac{1}{2^n} \cos nx.$$

1. Is the function f even, odd or neither? Justify your answer.
2. Determine the Fourier coefficients in complex form.
3. Determine a value $N \in \mathbb{N}$ such that the Nth partial sum S_N satisfies
$$|f(x) - S_N(x)| \leq 0.01, \ \forall x.$$
4. Calculate $f(\frac{\pi}{2})$.

Problem 529 Let $a \in \mathbb{R}$, and consider the system of differential equations

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}, \text{ where } \mathbf{A} = \begin{pmatrix} -4 & 5 \\ -a & a \end{pmatrix}.$$

(i) Determine the values of $a \in \mathbb{R}$ for which the system is asymptotically stable.

(ii) Check whether the system is stable for $a = 0$.

It is given (you do not need to show this) that the system

$$\begin{cases} \dot{\mathbf{x}} = \begin{pmatrix} -4 & 5 \\ -2 & 2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u, \\ y = \begin{pmatrix} 0 \\ 1 \end{pmatrix}^{\top} \mathbf{x}. \end{cases} \tag{F.26}$$

has transfer function

$$H(s) = \frac{4+s}{s^2+2s+2}, \quad s \neq -1 \pm i.$$

It is also given (you do not need to show) that the 2π-periodic function given by $u(t) = t^2, \ t \in [-\pi, \pi[$ has the Fourier series

$$u(t) \sim \frac{\pi^2}{3} + 2\sum_{n\neq 0} \frac{(-1)^n}{n^2} e^{int}.$$

(iii) Find the Fourier series for a solution to the system (F.26) with this forcing function.

Problem 530 This problem concerns the computation of the integral

$$\int_0^1 \sin(x^2)\, dx.$$

(i) Using the power series for $\sin(x)$, show that

$$\sin(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{4n+2}, \ x \in \mathbb{R}.$$

(ii) Show that

$$\int_0^1 \sin(x^2)\, dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!(4n+3)}.$$

(iii) Find an approximate value for the integral

$$\int_0^1 \sin(x^2)\, dx,$$

with an error at most 10^{-4}.

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Written 2 hours exam, 8.12.2008

Course: Mathematics 2

Course no.: 01035

Allowed aids: All
Weight: Problem 537: 16%, Problem 538: 12%,
Problem 539: 24%, Problem 540: 24%, Problem 541: 24%

Full credit requires that all arguments are included in Problem 539, 540 and 541.

Problem 537 and 538 are multiple-choice Problems and the answer should be given on the attached sheet. The answer "Don't know" gives 0 point, and a wrong answer -1 point.

Problem 537 Consider the differential equation

$$y''' + 2y'' + y' = u'' + 2u' + u \ , \tag{F.27}$$

where u is the forcing function and y is the solution.

(i) Find the transfer function. The answer is
a) $H(s) = \dfrac{1}{s}$, $s \notin \{0, -1\}$; b) $H(s) = \dfrac{1}{s}$, $s \neq 0$
c) $H(s) = \dfrac{1}{s+1}$, $s \neq -1$; d) Don't know.

(ii) Find the stationary solution associated to the forcing function $u(t) = \cos t$. The answer is
a) $y(t) = \cos t$; b) $y(t) = \sin t$; c) $y(t) = \sin t - \cos t$; d) Don't know.

(iii) Find the general real solution to the homogeneous differential equation corresponding to (F.27). The answer is
a) $y(t) = c_1 e^{-t} + c_2 te^{-t} + c_3 t^2 e^{-t}$, $c_1, c_2, c_3 \in \mathbb{R}$
b) $y(t) = c_1 e^{-t} + c_2 e^{-t} + c_3$, $c_1, c_2, c_3 \in \mathbb{R}$
c) $y(t) = c_1 e^{-t} + c_2 te^{-t} + c_3$, $c_1, c_2, c_3 \in \mathbb{R}$
d) Don't know.

(iv) We check whether (F.27) has a solution of a special form when $u(t) = e^{-t}$. Which one of the following statements is true?
a) The differential equation has precisely one solution of the form $y(t) = ce^{-t}$, where c is a real constant.
b) The differential has more than one solution of the form $y(t) = ce^{-t}$, where c is a real constant
c) The differential equation has no solutions.
d) Don't know.

Problem 538 Consider the infinite series

$$\sum_{n=1}^{\infty} \ln(1 + \frac{1}{n}) \ . \tag{F.28}$$

(i) Which information does the nth term test gives concerning the convergence of the infinite series? The answer is
a) The infinite series is convergent;
b) The infinite series is divergent;
c) The nth term test does not lead to any conclusion;
d) Don't know.

(ii) We will now use that (you are NOT expected to prove this)) that the Nth partial sum for (F.28) is given by

$$S_N = \ln(N + 1) \ , \quad N \in \mathbb{N} \ . \tag{F.29}$$

Which information can we extract from (F.29) concerning (F.28)? The answer is
a) The infinite series is convergent;
b) The infinite series is divergent;
c) We can not conclude anything;
d) Don't know.

Now consider the infinite series

$$\sum_{n=1}^{\infty} (-1)^{n-1} \ln(1 + \frac{1}{n}) \ . \tag{F.30}$$

(iii) Which information does the Leibniz' test give concerning convergence of the infinite series (F.30)? The answer is
a) The infinite series is convergent;
b) The infinite series is divergent;
c) We can not conclude anything;
d) Don't know.

Problem 539 It is know (you are NOT expected to prove this) that the power series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(n+1)(n+2)} x^{n+2} \tag{F.31}$$

has radius of convergence $\rho = 1$. Let $f(x)$ denote the sum of the power series (F.31).

(i) Use the power series for $\ln(1+x)$ (see Appendix B) to prove that

$$f'(x) = x - \ln(1+x), \quad x \in \left]-1, 1\right[\ .$$

(ii) Use (i) to find the sum $f(x)$.

(iii) Check whether (F.31) is uniformly convergent on the interval $[-1, 1]$.

Problem 540 Let f denote the 2π-periodic function, which in the interval $]0, 2\pi]$ is given by

$$f(x) = \frac{1}{2}x \ , \quad 0 < x \leq 2\pi.$$

(i) Show that the Fourier series for f is given by

$$\frac{\pi}{2} - \sum_{n=1}^{\infty} \frac{\sin(nx)}{n} \ . \tag{F.32}$$

Hint: You might use that for all $n \in \mathbb{N}$,

$$\int_0^{2\pi} x\cos(nx)dx = 0 \quad \text{and} \quad \int_0^{2\pi} x\sin(nx)dx = -\frac{2\pi}{n} \ .$$

(ii) Find for $x = 2\pi$ and for $x = 8$ the sum of the Fourier series (F.32).

(iii) Find the sum of the infinite series

$$\sin(4) + \frac{\sin(8)}{2} + \frac{\sin(12)}{3} + \frac{\sin(16)}{4} + \cdots$$

by inserting an appropriate x-value in the Fourier series (F.32).

Problem 541 Consider the linear system

$$\dot{\mathbf{x}}(t) = \begin{pmatrix} -2 & a & 1 \\ -1 & -3 & 2 \\ 0 & 0 & a \end{pmatrix} \mathbf{x}(t) \ , \tag{F.33}$$

where a is a real number. It is given that the characteristic polynomial can be written in the form

$$P(\lambda) = -(\lambda - a)(\lambda^2 + 5\lambda + 6 + a).$$

(i) Determine the values of a for which the system (F.33) is asymptotically stable.

(ii) Check whether the system (F.33) is stable for $a = -6$.

(iii) Find for $a = -6$ the general real solution to the system (F.33).
Hint: you might use that

$$\begin{pmatrix} -5 \\ -3 \\ 2 \end{pmatrix}, \quad \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}$$

are linearly independent eigenvectors for the system matrix when $a = -6$.

The Problems are finished!

Problem 564 (From the Exam 16 May 2011)

(1) The series $\sum_{n=1}^{\infty}(-1)^n\sqrt{n}$ is
a) conditionally convergent.
b) absolutely convergent.
c) divergent.

(2) The function $f(x){=}e^{\sin(x)}$, $x \in \mathbb{R}$ is
a) an odd function.
b) an even function.
c) neither odd nor even.

(3) The radius of convergence for the power series $\sum_{n=1}^{\infty} \frac{n+1}{n^3+4}x^n$ is:
a) $\rho = 5$.
b) $\rho = 1$.
c) $\rho = \infty$.

(4) For the inhomogeneous linear differential equation

$$y'' + y' + 4y = u(t) ,$$

where $y = y(t)$, $t \geq 0$, the corresponding transfer function is

$$H(s) = \frac{1}{s^2+s+4}, \qquad s \in \mathbb{C}\backslash\{-\frac{1}{2} + \frac{i}{2}\sqrt{15}, -\frac{1}{2} - \frac{i}{2}\sqrt{15}\}.$$

What is the stationary solution for the forcing function $u(t) = 2\sin(t)$?
a) $y(t) = -\frac{1}{5}\cos(t) + \frac{3}{5}\sin(t)$.
b) $y(t) = \frac{3}{5}\cos(t) - \frac{1}{5}\sin(t)$.
c) $y(t) = \frac{3}{5}\cos(t) + \frac{1}{5}\sin(t)$.

(5) For the homogeneous linear system of differential equations

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) , \qquad \text{where} \qquad \mathbf{A} = \begin{bmatrix} -1 & 0 & 2 \\ 4 & -1 & 0 \\ 0 & 0 & -3 \end{bmatrix} ,$$

it is given that the characteristic polynomial is $P(\lambda) = (\lambda{+}1)^2(\lambda{+}3)$. $\mathbf{x}(t)$ is a 3-dimensional vector-valued function. It is also given that the three vector functions below are a linearly independent set of solutions for the system.

$$\mathbf{x}_1(t) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^{-t} \ , \quad \mathbf{x}_2(t) = \begin{bmatrix} 1/4 \\ 1 \\ 0 \end{bmatrix} e^{-t} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} te^{-t} \ , \quad \mathbf{x}_3(t) = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} e^{-3t} \ ,$$

For the eigenvalue $\lambda = -1$:
a) The algebraic multiplicity is 2 and the geometric multiplicty is 1.
b) The algebraic multiplicity is 2 and the geometric multiplicty is 2.
c) The algebraic multiplicity is 3 and the geometric multiplicty is 2.

(6) We consider the differential equation

$$t\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + \frac{\mathrm{d}y}{\mathrm{d}t} + ty = 0$$

and seek a power series solution of the form

$$y(t) = \sum_{n=0}^{\infty} a_n t^n \ .$$

Which recursion formula will the unkown coefficients a_n satisfy?
a) $a_n = -\frac{1}{n^2} a_{n-2}$ for $n = 2, 3, 4, \ldots$, with $a_1 = 0$ and $a_n \neq 0$ for $n = 3, 5, 7, \ldots$.

b) $a_{n+2} = -\frac{1}{(n+2)^2} a_n$ for $n = 0, 1, 2, 3, \ldots$. The coefficients a_0 and a_1 can both be non-zero.

c) $a_{n+2} = -\frac{1}{(n+2)^2} a_n$ for $n = 0, 2, 4, \ldots$, and $a_n = 0$ for $n = 1, 3, 5, \ldots$.

F.6 Extra problems

Problem 601 Consider the system:

$$\begin{cases} \dot{x} = 3x + 4z \\ \dot{y} = -x - y \\ \dot{z} = -2x - 3z \end{cases}$$

(i) Use Maple to find the eigenvalues and the corresponding eigenvectors for the system matrix.

(ii) Determine the general real solution of the system.

Problem 602 Consider the inhomogeneous system

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -1 & 3 \\ -3 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

(i) Find the general complex solution to the associated homogeneous system.

(ii) Find the general real solution to the associated homogeneous system.

(iii) Find the general real solution to the given inhomogeneous system

Problem 603 Use Maple to determine whether the following systems are stable, assymptotically stable or neither.

(i) $\dot{x}(t) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix} x(t)$

(ii) $\dot{x}(t) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix} x(t)$

(iii) $\dot{x}(t) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 2 & -1 \end{pmatrix} x(t)$

Problem 604

(i) Find the transfer function for the linear system

$$\begin{cases} \dot{x}(t) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} x(t) + \begin{pmatrix} 1 \\ -1 \end{pmatrix} u(t) \\ y(t) = x_1(t) + x_2(t). \end{cases}$$

(ii) Next find the stationary solution corresponding to $u(t) = \sin 2t$.

Problem 606 For each of the following power series, find the values of x for which the series converges, and the sum of the series for these values of x:

(a) $e^{-x} + e^{-2x} + e^{-3x} + \cdots + e^{-nx} + \ldots$.

(b) $\ln(x) + \ln(x^2) + \ln(x^3) + \cdots + \ln(x^n) + \ldots$.

Problem 607 Use Maple to investigate whether the series

$$\sum_{n=1}^{\infty} \frac{2^n n!}{n^n},$$

is convergent or divergent.

(a) Does the nth term criterion give any conclusion?

(b) What conclusion does the ratio test give?

Problem 608 For each number x, consider the series

$$\frac{x}{2 \cdot 3} + \frac{x^2}{3 \cdot 3^2} + \frac{x^3}{4 \cdot 3^3} + \frac{x^4}{5 \cdot 3^4} + \ldots.$$

(a) Show that the series is absolutely convergent for $|x| < 3$ and divergent for $|x| > 3$. Solve this problem both by hand and using Maple.

(b) Investigate for $x = 3$ and for $x = -3$ whether the series is convergent or divergent.

Problem 609 Consider the series

$$\sum_{n=1}^{\infty} \frac{1}{2^n - 1}.$$

(The sum is called the Erdos-borwein constant, denoted E).

(a) Show that the series is convergent.

(b) Show that the sum of the series lies between 1 and 2.

(c) Calculate an approximation for the sum of the series with an error at most 0.01.

Problem 610 Find the radius of convergence for the power series

$$\sum_{n=0}^{\infty} \sqrt{2n+1}\, x^{3n}.$$

Problem 611 Consider the power series

$$\sum_{n=1}^{\infty} \frac{x^n}{n \cdot 2^n}.$$

(a) Find the radius of convergence for the series.

(b) Let $f(x)$ denote the power series' sum function. Show that the derivative of f is

$$f'(x) = \frac{1}{2-x}, \qquad |x| < 2.$$

(c) Show that

$$f(x) = \ln 2 - \ln(2-x), \qquad |x| < 2.$$

Problem 612

(a) Show that the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n \ln(n) + 1} \tag{F.34}$$

is convergent.

(b) Consider the N'th partial sum

$$S_N = \sum_{n=1}^{N} \frac{(-1)^{n-1}}{n \ln(n) + 1}.$$

With the help of Maple or a calculator, find an N such that S_N gives an approximation for the sum of the series (F.34) with an error at most 0.0001.

(c) Calculate the sum of the series (F.34) with an error at most 0.0001.

Problem 621 Consider the system of differential equations

$$\dot{x}(t) = \begin{pmatrix} a & 0 & 0 \\ 7 & -2 & 5 \\ 3 & 2 & -6 \end{pmatrix} x(t),$$

where a is a real parameter.

(a) Determine the values of a for which the system is asymptotically stable.

(b) For which values of a is the system stable but not asymptotically stable?

Problem 623 Consider the infinite series

$$\sum_{n=1}^{\infty} \frac{1}{a^n + 1} ,$$

where a is a positive constant.

(a) Find the values of $a > 0$ for which the infinite series is convergent.

(b) Consider now $a = 3$, i.e., the convergent infinite series

$$\sum_{n=1}^{\infty} \frac{1}{3^n + 1} .$$

Find an approximate value for the sum within a tolerance of 0.0001.

Problem 626 Consider the power series

$$\sum_{n=0}^{\infty} \frac{x^n}{\sqrt{n+1}} .$$

(a) Find the radius of convergence ρ.

(b) Show that the power series is divergent for $x = \rho$.

(c) Show that the power series is convergent for $x = -\rho$.

(d) Find an approximate value for the sum for $x = -1$, within a tolerance of 0.1.

Problem 631 (Linear differential equations, real roots). Consider the differential equation:

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} - 3\frac{\mathrm{d}y}{\mathrm{d}t} + 2y = 0. \tag{F.35}$$

(i) Write down the characteristic equation.

(ii) Find the general real solution to (F.35).

Now consider the inhomogeneous equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} - 3\frac{\mathrm{d}y}{\mathrm{d}t} + 2y = u. \tag{F.36}$$

(iii) Set $u = e^{3t}$. Using the guess and test method, find a solution to (F.36) of the form $y(t) = Ce^{3t}$.

(iv) Using parts (ii) and (iii), write down the general real solution to (F.36) for the case $u(t) = e^{3t}$.

Problem 632 (Linear differential equations, complex roots). Consider the differential equation:

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + 2\frac{\mathrm{d}y}{\mathrm{d}t} + 2y = 0. \tag{F.37}$$

(i) Write down the characteristic equation.

(ii) Find the general complex solution to (F.37).

(iii) Find the general real solution to (F.37).

Now consider the inhomogeneous equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + 2\frac{\mathrm{d}y}{\mathrm{d}t} + 2y = u. \tag{F.38}$$

(iii) Set $u = e^{2t}$. Using the guess and test method, find a solution to (F.38) of the form $y(t) = Ce^{2t}$.

(iv) Using parts (ii) and (iii), write down the general real solution to (F.38) for the case $u(t) = e^{2t}$.

Problem 633 (Systems of differential equations) Consider the system of differential equations

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -4 & 2 \\ 2 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

(i) Find all the eigenvalues and eigenvectors for the matrix $A = \begin{pmatrix} -4 & 2 \\ 2 & -4 \end{pmatrix}$.

(ii) For each eigenvalue, determine its algebraic and geometric multiplicity.

(iii) Determine the general real solution to the system of differential equations.

Problem 636 (Uniform convergence) Consider the function

$$f(x) = \frac{1}{1-x}, \ x \in [0,1[. \tag{F.39}$$

For $n \in \mathbb{N}$ we also consider the functions

$$f_n(x) = \frac{1-x^{n+1}}{1-x}, \ x \in [0,1[. \tag{F.40}$$

(i) Show that for all $x \in [0, 1/2]$,

$$|f(x) - f_n(x)| \leq \left(\frac{1}{2}\right)^n .$$

(ii) Plot the graphs of the functions f, f_2, f_5, f_{10}.

The result in (i) (and the graphs in (ii)) show that the functions f_n *are close to the function* f *over the entire interval* $[0, 1/2]$ *whenever* n *is sufficiently large. We say that the functions* f_n *converge uniformly to* f *on the interval* $[0, 1/2]$ *(check the book for the exact definition!). If we plot the graphs in (ii) on the entire interval* $[0, 1[$ *we can see that we do not have uniform convergence on this interval.*

Problem 638 (Complex numbers, trigonometric functions). Recall that the complex exponential function is defined by $e^{ix} = \cos x + i \sin x$, for $x \in \mathbb{R}$.

(i) Show that, for any $x \in \mathbb{R}$, we have

$$e^{-ix} = \cos x - i \sin x.$$

(ii) Show that, for any $x \in \mathbb{R}$, we have

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}, \qquad \sin x = \frac{e^{ix} - e^{-ix}}{2i}.$$

F.7 Multiple choice problems

Problem 700 Consider the differential equation

$$\frac{d^3y}{dt^3} + \frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 2y = u' + u \ , \tag{1}$$

where u is the forcing function and y is the solution.

Question 1: The transfer function is

1 ☐ $H(s) = \frac{s+1}{s^3+s^2+s+2}$, $s \notin \{1, \pm 2i\}$
2 ☐ $H(s) = \frac{s+1}{s^3+s^2+2s+2}$, $s \notin \{-1, \pm 2i\}$
3 ☐ $H(s) = \frac{1}{s^2+2}$, $s \notin \left\{-1, \pm i\sqrt{2}\right\}$
4 ☐ $H(s) = \frac{1}{s^2+2}$, $s \notin \{-1, \pm 2i\}$
5 ☐ $H(s) = \frac{1}{s^2+2}$, $s \neq \pm 2i$
6 ☐ Don't know

Question 2: The stationary solution associated to the forcing function

$$u(t) = \sin(2t)$$

is

1 ☐ $y(t) = \sin t$
2 ☐ $y(t) = \frac{1}{4}\sin(2t)$
3 ☐ $y(t) = \frac{1}{2}\sin(2t)$
4 ☐ $y(t) = -\frac{1}{2}\sin(2t)$
5 ☐ $y(t) = -\frac{1}{2}\cos(2t)$
6 ☐ Don't know

Problem 701 Consider the differential equation

$$\frac{d^3y}{dt^3} + \frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 2y = u' + u \ , \tag{1}$$

where u is the forcing function and y is the solution.

Question 1: The transfer function for (1) is

1 ☐ $H(s) = \frac{s+1}{s^3+s^2+s+2}$, $s \notin \{1, \pm 2i\}$
2 ☐ $H(s) = \frac{s+1}{s^3+s^2+2s+2}$, $s \notin \{-1, \pm 2i\}$
3 ☐ $H(s) = \frac{1}{s^2+2}$, $s \notin \left\{-1, \pm i\sqrt{2}\right\}$
4 ☐ $H(s) = \frac{1}{s^2+2}$, $s \notin \{-1, \pm 2i\}$
5 ☐ $H(s) = \frac{1}{s^2+2}$, $s \neq \pm 2i$
6 ☐ Don't know

Question 2: The stationary solution associated with

$$u(t) = \sin(2t)$$

is

1 ☐ $y(t) = \sin t$
2 ☐ $y(t) = \frac{1}{4}\sin(2t)$
3 ☐ $y(t) = \frac{1}{2}\sin(2t)$
4 ☐ $y(t) = -\frac{1}{2}\sin(2t)$
5 ☐ $y(t) = -\frac{1}{2}\cos(2t)$
6 ☐ Don't know

Problem 702 Consider the system of differential equations given by

$$\begin{cases} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} u(t), \\ y = x_2 \end{cases} \tag{2}$$

where u is the forcing function and y is the solution.

Question 1: The transfer function is
1 ☐ $H(s) = \frac{1}{s+1}\ ,\ s \neq -1$
2 ☐ $H(s) = \frac{-1}{s+1}\ ,\ s \neq -1$
3 ☐ $H(s) = \frac{1}{s-1}\ ,\ s \neq 1$
4 ☐ $H(s) = \frac{-1}{s-1}\ ,\ s \neq 1$
5 ☐ $H(s) = \frac{-s-1}{s^2+1}\ ,\ s \neq \pm i$
6 ☐ Don't know

Question 2: The amplitude characteristic for the system (2) is
1 ☐ $A(\omega) = \frac{1}{1+\omega^2}\ ,\ \ \omega > 0$
2 ☐ $A(\omega) = \frac{-1}{1+\omega^2}\ ,\ \ \omega > 0$
3 ☐ $A(\omega) = \frac{1}{\sqrt{1+\omega^2}}\ ,\ \ \omega > 0$
4 ☐ $A(\omega) = \frac{-1}{\sqrt{1+\omega^2}}\ ,\ \ \omega > 0$
5 ☐ $A(\omega) = \frac{1}{\sqrt{1-\omega^2}}\ ,\ \ \omega > 0$
6 ☐ Don't know

Question 3: The phase characteristic for the system (2) is
1 ☐ $\varphi(\omega) = \text{Arctan}\ \omega\ ,\ \ \omega > 0$
2 ☐ $\varphi(\omega) = -\text{Arctan}\ \omega\ ,\ \ \omega > 0$
3 ☐ $\varphi(\omega) = \text{Arctan}\ \omega + \pi\ ,\ \ \omega > 0$
4 ☐ $\varphi(\omega) = \text{Arctan}\ \omega - \pi\ ,\ \ \omega > 0$
5 ☐ $\varphi(\omega) = \pi - \text{Arctan}\ \omega\ ,\ \ \omega > 0$
6 ☐ Don't know

Problem 703 Consider the system

$$\dot{\mathbf{x}} = \begin{pmatrix} -2 & 3 \\ -3 & -2 \end{pmatrix} \mathbf{x} \ , \tag{3}$$

where

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} .$$

Question 1: The system (3) is

1 ☐ Instabile.
2 ☐ Unstable.
3 ☐ Stable, but not asymptotically stable.
4 ☐ Asymptotically stable.
5 ☐ BIBO-stable.
6 ☐ Don't know

Question 2: The general real solution to (3) is

1 ☐ $\mathbf{x} = c_1 e^{-2t} \begin{pmatrix} \sin 3t \\ \cos 3t \end{pmatrix} + c_2 e^{-2t} \begin{pmatrix} -\cos 3t \\ \sin 3t \end{pmatrix}, \quad c_1, \ c_2 \in \mathbb{R}$
2 ☐ $\mathbf{x} = c_1 e^{-2t} \begin{pmatrix} \sin 3t \\ \cos 3t \end{pmatrix} + c_2 e^{-2t} \begin{pmatrix} -\cos 3t \\ \sin 3t \end{pmatrix}, \quad c_1, \ c_2 \in \mathbb{C}$
3 ☐ $\mathbf{x} = c_1 e^{-3t} \begin{pmatrix} \sin 2t \\ \cos 2t \end{pmatrix} + c_2 e^{-3t} \begin{pmatrix} -\cos 2t \\ \sin 2t \end{pmatrix}, \quad c_1, \ c_2 \in \mathbb{R}$
4 ☐ $\mathbf{x} = c_1 e^{-2t} \begin{pmatrix} 3 \\ -3 \end{pmatrix} + c_2 e^{-2t} \begin{pmatrix} -3 \\ 3 \end{pmatrix}, \quad c_1, \ c_2 \in \mathbb{R}$
5 ☐ $\mathbf{x} = c_1 e^{-3t} \begin{pmatrix} 2 \\ -2 \end{pmatrix} + c_2 e^{-3t} \begin{pmatrix} -2 \\ 2 \end{pmatrix}, \quad c_1, \ c_2 \in \mathbb{R}$
6 ☐ Don't know

Problem 704 Consider the system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \ ,$$

where

$$\mathbf{A} = \begin{pmatrix} a & 0 & 4 \\ -1 & -1 & 0 \\ -2-a & 0 & -3 \end{pmatrix} \ .$$

The system is
1 ☐ Asymptotically stable for $-8 \leq a \leq 3$
2 ☐ Asymptotically stable for $-8 < a \leq 3$
3 ☐ Asymptotically stable for $-8 \leq a < 3$
4 ☐ Asymptotically stable for $-8 < a < 3$ and stable for $a \in \{-8, 3\}$
5 ☐ Asymptotically stable for $-8 < a < 4$ and stable for $a = -8$
6 ☐ Don't know

Problem 705 Consider the infinite series

$$\sum_{n=2}^{\infty} \left(\frac{4}{5}\right)^{n-2} .$$

The infinite series is
1 ☐ Divergent
2 ☐ Convergent with sum 5
3 ☐ Convergent with sum $\frac{16}{5}$
4 ☐ Don't know.

Problem 706 Consider the infinite series

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln(n)} .$$

The infinite series is
1 ☐ Divergent
2 ☐ Absolutely convergent
3 ☐ Conditionally convergent
4 ☐ Don't know.

Problem 707 Consider the power series

$$\sum_{n=0}^{\infty} \cosh(n) \cdot x^n.$$

The radius of convergence is

1 ☐ $\rho = 0$
2 ☐ $\rho = e^{-1}$

3 ☐ $\rho = 1$
4 ☐ $\rho = e$
5 ☐ $\rho = \infty$
6 ☐ Don't know

Problem 708 Consider the power series

$$\sum_{n=0}^{\infty} \frac{3^n}{2^n + 1} x^n.$$

The power series is

1 ☐ Absolutely convergent for $|x| < \frac{2}{3}$ and divergent for $|x| \geq \frac{2}{3}$
2 ☐ Absolutely convergent for $|x| \leq \frac{2}{3}$ and divergent for $|x| > \frac{2}{3}$
3 ☐ Absolutely convergent for $|x| < \frac{2}{3}$ and conditionally convergent for $x = -\frac{2}{3}$
4 ☐ Absolutely convergent for $|x| < \frac{3}{2}$ and divergent for $|x| \geq \frac{3}{2}$
5 ☐ Absolutely convergent for $|x| \leq \frac{3}{2}$ and divergent for $|x| > \frac{3}{2}$
6 ☐ Don't know

Problem 709 Consider the infinite series

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{n^2 + 1}. \qquad (*)$$

Question 1: The series is
1 ☐ Divergent
2 ☐ Absolutely convergent
3 ☐ Conditionally convergent
4 ☐ Don't know

Question 2: The partial sum

$$S_N = \sum_{n=1}^{N} (-1)^{n-1} \frac{n}{n^2 + 1}$$

gives an approximated value for the sum of the infinite series $(*)$ within a tolerance of 0.01 for
1 ☐ $N = 90$
2 ☐ $N = 95$
3 ☐ $N = 99$
4 ☐ Don't know

F.8 More problems

Problem 801 Assume that the sequence $(x_n)_{n=1}^{\infty}$ converges to x and the sequence $(y_n)_{n=1}^{\infty}$ converges to y. We show that the sequence $(x_n + y_n)_{n=1}^{\infty}$ converges to $x + y$:

(i) Show that

$$|(x+y) - (x_n + y_n)| \leq |x - x_n| + |y - y_n|, \quad \forall n \in \mathbb{N}.$$

(ii) Given $\epsilon > 0$, explain why we can find an $N \in \mathbb{N}$ such that

$$|x - x_n| \leq \epsilon/2 \quad \text{and} \quad |y - y_n| \leq \epsilon/2, \qquad \forall n \geq \mathbb{N}.$$

(iii) Use (i) and (ii) to show that $\lim_{n\to\infty}(x_n + y_n) = x + y$.

Problem 802 Assume that the series $(x_n)_{n=1}^{\infty}$ converges to x, and let $c \in \mathbb{C}$ Show, using the definition of convergence, that the serices $(cx_n)_{n=1}^{\infty}$ converges to cx.

Problem 803 It is well known that polynomial growth is faster than logarithmic growth, and exponential growth is faster than both of these. More precisely, for any real number $a > 0$ and any non-constant polynomial $P(x)$, we have

$$\lim_{x\to\infty} \frac{\log_a(x)}{P(x)} = 0, \quad \lim_{x\to\infty} \frac{\log_a(x)}{a^x} = 0, \quad \lim_{x\to\infty} \frac{P(x)}{a^x} = 0.$$

(You can prove these formulae using L'Hôspital's rule for limits involving quotients of differentiable functions that approach zero or infinity).
Using the formulae above, find the limits of the following sequences:

(i) $x_n = \frac{n^3}{2^n}$,

(ii) $x_n = \frac{n^2+1}{2^n+1}$,

(iii) $x_n = \frac{\ln(n^2)}{n^2}$.

Problem 804 A sequence $(x_n)_{n=1}^{\infty}$ is said to be *bounded* if there exists a real number $C \geq 0$, such that

$$|x_n| \leq C, \quad \forall n \in \mathbb{N}.$$

Assume now that a given sequence x_n is convergent. We will prove that this sequence is automatically bounded.

(i) Given that $\lim_{n\to\infty} x_n = x$, show that there exists a number $N \in \mathbb{N}$ such that

$$|x_n| \leq |x| + 1, \qquad \forall n \geq N.$$

(Hint: take $\epsilon = 1$ in the definition of limit, and use the triangle inequality on $|x_n| = |x_n - x + x|$).

(ii) Conclude that $(x_n)_{n=1}^{\infty}$ is bounded.

Problem 810 Lemma 4.17 states that, for a pair of convergent series, a series constructed as a linear combination of these series converges, with sum equal to the same linear combination of the limits. Show, with appropriately chosen examples, that one cannot conclude anything if the assumption on convergence is dropped.

Problem 811 Assume that the sequence $\{a_n\}_{n=1}^{\infty}$ satisfies

$$|a_{n+1}| \leq \frac{1}{2}|a_n| \text{ for all } n \in \mathbb{N}. \tag{F.41}$$

(i) Show that

$$|a_n| \leq |a_1| \left(\frac{1}{2}\right)^{n-1} \quad \text{for all } n \in \mathbb{N}.$$

(ii) Using (i), show that $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

In the rest of the problem we replace the assumption (F.41) with the more general assumption that there exists a number $C \geq 0$ such that

$$|a_{n+1}| \leq C\,|a_n| \text{ for all } n \in \mathbb{N}. \tag{F.42}$$

(iii) Assume that (F.42) is satisfied with $C < 1$. Show that

$$\sum_{n=1}^{\infty} a_n \tag{F.43}$$

is absolutely convergent.

(iv) Can one conclude anything on the convergence of the series at (F.43) if (F.42) is satisfied with $C = 1$?

(v) Why is the result (iii) more general than the quotient criterion given in the book?

Problem 864 Consider the extensions to 2π-periodic functions of the following functions

(i) $f_1(x) = |\cos(x)|,\ x \in [-\pi, \pi[$.

(ii) $f_2(x) = x^3,\ x \in [-\pi, \pi[$.

(iii) $f_3(x) = \sqrt{|x|},\ x \in [-\pi, \pi[$.

(iv) $f_4(x) = \begin{cases} \sqrt{|x|}, & x \in]-\pi, \pi[\\ 0, & x = -\pi. \end{cases}$

(i) Sketch each of the functions f_1, f_2, f_3, f_4.

(ii) Which ones of the functions f_1, f_2, f_3, f_4 are piecewise differentiable?

Problem 891 Consider the differential equation

$$t\frac{d^2y}{dt^2} - (t+1)\frac{dy}{dt} + y = 0,\ t \in \mathbb{R}. \tag{F.44}$$

Suppose that a function of the type

$$y(t) = \sum_{n=0}^{\infty} c_n t^n$$

is a solution.

(i) Show that

$$\sum_{n=2}^{\infty} c_n n(n-1)t^{n-1} - \sum_{n=1}^{\infty} c_n n t^n - \sum_{n=1}^{\infty} c_n n t^{n-1} + \sum_{n=0}^{\infty} c_n t^n = 0. \tag{F.45}$$

(ii) Via suitable changes of summation index, bring all terms on the left-hand side in the form $\sum d_n t^n$ for some coefficients d_n, and show hereby that (F.45) takes the form

$$c_0 - c_1 + \sum_{n=1}^{\infty} (n-1)[(n+1)c_{n+1} - c_n]t^n = 0. \tag{F.46}$$

(iii) Show using (F.46) that

$$c_0 = c_1 \text{ and } c_{n+1} = \frac{1}{n+1} c_n,\ n \geq 2. \tag{F.47}$$

We will now search for a solution satisfying the initial conditions

$$y(0) = 1,\ y''(0) = 1.$$

(iv) The initial conditions determine the coefficients c_0 and c_2. Find now a formula for c_n using (F.47).

(v) Find the radius of convergence for the resulting power series

$$y(t) = \sum_{n=0}^{\infty} c_n t^n,$$

and express the function in terms of elementary functions (check, e,g., the power series representations in Appendix B).

(vi) Find a solution y_2 to (F.44) such that y_2 and y in (v) are linearly independent.

(vii) Find a solution to the differential equation

$$t\frac{d^2y}{dt^2} - (t+1)\frac{dy}{dt} + y = t^2e^t,\ t > 0. \tag{F.48}$$

(viii) Find now the general solution to the differential equation (F.48).

Problem 897 We consider the differential equation

$$x\frac{d^2y}{dx^2} + (2+x)\frac{dy}{dx} + y = 0\,, \tag{F.49}$$

and seek a solution in the power series form

$$y(x) = \sum_{n=0}^{\infty} c_n x^n\,. \tag{F.50}$$

(i) Show, by substituting the expression (F.50) into the left hand side of the differential equation, that

$$x\frac{d^2y}{dx^2} + (2+x)\frac{dy}{dx} + y(x) = \sum_{n=0}^{\infty} \left((n+1)c_n + (n+1)(n+2)c_{n+1}\right)x^n\,.$$

(ii) Verify that, if the power series (F.50) satisfies the differential equation, then

$$c_{n+1} = -\frac{1}{n+2}c_n \quad \text{for} \quad n = 0, 1, 2, \dots\,. \tag{F.51}$$

(iii) Using (F.51), determine the radius of convergence for an arbitrary power series solution to the differential equation (F.49).

Hint: use the quotient criteria. Note that, to investigate $|c_{n+1}|/|c_n|$ for $n \to \infty$, it is not necessary to calculate the coefficients c_n.

Problem 901 Let $\lambda \in \mathbb{R}$ and define a function by

$$f(t) = te^{\lambda t}.$$

Show that, for all $n \in \mathbb{N}$,

$$\frac{\mathrm{d}^n}{\mathrm{d}t^n} f(t) = n\lambda^{n-2}e^{\lambda t} + t\lambda^n e^{\lambda t}.$$

Problem 902 Let λ_1 and λ_2 be two distinct real numbers. We will show that the functions

$$e^{\lambda_1 t}, \quad te^{\lambda_1 t}, \quad e^{\lambda_2 t}, \quad te^{\lambda_2 t},$$

are linearly independent as follows: assume that

$$c_1 e^{\lambda_1 t} + c_2 te^{\lambda_1 t} + c_3 e^{\lambda_2 t} + c_4 te^{\lambda_2 t} = 0.$$

We need to show that this implies that $c_i = 0$, for $i = 1...4$. Assume that $\lambda_1 > \lambda_2$.

(i) Show that $c_1 + c_2 = 0$.

(ii) Show that

$$c_1 e^{(\lambda_1 - \lambda_2)t} + c_2 te^{(\lambda_1 - \lambda_2)t} + c_3 + c_4 t = 0, \quad \forall t \in \mathbb{R}.$$

(iii) Now let $t \to \infty$. What conclusion can you deduce from (ii)?

(iv) Show, using the above, that the given functions are linearly independent.

(v) How does the proof need to be modified if $\lambda_1 < \lambda_2$?

Problem 903 Consider the homogeneous 2nd order differential equation

$$a_0 y'' + a_1 y' + a_2 y = 0.$$

Assume that $\lambda_0 \in \mathbb{R}$ is a root of the corresponding characteristic equation, and define the function

$$y(t) = te^{\lambda_0 t}. \tag{F.52}$$

(i) Show by substituting into the equation that y at (F.52) satisfies

$$a_0 y'' + a_1 y' + a_2 y = (2a_0\lambda_0 + a_1)e^{\lambda_0 t}.$$

(ii) Suppose now that λ_0 is a double root. Show that the function y above is then a solution of the homogeneous differential equation above. (Hint: write down the solution to the characteristic equation using the quadratic formula).

(iii) Assume that λ_0 is *not* a double root. Show that the function y at (F.52) is *not* a solution to the homogeneous equation. (Hint: use the quadratic formula, to deduce that $\lambda_0 \neq -a_1/(2a_0)$.)

Problem 905

(i) Show that the functions

$$y(t) = e^{t}, \quad y(t) = e^{2t}, \quad y(t) = e^{3t},$$

are linearly independent.

(ii) Let $\lambda_1 < \lambda_2 < \cdots < \lambda_n$ be n distinct real positive numbers. Show that

$$y(t) = e^{\lambda_1 t}, \quad y(t) = e^{\lambda_2 t}, \quad \ldots \quad y(t) = e^{\lambda_n t},$$

are linearly independent.

(iii) Show that the result in (ii) is still valid even without the assumption that all λ_i are positive.

(iv) Show that the functions

$$y(t) = e^{2t}, \quad y(t) = e^{3t}, \quad y(t) = te^{2t}, \quad y(t) = te^{3t},$$

are linearly independent.

(v) Let $k \in \mathbb{N}$ and let $\lambda_1 < \lambda_2 < \cdots < \lambda_n$ be n distinct real positive numbers. Show that the functions

$$e^{\lambda_i t}, \quad te^{\lambda_i t}, \quad \ldots \quad t^k e^{\lambda_i t}, \quad i = 1 \ldots n,$$

are linearly independent. [(vi)] Show that the assumption that all λ_i is not necessary.

Problem 914 (Proof of Theorem 2.5 (b)). We are dealing with the equation $\dot{x} = Ax$, where A is a real matrix.

(i) Explain why (non-real) complex eigenvalues come in pairs, $a \pm i\omega$.

Let $v = \mathrm{Re}\, v + i\mathrm{Im}\, v$ be an eigenvector corresponding to the eigenvalue $\lambda = a + i\omega$.

(ii) Show that

$$A(\mathrm{Re}\, v) = a\mathrm{Re}\, v - \omega \mathrm{Im}\, v, \qquad A(\mathrm{Im}\, v) = a\mathrm{Im}\, v + \omega \mathrm{Re}\, v.$$

(iii) Show that $\mathrm{Re}\, v - i\mathrm{Im}\, v$ is an eigenvector corresponding to the eigenvalue $a - i\omega$.

(iv) According to Theorem 2.3, the system has two solutions

$$u_1(t) = e^{(a+i\omega)t}(\mathrm{Re}\, v + i\mathrm{Im}\, v) \quad \text{and} \quad u_2(t) = e^{(a-i\omega)t}(\mathrm{Re}\, v - i\mathrm{Im}\, v)$$

Expand these solutions into their real and imaginary parts.

(v) Show that the functions

$$x(t) = \mathrm{Re}\,(e^{\lambda t} v), \quad \text{and} \quad x(t) = \mathrm{Im}\,(e^{\lambda t} v),$$

are solutions to the system and linearly independent.

Problem 915 Consider the differential equation

$$\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{b}u.$$

Set $u(t) = e^{st}$ and assume that $\det(A - sI) \neq 0$. Let $\mathbf{x}_0$ be some vector, and set

$$\mathbf{x}(t) = \mathbf{x}_0 e^{st}.$$

Show that $\mathbf{x}(t)$ is a solution to the differential equation if and only if

$$\mathbf{x}_0 = -(A - sI)^{-1}\mathbf{b}.$$

Problem 916 (The roots of the characteristic equation are the eigenvalues of the system matrix for the corresponding first order system). For an nth order differential equation,

$$y^{(n)}(t) + b_1 y^{(n-1)}(t) + \cdots + b_n y(t) = u(t),$$

rewritten as a first order system, call the system matrix (page 30) A_n.

(i) Show that, for a 2nd order equation, we have

$$\det(A_2 - \lambda I) = \lambda^2 + b_1\lambda + b_2.$$

(ii) Using (i), show that, for a 3rd order equation, we have

$$\det(A_3 - \lambda I) = -(\lambda^3 + b_1\lambda^2 + b_2\lambda + b_3).$$

(iii) Show that, for $n = 2, 3, \ldots,$ we have

$$\det(A_{n+1} - \lambda I) = -\lambda \det(A_n - \lambda I) + (-1)^{n+1} b_{n+1}.$$

(iv) Using (iii) and mathematical induction, show that, for all $n \in \mathbb{N}$,

$$\det(A_{n+1} - \lambda I) = (-1)^n(\lambda^n + b_1\lambda^{n-1} + b_2\lambda^{n-2} + \cdots + b_n).$$

Problem 917 A Sallen-Key lowpass filter is an electric circuit consisting of some resistors and capacitors, which we denote by R_1, R_2, C_1 and C_2. One can show that the relationship between the input voltage $x(t)$ and output voltage $y(t)$ is given by

$$y'' + a_1 y' + a_0 y = bx, \tag{F.53}$$

where

$$a_1 = \frac{1}{R_1C_1} + \frac{1}{R_2C_2} + \frac{1}{R_2C_1} - \frac{K}{R_2C_2}, \quad a_0 = \frac{1}{R_1R_2C_1C_2},$$
$$b = \frac{K}{R_1R_2C_1C_2}, \quad K = 1 + \frac{R_4}{R_3}.$$

We consider the case $R_1 = R_2 = C_1 = C_2 = 1$. Show that the system (F.53) is asymptotically stable if and only if we choose $R_4 < 2R_3$.

Problem 919 In Theorem 1.23 the equation (1.15) is solved under the assumption that s is not a root of the charateristic equation. In this problem we investigate in a particular case what happens when s *is* a root. Consider the differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} - y = u' + u. \tag{F.54}$$

(i) Determine the transfer function.

(ii) For $s \neq \pm 1$, write down the stationary solution corresponding to $u(t) = e^{st}$.

(iii) Find the general real solution to (F.54) for $u(t) = e^{-t}$.

(iv) Find the general real solution to (F.54) for $u(t) = e^{t}$ (you can use Thereom 1.30).

(v) Determine a solution to (F.54) for

$$u(t) = \sum_{n=1}^{N} n e^{-nt},$$

for a given $N = 2, 3, 4, ...$

(vi) Determine a solution for

$$u(t) = \sum_{n=-N}^{N} c_n e^{int},$$

for a given $N \in \mathbb{N}$ and arbitrary coefficients $c_n \in \mathbb{C}$.

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