Math2 Assignment

Fedir Vasyliev

s234542

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1 Problem 1

$$x_1'(t) = x_2(t) \tag{1}$$

$$x_2'(t) = -9x_1 - 6x_2 + e^{it} \tag{2}$$

To find a general solution for the system we need to separately find the solution for the homogeneous and the solution for inhomogeneous parts.

First, we will represent the system in matrix form.

$$\mathbf{x}'(t) = \begin{pmatrix} 0 & 1 \\ -9 & -6 \end{pmatrix} \mathbf{x}(t) \tag{3}$$

Now we find the eigenvalues and eigenvectors of the coefficient matrix.

$$det(A - \lambda I) = det \begin{pmatrix} -\lambda & 1 \\ -9 & -6-\lambda \end{pmatrix} = \lambda(6+\lambda) + 9 = 0$$

$$\lambda = \frac{-6 \pm \sqrt{36-36}}{2} = -3$$

Corresponding eigenvector to $\lambda = -3$ is

$$\begin{pmatrix} 3 & 1 \\ -9 & -3 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ -3 & -1 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} v1 \\ v2 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \end{pmatrix} t$$

We got the eigenvector and the eigenvalue, however, the algebraic multiplicity of the eigenvalue is 2, but we only have one eigenvector. Therefore, we can use Theorem 2.11

to find the whole solution.

$$\mathbf{x}_{hom}(t) = \underline{b}_{2,1} e^{-3t} + \underline{b}_{2,2} t e^{-3t}$$

$$\underline{b}_{2,2} = \begin{pmatrix} 1 \\ \\ -3 \end{pmatrix}, \ \underline{b}_{2,1} = \begin{pmatrix} a \\ \\ b \end{pmatrix}$$

$$\mathbf{x}_{hom}(t) = \begin{pmatrix} a \\ \\ b \end{pmatrix} e^{-3t} + \begin{pmatrix} 1 \\ \\ -3 \end{pmatrix} t e^{-3t}$$

To find a and b we can substitute this equation into the system's first equation(1).

$$be^{-3t} - 3te^{-3t} = -3ae^{-3t} - 3te^{-3t} + e^{-3t}$$

$$be^{-3t} = -3ae^{-3t} + e^{-3t}$$

$$a = 0 \Rightarrow b = 1$$

$$\mathbf{x}_{hom}(t) = \begin{pmatrix} 0 \\ \\ 1 \end{pmatrix} e^{-3t} + \begin{pmatrix} 1 \\ \\ -3 \end{pmatrix} t e^{-3t} \tag{4}$$

In this way, we have found the general solution for the homogeneous part. Now we can proceed to finding the particular solution of the system. We will use Theorem 2.23 to find the transfer function and then the solution.

$$\begin{cases} \mathbf{x}'(t) = \begin{pmatrix} 0 & 1 \\ -9 & -6 \end{pmatrix} \mathbf{x}(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{it} \\ x_1(t) = \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x}(t) \\ x_2(t) = \begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x}(t) \end{cases}$$

$$det(A - sI) = det \begin{pmatrix} -s & 1 \\ -9 & -6-s \end{pmatrix} = s(6+s) + 9$$

$$(A - sI)^{-1} = \frac{1}{s^2+6s+9} \begin{pmatrix} -6-s & -1 \\ 9 & -s \end{pmatrix}$$

$$H_1(s) = -\begin{pmatrix} 1 & 0 \end{pmatrix} \frac{1}{s^2+6s+9} \begin{pmatrix} -6-s & -1 \\ 9 & -s \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{s^2+6s+9}$$

$$H_2(s) = -\begin{pmatrix} 0 & 1 \end{pmatrix} \frac{1}{s^2+6s+9} \begin{pmatrix} -6-s & -1 \\ 9 & -s \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{s}{s^2+6s+9}$$

The solution then is $x_k(t) = H_k(s)e^{st}$, $k \in [1, 2]$. As we have $e^{st} = e^{it} \Rightarrow s = i$. So

$$\mathbf{x}_{part}(t) = \frac{1}{100} \begin{pmatrix} 8-6i \\ 6+8i \end{pmatrix} e^{it} \tag{5}$$

Then the final answer is

$$\mathbf{x}(t) = \mathbf{x}_{hom}(t) + \mathbf{x}_{part}(t) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-3t} + \begin{pmatrix} 1 \\ -3 \end{pmatrix} te^{-3t} + \frac{1}{100} \begin{pmatrix} 8-6i \\ 6+8i \end{pmatrix} e^{it} \tag{6}$$

2 Problem 2

$$f(t) = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}(3sin(2nt) - 2cos(nt)) \tag{7}$$

2.1

We need to find k s.t. $|\frac{1}{n^{3/2}}(3sin(2nt) - 2cos(nt))| \leq \frac{k}{n^{3/2}}$. As sin and cos functions can only take values in [-1,1] one can think of an extreme situation when $sin = 1$ and $cos = -1$, which is not exactly the case in our combination but it is still sufficient estimation to say that $|(3sin(2nt) - 2cos(nt))| \leq 5$. This also can be checked graphically. Thus, $k = 5$.

2.2

To determine if the series (7) is uniformly convergent we will first estimate if its majorant series is convergent. To do this the integral test can be used.

$$\int_1^{\infty} \frac{5}{n^{3/2}}\, dn = [\frac{-10}{n^{1/2}}]_1^{\infty} = \lim_{n\to\infty}(\frac{-10}{n^{1/2}} + 10) = 10$$

As the indefinite integral has the limit as $n \Rightarrow \infty$ we can conclude that the majorant series is convergent. Thus, we can conclude using Theorems 5.33, 5.35 that the series (7) is uniformly convergent and continuous.

2.3

Continuity was proven in the previous section.

2.4

To find N we will use the majorant series along with Equation (4.33) and following it Method (1) from the book.

$$\int_N^\infty \frac{5}{n^{3/2}}\,dn = [\frac{-10}{n^{1/2}}]_N^\infty \frac{-10}{N^{1/2}} \leq \varepsilon$$

$$\frac{10}{\varepsilon} \leq N^{1/2} \Rightarrow N \geq (\frac{10}{\varepsilon})^2$$

$$N \geq (\frac{10}{2 \cdot 10^{-3}})^2 = 25 \cdot 10^6$$

Such a high number can be explained by the slow convergence of $\frac{1}{n^{3/2}}$ and the fact that the majorant series chosen is not the lowest one, the better choice of k would be the true value 4.48874. Also, the choice of the majorant series as the estimate for the sum is not the best one.