Homework Assignment 2

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a)

Let $z_1 = r_1 e^{i\frac{\pi}{3}}$ and $z_2 = r_2 e^{i\frac{\pi}{6}}$, then $z_1 z_2 = r_1 e^{i\frac{\pi}{3}} * r_2 e^{i\frac{\pi}{6}}$.

$$z_1 z_2 = r_1 r_2 e^{i(\frac{\pi}{3}+\frac{\pi}{6})} = r_1 r_2 e^{i\frac{\pi}{2}} = r_1 r_2 i$$

Which is a purely imaginary number.

b)

$$e^z = 1 + i$$

$$z = \ln(1+i) = \ln \sqrt{2} e^{i\frac{\pi}{4}}$$

$$z = \ln \sqrt{2} + i\frac{\pi}{4} \text{ - first solution}$$

But there are an infinite number of solutions as we can rotate our complex number by 2π and get the same number. We do it by multiplying our $e^{ln\sqrt{2}+i\frac{\pi}{4}}$ with $e^{i2k\pi}$, where $k \in \mathbb{Z}$.

$$z = \ln \sqrt{2} + i\pi\frac{8k+1}{4},\ k \in \mathbb{Z}$$

Then our second solution could be

$$z = \ln \sqrt{2} + i\pi\frac{9}{4}.$$

c)

1)

$$
\begin{array}{r|l}
Z^4 + 3Z^3 + 4Z^2 + 6Z + 4 & Z^2 + 3Z + 2 \\ \cline{2-2}
\underline{- Z^4 - 3Z^3 - 2Z^2 \qquad\qquad\quad} & Z^2 + 2 \\
2Z^2 + 6Z + 4 & \\
\underline{- 2Z^2 - 6Z - 4} & \\
0 &
\end{array}
$$

2)

$$(Z^4 + 3Z^3 + 4Z^2 + 6Z + 4) = (Z^2 + 3Z + 2)(Z^2 + 2) = 0$$

$$Z^2 + 3Z + 2 = 0 \lor Z^2 + 2 = 0$$

$$Z = \frac{-3 \pm \sqrt{9-8}}{2} = -1 \text{ or } -2$$

$$Z = \sqrt{-2} = i\sqrt{2} \text{ or } -i\sqrt{2}$$

Roots are: $-1, 2, i\sqrt{2}, -i\sqrt{2}$

d)

1) We can perform a division to show that 2 is the root of the polynomial.

$$
\begin{array}{rrr|l}
Z^3 - 3Z^2 & & +4 & Z-2 \\ \cline{4-4}
-Z^3 + 2Z^2 & & & Z^2 - Z - 2 \\ \cline{1-1}
-Z^2 & & & \\
Z^2 & -2Z & & \\ \cline{1-2}
 & -2Z & +4 & \\
 & 2Z & -4 & \\ \cline{2-3}
 & & 0 &
\end{array}
$$

The remaining is zero, which means 2 is one of the roots.

2)

$$P(Z) = (Z^3 - 3Z^2 + 4)(Z - 2)$$

To find multiplicity of the root 2 we will keep dividing $Z^3 - 3Z^2 + 4$ by

$Z-2$ until the remaining won't be 0.

$$\begin{array}{r|l} Z^2 \quad - Z - 2 & Z-2 \\ \cline{2-2} -Z^2 + 2Z \qquad & Z+1 \\ \cline{1-1} Z - 2 & \\ -Z+2 & \\ \cline{1-1} 0 & \end{array}$$

Obviously, $Z+1$ cannot be further divided by $Z-2$. Now our polynomial can be expressed as:

$$P(Z) = (Z-2)^2(Z+1)$$

The multiplicity of the root 2 in the polynomial is 2.

e)

$$f(1000) \to f(500) \to f(250) \to f(125) \to f(124) \to f(62) \to f(31) \to f(30) \to$$
$$f(15) \to f(14) \to f(7) \to f(6) \to f(3) \to f(2) \to f(1)$$
$$f(1000) = 2*2*2*2*2*2*2*2*2*f(1) = 2^9 * f(1) = 512$$

f)

Base case:

$$n = 0 \Rightarrow 1 \geq 1$$

Induction step:

$$(1+h)^{n+1} \geq 1 + (n+1)h$$

$$(1+h)^{n+1} \geq 1 + hn + h$$

$$1 + (n+1)h + \binom{n+1}{2}h^2 + \binom{n+1}{3}h^3 + ... + h^{n+1} \geq 1 + (n+1)h$$

Cancel out common additives,

$$\binom{n+1}{3}h^3 + ... + h^{n+1} \geq 0$$

Once $\{h \in \mathbb{R} \,|\, h > 0\}$ and $\{n \in \mathbb{Z} \,|\, n \geq 0\}$, $\binom{n+1}{3}h^3 + ... + h^{n+1} \geq 0$ is **true**.

This implies that $(1+h)^n \geq 1 + nh$ is also **true** for all $\{h \in \mathbb{R} \,|\, h > 0\}$ and $\{n \in \mathbb{Z} \,|\, n \geq 0\}$.