Homework set 1: The Essay Exercise

Fedir Vasyliev

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a)

Show that: $((P \Leftrightarrow Q) \vee P) \Leftrightarrow (Q \Rightarrow P)$

P	Q	P $\Leftrightarrow Q$	(P $\Leftrightarrow Q) \vee P$	Q $\Rightarrow P$
T	T	T	T	T
T	F	F	T	T
F	T	F	F	F
F	F	T	T	T

As you can see the $(P \Leftrightarrow Q) \vee P$ is equivalent to the $Q \Rightarrow P$.

b)

Solve the equation $3|x| = x^2 + x - 2$ in real numbers.

$$3|x| = x^2 + x - 2 \Leftrightarrow$$

$$\Leftrightarrow (3x = x^2 + x - 2 \wedge x \geq 0) \vee (-3x = x^2 + x - 2 \wedge x < 0) \Leftrightarrow$$

$$\Leftrightarrow (0 = x^2 - 2x - 2 \wedge x \geq 0) \vee (0 = x^2 + 4x - 2 \wedge x < 0) \Leftrightarrow$$

$$\Leftrightarrow (x = \frac{2 \pm \sqrt{12}}{2} \wedge x \geq 0) \vee (x = \frac{-4 \pm \sqrt{24}}{2} \wedge x < 0) \Leftrightarrow$$

$$\Leftrightarrow (x = 1 + \sqrt{3} \vee x = -2 - \sqrt{6})$$

c)

$$f(x) = x^2 - x - 3|x|$$

$$f : \mathbb{R} \to \mathbb{R}$$

1. Determine whether the function is injective.

To determine if the function is injective or not we can split this function into 2 intervals for $x \geq 0$ and $x < 0$ then find the local extreme values: if the function has at least 1 extreme value then the function is not monotonic, hence not injective.

$$x \geq 0 \Rightarrow f(x) = x^2 - 4x \Rightarrow f'(x) = 2x - 4 \Leftrightarrow x_0 = 2$$

There is no point in checking the $x < 0$ interval as we already have the extreme point. So the function is not injective, indeed it is surjective.

2. Compute the image set of the function.

To find the image of the function we need to find all extreme points.

$$x < 0 \Rightarrow f(x) = x^2 + 2x \Rightarrow f'(x) = 2x + 2 \Leftrightarrow x_0 = -1$$

Now we will look at monotone intervals:

$$f \downarrow \text{on}\, x\ \in]-\infty; -1] \cup [0; 2] \,\text{and}\, f \uparrow \text{on}\, x\ \in [-1; 0] \cup [2; +\infty[$$

$$f(-1) > f(2)\ \Rightarrow Image(f) \subseteq [-4; +\infty]$$

d)

$$A = \{z \in \mathbb{C} |\, |z-1| = 1\}$$

$$B = \{z \in \mathbb{C} |\, Re(z) = 1\}$$

1. Draw the sets A and B in the complex plane.

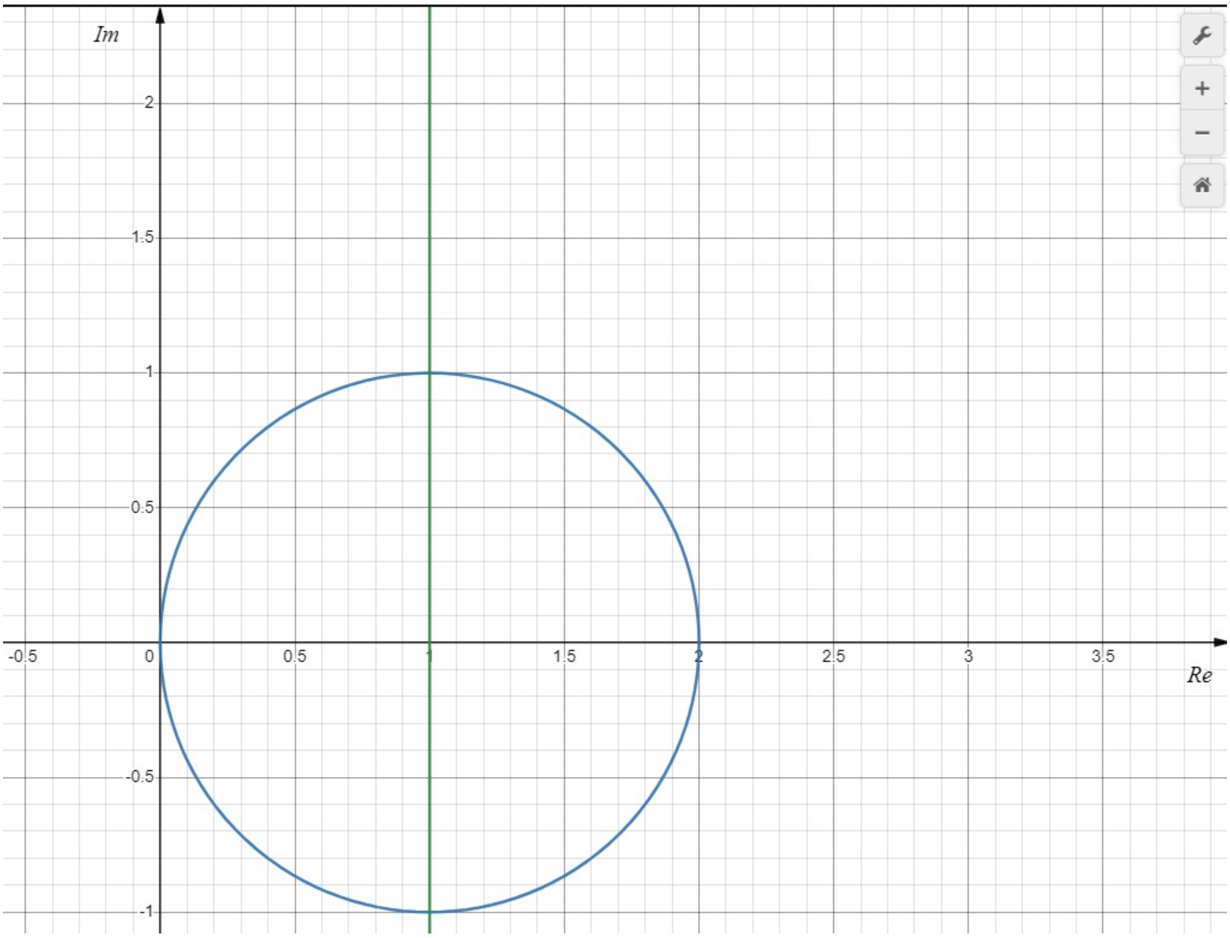

Figure 1: Set A is the blue line, set B is the green line.

2. Compute $A \cap B$.

We can see 2 intersection points on the graph: $1+i$ and $1-i$, meaning

$$A \cap B = \{1+i, 1-i\}$$

e)

Show that the complex number $(1+i)^{300}$ is a real number.

Let's express the $1+i$ in polar form:

$$r = \sqrt{1+1} = 2 \Rightarrow 1+i = \sqrt{2}(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}) \Leftrightarrow \sqrt{2}e^{i\frac{\pi}{4}}$$

$$(1+i)^{300} = (\sqrt{2}e^{i\frac{\pi}{4}})^{300}$$

$$(1+i)^{300} = 2^{150}e^{i75\pi}$$

$$e^{i75\pi} = e^{i\pi} = -1$$

$$(1+i)^{300} = -2^{150}, \text{ which is the real number.}$$

f)

Determine whether the following propositions are true:

$$1\colon Arg(z) = 0 \Rightarrow z \in \mathbb{R}$$

$$2\colon z \in \mathbb{R} \Rightarrow Arg(z) = 0$$

1)

$$z = r(\cos 0 + i \sin o),\ r \in \mathbb{R} \Rightarrow$$

$$z = r, \text{ which means that z is a real number}$$

2)

$$z = a + bi,\ a \in \mathbb{R},\ b = 0$$

$$\tan \alpha = \frac{b}{a} = 0$$

$$\arctan \alpha = 0 \Rightarrow arg(z) = 0$$

Indeed, $Arg(z) = 0 \Leftrightarrow z \in \mathbb{R}$.