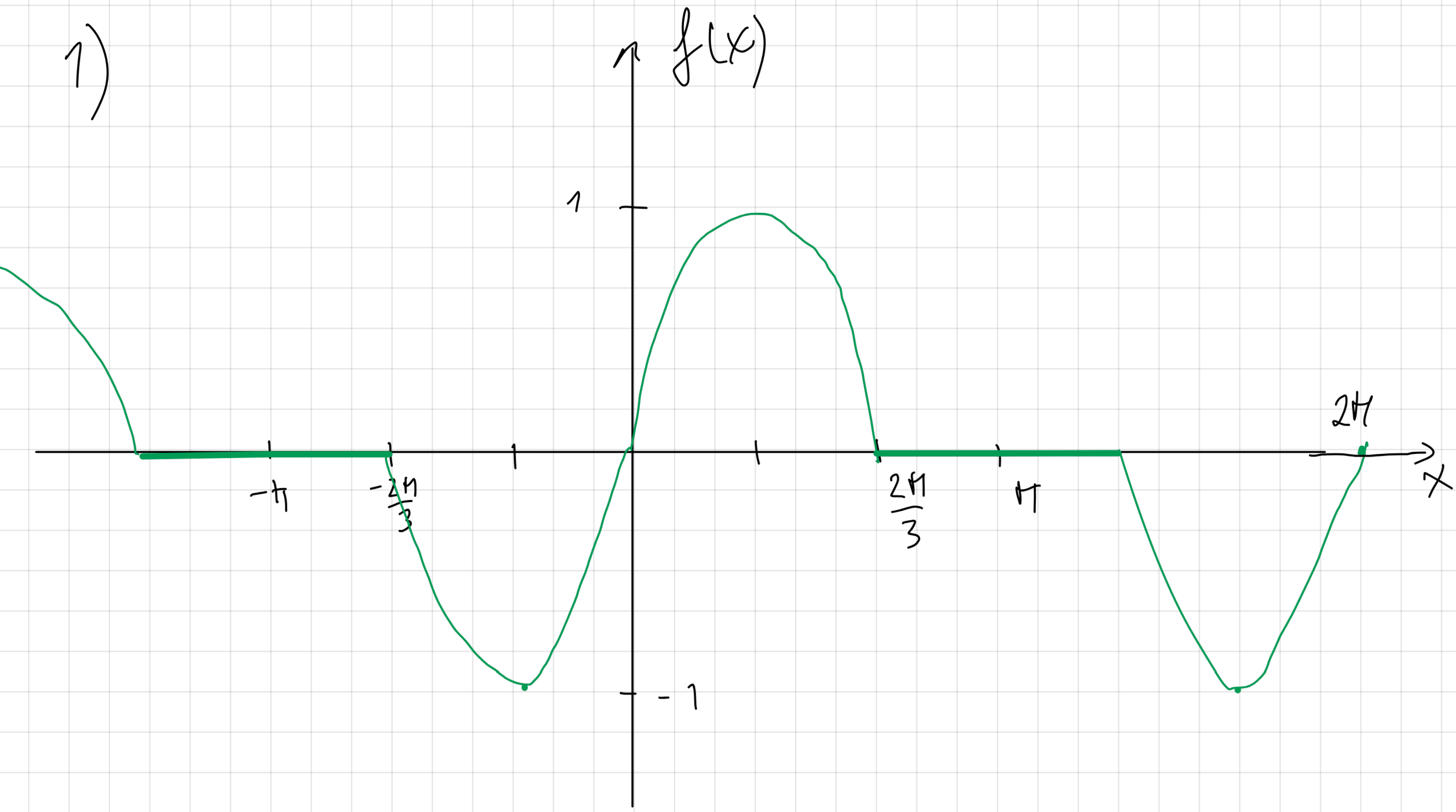
Problem A

$$f(x) = \begin{cases} \sin\left(\frac{3}{2}x\right) & \text{for } x \in \left[0, \frac{2\pi}{3}\right] \\ 0 & \text{for } x \in \left]\frac{2\pi}{3}, \pi\right] \end{cases}$$

1)




2) From Corollary 6.17 the FS converges uniformly iff $f(x)$ is continuous and piecewise diff-able.

i) It is continuous as can be seen on the graph

ii) To prove it is piecewise diff-able we will use def. 6.13.

$\sin\left(\frac{3}{2}x\right)$ is diff-able on $\left[0, \frac{2\pi}{3}\right]$ &

$\sin'\left(\frac{3}{2}x\right)$ is continuous on $\left[0, \frac{2\pi}{3}\right]$

as it is cos function.

As 0 is the constan ∴ is diff-able and continuous on $[\frac{2\pi}{3}, \pi]$.

$\Rightarrow$ FS of $f$ converges uniformly to $f$

3) As $f$ is an odd function $a_n = 0$ for all $n$ and $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx \Rightarrow$

$$b_n = \frac{2}{\pi}\int_0^{\frac{2\pi}{3}} \sin(\tfrac{3}{2}x)\sin(nx)\,dx =$$

$$= \frac{2}{\pi}\left[\frac{\sin((\frac{3}{2}-n)x)}{2(\frac{3}{2}-n)} - \frac{\sin((\frac{3}{2}+n)x)}{2(\frac{3}{2}+n)}\right]_0^{\frac{2}{3}\pi} =$$

$$= \frac{2}{\pi}\left(\frac{\sin((\frac{3}{2}-n)\frac{2}{3}\pi)}{3-2n} - \frac{\sin((\frac{3}{2}+n)\frac{2}{3}\pi)}{3+2n}\right) =$$

$$= \frac{2}{\pi}\left(\frac{\sin(\pi - \frac{2}{3}n\pi)}{3-2n} - \frac{\sin(\pi + \frac{2}{3}n\pi)}{3+2n}\right) =$$

$$= \frac{2}{\pi}\left(\frac{\sin(\frac{2}{3}n\pi)}{3-2n} + \frac{\sin(\frac{2}{3}n\pi)}{3+2n}\right) =$$

$$= \frac{2\sin(\frac{2}{3}n\pi)}{\pi}\left(\frac{6}{9-4n^2}\right) \Rightarrow$$

$$FS = \frac{12}{\pi} \sum_{n=1}^{\infty} \frac{\sin\left(\frac{2}{3} n\pi\right)}{9-4n^2} \left(\sin(nx)\right)$$

Part 2

I first did part 2 and the 1 & 3 over fundamental matrix.
I used python for computations.

Theorem 2.15, 2.18.

2) solution to homogeneous part

$$\det\begin{pmatrix} -1-\lambda & 1 \\ 0 & -2-\lambda \end{pmatrix} = (\lambda+2)(\lambda+1) \Rightarrow \lambda = -2/-1$$

eigenvector for $\lambda = -2$ : $v = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

eivenvector for $\lambda = -1$ : $v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\Rightarrow$

$$x_{hom}(t) = c_1 e^{-2t}\begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 e^{-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Fundamental matrix solution

$$\phi = \begin{bmatrix} -e^{-2t} & e^{-t} \\ e^{-2t} & \end{bmatrix}$$

$$\underline{y}(t) = \phi(t)\,\underline{c} + \phi(t)\int_{t_0}^{t} [\phi(\tau)]^{-1} \begin{bmatrix} \sin(3t) \\ 0 \end{bmatrix} d\tau$$

$$y(t) = \begin{bmatrix} -c_1 e^{-2t} + c_2 e^{-t} + \left( \frac{e^{t}\sin(3t)}{10} - \frac{3e^{t}\cos(3t)}{10} \right) e^{-t} \\ c_1 e^{-2t} \end{bmatrix}$$

$$= \begin{bmatrix} -c_1 e^{-2t} + c_2 e^{-t} + \frac{\sin(3t)}{10} - \frac{3}{10}\cos(3t) \\ c_1 e^{-2t} \end{bmatrix}$$

We can see that $x_p$ indeed a solution of the system.

3. $$\begin{bmatrix} -c_1 + c_2 - \frac{3}{10} \\ c_1 \end{bmatrix} = \underline{0}$$

$$c_1 = 0 \Rightarrow c_2 = \frac{3}{10}$$

$$y(t) = \begin{pmatrix} 1 & 2 \end{pmatrix} \begin{bmatrix} \frac{3}{10} e^{-t} + \frac{\sin(3t)}{10} - \frac{\cos(3t)}{10} \\ 0 \end{bmatrix} =$$

$$= \frac{3e^{-t} + \sin(3t) - \cos(3t)}{10}$$

Problem 3

$$t y' - y = \frac{t^2}{1+t}$$

1) We know that $\sum_{n=N}^{\infty} x^n = \frac{x^N}{1-x}$, $|x|<1$

we have $\frac{t^2}{1+t}$, it is the same as put $x=-t$. $\sum_{n=N}^{\infty}(-t)^n = \frac{t^N}{1+t}$, $N=2$ for us $\Rightarrow$

$$\sum_{n=2}^{\infty}(-1)^n t^n = \frac{t^2}{1+t}$$

2) $$t\sum_{n=1}^{\infty} n a_n t^{n-1} - \sum_{n=0}^{\infty} a_n t^n = \sum_{n=2}^{\infty}(-1)^n t^n$$

$$\sum_{n=1}^{\infty} n a_n t^n - \sum_{n=0}^{\infty} a_n t^n = \sum_{n=2}^{\infty}(-1)^n t^n$$

$$a_1 t + \sum_{n=2}^{\infty} n a_n t^n - a_0 - a_1 t - \sum_{n=2}^{\infty} a_n t^n = \sum_{n=2}^{\infty}(-1)^n t^n$$

$$-a_0 + \sum_{n=2}^{\infty} n a_n t^n - a_n t^n - (-1)^n t^n = 0$$

$$\begin{cases} \sum_{n=2}^{\infty} t^n \left(n a_n - a_n - (-1)^n\right) = 0 \\ a_0 = 0 \end{cases}$$

$$a_n(n-1) = (-1)^n \Rightarrow a_n = \frac{(-1)^n}{n-1}, \quad n \geq 2$$

replace $n$ with $n+2$ $\Rightarrow a_{n+2} = \frac{(-1)^{n+2}}{n+1} = \frac{(-1)^n \cdot 1}{n+1}$

for $n \geq 0$

3) $$\sum_{n=2}^{\infty} \frac{(-1)^n}{n-1} t^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} t^{n+2}$$

from Appendix B

$$\ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1}, \quad |x| < 1$$ , as we have $t^{n+2}$ $\Rightarrow$

$$t \ln(1+t) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} t^{n+2}, \quad |t| < 1.$$