# Exam solution for Mathematics 1b (F24)

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May 13, 2024

```
[1]: from sympy import *
     init_printing()
     from dtumathtools import *
```

## 1  Exercise 1

Given quadratic form:

```
[2]: x1,x2,x3 = symbols("x_1 x_2 x_3")
     xvec = Matrix([x1, x2, x3])
     q = 5*x1**2 + 8*x1*x2 - 4*x1*x3 - 22*x1 + 5*x2**2 + 4*x2*x3 - 32*x2 + 8*x3**2 -␣
      ↪20*x3 + 53
     q
```

[2]:
$$5x_1^2 + 8x_1x_2 - 4x_1x_3 - 22x_1 + 5x_2^2 + 4x_2x_3 - 32x_2 + 8x_3^2 - 20x_3 + 53$$

### 1.1  a

The partial derivatives are:

```
[3]: qx1=diff(q,x1)
     qx2=diff(q,x2)
     qx3=diff(q,x3)
```

hence, the gradient is:

```
[4]: nabla_q = Matrix([qx1,qx2,qx3])
     nabla_q
```

[4]:
$$\begin{bmatrix} 10x_1 + 8x_2 - 4x_3 - 22 \\ 8x_1 + 10x_2 + 4x_3 - 32 \\ -4x_1 + 4x_2 + 16x_3 - 20 \end{bmatrix}$$

for any $\boldsymbol{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$. The gradient can also be found by:

```
[5]: nabla_q = dtutools.gradient(q,[x1,x2,x3])
     nabla_q
```

[5]:

$$\begin{bmatrix} 10x_1 + 8x_2 - 4x_3 - 22 \\ 8x_1 + 10x_2 + 4x_3 - 32 \\ -4x_1 + 4x_2 + 16x_3 - 20 \end{bmatrix}$$

## 1.2  b

The Hessian matrix is computed by:

```
[6]: H_q = hessian(q, [x1,x2,x3])    # or dtutools.hessian(q, [x1,x2,x3])
     H_q
```

$$[6]: \begin{bmatrix} 10 & 8 & -4 \\ 8 & 10 & 4 \\ -4 & 4 & 16 \end{bmatrix}$$

This completes the answer to question b. Using the Hessian matrix, it is easy to write the quadratic form in matrix form as:

```
[7]: A = S(1)/2 * H_q
     b = Matrix([-22, -32, -20])
     c = 53
     A, b, c
```

$$[7]: \left( \begin{bmatrix} 5 & 4 & -2 \\ 4 & 5 & 2 \\ -2 & 2 & 8 \end{bmatrix}, \begin{bmatrix} -22 \\ -32 \\ -20 \end{bmatrix}, 53 \right)$$

```
[8]: q_check = xvec.T * A * xvec + b.T * xvec + Matrix([c])
     q_check[0].simplify()
```

$$[8]: 5x_1^2 + 8x_1x_2 - 4x_1x_3 - 22x_1 + 5x_2^2 + 4x_2x_3 - 32x_2 + 8x_3^2 - 20x_3 + 53$$

Indeed, the matrix form of $q$ based on the Hessian matrix agrees with the given $q$:

```
[9]: q_check[0].simplify() == q.simplify()
```

```
[9]: True
```

## 1.3  c

The Hessian is a real, symmetric matrix so we know according to the spectral theorem that there exists an orthonormal basis of eigenvectors. We find the eigenvalues and eigenvectors by:

```
[10]: eig = H_q.eigenvects()
      eig
```

$$[10]: \left[ \left( 0, 1, \left[ \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \right] \right), \left( 18, 2, \left[ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix} \right] \right) \right]$$

The eigenvectors associated with different eigenvalues are orthogonal to each other since the matrix is real symmetric. However, since the algebraic multiplicity of the eigenvalue 18 is two, we need

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to make sure that the two associated linearly independent eigenvectors are orthogonal. We use the Gram-Schmidt procedure to obtain two orthonormal eigenvectors:

```
[11]: GramSchmidt(eig[1][2], True)
```

[11]: 
$$\left[\begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{\sqrt{2}}{6} \\ \frac{\sqrt{2}}{6} \\ \frac{2\sqrt{2}}{3} \end{bmatrix}\right]$$

The eigenvector associated with 0 only needs to be normalized. There are several ways to do this, e.g.,

```
[12]: GramSchmidt(eig[0][2], True)
```

[12]: 
$$\left[\begin{bmatrix} \frac{2}{3} \\ -\frac{2}{3} \\ \frac{1}{3} \end{bmatrix}\right]$$

or,

```
[13]: Matrix(eig[0][2]).normalized()
```

[13]: 
$$\begin{bmatrix} \frac{2}{3} \\ -\frac{2}{3} \\ \frac{1}{3} \end{bmatrix}$$

Finally, we combine the three orthonormal eigenvectors in a basis denoted $\beta$:

```
[14]: beta = GramSchmidt(eig[0][2], True)[0], GramSchmidt(eig[1][2], True)[0],␣
      ↪GramSchmidt(eig[1][2], True)[1]
      beta
```

[14]: 
$$\left(\begin{bmatrix} \frac{2}{3} \\ -\frac{2}{3} \\ \frac{1}{3} \end{bmatrix}, \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{\sqrt{2}}{6} \\ \frac{\sqrt{2}}{6} \\ \frac{2\sqrt{2}}{3} \end{bmatrix}\right)$$

The assoicated change-of-basis matrix $Q$ is:

```
[15]: Q = Matrix([beta])
      Q
```

[15]: 
$$\begin{bmatrix} \frac{2}{3} & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{6} \\ -\frac{2}{3} & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{6} \\ \frac{1}{3} & 0 & \frac{2\sqrt{2}}{3} \end{bmatrix}$$

Let's check that $Q$ is indeed an orthogonal matrix:

```
[16]: Q * Q.T
```

[16]: 
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note that $Q$ is not unique (not even up to multiplication by $-1$) as there are infinitely many ways to obtain an orthonormal basis for the eigenspace associated with 18. Here is another change-of-basis matrix:

```
[17]: V = Matrix([[S(1)/3, S(-2)/3, S(2)/3], [S(2)/3, S(-1)/3, S(-2)/3], [S(2)/
      →3, S(2)/3, S(1)/3]])
      V * V.T, H_q * V, V
```

[17]:
$$\left( \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 6 & -12 & 0 \\ 12 & -6 & 0 \\ 12 & 12 & 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{bmatrix} \right)$$

### 1.4   d

We insert $\boldsymbol{x} = (1, 2, 1)$ in the expresseion of $\nabla q$ which yields:

```
[18]: nabla_q.subs({x1:1,x2:2,x3:1})
```

[18]:
$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Since the gradient at the given point is the zero vector, then $\boldsymbol{x} = (1, 2, 1)$ is a stationary point. To find all stationary points, we find all solutions to the equation system $\nabla q(\boldsymbol{x}) = (0, 0, 0)$:

```
[19]: solve(list(nabla_q),[x1,x2,x3])
```

[19]:
$$\{x_1 : 2x_3 - 1, \ x_2 : 4 - 2x_3\}$$

Setting $x_3 = t, t \in \mathbb{R}$, we see that all stationary points are given by:

$$(x_1, x_2, x_3) = (2t - 1, -2t + 4, t) = (2, -2, 1)t + (-1, 4, 0)$$

where $t \in \mathbb{R}$.

### 1.5   e

Since the function $q$ is a quadratic form, the line found in question d above is a line along which there is no increase nor decrease of $q$. In fact, this line is $(1, 2, 1) + \text{span}(v_1)$ where $v_1$ is an eigenvector of 0. So, the function $q$ is constant in the direction of $v_1$. *Remark:* The Hessian is not really useful here since:

```
[20]: H_q.eigenvals()
```

[20]:
$$\{0 : 1, \ 18 : 2\}$$

### 1.6   f

Given $x_0$, and restating the gradient:

```
[21]: x0 = Matrix([1, 2, 1]) + 3 * V[:, 1]
      nabla_q, x0
```

[21]:
$$
\left(
\begin{bmatrix}
10x_1 + 8x_2 - 4x_3 - 22 \\
8x_1 + 10x_2 + 4x_3 - 32 \\
-4x_1 + 4x_2 + 16x_3 - 20
\end{bmatrix}
,
\begin{bmatrix}
-1 \\
1 \\
3
\end{bmatrix}
\right)
$$

We carry out the gradient method to compute $x_{10}$, where $x_{n+1} = x_n - \alpha \nabla q(x_n)$ for $n = 0, 1, 2, \ldots$:

```
[22]: alpha = 0.02
      x = x0
      for n in range(11):
          x = x - alpha * nabla_q.subs({x1: x[0], x2: x[1], x3: x[2]})
      x
```

[22]:
$$
\begin{bmatrix}
0.985242604741033 \\
1.99262130237052 \\
1.01475739525897
\end{bmatrix}
$$

The gradient method converges towards $(1, 2, 1)$ since `3 * V[:,1]` belongs to the orthogonal complement of the null-space of $A$. The gradient method progresses along this direction and intersects the "stationary line" at the point $(1, 2, 1)$.

## 2   Exercise 2

Given quadratic form:

```
[23]: x1, x2, x3, x4 = symbols("x_1 x_2 x_3 x_4")
      q = 2 * x1 * x3 + 4 * x2 * x4
      q
```

[23]:
$$2x_1x_3 + 4x_2x_4$$

### 2.1   a

The Hessian matrix is given by:

```
[24]: H = hessian(q, [x1, x2, x3, x4])
      H
```

[24]:
$$
\begin{bmatrix}
0 & 0 & 2 & 0 \\
0 & 0 & 0 & 4 \\
2 & 0 & 0 & 0 \\
0 & 4 & 0 & 0
\end{bmatrix}
$$

We note that the Hessian matrix is symmetric. A symmetric matrix $A$ that fulfills $q = \boldsymbol{x}^T A \boldsymbol{x}$, where $\boldsymbol{x} = [x_1, x_2, x_3, x_4]^T$ is thus:

```
[25]: A = S(1)/2 * H
      A
```

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$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{bmatrix}$$

## 2.2  b

An orthogonal matrix $Q$ that reduces the quadratic form is found as the change-of-basis matrix that diagonalizing $A$:

```
[26]: Q, Lamda = A.diagonalize(normalize=True)
      Q, Lamda
```

[26]:
$$\left( \begin{bmatrix} 0 & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & 0 & 0 & \frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & 0 & 0 & \frac{\sqrt{2}}{2} \end{bmatrix}, \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \right)$$

Denote the $i$th column vector of $Q$ by $\boldsymbol{q}_i$, and let $\beta = \boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3, \boldsymbol{q}_4$.
Defining new coordinates as $\boldsymbol{k} = [k_1, k_2, k_3, k_4]^T$, where $\boldsymbol{k} = Q^T \boldsymbol{x}$, we have $q$ expressed in the reduced form:

```
[27]: k1, k2, k3, k4 = symbols("k_1,k_2,k_3,k_4")
      kvec = Matrix([k1, k2, k3, k4])
      q_new = Matrix([kvec.T * Lamda * kvec])[0]
      q_new
```

[27]:
$$-2k_1^2 - k_2^2 + k_3^2 + 2k_4^2$$

## 2.3  c

We now consider $q$ restricted to the set:

$$B = \{\boldsymbol{x} \in \mathbb{R}^4 \mid x_1^2 + x_2^2 + x_3^2 + x_4^2 \le 1\}.$$

Since $q : B \to \mathbb{R}$ is continuous on a bounded and closed domain, it has a global minimum and maximum value by Theorem 5.2.1.

## 2.4  d

Extrema of $q$ over the domain $B$ are to be found in the stationary points in the interior of $B$, at the boundary of $B$, or at exceptional points (see Theorem 5.2.2). Since the function is smooth, there are no exceptional points. Hence, we must investigate the interior and the boundary. First, the interior:

The partial derivatives are given by:

```
[28]: qx1 = diff(q, x1)
      qx2 = diff(q, x2)
```

```
qx3 = diff(q, x3)
qx4 = diff(q, x4)
qx1, qx2, qx3, qx4
```

[28]: $(2x_3, \ 4x_4, \ 2x_1, \ 4x_2)$

Thus, the gradient is:

[29]:
```
nabla_q = Matrix([qx1, qx2, qx3, qx4])
nabla_q
```

[29]:
$$\begin{bmatrix} 2x_3 \\ 4x_4 \\ 2x_1 \\ 4x_2 \end{bmatrix}$$

Alternatively, it can be found by:

[30]:
```
nabla_q = dtutools.gradient(q, [x1, x2, x3, x4])
nabla_q
```

[30]:
$$\begin{bmatrix} 2x_3 \\ 4x_4 \\ 2x_1 \\ 4x_2 \end{bmatrix}$$

Hence, obviuosly, the gradient equals the zero vector if and only if $\boldsymbol{x} = 0$. We can check this in Python by:

[31]:
```
solve(list(nabla_q), [x1, x2, x3, x4])
```

[31]: $\{x_1 : 0, \ x_2 : 0, \ x_3 : 0, \ x_4 : 0\}$

So, there exists only one stationary point in the interior of $q$ on $B$, and it is found at $(0, 0, 0, 0)$. The corresponding function value is:

[32]:
```
q.subs({x1: 0, x2: 0, x3: 0, x4: 0})
```

[32]: $0$

Now, investigating boundary points. We see from the given set that the boundary of $B$ is a unit sphere centred at the origin with a radius of 1, so all points that fulfill $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$:

$$\partial B = \{\boldsymbol{x} \in \mathbb{R}^4 \mid \|\boldsymbol{x}\| = 1\}.$$

We note that this unit sphere $\partial B$ is invariant under $Q^T$ (and $Q$) since $Q^T$ is orthogonal and therefore satisfies $\|Q\boldsymbol{x}\| = \|\boldsymbol{x}\|$ for all $\boldsymbol{x}$ (see Theorem 2.6.1(vi)). Alternatively, you may argue using that $Q^T$ only causes rotation and refleksion of the coordinate system and does not alter distances. Hence, we may use the easier new coordinates, which we above denoted by $\boldsymbol{k}$, and the unit sphere is thus described by $\|\boldsymbol{k}\| = k_1^2 + k_2^2 + k_3^2 + k_4^2 = 1$.

Recall that $q$ in the new coordinates was:

[33]: `q_new`

[33]: $-2k_1^2 - k_2^2 + k_3^2 + 2k_4^2$

Note that $k_1, k_2, k_3, k_4$ are squared in this expression, so $k_1, k_2$ have negative contributions, while $k_3, k_4$ have positive contributions. Maximum of $q$ on the sphere is thus found, firstly where $k_1 = 0, k_2 = 0$, and secondly where $k_4$ is largest due to its larger factor, meaning that $k_4 = 1$ while $k_3 = 0$. Maximum on the boundary is thus found at $\boldsymbol{k} = [0, 0, 0, 1]^T$ with a function value of:

[34]: `q_new.subs({k1: 0, k2: 0, k3: 0, k4: 1})`

[34]: $2$

Equivalently, a minimum is found where $k_3, k_4$ are zero and $k_1$ is maximum, so $k_1 = 1$, thus requiring $k_2 = 0$. Minimum on the boundary is thus found at $(1, 0, 0, 0)$ with a function value of:

[35]: `q_new.subs({k1: 1, k2: 0, k3: 0, k4: 0})`

[35]: $-2$

The stationary point is seen to not be an extremum. Global extrema of $q$ on $B$ are hence on the boundary as found above. We are not asked to find the location of the extremas, but they are easily found in the standard basis using $Q$

[36]: `Q * kvec.subs({k1: 0, k2: 0, k3: 0, k4: 1}), Q * kvec.subs({k1: 1, k2: 0, k3: 0,␣`
`↪k4: 0})`

[36]: $$\left( \begin{bmatrix} 0 \\ \frac{\sqrt{2}}{2} \\ 0 \\ \frac{\sqrt{2}}{2} \end{bmatrix}, \begin{bmatrix} 0 \\ -\frac{\sqrt{2}}{2} \\ 0 \\ \frac{\sqrt{2}}{2} \end{bmatrix} \right)$$

which is the location of the maximum and minimum, respectively.

## 3   Exercise 3

Given function $f : \mathbb{R}^2 \to \mathbb{R}$ which for $(x, y) = (0, 0)$ is $f(0, 0) = 0$, and for $(x, y) \in \mathbb{R}^2 \setminus (0, 0)$ is:
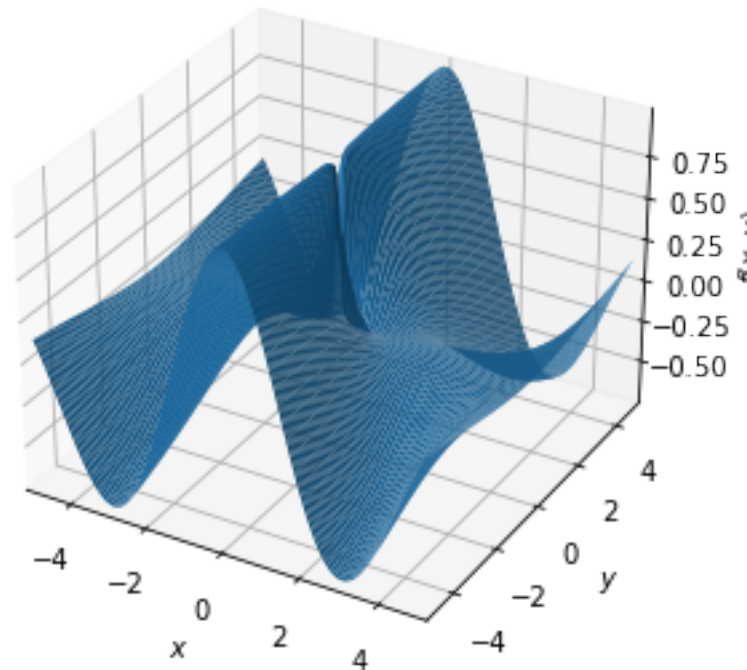
[37]: 
```
x,y = symbols("x y", real=True)
f = y**2 * cos(x) / (x**2 + y**2)
f
```

[37]: $\dfrac{y^2 \cos(x)}{x^2 + y^2}$

### 3.1   a

Plot for $(x, y) \in \mathbb{R}^2 \setminus (0, 0)$:

[38]: `dtuplot.plot3d(f, (x, -5, 5), (y, -5, 5))`

## 3.2 b

First-order partial derivatives for $(x, y) \in \mathbb{R}^2 \setminus (0,0)$ are found to be:

[39]: `f.diff(x), f.diff(y)`

[39]: $$\left( -\frac{2xy^2 \cos(x)}{(x^2 + y^2)^2} - \frac{y^2 \sin(x)}{x^2 + y^2}, \quad -\frac{2y^3 \cos(x)}{(x^2 + y^2)^2} + \frac{2y \cos(x)}{x^2 + y^2} \right)$$

The is not asked for in the exercise, but the partial derivatives exists at $(0,0)$ and are continuous:

[40]: `f.diff(x).subs({x:0,y:0}), f.diff(y).subs({x:0,y:0})`

[40]: $(0, 0)$

## 3.3 c

2nd-degree Taylor polynomial $P_2$ of $\cos(x)$ expanded from $x_0 = 0$:

[41]: `series(cos(x), x, 0, 3).removeO()`

[41]: $$1 - \frac{x^2}{2}$$

9

### 3.4  d

Note first that $f(x, x)$ is just a special case of $f(x, y)$ where $x = y$. The limit value is then easily computed:

$$\lim_{x \to 0} f(x, x) = \lim_{x \to 0} \frac{x^2 \cos(x)}{x^2 + x^2} = \lim_{x \to 0} \frac{\cos(x)}{2} = \frac{1}{2}$$

Check:

```
[42]: f.subs(y,x).limit(x,0)
```

[42]: $\dfrac{1}{2}$

or (using Taylor's limit formula):

```
[43]: (y**2 * series(cos(x), x, 0, 3).removeO() / (x**2 + y**2)).subs(y,x)
```

[43]: $\dfrac{1}{2} - \dfrac{x^2}{4}$

### 3.5  e

The limit value:

$$\lim_{x \to 0} f(x, 2x) = \lim_{x \to 0} \frac{(2x)^2 \cos(x)}{x^2 + (2x)^2} = \lim_{x \to 0} \frac{4x^2 \cos(x)}{x^2 + 4x^2} = \lim_{x \to 0} \frac{4 \cos(x)}{5} = \frac{4}{5}$$

Check:

```
[44]: f.subs(y,2*x).limit(x,0)
```

[44]: $\dfrac{4}{5}$

### 3.6  f

The first-order partial derivatives are continuously differentiable on $\mathbb{R}^2 \setminus \{(0, 0)\}$ so $f$ is differentiable on this domain. Since $f$ is not continuous at $(0, 0)$, it is not differentiable here.

## 4  Exercise 4

Given vector field $\boldsymbol{V}$ and parametrization $\boldsymbol{r}(u)$ of a curve $K_1$:

```
[45]: x, y, z, u, t = symbols("x y z u t", real=True)
      V = Matrix([-x, x*y**2, x+z])
      r = Matrix([u, u**2, u +1])
      V, r
```

[45]:

10

$$
\left( \begin{bmatrix} -x \\ xy^2 \\ x+z \end{bmatrix}, \begin{bmatrix} u \\ u^2 \\ u+1 \end{bmatrix} \right)
$$

where $u \in [0, 2]$.

### 4.1  a

Tangent vector $\boldsymbol{r}'(u)$:

```
[46]: rd = diff(r,u)
      rd
```

[46]:
$$
\begin{bmatrix} 1 \\ 2u \\ 1 \end{bmatrix}
$$

This vector is never the zero vector (and $\boldsymbol{r}$ is obviously injective), hence the parametrization is regular.

### 4.2  b

The inner product of $\boldsymbol{V}(\boldsymbol{r}(u))$ with $\boldsymbol{r}'(u)$ can be carried out as a usual dot product since all elements are real, $\langle \boldsymbol{V}(\boldsymbol{r}(u)), \boldsymbol{r}'(u) \rangle = \boldsymbol{V}(\boldsymbol{r}(u)) \cdot \boldsymbol{r}'(u)$. Hence, we get:

```
[47]: innerproduct = V.subs({x: r[0], y: r[1], z: r[2]}).dot(rd)
      innerproduct.simplify()
```

[47]:
$$
2u^6 + u + 1
$$

The tangential line integral $\int_{K_1} \boldsymbol{V} \cdot \mathrm{d}\boldsymbol{s}$ is given by:

```
[48]: integrate(innerproduct, (u, 0, 2))
```

[48]: $\dfrac{284}{7}$

### 4.3  c

As a parametrization $K_2 = \boldsymbol{p}([0,1])$ of the straight line segment $K_2$ from $(0,0,1)$ to $(2,4,3)$ we use:

```
[49]: xstart = Matrix([0, 0, 1])
      xend = Matrix([2, 4, 3])

      p = xstart + t * (xend - xstart)
      p
```

[49]:
$$
\begin{bmatrix} 2t \\ 4t \\ 2t+1 \end{bmatrix}
$$

where $t \in [0, 1]$.

The tangent vector $\boldsymbol{p}'(t)$:

```
[50]: pd = diff(p,t)
      pd
```

[50]: $\begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}$

The (tangential) line integral $\int_{K_2} \boldsymbol{V} \cdot \mathrm{d}\boldsymbol{s}$, where the inner product again is a dot product, is:

```
[51]: innerproduct2 = V.subs({x: p[0], y: p[1], z: p[2]}).dot(pd)
      innerproduct2.simplify()
```

[51]: $128t^3 + 4t + 2$

```
[52]: integrate(innerproduct2, (t, 0, 1))
```

[52]: $36$

### 4.4  d

Looking back at the parametrization $\boldsymbol{r}(u), u \in [0, 2]$ of $K_1$, we evaluate the $(x, y)$ coordinates of the end points:

```
[53]: r.subs({u:0}),r.subs({u:2})
```

[53]: $\left( \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix} \right)$

We see that $K_1$ and $K_2$ are two curves with the same starting and end points. Thus, $\boldsymbol{V}$ is not a gradient vector field since the (tangential) line integral from $(0, 0, 1)$ to $(2, 4, 3)$ depends on the path, since we got different values in questions b and c (according to Lemma 7.4.1).

Alternatively, we can arrive at the same conclusion by showing that the Jacobian matrix is not symmetric (according to Lemma 7.3.2):

```
[54]: V.jacobian([x,y,z])
```

[54]: $\begin{bmatrix} -1 & 0 & 0 \\ y^2 & 2xy & 0 \\ 1 & 0 & 1 \end{bmatrix}$

### 4.5  Some illustrative plots (not asked for)

```
[55]: K1 = dtuplot.plot3d_parametric_line(
          *r, (u,0,2), show=False, rendering_kw={"color": "red"}, colorbar=False
      )
      K2 = dtuplot.plot3d_parametric_line(
          *p, (t,0,1), show=False, rendering_kw={"color": "blue"}, colorbar=False
```

```
)
vektorfelt_V = dtuplot.plot_vector(
    V,
    (x, -.1, 2.1),
    (y, -.1, 4.1),
    (z, 0, 3),
    n=5,
    quiver_kw={"alpha": 0.5, "length": 0.05, "color": "black"},
    colorbar=False,
    show=False,
)

combined = K1 + K2 + vektorfelt_V
combined.legend = False
combined.show()
```
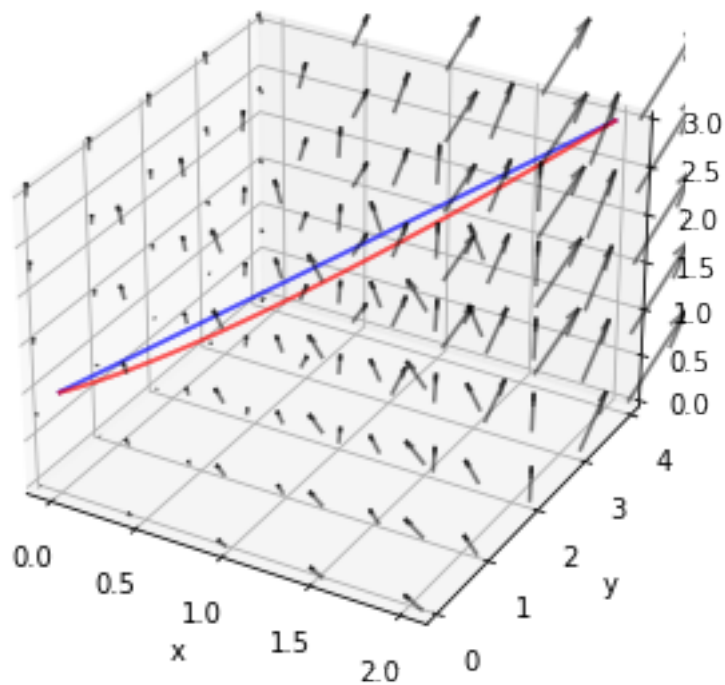


## 5   Exercise 5

Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x_1, x_2) = x_1^2 + x_2^2 + x_1 + 1$$

and the subset $A \subset \mathbb{R}^2$ given by:

$$A = \{(x_1, x_2) \in \mathbb{R}^2 \mid -2 \leq x_1 \leq 2 \wedge -1 \leq x_2 \leq 1\}.$$

## 5.1  a

We compute the integral $\int_A f(x_1, x_2) \, d(x_1, x_2)$ by:

```
[56]:  x1, x2 = symbols("x_1 x_2", real=True)
       f = x1**2 + x2**2 + x1 + 1
       integrate(f, (x1,-2,2), (x2,-1,1))
```
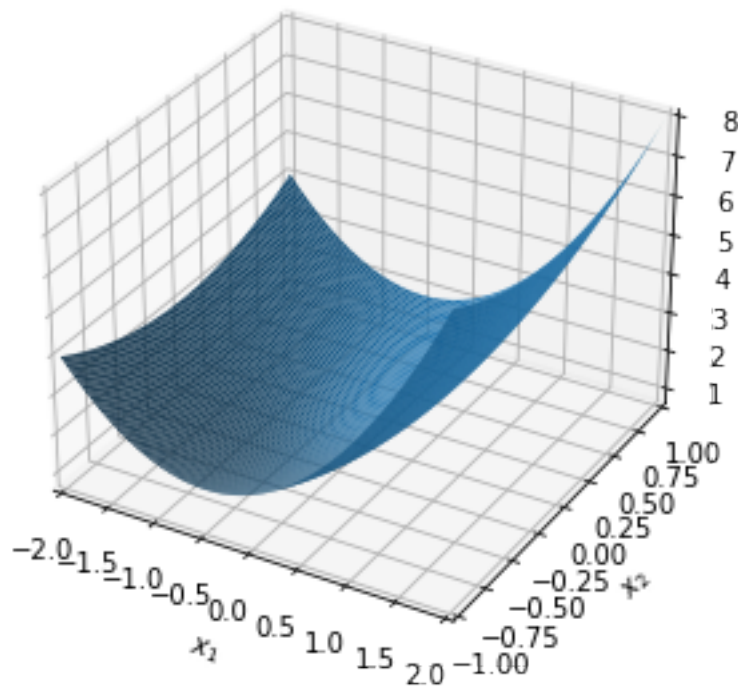
[56]: $\dfrac{64}{3}$

## 5.2  b

Determining the volume of the set

$$\{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid (x_1, x_2) \in A \wedge 0 \leq x_3 \leq f(x_1, x_2)\}.$$

We first plot the graph of $f$:

```
[57]:  dtuplot.plot3d(f, (x1, -2, 2), (x2, -1, 1))
```



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and note that $f$ is positive on $A$. Since $f$ is positive on $A$, then $f$ resembles an elevation function. The volume is thus equal to the integral found in question a, i.e. $64/3$.

### 5.3  c

Let $a > 0$. Let $B \subset \mathbb{R}^2$ denote the circular disk with center at the origin and radius $a$:

$$B = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 \le a^2\}.$$

Parametrizing $B$ using polar coordinates $(r, \theta)$ as:

$$s(r, \theta) = (r\cos(\theta), r\sin(\theta))$$

where $r \in [0, a], \theta \in [0, 2\pi[$. Hence, we can write $B$ as:

$$B = \{(r\cos(\theta), r\sin(\theta)) \in \mathbb{R}^2 \mid r \in [0, a] \wedge \theta \in [0, 2\pi[\}.$$

```
[58]: r, theta = symbols("r theta", real=True)
      s = Matrix([r * cos(theta), r * sin(theta)])
      s
```

[58]: $\begin{bmatrix} r\cos(\theta) \\ r\sin(\theta) \end{bmatrix}$

The partial derivatives of $s$ is given by:

```
[59]: sr = s.diff(r)
      stheta = s.diff(theta)
      sr, stheta
```

[59]: $\left( \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}, \begin{bmatrix} -r\sin(\theta) \\ r\cos(\theta) \end{bmatrix} \right)$

and the Jacobian determinant is therefore:

```
[60]: Jac_det = Matrix.hstack(sr, stheta).det().simplify()
      Jac_det
```

[60]: $r$

### 5.4  d

To determine the value of $a$ to 3 decimal places such that

$$\int_A f(x_1, x_2)\, \mathrm{d}(x_1, x_2) = \int_B f(x_1, x_2)\, \mathrm{d}(x_1, x_2)$$

we must compute the right-hand side's plane integral $\int_B f(x_1, x_2)\, \mathrm{d}(x_1, x_2)$.

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Since the Jacobian determinant is non-zero in the interior of $B$, and since the parametrization is regular (injective and never the zero vector on the interior of $B$), we can carry out the integral over the parameter region with the Jacobian function as a correction factor.

The Jacobian function is the absolute value of the Jacobian determinant. The determinant is $r$ and thus always non-negative, so the Jacobian function is just:

```
[61]: Jac = r
```

The integrand of the plane integral of $f$ over $B$:

```
[62]: a = symbols("a", real=True)
      integrand = Jac_det * f.subs({x1: r * cos(theta), x2: r * sin(theta)}).simplify()
      integrand
```

[62]: $r\left(r^2 + r\cos\left(\theta\right) + 1\right)$

The plane integral $\int_B f(x_1, x_2)\,\mathrm{d}(x_1, x_2)$:

```
[63]: int_value = integrate(integrand, (theta, 0, 2* pi), (r,0,a))
      int_value
```

[63]: $\dfrac{\pi a^4}{2} + \pi a^2$

Hence, $\int_A f(x_1, x_2)\,\mathrm{d}(x_1, x_2) = \int_B f(x_1, x_2)\,\mathrm{d}(x_1, x_2)$ happens exactly when $64/3 = \frac{\pi a^4}{2} + \pi a^2$. Thus, we can find the value of $a > 0$ that results in $\int_A f(x_1, x_2)\,\mathrm{d}(x_1, x_2) = \int_B f(x_1, x_2)\,\mathrm{d}(x_1, x_2)$ by:

```
[64]: sols = solve(Eq(int_value, S(64) / 3))
      sols
```

[64]: $$\left[-\frac{\sqrt{3}\sqrt{-3\sqrt{\pi} + \sqrt{3}\sqrt{3\pi + 128}}}{3\sqrt[4]{\pi}}, \ \frac{\sqrt{3}\sqrt{-3\sqrt{\pi} + \sqrt{3}\sqrt{3\pi + 128}}}{3\sqrt[4]{\pi}}\right]$$

```
[65]: sols[0].evalf(),sols[1].evalf()
```

[65]: $(-1.67884978657658, \ 1.67884978657658)$

Hence, since $a > 0$, we conclude that $a = 1.679$.