Miniproject 1
Robot Control

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In this miniproject for Mathematics 2 we use the theory of linear differential equations to model the control of a simple robot. The robot consists of a DC motor connected to a gearbox which translates the rotary motion of the motor to a simple rectilinear motion of an arm with a grip that can move an object along a straight line. The robot is sketched in Figure 1. The figure shows the electrical circuit for the motor together with the mechanical movement of the robot arm.

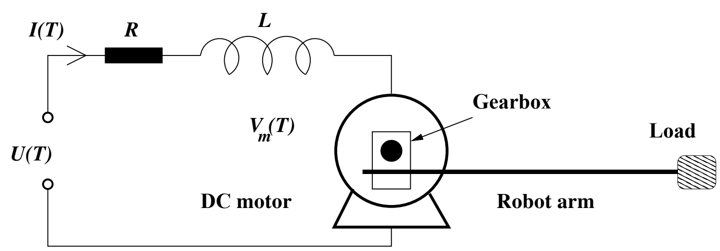

Figure 1: Sketch of a simple robot, consisting of a DC-motor, gearbox and robot arm.

1.1 Mathematical model for a simple robot

The model for the robot shown in Figure 1 is given by a coupled pair of differential equations of the form (see [1]):

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \alpha \frac{\mathrm{d}x}{\mathrm{d}t} = j\,. \tag{1}$$

$$\frac{\mathrm{d}j}{\mathrm{d}t} = -\beta j - \frac{dx}{dt} + u(t)\,, \tag{2}$$

Here $j = j(t)$ denotes the current through the motor's coils and $x = x(t)$ is the position of the load, both given as functions of time $t \in [0, \infty[$. The voltage $u = u(t)$, across the motor, controls the movement of the robot arm. The damping term $\alpha \dot{x}$ accounts for the friction in the motor and the loaded robot arm. The term β describes the electrical resistance in the circuit. We assume that α and β are positive real constants, that are very small compared to 1. It folllows from this that $|\alpha - \beta| < 2$, which we will use later.

Those that are interested in learning about how the model is derived can consult the appendix. The model in (1)-(2) are written in dimensionless variables and parameters. The way to do this is also described in the appendix.

1.2 Miniproject 1, Part 1

1) Eliminate $j(t)$ from the system (1)-(2) and show that

$$\frac{\mathrm{d}^3x}{\mathrm{d}t^3} + (\alpha+\beta)\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + (1+\alpha\,\beta)\frac{\mathrm{d}x}{\mathrm{d}t} = u\,. \tag{3}$$

2) Find the transfer function $H(s)$ for (3) with $u(t) = e^{st}$.

3) With $u(t) = \sin(2\,t)$, and using the transfer function derived in part 2), show that the general real solution to (3) is given in the form:

$$x(t) = c_1 + c_2e^{at}\cos(\omega t) + c_3e^{at}\sin(\omega t) + A\cos(2\,t) + B\sin(2\,t)\,, \tag{4}$$

where $c_1, c_2, c_3 \in \mathbb{R}$. Determine the real parameters a, ω, A and B. (You must find the solution using the methods from the textbook - it is not enough to simply check that the given expression for $x(t)$ is a solution to (3)).

4) Substitute $v(t) = \frac{\mathrm{d}x}{\mathrm{d}t}$ into the system (1)-(2) and rewrite the system as 3 coupled linear differential equations of the same form as in equation (2.35) on page 47 of the textbook (with $y(t) = x(t)$ the solution sought), with the time-dependent vector function $\mathbf{x}(t)$=$(x(t), v(t), j(t))$. Write down the system matrix $\mathbf{A}$, and the vectors $\mathbf{b}$ and $\mathbf{d}$ explicitly.

1.3 Miniproject 1, Part 2

In the following we will introduce a realistic control for the robot. The robot arm is required to steer an object, e.g., a glass, along a straight line from position $x = 0$ at time $t = 0$ to a position $x = \ell$ at time $t = t_1$. It needs to be done gently so that the robot does not destroy the object being moved. The desired trajectory for the object is given by a function $z = z(t)$, i.e., the robot must be controlled with a voltage $u = u(t)$ chosen such that $x(t)$ is as close as possible to $z(t)$ at each time-point t. The function $z(t)$ should, naturally, satisfy $z(0) = 0$ and $z(t_1) = \ell$. Additionally, the start and finish speed is to be zero. Gentle motion means that the acceleration at $t = 0$ and $t = t_1$ should also be zero. This gives 6 conditions that can be fulfilled by a 5th degree polynomial $z(t)$. We will not go deeper into the planning of a trajectory $z(t)$, but will concentrate on a control strategy for $u(t)$. We choose a control strategy of the form:

$$u(t) = g_1 f(t) + g_2 f'(t)\,. \tag{5}$$

where $g_1, g_2 \in \mathbb{R}$ and $f(t) = z(t) - x(t)$ describes the distance by which $z(t)$ leads $x(t)$. Our control strategy means that if $z(t) > x(t)$ then the voltage $u(t)$ increases for positive values of g_1. Thus, the robot seeks to “catch up” with $z(t)$. On the other hand, if $z(t) < x(t)$, the voltage $u(t)$ becomes negative and the robot arm is slowed down. The corresponding statements also apply to the speed and the term g_2.

5) Show that $f(t)$ satisfies the differential equation

$$f''' + (\alpha+\beta)f'' + (1+\alpha\,\beta+g_2)f' + g_1 f = z''' + (\alpha+\beta)z'' + (1+\alpha\,\beta)z'\,. \tag{6}$$

6) Use Routh-Hurwitz’ criterion to determine for which values of g_1 and g_2 the differential equation (6) is asymptotically stable. Sketch the area in the (g_1, g_2)-plane where (6) is asymptotically stable.

In robot control, it is important that the control is asymptotically stable. Otherwise the behaviour of the robot may be erratic, with risk of injury. Therefore, it matters how we choose g_1 and g_2.

1.4 Appendix: Derivation of a mathematical model for a simple robot.

In this appendix we derive the mathematical model (1)-(2) in dimension-free variables for the robot sketched in Figure 1. This model describes the electric circuit for the motor coupled to the mechanical robot arm. In the circuit, L is the inductance in the motor's rotor coils and R is the resistance in the coils and the supply lines. The current through the motor's coils is $I = I(T)$, and $U = U(T)$ is the voltage applied, both given as functions of $T \in [0, \infty[$. The rotor coils plass close by the motor's magnets and this introduces an electromotive force $V_m = V_m(T)$ in the circuit. $\Phi(T)$ measures the rotor's angular motion as a function of time, and the rotation of the motor induces a voltage proportional to the time derivative of the angular velocity (see [1])

$$V_m(T) = K_e \frac{\mathrm{d}\Phi}{\mathrm{d}T}\,, \tag{7}$$

where K_e is a real constant. All parameters are real positive constants. If you have studied electric circuits and electromagnetism, you can easily observe that the voltage over the motor is given by

$$U(T) = RI(T) + L\frac{\mathrm{d}I(T)}{\mathrm{d}T} + V_m(T)\,. \tag{8}$$

We have now modelled the motor, and turn to the mechanical arm and gearbox. The angle between the motor's rotation angle and the position of the robots grab bar $X(T)$ is given by a positive real constant G, such that $\Phi = GX$. The motor provides a torque that we call M and this is proportional to the current I with proportionality constant K_T, which is to say (see [1])

$$M(T) = K_T I(T)\,. \tag{9}$$

Introducing the friction in the robot arm motion proportional to the speed of the arm and the friction in the rotors all in a single term, $B\,\mathrm{d}\Phi/\mathrm{d}T$, we obtain the mechanical model (Newton's Second Law for Rotary Machines):

$$J\frac{\mathrm{d}^2\Phi(T)}{\mathrm{d}T^2} = -B\frac{\mathrm{d}\Phi(T)}{\mathrm{d}T} + M(T)\,. \tag{10}$$

Here J is the moment of inertia, which is the sum of the moment of inertia I_m for the rotor of the engine and the mass m of the robot arm including a possible load, divided by G squared, i.e., $J = I_m + m/G^2$.

The two linear differential equations (8)) and (10) describe respectively the dynamics of the motor's electrical circuit and of the mechanics of the robot arm. The two equations are coupled via the relations (7) and (9). Substituting these coupling terms into (8) together with (10) we obtain ([1])

$$L\frac{\mathrm{d}I}{\mathrm{d}T} = -RI - K_e\frac{\mathrm{d}\Phi}{\mathrm{d}T} + U(T)\,, \tag{11}$$

$$J\frac{\mathrm{d}^2\Phi}{\mathrm{d}T^2} + B\frac{\mathrm{d}\Phi}{\mathrm{d}T} = K_T I\,. \tag{12}$$

The system (11)-(12) consists of two coupled differential equations, one of second order and one of first order.

1.5 Dimension-free formulation

So far we have used capital letters to describe physical variables and parameters with corresponding units in the SI system. For example, the current I has the units $[I] = A$ (Ampere) and T has units $[T] = s$ (seconds). All variables and parameters have units except for the angular velocity Φ, which is dimension-free. We can introduce new variables without units with a linear scaling as follows:

$$T = k_t t\,, \quad I(T) = k_j j(t)\,, \quad \Phi(T) = k_\phi x(t)\,. \tag{13}$$

The scaling factors k_t, k_j and k_ϕ can be chosen freely and we choose them so that the differential equations for the robot's dynamics become as simple as possible. We attach the unit s to k_t in the above scaling, i.e., ($[k_t] = s$) and from this it follows that the time t is dimensionless. The corresponding k_j has units Ampere and the current $j(t)$ is dimensionless. Note that both k_ϕ and $x(t)$ are dimensionless. The first derivative of $I(T)$ with respect to time T is deduced from the chain rule:

$$\frac{\mathrm{d}I}{\mathrm{d}T} = k_j \frac{\mathrm{d}j}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}T} = \frac{k_j}{k_t}\frac{\mathrm{d}j}{\mathrm{d}t} \,. \tag{14}$$

For Φ we obtain

$$\frac{\mathrm{d}\Phi}{\mathrm{d}T} = \frac{k_\phi}{k_t}\frac{\mathrm{d}x}{\mathrm{d}t} \,, \qquad \frac{\mathrm{d}^2\Phi}{\mathrm{d}T^2} = \frac{k_\phi}{k_t^2}\frac{\mathrm{d}^2x}{dt^2} \,. \tag{15}$$

Substituting (13)-(15) into the dynamic equations (11)-(12) we obtain

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \frac{Bk_t}{J}\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{K_T k_j k_t^2}{J k_\phi} j \,, \tag{16}$$

$$\frac{\mathrm{d}j}{\mathrm{d}t} = -\frac{Rk_t}{L} j - \frac{k_e k_\phi}{L k_j}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{k_t}{L k_j} U(T) \,, \tag{17}$$

after dividing by the coefficients of $\mathrm{d}j/\mathrm{d}t$ and $\mathrm{d}^2x/\mathrm{d}t^2$. The scaling coefficients k_t, k_j and k_ϕ can be freely chosen and, to make the system as simple as possible, we attempt to set as many coefficients in the above model equations to 1. If we choose

$$\frac{K_T k_j k_t^2}{J k_\phi} = 1 \,, \qquad \frac{k_e k_\phi}{L k_j} = 1 \,, \tag{18}$$

we obtain

$$k_t = \sqrt{\frac{JL}{K_T K_e}} \,, \qquad \frac{k_\phi}{k_j} = \frac{L}{K_e} \,. \tag{19}$$

Here we can choose k_ϕ freely and then deduce k_j. But one choice of k_ϕ could be $k_\phi = G$. Setting

$$\alpha = \frac{Bk_t}{J} \,, \quad \beta = \frac{Rk_t}{L} \,, \quad u(t) = \frac{k_t}{L k_j} U(T) \,, \tag{20}$$

we obtain the two coupled model equations expressed with dimensionless quantities:

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \alpha\frac{\mathrm{d}x}{\mathrm{d}t} = j \,. \tag{21}$$

$$\frac{\mathrm{d}j}{\mathrm{d}t} = -\beta j - \frac{\mathrm{d}x}{\mathrm{d}t} + u(t) \,, \tag{22}$$

With this scaling we have managed to reduce the number of parameters from five, namely L, R, K_e, J, B, K_T, to only two parameters α andβ. This is a clear simplification of the model, making it much more comprehensible. We deduce in fact that the six original independent parameters really consist of effectively two independent parameters. This means that if we do experiments or simulations by varying, e.g., L and then do experiments or simulations varying J we have not carried out two independent sets of experiments, because the first set of experiments can be transfered to the second set with a simple linear transformation.

A second advantage is that, when implementing a model on a computer, we do not need to worry about units of variables and parameters. Even if we do not have a physical interpretation for the parameters involved, we can nevertheless do meaningful simulations, relevant for practice.

References

[1] A. Friedman, "An adaptive feedforward approach to robot control" in *Mathematics in Industrial Problems, Part 4*, IMA volumes in mathematics and its applications, Springer-Verlag, pp. 186–193, 1991.