appendix_exam_F24

May 13, 2024

[154]:
```
from sympy import *
init_printing()
from dtumathtools import *
```

[155]:
```
x1,x2,x3 = symbols("x1 x2 x3")
x, y, z = symbols("x, y, z")
#r, u, w = symbols("r, u, w")
```

1 Exercise 1

Consider the quadratic form $q : \mathbb{R}^3 \to \mathbb{R}$ given by

$$q(\boldsymbol{x}) = 5x_1^2 + 5x_2^2 + 8x_3^2 + 8x_1x_2 - 4x_1x_3 + 4x_2x_3 - 22x_1 - 32x_2 - 20x_3 + 53,$$

where $\boldsymbol{x} = [x_1, x_2, x_3]^T \in \mathbb{R}^3$.

[156]:
```
xvec = Matrix([x1,x2,x3])
q = 5*x1**2 + 8*x1*x2 - 4*x1*x3 - 22*x1 + 5*x2**2 + 4*x2*x3 - 32*x2 + 8*x3**2 -
↪20*x3 + 53
```

1.1 a

Compute the gradient $\nabla q(\boldsymbol{x})$ for any $\boldsymbol{x} \in \mathbb{R}^3$.

To find the gradient we need to compute the partial derivatives.

[157]:
```
grad = Matrix([q.diff(x1), q.diff(x2), q.diff(x3)]).transpose()
grad
```

[157]:

$$\begin{bmatrix} 10x_1 + 8x_2 - 4x_3 - 22 & 8x_1 + 10x_2 + 4x_3 - 32 & -4x_1 + 4x_2 + 16x_3 - 20 \end{bmatrix}$$

1.2 b

Compute the Hessian matrix $\boldsymbol{H}_q$. *Hint: The Hessian matrix should not depend on* $\boldsymbol{x}$.

The hessian matrix of the given function will look like this, but it is going to be 3x3 matrix as we need to add partial derivatives with respect to the third variable:

$$H_f(x, y) = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix} \tag{1}$$

[158]:
```python
H_q = dtutools.hessian(q)
H_q
```

[158]:

$$\begin{bmatrix} 10 & 8 & -4 \\ 8 & 10 & 4 \\ -4 & 4 & 16 \end{bmatrix}$$

1.3 c

Find an orthonormal basis of eigenvectors for the Hessian matrix $\boldsymbol{H}_q$.

According to the spectral theorem if matrix H_q is symmetric(it is), a real orthogonal matrix Q exists such that $Q^T AQ = D$, where D is a diagonal matrix of eigenvalues.

[159]:
```python
H_q.diagonalize()
```

[159]:

$$\left(\begin{bmatrix} 2 & 1 & -1 \\ -2 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 18 & 0 \\ 0 & 0 & 18 \end{bmatrix} \right)$$

We can see that $\lambda = 18$ has multiplicity of 2, so we need to check if the eigenvectors are orthogonal and normalize them. Also eigenvectors that belong to different eigenvalues are already orthonormal.

[160]:
```python
v1 = Matrix([2, -2, 1])
v2 = Matrix([1, 1, 0])
v3 = Matrix([-1, 0, 2])
```

[161]:
```python
v2.dot(v3)
```

[161]:

$$-1$$

As we now know they are not orthogonal we can apply gram-schmidt process to orthonormalize them.

[162]:
```python
v3, v2 = GramSchmidt([v3,v2], orthonormal=True)
v3, v2
```

[162]:

$$\left(\begin{bmatrix} -\frac{\sqrt{5}}{5} \\ 0 \\ \frac{2\sqrt{5}}{5} \end{bmatrix}, \begin{bmatrix} \frac{4\sqrt{5}}{15} \\ \frac{\sqrt{5}}{3} \\ \frac{2\sqrt{5}}{15} \end{bmatrix} \right)$$

And we also need no normalize the first vector.

[163]:
```python
v1 = v1/v1.norm()
```

[164]:
```python
Q = Matrix.hstack(v1, v2, v3)
Q
```

[164]:

$$\begin{bmatrix} \frac{2}{3} & \frac{4\sqrt{5}}{15} & -\frac{\sqrt{5}}{5} \\ -\frac{2}{3} & \frac{\sqrt{5}}{3} & 0 \\ \frac{1}{3} & \frac{2\sqrt{5}}{15} & \frac{2\sqrt{5}}{5} \end{bmatrix}$$

So, the Q is the orthonormal basis of eigenvectors for the Hessian matrix $\boldsymbol{H}_q$.

1.4 d

Show that $(1, 2, 1)$ is a stationary point of q. Find all stationary points of q.

To find stationary points of q we need to equate gradient to 0.

[165]:
```
eq1 = Eq(grad[0],0)
eq2 = Eq(grad[1],0)
eq3 = Eq(grad[2],0)
```

[166]:
```
solve([eq1, eq2, eq3], x1, x2, x3)
```

[166]:

$$\{x_1 : 2x_3 - 1,\ x_2 : 4 - 2x_3\}$$

We can write it as

$$\begin{bmatrix} x1 \\ x2 \\ x3 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} t + \begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix} \tag{2}$$

, where $t \in \mathbb{R}$

1.5 e

State a direction from the stationary point $(1, 2, 1)$ in which the function q neither increases nor decreases (i.e., a direction where the function is constant).

To find this direction we need find a unit vector such that the dot product of this vector and the gradient vector at the given point is 0

[167]:
```
grad.subs({x1: 1, x2: 2, x3: 1})
```

[167]:

$$\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$$

As the gradient itself equals to 0 we can state that the point $(1, 2, 1)$ is stationary point and the function at this point is neither increasing nor decreasing in any direction.

1.6 f

We now consider the gradient method with learning rate $\alpha = 0.02$ and initial guess $\boldsymbol{x}_0$ given by:

$$\boldsymbol{x}_0 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} -\frac{2}{3} \\ -\frac{1}{3} \\ \frac{2}{3} \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}.$$

Compute $\boldsymbol{x}_{10}$, where

$$\boldsymbol{x}_{n+1} = \boldsymbol{x}_n - \alpha \nabla q(\boldsymbol{x}_n), \quad n = 0.1, 2, \ldots.$$

You should state $\boldsymbol{x}_{10}$ as a vector of *decimal numbers* with an appropriate number of decimals. Which point does the gradient method converge to? (you do not have to provide a proof, just a qualified guess based on your computations).

[168]:
```
x_0 = Matrix([-1, 1, 3])
for i in range(10):
    if i == 0:
        x_a = x_0
    x_b = x_a - 0.02*grad.subs({x1: x_a[0], x2: x_a[1], x3: x_a[2]}).transpose()
    x_a = x_b
display(x_b)
```

$$\begin{bmatrix} 0.976941569907863 \\ 1.98847078495393 \\ 1.02305843009214 \end{bmatrix}$$

The gradient converges to the point $(1, 2, 1)$.

2 Exercise 2

Consider the quadratic form $q : \mathbb{R}^4 \to \mathbb{R}$ given by

$$q(\boldsymbol{x}) = 2x_1x_3 + 4x_2x_4,$$

where $\boldsymbol{x} = [x_1, x_2, x_3, x_4]^T \in \mathbb{R}^4$.

[169]:
```
x1,x2,x3,x4 = symbols("x_1 x_2 x_3 x_4")
q =  2*x1*x3 + 4*x2*x4
```

2.1 a

State a *symmetric* matrix $A \in \mathbb{R}^{4\times 4}$ such that $q(\boldsymbol{x}) = \boldsymbol{x}^T A\boldsymbol{x}$, where $\boldsymbol{x} = [x_1, x_2, x_3, x_4]^T$.

[170]:
```
# Define the matrix A
A = Matrix([[0,0,1,0],[0,0,0,2],[1,0,0,0],[0,2,0,0]])

# Define a vector of the variables
var = Matrix([x1, x2, x3, x4])

# Display all the matrix A and the quadratic equation
display(A, var.transpose()*A*var)
```

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 2x_1x_3 + 4x_2x_4 \end{bmatrix}$$

2.2 b

Find an orthogonal (change-of-basis) matrix $Q \in \mathbb{R}^{4\times 4}$ that "reduces" the quadratic form q in the sense that q, in the new coordinates, does not contain "mixed terms" of the form x_ix_j (where $i \neq j$). Express q in the new coordinates.

In orded to achive reduced quadratic form we need to find orthonormal basis $\beta = u_1, u_2, u_3, u_4$ for $\underline{\underline{A}}$ such that

$$
\begin{aligned}
&(\underline{\underline{U}} * \underline{\tilde{x}})^T * \underline{\underline{A}} * (\underline{\underline{U}} * \underline{\tilde{x}}) + (\underline{\underline{U}} * \underline{\tilde{x}})^T * \underline{b} + c = 0 \\
&\underline{\tilde{x}}^T * \underline{\underline{U}}^T * \underline{\underline{A}} * \underline{\underline{U}} * \underline{\tilde{x}} + \underline{\tilde{x}}^T * \underline{\underline{U}}^T * \underline{b} + c = 0 \\
&\underline{\tilde{x}}^T * \underline{\underline{D}} * \underline{\tilde{x}} + \underline{\tilde{x}}^T * \underline{\underline{U}}^T * \underline{b} + c = 0,
\end{aligned}
$$

where $\underline{\underline{U}} = \begin{bmatrix} u_1 & u_2 & u_3 & u_4 \end{bmatrix}$ and $\underline{\underline{D}}$ is a diagonal matrix of corresponding eigenvalues of $\underline{\underline{U}}$.

[171]:
```
#Find the eigenvectors and eigenvalues of A
eigens = A.eigenvects() #returns list of tuples with eigenvalue, its␣
↪multiplicity and eigenvectors

eigens
```

[171]:

$$\left[\left(-2, \ 1, \ \left[\left[\begin{matrix}0\\-1\\0\\1\end{matrix}\right]\right]\right), \ \left(-1, \ 1, \ \left[\left[\begin{matrix}-1\\0\\1\\0\end{matrix}\right]\right]\right), \ \left(1, \ 1, \ \left[\left[\begin{matrix}1\\0\\1\\0\end{matrix}\right]\right]\right), \ \left(2, \ 1, \ \left[\left[\begin{matrix}0\\1\\0\\1\end{matrix}\right]\right]\right)\right]$$

[172]:
```
lamda1 = eigens[0][0] #first eigenvalue
lamda2 = eigens[1][0] #second eigenvalue
lamda3 = eigens[2][0] #third eigenvalue
lamda4 = eigens[3][0] #forth eigenvalue

u1 = eigens[0][2][0] #first eigenvector lamda1
u2 = eigens[1][2][0] #second eigenvector lamda2
u3 = eigens[2][2][0] #third eigenvector lamda3
u4 = eigens[3][2][0] #forth eigenvector lamda4
```

As $\underline{\underline{A}}$ is symmetric then 1) Diagonalization is possible. 2) The different E_λ(eigenvectors belonging to different λ) are orthogonal.

As all of 4 eigenvalues have only one corresponding eigenvector the eigenvectors are already orthogonal, and we only need to normalize them.

[173]:
```
u1 = u1/u1.norm()
u2 = u2/u2.norm()
u3 = u3/u3.norm()
u4 = u4/u4.norm()

Q = Matrix([u1.T, u2.T, u3.T, u4.T]).T
D = diag(lamda1, lamda2, lamda3, lamda4)
```

The matrix Q is

[174]:
```
Q
```

[174]:

$$\begin{bmatrix} 0 & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & 0 & 0 & \frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & 0 & 0 & \frac{\sqrt{2}}{2} \end{bmatrix}$$

The q expressed in new coordinates is

[175]:
```
q_new = var.transpose()*D*var
q_new
```

[175]: $\left[-2x_1^2 - x_2^2 + x_3^2 + 2x_4^2\right]$

We now consider q restricted to the set

$$B = \{\boldsymbol{x} \in \mathbb{R}^4 \mid x_1^2 + x_2^2 + x_3^2 + x_4^2 \leq 1\}.$$

2.3 c

Explain why the function $q : B \to \mathbb{R}$ has a minimal and maximal value.

As we can see from the previos question the q can take arbitrary large and small values, we can see it by setting certaine variables to 0 and playing around with others. As the function is now bounded to B we can definatelly say that it will have maximum and minimum values.

Also according to theorem 5.2.1, as B is closed and bounded set and f is continuous, f has global max and min on the boundary.

2.4 d

Determine the minimum value and the maximum value of $q : B \to \mathbb{R}$.

The first step would be finding stationary points by equation gradient to 0

[176]:
```
grad = Matrix([q.diff(x1), q.diff(x2), q.diff(x3), q.diff(x4)])
grad
```

[176]:
$$\begin{bmatrix} 2x_3 \\ 4x_4 \\ 2x_1 \\ 4x_2 \end{bmatrix}$$

So the stationary point is $(0, 0, 0, 0)$

We now can asses if this point is local maxim or min, or saddle point

[177]:
```
dtutools.hessian(q).diagonalize()
```

[177]:
$$\left(\begin{bmatrix} 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -4 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} \right)$$

It is the saddle point, because the eigenvalues have different signs.

The second step is boundary investigation. To do it we need find the parametrization for our boundary and then find the points where our function intercepts this boundary. And by substitution of those values we can find the maximum and the minimum points.

3 Exercise 3

Let the function $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

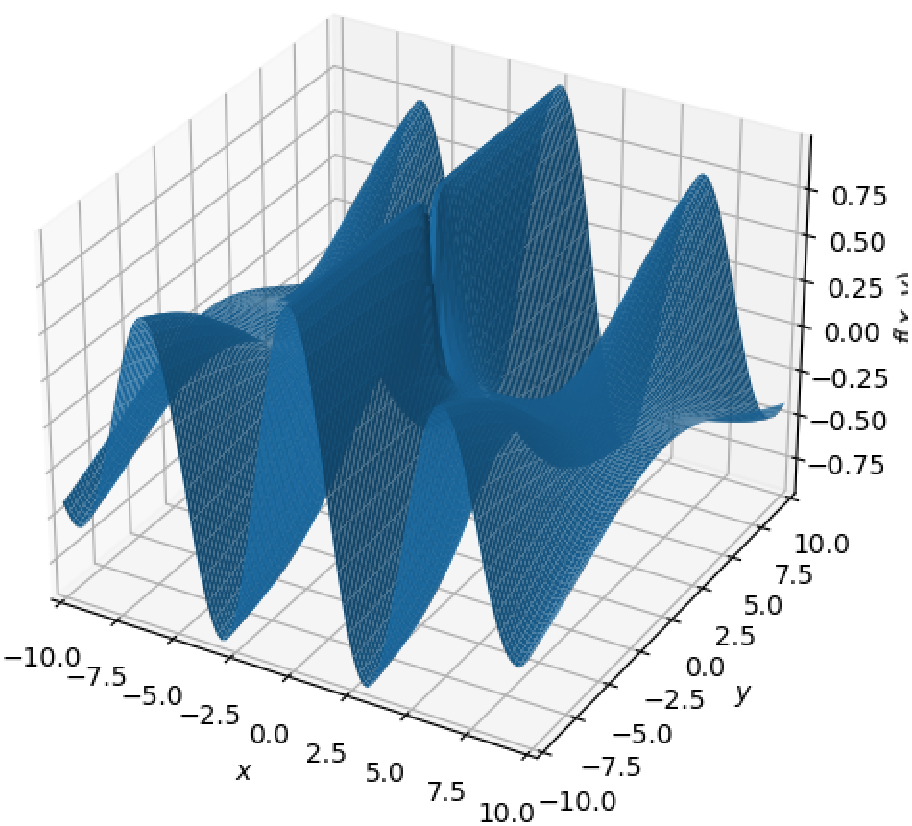
$$f(x,y) = \begin{cases} \frac{y^2 \cos(x)}{x^2+y^2} & \text{for } (x,y) \in \mathbb{R}^2 \quad \{(0,0)\}, \\ 0 & \text{for } (x,y) = (0,0). \end{cases}$$

[178]:
```
x,y = symbols("x y", real=True)
f = y**2 * cos(x) / (x**2 + y**2)
```

3.1 a

Plot the graph of the function f.

[179]:
```
dtuplot.plot3d(f, (x, -10, 10), (y, -10, 10))
```



[179]: <spb.backends.matplotlib.matplotlib.MatplotlibBackend at 0x7b76b394a830>

3.2 b

Compute the two first-order partial derivatives of f for $(x, y) \in \mathbb{R}^2 \quad \{(0,0)\}$.

[180]:
```
df_dx = f.diff(x).simplify()
df_dy = f.diff(y).simplify()
display(df_dx, df_dy)
```

$$\frac{y^2\left(-2x\cos(x) + \left(-x^2 - y^2\right)\sin(x)\right)}{\left(x^2 + y^2\right)^2}$$

$$\frac{2x^2y\cos(x)}{\left(x^2 + y^2\right)^2}$$

3.3 c

Find the degree-two Taylor polynomial $P_2(x)$ of $\cos(x)$ from the expansion point $x_0 = 0$.

[181]:
```
tay_pol = series(cos(x), x, 0, 3).removeO()
display(tay_pol)
```

$$1 - \frac{x^2}{2}$$

3.4 d

Show that

$$\lim_{x\to 0} f(x, x) = \frac{1}{2}.$$

To find the limit we can use following:

one can say that $cos(x) = P_2(x) + R_2(x) = P_2(x) + \epsilon(x)x^2$. An ϵ is a category of function that fulfill: 1) $\epsilon(0) = 0$ 2) $\epsilon(x) \to 0$

So, using the result from the previous question we can write

$$\lim_{x\to 0} f(x, x) = \lim_{x\to 0} \frac{x^2(1 - \frac{x^2}{2} + \epsilon(x)x^2)}{x^2 + x^2} = \frac{1}{2}(1 - x^2 + \epsilon(x)x^2)) \tag{3}$$

As x goes to 0, we can conclude:

$$\lim_{x\to 0} f(x, x) = \frac{1}{2}(1) = \frac{1}{2} \tag{4}$$

3.5 e

Determine the limit

$$\lim_{x\to 0} f(x, 2x).$$

Using the same logic,

$$\lim_{x\to 0} f(x,2x) = \lim_{x\to 0} \frac{4x^2(1-\frac{x^2}{2}+\epsilon(x)x^2)}{x^2+4x^2} = \lim_{x\to 0} \frac{4x^2(1-\frac{x^2}{2}+\epsilon(x)x^2)}{5x^2} = \lim_{x\to 0} \frac{4(1-\frac{x^2}{2}+\epsilon(x)x^2)}{5} = \frac{4}{5} \tag{5}$$

3.6 f

Argue that f is differentiable on $\mathbb{R}^2 \ \{(0,0)\}$, but not at $(0,0)$.

It is self evident that both partial derivatives of the function f is defined on the $\mathbb{R}^2 \ \{(0,0)\}$, but at the point $(0,0)$ both partial derivatives are not defined because of the denominator. So, according to the Theorem 3.6.2 the function f is not defferentiable at this point.

4 Exercise 4

Consider the vector field $\boldsymbol{V} : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$\boldsymbol{V}(x,y,z) = (-x, xy^2, x+z)$$

and the curve $\mathcal{K}_1$ given by the parametrization:

$$\boldsymbol{r}(u) = (u, u^2, u+1), \quad u \in [0,2].$$

Thus, $\mathcal{K}_1 = \mathrm{im}(\boldsymbol{r})$.

[182]:
```
x, y, z, u, t = symbols("x y z u t", real=True)
V = Matrix([-x, x*y**2, x+z])
r = Matrix([u, u**2, u +1])
```

4.1 a

Find the tangent vector $\boldsymbol{r}'(u)$ and argue that the parameterization is regular.

[183]:
```
r_t = r.diff(u)
r_t
```

[183]:

$$\begin{bmatrix} 1 \\ 2u \\ 1 \end{bmatrix}$$

According to the Definition 3.1.4 if $\mathbf{r}'(t) \neq \mathbf{0}$ the parametrization is regular.

So, it is evident that this parametrization is regular.

4.2 b

Calculate $\langle \boldsymbol{V}(\boldsymbol{r}(u)), \boldsymbol{r}'(u)\rangle$ for any $u \in [0,2]$ and compute the line integral $\int_{\mathcal{K}_1} \boldsymbol{V} \cdot \mathrm{d}\boldsymbol{s}$.

According to the Definition 7.4.1 the dot product $\langle \boldsymbol{V}(\boldsymbol{r}(u)), \boldsymbol{r}'(u)\rangle$ is the integrand itself. To compute the line integral $\int_{\mathcal{K}_1} \boldsymbol{V} \cdot \mathrm{d}\boldsymbol{s}$ we need integrate the dot product with respect to u.

[184]:
```
# The dot product
integrand = V.subs({x: r[0], y: r[1], z: r[2]}).dot(r_t)

# Integration
int_V = integrate(integrand, (u, 0, 2))

display(integrand, int_V.evalf())
```

$2u^6 + u + 1$

40.5714285714286

4.3 c

Find a parametrization $\boldsymbol{p} : [0,1] \to \mathbb{R}^3$ of the straight line from $(0,0,1)$ to $(2,4,3)$. We denote the line segment by $\mathcal{K}_2 = \mathrm{im}(\boldsymbol{p})$. Compute the line integral $\int_{\mathcal{K}_2} \boldsymbol{V} \cdot \mathrm{d}\boldsymbol{s}$.

First we parametrize the line. We can do it by thinking of the line in terms of the vector $\begin{bmatrix}2\\4\\3\end{bmatrix} - \begin{bmatrix}0\\0\\1\end{bmatrix}$, which we can multiply by the coefficient u, $u \in [0,1]$, the only step we are missing is the starting point, which can be achieved by adding $\begin{bmatrix}0\\0\\1\end{bmatrix}$ to our vector.

$$p = \begin{bmatrix}2u\\4u\\2u+1\end{bmatrix} \tag{6}$$

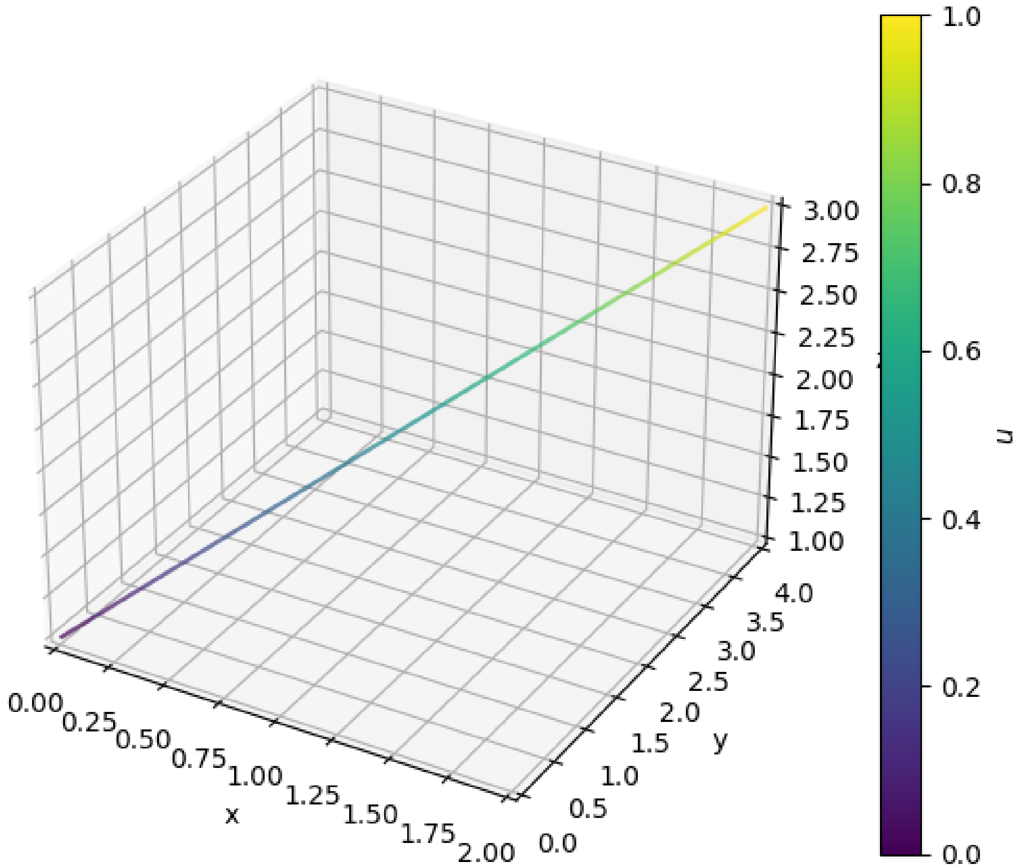
[185]:
```
p = Matrix([u*2, u*4, u*2 + 1])
p
```

[185]:

$$\begin{bmatrix}2u\\4u\\2u+1\end{bmatrix}$$

We can varify it by plotting

[186]:
```
dtuplot.plot3d_parametric_line(*p,(u,0,1))
```



[186]:
```
<spb.backends.matplotlib.matplotlib.MatplotlibBackend at 0x7b7710591870>
```

The line integral can be computed in the same manner:

[187]:
```python
#The tangential vector
p_dif = p.diff(u)

#dot product of the vector field and the tangential vector
integrand = V.subs({x: p[0], y: p[1], z: p[2]}).dot(p_dif)

integrate(integrand, (u, 0, 1))
```

[187]: 36

4.4 d

Determine if $\boldsymbol{V}$ is a gradient field.

To determine if the given gradient field is if fact vector field we can use one of the requirements for the vector field to be a gradient field: the Jacobian matrix must be symmetric.

[188]:
```
V.jacobian([x,y,z])
```

[188]:

$$\begin{bmatrix} -1 & 0 & 0 \\ y^2 & 2xy & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

As the Jacobian matrix is not symmetic we can conclude that the given vector field is not gradient field.

5 Exercise 5

Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x_1, x_2) = x_1^2 + x_2^2 + x_1 + 1$$

and the subset $A \subset \mathbb{R}^2$ given by:

$$A = \{(x_1, x_2) \in \mathbb{R}^2 \mid -2 \leq x_1 \leq 2 \wedge -1 \leq x_2 \leq 1\}.$$

[189]:
```
x1, x2, r, theta, a = symbols("x_1 x_2 r theta a", real=True)
f = x1**2 + x2**2 + x1 + 1
```

5.1 a

Compute the integral $\int_A f(x_1, x_2)\,\mathrm{d}(x_1, x_2)$.

As the given subset has the form of ractangle, we can integrate it without parametrization.

[190]:
```
vol = integrate(f, (x1, -2, 2), (x2, -1, 1)).evalf()
display(vol)
```

21.3333333333333

5.2 b

Determine the volume of the set

$$\{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid (x_1, x_2) \in A \wedge 0 \leq x_3 \leq f(x_1, x_2)\}.$$

In previous exercise we have already found th volume of the given set: we integrated over the surfase given by A and for each tiny area we multiplied by the height given by the function f.

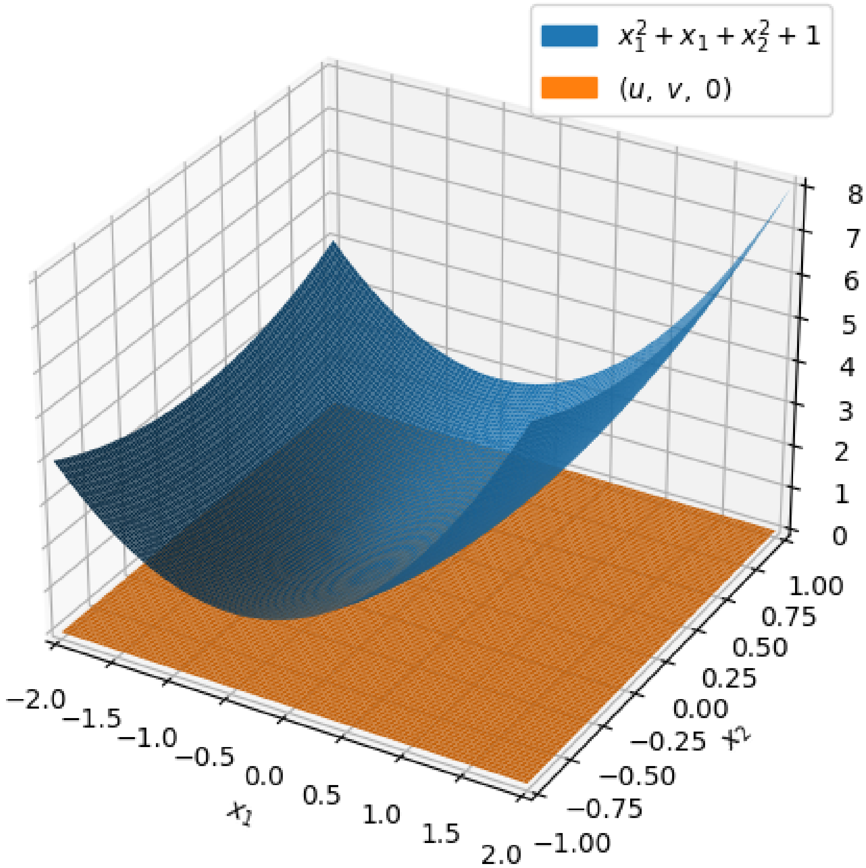
The bottom surface A and the top surface given by the function f is visualized below.

[191]:
```
u, v = symbols("u, v")
p = Matrix([u, v, 0])
p
```

[191]:

$$\begin{bmatrix} u \\ v \\ 0 \end{bmatrix}$$

[192]:
```
plt = dtuplot.plot3d(f, (x1, -2, 2), (x2, -1, 1), show=False)
plt.extend(dtuplot.plot3d_parametric_surface(*p, (u, -2, 2), (v, -1,1),␣
 ↪show=False))
plt.show()
```



Let $a > 0$. Let $B \subset \mathbb{R}^2$ denote the circular disk with center at the origin and radius a:

$$B = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 \leq a^2\}.$$

5.3 c

Specify a parametrization of the circular disc B and find the associated Jacobian determinant. *Hint: Polar coordinates.*

As it is a surface in 2d the Jacobian determinant will be the absolute value of the determinant of the jacobian matrix.

[193]:
```
a = symbols("a", nonnegative=True)
p = Matrix([r*cos(w), r*sin(w)])
p_jac = abs(p.jacobian([r,w]).det()).simplify()
display(p_jac)
```

$|r|$

5.4 d

Determine the value of a to 3 decimal places such that

$$\int_A f(x_1, x_2)\,\mathrm{d}(x_1, x_2) = \int_B f(x_1, x_2)\,\mathrm{d}(x_1, x_2).$$

[194]:
```
integrand = f.subs({x1: p[0], x2: p[1]}) * p_jac
val = integrate(integrand, (r, 0, a), (w, 0, 2*pi))
display(val)
```

$$\frac{\pi a^4}{2} + \pi a^2$$

[195]:
```
solve(Eq(val, vol), a)[0]
```

[195]: 1.67884978657658

$a = 1.679$