Written exam, E23, 01003 shsp

5/12-23

Q1

a)

P	R	Q	$P \Rightarrow R$	$(P \Rightarrow R) \Leftrightarrow Q$
T	T	T	T	T
T	T	F	T	F
T	F	T	F	F
T	F	F	F	T
F	T	T	T	T
F	T	F	T	F
F	F	T	T	T
F	F	F	T	F

b) continuing the truth table for $R \Rightarrow P$ gives:

...	$R \Rightarrow P$	$(R \Rightarrow P) \Leftrightarrow Q$
	T	T
	T	F
	T	T
	T	F
	F	F
	F	T
	T	T
	T	F

we see that $(R \Rightarrow P) \Leftrightarrow Q$ and $(P \Rightarrow R) \Leftrightarrow Q$ are NOT logically equivalent since they have different truth table

Q2

a) $$e^{i\pi/2} \cdot (2+i) = (\cos(\tfrac{\pi}{2}) + i \sin(\tfrac{\pi}{2}))(2+i)$$

using Euler's formula

$$= i(2+i) = \underline{\underline{-1+2i}}$$

b) $$\left(\frac{\sqrt{2}}{2} + i \frac{\sqrt{2}}{2}\right)^4 = \left(\cos(\tfrac{\pi}{4}) + i \sin(\tfrac{\pi}{4})\right)^4$$

$$= \left(e^{i\frac{\pi}{4}}\right)^4 = e^{4i\pi/4} = e^{i\pi} = \cos(\pi) + i \sin(\pi) = \underline{\underline{-1}}$$

Q3 Given for all $n \in \mathbb{Z}_{\geq 2}$:

$$S_n = \sum_{k=2}^{n} \frac{k}{2}.$$

a) $S_2 = \sum_{k=2}^{2} \frac{k}{2} = \frac{2}{2} = 1$

$S_3 = \sum_{k=2}^{3} \frac{k}{2} = \frac{2}{2} + \frac{3}{2} = \frac{5}{2}$

$S_4 = \sum_{k=2}^{4} \frac{k}{2} = \frac{2}{2} + \frac{3}{2} + \frac{4}{2} = \frac{9}{2}$

b) Claim: $S_n = \frac{n^2+n-2}{4}$ for all $n \in \mathbb{Z}_{\geq 2}$.

showing using induction on n:

Base case: For $n=2$ we have:
$S_2 = \frac{2^2+2-2}{4} = \frac{4}{4} = 1$, so the expression is true for $n=2$.

Induction step:
Assuming true for $n-1$ for $n \in \mathbb{Z}_{>2}$, so:
$$S_{n-1} = \frac{(n-1)^2 + (n-1) - 2}{4}$$

Rewriting:
$$S_n = \sum_{k=2}^{n} \frac{k}{2} = \sum_{k=2}^{n-1} \frac{k}{2} + \frac{n}{2} = S_{n-1} + \frac{n}{2}$$
$$= \frac{(n-1)^2 + (n-1) - 2}{4} + \frac{n}{2}$$
$$= \frac{n^2 + \cancel{1} - 2n + n - \cancel{1} - 2}{4} + \frac{2n}{4}$$
$$= \frac{n^2 + n - 2}{4}$$

We here see that if the expression is true for $n-1$, it is also true for n, for any $n \in \mathbb{Z}_{>}$.
Along with the base step, we thus have that the expression is true for all $n \in \mathbb{Z}_{\geq 2}$

Q4 Given $\underline{\underline{A}} = \begin{bmatrix} 1 & -2 \\ -3 & 7 \end{bmatrix}$

a) $\underline{\underline{A}}$ is invertible if $\det(\underline{\underline{A}}) \neq 0$.

$\det(\underline{\underline{A}}) = 1 \cdot 7 - (-3)\cdot(-2) = 7 - 6 = 1 \neq 0$, so $\underline{\underline{A}}$ is invertible.

Finding the inverse:

$$[\underline{\underline{A}} \mid \underline{\underline{I}}_2] = \left[\begin{array}{cc|cc} 1 & -2 & 1 & 0 \\ -3 & 7 & 0 & 1 \end{array}\right] \xrightarrow{R_2 : R_2 + 3 \cdot R_1} \left[\begin{array}{cc|cc} 1 & -2 & 1 & 0 \\ 0 & 1 & 3 & 1 \end{array}\right]$$

$$\xrightarrow{R_1 : R_1 + 2 \cdot R_2} \left[\begin{array}{cc|cc} 1 & 0 & 7 & 2 \\ 0 & 1 & 3 & 1 \end{array}\right], \quad \text{so } \underline{\underline{A}}^{-1} = \begin{bmatrix} 7 & 2 \\ 3 & 1 \end{bmatrix}$$

b) $\underline{\underline{B}} \in \mathbb{R}^{n \times n}$ is an invertible square matrix for an $n \in \mathbb{N}$.

Assuming $\lambda = 0$ is an eigenvalue of $\underline{\underline{B}}$, then a corresponding eigenvector $\underline{v}$ must exist such that

$$\underline{\underline{B}} \cdot \underline{v} = 0 \cdot \underline{v} = \underline{0}, \quad \text{where} \quad \underline{v} \neq \underline{0}.$$

Since an eigenvector is a proper (non-zero) vector, then not only the zero-vector is a solution, and thus $\underline{\underline{B}}$ does not have full rank, so $\rho(\underline{\underline{B}}) \neq 2$. According to corollary 7.25 a matrix is invertible if and only if it has full rank, $\rho(\underline{\underline{B}}) = 2$. Hence, we conclude that $\underline{\underline{B}}$ does not have an eigenvalue of 0.

Alternatively: Since $\underline{\underline{B}}$ is invertible, then $\underline{\underline{B}}^{-1}$ exists, and then

$$\underline{\underline{B}}\underline{v} = \underline{0} \Leftrightarrow \underline{\underline{B}}^{-1}\underline{\underline{B}}\underline{v} = \underline{\underline{B}}^{-1}\underline{0} \Leftrightarrow \underline{v} = \underline{0}$$

This is a contradiction since $\underline{v}$ must be non-zero. Hence, 0 is not an eigenvalue of B.

Q5 V is the subspace in $\mathbb{R}(Z)$ consisting of polynomials of up to degree 2.

An ordered basis γ is chosen for V:

$$\gamma = (1+2Z,\ 2+Z-Z^2,\ Z^2)$$

A linear map $L: V \to V$ is given by:

$$_\gamma[L]_\gamma = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

a) The mapping matrix consists of the images of the γ-basis vectors as columns. We thus see from the first two columns, that the first and second γ-basis vectors, $1+2Z$ and $2+Z-Z^2$, map to the 0-polynomial and hence belong to the kernel of L, $\ker(L)$. The third γ-basis vector, Z^2, does not.

According to Theorem 10.31

For the polynomial $1+2Z+Z^2$, which has the γ-coordinate vector: $[1+2Z+Z^2]_\gamma = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, we see:

$$_\gamma[L]_\gamma \cdot [1+2Z+Z^2]_\gamma = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

So, it does not belong to the kernel of L.

b) We have $\dim(V)=3$. According to the rank-nullity theorem, $\dim(V) = \dim(\ker(L)) + \dim(\mathrm{image}(L))$.

Since the first two γ-basis vectors, which are linearly independent, belong to $\ker(L)$, then $\dim(\ker(L))$ is at least 2. Since we from a) have a polynomial with a non-zero image, then $\dim(\mathrm{image}(L))$ is at least 1.

We conclude that $\dim(\ker(L))=2$ and $\dim(\mathrm{image}(L))=1$ so that the rank-nullity theorem is fulfilled.

A basis for the kernel must thus consist of two polynomials, which can be:

$$(1 + 2z \, , \, 2 + z - z^2)$$

A basis for the image space consists of one polynomial, which can be:

$$(1 + 2z + z^2)$$

Q 6 Given real system of differential equations:

$$\begin{bmatrix} f_1'(t) \\ f_2'(t) \end{bmatrix} = \underbrace{\begin{bmatrix} 2 & 1 \\ 5 & -2 \end{bmatrix}}_{\underline{\underline{A}}} \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix}$$

a)

We can identify a coefficient matrix $\underline{\underline{A}}$ but no forcing functions, so $\underline{q}(t) = \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. According to Definition 12.10 the system is homogeneous.

b) Eigenvalues found by solving the characteristic equation:

$$\det(\underline{\underline{A}} - \lambda \cdot \underline{\underline{I}}_2) = \det\left(\begin{bmatrix} 2-\lambda & 1 \\ 5 & -2-\lambda \end{bmatrix}\right) = (2-\lambda)(-2-\lambda) - 5 \cdot 1$$

$$= \lambda^2 - 4 - 5 = \lambda^2 - 9 = 0 \quad \Leftrightarrow \quad \lambda^2 = 9 \quad \Leftrightarrow \quad \lambda = \begin{cases} 3 \\ -3 \end{cases}$$

Corresponding eigenvectors:

For $\lambda = 3$: $\left[\begin{array}{cc|c} 2-3 & 1 & 0 \\ 5 & -2-3 & 0 \end{array}\right] = \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 5 & -5 & 0 \end{array}\right] \xrightarrow{R_2 : R_2 + 5R_1} \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$\xrightarrow{R_1 : (-1) \cdot R_1} \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \Leftrightarrow v_1 = v_2 \Leftrightarrow \underline{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} t, \; t \in \mathbb{R}$

where $gm(3) = 1$

For $\lambda = -3$: $\left[\begin{array}{cc|c} 2+3 & 1 & 0 \\ 5 & -2+3 & 0 \end{array}\right] = \left[\begin{array}{cc|c} 5 & 1 & 0 \\ 5 & 1 & 0 \end{array}\right] \xrightarrow{R_2 : R_2 - R_1} \left[\begin{array}{cc|c} 5 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$\xrightarrow{R_1 : \frac{1}{5} \cdot R_1} \left[\begin{array}{cc|c} 1 & \frac{1}{5} & 0 \\ 0 & 0 & 0 \end{array}\right] \Leftrightarrow v_1 = -\frac{1}{5} v_2 \Leftrightarrow \underline{v} = \begin{bmatrix} -1/5 \\ 1 \end{bmatrix} t, \; t \in \mathbb{R}$

where $gm(-3) = 1$

Since $gm(3) = am(3) = 1$ and $gm(-3) = am(-3) = 1$, then an eigenbasis exists, and according to Theorem 12.16 the general real solution is:

$$\begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix} = c_1 e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} -1/5 \\ 1 \end{bmatrix}, \; c_1, c_2 \in \mathbb{R}.$$

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