Technical University of Denmark

3-hour written exam, December 08, 2024
Course: Advanced Engineering Mathematics 2 01034 / 01035

Allowed aids: All aids allowed by DTU.
Weighting: Multiple-choice(administrated electronically): 55%,
Part B: Problem 1: 15%, Problem 2: 15% and Problem 3: 15%.

The weighting is only a guide. The exam is evaluated as a whole. In order to receive full points in Part B, all answers must be substantiated, if necessary with reference to the text book, and a reasonable number of intermediate steps must be included.

The exam consists of 2 parts: an electronic multiple choice section (**Part A**) and this (**Part B**).

- **Part A must be answered electronically.**.
- **Part B follows below, and can be handed in either electronically or on paper.**

Problem 1

A function $f : \mathbb{R} \to \mathbb{R}$ has the following properties: it is an odd function, it is 2π-periodic, and on the interval $[0, \pi]$ the function is given by

$$f(x) = \begin{cases} \sin(\frac{3}{2}x) & \text{for } x \in \left[0, \frac{2\pi}{3}\right] \\ 0 & \text{for } x \in \left]\frac{2\pi}{3}, \pi\right] \end{cases} .$$

1) Sketch the function f on the interval $[-2\pi, 2\pi]$.
2) Argue that the Fourier series of f converges uniformly to f.
3) Find the Fourier series for f.

Hint: It can be used that for every $n \in \mathbb{N}$ and $a \in \mathbb{R}$ which is not integer,

$$\int \sin(ax)\sin(nx)dx = \frac{\sin((a-n)x)}{2(a-n)} - \frac{\sin((a+n)x)}{2(a+n)} .$$

The problem set continues!

Problem 2

Consider the system of differential equations given by

$$\frac{d\mathbf{x}(t)}{dt} = \begin{pmatrix} -1 & 1 \\ 0 & -2 \end{pmatrix} \mathbf{x}(t) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \sin(3t) \ , \tag{1}$$

$$y(t) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}^T \mathbf{x}(t) \ . \tag{2}$$

Moreover, consider the function

$$\mathbf{x}_p(t) = \frac{1}{10} \begin{pmatrix} -3\cos(3t) + \sin(3t) \\ 0 \end{pmatrix} .$$

1) Show that $\mathbf{x}_p(t)$ is a solution to the inhomogeneous system (1).
2) Determine the complete solution to the homogeneous system associated with (1).
3) Determine $y(t)$ in (2), when $\mathbf{x}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

Problem 3

For $t > 0$ consider the inhomogeneous differential equation

$$t\frac{dy}{dt} - y = \frac{t^2}{1+t} \ . \tag{3}$$

Assume that the differential equation (3), has a solution that can be written as a power series, $y(t) = \sum_{n=0}^{\infty} a_n t^n$, with radius of convergence $\rho > 0$.

1) Show that the right-hand side of (3), that is the function $\frac{t^2}{1+t}$, has the power series representation $\sum_{n=2}^{\infty} (-1)^n t^n$ for $|t| < 1$. *Hint*: See Corollary 5.5 in the textbook.
2) Insert the power series for $y(t)$ into the differential equation (3) and show that $a_{n+2} = \frac{(-1)^n}{n+1}$ for all $n \geq 0$.
3) Determine the sum function and the radius of convergence for the power series solution $\sum_{n=2}^{\infty} a_n t^n$. *Hint*: See Appendix B in the textbook.

Part B is finished. Remember to answer part A (Multiple Choice).